وزارة التعليم العالي والبحث العلمي

Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

BADJI MOKHTAR –ANNABA UNIVERSITY
UNIVERSITE BADJI MOKHTAR ANNABA

جامعة باجي مختار - عنابة -

Faculté des Sciences

Département de Mathématiques

Année : 2023/2024

THÈSE

Présenté en vue de l'obtention du diplôme de Doctorat en mathématiques

SUR CERTAINS PROBLÈMES DIRECTS ET INVERSES LIÉS AU BILAPLACIEN

Filière
Mathématiques
Spécialité
Mathématiques et Applications
Par

BOUSLAH ZINEB

DIRECTEUR DE THÈSE : Hacene SAKER Prof . U.B.M. ANNABA

Devant le jury

PRESIDENT :	Ahmed Salah CHIBI	Prof.	U.B.M. ANNABA
EXAMINATEUR :	Faouzia REBBANI	Prof.	ENSTI. ANNABA
EXAMINATEUR :	Lahcene CHORFI	Prof.	U.B.M. ANNABA
EXAMINATEUR :	Abdelhak DJEBABLA	Prof.	U.B.M. ANNABA
INVITÈ:	Abdelhak HADJ	M.C.B.	HNS-RE2SD. BATNA

وزارة التعليم العالي والبحث العلمي

Ministry of Higher Education and Scientific Research

BADJI MOKHTAR-ANNABA UNIVERSITY
UNIVERSITE BADJI MOKHTAR-ANNABA

جامعة باجي مختار - عنابة -

Faculty of Sciences
Department of Mathematics

Year: 2023/ 2024

THESIS

Presented with a view to obtaining the doctorate degree

ON CERTAIN DIRECT AND INVERSE PROBLEMS RELATED TO THE BILAPLACIAN

Stream
Mathematics
Specialty
Mathematics and Applications
Presented by:

Zineb Bouslah

Supervisor: SAKER Hacene Prof. U.B.M. ANNABA

In front of the jury

CHAIRMAN:	CHIBI Ahmed Salah	Prof.	U.B.M. ANNABA
EXAMINER:	REBBANI Faouzia	Prof.	ENSTI. ANNABA
EXAMINER:	CHORFI Lahcene	Prof.	U.B.M. ANNABA
EXAMINER:	DJEBABLA Abdelhak	Prof.	U.B.M. ANNABA
INVITED:	HADJ Abdelhak	M.C.B.	HNS-RE2SD. BATNA

Acknowledgements

*First and foremost, I would like to express my gratitude to **Allah** above all, for the strength and determination He has bestowed upon me to complete this work.*

*I extend my deepest gratitude to my thesis advisor, **M.Hacene Saker**, professor at the University of Annaba, for his assistance, support, guidance, and the trust he placed in me by agreeing to supervise my work.*

*I also extend my sincere thanks to all the members of the jury for the honor they to upon me by agreeing to examine this thesis. I would like to express my gratitude **M. Chibi Ahmed Salah** professor at the University of Annaba, for agreeing to chair this thesis committee.*

*Also **Mme. REBBANI Faouzia**, professor at the University of Annaba, **M. Chorfi Lahcene,** professor at the University of Annaba, **M. Djebabla Abdelhak,** professor at the University of Annaba, for the honor they bestowed upon me by participating in this committee as examiners, and for the time devoted for reading this thesis.*
*I want to express my gratitude to **M. Hadj Abdelhak,** MCB at the Higher National School of Renewable Energies, for his valuable help and insightful advice, as well as for agreeing to participate in the defense of my thesis defense.*

I extend my sincere thanks to my parents, my Husband, my son and my daughter and all my family for their continuous companionship, assistance, support, and encouragement during the entire process of preparing this thesis.
I extend my sincere thanks to my friends from all over the world, especially to Khélifa Imène, who supported me spiritually and morally during this journey, and also to Meriem Boussebha and Radia Halilou.

Finally, I would like to thank everyone who has contributed, directly or indirectly, to the completion of this thesis.

ABSTRACT

This thesis is concerned with the Robin's inverse problem of determining the geometric shape of a missing part Γ_c of the boundary of a $2D$ simply connected domain Ω from a single Riquier-Neumann data on the accessible part Γ_m of a biharmonic function u in Ω. Our approach extends the method that has been suggested by Kress and Cakoni for the Laplacian [10]. The identification of Γ_c is based on a system of nonlinear ill-posed integral equations which is solved iteratively by linearisation using Fréchet derivatives. We present the mathematical spirit of the proposed method and, in particular, establish the injectivity for the linearised system for Robin's coefficients. The performance of the method is illustrated by numerical examples.

KEYWORDS: Biharmonic Equation, Boundary Value Problem, Nonlinear Inverse Problem.

MSC CLASSIFICATION: 31A30, 45Q05, 35R30.

RÉSUMÉ

Cette thèse s'intéresse au problème inverse de Robin consistant à déterminer la forme géométrique d'une partie manquante Γ_c de la frontière d'un domaine $2D$ simplement connexe Ω à partir d'une seule mesure de données de Riquier-Neumann sur la partie accessible Γ_m d'une fonction biharmonique u dans Ω. Notre approche étend la méthode qui a été suggérée par Kress et Cakoni pour le Laplacien [10].L'identification de Γ_c est basée sur un système d'équations intégrales non linéaires et mal posées, résolu itérativement par linéarisation à l'aide des dérivées de Fréchet. Nous présentons l'idée mathématique de la méthode proposée et, en particulier, établissons l'injectivité du système linéarisé des coefficients de Robin. Les performances de la méthode sont illustrées par des exemples numériques.

Mots clés: Équation biharmonique, Problème de valeur aux limites, Problème inverse nonlinéaire.

ملخص

موضوع هذه الأطروحة يرتكز أساسا على دراسة مسألة Robin العكسية المتمثلة في التعرف على قطعة من الحدود Γ_c من حدود المجال المتصل ببساطة $\Omega \subset 2D$ ذو الحدود $\partial\Omega = \Gamma_c \cup \Gamma_m$، من خلال معطيات Riquier-Neumann للدالة ثنائية التوافق u في Ω. يوسع منهجنا الطريقة التي اقترحها Kress و Cakoni من أجل معادلة Laplace. يعتمد تحديد Γ_c على نظام من المعادلات التكاملية الغير خطية سيئة الطرح بمعنى Hadamard مكافئة للمسألة العكسية. يتم توضيح أداء الطريقة من خلال مجموعة من الأمثلة العددية.

الكلمات المفتاحية: المعادلة الثنائية التوافقية، مشكلة القيمة الحدودية، المسألة العكسية للمعادلات التكاملية .

Contents

General Introduction

Many problems in physics, chemistry, biology and geophysics are modeled by systems of differential or partial differential equations. When all the parameters of the system are known, such as the initial and boundary conditions, the coefficients involved in the equations, as well as the spatial domain, we can solve directly the problem. These problems are called direct problems. When one (or more) of the components of the problem is missing the equation can no longer be solved without additional information and the resolution of the equation is no longer direct but inverse.

Inverse problems are some of the most important mathematical problems in science and mathematics because they tell us about parameters that we cannot directly observe. They have wide application, which can be found in many areas for example [12] (cf. P.C.Sabatier [1], Isakov [2, 3], ofta [4])

- Engineering (detect a corrosion surface).
- Medical imaging (ultrasound, scanners, X-rays).
- Petroleum engineering (prospecting using seismic, magnetic methods, identification of permeabilities in a reservoir).
- Hydrogeology (identification of hydraulic permeabilities).
- Chemistry (determination of reaction constants).
- Image processing (restoration of blurred images).
- Radar and underwater acoustics (determination of the shape of an obstacle).

- Seismology (locate the origin of an earthquake based on measurements made by several seismic stations).

- Meteorology (the evolution of the weather, which is based on the identification of initial conditions that we can find using all the available temperature data, etc.) and data from previous weather forecasts, because of these known information, we go back to the initial conditions which will then make it possible to predict the evolution of different meteorological parameters (temperature, humidity, *etc*).

Direct and Inverse Problems

An inverse problem consists of finding the causes of a system based on the observation of its effects. These problems are difficult to solve in view of the fact of perturbation in the experimental data. Thus this problem is the opposite of the one called direct, consisting of finding the effects, the causes being known.
We can consider various inverse problems, for example reconstruction of the boundary conditions, reconstitution of the past state of a system knowing its current state, identification of parameters (the coefficients of an equation or the source term), reconstruction of the geometry of a domain.

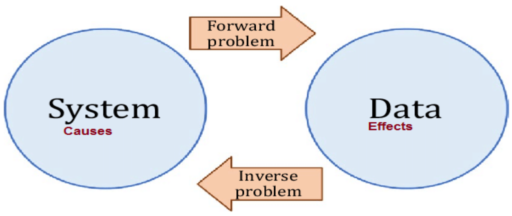

Figure 1: Direct and inverse problems

From a mathematical point of view, these problems fall into two large groups. On the one hand, there are linear problems which boil down to the resolution of a first-kind integral equation in continuous case or to the resolution of a system in the discrete case. The use of functional analysis and linear algebra makes it possible to obtain precise results and efficient algorithms. On the other hand, there are nonlinear problems, where the most known question is the estimation of parameters in differential or partial differential equations. Non-linear problems can be divided into two categories depending on whether the parameter we are trying to estimate is a vector or a function. Therfore nonlinear problems are more difficult and there are fewer general results (Kern, [37]). Typically, inverse problems pose challenges as ill-posed problems, and the definition of the latter will be provided in the subsequent section.

Ill-posed Problems and Regularisation

In 1923, the renowned French mathematician **J. HADAMARD** wrote his famous book on partial differential equations and their physical significance [20]. This work was the starting point for the development of the notion of well-posed problem in mathematical physics. A well-posed problem is the one where these conditions are satisfied:

- a solution exist,
- the solution is unique,
- the solution depends continuously upon the initial or boundary data.

If any of these conditions is not satisfied, the problem becomes ill-posed. If we contemplate the given operational equation:

$$\mathcal{K}u = v, \quad u \in E, \quad v \in F, \tag{1}$$

Considering normed spaces E and F and a defined operator $\mathcal{K}$ from E to F.

The problem (1) is said to be well posed if $\mathcal{K} : E \longrightarrow F$ is bijective. In such a cases, the inverse operator $\mathcal{K}^{-1} : F \longrightarrow E$ is continuous. Alternatively, if these conditions are not met, we characterize the equation as ill-posed.

Establishing existence is not typically a significant challenge, it is usually feasible to restore existence by relaxing the solution set (restricting to $N(\mathcal{K})$). Non-uniqueness of the solution is a more serious problem, if a problem has several solutions, we need a way of choosing between them. This requires additional information (a priori information). The lack of continuity of $\mathcal{K}^{-1}$ is unquestionably the most challenging aspect, meaning that a slight perturbation in the data can result in a change in the solution u.

Mathematical physics has long ignored ill-posed problems, considering them either devoid of physical meaning or reflecting inadequate modeling. The current reality is quite different: the fundamentally ill-posed nature of certain problems is recognized and motivates numerous research in mathematics [5].

The general methods of mathematical analysis have been well adapted for the solutions of well-posed problems. However, it was not clear in what sense ill-posed problems can have solutions. Several mathematicians like Tikhonov, John, Lavrentiev, Ivanov and others worked to develop theory and methods for solving ill-posed problems. They were able to give a precise mathematical definition of "approximate solutions" for a fairly large class of problems. Today, these problems are a very rich area of research and full of mathematical questions. To overcome this ill-posed character, there are what we call regularization techniques.

The regularization of an ill-posed problem consists of replacing the initial ill-posed problem with an approximate well-posed problem. There are several approaches to solve ill-posed problems, namely the Tikhonov method, spectral truncation for linear problems, Landweber method and iterative methods for non-linear problems *etc*.

The primary focus of this research is to investigate a geometric inverse problem associated with the bilaplace equation, employing the boundary integral equations method. The structure of this thesis is organized as follows:
In the initial chapter, titled "Preliminaries," we recall the essential tools essential for a

comprehensive comprehension of this research.

Chapter two is divided into two distinct sections. The first section introduces an approach to the theory of integral equations, with a particular focus on the direct method applied to the biharmonic operator in a simply connected domain in $\mathbb{R}^2$. By using the Green's formula, we prove that the associated system of integral equations is strongly elliptic and possesses the coercivity property. This in-depth analysis provides a thorough understanding of the mathematical properties of the given problem, laying the necessary groundwork for an efficient resolution of this system of integral equations. The second section of the chapter focuses on the study of an inverse problem associated with the biharmonic operator. We recall the indirect boundary integral equation method, and at the end, we prove that the inverse problem can be equivalently formulated as a system of the nonlinear integral equations.

In the third and last chapter, we propose an algorithm to solve the inverse problem, using a Newton-type method in combination with the Tikhonov regularization technique to achieve stability. A series of numerical examples is discussed in the final section.

Chapter 1

Preliminaries

This chapter contains basic knowledge on the notions of functional analysis and some elements of the theory of inverse problems which will be used throughout this work. [6, 39, 36, 53, 21, 38, 55, 23, 45, 46, 57]

1.1 Functional Spaces

Consider an open subset Ω of $\mathbb{R}^2$. It is denoted by:

- $C(\Omega)$: represent the set of functions $f : \Omega \to \mathbb{R}$ that are continuous. If f is bounded, we define $\|f\|_\infty = \max_{x\in\Omega} |f(x)|$.
- $C^m(\Omega)$, where $m \in \mathbb{N}$, denotes the space of functions on Ω that are continuously differentiable up to order m.
- $C^\infty(\Omega) = \cap_{m\in\mathbb{N}} C^m(\Omega)$: denotes the space of infinitely differentiable functions on Ω.
- $C_c(\mathbb{R})$: the set of functions $f \in C(\mathbb{R})$ such that $f(x) = 0$ for all $x \in \mathbb{R} \setminus K$, where K is a compact set.
- $C_B(\mathbb{R})$: the space of continuous and bounded functions.
- $\mathcal{D}(\mathbb{R})$: defined as the space of C^∞ functions on $\mathbb{R}$ with compact support.
- $\mathcal{D}'(\mathbb{R})$: the topological dual of $D(\mathbb{R})$.
- $S(\mathbb{R})$: designates the space of rapidly decreasing functions on $\mathbb{R}$.

- $S'(\mathbb{R})$: the topological dual of $S(\mathbb{R})$ or space of tempered distributions on $\mathbb{R}$ (Schwartz space).

Definition 1.1.1. *Let $p \in \mathbb{R}$ with $1 \leq p < \infty$,*

$$\mathbb{L}^p = \{f : \Omega \to \mathbb{R},\ \text{measurable for which} \int_\Omega |f(x)|^p\, dx < \infty\}$$

When $p = \infty$, the set $\mathbb{L}^\infty$ consists of measurable functions $f : \Omega \to \mathbb{R}$ such that there exists a non-negative constant C with the property that $|f(x)| \leq C$ almost everywhere on Ω.

Theorem 1.1.1. *The vector space $\mathbb{L}^p$ forms a Banach space equipped with the norm:*

$$\|f\|_{\mathbb{L}^p} = \left(\int_\Omega |f(x)|^p\right)^{\frac{1}{p}}.$$

Theorem 1.1.2. *Consider $1 \leq q < p \leq \infty$. The space $(\mathbb{L}^p(\Omega), \|.\|_p)$ is defined. If Ω has finite measure, then we have the inclusion relationship:*

$$\mathbb{L}^\infty(\Omega) \subset \mathbb{L}^q(\Omega) \subset \mathbb{L}^p(\Omega) \subset \mathbb{L}^1(\Omega)$$

Definition 1.1.2. *For $f \in \mathbb{L}^1(\mathbb{R})$, the Fourier transform is defined as follows:*

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(s) e^{-is\xi} ds, \quad \xi \in \mathbb{R}$$

and the Fourier inversion formula on $\mathbb{R}$ is given by

$$\mathcal{F}(\hat{f})(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(s) e^{is\xi} d\xi, \quad s \in \mathbb{R}.$$

Remark 1.1.1. *[50] $\mathcal{F} : S \to S$ is an isomorphism and*

$$\left\|\hat{f}\right\|_{\mathbb{L}^2} = \|f\|_{\mathbb{L}^2}$$

Definition 1.1.3. *(**Fourier Series**) Consider $\varphi \in \mathbb{L}^2[0, 2\pi]$. The series*

$$\sum_{m=-\infty}^{\infty} a_m e^{imt} \tag{1.1}$$

with

$$a_m := \frac{1}{2\pi}\int_0^{2\pi} \varphi(t)e^{-imt}dt,$$

is termed as the Fourier series of φ, and its coefficients a_m are referred to as the Fourier coefficients of φ.

1.2 Bounded and Compact Linear Operators

We denote by H a Hilbert space on $\mathbb{C}$ or $\mathbb{R}$ with the inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$, and let X and Y be two Banach spaces. The space of continuous linear maps from X to Y is denoted by $\mathcal{L}(X,Y)$, and it is equipped with the norm:

$$\|\mathcal{A}\|_{\mathcal{L}(X,Y)} = \sup_{x\neq 0}\frac{\|\mathcal{A}x\|_Y}{\|x\|_X}.$$

Definition 1.2.1. *A linear operator $\mathcal{A} : X \to Y$ is considered bounded if there exists a positive constant $C > 0$ such that*

$$\|\mathcal{A}x\|_Y \leq C\,\|x\|_X\,,\ \ for\ all\ x \in X.$$

Theorem 1.2.1. *Consider a continuous and bijective linear operator $\mathcal{A}$ from X to Y. Then $\mathcal{A}$ is invertible, and furthermore, $\mathcal{A}^{-1}$ is a continuous linear operator from Y to X.*

The graph of an operator $\mathcal{A} : X \to Y$, is the vector subspace of $X\times\ Y$ defined by:

$$G(\mathcal{A}) = \{(u,\mathcal{A}u) \in X\times\ Y,\ u \in D(\mathcal{A})\}$$

Definition 1.2.2. *The operator $\mathcal{A}$ is termed closed if its graph is a closed subset of $X\times Y$.*

Definition 1.2.3. *Consider a linear operator $\mathcal{A}$ with a definition domain $D(\mathcal{A})$ that is dense in H. Let $D(\mathcal{A}^*)$ represent the set of the vectors $w \in H$ for which there exists $f \in H$ such that*:

$$(\mathcal{A}u,w) = (u,f),\ for\ all\ u \in D(\mathcal{A}).$$

For all $w \in D(\mathcal{A}^*)$, we pose

$$\mathcal{A}^*w = f.$$

We call $\mathcal{A}^*$ the adjoint operator of $\mathcal{A}$.

Definition 1.2.4. *Let $\mathcal{A}$ be a linear operator of definition domain $D(\mathcal{A})$ dense in H. We call $\mathcal{A}$ self-adjoint if $\mathcal{A} = \mathcal{A}^*$ i.e;*

$$D(\mathcal{A}) = D(\mathcal{A}^*) \text{ and } (\mathcal{A}x, z) = (x, \mathcal{A}z), \quad \forall x, z \in D(\mathcal{A}).$$

Proposition 1.2.1. *A self-adjoint operator is characterized as being closed.*

Proposition 1.2.2. *Let A be an invertible self-adjoint operator. Then $\mathcal{A}^{-1}$ is self-adjoint.*

Definition 1.2.5. *We say that a self-adjoint operator is positive if:*

$$\forall x \in D(\mathcal{A}),\ (\mathcal{A}x, x) \geq 0.$$

1.2.1 Compact Operators

Definition 1.2.6. *Let $\mathcal{A} \in \mathcal{L}(X, Y)$. is said to be compact if the image of any bounded subset of X is a relatively compact subset (with compact closure) in Y.*

Theorem 1.2.2. *Consider $\mathcal{A} \in \mathcal{L}(X, Y)$. If $\mathcal{A}$ has finite rank, it implies that $\mathcal{A}$ is compact.*

Theorem 1.2.3. *Let X, Y and Z be Banach spaces and let $\mathcal{A} : X \to Y$ and $\mathcal{B} : Y \to Z$ be bounded linear operators. Then, the composition $\mathcal{B}\mathcal{A} : X \to Z$ is compact if either $\mathcal{A}$ or $\mathcal{B}$ is compact.*

Corollary 1.2.1. *If the dimension of X is infinite and $\mathcal{A}$ is compact, then $\mathcal{A}^{-1}$ does not belong to $\mathcal{L}(X)$.*

Definition 1.2.7. *(**Jordan Measures**) [19] Consider a subset $\mathcal{U}$ of $\mathbb{R}^n$. It is measurable in the Jordan sense if its interior and exterior Jordan measures are equal.*

$$\eta_*^J(U) = \sup_{E \subset U} \quad measure(E) = \inf_{E \subset U} \quad measure(E) = \eta_J^*(U)$$

Theorem 1.2.4. *[19] Let $U \subset \mathbb{R}^m$ be a compact, nonempty, and measurable set in the Jordan sense, with its boundary aligning with the closure of its interior. Consider $E : U \times U \longrightarrow \mathbb{R}$*

as a continuous function. The linear operator $\mathcal{A} : C(U) \longrightarrow C(U)$, defined by

$$(\mathcal{A}\psi)(x) = \int_U E(x,z)\psi(z)dz, \quad x \in U$$

is referred to as an integral operator with a continuous kernel E. It is a bounded linear operator with

$$\|\mathcal{A}\|_\infty = \max_{x \subset U} \int_U |E(x,z)| dz.$$

Moreover, the operator $\mathcal{A}$ can be extended to the integral operator $\mathcal{A} : C(U) \longrightarrow C(M)$ given by

$$(\mathcal{A}\psi)(x) = \int_U E(x,z)\psi(z)dz, \quad x \in M$$

where $E : M \times U \longrightarrow \mathbb{R}$ is a continuous function, $M \subset \mathbb{R}^n$ is a compact set, and $m \neq n$.

Proof. see [40] □

Definition 1.2.8 (**Weakly Singular Kernel**). *E is characterized as a weakly singular kernel if it is defined and continuous for all $x, z \in U \subset \mathbb{R}^n$, where $x \neq z$, and there exists $c > 0$ such that*

$$|E(x,z)| \leq c|x-z|^{\alpha - n}, \forall x \neq z; 0 < \alpha \leq n$$

Theorem 1.2.5.

