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Retrial Queueing Systems Modelling with Interrupted Service and Orbital Search

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Modélisation par les systèmes de files d'attente avec rappels, service interrompu et recherche en orbite

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DEDICATION

I want to express my special feeling of gratitude

To my **parents** who were my earliest and best teachers.

To my dearest sisters **Imene**, **Zineb**, **Fadia** and **Anfel**.

To my awesome brother **Hachem**.

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Retrial queueing systems modelling with interrupted service and orbital search

Abstract

In this work, we first propose an M/M/1 retrial queue system with interrupted service by the customer being served, where the access to the orbit is done after a first pass through the server, and we also consider a single server, whose orbit and queue have infinite capacity. Our interest is in the customers who can decide between leaving the system forever or joining the orbit for coming back again to the server after a random time in order to get another service, so we present the generation functions of different stationary distributions and the calculation of some performance measures. Using the infinitesimal generator, we obtain the stationary distributions of this model, and we also use the matrix analytical method to provide some numerical results to illustrate the impact of different parameters on the model's characteristics.

In addition to the above assumptions, we assume that the retrial policy is linear. Then, we generalize the first model, taking into account the fact that the access to the server can be made in the following three cases: from the queue by a primary customer, or from the orbit by a secondary customer (who have already made at least one pass through the server), or the server itself searches for customers in orbit immediately after the end of a service (assuming the queue is free).

Given the complexity of the stochastic analysis of the second model, we again use the matrix analytic method, which allows us to obtain an approximation of the limiting probabilities. Some useful performance measures are computed. These results are supported by numerical examples and simulations to study the influence of some parameters on the characteristics of the system.

Keywords Retrial queue, interrupted service, orbital search, linear retrial policy, matrix-analytic method, performance measures.

Modélisation par les systèmes de files d’attente avec rappels, service interrompu et recherche en orbite

Résumé

Dans ce travail, nous proposons tout d'abord un système de file d'attente M/M/1 avec interruption du service par le client, où l'accès à l'orbite se fait après un premier passage par le serveur, et nous considérons également un serveur unique, dont l'orbite et la file d'attente ont une capacité infinie. Nous nous intéressons aux clients qui peuvent décider de quitter le système pour toujours ou de rejoindre l'orbite pour revenir au serveur après un temps aléatoire afin d'obtenir un autre service, nous présentons donc les fonctions génératrices de la distribution stationnaire et le calcul de quelques mesures de performance. En utilisant le générateur infinitésimal, nous obtenons les distributions stationnaires de ce modèle, et nous utilisons également la méthode analytique matricielle pour fournir quelques résultats numériques afin d'illustrer l'impact des différents paramètres sur les caractéristiques du modèle.

En plus des hypothèses précédentes, nous supposons que la politique de rappels est linéaire. Alors, nous généralisons le premier modèle, en prenant en considération le fait que l'accès au serveur peut se faire dans les trois cas suivants : depuis la file d'attente par des clients primaires, ou depuis l'orbite par des clients secondaires (qui ont déjà fait au moins un passage par le serveur), ou encorelorsque le serveur lui-même cherche des clients en orbiteimmédiatement après la fin d’un service (en supposant que la file d'attente est libre).

Étant donné la complexité de l'analyse du second modèle, nous utilisons à nouveau la méthode analytique matricielle, qui nous permet d'obtenir une approximation des probabilités limites. Quelques mesures de performance utiles sont calculées. Ces résultats sont étayés par des exemples numériques et des simulations pour étudier l'influence de certains paramètres sur les caractéristiques du système.

Mots-clés File d'attente avec rappel, service interrompu, recherche en orbite, politique de rappels linéaire, méthode d'analyse matricielle, mesures de performance.