- *The continuous-kernel of the integral operators form a class of compact linear operators $C(U)$.*
- *Integral operators with a continuous kernel are considered compact linear operators on $L^2(U)$.*
- *The integral operator with a weakly singular kernel is a compact operator on $C(U)$.*

Proof. See [39, 40] □

Theorem 1.2.6. *(**Riez Theorem**) Consider X as a normed space, and let $\mathcal{A} : X \longrightarrow X$ be a compact linear operator.*

i) The null space $N(I - \mathcal{A}) = \{x \in X : x = \mathcal{A}x\}$ is finite-dimensional and the range $(I - \mathcal{A})(x)$ is closed in X.

ii) *If the homogeneous equation $x - \mathcal{A}x = 0$ has only trivial solution $x = 0$, then the inhomogeneous equation $x - \mathcal{A}x = y$ is uniquely solvable for any $y \in Y$ and x depends continuously on y.*

1.3 Fredholm Operators

Fredholm operators are an important concept in functional analysis and operator theory. A Fredholm operator is a class of bounded linear operators acting on function spaces. Here is a formal definition and some associated properties:

Definition 1.3.1. *Let H and K be two Hilbert spaces (Banach spaces equipped with a complete inner product), and let $\mathcal{A} : H \longrightarrow K$ be a bounded linear operator. The operator $\mathcal{A}$ is said to be Fredholm if the following two conditions are satisfied:*

i) *Finite Kernel Property: The kernel of the operator $\mathcal{A}$, denoted by $Ker(\mathcal{A})$, is a Hilbert space of finite dimension.*

ii) *Closed Image Property: The image of the operator $\mathcal{A}$, denoted by $Im(\mathcal{A})$, is a closed subspace of K.*

Properties:

- Fredholm Index: The Fredholm index is defined as the difference between the dimension of the image and the dimension of the kernel. Mathematically, the index $n(\mathcal{A})$ is given by:

$$n(\mathcal{A}) = dim(Im(\mathcal{A})) - dim(Ker(\mathcal{A}))$$

- Fredholm Characteristic: A Fredholm operator is said to be of finite order if it has finite-dimensional kernel and image.

- Fredholm Equation: A Fredholm equation problem is an equation of the form:

$$\varphi(x) - \lambda \int_{\Omega} E(x,z)\varphi(z)dz = f(x)$$

where φ is unknown, E is a given kernel, λ is a parameter, and f is a given function. Solutions to such equations are often key in the study of Fredholm operators.

- Stability: Fredholm operators are often studied in terms of stability, exploring how small changes in the operator lead to changes in kernel and image spaces.

Remark 1.3.1. *Fredholm operators have applications in various fields of applied mathematics, including integral equations, differential equations theory, and other branches of functional analysis. They provide a useful framework for studying complex problems involving integral equations and differential operators.*

1.4 Pseudo-Differential Operator

Pseudo-differential operators are a class of linear operators that generalize differential operators. They are widely used in harmonic analysis, mathematical physics, and distribution theory. Here is an informal definition of these operators:

Definition 1.4.1. *Let $m \in \mathbb{R}$ and S^m be the set of symbols of class m. A pseudo-differential operator of order m is defined as follows: Let $h > 0$ and let $a(x,\xi,h)$ be a complex-valued C^∞ function defined for $x,\xi \in \mathbb{R}^n$ and $h > 0$. This function is called the symbol of the pseudo-differential operator.*
The pseudo-differential operator P_h associated with the symbol a is given by:

$$P_h\varphi(x) = \frac{1}{(2\pi h)^2}\int e^{\frac{i(x-z).\xi}{h}} a(\frac{x-z}{2},\xi,h)\varphi(z)dzd\xi$$

Here, φ is a test function in $\mathcal{D}(\mathbb{R}^n)$.
This integral is an oscillatory integral that depends on the constant h and the variables x and ξ. The parameter h is often referred to as the "Planck constant [1] " in the context of quantum mechanics.

Pseudo-differential operators are defined to generalize differential operators and incorporate dispersion effects, making them useful in the study of partial differential

[1] The Planck constant (h) is a fundamental value in quantum physics, with a value of approximately 6.626×10^{-34}

equations, integral equations, and other mathematical areas. Analytical properties of symbols $a(x,\xi,h)$ play a crucial role in the analysis and theory of pseudo-differential operators.

1.5 Sobolev Spaces

Definition 1.5.1. *(**Lax-Milgram Theorem**) Consider V as a Hilbert space, and let $a : V \times V \longrightarrow \mathbb{R}$ be a bilinear form. Then, $a(u,w)$ can be expressed as:*

a) There exists a positive constant θ such that

$$|a(u,w)| \leq \theta \|u\|_V \|w\|_V \quad \forall \quad u,w \in V.$$

b) Coercive or elliptic if there exists a positive constant η such that

$$a(w,w) \geq \eta \|w\|_V^2, \quad w \in V.$$

Additionally, consider $l : V \longrightarrow \mathbb{R}$ as a bounded linear functional. Then, there exists a unique solution u in V satisfying

$$a(u,w) = l(w),$$

for all w in V.

Definition 1.5.2. *(**Sobolev Spaces** $W^{m,p}$) Let $W^{m,p}$ be a sobolev space given as*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : \exists g_\beta \in L^p(\Omega), D^\beta u = g_\beta \quad \forall \quad |\beta| \leq m \quad in \quad D'(\Omega)\}$$

where $\beta = (\beta_1, \beta_2, ...)$ with β_i are positive integers,

$$|\beta| = \sum_{i=1}^{n} \beta_i \quad and \quad D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} ... \partial x_n^{\beta_n}}$$

with the norm

$$\|u\|_W^{m,p}(\Omega) = (\sum_{0 \leq |\beta| \leq m} \|D^\beta u\|_{L^p(\Omega)})^{\frac{1}{2}}$$

is a Banach space. For $m \in \mathbb{N}$, we represent the Sobolev space $H^m(\Omega)$ as:

$$H^m(\Omega) = \left\{ u \in \mathbb{L}^2(\Omega) : \text{for all} \beta, \left|\beta\right| \leq m, \exists g_\beta \in \mathbb{L}^2(\Omega) : g_\beta = \partial^\beta u \text{ in weak sense} \right\}.$$

We introduce on $H^m(\Omega)$ the inner product

$$\langle u, w \rangle_m = \sum_{\left|\beta\right| \leq m} \left\langle \partial^\beta u, \partial^\beta w \right\rangle, \tag{1.2}$$

is a Hilbert space

Definition 1.5.3. *(**Sobolev Space** $H^s(\Omega)$) Let $s \in \mathbb{R}$. We define the Sobolev space*

$$H^s(\mathbb{R}) = \{ f \in S^{'}(\mathbb{R}), \ \left(1 + \xi^2\right)^{\frac{s}{2}} \widehat{f} \in \mathbb{L}^2(\mathbb{R}) \},$$

with the norm

$$\|f\|_{\mathbb{H}^s} = \left(\int_{\mathbb{R}} \left(1 + \xi^2\right)^s \left|\widehat{f}(\xi)\right|^2 d\xi \right)^{\frac{1}{2}}.$$

For $s = 0$ we note

$$\|f\|_{L^2(\mathrm{R})} = \|f\|_0.$$

Remark 1.5.1. *If $s \geq 0$, then $H^s(\mathbb{R}) \subset \mathbb{L}^2(\mathbb{R})$.*

Furthermore, $H^{-s}(\mathbb{R}^n)$ can be considered as an isometric realization of the dual space to $H^s(\mathbb{R}^n)$, denoted by $(H^s(\mathbb{R}^n))'$. In other words, $H^{-s}(\mathbb{R}^n) = (H^s(\mathbb{R}^n))' \quad \forall s \in \mathbb{R}$ with the corresponding norm given by

$$\|u\|_{H^{-s}(\mathbb{R}^n)} = \sup_{\|w\|_{H^s(\mathbb{R}^n)} \neq 0} \frac{|(u, w)|}{\|w\|_{H^s(\mathbb{R}^n)}}$$

We can define the space $H^s(\Omega)$ of order s as follows:

$$H^s(\Omega) = \{ u \in D'(\Omega) : u = w|_\Omega \quad \forall w \in H^s(\mathbb{R}^n) \}$$

with the induced norm given by

$$\|u\|_{H^s(\Omega)} = ((u, u)_{H^s(\Omega)})^{\frac{1}{2}} = \inf_{w \in H^s(\mathbb{R}^n)} \{ \|w\|_{H^s(\mathbb{R}^n)} | \quad u = w|_\Omega \}$$

Definition 1.5.4. *We define the Sobolev spaces on* Ω

$$\begin{aligned} \tilde{H}^s(\Omega) &= \overline{D(\Omega)} \quad in \quad H^s(\mathbb{R}^n), \\ H_0^s(\Omega) &= \overline{D(\Omega)} \quad in \quad H^s(\Omega) \end{aligned} \tag{1.3}$$

This definition entails that

$$\begin{aligned} \tilde{H}^s(\Omega) &= \{u \in H^s(\mathbb{R}^n) | \quad supp \quad u \subset \bar{\Omega}\}, \\ H_0^s(\Omega) &= \{u \in H^s(\Omega) | \quad supp \quad u \subset \bar{\Omega}\} \end{aligned} \tag{1.4}$$

The following inclusions are valid:

$$\tilde{H}^s(\Omega) \subseteq H^s_{\bar{\Omega}}(\Omega), \quad \tilde{H}^s(\Omega) \subseteq H^s_0(\Omega)$$

where

$$H^s_{\bar{\Omega}}(\Omega) = \{u \in H^s(\Omega) : \quad supp \quad u \subset \bar{\Omega}\}$$

and

$$D(\bar{\Omega}) = \{u : u = w|_{\Omega} \quad \forall \quad w \in D(\mathbb{R}^n)\}$$

It is noteworthy that $D(\bar{\Omega})$ is dense in $H^s(\Omega)$ due to the density of $D(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$.

Definition 1.5.5 (Lipschitz Domains). *The open set* Ω *is considered a Lipschitz domain if its boundary, denoted by* $\partial\Omega$ *and represented by* $\Gamma = \overline{\Omega} \cap (\mathbb{R}^n \backslash \Omega)$, *is compact, and if there exist finite families* $\{F_i\}$ *and* $\{\Omega_i\}$ *that satisfy the following conditions:*

- *forms a finite open cover of* Γ, *in other words, each* F_i *constitutes an open subset of* $\mathbb{R}^n$, *with* Γ *being a subset of the union* $\cup_i F_i$.
- *Every* Ω_i *can be converted to a Lipschitz hypograph By means of a rigid motion.*
- *The set* Ω_i *satisfies* $F_i \cap \Omega = \Omega_i \cap F_i$ *for all* i.

Note that if Ω *is a Lipschitz hypograph, as discussed in [44], then*

$$\Gamma = \left\{x \in \mathbb{R}^{n-1} \; : \; x_n = \kappa(x') \quad \forall \quad x' = (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}\right\}$$

with κ is Lipschitz, i.e., if there exists constant ϑ such that

$$|\kappa(x') - \kappa(z')| \leq \vartheta |x' - z'| \,, \; \forall x', z' \in \mathbb{R}^{n-1}$$

For any Lipschitz domain Ω, there exists a surface measure ρ and an outward unit normal n that exist ρ-almost everywhere on Γ. If Ω is the Lipschitz hypograph, (this is established in Theorem 3.34 of [44]) then.

$$d\rho_x = \sqrt{1 + |\nabla\kappa(x')|}dx' \,, \quad n(x) = \frac{(-\nabla\kappa(x'), 1)}{\sqrt{1 + |\nabla\kappa(x')|}} \,, \qquad for\ x \in \Gamma \tag{1.5}$$

Example 1.5.1. ▶ *Similarly, for $0 < \alpha < 1$, we define a $C^{m,\alpha}$ domain by imposing the condition that the mth-order partial derivatives of be Holder-continuous with exponent α, i.e.,*

$$|\partial^\beta \kappa(x') - \partial^\beta \kappa(z')| \leq \vartheta |x' - z'|^\alpha \,, \; \forall \quad x', z' \in \mathbb{R}^{n-1} \text{ and } |\beta| = m$$

A Lipschitz domain is equivalent to a $C^{0,1}$ domain. When $\Gamma \in C^{0,1}$, the boundary is referred to as a Lipschitz boundary possessing a strong Lipschitz property, and Ω is designated as a robust Lipschitz domain according to [26]. Moreover:

- *In $\mathbb{R}^2$, any polygon qualifies as a Lipschitz domain.*
- *For a simply-connected domain, an ellipse is considered a Lipschitz domain.*
- *If $\Gamma \in C^\infty$, we refer to Γ as having a smooth boundary.*

Example 1.5.2 (Parametrization). *A planar curve that is parameterized represents the path traced out by a point in the plane.*

$$w(t) = (w_1(t), w_2(t)), \qquad 0 \leq t \leq L \tag{1.6}$$

Considering a smooth boundary denoted by $\partial\Omega$ and a parameterized curve $w \in C^2(0, L)$, where $|w'(t)| \neq 0$, and t spans an interval from 0 to L, we introduce the surface measure dt and define the exterior unit normal $n(t)$ orthogonal to the curve $\partial\Omega$ at $w(t)$. The expressions *are given by:*

$$d\sigma_x = \sqrt{w_1'(t)^2 + w_2'(t)^2}dt\ , \quad n(t) = \frac{[w'(t)]^{\perp}}{\sqrt{w_1(t)^2 + w_2(t)^2}}\ , \quad for\ t \in [0, L] \tag{1.7}$$

where $[w'(t)]^{\perp}$ *represents the vector orthogonal to* $[w'(t)]$.

Theorem 1.5.1. *For* Ω *as a Lipschitz domain, the following equivalences hold:*

$$H^{-s}(\Omega) = \tilde{H}^s(\Omega)' \quad and \quad \tilde{H}^{-s}(\Omega) = H^s(\Omega)'. \quad \forall s \in \mathbb{R}.$$

For $s \geq 0$, *we define*

$$\tilde{H}^s(\Omega) = \left\{u \in L^2(\Omega)\ :\ \tilde{u} \in H^s(\mathbb{R}^n)\right\} \subseteq H_0^s(\Omega)$$

where $\tilde{u}$ *is the extension of* u *by zero:*

$$\tilde{u}(x) = \begin{cases} u(x) & if\ x \in \Omega \\ 0 & if\ x \in \mathbb{R}^n \backslash \Omega \end{cases} \tag{1.8}$$

Moreover,

$$\tilde{H}^s(\Omega) = H_0^s(\Omega)\ provided\ s \notin \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\right\}$$

Proof. Refer to Theorems 3.30 and 3.33 in [44]. □

Definition 1.5.6 (The Spaces $L^2(\Gamma)$). *By defining* $L(\Gamma)$ *as the completion of* $C(\Gamma)$, *the space of all continuous functions on* Γ, *with respect to the norm:*

$$\|u\|_{L^2(\Gamma)} = \left\{\int_{\Gamma} |u(x)|^2 ds_x\right\}^{\frac{1}{2}}$$

Definition 1.5.7 (The Trace Spaces $H^s(\Gamma)$). *Consider a bounded domain* $\Omega \subset \mathbb{R}^n$ *with* $\Gamma = \partial\Omega$. *The spaces* $H^s(\Gamma)$ *are defined as follows:*

$$H^s(\Gamma) = \begin{cases} \{u|_{\Gamma} : u \in H^{s+\frac{1}{2}}(\mathbb{R}^n)\}, s > 0 \\ (H^{-s}(\Gamma))' \quad , s < 0 \\ L^2(\Gamma), \quad s = 0 \end{cases}$$

For $s \geq 0$ and $\Gamma \subset \partial\Omega$ as an open subset of the boundary, according to 1.5.4 and [9, 8],

$$\begin{aligned} H^s(\Gamma) &= \{u|_\Gamma : u \in H^s(\partial\Omega)\}, \\ \tilde{H}^s(\Gamma) &= \left\{u \in H^s(\partial\Omega) : Supp\ u \subset \overline{\Gamma}\right\}. \\ \tilde{H}^{-s}(\Gamma) &= H^{-s}_{\overline{\Gamma}}(\partial\Omega) \ = \ \left\{u \in H^{-s}(\partial\Omega) : supp\ u \subset \overline{\Gamma}\right\} \end{aligned}$$

with the norm

$$\|u\|_{H^s(\Gamma)} \ = \inf_{\{w \in H^s(\partial\Omega),\ w|_\Gamma = u\}} \left\{\|w\|_{H^s(\partial\Omega)}\right\}$$

We define $H^{-s}(\Gamma)$ as the dual space of $\tilde{H}^s(\Gamma)$. Additionally, the following inclusions are satisfied:

$$\tilde{H}^s(\Gamma) \subset H^s(\Gamma), \quad for\ s \geq 0.$$

$$\tilde{H}^s(\Gamma) \subset H^s(\Gamma) \subset L^2(\Gamma) \subset \tilde{H}^{-s}(\Gamma) \subset H^{-s}(\Gamma), \quad for\ s > 0.$$

For $s < 0$, the boundary spaces of negative orders $H^s(\Gamma)$ can be defined as the dual of $H^{-s}(\Gamma)$ with respect to the $L^2(\Gamma)$ scalar product. In other words, it is the completion of $L^2(\Gamma)$ with respect to the norm given by:

$$\|u\|_{H^s(\Gamma)} = \sup_{\|w\|_{H^{-s}(\Gamma)}=1} |(u, w)|$$

Theorem 1.5.2 (**The Trace Operator**). *Define the trace operator $\gamma : \mathcal{D}(\overline{\Omega}) \longrightarrow \mathcal{D}(\Gamma)$ by*

$$\gamma u = u|_\Gamma$$

where u belongs to the domain of $\overline{\Omega}$. If Ω is a $C^{k-1,1}$ domain, and $\frac{1}{2} < s < k$, then γ has a unique extension to a bounded linear operator

$$\gamma \ : \ H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\Gamma) \subset L^2(\Gamma)$$

and this extension possesses a continuous right inverse, as measured by the norm

$$\|u\|_{H^{s-\frac{1}{2}}(\Gamma)} = \inf_{\gamma\tilde{u}=u} \|\tilde{u}\|_{H^s(\Omega)}$$

Proof. See [44] □

Definition 1.5.8 (Trace Spaces on Curved Polygons in $\mathbb{R}^2$). *Consider a curved polygon $\partial\Omega$ in $\mathbb{R}^2$, where $\partial\Omega$ is composed of m simple C^∞-arcs denoted by $\Gamma_j,, j = 1,\ldots,m$. The closures $\overline{\Gamma} j$ of these arcs are C^∞, and the curve $\Gamma j+1$ follows Γ_j in the positive orientation. Let w_j represent the vertex at the endpoint of Γ_j and the starting point of Γ_{j+1}. For any $s \in \mathbb{R}$, let $H^s(\Gamma_j)$ be the standard Sobolev spaces defined on the pieces $\Gamma_j,, j = 1,\ldots,m$. Without loss of generality, assume that each Γ_j has a parametric representation given by*

$$x = w_j(t) \ for \ t \in \Omega_j = [a_j, b_j] \subset \mathbb{R}$$

where $w_j(a_j) = w_{j-1}$, $w_j(b_j) = w_j$, and $\Omega_j = (a_j, b_j)$. Here, $w_j \in C^\infty(\overline{\Omega})$.
Define the space (see [26] pp 186)

$$\tilde{H}^s(\Gamma_j) = \left\{\varphi \mid \varphi \circ w_j \in H^s(\Omega_j)\right\}$$

with the norm

$$\|\varphi\|_{\tilde{H}^s(\Gamma_j)} := \|\varphi \circ w_j\|_{H^s(\Omega_j)}$$

where $\|\cdot\|_{H^s(\Omega_j)}$ is defined in definition 1.5.3. Then, $\tilde{H}^s(\Gamma_j)$ is a Hilbert space with the inner product

$$(\varphi,\psi)_{\tilde{H}^s(\Gamma_j)} := \left(\varphi \circ w_j, \psi \circ w_j\right)_{H^s(\Omega_j)}$$

1.6 Potential Theory

Solving boundary problems for partial differential equations stands as a pivotal realm where integral equations find significant application. The systematic exploration of boundary problems in the latter half of the 19th century played a foundational role in shaping the theory of integral equations. This period marked the initiation of a productive interplay between these two domains within applied mathematics. The aim of this section is to expound the fundamental concepts in this field, particularly by introducing harmonic functions and surface potentials. For a more thorough exploration of potential theory, readers are encouraged to refer to the works of ([41],[14]).

1.6.1 Harmonic and Biharmonic Functions

We begin by providing a concise overview of the fundamental characteristics of harmonic and biharmonic functions, rooted in the early 19th-century potential theory. This field saw influential contributions from eminent figures such as Dirichlet, Gauss, Green, Riemann, and Weierstrass.

Definition 1.6.1. *A harmonic function is a function $u : \Omega \longrightarrow \mathbb{R}$, where Ω is an open subset of $\mathbb{R}^2$, that is twice continuously differentiable and satisfies Laplace's equation, given by $\Delta u = 0$. Here, Δu is defined as*

$$\Delta u = \sum_{i=1}^{2} \frac{\partial^2 u}{\partial x_i^2}$$

.

Definition 1.6.2. *A biharmonic function is a function $u : \Omega \longrightarrow \mathbb{R}$, where Ω is an open subset of $\mathbb{R}^2$, that is twice continuously differentiable and satisfies the biharmonic equation, $\Delta^2 u = 0$. The biharmonic operator $\Delta^2 u$ is defined as*

$$\Delta^2 u = \sum_{i=1}^{2} \frac{\partial^4 u}{\partial x_i^4} + 2 \sum_{i<j} \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2}.$$

Harmonic functions find applications in characterizing time-independent temperature distributions, as well as representing the electrostatic and magnetostatic field potentials, and velocity potentials of incompressible irrotational fluid flows. On the other hand, biharmonic functions emerge in the realm of continuum mechanics and elasticity theory. They play a crucial role in modeling thin structures that exhibit elastic responses to external forces.

The fundamental solution presented in the following theorem serves as a key tool for deducing many of the fundamental properties of harmonic and biharmonic functions. It's worth noting that throughout, we denote the Euclidean norm in $\mathbb{R}^2$ by $|.|$.

Theorem 1.6.1. *For all $x \neq z$ in $\mathbb{R}^2$, we introduce the fundamental solution of the Laplace equation and the bi-Laplace equation, denoted as harmonic and biharmonic in $\mathbb{R}^2 \setminus \{z\}$ for a fixed point $z \in \mathbb{R}^2$. The functions are defined as follows:*

$$E_1(x,z) := \frac{1}{2\pi} \ln \frac{1}{|x-z|}, \quad n = 2 \tag{1.9}$$

$$E_2(x,z) := \frac{1}{8\pi} |x-z|^2 \ln |x-z|, \quad n = 2 \tag{1.10}$$

The functions E_1 and E_2 satisfy, in the sense of distributions,

$$\Delta_x E_1(x,z) = \delta(x-z)$$

$$\Delta_x^2 E_2(x,z) = \delta(x-z).$$

Remark 1.6.1. *Every harmonic function can be viewed as a special case of a biharmonic function.*

Let Ω represent a bounded domain with a C^1 class, and let n denote the unit normal vector to the boundary $\partial\Omega$, directed outward from Ω.

Theorem 1.6.2. *[41] Let $u \in C^1(\bar{\Omega})$ and $w \in C^2(\bar{\Omega})$ we have Green's first theorem:*

$$\int_\Omega (u\Delta w + \nabla u \nabla w) dx = \int_{\partial\Omega} u \frac{\partial w}{\partial n} ds, \tag{1.11}$$

and for $u, w \in C^2(\bar{\Omega})$ we have Green's second theorem:

$$\int_\Omega (u\Delta w - w\Delta u) dx = \int_{\partial\Omega} (u \frac{\partial w}{\partial n} - w \frac{\partial u}{\partial n}) ds. \tag{1.12}$$

Remark 1.6.2. *Consider $u, w \in C^4(\bar{\Omega})$, the Green's formula for the biharmonic operator is expressed as:*

$$\int_\Omega (u\Delta^2 w - w\Delta^2 u) dx = \int_{\partial\Omega} (u \frac{\partial \Delta w}{\partial n} - \Delta w \frac{\partial u}{\partial n} + w \frac{\partial \Delta u}{\partial n} - \Delta u \frac{\partial w}{\partial n}) ds. \tag{1.13}$$

Theorem 1.6.3. ***(Green's Formula)***

Let $u \in C^2(\bar{\Omega})$ a harmonic function in Ω. Then

$$u(x) = \int_{\partial\Omega} \{\frac{\partial u}{\partial n}(z)E(x,z) - u(z)\frac{\partial E(x,z)}{\partial n(z)}\} ds_z, \quad x \in \Omega. \tag{1.14}$$

Remark 1.6.3. *Integral equation methods rely on an explicit formula that facilitates the deduction of the solution throughout the entire domain, leveraging boundary information. By employing Green's theorems (1.11, 1.12), we derive the integral representation formula for the solution of Laplace's equation.*

Theorem 1.6.4. ***(Integral Representation Formula)***
Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be the solution of Laplace's equation

$$\Delta u = 0 \quad dans \ \ \Omega.$$

Then

$$u(x) = \int_{\partial\Omega} \left\{ u(z)\frac{\partial E(x,z)}{\partial n_z} - E(x,z)\frac{\partial u(z)}{\partial n_z} \right\} ds_z, \tag{1.15}$$

where $E(x,z)$ is the fundamental solution of the Laplace equation defined by (1.9).

Theorem 1.6.5. ***(Cauchy-Kovalevskaya)*** *(see [41]) Let Ω be a simply connected domain, and suppose there exists a real-analytic nontrivial arc Σ contained in $\partial\Omega$. If f_k for all $k = 1,...,2N$ are real-analytic functions on Σ, then there exists a function u in a neighborhood of Σ satisfying $\Delta^N u = 0$ and $\frac{\partial^{k-1} u}{\partial n} = f_k$ for all $k = 1,...,2N$. The solution u is unique among real-analytic functions.*

Theorem 1.6.6. ***(Holmgren's)*** *(see [41]) Let Ω be a simply connected domain, and consider a real-analytic nontrivial arc Σ such that $\Sigma \subset \partial\Omega$. If u is smooth in a planar neighborhood ε of Σ and satisfies $\Delta^N u = 0$ for all $N \geq 1$ in $\varepsilon \cap \Omega$, with $\frac{\partial^{k-1} u}{\partial n} = 0$ on Σ for every $k = 1,...,2N$, then $u = 0$ in $\Omega \cap \varepsilon$ under the condition that the open set $\Omega \cap \varepsilon$ is connected.*

Theorem 1.6.7. ***(Maximum-Minimum Principle)*** *For any connected open set Ω, a harmonic function within Ω cannot attain its maximum or minimum values unless it is constant throughout.*

Corollary 1.6.1. *Consider a bounded domain Ω and let u be a harmonic function in Ω that is continuous on $\bar{\Omega}$. In such a case, the extremal values of u are reached on the boundary $\partial\Omega$.*

1.6.2 Simple and Double Layer Potentials

Definition 1.6.3. *For a function $\varphi \in C(\partial\Omega)$, the single-layer potential with density φ is defined as:*

$$u(x) := \int_{\partial\Omega} \varphi(z) E_1(x,z) ds_z, \;\; x \in \mathbb{R}^m \setminus \partial\Omega, \tag{1.16}$$

Definition 1.6.4. *or a function $\varphi \in C(\partial\Omega)$, the double-layer potential with density φ is defined as:*

$$v(x) := \int_{\partial\Omega} \varphi(z) \frac{\partial E_1(x,z)}{\partial n_z} ds_z, \;\; x \in \mathbb{R}^m \setminus \partial\Omega, \tag{1.17}$$

Theorem 1.6.8. *Assuming $\partial\Omega$ is of class C^2 and $\varphi \in C(\partial\Omega)$, the single-layer potential u with density φ is continuous throughout $\mathbb{R}^n$. On the boundary, we obtain:*

$$u(x) = \int_{\partial\Omega} E_1(x,z)\varphi(z) ds_z, \qquad x \in \partial\Omega,$$

where the integral is well-defined as an improper integral

Definition 1.6.5. *([26], [39]) We introduce the integral operators $\mathcal{A}$, $\mathcal{S}$, $\mathcal{B}'$, and $\mathcal{K}'$, mapping $C(\partial\Omega)$ to $C(\partial\Omega)$. For all $x \in \partial\Omega$, these operators are defined as follows:*

$$\begin{aligned}
(\mathcal{A}\varphi)(x) &= \int_{\partial\Omega} E_2(x,z)\varphi(z) ds_z, \qquad x \in \partial\Omega, \\
(\mathcal{S}\varphi)(x) &= \int_{\partial\Omega} E_1(x,z)\varphi(z) ds_z, \qquad x \in \partial\Omega, \\
(\mathcal{B}'\varphi)(x) &= \int_{\partial\Omega} \frac{\partial E_2(x,z)}{\partial n_x}\varphi(z) ds_z, \quad x \in \partial\Omega, \\
(\mathcal{K}'\varphi)(x) &= \int_{\partial\Omega} \frac{\partial E_1(x,z)}{\partial n_x}\varphi(z) ds_z, \quad x \in \partial\Omega,
\end{aligned} \tag{1.18}$$

Lemma 1.6.1. *The operators $\mathcal{A}, \mathcal{B}'$ and $\mathcal{S}$ are continuous.*

Proof. See [51]. □

Lemma 1.6.2. *The integral operators $\mathcal{S}, \mathcal{A}, \mathcal{B}'$ and $\mathcal{K}'$ possess kernels with weak singularity.*

Proof. See [40]. □

Remark 1.6.4. *[40] For the double-layer potential with a constant density, the following holds:*

$$\int_{\partial\Omega} \frac{\partial E_1(x,z)}{\partial n_z} ds_z = \begin{cases} \frac{1}{2}, & x \in \partial\Omega \\ 1, & x \in \Omega \\ 0, & x \in \mathbb{R}^2 \setminus \overline{\Omega} \end{cases}$$

The behavior on the boundary is described by the theorems presented in [40].

Theorem 1.6.9. *(**Jump Relations 1**) [40] Assuming $\partial\Omega$ is of class C^2, the double-layer potential v with a continuous density φ exhibits continuous extension from Ω to $\bar{\Omega}$ and from $\mathbb{R}^m \setminus \bar{\Omega}$ to $\mathbb{R}^m \setminus \Omega$ with limiting values*

$$v_{\pm}(x) := \int_{\partial\Omega} \varphi(z) \frac{\partial E_1(x,z)}{\partial n_z} ds_z \pm \frac{1}{2}\varphi(x), \ \ x \in \partial\Omega, \tag{1.19}$$

where

$$v_{\pm}(x) := \lim_{h\to+0} v(x \pm h n_x),$$

and the integral is well-defined as an improper integral.

Theorem 1.6.10. *(**Jump Relations 2**) [40] Assuming $\partial\Omega$ is of class C^2, for the single-layer potential u with a continuous density φ, the following relations hold on the boundary:*

$$\frac{\partial u_{\pm}}{\partial n}(x) := \int_{\partial\Omega} \varphi(z) \frac{\partial E(x,z)}{\partial n_x} ds_z \mp \frac{1}{2}\varphi(x), \ \ x \in \partial\Omega, \tag{1.20}$$

where

$$\frac{\partial u_{\pm}}{\partial n}(x) := \lim_{h\to+0} n_x.\nabla u(x \pm h n_x),$$

the integral is well-defined as an improper integral.
If $n = 2$, it also satisfies:

$$\int_{\partial\Omega} \varphi(z) ds_z = 0$$

Proof. See [40] □

Theorem 1.6.11 (Theorem 7.38. [39] and 3.16. [38]). *Under the assumption that there exists $x_0 \in \Omega$ such that $|x - x_0| \neq 1$ for all $x \in \partial\Omega$, the single-layer operator $\mathcal{S} : C(\partial\Omega) \to C(\partial\Omega)$ is injective.*

Proof. See [39]. □

Theorem 1.6.12. *The operators $I - \mathcal{K}'$ and $I - \mathcal{K}$ have trivial null-spaces*

$$N(I - \mathcal{K}') = N(I - \mathcal{K}) = \{0\}$$

Proof. Refer to [39] □

1.7 Ill-Posed and Inverse Problems

1.8 Regularization Methods

The regularization of ill-posed problems, initially introduced by Tikhonov [55], involves redefining concepts such as inversion and solution (quasi-solution, approximate solution, etc.). This redefinition aims to ensure that the "regularized solution" obtained through "regularized inversion" exhibits continuous variation with the data and closely approximates the exact solution. We assume the existence of such a solution for data close to the values obtained through measurement.

In essence, this approach replaces the initially poorly posed problem with another problem that is "close in a certain sense" to the original problem and is well-posed. Consider the inverse problem $\mathcal{K}u = v$ where $\mathcal{K} : H_1 \to H_2$ is an injective[2] bounded linear operator. We assume that $v \in \mathbf{R}(\mathcal{K})$, indicating that the inverse problem has a unique solution[3].

Definition 1.8.1. *A family of bounded linear operators $R_\alpha : H_2 \to H_1$, where $\alpha > 0$ is called "regularizing family" for the operator $\mathcal{K}$ if*

$$\forall x \in H_1, \ \lim_{\alpha \to 0} (R_\alpha \mathcal{K}) u = u, \ \textit{i.e.}, \ R_\alpha \mathcal{K} \to I \ \textit{simply.}$$

[2]The choice of an injective $\mathcal{K}$ is not overly restrictive, as we can always restrict the space H_1 to the orthogonal complement of $\mathbf{N}(\mathcal{K})$, where $\mathbf{N}$ denotes the kernel.

[3]It is noteworthy that our inverse problem $\mathcal{K}u = v$ is inherently ill-posed due to the non-continuity of $\mathcal{K}^{-1}$.

Remark 1.8.1. *[53] If R_α is a regularizing family for the operator $\mathcal{K}: H_1 \to H_2$, where H_1 is of infinite dimension, then the operators R_α are not uniformly bounded, i.e., there exists a sequence $(\alpha_n) \subset \mathbb{R}_+$ such that $\lim_{n\to\infty} \left\|R_{\alpha_n}\right\| = +\infty$.*

- ♦ Usually we do not have the exact data but only noisy data. We note by v^δ the noisy data of v where the $\delta > 0$ represents the noise level, amount of noise noiselevel:

$$\left\|v - v^\delta\right\| \leq \delta.$$

- ♦ Let $u^{\alpha,\delta} = R_\alpha v^\delta$ represent the approximation of the solution to the inverse problem $Ku = v$ obtained through the regularization operator and perturbed data. Utilizing the triangular inequality, we derive the following expression:

$$\left\|u - u^{\alpha,\delta}\right\| = \left\|(u - R_\alpha v) + \left(R_\alpha v - u^{\alpha,\delta}\right)\right\| \leq \delta \|R_\alpha\| + \|u - R_\alpha v\|. \tag{1.21}$$

- ♦ The first term on the right side of the inequality (1.21) signifies the escalation of error attributable to the level of noise. As indicated in Remark (1.8.1), the fact that $\left\|R_{\alpha_n}\right\| \to +\infty$ as $\alpha \to 0$. Implies that selecting an excessively small α leads to a substantial increase in the error. Conversely, the second term on the right side of the inequality (1.21) diminishes to 0 as α approaches 0, as per the definition of R_α. The objective is to adopt a regularization strategy that minimizes the error in the exact solution u as the noise level δ tends to 0.

Definition 1.8.2. *A regularization strategy $\alpha(\delta)$ is admissible if for all $u \in H_1$, we have*

$$\lim_{\delta\to 0} \alpha(\delta) = 0 \text{ et } \lim_{\delta\longrightarrow 0}\left(\sup_{v^\delta\in H_2}\left\{\left\|R_{\alpha(\delta)}v^\delta - u\right\| \text{ tel que } \left\|\mathcal{K}u - v^\delta\right\| \leq \delta\right\}\right) = 0. \tag{1.22}$$

One of the best-known regularization methods for inverse problems and ill-conditioned matrix calculus is Tikhonov's method.

Definition 1.8.3. *(Discrepancy principle) The regularization parameter α corresponding to the error level δ is selected such that*

$$\|\mathcal{K}R_\alpha v^\delta - v^\delta\| = \epsilon\delta$$

with a predefined parameter $\epsilon \geq 1$.

There is variety of regularization schemes, for an overview see e.g. [28], but we will concentrate on one of the most commonly used, the Tikhonov regularization.

1.8.1 Tikhonov Method

- The principle of Tikhonov regularization to stabilize the ill-posed inverse problem $\mathcal{K}u = v$ is to choose as a solution the element u_α which minimizes the functional *i.e,*

$$\|Ku_\alpha - v\|^2 + \alpha \|u_\alpha\|^2, \alpha > 0. \tag{1.23}$$

  The existence and uniqueness of the minimum are ensured by the coercivity and strict convexity of $u \longmapsto \|u\|^2$. The parameter α is called the regularization parameter and the term $\|u\|^2$ is called the correction term. The choice of the parameter α is based on a criterion of equilibrium between the error due to the correction term and the gain in stability.

We can state the following theorem based on Kirsh [38]

Theorem 1.8.1. *For $\mathcal{K} \in \mathcal{L}(H_1, H_2)$, the Tikhonov functional admits a unique minimum f_α. The solution f_α is determined by the normal equation:*

$$(\alpha I + \mathcal{K}^*\mathcal{K})\, u_\alpha = \mathcal{K}^* v. \tag{1.24}$$

The family of operators $R_\alpha = (\alpha I + \mathcal{K}^*\mathcal{K})^{-1} \mathcal{K}^* : H_2 \to H_1$ is referred to as the regularizing family of Tikhonov. It is noteworthy that $\|R_\alpha\| \leq \frac{1}{2\sqrt{\alpha}}$, and any choices of $\alpha(\delta) \longrightarrow 0$ with

$$\delta^2 \alpha(\eta) \longrightarrow 0,$$

is admissible. For further insights into convergence results, pertinent references include [23, 46].

The family of operators $R_\alpha = (\alpha I + \mathcal{K}^*\mathcal{K})^{-1} \mathcal{K}^* : H_2 \to H_1$ is called regularizing family of Tikhonov. We have $\|R_\alpha\| \leq \frac{1}{2\sqrt{\alpha}}$ and all choices of $\alpha(\delta) \longrightarrow 0$ with

$$\delta^2 \alpha(\eta) \longrightarrow 0,$$

is admissible. For further insights into convergence results, you can consult the following references [23, 46].

- The regularization parameter $\alpha > 0$ is determined using Morozov's discrepancy principle [28]. This principle involves selecting the parameter in such a way that the resultant solution exhibits an error equal to the noise level (cf. [45], [57]).

- The optimal choice is extremely difficult and the criteria that exist are difficult to apply, and require iterative methods to be implemented. [4, 5]

- In practice, we assume that a parameter α is valid if the error belongs to a small interval containing the noise level value $\delta > 0$ (see [46], page 172).

1.9 Numerical Methods

The Nyström method is a quadrature technique that involves employing numerical methods to evaluate integrals and formulate a linear system. Essentially, it amounts to approximating the integral operator $\mathcal{A}$ (with the kernel E) using a finite-dimensional operator, akin to a matrix. Being inherently discrete, this method serves as an initial and efficient approach to numerically solving a second-kind integral equation. Given its significance and reliance on numerical integration methods, a detailed presentation of the Nyström method will be provided.

[4] A-priori methods: use of error level and operator information $\mathcal{K}$.

[5] A-posteriori methods: also use data v_δ. $\alpha_{opt} := \max\{\alpha : \|\mathcal{K}u_\alpha - v_\delta\| \leq \delta\}$, where $u_\alpha = \inf_u \left\{\|\mathcal{K}u_\alpha - v_\delta\|^2 + \alpha\|u_\alpha\|^2\right\}$.

1.9.1 Quadrature Method

In this section, we recall the notation used in numerical integration [40].

Definition 1.9.1. *Let h be a function defined on a compact subset of Ω, and let μ be a nonempty Jordan measure in $\mathbb{R}^d$. Any method of approximating the integral $\int_\Omega h(x)d\mu$ is referred to as a quadrature formula. A standard quadrature formula involves the values $h(x_i)$ of h at the points $x_i \in \Omega$ for $i = 1, ..., n$. We denote the quadrature formula as $Q_n(h)$, given by:*

$$Q_n(h) = \sum_{i=1}^{n} a_{i,n} h(x_{i,n}).$$

The coefficients $a_{i,n}$ are the quadrature points.

The quadratic error, denoted as $R(h)$, is expressed as:

$$Q(h) = \int_\Omega h(x)d\mu = Q(h) - R(h) \quad pour \quad h \in C(\Omega).$$

Remark 1.9.1. *If $\Omega = \partial\Omega$ is a surface of $\mathbb{R}^3$ or a curve of $\mathbb{R}^2$, then $d\mu = ds$ or dl (surface element or length element).*

Definition 1.9.2. *The quadrature method $\{Q_n\}_{n\geq 1}$ is considered convergent if it converges pointwise, i.e., if :*

$$\lim_{n\to+\infty} Q_n(h) = \int_\Omega h(x)dx, \quad for \quad all \quad h \in C(\Omega).$$

Definition 1.9.3. *The quadrature method $\{Q_n\}_{n\geq 1}$ is considered consistent if there exists a dense subset $V \subset C(\Omega)$ such that $Q_n(h)$ converges to $Q(h)$, $\quad \forall h \in V$.*

Definition 1.9.4. *The quadrature method $\{Q_n\}_{n\geq 1}$ is said to be stable if*

$$\sup\left\{\sum_{i=1}^{n} \left|a_{i,n}\right| : n \in \mathbb{N}^*\right\} < \infty.$$

Theorem 1.9.1. *A quadrature rule is convergent if and only if it is consistent and stable.*

Theorem 1.9.2. *If $Q_n(1) \underset{n\to\infty}{\longrightarrow} Q(1)$ and the quadrature points are positive, then the quadratic* *form* $\{Q_n\}_{n\geq 1}$ *converges whenever* $Q_n(f) \underset{n\to\infty}{\to} Q(f)$ *for all* f *in the set* U *that is dense in* $C(\Omega)$.

The proofs of these theorems can be referenced in the book [41].

1.9.2 Nyström Method

Consider a sequence of convergent quadrature rules $\{Q_n\}_{n\in\mathbb{N}}$ for the integral $\int_\Omega h(x)dx$ and the approximation of the integral operator

$$\mathcal{A}\phi(x) = \int_\Omega E(x,z)\phi(z)dz, \quad x \in \Omega$$

where E is a continuous kernel, through a sequence of linear discrete operators

$$\mathcal{A}_n\phi(x) = \sum_{k=1}^{n} a_k E(x,x_k)\phi(x_k), \quad x \in \Omega$$

then the solution to the second-kind integral equation

$$\phi(x) - \mathcal{A}\phi(x) = h(x),$$

is approximated by the solution to the equation

$$\phi_n(x) - \mathcal{A}_n\phi_n(x) = h(x).$$

This process reduces the problem to solving a system of equations with finite dimensions.

Approached Method

Consider ϕ_n as the solution of the equation

$$\phi_n(x) - \sum_{k=1}^{n} a_{k,n} E(x,x_{k,n})\phi_n(x_k) = h(x), \quad x \in \Omega \tag{1.25}$$

The values $\phi_{i,n}(x) = \phi_n(x_i)$, $i = 1,...,n$. at the quadrature points $a_{k,n}$ are solutions of the system

$$\phi_{i,n}(x) - \sum_{k=1}^{n} a_{k,n} E(x_{i,n}, x_{k,n}) \phi_{i,n}(x_k) = h(x_{i,n}), \quad x \in \Omega \tag{1.26}$$

Thus, ϕ_n is defined by

$$\phi_n(x) = h(x) + \sum_{k=1}^{n} a_{k,n} E(x, x_{k,n}) \phi_{k,n}(x_k), \quad x \in \Omega \tag{1.27}$$

and is a solution of the equation (1.25)

Remark 1.9.2. *In the case of a symmetric kernel, where $E(x,z) = E(z,x)$, the matrix $\left(a_{k,n}E(x,x_{k,n})\right)_{n\geq 1}$ is generally non-symmetrical.*

Theorem 1.9.3. *For all solution to a second-kind integral equation with a continuous kernel and a continuous function h, the quadrature formula of the Nyström method exhibits uniform convergence.*

1.9.3 The Case of a Weakly Singular Kernel

We will describe the Nyström method applied to approximate the solution of a second-kind integral equation with a weakly singular kernel in the following manner:

$$\mathcal{A}\phi(x) = \int_{\Omega} \nu(|x-z|) E(x,z) \phi(z) dz, \quad x \in \Omega$$

where the function $\nu : (0,\infty) \to \mathbb{R}$ represents the kernel singularity.
Suppose that ν is continuous and satisfies $|\nu(t)| \leq \zeta t^{\alpha-d}$ for all $t > 0$, where ζ and α are constants, $\alpha > 0$, and $E \in C(\Omega \times \Omega)$. We choose $\{Q_n\}_{n\geq 1}$ as the quadrature sequence defined by

$$Q_n g(x) = \sum_{k=1}^{n} a_{k,n}(x) g(x_{k,n}), \quad x \in \Omega$$

for the integral equation

$$Qg(x) = \int_{\Omega} \nu(|x-z|) g(z) dz, \quad x \in \Omega.$$

The solution with the quadrature points depends continuously on x. Then, We approach the singular integral operator through the sequence of discrete integral operators :

$$\mathcal{A}\phi_n(x) = \sum_{k=1}^{n} a_{k,n} E(x, x_{k,n})\phi_n(x_k), \;\; x \in \Omega.$$

The solution to the approximate second-kind equation is expressed as (1.27). The linear system (1.25) is formulated as follows:

$$\phi_{j,n} - \sum_{k=1}^{n} a_{k,n} E(x, x_{k,n})\phi_n(x_k) = h(x_j), \;\; j = 1, ..., n. \tag{1.28}$$

Theorem 1.9.4. *If the quadrature formula $\{Q_n\}_{n\geq 1}$ converges and satisfies*

$$\lim_{z\to x} \sup_{n\in\mathbb{N}^*} \sum_{k=1}^{n} \left| a_{k,n}(z) - a_{k,n}(x) \right| = 0 \tag{1.29}$$

then the sequence $\mathcal{A}_n\phi \underset{n\to\infty}{\to} \mathcal{A}\phi$ for all $\phi \in C(\Omega)$; but not uniformly.