النمذجة لقوائم الانتظار ذات النداء المتكرر والخدمة المتقطعة والبحث في المدار

ملخص

في هذا العمل، نقترح أولاً نظام انتظار ذات النداء المتكرر M/M/1 مع الخدمة المتقطعة من قبل طالب الخدمة، ولا يتم الدخول إلى المدار إلا بعد المرور الأول عبر الخادم، كما نعتبر خادمًا واحدًا، يتمتع مداره وقائمة انتظاره بسعة غير محدودة. في هذا النموذج نهتم بالطلبات التي يمكنها بعد حصولها على خدمة، الاختيار بين ترك النظام إلى الأبد أو الانضمام إلى المدار للعودة مرة أخرى إلى الخادم بعد وقت عشوائي للحصول على خدمة أخرى، لذلك نقدم معادلات التوليد للتوزيعات أو الاحتمالات الثابتة لهذا النموذج باستخدام المولد المتناهي الصغر للعملية، كما نقوم بحساب مقاييس الأداء. علاوة على ذلك، تم استخدام طريقة التحليل للمصفوفات لتقديم بعض النتائج العددية لتوضيح تأثير المعاملات المختلفة على خصائص النموذج.

بالإضافة إلى الفرضيات السابقة، نفترض أن سياسة تكرار النداء خطية. فنقوم بتعميم النموذج الأول، بالأخذ في الاعتبار أن الوصول إلى الخادم يمكن أن يتم في الحالات الثلاث التالية: من قائمة الانتظار من طرف الطلبات الأولية، أو من المدار من طرف الطلبات الثانوية، أو عندما يبحث الخادم نفسه عن طلبات في المدار مباشرة بعد انتهاء الخدمة (على افتراض أن قائمة الانتظار شاغرة).

ونظرًا لتعقيد التحليل العشوائي للنموذج الثاني، فإننا نستخدم مرة أخرى طريقة تحليل المصفوفة التي تتيح لنا الحصول على تقريب للاحتمالات المحددة. يتم حساب بعض مقاييس الأداء المفيدة و دعم هذه النتائج بأمثلة عددية وعمليات محاكاة لدراسة تأثير بعض المعاملات على خصائص النظام .

كلمات البحث قوائم الانتظار ذات النداء المتكرر، الخدمة المتقطعة، البحث المداري، السياسة الخطية للنداء المتكرر، طريقة تحليل المصفوفة، مقاييس الأداء.

INTRODUCTION

The word queue comes from the French interpretation of Latin *cauda*, meaning a tail. According the Funk and Wagnall's New International Dictionary, a queue is "a line of persons or vehicles waiting in the order of their arrival". The word queue is the common way to refer to a line in England.

In fact, Queueing theory is a branch of applied probability theory that pertains to the study of waiting lines (queues) and service system prone to congestion, including the arrival of units (customers, calls, messages, etc...) to servers, the waiting of units for servers, the processing of units by servers and the departure of units.

As a literature survey, which covers the topic of queueing system, we suggest classifying them into the following four categories:

Early Literature with Standard queues: Agner Krarup Erlang (1878-1929), a Danish mathematician, invented the fields of traffic engineering and queueing theory starting in the 1900s. While working for the Copenhagen Telephone Company, he was confronted with the classic problem of determining how many circuits were needed to provide an acceptable telephone service. Then, he formed the mathematical way of determining how many telephone operators were needed to handle a given volume of calls. Besides, he is the founder on the theory of telephone traffic and he published, over his career, papers, starting in 1909 that became the foundation of queueing theory. Likewise, he developed the Erlang probability distribution, which plays a significant role in various queueing applications.

Research into the application of the idea to telephony continued after Erlang. In 1927, E. C. Molina published his paper "Application of the Theory of Probability to Telephone Trunking Problems", which was accompanied 12 months later by Thornton Fry's book "Probability and Its Engineering Uses", which elevated a variety of Erlang's earlier research. Within the early nineteen thirties, Felix Pollaczek made a few extra studies on Poisson input, arbitrary output, and one/more-channel issues. Extra researches became Further research was carried out at that time in Russia by Kolmogorov and Khintchine, in France by Crommelin's studies, and in Sweden by Palm.