Example:

We consider the weakly singular operator of the form

$$\mathcal{A}\phi(\rho) = \frac{1}{2\pi}\int_0^{2\pi} \ln\left(4\sin^2\frac{\rho-\sigma}{2}\right) E(\rho,\sigma)\phi(\sigma)d\sigma, \;\; 0 \leq \rho \leq 2\pi. \tag{1.30}$$

Here, $\mathcal{A}$ is defined in the space $C_{2\pi} \subset C(\mathbb{R})$ of 2π-periodic continuous functions. We construct a numerical quadrature formula for the improper integral

$$Qg(\rho) = \frac{1}{2\pi}\int_0^{2\pi} \ln\left(4\sin^2\frac{\rho-\sigma}{2}\right) g(\sigma)d\sigma$$

By employing the Lagrange basis L_j, we determine

$$(Q_n g)(\rho) = \sum_{j=0}^{2n-1} R_j^{(n)}(\rho) g(\rho_j) \tag{1.31}$$

with $\rho_j = j\pi/n$ and the quadrature powers

$$R_j^{(n)}(\rho) = \frac{1}{2\pi}\int_0^{2\pi} \ln\left(4\sin^2\frac{\rho-\sigma}{2}\right)L_j(\tau)d\tau, \quad j = 0,\dots,2n-1.$$

According to the properties of Lagrange polynomials, we obtain

$$R_j^{(n)}(\rho) = -\frac{1}{n}\left\{\sum_{m=1}^{n-1}\frac{1}{m}\cos(\rho-\rho_j) + \frac{1}{2n}\cos(\rho-\rho_j)\right\}, \quad j = 0,\dots,2n-1.$$

Hence the following

$$A_n\phi_n(\rho) = \sum_{j=0}^{2n-1} R_j^{(n)}(\rho)E(\rho,\rho_j)\phi(\rho_j) \tag{1.32}$$

is generated by the quadrature rules (1.31). More precisely, we have

$$R_j^{(n)}(\rho_j) = R_{|j-k|}^{(n)}, \quad j,k = 0,\dots,2n,$$

with the weights

$$R_j^{(n)}(\rho) = -\frac{1}{n}\left\{\sum_{m=1}^{n-1}\frac{1}{m}\cos\frac{mj\pi}{n} + \frac{(-1)^j}{2n}\right\}, \quad j = 0,\dots,2n-1.$$

Theorem 1.9.5. *The sequence $(\mathcal{A}_n)$ defined in (1.32) is compact and locally convergence towards the integral operator $\mathcal{A}$ featuring the logarithmic singularity expressed in (1.30).*

CHAPTER

2

DIRECT AND INVERSE PROBLEM FOR THE BIHARMONIC EQUATION

The Bi-Laplace operator Δ^2 is the prototype of fourth-order elliptic operator that appears in many practical areas of science and engineering [16]. Several scientific studies have been devoted to the application of biharmonic problems in science and engineering, such as the deformation of thin plates, the motion of fluids, determining an unknown boundary, detecting the corrosion and the problems related to blending surface [3, 4, 11, 60, 54, 42].

Over the past few years, several types of boundary conditions have been actively investigated for the application of the biharmonic equation, including, Dirichlet problem, Neumann problem, Robin conditions, Navier boundary conditions and Riquier-Neumann conditions [34, 35, 52].

In $2D$ domains, the inverse problems related to the detection of geometrical shape of the boundary are extremely important for several engineering applications, such as evaluating material losses due to corrosion, which increases considerably the lifetime of the beam structure [60]. Many scientific researches are devoted to this problem, (see [36, 13, 8, 10, 56]). This chapter is structured as follows:

This chapter is structured as follows:

The first section is dedicated to the study of the direct problem, focusing on investigating the existence and uniqueness of the solution.

In the second section, we define the inverse problem of interest and establish a uniqueness result. Moreover, we prove the equivalence between the inverse problem and a system of nonlinear integral equations, as investigated in [19].

2.1 Direct Problem

2.1.1 Problem Formulation

Consider a 2D bounded, simply connected domain $\Omega \subset \mathbb{R}^2$ with a piecewise smooth boundary $\partial\Omega$. Assume that $\partial\Omega = \Gamma_m \cup \Gamma_c \cup \Gamma_N$, where Γ_m and Γ_c are two non-empty, open, connected, and disjoint parts of class C^2. Now, let us examine the following interior problem for a function u.

$$\Delta^2 u = 0 \qquad in \qquad \Omega, \tag{2.1}$$

with the Navier condition

$$u = u_0 \quad \text{and} \quad \Delta u = u_2, \qquad \text{on } \Gamma_m, \tag{2.2}$$

and Riquier-Neumann condition

$$\frac{\partial u}{\partial n} = u_1 \quad \text{and} \quad \frac{\partial \Delta u}{\partial n} = u_3 \qquad \text{on } \Gamma_c. \tag{2.3}$$

We denote by n the outward unit normal to $\partial\Omega$.

We have a keen interest in examining the weak solutions pertaining to the Navier-Riquier Neumann boundary value problem outlined in equations (2.1)-(2.3).
The standard Sobolev space $H^2(\Omega)$ serves as our solution space for the biharmonic equation (2.1)

$$H^2(\Omega) = \{u \in L^2(\Omega) : \frac{\partial^2 u}{\partial x_i^2} \in L^2(\Omega) \quad for \quad i = 1, 2\}$$

Initially, we note that due to the boundary $\partial\Omega$ is C^2, the well-defined nature of the trace spaces $H^{\frac{3}{2}}(\Gamma)$ and $H^{\frac{-1}{2}}(\Gamma)$ is evident [33], and furthermore, for u in $H^2(\Omega)$ we have that $u|_{\partial\Omega} \in H^{\frac{3}{2}}(\partial\Omega)$ and $\Delta u|_{\partial\Omega} \in H^{\frac{-1}{2}}(\partial\Omega)$. To delve into the boundary value problem for (2.1) in Ω, it is most effective to begin with the Green's formula.

We integrate over the domain Ω after multiplying the partial differential equation (2.1) by a real-valued test function w defined on Ω.

$$\int_{\Omega} (\Delta^2 u) w dx = 0. \tag{2.4}$$

Afterward, we employ Green's theorem twice. It can be proved that the subsequent Green's formula is valid

$$\int_{\Omega} (\Delta^2 u) w dx = \int_{\Omega} \Delta(\Delta u) w dx = \int_{\Omega} \Delta u \Delta w dx + \int_{\partial\Omega} w \frac{\partial \Delta u}{\partial n} - \Delta u \frac{\partial w}{\partial n} ds. \tag{2.5}$$

Considering smooth functions, where in the bilinear form $a(u,w)$ is defined as follows:

$$a(u,w) = \int_{\Omega} \Delta u \Delta w dx. \tag{2.6}$$

It's worth mentioning that the bilinear form in (2.6) is appropriately defined for functions within the context of $H^2(\Omega)$.
Let's consider u in $H^2(\Omega, \Delta^2)$ such that:

$$H^2(\Omega, \Delta^2) = \left\{ u \in H^2(\Omega) : \Delta^2 u \in \tilde{H}^{-2}(\Omega) \right\}.$$

We choose w in $H^2(\Omega)$ and $\tilde{H}^{-2}(\Omega)$ represent the dual space of $H^2(\Omega)$. Subsequently, the aforementioned Green's formula is applicable, and through a duality argument, it can be established that $\frac{\partial u}{\partial n} \in H^{\frac{1}{2}}(\partial\Omega)$ and $\frac{\partial \Delta u}{\partial n} \in H^{\frac{-3}{2}}(\partial\Omega)$ are properly defined [33]. Here, $H^{\frac{-3}{2}}(\partial\Omega)$ denote the dual spaces of $H^{\frac{3}{2}}(\partial\Omega)$.

Lemma 2.1.1. *[33] The bilinear form $a(.,.)$ defined by (2.6) exhibits a Garding inequality in the following manner:*

$$a(w,w) \geq \eta \|w\|^2_{H^2(\Omega)} - \vartheta \|w\|^2_{L^2(\Omega)},$$

$\forall w$ in $H^2(\Omega)$, with the constants $\eta > 0$ and $\vartheta \geq 0$.

Now, we are prepared to precisely state the Navier-Riquier Neumann boundary value problem for the biharmonic equation. Given u_0 on $H^{\frac{3}{2}}(\Gamma_m)$, u_2 on $H^{\frac{-1}{2}}(\Gamma_m)$, u_1 on

$H^{\frac{1}{2}}(\Gamma_c)$, u_3 on $H^{\frac{-3}{2}}(\Gamma_c)$, the objective is to find u in $H^2(\Omega)$ such that satisfies equations (2.1)-(2.3).

Theorem 2.1.1. *[33] There exists, at most, a unique solution to the problem described by equations (2.1)-(2.3).*

Proof. Assuming u is the solution to equations (2.1)-(2.3) with $u_0 = u_2 = u_1 = u_3 = 0$, applying Green's formula to u and $\tilde{u}$ results in:

$$\int_\Omega |\Delta u| dx = 0$$

we deduce that $\Delta u = 0$ in Ω and $\frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0$. The Holmgren theorem (see theorem 1.6.6 in chapter 1) implies that $u = 0$ in Ω. □

2.1.2 System of Integral Equations

To establish the existence of a solution for (2.1)-(2.3) and derive a solution formula, we will reframe the problem as a system of boundary integral equations of the first kind.

If we take $w = E_2$, where $E_2(x,z)$ is the elementary solution associated with the biharmonic operator provided by

$$E_2(x,z) = \frac{1}{8\pi}|x-z|^2 \log|x-z|$$

which satisfies

$$\Delta_z^2 E_2(x,z) = \delta(z-x), \quad \in D'(\mathbb{R}^2)$$

We initiate this process by using the Green representation formula for a weak solution in $H^2(\Omega)$ [26].

$$u(x) = \int_{\partial\Omega} \left\{ u(z)\frac{\partial \Delta E_2(x,z)}{\partial n_z} - \Delta E_2(x,z)\frac{\partial u(z)}{\partial n_z} \right\} ds_z - \int_{\partial\Omega} \left\{ E_2(x,z)\frac{\partial \Delta u(z)}{\partial n_z} - \Delta u \frac{\partial E_2(x,z)}{\partial n_z} \right\} ds_z, \quad x \in \Omega. \tag{2.7}$$

The single-layer and double-layer potentials are defined, respectively, by

$$\mathbf{S}(\frac{\partial \Delta u}{\partial n_z}, \Delta u) = \int_{\partial\Omega} \left\{ E_2(x,z)\frac{\partial \Delta u(z)}{\partial n_z} - \Delta u \frac{\partial E_2(x,z)}{\partial n_z} \right\} ds_z, \quad x \in \Omega,$$

and
$$\mathbf{D}(u, \frac{\partial u}{\partial n_z}) = \int_{\partial\Omega} \left\{ u(z)\frac{\partial \Delta E_2(x,z)}{\partial n_z} - \Delta E_2(x,z)\frac{\partial u(z)}{\partial n_z} \right\} ds_z, \quad x \in \Omega.$$

Therefore, the representation formula (2.7) can be expressed as

$$u(x) = \mathbf{D}(u, \frac{\partial u}{\partial n_z}) - \mathbf{S}(\frac{\partial \Delta u}{\partial n_z}, \Delta u), \quad x \in \Omega. \tag{2.8}$$

As we approach $\partial\Omega$ from inside Ω by letting $x \to \partial\Omega$ and following the standard procedure in potential theory, which involves jump relations, we derive the following integral equations on $\partial\Omega$

$$\begin{aligned} u|_{\partial\Omega} &= \left[\frac{1}{2}u(x) + \int_{\partial\Omega} \frac{\partial \Delta E_2(x,z)}{\partial n_z} u(z) ds_z\right] - \int_{\partial\Omega} \Delta_z E_2(x,z)\frac{\partial u(z)}{\partial n} ds_z \\ &- \int_{\partial\Omega} \left\{ E_2(x,z)\frac{\partial \Delta u(z)}{\partial n} - \frac{\partial E_2(x,z)}{\partial n_z}\Delta u(z) \right\} ds_z, \end{aligned} \tag{2.9}$$

$$\begin{aligned} \frac{\partial u}{\partial n}|_{\partial\Omega} &= \int_{\partial\Omega} \frac{\partial^2 \Delta E_2(x,z)}{\partial n_x \partial n_z} u(z) ds_z + \left[\frac{1}{2}\frac{\partial u(x)}{\partial n} - \int_{\partial\Omega} \frac{\partial}{\partial n_x}\Delta_z E_2(x,z)\frac{\partial u(z)}{\partial n_z} ds_z\right] \\ &- \int_{\partial\Omega} \left\{ \frac{\partial E_2(x,z)}{\partial n_x}\frac{\partial \Delta u(z)}{\partial n} - \frac{\partial^2 E_2(x,z)}{\partial n_x \partial n_z}\Delta u(z) \right\} ds_z, \end{aligned} \tag{2.10}$$

$$\begin{aligned} \Delta u|_{\partial\Omega} &= \int_{\partial\Omega} \left\{ \Delta_x \frac{\partial \Delta E_2(x,z)}{\partial n_z} u(z) - \Delta_x \Delta_z E_2(x,z)\frac{\partial u(z)}{\partial n_z} \right\} ds_z - \int_{\partial\Omega} \Delta_x E_2(x,z)\frac{\partial \Delta u(z)}{\partial n} ds_z \\ &+ \left[\frac{1}{2}\Delta u(x) - \int_{\partial\Omega} \Delta_x \frac{\partial E_2(x,z)}{\partial n_z}\Delta u(z) ds_z\right], \end{aligned} \tag{2.11}$$

$$\begin{aligned} \frac{\partial \Delta u}{\partial n}|_{\partial\Omega} &= \int_{\partial\Omega} \left\{ \frac{\partial \Delta}{\partial n_x}\frac{\partial \Delta E_2(x,z)}{\partial n_z} u(z) - \frac{\partial \Delta}{\partial n_x}\Delta_z E_2(x,z)\frac{\partial u(z)}{\partial n_z} \right\} ds_z + \int_{\partial\Omega} \frac{\partial \Delta}{\partial n_x}\frac{\partial E_2(x,z)}{\partial n_z}\Delta u \, ds_z \\ &+ \left[\frac{1}{2}\frac{\partial \Delta u(x)}{\partial n} - \int_{\partial\Omega} \frac{\partial \Delta E_2(x,z)}{\partial n_x}\frac{\partial \Delta u(z)}{\partial n_z} ds_z\right]. \end{aligned} \tag{2.12}$$

To understand the mapping characteristics of the 16 boundary integral operators mentioned above, we can express equations (2.9) to (2.12) in the following manner:

$$\begin{pmatrix} u \\ \frac{\partial u}{\partial n} \\ \Delta u \\ \frac{\partial \Delta u}{\partial n} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I + A_{11} & B_{12} & B_{13} & B_{14} \\ C_{21} & \frac{1}{2}I - A_{22} & B_{23} & B_{24} \\ C_{31} & C_{32} & \frac{1}{2}I - A_{33} & B_{34} \\ C_{41} & C_{42} & C_{43} & \frac{1}{2}I - A_{44} \end{pmatrix} \begin{pmatrix} u \\ \frac{\partial u}{\partial n} \\ \Delta u \\ \frac{\partial \Delta u}{\partial n} \end{pmatrix}$$

The integral operator matrix mentioned above serves as the Calderón projector specific to the

biharmonic equation in relation to the domain Ω, and it will be identified as K_{Ω}^{lm}.
The Calderón projector, indeed, is composed of pseudodifferential operators on $\partial\Omega$ and has been extensively investigated in [26]. Specifically, it continuously maps $H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \times H^{\frac{-1}{2}}(\partial\Omega) \times H^{\frac{-3}{2}}(\partial\Omega)$ into itself.
The mapping characteristics of each operator present in K_{Ω} can be readily derived from its principal symbol as follows [33]

$$\begin{pmatrix} 0 & -1 & -3 & -3 \\ 1 & 0 & -1 & -3 \\ 1 & 1 & 0 & -1 \\ 3 & 1 & 1 & 0 \end{pmatrix}.$$

2.1.3 Representation of the Problem (2.1)-(2.3) in Boundary Integral Equations

In this section, we begin by introducing the systems of integral equations on Γ_m and Γ_c. Subsequently, we will focus on the properties of the operators applied to the boundaries, as well as the solvability of these systems.
In the previous section, we derived the system (2.9)-(2.12) for the Cauchy data $u|_{\partial\Omega}$, $\frac{\partial u}{\partial n}|_{\partial\Omega}$, $\Delta u|_{\partial\Omega}$ and $\frac{\partial \Delta u}{\partial n}|_{\partial\Omega}$.
Let $\tilde{u}_0$, $\tilde{u}_2$, $\tilde{u}_1$ and $\tilde{u}_3$ represent bounded extensions by zero to the entire $\partial\Omega$ of the respective boundary data u_0, u_2, u_1 and u_3. Subsequently, we express them as follows:

$$u|_{\partial\Omega} = \omega_c + \tilde{u}_0, \quad \Delta u|_{\partial\Omega} = \epsilon_c + \tilde{u}_2, \tag{2.13}$$

$$\frac{\partial u}{\partial n}|_{\partial\Omega} = \varphi_m + \tilde{u}_1, \quad \frac{\partial \Delta u}{\partial n}|_{\partial\Omega} = \psi_m + \tilde{u}_3. \tag{2.14}$$

Certainly, $\omega_c \in \tilde{H}^{\frac{3}{2}}(\Gamma_c)$, $\epsilon_c \in \tilde{H}^{\frac{-1}{2}}(\Gamma_c)$, $\varphi_m \in \tilde{H}^{\frac{1}{2}}(\Gamma_m)$ and $\psi_m \in \tilde{H}^{\frac{-3}{2}}(\Gamma_m)$ as $\omega_c = \epsilon_c = 0$ on Γ_m and $\varphi_m = \psi_m = 0$ on Γ_c.
By limiting equations (2.9) and (2.11) to Γ_m and equations (2.10) and (2.12) to Γ_c, we derive the system of first-kind boundary integral equations for ω_c, ϵ_c, φ_m and ψ_m.

$$
\begin{aligned}
-\int_{\Gamma_m} E_2(x,z)\psi_m(z)ds_z - \int_{\Gamma_m} \Delta_z E_2(x,z)\varphi_m(z)ds_z + \int_{\Gamma_c} \frac{\partial E_2(x,z)}{\partial n}\epsilon_c(z)ds_z \\
+ \int_{\Gamma_c} \frac{\partial \Delta E_2(x,z)}{\partial n_z}\omega_c(z)ds_z = \mathcal{H}_1(x), \quad x \in \Gamma_m.
\end{aligned} \tag{2.15}
$$

$$
\begin{aligned}
-\int_{\Gamma_m} \Delta_x E_2(x,z)\psi_m(z) - \int_{\Gamma_m} \Delta_x \Delta_z E_2(x,z)\varphi_m(z)ds_z - \int_{\Gamma_c} \Delta_x \frac{\partial E_2(x,z)}{\partial n_y}\epsilon_c(z)(z)ds_z \\
+ \int_{\Gamma_c} \Delta_x \frac{\partial \Delta E_2(x,z)}{\partial n_z}\omega_c(z)ds_z = \mathcal{H}_2(x), \quad x \in \Gamma_m.
\end{aligned} \tag{2.16}
$$

$$
\begin{aligned}
-\int_{\Gamma_m} \frac{\partial E_2(x,z)}{\partial n_x}\psi_m(z)ds_z - \int_{\Gamma_m} \frac{\partial}{\partial n_x}\Delta_z E_2(x,z)\varphi_m(z)ds_z + \int_{\Gamma_c} \frac{\partial^2 E_2(x,z)}{\partial n_x \partial n_z}\epsilon_c(z)ds_z \\
+ \int_{\Gamma_c} \frac{\partial^2 \Delta E_2(x,z)}{\partial n_x \partial n_z}\omega_c(z)ds_z = \mathcal{H}_3(x), \quad x \in \Gamma_c.
\end{aligned} \tag{2.17}
$$

$$
\begin{aligned}
-\int_{\Gamma_m} \frac{\partial \Delta E_2(x,z)}{\partial n_x}\psi_m(z)ds_z - \int_{\Gamma_m} \frac{\partial \Delta}{\partial n_x}\Delta_z E_2(x,z)\varphi_m(z)ds_z + \int_{\Gamma_c} \frac{\partial \Delta}{\partial n_x}\frac{\partial E_2(x,z)}{\partial n_z}\epsilon_c(z)ds_z \\
+ \int_{\Gamma_c} \frac{\partial \Delta}{\partial n_x}\frac{\partial \Delta E_2(x,z)}{\partial n_z}\omega_c(z)ds_z = \mathcal{H}_4(x), \quad x \in \Gamma_c.
\end{aligned} \tag{2.18}
$$

where

$$
\begin{aligned}
\mathcal{H}_1(x) = \frac{1}{2}\tilde{u}_0 - \int_{\partial\Omega} \frac{\partial \Delta E_2(x,z)}{\partial n_z}\tilde{u}_0(z)ds_z - \int_{\partial\Omega} \frac{\partial E_2(x,z)}{\partial n_z}\tilde{u}_2(z)ds_z + \int_{\partial\Omega} \Delta_z E_2(x,z)\tilde{u}_1(z)ds_z \\
+ \int_{\partial\Omega} E_2(x,z)\tilde{u}_3(z)ds_z, \quad x \in \Gamma_m,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_2(x) = -\int_{\partial\Omega} \Delta_x \frac{\partial \Delta E_2(x,z)}{\partial n_z}\tilde{u}_0(z)ds_z - \frac{1}{2}\tilde{u}_2 + \int_{\partial\Omega} \Delta_x \frac{\partial E_2(x,z)}{\partial n}\tilde{u}_2(z)ds_z + \int_{\partial\Omega} \Delta_x \Delta_z E_2(x,z)\tilde{u}_1(z)ds_z \\
+ \int_{\partial\Omega} \Delta_x E_2(x,z)\tilde{u}_3 ds_z, \quad x \in \Gamma_m,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_3(x) = -\int_{\partial\Omega} \frac{\partial^2 \Delta E_2(x,z)}{\partial n_x \partial n_z}\tilde{u}_0(z)ds_z + \frac{1}{2}\tilde{u}_1(x) + \int_{\partial\Omega} \frac{\partial}{\partial n_x}\Delta_z E_2(x,z)\tilde{u}_1(z)ds_z + \int_{\partial\Omega} \frac{\partial E_2(x,z)}{\partial n_x}\tilde{u}_3(z)ds_z \\
- \int_{\partial\Omega} \frac{\partial^2 E_2(x,z)}{\partial n_x \partial n_z}\tilde{u}_2(z)ds_z, \quad x \in \Gamma_c,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_4(x) = -\int_{\partial\Omega} \frac{\partial \Delta}{\partial n_x}\frac{\partial \Delta E_2(x,z)}{\partial n_z}\tilde{u}_0(z)ds_z + \int_{\partial\Omega} \frac{\partial \Delta}{\partial n_x}\Delta_z E_2(x,z)\tilde{u}_1(z)ds_z - \int_{\partial\Omega} \frac{\partial \Delta}{\partial n_x}\frac{\partial E_2(x,z)}{\partial n_z}\tilde{u}_2(z)ds_z \\
+ \int_{\partial\Omega} \frac{\partial \Delta E_2(x,z)}{\partial n_x}\tilde{u}_3(z)ds_z + \frac{1}{2}\tilde{u}_3(x), \quad x \in \Gamma_c.
\end{aligned}
$$

The set of equations (2.15)-(2.18) can be expressed in matrix form as follows

$$\mathcal{T}\begin{pmatrix}\psi_m\\ \varphi_m\\ \epsilon_c\\ \omega_c\end{pmatrix}=\begin{pmatrix}B_{14}^{mm} & B_{12}^{mm} & B_{13}^{mc} & A_{11}^{mc}\\ B_{34}^{mm} & C_{32}^{mm} & A_{33}^{mc} & C_{31}^{mc}\\ B_{24}^{cm} & A_{22}^{cm} & B_{23}^{cc} & C_{21}^{cc}\\ A_{44}^{cm} & C_{42}^{cm} & C_{43}^{cc} & C_{41}^{cc}\end{pmatrix}\begin{pmatrix}\psi_m\\ \varphi_m\\ \epsilon_c\\ \omega_c\end{pmatrix}=\mathcal{H} \tag{2.19}$$

which can be expressed as

$$\mathcal{T}U=\mathcal{H}, \quad on \quad \partial\Omega \tag{2.20}$$

with $\mathcal{H}=(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3,\mathcal{H}_4)^{\perp}$. The operators A_{lm},B_{lm},C_{lm}, $l=1...4$ and $m=1...4$ are the operators involved in the Calderon operator. The operator B_{34}^{mc} is defined by:

$$B_{34}^{mc}u(x)=\int_{\Gamma_c}\Delta_x E_2(x,z)u(z)ds_z, \quad x\in\Gamma_m$$

The other operators are defined in a similar manner.

2.1.4 Existence and Uniqueness

2.1.5 Properties of Boundary Integral Operators

i) The single-layer and double-layer operators **S** and **D** are continuous operators such that:

$$\mathbf{S}: H^{\frac{-3}{2}}(\partial\Omega)\times H^{\frac{-1}{2}}(\partial\Omega)\longrightarrow H^2(\Omega)$$
$$\mathbf{D}: H^{\frac{3}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)\longrightarrow H^2(\Omega)$$

ii) Considering the mapping properties of the Calderon operator [33], it is evident that the operator $\mathcal{T}$ establishes a continuous mapping $\mathcal{T}:\mathcal{F}\rightarrow\mathcal{F}^*$ with $\mathcal{F}=\tilde{H}^{\frac{-3}{2}}(\Gamma_m)\times\tilde{H}^{\frac{1}{2}}(\Gamma_m)\times\tilde{H}^{\frac{-1}{2}}(\Gamma_c)\times\tilde{H}^{\frac{3}{2}}(\Gamma_c)$ and $\mathcal{F}^*=H^{\frac{3}{2}}(\Gamma_m)\times H^{\frac{-1}{2}}(\Gamma_m)\times H^{\frac{1}{2}}(\Gamma_c)\times H^{\frac{-3}{2}}(\Gamma_c)$ represent the dual space of $\mathcal{T}$.

It is observed that if $\varphi_m,\psi_m,\epsilon_c,\omega_c$, satisfy (2.19), then, after defining $\frac{\partial u}{\partial n},\frac{\partial\Delta u}{\partial n},\Delta u,u$ on $\partial\Omega$ in (2.13)-(2.14), the representation formulas (2.7) provide a solution for problem (2.1)-(2.3), which, according to Theorem 2.1.1, is the unique solution. Therefore, it is imperative to investigate the solvability of the system of integral equations of the first kind (2.20).
To achieve this, we will initially introduce the matrix operators. $\mathcal{B}: H^{\frac{-3}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)\longrightarrow$

$H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{-1}{2}}(\partial\Omega)$ and $\mathcal{C} : H^{\frac{-1}{2}}(\partial\Omega) \times H^{\frac{3}{2}}(\partial\Omega) \longrightarrow H^{\frac{1}{2}}(\partial\Omega) \times H^{\frac{-3}{2}}(\partial\Omega)$ defined by:

$$\mathcal{B} = \begin{pmatrix} B_{14} & B_{12} \\ B_{34} & C_{32} \end{pmatrix} \tag{2.21}$$

and

$$\mathcal{C} = \begin{pmatrix} B_{23} & C_{21} \\ C_{43} & C_{41} \end{pmatrix} \tag{2.22}$$

The following lemma is valid.

Lemma 2.1.2. *[33]*

i) There is a compact operator $C_{\mathcal{B}} : H^{\frac{-3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{-1}{2}}(\partial\Omega)$ such that

$$Re\langle(\mathcal{B} + C_{\mathcal{B}})\Lambda, \bar{\Lambda}\rangle \geq \vartheta\|\Lambda\|^2_{H^{\frac{-3}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} \quad for \quad \Lambda \in H^{\frac{-3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega).$$

ii) There is a compact operator $C_{\mathcal{C}} : H^{\frac{-1}{2}}(\partial\Omega) \times H^{\frac{3}{2}}(\partial\Omega) \longrightarrow H^{\frac{1}{2}}(\partial\Omega) \times H^{\frac{-3}{2}}(\partial\Omega)$ such that

$$Re\langle(\mathcal{C} + C_{\mathcal{C}})\Upsilon, \bar{\Upsilon}\rangle \geq \vartheta\|\Lambda\|^2_{H^{\frac{-1}{2}}(\partial\Omega)\times H^{\frac{3}{2}}(\partial\Omega)} \quad for \quad \Upsilon \in H^{\frac{-1}{2}}(\partial\Omega) \times H^{\frac{3}{2}}(\partial\Omega).$$

In this context, the notation $< \cdot, \cdot >$ signifies the duality pairing between $H^{\frac{-3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{-1}{2}}(\partial\Omega)$, as well as between $H^{\frac{-1}{2}}(\partial\Omega) \times H^{\frac{3}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega) \times H^{\frac{-3}{2}}(\partial\Omega)$.

Proof. see [33, 27, 17]. □

Theorem 2.1.2. *The operator $\mathcal{T} : \mathcal{F} \longrightarrow \mathcal{F}^*$ is Fredholm with index zero, where $\mathcal{F} = \tilde{H}^{\frac{-3}{2}}(\Gamma_m) \times \tilde{H}^{\frac{1}{2}}(\Gamma_m) \times \tilde{H}^{\frac{-1}{2}}(\Gamma_c) \times \tilde{H}^{\frac{3}{2}}(\Gamma_c)$ and its dual is denoted by $\mathcal{F}^* = H^{\frac{3}{2}}(\Gamma_m) \times H^{\frac{-1}{2}}(\Gamma_m) \times H^{\frac{1}{2}}(\Gamma_c) \times H^{\frac{-3}{2}}(\Gamma_c)$.*

Proof. Using Lemma 2.1.2, let us define $\mathcal{B}_0 = \mathcal{B} + C_{\mathcal{B}}$ and $\mathcal{C}_0 = \mathcal{C} + C_{\mathcal{C}}$, where $\mathcal{B}$ and $\mathcal{C}$ are given by (2.21)-(2.22). It is established that $\mathcal{B}_0$ and $\mathcal{C}_0$ are both bounded below and positive.
Considering $\Pi = (\psi_m, \varphi_m, \epsilon_c, \omega_c) \in \tilde{H}^{\frac{-3}{2}}(\Gamma_m) \times \tilde{H}^{\frac{1}{2}}(\Gamma_m) \times \tilde{H}^{\frac{-1}{2}}(\Gamma_c) \times \tilde{H}^{\frac{3}{2}}(\Gamma_c)$, we can extend it by zero to function $\tilde{\Pi} = (\tilde{\psi}_m, \tilde{\varphi}_m, \tilde{\epsilon}_c, \tilde{\omega}_c) \in H^{\frac{-3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \times H^{\frac{-1}{2}}(\partial\Omega) \times H^{\frac{3}{2}}(\partial\Omega)$. Thus, we can express $\mathcal{T}$ in the form

$$\mathcal{T} = \mathcal{T}_0 + C_{\mathcal{B}} = \begin{pmatrix} \mathcal{B}_0 & \mathcal{N}_{mc} \\ \mathcal{N}_{cm} & \mathcal{C}_0 \end{pmatrix} + C_{\mathcal{B}},$$

where $C_{\mathcal{B}}$ is compact, and

$$\mathcal{N}_{mc} = \begin{pmatrix} B_{13}^{mc} & A_{11}^{mc} \\ A_{33}^{mc} & C_{31}^{mc} \end{pmatrix} \quad and \quad \mathcal{N}_{cm} = \begin{pmatrix} B_{24}^{cm} & A_{22}^{cm} \\ A_{44}^{cm} & C_{42}^{cm} \end{pmatrix}.$$

Moreover, considering (2.15)-(2.18), we can observe that

$$\begin{aligned} \langle B_{13}^{mc}\epsilon_c, \bar{\psi}_m\rangle &= \int_{\Gamma_m} \bar{\psi}_m(x) \int_{\Gamma_c} \frac{\partial E_2(x,z)}{\partial n_z} \epsilon_c(z) ds_z ds_x \\ &= \int_{\partial\Omega} \bar{\tilde{\psi}}_m(x) \int_{\partial\Omega} \frac{\partial E_2(x,z)}{\partial n_z} \tilde{\epsilon}_c(z) ds_z ds_x \\ &= -\int_{\partial\Omega} \tilde{\epsilon}_c(z) \int_{\partial\Omega} \frac{\partial E_2(x,z)}{\partial n_x} \bar{\tilde{\psi}}_m(x) ds_x ds_z \\ &= -\int_{\Gamma_c} \epsilon_c(z) \int_{\Gamma_c} \frac{\partial E_2(x,z)}{\partial n_x} \bar{\psi}_m(x) ds_x ds_z \\ &= -\overline{\langle B_{24}^{cm}\psi_m, \bar{\epsilon}_c\rangle}. \end{aligned}$$

Similarly, it can be proved that

$$\begin{aligned} \langle A_{11}^{mc}\omega_c, \bar{\psi}_m\rangle &= -\overline{\langle A_{44}^{cm}\psi_m, \bar{\omega}_c\rangle}, \\ \langle A_{33}^{mc}\epsilon_c, \bar{\varphi}_m\rangle &= \overline{\langle A_{22}^{cm}\varphi_m, \bar{\epsilon}_c\rangle}, \\ \langle C_{31}^{mc}\omega_c, \bar{\varphi}_m\rangle &= \overline{\langle C_{42}^{cm}\varphi_m, \bar{\omega}_c\rangle}. \end{aligned}$$

Finally, by combining these equations and applying Lemma 2.1.2, we obtain

$$\begin{aligned} Re\langle \mathcal{T}_0\Pi, \Pi\rangle_{\mathcal{F},\mathcal{F}^*} &= Re\langle \mathcal{B}_0(\psi_m, \varphi_m), (\bar{\psi}_m, \bar{\varphi}_m)\rangle + Re\langle \mathcal{C}_0(\epsilon_c, \omega_c), (\bar{\epsilon}_c, \bar{\omega}_c)\rangle \\ &\geq \vartheta_1 \|(\psi_m, \varphi_m)\|^2_{H^{\frac{-3}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} + \vartheta_2 \|(\epsilon_m, \omega_m)\|^2_{H^{\frac{-1}{2}}(\partial\Omega)\times H^{\frac{3}{2}}(\partial\Omega)} \\ &\geq \vartheta \|\Pi\|^2_{\mathcal{F}}. \end{aligned}$$

For every $\Pi \in \mathcal{F}$ with a positive constant $\vartheta > 0$, we deduce that $\mathcal{T}$ is a Fredholm operator with index zero. Importantly, the uniqueness of the solution to (2.19) implies the existence of a solution for (2.19). □

The following theorem confirms the uniqueness of a solution for the equation (2.19).

Theorem 2.1.3. *The kernel of the operator $\mathcal{T} : \mathcal{F} \longrightarrow \mathcal{F}^*$ is zero.*

Proof. Consider $\Pi = (\psi_m, \varphi_m, \epsilon_c, \omega_c) \in \tilde{H}^{\frac{-3}{2}}(\Gamma_m) \times \tilde{H}^{\frac{1}{2}}(\Gamma_m) \times \tilde{H}^{\frac{-1}{2}}(\Gamma_c) \times \tilde{H}^{\frac{3}{2}}(\Gamma_c)$ as a solution to the homogeneous equation $\mathcal{T}\Pi = 0$, and $\tilde{\Pi} = (\tilde{\psi}_m, \tilde{\varphi}_m, \tilde{\epsilon}_c, \tilde{\omega}_c) \in H^{\frac{-3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \times H^{\frac{-1}{2}}(\partial\Omega) \times$

$H^{\frac{3}{2}}(\partial\Omega)$ be its extension by zero.

Define the potential v as follows:

$$v(x) = \int_{\partial\Omega} \left\{ \omega_c \frac{\partial \Delta E_2(x,z)}{\partial n_z} - \Delta E_2(x,z)\varphi_m \right\} ds_z - \int_{\partial\Omega} \left\{ E_2(x,z)\psi_m - \epsilon_c \frac{\partial E_2(x,z)}{\partial n_z} \right\} ds_z, \quad x \in \Omega \tag{2.23}$$

is in $H^2(\Omega, \Delta^2)$ and satisfies the biharmonic equation. Now, let $x \longrightarrow \partial\Omega$ from inside Ω, using the Jump relations (2.9)-(2.12), we obtain:

$$\begin{aligned} v|_{\partial\Omega} &= B_{14}\tilde{\psi}_m + B_{12}\tilde{\varphi}_m + B_{13}\tilde{\epsilon}_c + [\frac{1}{2}\tilde{\omega}_c + A_{11}\tilde{\omega}_c], \\ \frac{\partial v}{\partial n}|_{\partial\Omega} &= B_{24}\tilde{\psi}_m + [\frac{1}{2}\tilde{\varphi}_m - A_{22}\tilde{\varphi}_m] + B_{23}\tilde{\epsilon}_c + C_{23}\tilde{\omega}_c, \\ \Delta v|_{\partial\Omega} &= B_{34}\tilde{\psi}_m + C_{32}\tilde{\varphi}_m + [\frac{1}{2}\tilde{\epsilon}_c - A_{33}\tilde{\epsilon}_c] + C_{31}\tilde{\omega}_c, \\ \frac{\partial \Delta v}{\partial n}|_{\partial\Omega} &= [\frac{1}{2}\tilde{\psi}_m - A_{44}\tilde{\psi}_m] + C_{42}\tilde{\varphi}_m + C_{43}\tilde{\epsilon}_c + C_{41}\tilde{\omega}_c. \end{aligned}$$

Utilizing the fact that the supports of ψ_m, φ_m are on Γ_m and the supports of ϵ_c, ω_c are on Γ_c, the integral equation $\mathcal{T}\Pi = 0$ implies that

$$v|_{\Gamma_m} = 0, \quad \Delta v|_{\Gamma_m} = 0, \quad \frac{\partial v}{\partial n}|_{\Gamma_c} = 0, \quad \frac{\partial \Delta v}{\partial n}|_{\Gamma_c} = 0.$$

This implies that (2.23) is a weak solution to the homogeneous interior Navier-Riquier Neumann boundary value problem for the biharmonic equation. Therefore, from Theorem 2.1.1, $v = 0$ in Ω.

□

Summarizing the aforementioned analysis, we have established the following result:

Theorem 2.1.4. *For given functions $u_0 \in H^{\frac{3}{2}}(\Gamma_m)$, $u_2 \in H^{\frac{-1}{2}}(\Gamma_m)$, $u_1 \in H^{\frac{1}{2}}(\Gamma_c)$ and $u_3 \in H^{\frac{-3}{2}}(\Gamma_c)$, the Navier-Riquier Neumann boundary value problem (2.1)-(2.3) admits a weak solution in $H^2(\Omega)$. Furthermore, the solution satisfies the estimate:*

$$\|u\|_{H^2(\Omega)} \leq \theta \left(\|u_0\|_{H^{\frac{3}{2}}(\Gamma_m)} + \|u_2\|_{H^{\frac{-1}{2}}(\Gamma_m)} + \|u_1\|_{H^{\frac{1}{2}}(\Gamma_c)} + \|u_3\|_{H^{\frac{-3}{2}}(\Gamma_c)} \right)$$

where θ is a positive constant.

2.2 The Inverse Problem

2.2.1 Modeling and Formulation of the Problem

Let $\Omega \subset \mathbb{R}^2$ be a 2D bounded simply connected domain with piecewise smooth boundary $\partial\Omega$, and assume that $\partial\Omega = \overline{\Gamma}_m \cup \overline{\Gamma}_c$, where Γ_m and Γ_c are two non-empty, open, connected and disjoint parts of class C^2 without cusps at the two intersection points. We denote by n the outward unit normal to $\partial\Omega$, and u is a solution of the following boundary value problem:

$$\Delta^2 u = 0 \qquad in\,\Omega, \tag{2.24}$$

with the Navier-boundary condition [52]

$$\begin{cases} u &= u_0, \qquad \text{on } \Gamma_m \\ \Delta u &= u_2, \qquad \text{on } \Gamma_m \end{cases}, \tag{2.25}$$

and a Robin-conditions [18, 34]

$$\begin{cases} \frac{\partial u}{\partial n} + \mu u &= 0, \qquad \text{on } \Gamma_c \\ \frac{\partial(\Delta u)}{\partial n} + \lambda \Delta u &= 0, \qquad \text{on } \Gamma_c \end{cases}, \tag{2.26}$$

with $(\lambda, \mu) \in (L^\infty(\Gamma_c) \times L^\infty(\Gamma_c))$, $\lambda \geq 0$ and $\mu \geq 0$. The formulation of our inverse problem can be articulated as follows: given the observed (Riquier-Neumann) data $u_1 \in H^{\frac{1}{2}}(\Gamma_m)$, $u_3 \in H^{\frac{-3}{2}}(\Gamma_m)$ [35]

$$\begin{cases} \frac{\partial u}{\partial n} &= u_1, \qquad \text{on } \Gamma_m \\ \frac{\partial(\Delta u)}{\partial n} &= u_3, \qquad \text{on } \Gamma_m \end{cases}, \tag{2.27}$$

we would like to determine the shape Γ_c part of the boundary $\partial\Omega$.

This problem arises in the static deflection of an elastic bending beam, where a part of the boundary of that beam is not accessible for measurements and has specific coefficients (generalised by Robin coefficients, for more details see [19, 47]). We assume that these specified coefficients are known and we seek an effective numerical algorithm for a method to recover the non-accessible part Γ_c from a single measurement of the bending moment $\frac{\partial u}{\partial n}|_{\Gamma_m}$, and the effective shear force $\frac{\partial \Delta u}{\partial n}|_{\Gamma_m}$ on the accessible part Γ_m of the boundary of that beam. It should be pointed out that, if $u = 0$ on Γ_c this is a typical example of detection of the corrosion surface of complex metal assemblies in aircraft structures in nondestructive testing. Indeed, unknown parts of the boundary that have been subjected to corrosion cannot be reached through direct inspection. The goal of this approach is to reveal the presence of such defects in a nondestructive way from the knowledge of measurements (u_1, u_3) on the accessible part Γ_m of the boundary [60].

Remark 2.2.1. *It is noteworthy that Karachik [34] provides specific sufficient conditions for the resolvability of the Robin-type problem for the Biharmonic Equation. Notably, the Robin's problem is unconditionally resolvable in the unit ball, and its solution is unique. This can be proved by confirming that the hypotheses $\mu \geq 0$ and $\lambda \geq 0$ meet the criteria outlined in Theorem 1 of [34].*

Many works are devoted to these problems and modified versions to the determination of sub-boundary for the Laplace case using the direct and indirect boundary integral equation method, detection of corrosion [10, 8, 9], in Wentzell-type (GIBC) boundary condition [58]. Various methods exist for solving the problems in applied science, to complete the missing Cauchy data there are several methods, such as the iterative method [54], Tikhonov method [11], the method of fundamental solution (MFS) in combination with the Tikhonov method [60]. For the reconstruction of the boundary (see [8, 9]).

- It is well known that, see [54, 19], for $(u_0, u_2) \in H^{\frac{3}{2}}(\Gamma_m) \times H^{-\frac{1}{2}}(\Gamma_m)$, there exists a unique solution $u \in H^2(\Omega)$ to the problem (2.24)-(2.25).

- In both cases, when $(\lambda = \mu = 0)$ and $(\lambda = \mu = \infty)$, the direct problem (2.24)-(2.26) has a unique solution in $H^2(\Omega)$ (see Section 2.1). For the other two cases, namely, $\lambda = 0, \mu = \infty$ and $\lambda = \infty, \mu = 0$, the problem (2.24)-(2.26) also has a unique solution, as discussed in [33].

- The inverse problem of determining the Robin coefficients λ and μ has been investigated in [19].

- One cannot determine λ, μ and Γ_c simultaneously from a single quad of Cauchy data (u_0, u_1, u_2, u_3) see [19]. On the other hand, the inverse problem of determining Γ_c from a single quad of the Cauchy data (u_0, u_1, u_2, u_3) on Γ_m can be understood by assuming that the coefficients λ and μ are known. This has the following meaning: given λ, μ and $u_0 \in H^{\frac{3}{2}}(\Gamma_m)$, $u_1 \in H^{\frac{1}{2}}(\Gamma_m)$, $u_2 \in H^{-\frac{1}{2}}(\Gamma_m)$ and $u_3 \in H^{-\frac{3}{2}}(\Gamma_m)$, we determine Γ_c.

- The existence of the non-accessible part Γ_c cannot be guaranteed based on arbitrary Cauchy data (u_0, u_1, u_2, u_3). Furthermore, for fixed constants λ and μ, the existence of Γ_c cannot be ensured from arbitrary Cauchy data (u_0, u_1, u_2, u_3) see [19].

2.2.2 Uniqueness of the Solution to the Inverse Problem

The following two theorems below establish the uniqueness of the part Γ_c in the cases $\mu = \lambda = \infty$ and $\lambda = \infty, \mu = 0$ (see [19]).

Theorem 2.2.1. *Consider u as a solution to the problem* (2.24) – (2.26). *If $u = \Delta u = 0$ on Γ_c then, the Cauchy data (u_0, u_1, u_2, u_3) on Γ_m uniquely determine the portion Γ_c, provided that* $|u_0| + |u_2| \neq 0$.

Proof. Consider Ω_1 and Ω_2, bounded domains such that $\partial\Omega_1 = \bar{\Gamma}_m \cup \bar{\Gamma}_c^1$ and $\partial\Omega_2 = \bar{\Gamma}_m \cup \bar{\Gamma}_c^2$ in which their corresponding solutions f_1 and f_2, respectively, verify $\Delta^2 f_i = 0$ in Ω_i, $i = 1, 2$ and satisfy $f_i = \Delta f_i = 0$ on Γ_c^i, $f_1 = f_2 = u_0$ on Γ_m, $\frac{\partial f_1}{\partial n} = \frac{\partial f_2}{\partial n} = u_1$ on Γ_m, $\Delta f_1 = \Delta f_2 = u_2$ on Γ_m, $\frac{\partial \Delta f_1}{\partial n} = \frac{\partial \Delta f_2}{\partial n} = u_3$ on Γ_m. Then according to Holmgren's theorem (1.6.6) we have $f_1 = f_2$ in $\Omega_1 \cap \Omega_2$.