More to the point, we refer to one of the comprehensive books authored by Donald Gross et al

(2008), another informative publication on Fundamentals of Queueing Systems Statistical Methods for Analyzing Queueing Models was by Thomopoulos (2012).

Retrial queues: The old queueing models don't consider the repetitions phenomenon and therefore cannot be used to solve a variety of vital real-life situations. Kosten (1973) (p.33) points out that any theoretical result should be considered suspect if it does not take into account the effect of repetition. Retrial queues (or queues with repeated attempts, repeated calls, etc...) were introduced to deal with specific situations or to understand the fundamental stochastic processes.

One of the earliest papers on retry queues was done by Kosten (1947), on the effect of repeated calls in the theory of blocking probabilities. Wilkinson (1956) encouraged the use of a truncated model to solve numerically the Kolmogorov equations of the main model in the case of unlimited orbit capacity. Cohen (1957) was the author who treated the case of a $M/M/C$ queue, taking into account retries and impatient customers. He also obtained the essential and sufficient conditions for the ergodicity of retrial queues. However, the technique was primarily based on the specific solution of the Kolmogorov equations for the stationary distribution, leading to complicated arguments.

Two extensive survey articles on retrial queues are by Yang & Templeton (1987) and Falin (1990), covering, respectively, the developments up to mid 80's and late $80'$s. Falin & Templeton (1997) published a monograph on the subject, providing an excellent scenario of retrial queues.

Various techniques and results have been developed since the early work of Kosten (1947), Wilkinson (1956), and Cohen (1957) for solving particular problems or understanding the basic stochastic processes. Analytic results are generally difficult to obtain due to the complicity of retrial queueing models. That is why there are a large number of numerical and approximation methods.

The first stochastic analysis bounding mean response time of the $M/G/k$ under the Shortest Remaining Processing Time ($SRPT$) scheduling policy by Isaac Grosof et al (2018), by comparing the multi-server system with a single server system of the same service capacity, whereat they showed that even in the worst case, the steady state amount of relevant work under $SRPT-k$ (the policy which uses multi-server $SRPT$ on k servers) close to the steady state amount of relevant work under $SRPT{-}1$ ($SRPT$ on a single server). However, beyond $SRPT$, they proved similar response time bounds and optimality results for several other multi-server scheduling policies which include $PSJF$ (Pre-emptive Shortest Job First), RS (Remaining Size) and FB (Foreground-Background, also known as Least Attained Service (LAS)).

Queues with interruptions: In traditional models of queuing, the service facility used to be available to serve clients, either at normal or reduced prices (caused by wear and tear or congestion) throughout its availability. As a variant of classical queuing models, the service facility is not continuously available because of:

1. The server goes on holiday when the system is empty, and resumes service if the queue has built up to a certain threshold; such a case is called a vacation queueing model. (see Doshi (1986), Tian & Zhang (2006) and Takagi (1991))

2. Failure of a server (or a device) or an unplanned stop (e.g. answering an urgent incoming call, communicating with senior management, helping a co-worker) that needs an immediate response, in this case, the variant stops the current service, and after which the service can be resumed. Models with this variant are described as queuing models with service Interruptions.

3. A client can interrupt its service due to an external event. The models described in variant (3) are termed queueing models with client-interrupted service. and

4. An external event may cause a catastrophic failure, leading the system to become empty and the server to wait for the next arrival to resume service. Besides, the models under variant (4) go under the caption queueing models with catastrophic emergencies, where not only does the existing service get interrupted in these models, but also all the clients present in the system also deleted.

5. Note that in pre-emptive priority queues, the services of lower-priority clients are interrupted by higher-priority clients. (see Jaiswal (1961))

6. Client-induced service interruptions are possible, although they are not common in many applications. The notable features of this type of interruption, as opposed to server interruption, are that (a) the system can have more interrupted clients than the number of servers in the system, and (b) the system can provide services to other clients while one or more clients are being interrupted.