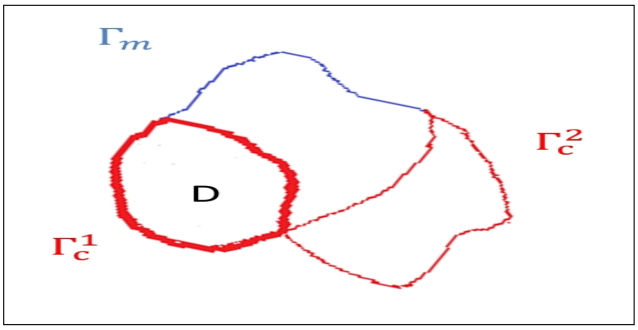

Figure 2.1: Domain D

Now we use the coupled equation technique, taking

$$\begin{cases} \Delta f_1 = h_1, & in \quad \Omega_1 \\ \Delta h_1 = 0, & in \quad \Omega_1 \end{cases} (P1), \quad and \quad \begin{cases} \Delta f_2 = h_2, & in \quad \Omega_2 \\ \Delta h_2 = 0, & in \quad \Omega_2 \end{cases} (P2)$$

and the harmonic fonctions h_1 and h_2 satisfy:

$$\begin{cases} h_1 = 0, & on \quad \Gamma_c^1 \\ h_1 = u_2, & on \quad \Gamma_m \ (P1), \\ \dfrac{\partial h_1}{\partial n} = u_3, & on \quad \Gamma_m \end{cases} \quad and \quad \begin{cases} h_2 = 0, & on \ \Gamma_c^2 \\ h_2 = u_2, & on \quad \Gamma_m \ (P2) \\ \dfrac{\partial h_2}{\partial n} = u_3, & on \quad \Gamma_m \end{cases}$$

In particular, we assume the existence of a non-empty connected part D of $\Omega_1 \setminus \bar{\Omega}_2$ (see fig. 2.1). From $f_1 = f_2$ in $\Omega_1 \cap \Omega_2$ and the boundary conditions of f_1 and f_2. Thus, we can deduce that $f_1 = h_1 = 0$ on the boundary of D (see [10]). Thus, using the Maximum-Minimum Principle (see Corollary 1.6.1) we obtain $h_1 = 0$ in D, by replacing in the above equation, therefore we get $\Delta f_1 = 0$ in D, and satisfied $f_1 = 0$ on ∂D, then $f_1 = 0$ in D and therefore, according to analyticity (see Theorem 1.27 in [2]) $f_1 = h_1 = 0$ in Ω_1 and $u_0 = u_2 = 0$. However, this leads to a contradiction with our hypothesis $|u_0| + |u_2| \neq 0$. □

Theorem 2.2.2. *Consider u as a solution to the problem* (2.24) − (2.26). *If* $\dfrac{\partial u}{\partial n} = \Delta u = 0$ *on* Γ_c *then, the Cauchy data* (u_0, u_1, u_2, u_3) *on* Γ_m *uniquely determine the portion* Γ_c, *provided that* $u_0 \neq constant$ *or* $u_2 \neq 0$.

Proof. Consider Ω_1 and Ω_2, bounded domains such that $\partial\Omega_1 = \bar{\Gamma}_m \cup \bar{\Gamma}_c^1$ and $\partial\Omega_2 = \bar{\Gamma}_m \cup \bar{\Gamma}_c^2$ in which their respective solutions f_1 and f_2, satisfy $\Delta^2 f_i = 0$ in Ω_i, $i = 1,2$ and satisfy the conditions $\frac{\partial f_1}{\partial n} = \Delta f_1 = 0$ on Γ_c^1, $\frac{\partial f_2}{\partial n} = \Delta f_2 = 0$ on Γ_c^2 and $f_1 = f_2 = u_0$ on Γ_m, $\frac{\partial f_1}{\partial n} = \frac{\partial f_2}{\partial n} = u_1$ on Γ_m, $\Delta f_1 = \Delta f_2 = u_2$ on Γ_m, $\frac{\partial \Delta f_1}{\partial n} = \frac{\partial \Delta f_2}{\partial n} = u_3$ on Γ_m. Then according to Holmgren's theorem 1.6.6 we have $f_1 = f_2$ in $\Omega_1 \cap \Omega_2$.

We take $\Delta f_i = h_i$ in Ω_i, where $\Delta h_i = 0$ in Ω_i for $i = 1,2$. In particular, and without loss of generality, we assume the existence of a nonempty connected part D of $\Omega_1 \setminus \bar{\Omega}_2$ (see fig. 2.1). From $f_1 = f_2$ in $\Omega_1 \cap \Omega_2$, and taking the boundary conditions of f_1 and f_2 then we can deduce that $\dfrac{\partial f_1}{\partial n} = h_1 = 0$ on ∂D.

Thus, $h_1 = 0$ in D, substituting in the above equation, we obtain $\Delta f_1 = 0$ in D, and satisfy $\dfrac{\partial f_1}{\partial n} = 0$ on ∂D. Thus (see proof of Theorem 2.2.1), $h_1 = 0$ in Ω_1 and $f_1 = constant$ in D, and, due to the unique continuation property for solutions of elliptic equations, $f_1 = constant$ in Ω_1 (see [49]). Hence, we deduce that $u_0 = constant$ and $u_2 = 0$, which contradicts our hypothesis. Therefore, we can infer that, at most one part Γ_c provided $u_2 \neq 0$ or $u_0 \neq constant$. □

2.2.3 Indirect Scalar Boundary Integral Equation Representation

The resolution method of our inverse problem is based on the indirect boundary integral equation method [19, 59], and aims to avoid the hyper-singularity of the boundary integral. Any biharmonic function u, as stated by Jaswon and Symm [29], can be expressed as follows.

$$u(\varphi,\psi) = \mathcal{B}_1(\varphi) + \mathcal{B}_2(\psi) \tag{2.28}$$

Where $\mathcal{B}_1(\varphi)$ and $\mathcal{B}_2(\psi)$ are the representation of the harmonic and biharmonic parts of u, respectively, such that

$$\mathcal{B}_1(\varphi) = \int_{\partial\Omega} E_1(x,z)\varphi(z)ds_z, \quad x \in \Omega,$$

and

$$\mathcal{B}_2(\psi) = \int_{\partial\Omega} E_2(x,z)\psi(z)ds_z, \quad x \in \Omega,$$

E_1 and E_2 represent the fundamental solutions corresponding to the Laplacian and Bilaplacian operators, respectively. It is noteworthy that the expression (2.28) is referred to as the Chakrabarty representation. Consequently, we define the solution u for (2.24) as:

$$u(x) = \int_{\partial\Omega} E_1(x,z)\varphi(z)ds_z + \int_{\partial\Omega} E_2(x,z)\psi(z)ds_z, \qquad x \in \Omega, \tag{2.29}$$

where

$$E_1 = \frac{1}{2\pi}\log r, \qquad E_2 = \frac{1}{8\pi}r^2\log r, \qquad r = |x-z|, \qquad x \neq z.$$

The densities $(\psi,\varphi) \in H^{\frac{-3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ are assumed continuous and verify the condition in ([40], theorem 6.28) as follows

$$\int_{\partial\Omega} \varphi(z)ds_z = \int_{\partial\Omega} \psi(z)ds_z = 0.$$

By substituting the value of u in (2.29) inside $\partial\Omega$, in boundary conditions (2.25)-(2.26) and (2.27) and using the theorems 1.6.9 and 1.6.10, we get

$$\begin{array}{rcll} \mathcal{A}\psi + \mathcal{S}\varphi &=& u_0, & \text{on } \Gamma_m \\ \mathcal{B}'\psi + \mathcal{K}'\varphi - \frac{1}{2}\varphi &=& u_1, & \text{on } \Gamma_m \\ \mathcal{S}\psi &=& u_2, & \text{on } \Gamma_m \\ \mathcal{K}'\psi - \frac{1}{2}\psi &=& u_3, & \text{on } \Gamma_m \end{array}, \tag{2.30}$$

and

$$\begin{aligned} \mathcal{B}'\psi + \mathcal{K}'\varphi - \tfrac{\varphi}{2} + (\mathcal{A}\psi + \mathcal{S}\varphi)\mu &= 0, \qquad \text{on } \Gamma_c \\ \mathcal{K}'\psi - \tfrac{\psi}{2} + (\mathcal{S}\psi)\lambda &= 0, \qquad \text{on } \Gamma_c \end{aligned}, \tag{2.31}$$

where

$$\begin{aligned} &\mathcal{A} : H^{s-1}(\partial\Omega) \longrightarrow H^{s+2}(\partial\Omega), \quad \mathcal{B}' : H^{s-1}(\partial\Omega) \longrightarrow H^{s+1}(\partial\Omega), \\ &\mathcal{S} : H^{s-1}(\partial\Omega) \longrightarrow H^{s}(\partial\Omega), \qquad \mathcal{K}' : H^{s-1}(\partial\Omega) \longrightarrow H^{s-1}(\partial\Omega). \end{aligned}$$

In the range of $-\frac{1}{2} \leq s \leq \frac{3}{2}$, the subsequent boundary integral operators are continuous [51]. They are defined for all $x \in \partial\Omega$ and expressed as:

$$\begin{aligned} (\mathcal{A}\psi)(x) &= \int_{\partial\Omega} E_2(x,z)\psi(z)ds_z, \qquad (\mathcal{S}\varphi)(x) = \int_{\partial\Omega} E_1(x,z)\varphi(z)ds_z, \\ (\mathcal{B}'\psi)(x) &= \int_{\partial\Omega} \frac{\partial E_2(x,z)}{\partial n_x}\psi(z)ds_z, \quad (\mathcal{K}'\psi)(x) = \int_{\partial\Omega} \frac{\partial E_1(x,z)}{\partial n_x}\psi(z)ds_z. \end{aligned} \tag{2.32}$$

with the following kernels, as outlined in G.C. Hsiao [26], Chapter 10.4.4.

$$\begin{aligned} \frac{\partial E_1(x,z)}{\partial n_x} &= \frac{1}{2\pi}.\frac{n_x.(x-z)}{r^2}, \\ \frac{\partial E_2(x,z)}{\partial n_x} &= \frac{1}{8\pi} n_x.(x-z)(2\ln r + 1), \\ \frac{\partial \Delta_x E_2(x,z)}{\partial n_x} &= \frac{\partial E_1(x,z)}{\partial n_x}, \\ \Delta_x E_2(x,z) &= \frac{1}{8\pi}(4\ln r + 4), \end{aligned} \tag{2.33}$$

Parametrization of the Integral Equations

To study the numerical solution of the nonlinear integral equations, a parametrization approach is essential [10, 9, 8]. For the sake of simplicity, let us assume that the boundary $\partial\Omega$ is smooth of class C^2. In this context, we define: $\partial\Omega = \{w(t) : t \in [0, 2\pi]\}$, and

$$\Gamma_m = \{w(t), 0 \leq t \leq \pi\},$$

$$\Gamma_c = \{w(t), \pi \leq t \leq 2\pi\},$$

where $w : \mathbb{R} \longmapsto \mathbb{R}^2$ is an injective twice continuously differentiable function, and 2π-periodic, such that $w'(t) \neq 0$, for all $t \in \mathbb{R}$. Furthermore, we assume that the orientations of Γ_c and Γ_m are counter–clockwise. For convenience, we introduce the vectors

$$[w]^{\perp} = (w_2, -w_1)^{\perp}.$$

that are exterior normal vectors to $\partial\Omega$. We introduce the parametrized operators

$$
\begin{aligned}
(\tilde{\mathcal{A}}\psi)(t) &= \int_0^{2\pi} E_2(w(t), w(\tau))\psi(w(\tau))|w'(\tau)|d\tau, && t \in [0, 2\pi] \\
(\tilde{\mathcal{B}}'\psi)(t) &= \int_0^{2\pi} \frac{\partial E_2(w(t), w(\tau))}{\partial n(z(t))}\psi(w(\tau))|w'(\tau)|d\tau, && t \in [0, 2\pi] \\
(\tilde{\mathcal{S}}\varphi)(t) &= \int_0^{2\pi} E_1(w(t), w(\tau))\varphi(w(\tau))|w'(\tau)|d\tau, && t \in [0, 2\pi] \\
(\tilde{\mathcal{K}}'\psi)(t) &= \int_0^{2\pi} \frac{\partial E_1(w(t), w(\tau))}{\partial n(w(t))}\psi(w(\tau))|h'(\tau)|d\tau, && t \in [0, 2\pi]
\end{aligned}, \tag{2.34}
$$

where $n(w(t)) = \frac{[w_2', -w_1']}{|w'(t)|} = \frac{[w'(t)]^\perp}{|w'(t)|}$. By considering the derivative of the fundamental solutions E_1 and E_2 with respect to $w(t)$, and introducing the transformation:

$$
\tilde{\psi}(t) = |w'(t)|\psi(w(t)), \quad \textit{and} \quad \tilde{\varphi}(t) = |w'(t)|\varphi(w(t)), \tag{2.35}
$$

we obtain, from (2.34)

$$
\begin{aligned}
(\tilde{\mathcal{A}}\tilde{\psi})(t) &= \frac{1}{8\pi}\int_0^{2\pi} |w(t) - (\tau)|^2 \ln|w(t) - w(\tau)|\tilde{\psi}(\tau)d\tau, && t \in [0, 2\pi], \\
(\tilde{\mathcal{B}}'\tilde{\psi})(t) &= \frac{1}{8\pi|w'(t)|}\int_0^{2\pi} [w'(t)]^\perp \cdot [w(t) - w(\tau)](2\ln|w(t) - w(\tau)| + 1)\tilde{\psi}(\tau)d\tau, && t \in [0, 2\pi], \\
(\tilde{\mathcal{S}}\tilde{\varphi})(t) &= \frac{1}{2\pi}\int_0^{2\pi} \ln|w(t) - w(\tau)|\tilde{\varphi}(\tau)d\tau, && t \in [0, 2\pi], \\
(\tilde{\mathcal{K}}'\tilde{\psi})(t) &= \frac{1}{2\pi|w'(t)|}\int_0^{2\pi} \frac{[w'(t)]^\perp \cdot [w(t) - w(\tau)]}{|w(\tau) - w(t)|^2}\tilde{\psi}(\tau)d\tau, && t \in [0, 2\pi].
\end{aligned} \tag{2.36}
$$

The parametrization of the nonlinear integral equations (2.30)-(2.31) is expressed by:

$$
\begin{aligned}
\tilde{\mathcal{A}}\tilde{\psi} + \tilde{\mathcal{S}}\tilde{\varphi} &= u_0 \circ w, && \text{on } [0, \pi], \\
\tilde{\mathcal{B}}'\psi + \tilde{\mathcal{K}}'\tilde{\varphi} - \frac{1}{2}\tilde{\varphi} &= u_1 \circ w, && \text{on } [0, \pi], \\
\tilde{\mathcal{S}}\tilde{\psi} &= u_2 \circ w, && \text{on } [0, \pi], \\
\tilde{\mathcal{K}}'\tilde{\psi} - \frac{1}{2}\psi &= u_3 \circ w, && \text{on } [0, \pi],
\end{aligned} \tag{2.37}
$$

and

$$
\begin{aligned}
\tilde{\mathcal{B}}'\tilde{\psi} + \tilde{\mathcal{K}}'\tilde{\varphi} - \frac{\tilde{\varphi}}{2} + (\tilde{\mathcal{A}}\tilde{\psi} + \tilde{\mathcal{S}}\tilde{\varphi})\mu &= 0, && \text{on } [\pi, 2\pi], \\
\tilde{\mathcal{K}}'\tilde{\psi} - \frac{\tilde{\psi}}{2} + (\tilde{\mathcal{S}}\tilde{\psi})\lambda &= 0, && \text{on } [\pi, 2\pi].
\end{aligned} \tag{2.38}
$$

For the discretization of the integral operators, it is essential to note that the 2π-periodic kernel has the following decomposition (see [8, 40]):

$$\begin{aligned}\ln|w(t)-w(\tau)| &= \ln\left(|w(t)-w(\tau)| \times \frac{\sin\frac{t-\tau}{2}}{\sin\frac{t-\tau}{2}}\right)\\ &= \ln\frac{|w(t)-w(\tau)|}{\sin\frac{t-\tau}{2}} + \ln\sin\frac{t-\tau}{2}.\end{aligned} \tag{2.39}$$

The initial term is smooth with a diagonal value:

$$\ln\frac{|w(t)-w(\tau)|}{|\sin\frac{t-\tau}{2}|} = \ln 2|w'(t)|.$$

The kernel $k(t,\tau)$ is 2π-periodic and smooth with diagonal value [1]

$$k(t,\tau) = \begin{cases} \dfrac{[w'(t)]^{\perp}\cdot[w(t)-w(\tau)]}{|w(t)-w(\tau)|^2}, & t \neq \tau \\ \\ \dfrac{[w'(t)]^{\perp}\cdot w''(t)}{|w'(t)|^2}, & t = \tau \end{cases},$$

and the kernels $a(t,\tau)$, $b(t,\tau)$ of the integral operators $\tilde{A}$ and $\tilde{B}'$ respectively, are smooth with vanishing diagonal values.

The system (2.37)-(2.38) is nonlinear with respect to w.

Remark 2.2.2. *The inverse problem we are addressing is equivalent to the system (2.37)-(2.38) [19, 7]. The resolution will be achieved through an iterative method in the subsequent chapter.*

CHAPTER

3 NUMERICAL METHOD AND EXAMPLES

In this chapter, our emphasis is on the **Newton-type** iterative method used to solve the parametrized system of nonlinear integral equations (2.37)-(2.38) which is equivalent to our inverse problem (see Chapter 2). It involves a partial linearization of the system with respect to the variable w by considering the Fréchet derivative of the integral operators on the boundary, as described in [48, 10, 7].

We recall the inverse problem that interests us: given the functions λ, μ and $u_0 \in H^{\frac{3}{2}}(\Gamma_m)$, $u_1 \in H^{\frac{1}{2}}(\Gamma_m)$, $u_2 \in H^{\frac{-1}{2}}(\Gamma_m)$, $u_3 \in H^{\frac{-3}{2}}(\Gamma_m)$. We would like to determine $w(t)$ the parametrization of Γ_c, such that

$$\Gamma_c = \{y = w(t); \quad \pi \leq t \leq 2\pi\}.$$

3.1 Fréchet Derivatives

We briefly review key findings pertaining to nonlinear applications within normalized spaces, as documented in [38].

Definition 3.1.1. *(Fréchet derivative [38]) Consider normed spaces X and Y, and let $M \subset X$ be an open subset of X. The mapping $\mathcal{T} : M \to Y$ is called Fréchet differentiable in $w \in M$ if there exist a linear bounded operator $\mathcal{DT}(w,\xi) : X \to Y$ such that*

$$\|\mathcal{T}(w+\xi) - \mathcal{T}(w) - \mathcal{DT}(w,\xi)\| = o(\|\xi\|) \quad uniformely \quad when \quad \|\xi\| \to 0.$$

Here, $\mathcal{DT}(w,\xi)$ is the Fréchet derivative of $\mathcal{T}$ at w in the direction of ξ. If $\mathcal{T}$ is Fréchet differentiable at all points $w \in M$, it is termed Fréchet differentiable on M.

Remark 3.1.1. *In the case of finite dimensions, where $X = \mathbb{K}^n$ and $Y = \mathbb{K}^m$, the linear bounded* *mapping $\mathcal{DT}(w,\xi)$ can be expressed as*

$$\mathcal{DT}(w,\xi) = J\xi(t),$$

Here, J represents the Jacobian matrix (in terms of Cartesian coordinates), with its elements given by

$$J_{ij} = \frac{\partial \mathcal{T}_i}{\partial w_j}.$$

The subsequent theorem compiles additional properties of the Fréchet derivative.

Theorem 3.1.1. *Let $\mathcal{T} : M \subset X \to Y$ Fréchet-différentiable, and V be a normed space.*

- *The Fréchet derivative of $\mathcal{T}$ is uniquely determined.*
- *If $\mathcal{V} : M \to Y$ is Fréchet-differentiable, then $\lambda_1\mathcal{T} + \lambda_2\mathcal{V}$ is Fréchet-differentiable for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and*

$$\mathcal{D}(\lambda_1\mathcal{T} + \lambda_2\mathcal{V})(w) = \lambda_1\mathcal{DT}(w) + \lambda_2\mathcal{DV}(w), \quad w \in M.$$

- *If $\mathcal{V} : Y \to V$ is Fréchet-differentiable, then $\mathcal{V} \circ \mathcal{T} : M \to N$ is Fréchet-differentiable and*

$$\mathcal{D}(\mathcal{V} \circ \mathcal{T})(w) = \mathcal{DV}[\mathcal{T}(w)]\mathcal{DT}(w), \quad w \in M.$$

Proof. Proofs can be found in [30] on pp. 103-106. □

The subsequent theorem proves that the ill-posed nature of a nonlinear problem is due to its linearization.

Theorem 3.1.2. *Consider the operator $\mathcal{T} : M \subset X \longrightarrow Y$, where $\mathcal{T}$ is a completely continuous operator from an open subset M of a normed space X into a Banach space Y. Assume that $\mathcal{T}$ is Fréchet differentiable at $\phi \in M$. Then, the derivative $\mathcal{DT}(\phi)$ is compact.*

Proof. The proof of this theorem relies on the observation that a subset of a Banach space is relatively compact if and only if it is totally bounded. For a detailed explanation, refer to Theorem 4.19 in [15]. □

Example 3.1.1. *Let us compute the Fréchet derivative of $F(w) = \dfrac{1}{|w'(t)|}$.*
The update (perturbation) of the functions $w(t), w'(t)$ are denoted by $w(t) + \xi(t)$ and $w'(t) + \xi'(t)$, respectively. In this case, $w(t) = (w_1(t), w_2(t))$, and the Jacobian has an order of 1×2, given by

$$J = \left(\frac{\partial F}{\partial w_1}, \frac{\partial F}{\partial w_2}\right).$$

Thus, we have

$$\mathcal{D}(\frac{1}{|w'(t)|}) = \frac{\partial}{\partial w_1}(\frac{1}{|w'(t)|})\xi_1'(t) + \frac{\partial}{\partial w_2}(\frac{1}{|w'(t)|})\xi_2'(t).$$

Hence, we obtain $\mathcal{D}F(w,\xi) = \dfrac{-w'(t).\xi'(t)}{|w'(t)|^3}$.