Depending on the situation, there are several possible ways to restore an interrupted service. Such ways include (i) starting a service from the very beginning (repeat), (ii) starting a service from where it got interrupted (resume), (iii) a combination of both (i) and (ii), and the selection is done by looking at the way the (random) clocks (which are simultaneously started at the time of onset of interruptions) expire, and (iv) denying a service to the one whose service got interrupted.

Survey work by Krishnamoorthy et al. (2014) summarized many models that take into account the occurrence due to many reasons, in respect such as server breakdowns, servers taking emergency brakes and customers having incomplete information or getting distracted, by grouping them into various categories depending on (1) the nature of service times at both discrete and continuous time; (2) the number of servers, by involving single or multiple or infinite server cases; and (3) regular interruptions or interruptions with retrials.

Queues with orbital search: They were interested in designing retrial queues that reduce the server(s) idle time and achieve this by the introduction of search of orbital customers immediately after a service completion (we associate a probability with search) as follows: after completing a service, the server either immediately picks up a customer from the orbit if any with probability p or remains idle with probability $1 - p$. In this case, as in the classical retrial queue, a competition takes place in between primary and orbital customers for service. Thus, if the orbital search is done, a service is followed by another service. Otherwise, if no orbital search is done, a service is followed by an idle time. Other related works are performed in references: Nila & Sumitha (2020) and Pazhani Bala Murugan & Vijaykrishnaraj (2019).

In this thesis, our purpose is to investigate a much more generalized of a Markovian model by the concept of repeated attempts under a linear retrial policy with orbital search and taking into consideration the interruption service in order to leave the system forever or to rejoin the orbit for another service.

Indeed, we present a detailed approximation of the stationary distribution for a single server Markovian queueing model with several parameters, by using the matrix-analytic method. This method was developed by Neuts (1981), Neuts & Rao (1990) and Latouche & Ramaswami (1999), for solving Markov chain that are quite complex.

The present investigation includes many features simultaneously such as: (1) Retrials according to retrial linear policy; (2) Interruption service; (3) Orbital search. We note that all these realistic assumptions have not been gathered together in the existing literature. The analytical results have been obtained by using the Q-matrix (infinitesimal generating matrix) technique. Particularly, we have obtained approximated values of the steady-state distribution and some performance measures of the model. Moreover, some numerical results are presented to demonstrate how the different parameters of the model influence on the behaviour of the system.

Most importantly, our study has two main objectives. The first one is to link between the corresponding retrial queue with interruption service under several retrial policy (according to a constant retrial policy, classical retrial policy or linear retrial policy) and the classical queue. That is why our model can be considered as a generalized version of many existing queueing models associated with many practical situations. The second objective is to introduce orbital search in retrial queueing models which allows minimizing the idle time of the server. Whereat: if the holding costs and cost of using the search of customers will be introduced, the obtained results can be used for the optimal tuning of the parameters of the search mechanism.

The rest of this thesis is organized as follows: Chapter 1 highlights some advanced Queueing Systems like: networks queues , queuing system with feedback, retrial queues (without service interruption) and retrial queues with orbital search, by offering a theoretical framework that provides definitions, descriptions, examples and specific bibliographical notes to clarify the meaning of each one.

The generating functions of different stationary distributions for an $M/M/1$ retrial queue system with interrupted service and some special cases are illustrated in Chapter 2. Furthermore, based on the structure of the infinitesimal generator of the process of the model, we use the Matrix-analytic method to provide some numerical results to illustrate the impact of different parameters on the stationary distributions of the model. Likewise, a numerical analysis was performed for the characteristics of the system.

Chapter 3 is devoted to analyse an $M/M/1$ queueing system with service interrupted and orbital search. Due to the complexity of the analysis of this model, we present the matrix analytic method, which allows us to obtain an approximation of the limiting probabilities. Some useful performance measures are computed. These results are supported by numerical examples and simulations to study the influence of some parameters on the characteristics of the system.

A special cases are given in Chapter 4. It covers the previous model treated in chapter 3 for a constant retrial policy and a generalisation of the model treated in chapter 2 for a linear retrial policy.

In closing, we conclude this thesis by presenting a general conclusion and bibliographical remarks.

Chapter 1

Advanced Queueing Systems

This chapter is devoted to cover the description part of some advanced queueing systems and to mention some bibliographical notes used for modelling: Networks queues, queueing system with feedback, retrial queues (without service interruption) and retrial queues with orbital search, in order to clarify the meaning and applications thereof.

1.1 Networks Queues

In many of today's global structures, clients are served at multiple stations in a grid structure, which is a group of nodes linked by a series of routes. In grid queues (queues of networks), many servers working in the same installation are referred to as nodes.

Generally, clients can log on to the network at a particular node, move from one node to another within the network, and log off from a particular node, although not all clients necessarily log on and off at the same nodes, or follow the same path once they have logged on to the network. Clients can return to nodes they have already visited, can skip certain nodes altogether, and can even decide to still be on the network for all time.

The simplest example of a queueing network is very useful in modelling packet-routing computer networks or networks of manufacturing stations.

We consider some very basic concepts regarding queueing networks, known as Jackson networks.

1.1.1 Jackson Network Definition

A Jackson network is a very general form of queueing network. In which there are k servers, each with its own (unbounded) queue. Jobs at a server are served in FCFS order. The ith server has service rate $\mathbf{Exp}(\mu_i)$. Each server may receive arrivals from inside and outside the network. The external arrivals in the ith server follow a Poisson flow with rate r_i. The routing of jobs is probabilistic. Specifically, every job that completes at server i will be transferred to server j with probability P_{ij}, or will exit the system with probability

$$P_{i,out} = 1 - \sum_j P_{ij}.$$

The response time of a job is defined as the time from which the job arrives to the network until it leaves the network, including possibly visiting the same server or different servers multiple times. For each server i, we denote the total arrival rate into server i by λ_i. Where the total arrival rate into server i is the sum of the outside arrival rate (rate of jobs arriving to server i from outside the network), and the inside arrival rate (rate of jobs arriving to server i from inside the network):

$$\lambda_i = r_i + \sum_j \lambda_j P_{ji}, \tag{1.1.1}$$

equivalently, we can write

$$\lambda_i(1 - P_{ii}) = r_i + \sum_{j \neq i} \lambda_j P_{ji}, \tag{1.1.2}$$

where (1.1.2) is identical to (1.1.1), except that on both sides we are not including transitions from server i back to server i.

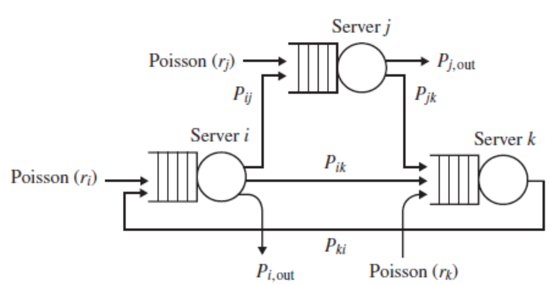


Figure 1.1: A simple Jackson network.

Figure 1.1 shows the general set up of a Jackson network.

1.1.2 Open Jackson Networks

Assuming that in the Markovian node network, each node constitutes a $M/M/s$ queue, having s_i servers at node i ($i = 1, 2, ..., k$), and there is no blocking for transitions between the nodes. Thus, each of these queues forms a $M/M/s$ system with an infinite buffer. Also, the external arrivals from outside the network at node i follow a Poisson flow with rate λ_i and service times at node i are exponential with mean $\frac{1}{\mu_i}$. Suppose α_{ij} is the probability that a client who completes service at node i, requests service at node j (j is different from i) and α_{i0} be the probability that the client leaves the network after service at node i. Denote $Q_1, Q_2, ..., Q_k$ how many clients are in each node, respectively, when t goes towards $+\infty$, and define:

$$p_{n_1,n_2,...,n_k} = P(Q_1 = n_1, Q_2 = n_2, ..., Q_k = n_k). \tag{1.1.3}$$

It is a common example of a so-called Jackson open network, first analysed by Jackson (1957). Jackson found for $p_{n_1,n_2,...,n_k}$ of (1.1.3) that:

$$p_{n_1,n_2,...,n_k} = \prod_{i=1}^{k} p_i(n_i), \tag{1.1.4}$$

where

$$p_i(r) = \begin{cases} p_i(0)\frac{(\frac{\gamma_i}{\mu_i})^r}{r!} & \text{If } r = 0, 1, 2, ..., s_i\ ; \\ p_i(0)\frac{(\frac{\gamma_i}{\mu_i})^r}{s_i! s_i^{r-s_i}} & \text{If } r = s_i, s_i + 1, ...\ . \end{cases}$$

and

$$\gamma_i = \lambda_i + \sum_j \alpha_{ji}\gamma_j, i = 1, 2, ..., k.$$

By having λ_i and α_{ij} $(i, j = 1, 2, ..., k)$, the γ_i's value may be found from $(1.1.4)$. Let γ_i be the rate of effective arrival at node i after accounting for traffic from the network's exterior and the resting $k-1$ nodes inside it.

Consequently, if $\rho_i = \gamma_i/\mu_i$ denotes the effective traffic intensity at any node i (for $i = 1, 2, ..., k$), ρ_i is less than 1 to obtain the limiting distribution. And $p_i(0)$ may be determined via

$$\sum_{n_1}\sum_{n_2} ... \sum_{n_k} p_{n_1,n_2,...,n_k} = 1.$$

The distribution structure $p_i(r)$ in $(1.1.3)$ looks like the one for the $M/M/s_i$ queue with arrival rate γ_i and service rate μ_i. This implies that the arrival process at the ith node is Poisson. This isn't the case, even if t goes towards ∞, because of the feedback from cross-node transitions. In a queueing series with only feedback transitions, we could use the result of Burke (1956), on the output process and deduce that if t goes towards ∞, the feedback transition generates a Poisson process. However, if the transition contains the feedback feature, the resulting arrival process isn't a Poisson process.

Burke (1976) showed that in a $M/M/1$ queue with feedback, the efficient waiting times distribution is a mixture of exponentials. Consequently, from the expression in $(1.1.3)$ that is given as the limiting distribution product of $M/M/s_i$ queueing systems, the only result we can obtain is that in the limit, Jackson's network behaves as if it were a series of $M/M/s_i$ queues. For a full review of these properties of queueing networks, we refer the reader to Disney & Kiessler (1987).

Markov network models applied to queueing are also represented as Markov population processes. A systematic approach to the analysis of these processes, with particular reference to their limit distributions, has been provided by Kingman (1969). Kingman's results verify those of Jackson, who also generalized his earlier result to include production systems composed of specialized service centers (see Jackson (1963)), and of Whittle (1967, 1968), who derived limit distributions for migration processes. See Bhat (1984) for further details. Deriving the limiting distribution (1.1.3) is complicated, even when there are only 2 nodes in the system, as can be seen from the following outline. Suppose $k = 2$ and $s_1 = s_2 = 1$. Using the properties of state transitions, we can write the state equations as follows:

$$
\begin{aligned}
(\lambda_1 + \lambda_2)p_{00} &= \mu_1\alpha_{10}p_{10} + \mu_2\alpha_{20}p_{01}, \\
(\lambda_1 + \lambda_2 + \mu_1)p_{10} &= \lambda_1 p_{00} + \mu_2\alpha_{21}p_{01} + \mu_1\alpha_{10}p_{20}, \\
(\lambda_1 + \lambda_2 + \mu_2)p_{01} &= \lambda_2 p_{00} + \mu_1\alpha_{12}p_{10} + \mu_2\alpha_{20}p_{02}, \\
(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)p_{11} &= \lambda_1 p_{01} + \lambda_2 p_{10} + \mu_1\alpha_{10}p_{21} + \mu_2\alpha_{20}p_{12} \\
&\quad + \mu_1\alpha_{12}p_{20} + \mu_2\alpha_{21}p_{02}, \\
&\vdots \\
(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)p_{n_1 n_2} &= \lambda_1 p_{n_1-1,n_2} + \lambda_2 p_{n_1,n_2-1} + \mu_1\alpha_{10}p_{n_1+1,n_2} \\
&\quad + \mu_2\alpha_{20}p_{n_1,n_2+1} + \mu_1\alpha_{12}p_{n_1+1,n_2-1} + \mu_2\alpha_{21}p_{n_1-1,n_2+1}, \\
&\quad n_1, n_2 > 0.
\end{aligned}
\tag{1.1.5}
$$

Both of γ_1 and γ_2 are expressed at each node as

$$
\begin{aligned}
\gamma_1 &= \lambda_1 + \alpha_{21}\gamma_2; \\
\gamma_2 &= \lambda_2 + \alpha_{12}\gamma_1.
\end{aligned}
\tag{1.1.6}
$$

Then, in (1.1.6), solve for γ_1 and γ_2,

$$\gamma_1 = \frac{\lambda_1 + \lambda_2\alpha_{21}}{1 - \alpha_{12}\alpha_{21}}; \tag{1.1.7}$$

$$\gamma_2 = \frac{\lambda_2 + \lambda_1\alpha_{12}}{1 - \alpha_{12}\alpha_{21}}. \tag{1.1.8}$$

We take $\rho_i = \frac{\gamma_i}{\mu_i}$, $i = 1, 2$. Assume

$$p_{n_1,n_2} = C\rho_1^{n_1}\rho_2^{n_2}. \tag{1.1.9}$$

It is not easy to check that (1.1.8) is a proper solution of the equilibrium state equation (1.1.5) satisfying $\sum_{n_1}\sum_{n_2} p_{n_1,n_2} = 1$. The reader is referred to Gross & Harris (1998) for more details on such a procedure in the general case, with k nodes and multiple servers at each node.

1.1.3 Closed Jackson Networks

Assume that $\lambda_i = 0$ and $\alpha_{i0} = 0$ under the assumptions made when defining Jackson's open network. Consider $Q = \sum_{i=1}^{k} Q_i$, the total number of clients in the network. We now have a closed Jackson network which can be used to model a queueing network with a fixed number of clients circulating in the network. Based on the same considerations as for open networks with k nodes and the ith node having s_i servers ($i = 1, 2, ...$), the limiting distribution $p_{n_1,n_2,...,n_k} = P(Q_1 = n_1, Q_2 = n_2, ..., Q_k = n_k)$ may be given as

$$p_{n_1,n_2,...,n_k} = C \prod_{i=1}^{k} \frac{\rho_i^{n_i}}{a_i(n_i)}, \tag{1.1.10}$$

where

$$a_i(n_i) = \begin{cases} n_i! & \text{For } n_i < s_i, \\ s_i! s_i^{n_i - s_i} & \text{For } n_i \geq s_i, \end{cases} \tag{1.1.11}$$

and $\rho_i = \frac{\gamma_i}{\mu_i}$ with γ_i satisfying the relation

$$\gamma_i = \sum_{i=1}^{k} \gamma_j \alpha_{ji}.$$

This formula may be expressed as

$$\mu_i \rho_i = \sum_{i=1}^{k} \mu_j \rho_j \alpha_{ji}. \tag{1.1.12}$$

In (1.1.10), the constant term C is determined by $\sum_{n_1} \sum_{n_2} ... \sum_{n_k} p_{n_1,n_2,...,n_k} = 1$. Note that the expression "product form" is only used for the part of the result that contains $n_1, n_2, ..., n_k$. In this case, the constant C is not factorized as a function of the nodes, as it was in the open lattice. Solving (1.1.12) to find $\rho_i, i = 1, 2, ..., k$, we notice that only the $k - 1$ equations are independent since the total traffic is given.