Example 3.1.2. ***(Integral operator)***

Consider $f : [c,d] \times [a,b] \times \mathbb{C} \longrightarrow \mathbb{C}$ *as a continuous function, and let* $w \in C^1([a,b])$. *We define the integral operator* $\mathcal{T}$ *by:*

$$\mathcal{T}(w) = \int_a^b f(t,s,w(s))ds, \quad t \in [c,d].$$

Then, $\mathcal{T}$ *is continuously differentiable with the Fréchet derivative:*

$$\mathcal{D}\mathcal{T}(w,\xi) = \int_a^b \frac{\partial}{\partial w} f(t,s,w(s))\xi(s)ds, \quad t \in [c,d]. \tag{3.1}$$

Remark 3.1.2. *The Fréchet derivatives of the integral operators* $\tilde{\mathcal{A}}$ *and* $\tilde{\mathcal{B}}'$ *with respect to* w *can be analytically determined by differentiating their kernels with respect to* w.

Computation of Fréchet Derivatives of $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}'$:

We will employ Definition 3.1.1 and the properties outlined in Theorem 3.1.1 to compute the Fréchet derivatives of the nonlinear integral operators $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}'$, considering $t \neq \tau$ and $h \in C^2[0,2\pi]$.

The computation of $\mathcal{D}\tilde{\mathcal{A}}(w,\xi,\tilde{\psi})(t)$:

The Fréchet derivative $\mathcal{D}(|w(t) - w(\tau)|^2)$ is given by:

$$2[w(t) - w(\tau)] \cdot [\xi(t) - \xi(\tau)]. \tag{3.2}$$

In [10], the Fréchet derivative $\mathcal{D}(\ln|w(t) - w(\tau)|)$ is expressed as:

$$\frac{[w(t) - w(\tau)] \cdot [\xi(t) - \xi(\tau)]}{|w(t) - w(\tau)|^2}. \tag{3.3}$$

Combining (3.2) and (3.3), we derive the Fréchet derivative of $\tilde{\mathcal{A}}$:

$$\begin{aligned}\mathcal{D}\tilde{\mathcal{A}}(w,\xi,\tilde{\psi}) = \frac{1}{8\pi}\int_0^{2\pi} \Big\{ & 2[w(t) - w(\tau)] \cdot [\xi(t) - \xi(\tau)] \ln|w(t) - w(\tau)|\tilde{\psi}(\tau) \\ & + [w(t) - w(\tau)] \cdot [\xi(t) - \xi(\tau)]\tilde{\psi}(\tau)\Big\} d\tau \end{aligned} \quad t \in [0,2\pi]. \tag{3.4}$$

The Compute of $\mathcal{D}\tilde{\mathcal{B}}'(w,\xi,\tilde{\psi})(t)$

In the same way, we compute the Fréchet derivative of $\tilde{\mathcal{B}}'$. The Fréchet derivative $\mathcal{D}(\frac{1}{8\pi|w'(t)|})$ can be obtained as:

$$\frac{-w'(t).\xi'(t)}{|w'(t)|^2}\frac{1}{8\pi|w'(t)|}. \tag{3.5}$$

on the other hand, the Fréchet derivative of $[w'(t)]^{\perp}\cdot[w(t)-w(\tau)](2\ln|w(t)-w(\tau)|+1)$ can be obtained as

$$\begin{aligned}&\frac{2[w(t)-w(\tau)]\cdot[\xi(t)-\xi(\tau)]\cdot[w'(t)]^{\perp}\cdot[w(t)-w(\tau)]}{|w(t)-w(\tau)|^2}+\\&+([w'(t)]^{\perp}\cdot[\xi(t)-\xi(\tau)]+[\xi'(t)]^{\perp}\cdot[w(t)-w(\tau)])(2\ln|w(t)-w(\tau)|+1).\end{aligned} \tag{3.6}$$

Then, the Fréchet derivative of the integral operator $\tilde{B}'$ is given by

$$\begin{aligned}\mathcal{D}\tilde{\mathcal{B}}'(w,\xi,\psi)=&\int_0^{2\pi}\{([\xi'(t)]^{\perp}.[w(t)-w(\tau)]+[w'(t)]^{\perp}.[\xi(t)-\xi(\tau)])\times(2\ln|w(t)-w(\tau)|+1))\\&+([w'(t)]^{\perp}.[w(t)-w(\tau)])\frac{2[w(t)-w(\tau)][\xi(t)-\xi(\tau)]}{|w(t)-w(\tau)|^2}\}\tilde{\psi}(\tau)d\tau\\&-\frac{w'(t).\xi'(t)}{|w'(t)|^2}\tilde{\mathcal{B}}'(\tilde{\psi},w)(t)\end{aligned} \tag{3.7}$$

For the discretization of the integral operators, the kernels of the integral operators $\mathcal{D}\tilde{\mathcal{A}}$ and $\mathcal{D}\tilde{\mathcal{B}}'$ are smooth when $t=\tau$, using the properties of the limit, the kernels are with vanishing diagonal values.

The Fréchet derivatives of $\tilde{S}(w,\xi,\tilde{\psi})(t)$ and $\tilde{K}'(w,\xi,\tilde{\psi})(t)$ with respect to w are already given by [10]:

$$\mathcal{D}\tilde{\mathcal{S}}(w,\xi,\tilde{\psi})=\frac{1}{2\pi}\int_0^{2\pi}\frac{[w(t)-w(\tau)]\cdot[\xi(t)-\xi(\tau)]}{|w(t)-w(\tau)|^2}\tilde{\psi}(\tau)d\tau,\quad t\in[0,2\pi] \tag{3.8}$$

and

$$\begin{aligned}\mathcal{D}\tilde{\mathcal{K}}'(h,\xi,\tilde{\psi})=-\frac{1}{2\pi|w'(t)|}\int_0^{2\pi}&\Big\{\frac{2[w'(t)]^{\perp}\cdot[w(t)-w(\tau)]\cdot[w(t)-w(\tau)]\cdot[\xi(t)-\xi(\tau)]}{|w(t)-w(\tau)|^4}\\&+\frac{[w'(t)]^{\perp}\cdot[\xi(t)-\xi(\tau)]+[\xi'(t)]^{\perp}\cdot[w(t)-w(\tau)]}{|w(t)-w(\tau)|^2}\Big\}\tilde{\psi}(\tau)d\tau+\frac{w'(t)\cdot\xi'(t)}{|w'(t)|^2}\tilde{\mathcal{K}}'(\psi,w)(t)\end{aligned},\quad t\in[0,2\pi]. \tag{3.9}$$

The kernels $\mathcal{D}s(w,\xi)$ and $\mathcal{D}k(w,\xi)$ of the operator $\mathcal{D}\tilde{S}$ and $d\tilde{K}$, respectively are smooth with their diagonal values [8] given by:

$$\mathcal{D}s(t,\tau)=\begin{cases}\dfrac{[w(t)-w(\tau)]\cdot[\xi(t)-\xi(\tau)]}{|w(t)-w(\tau)|^2}, & t\neq\tau\\ \dfrac{w'(t)\xi'(t)}{|w'(t)|^2}, & t=\tau\end{cases},$$

and

$$\mathcal{D}k(t,\tau)=\begin{cases}\dfrac{2[w'(t)]^{\perp}\cdot[w(t)-w(\tau)]\cdot[w(t)-w(\tau)]\cdot[\xi(t)-\xi(\tau)]}{|w(t)-w(\tau)|^4} \\ \quad -\dfrac{|w'(t)|^{\perp}\cdot[\xi(t)-\xi(\tau)]+[\xi'(t)]^{\perp}\cdot[w(t)-w(\tau)]}{|w(t)-w(\tau)|^2}, & t\neq\tau\\ -\dfrac{[w'(t)]^{\perp}\cdot w''(t)\cdot w'(t)\cdot\xi'(t)}{|w'(t)|^4}+\dfrac{[w'(t)]^{\perp}\cdot\xi''(t)+[\xi'(t)]^{\perp}\cdot w''(t)}{2|w'(t)|^2}, & t=\tau\end{cases}.$$

In the above expressions, we note that the perturbation ξ is nontrivial on the interval $[\pi,2\pi]$. Replace the integral operators by these linearizations, then the linearization of the system (2.37)-(2.38), leads to:

$$\begin{aligned}\tilde{\mathcal{A}}(w,\tilde{\psi})+\tilde{\mathcal{A}}(w,\chi)+d\tilde{\mathcal{A}}(w,\xi,\tilde{\psi})+\tilde{\mathcal{S}}(w,\tilde{\varphi})+\tilde{\mathcal{S}}(w,\phi)+\mathcal{D}\tilde{\mathcal{S}}(w,\xi,\tilde{\varphi})&=u_0\circ w &&\text{on } [0,\pi],\\ \tilde{\mathcal{B}}'(h,\tilde{\psi})+\tilde{\mathcal{B}}'(w,\chi)+\mathcal{D}\tilde{\mathcal{B}}'(w,\xi,\tilde{\psi})+\tilde{\mathcal{K}}'(w,\tilde{\varphi})+\tilde{\mathcal{K}}'(w,\phi)+\mathcal{D}\tilde{\mathcal{K}}'(w,\xi,\tilde{\varphi})-\frac{\tilde{\varphi}}{2}&=u_1\circ w &&\text{on } [0,\pi],\\ \tilde{\mathcal{S}}(w,\tilde{\psi})+\tilde{\mathcal{S}}(w,\chi)+\mathcal{D}\tilde{\mathcal{S}}(w,\xi,\tilde{\psi})&=u_2\circ w &&\text{on } [0,\pi],\\ \tilde{\mathcal{K}}'(w,\tilde{\psi})+\tilde{\mathcal{K}}'(w,\chi)+\mathcal{D}\tilde{\mathcal{K}}'(w,\xi,\tilde{\psi})-\frac{\tilde{\psi}}{2}&=u_3\circ w &&\text{on } [0,\pi],\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}&\tilde{\mathcal{B}}'(w,\tilde{\psi})+\tilde{\mathcal{B}}'(w,\tilde{\chi})+\mathcal{D}\tilde{\mathcal{B}}'(w,\xi,\tilde{\psi})+\tilde{\mathcal{K}}'(w,\tilde{\varphi})+\tilde{\mathcal{K}}'(w,\tilde{\phi})+\mathcal{D}\tilde{\mathcal{K}}'(w,\xi,\tilde{\varphi})\\ &-\frac{\tilde{\varphi}}{2}+[\tilde{\mathcal{A}}(w,\tilde{\psi})+\tilde{\mathcal{A}}(w,\tilde{\chi})+\mathcal{D}\tilde{\mathcal{A}}(w,\xi,\tilde{\psi})+\tilde{\mathcal{S}}(w,\tilde{\varphi})+\tilde{\mathcal{S}}(w,\tilde{\phi})+\mathcal{D}\tilde{\mathcal{S}}(w,\xi,\tilde{\varphi})]\mu=0 \qquad \text{on } [\pi,2\pi],\end{aligned}\tag{3.11}$$

$$\tilde{\mathcal{K}}'(w,\tilde{\psi})+\tilde{\mathcal{K}}'(w,\tilde{\chi})+\mathcal{D}\tilde{\mathcal{K}}'(w,\xi,\tilde{\psi})-\frac{\tilde{\psi}}{2}+[\tilde{\mathcal{S}}(w,\tilde{\psi})+\tilde{\mathcal{S}}(w,\tilde{\chi})+\mathcal{D}\tilde{\mathcal{S}}(w,\xi,\tilde{\psi})]\lambda=0 \qquad \text{on } [\pi,2\pi].\tag{3.12}$$

3.2 Reconstruction Algorithm

We will use an iterative method for the resolution, where each iteration involves solving the ill-posed system (2.37) for ψ and φ on Γ_m using the Tikhonov regularization method. After fixing ψ and φ, we then solve the linearized system (3.11)-(3.12) to obtain the update $w+\xi$ for the parameterization of the inaccessible part Γ_c. Updates $\tilde{\psi}+\chi$ and $\tilde{\varphi}+\phi$ can be obtained for the densities $\tilde{\psi}$ and $\tilde{\varphi}$, respectively. We assume that ξ is nonzero only on the inaccessible part Γ_c of $\partial\Omega$ and is zero otherwise, so the process naturally iterates.
Our algorithm includes the following steps:

Step 1: Given the exact data (u_0, u_2), we compute the Cauchy data (u_1, u_3) by solving integral equations.

Step 2: Perturb the data $u_1^\delta = u_1 + \delta\epsilon(t)$ and $u_3^\delta = u_3 + \delta\epsilon(t)$, where $\epsilon(t)$ is a Gaussian noise.

Step 3: Solve the ill-posed system (2.37), with respect to $\tilde{\psi}$ and $\tilde{\varphi}$ on Γ_m by the Tikhonov regularization method.

Step 4: Initialize $w = w_0$ and fix the parameters of the discrete problem.

Step 5:

- Solve the linear system (3.11)-(3.12) by Tikhonov regularization for ξ, χ and ϕ to get the updates $w+\xi$ for the parametrization of Γ_c, $\psi+\chi$, $\varphi+\phi$ for the densities ψ, φ respectively.

- put $w = w_0 + \xi$
  Fin

Remark 3.2.1. *The stopping criterion is an important issue for iterative regularisation method since the approximation will deteriorate for inexact data after a certain number of iterations [28, 22]. In our algorithm we compute the error at each step and we stop after a few iterations, when the error attains its minimum.*

The following theorem establishes the injectivity of the linearized system at the exact solution, considering the limiting case $\lambda = \mu = \infty$.

Theorem 3.2.1. *Let w be the parametrization of the exact boundary $\partial\Omega$, and let $\tilde{\psi} = |w'|\psi(w)$ and $\tilde{\varphi} = |w'|\varphi(w)$, where ψ and φ satisfy (2.30)-(2.31) for $\lambda = \mu = \infty$.*

Assume that $\chi \in H^{\frac{3}{2}}[0,2\pi]$, $\phi \in H^{-\frac{1}{2}}[0,2\pi]$ and $\xi \in C^2[0,2\pi]$ with $\xi = 0$ on $[0,\pi]$ and $\xi(t).n(w(t)) \neq 0$ for $t \in (\pi, 2\pi)$ satisfy the homogeneous system:

$$\begin{aligned}
\tilde{\mathcal{A}}(w,\chi) + \mathcal{D}\tilde{\mathcal{A}}(w,\xi,\tilde{\psi}) + \tilde{\mathcal{S}}(w,\phi) + \mathcal{D}\tilde{\mathcal{S}}(w,\xi,\tilde{\varphi}) = 0, \qquad & on\ [0,\pi] \\
\tilde{\mathcal{B}}'(w,\chi) + \mathcal{D}\tilde{\mathcal{B}}'(w,\xi,\tilde{\psi}) + \tilde{\mathcal{K}}'(w,\phi) + \mathcal{D}\tilde{\mathcal{K}}'(w,\xi,\tilde{\varphi}) = 0, \qquad & on\ [0,\pi] \\
\tilde{\mathcal{S}}(w,\chi) + \mathcal{D}\tilde{\mathcal{S}}(w,\xi,\tilde{\psi}) = 0, \qquad & on\ [0,\pi] \\
\tilde{\mathcal{K}}'(w,\chi) + \mathcal{D}\tilde{\mathcal{K}}'(w,\xi,\tilde{\psi}) = 0, \qquad & on\ [0,\pi]
\end{aligned}, \tag{3.13}$$

and

$$\tilde{\mathcal{A}}(w,\chi) + \mathcal{D}\tilde{\mathcal{A}}(w,\xi,\tilde{\psi}) + \tilde{\mathcal{S}}(w,\phi) + \mathcal{D}\tilde{\mathcal{S}}(w,\xi,\tilde{\varphi}) = 0, \qquad on\ [\pi,2\pi] \tag{3.14}$$

$$\tilde{\mathcal{S}}(w,\chi) + \mathcal{D}\tilde{\mathcal{S}}(w,\xi,\tilde{\psi}) = 0, \qquad on\ [\pi,2\pi]. \tag{3.15}$$

Then $\xi = 0$, $\chi = 0$ and $\phi = 0$.

Proof. Let us define

$$\begin{aligned}
P(x) = &\int_0^{2\pi} E_2(x,w(\tau))\chi(\tau)d\tau + \int_0^{2\pi} E_1(x,w(\tau))\phi(\tau)d\tau + \int_0^{2\pi} \psi(\tau).\,\nabla_x E_2(x,w(\tau)).\xi(\tau)d\tau \\
&+ \int_0^{2\pi} \varphi(\tau).\,\nabla_x E_1(x,w(\tau)).\xi(\tau)d\tau,
\end{aligned} \tag{*}$$

and

$$Q(x) = \int_0^{2\pi} E_1(x,w(\tau))\chi(\tau)d\tau + \int_0^{2\pi} \psi(\tau).\,\nabla_x E_1(x,w(\tau).\xi(\tau)d\tau. \tag{**}$$

Taking the boundary values and the normal derivative of P and Q when approaching the boundary, from (3.13) we obtain that $P|_{\Gamma_m} = \frac{\partial p}{\partial n}|_{\Gamma_m} = Q|_{\Gamma_m} = \frac{\partial Q}{\partial n}|_{\Gamma_m} = 0$. By applying Holmgren's theorem [25] in (**) we obtain $Q = 0$ in Ω. By substituting $Q = 0$ in (*) similarly, one can deduce that $P = 0$ in Ω.

Applying the coupled equation technique [54], u is a solution of the boundary value problem (2.24)-(2.27) for $\lambda = \mu = \infty$, we take $\Delta u = h$ in Ω where $\Delta h = 0$, and the harmonic function h satisfy $h = u_2$ on Γ_m, $\frac{\partial h}{\partial n} = u_3$ on Γ_m, subtracting $P = 0$ and $Q = 0$ on Γ_c, from (3.15) and (3.14) we observe that $\xi.\nabla_x u = 0$ on Γ_c and $\xi.\nabla_x h = 0$ on Γ_c. Given the boundary situation $u = 0$ and $h = 0$ on Γ_c and by the hypothesis $\xi.n(w(t)) \neq 0$ for $t \in (\pi, 2\pi)$, the Holmgren's theorem implies that $\xi = 0$. Due to our geometric assumption on Ω, ensuring the injectivity of the operator S in $\partial\Omega$ (see [39]) we obtain $\phi = 0$ and from (3.14) we deduce that $\chi = 0$. □

3.3 Example of Reconstructions

To address potential singularities in the solution of (2.30)-(2.31) at the two intersection points, a sigmoidal transformation with a parameter $p = 6$ is applied for the forward problem, and $p = 8$ is used for the inverse algorithm. This transformation is crucial to avoid issues related to the inverse crime.

To simulate the Riquier-Neumann data (u_1, u_3) with noise, random noise of a given level δ is added. This step is essential for a more realistic representation of the data, considering the practical limitations in obtaining noise-free measurements.

$$u_1^\delta = u_1 + \epsilon \frac{\|u_1\|_{L^2(\Gamma_m)}}{\|\xi\|_{L^2(\Gamma_m)}} \xi, \qquad u_3^\delta = u_3 + \epsilon \frac{\|u_3\|_{L^2(\Gamma_m)}}{\|\xi\|_{L^2(\Gamma_m)}} \xi, \tag{3.16}$$

with

$$\|u_1^\delta - u_1\|_{L^2(\Gamma_m)} \leq \delta, \qquad \|u_3^\delta - u_3\|_{L^2(\Gamma_m)} \leq \delta, \tag{3.17}$$

where ξ is a normally distributed random variable and ϵ is the relative noise level.

In each iteration of the inverse algorithm, we approximately solve the system (3.11)-(3.12) using Tikhonov regularization. with an H^2 penalty term on ξ with regularization parameter γ_0, an L^2 penalty term on the densities with parameter γ_1.

For the inverse problem, we employ the projection method, wherein we utilize the Fourier approximation of a function. This approximation is essentially the projection of a function onto the vector space spanned by the orthonormal set defined by (3.21). In our numerical examples, the iteration steps and regularization parameters are chosen by trial and errors.

The integral operators and densities were discretized using a grid with $2n = 64$ points on the boundary. Notably, it should be emphasized that the parametrization of the update ξ obtained from (3.11) and (3.12) is not unique. To address this ambiguity, we opt for a star-like parametrization, involving a non-negative function r that signifies the radial distance of Γ_c from the origin.

$$w(t) = r(t)(\cos t, \sin t), \qquad t \in [\pi, 2\pi] \tag{3.18}$$

Using a real function q, the update of the boundary part Γ_c can be expressed as:

$$\xi(t) = q(t)(\cos t, \sin t), \qquad t \in [\pi, 2\pi] \tag{3.19}$$

We consider an approximation space Q_m of a trigonometric polynomial basis [58] given as follows:

$$\{1, \cos(jt), \sin(jt), \quad j = 1, 2...\} \tag{3.20}$$

In our approach of iteration approximation, we assume that q can be represented as follows:

$$q(t) \approx a_0 + \sum_{j=1}^{m} \Big\{ a_j \cos(jt) + b_j \sin(jt) \Big\}, \qquad \text{for } m \in \mathbb{N} \tag{3.21}$$

The first two examples are in conformity with the homogeneous Navier boundary condition on the non-accessible part Γ_c, which corresponds to the limiting case $\mu = \lambda = \infty$. The Cauchy data (u_0, u_1, u_2, u_3) on Γ_m were obtained by solving the Robin problem in Ω, with the Navier boundary condition:

$$\Delta u(t) = \begin{cases} 1 + \sin^2 t, & t \in [0, \pi] \\ 0, & t \in [\pi, 2\pi] \end{cases}, \qquad u(t) = \begin{cases} 1 + \cos^4 t, & t \in [0, \pi] \\ 0, & t \in [\pi, 2\pi] \end{cases}.$$

In order to examine the influence of different choices of Navier boundary conditions and Robin coefficients on the reconstruction. Let the synthetic Cauchy data (u_0, u_1, u_2, u_3) on Γ_m be obtained by solving the Robin problem in Ω, with the boundary conditions

$$\frac{\partial u}{\partial n} + \mu u = h \quad , \quad \frac{\partial \Delta u}{\partial n} + \lambda \Delta u = g,$$

In the example 3.3.3, the profiles of Robin coefficients are given by:

$$\mu(t) = \begin{cases} 0, & t \in [0, \pi] \\ 1 + \cos^4 t, & t \in [\pi, 2\pi] \end{cases}, \quad \lambda(t) = \begin{cases} 0, & t \in [0, \pi] \\ 1 + \sin^4 t, & t \in [\pi, 2\pi] \end{cases},$$

$$h(t) = \begin{cases} 1 + \sin^4 t, & t \in [0, \pi] \\ 0, & t \in [\pi, 2\pi] \end{cases}, \quad g(t) = \begin{cases} 1 + \cos^4 t, & t \in [0, \pi] \\ 0, & t \in [\pi, 2\pi] \end{cases}$$

For the last example 3.3.4, we take $\mu = 5$ and $\lambda = 2$ with

$$h(t) = \begin{cases} 0, & t \in [\pi, 2\pi] \\ \sin^2 t, & t \in [0, \pi] \end{cases}, \quad g(t) = \begin{cases} 0, & t \in [\pi, 2\pi] \\ \cos^2 t, & t \in [0, \pi] \end{cases}$$

Example 3.3.1. *We start with an ellipse having the following parametrization:*

$$w_c(t) = (0.5\cos t; 0.3\sin t), \tag{3.22}$$

with $m = 8$ in (3.21) and we give the parametric form of accessible boundary Γ_m by:

$$w_m(t) = (0.5\cos t; 0.3\sin t). \tag{3.23}$$

Let the initial value of the inaccessible boundary Γ_c is $w_0(t) = (0.5cost, 0.5sint)$, $\quad t \in [\pi, 2\pi]$. In Figures 3.1 and 3.2 we show the reconstruction of the inaccessible part Γ_c with 8 iteration steps in the exact Cauchy data and the noisy data under noise $\delta = 3\%$ for 10 iteration steps. The regularization parameters are chosen by trial and errors as $\gamma_0 = \gamma_1 = 10^{-12}$ for exact data and $\gamma_0 = \gamma_1 = 10^{-5}$ for the noise data.

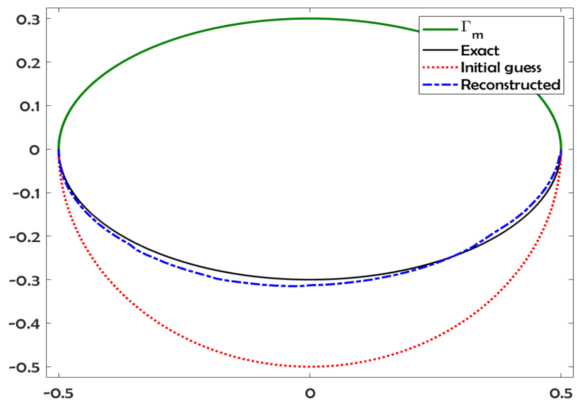

Figure 3.1: Reconstruction with 3% noise, 10 iteration steps, with respect to regularization parameters $\gamma_0 = \gamma_1 = 10^{-5}$

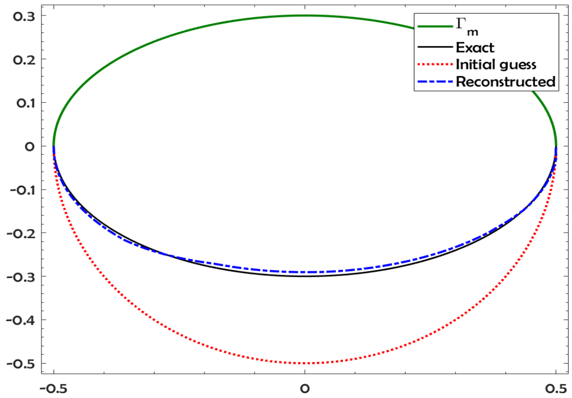

Figure 3.2: Reconstruction without noise, 8 iteration steps, with respect to regularization parameters $\gamma_0 = \gamma_1 = 10^{-12}$

Example 3.3.2. *We consider an apple shaped contour with the parameterization*

$$w_c(t) = 0.5\sqrt{\cos^2 t + 0.15\sin^2 t}\,(\cos t; \sin t), \tag{3.24}$$

with $m = 10$ in (3.21) and the accessible boundary Γ_m is parametrised by a half of a circle

$$w_m(t) = 0.5\,(\cos t; \sin t). \tag{3.25}$$

The initial value of the inaccessible boundary Γ_c is $w_0(t) = (0.5\cos t, 0.8\sin t)$, $\quad t \in [\pi, 2\pi]$. The Figures 3.3 and 3.4 show the reconstruction of the inaccessible part Γ_c under 10 iteration steps in the exact Cauchy data and 14 iteration steps in the noisy data with $\delta = 3\%$ added to the Riquier-Neumann data. The regularization parameters are chosen by trial and errors with $\gamma_0 = \gamma_1 = 10^{-12}$ for exact data and $\gamma_0 = \gamma_1 = 10^{-6}$ for the noise data.

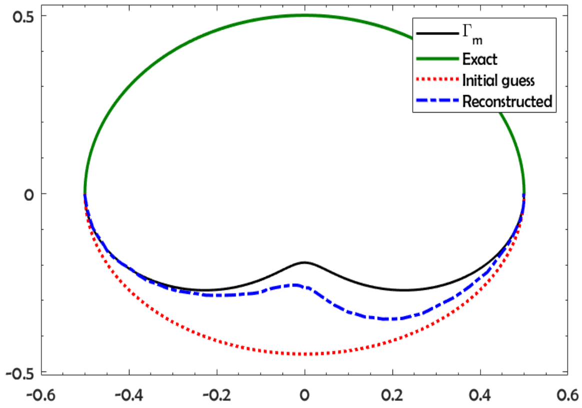

Figure 3.3: Reconstruction with 3% noise, 14 iteration steps, with respect to regularization parameters $\gamma_0 = \gamma_1 = 10^{-6}$

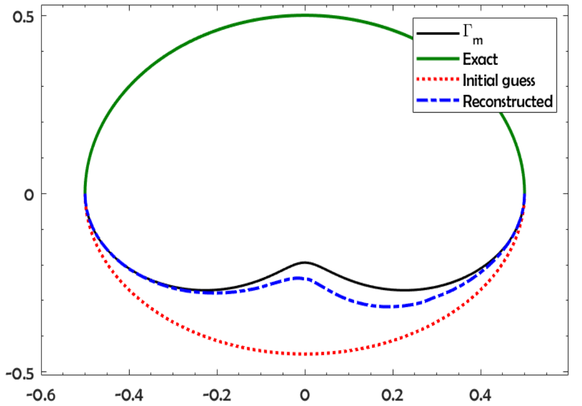

Figure 3.4: Reconstruction without noise, 10 iteration steps, with respect to regularization parameters $\gamma_0 = \gamma_1 = 10^{-12}$

Example 3.3.3. *In the third example, we consider a kite-shaped contour with the parameterization*

$$w_c(t) = (-0.3\cos t - 0.12\sin 2t; 0.3\sin t), \tag{3.26}$$

with $m = 12$ in (3.21), and we give the parametric form of accessible boundary Γ_m

$$w_m(t) = (-0.3\cos t - 0.12\sin 2t; 0.3\sin t). \tag{3.27}$$

The initial value of the inaccessible boundary Γ_c is $w_0(t) = (0.25cost; 0.25sint)$, $t \in [\pi, 2\pi]$. The Figures 3.5 and 3.6 we show the reconstruction of the inaccessible part Γ_c with 8 iteration steps in the exact Cauchy data and the noisy data under noise $\delta = 3\%$ for 10 iteration steps. The regularization *parameters are chosen by trial and errors as* $\gamma_0 = \gamma_1 = 10^{-13}$ *for exact data and* $\gamma_0 = \gamma_1 = 10^{-8}$ *for the noise data.*

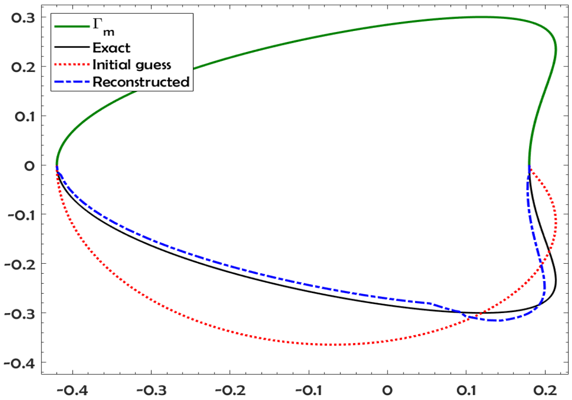

Figure 3.5: Reconstruction with 3% noise, 10 iteration steps, with respect to regularization parameters $\gamma_0 = \gamma_1 = 10^{-13}$

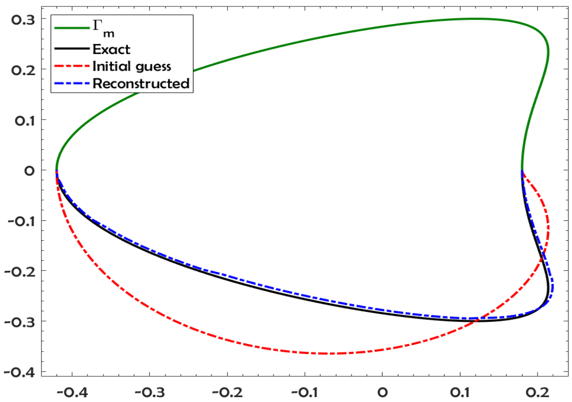

Figure 3.6: Reconstruction without noise, 8 iteration steps, with respect to regularization parameters $\gamma_0 = \gamma_1 = 10^{-8}$

Example 3.3.4. *In the last example we consider an apple contour represented by the parameterization on* Γ_c

$$w_c(t) = 0.5\frac{0.5 + 0.4\cos t + 0.1 sint}{1 + 0.8\cos t}(\cos t; \sin t), \tag{3.28}$$

with $m = 12$ *in (3.21) and the accessible boundary* Γ_m *is parametrised by:*

$$w_m(t) = 0.5\frac{0.5 + 0.4\cos t + 0.1 sint}{1 + 0.8\cos t}(\cos t; \sin t). \tag{3.29}$$

The initial value of the inaccessible boundary Γ_c *is* $w_0(t) = (-0.3\cos t - 0.12\sin 2t; 0.3\sin t + 0.12\sin 2t)$, $t \in [\pi, 2\pi]$. *The Figures 3.7 and 3.8 show the reconstruction of the inaccessible part* Γ_c *under 8 iteration steps in the exact Cauchy data and 10 iteration steps in the noisy data with* $\delta = 5\%$ *added to the Riquier-Neumann data . The regularization parameters are chosen by trial and errors with* $\gamma_0 = \gamma_1 = 10^{-13}$ *for exact data and* $\gamma_0 = \gamma_1 = 10^{-8}$ *for the noise data.*

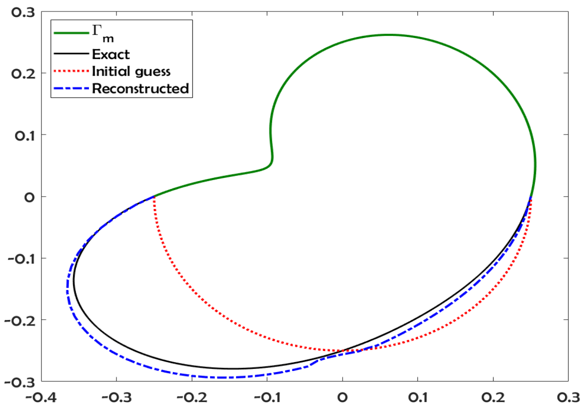

Figure 3.7: Reconstruction with 5% noise, 10 iteration steps, with respect to regularization parameters $\gamma_0 = \gamma_1 = 10^{-13}$

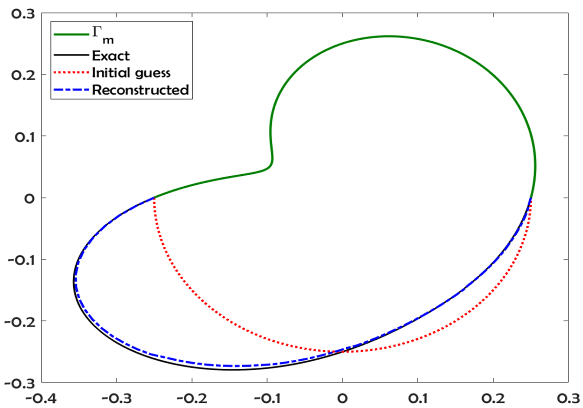

Figure 3.8: Reconstruction without noise, 8 iteration steps, with respect to regularization parameters $\gamma_0 = \gamma_1 = 10^{-8}$

Discussion

The Figures. 3.1 - 3.8 show the exact and the numerical results for Examples 3.3.1, 3.3.2, 3.3.3 and 3.3.4, involving both exact and noisy data. The reconstructions are obtained after 8 iterations for exact data for Examples 3.3.1, 3.3.3, 3.3.4 and 10 iterations for Example 3.3.2.

Regarding noisy data, 10 iterations for Examples 3.3.1, 3.3.3 and 4, and 14 iterations for Example 3.3.2. The numerical experiments show satisfying reconstructions under the restrictions:

- $-0.5 \leq w(t) \leq 0$,
- with an appropriate initial approximation w_0,
- the degree of the polynomial which interpolates w ($m = 8$ for Example 3.3.1, $m = 10$ for Example 2 and $m = 12$ for Examples 3.3.3-3.3.4),
- $\delta = 3\%$ for Example 3.3.1 - 3.3.3, $\delta = 5\%$ for Example 3.3.4.

We remark that with arbitrary initial guess w_0 the algorithm fails to recover the boundary.

3.4 Conclusion and Perspective

Reconstructing shapes from partial knowledge of solutions to corresponding boundary value problems presents a formidable class of inverse problems. In this study, we addressed the specific case of bi-harmonic differential equations with Navier boundary conditions on a known part of the domain boundary and Robin's conditions on an unknown part. The inverse problem under consideration involved recovering the shape of the unknown part of the boundary once Riquier-Neumann data on the known part are accessible through measurements.

The inverse problem was approximated using a Gauss-Newton method. We employed the iterative method in conjunction with the Tikhonov regularization technique to ensure stability. Numerical examples demonstrated the feasibility of recovery for both exact and noisy data with a suitable initial approximation. However, disturbances to the initial approximation posed challenges for the algorithm in accurately recovering the boundary. Further research is necessary to enhance the algorithm's performance in boundary reconstruction. This includes introducing additional regularization treatments such as the choice of the parametrization of w, the initial approximation w_0, and the stopping creteria to stop the iterations.

Future research should extend these studies to the case of the full inverse problem to simultaneously recover the shape and the impedance function. The adopted methodology relies on an inverse approach, aiming to determine the geometry of the missing boundary from available Cauchy data. This inverse approach is commonly employed in fields such as tomography and other inverse problems in applied sciences.

BIBLIOGRAPHY

[1] K.E. Atkinson, *The numerical solution of integral equations of the second kind*. Camb monog on app and comput math 2009.

[2] P. Bourdon, W. Ramey, *Harmonic Function Theory* Springer 2001.

[3] A.Benrabah and N. Boussetila, *Modifed nonlocal boundary value problem method for an ill-posed problem for the biharmonic equation*, Inv Probs in Sci and Engi (2018) pp. 340-368.

[4] A. Benrabah, N. Boussetila and F. Rebbani, *Modifed auxiliary boundary conditions method for an ill-posed problem for the homogeneous biharmonic equation*.43-1 Math Meth App Sci. (2019) pp 358-383.

[5] M. Bonnet *Problèmes inverses*, Ecole Centrale de Paris, 2008.

[6] H. Brézis, *Analyse fonctionnelle thèorie et applications*, Masson, Paris, 1983 .

[7] Z. Bouslah, A. Hadj, H. Saker, Shape reconstruction for an inverse biharmonic problem from partial Cauchy data, Math. Meth. Appl. Sci. (2023), 1–15, DOI 10.1002/mma.9712.

[8] F. Cakoni, R. Kress, C. Schuft, *Integral equations for shape and impedance reconstruction in corrosion detection*. Inverse Problems (2008) 095012.

[9] F. Cakoni, Rainer Kress, and Christian Schuft, *Simultaneous reconstruction of shape and impedance in corrosion detection*. Methods and application of analysis 2010; 357-378.

[10] F. Cakoni, Rainer Kress, *Integral equations for inverse problems in corrosion detection from partial Cauchy data*. Inverse Probl Imaging (2007); 1(2):229–45.

[11] R. Chapko, B.T. Johansson, *Integral equations for biharmonic data completion*, 13(5) Inv prob and Imag (2019), 1095-1111.

[12] C. Charles, *Introduction aux problèmes inverses* , Université de Liège – Gembloux Agro-Bio Tech Unité de Statistique, Informatique et Mathématique appliquées à la bioingénierie Gembloux (Belgique), (2014).

[13] Chein-Shan Liu, *A modified collocation Trefftz method for the inverse Cauchy problem of Laplace equation*. Engineering Analysis with Boundary Elements (2008) 778-785.

[14] C. Constanda, *Direct and indirect boundary integral equation methods*, Chapman et Hall/CRC, Boca Raton 2000.

[15] D. Colton, and R. Kress. *Inverse acoustic and electromagnetic scattering theory*, second ed., vol. 93 of Applied Mathematical Sciences. Springer-Verlag, Berlin, 1998.

[16] G. Filippo, G.H. Christoph, S. Guido, *Polyharmonic boundary value problems*. A monograph on positivity preserving and nonlinear higher order elliptic equations in bounded domains. Springer-Verlag 2010.

[17] M. Costabel, E. Stephan and W.L. Wendland, *On boundary integral equations of the first kind for the bi-Laplacian in a polygonal plane domain*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 10, (1986) 197-241.

[18] A. Gomez-Polanco, J.M. Guevara-Jordan. B. Molina, *A mimetic iterative scheme for solving biharmonic equations*. 57 Math. Comput. Model (2011) 2132-2139.

[19] A. Hadj, H. Saker, *Integral equations method for solving a Biharmonic inverse problem in detection of Robin coeffcients*. Applied Numerical Mathematics 160(2021)436-450.

[20] J. Hadamard, *Lecture note on Cauchy's problem in linear partial differential equations*. Yale Uni. Press, New Haven. 1923.

[21] A. Hadj, *Integral equations method for solving a Biharmonic inverse problem in detection of Robin coeffcients*. Doctoral thesis, Annaba 2021.

[22] M. Hanke *A regularizing Levenberg-Marquardt scheme, with applications to inverse groundwater filtration problems* Inverse Problems 13 79-95 (1997).

[23] HW. Engel, M. Hanke and A. Neubauer. *Regularization of Inverse Problems*. Kluwer Academic 2000.

[24] P. C. Hansen, *Regularization tools version 4.0 for Matlab 7.3*, Numerical algorithms, 46, 2:189–194, 2007, Springer.

[25] H. Hedenmalm, *On the uniqueness theorem of Holmgren*. 281 Mathematische Zeitschrift (2015) 357-378.

[26] G.C. Hsiao and W.L. Wendland, *Boundary Integral Equations*, Springer-Verlag 2008.

[27] G. C. Hsiao and W.L. Wendland, A finite element method for some integral equations of the first kind, J. Math. Anal. Appl. 34, (1977) 1-19.

[28] H W. Engl, M. Hanke and A. Neubauer, *Regularization of Inverse Problems* (Dordrecht: Kluwer), 1996.

[29] M. Jaswon, G. Symm, *Integral Equation Methods in Potential Theory and Elastostatics*, Academic Press, New York, 1977.

[30] J. Jost. *Postmodern analysis*. Springer-Verlag. Berlin, 2003.

[31] SI. Kabanikhin. *Inverse and ill-posed problems, theory and applications*. Novosibirsk, Russia /Almaty, Kazakhstan. 2011.

[32] F. Cakoni, G.C. Hsiao, W L. Wendland, *On the Boundary Integral Equation Method for a Mixed Boundary Value Problem of the Biharmonic Equation*, n Complex Variables Theory and Application An International Journal, June 2005.

[33] F. Cakoni, G.C. Hsiao and Wolfgang L. Wendland, *On the Boundary Integral Equation Method for a Mixed Boundary Value Problem of the Biharmonic Equation*, in Complex Variables Theory and Application An International Journal June 2005.

[34] V.V. Karachik, *Solving a Problem of Robin Type for biharmonic Equation*. 2 Izv Vyssh Uchebn Zaved Mat (2018) pp 39-53.

[35] V.V. Karachik, *Riquier-Neumann Problem for the Polyharmonic Equation in a Ball*. 54 Diff Eq (2018), 648-657.

[36] I. Khélifa, *Détermination d'un arc de frontière dans un problème aux limites Elliptique* Doctoral thesis, Annaba 2020.

[37] M. Kern, *Problèmes inverses* , École superieure d'ingénieurs Léonard de vinci, (2002-2003).

[38] Kirsch, *An Introduction to the Mathematical Theory of Inverse Problems*. Springer, New York (2011)

[39] R. Kress, *Linear Integral Equation*. Springer 2014.

[40] R. Kress, *Numerical Analysis*. Springer-Verlag New York (1998).

[41] R. Kress, *Linear integral equations*, Applied Mathematical Sciences, Volume 82, No 1, NY(2014)

[42] J. Li, *Application of radial basis meshless methods to direct and inverse biharmonic boundary value problems*. 21-4 Commun Numer Methods Eng 2005 pp 169-182.

[43] Ch. Sh. Liu. *A Highly Accurate MCTM for Direct and Inverse Problems of Biharmonic Equation in Arbitrary Plane Domains*. 30-2 CMES, (2008) pp 65-75.

[44] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press 2000.

[45] VA. Morozov, Methods for Solving Incorrectly Posed Problems. *Springer-Verlag*. 1984.

[46] MT. Nair. Linear operator equations : approximation and regularization. *World Scientific*. 2009.

[47] C.V. Pao, *On fourth order elliptic boundary value problem*. 4-128 Proc of the Am Math Soc (2000) pp. 1023-1030.

[48] R. Potthast, *Fréchet differentiability of boundary integral operators in inverse acoustic scattering*. Inverse Problems 10, 431–447 (1994)

[49] A.G. Ramm, *A geometrical inverse problem*. 2-2 Inv Prob (1986) 2 L19

[50] W. Rudin, *Analyse réelle et complexe*, Masson, 1975.

[51] H. Saker, *Sur une methode de decomposition du domaine pour les équations integrales au bord pour le bi-Laplacien*, Doctoral thesis, Annaba 2008.

[52] G. Sweers, A. survey, *on boundary conditions for the biharmonic*. Comp Var and Ell Eq, (2008) pp 79-93.

[53] R. Sassane, *Méthodes vartationnelles de régularisation pour une classe de problèmes inverses en EDP* Doctoral thesis, Annaba 2020.

[54] Ch. Tajani, H. Kajtih and A. Daanoun, *Iterative Method to Solve a Data Completion Problem for biharmonic Equation for Rectangular Domain*. 55-1 annals of West Univ of Timi-Math and Comp Sci (2017) 129-147.

[55] A.N. Tikhonov and V.I.A. Arsenin, *Solutions of ill-posed problems*, Scripta series in mathematics. Winston, 1977.

[56] T. Wei n and Y.G.Chen, *Numerical identification for impedance coefficient by a MFS-based optimization method*. Engineering Analysis with Boundary Elements, (2012) 1445–1452.

[57] ML. Whitney, *Theoretical and numerical study of tikhonov's regularization and Morozov's discrepancy principle*, Ph.D. Thesis, Georgia State University. 2009.

[58] Hu. Yuqing, B. Wang and T. Li, *Corrosion shape reconstruction of the mixed boundary in electrostatic imaging* Advances in Difference Equations (2020).

[59] Y. Jeon, *New indirect scalar boundary integral equation formulas for the biharmonic equation*, Journal of Comput and App Math, 135 (2001) 313–324.

[60] A. Zeb, D.B. Ingham and D. Lesnic, *The method of fundamental solutions for a biharmonic inverse boundary determination problem*. Comput Mech, 42 (2008) 371–379.