So, we first set one of the ρ_i to 1.

It is not easy to find $C \equiv C(Q)$. We have

$$C^{-1}(Q) = [C(Q)]^{-1} = \sum_{n_1+n_2+...+n_k=Q} \prod_{i=1}^{k} \frac{\rho_i^{n_i}}{a_i(n_i)}, \tag{1.1.13}$$

where the sum extends to all available ways of choosing $n_1, n_2, ..., n_k$ such that $\sum i = 1^k n_i = Q$. The number of possibilities is expressed by the combining term $(Q + k - 1Q)$ (the equivalent combinatorial problem is that of distributing Q balls in k cells, which in turn is equivalent to randomly assigning $k-1$ balls from $Q+k-1$ positions in a row). Therefore, computing $C^{-1}(Q)$ as a direct formula from (1.1.13) is only easy for small values of Q and k, even with the help of a computer. One of the first algorithms to systematically calculate $G(Q) = C^{-1}(Q)$ was given by Buzen (1973). He gives the following definition

$$f_i(n_i) = \frac{\rho_i^{n_i}}{a_i(n_i)}, \tag{1.1.14}$$

so that

$$G(Q) = \sum_{\sum_{i=1}^{k} n_r = Q} \prod_{i=1}^{k} f_i(n_i).$$

Consider

$$g_m(n) = \sum_{n_1+n_2+\cdots+n_k=Q} \prod_{i=1}^{m} f_i(n_i) \tag{1.1.15}$$

and $g_k(Q) = G(Q)(m = k$ and $n = Q)$. We can express

$$\begin{aligned} g_m(n) &= \sum_{r=0}^{n}[\sum_{\sum_{l=1}^{m-1} n_l = n-r} \prod_{i=1}^{m} f_i(n_i)]; \\ &= \sum_{r=0}^{n} f_m(r)[\sum_{\sum_{l=1}^{m-1} n_l = n-r} \prod_{i=1}^{m-1} f_i(n_i)]; \\ &= \sum_{r=0}^{n} f_m(r) g_{m-1}(n-r), \quad n = 0, 1, 2, ..., Q. \end{aligned} \tag{1.1.16}$$

Furthermore, $g_1(n) = f_1(n)$ and $g_m(0) = 1$. A recursive structure for computing $G(Q)$ is given by (1.1.16), according to an algorithm called the convolution algorithm.

There are many computational algorithms in the literature, some of them enhancements of Buzen's algorithm, to compute $G(Q)$ and $p_i(n)$ (see, for example, Gelenbe & Pujolle (1998)). The reader is referred to works on modelling the performance of computer networks, such as Sauer & Chandy (1981), for a discussion of their relative merits. An illustration of the use of recursive problem-solving can also be found in Gross & Harris (1998).

1.1.4 Cyclic Queues

Consider the special case of a closed network of queues, where

$$\alpha_{ij} = \begin{cases} 1 & \text{For } j = i+1, 1 \le i \le k-1 \,; \\ 1 & \text{For } i = k, j = 1 \,; \\ 0 & \text{Otherwise.} \end{cases} \tag{1.1.17}$$

It is a cyclic queue (see Koenigsberg (1958)), in which the service is provided cyclically by at least one server. In order to keep things simple, it is assumed that there is only a single server at every station. Following the same notation like in the previous section, corresponding to (1.1.12), we have the expressions:

$$\begin{aligned} \mu_1\rho_1 &= \mu_k\rho_k, \\ \mu_2\rho_2 &= \mu_1\rho_1, \\ &\vdots \\ \mu_k\rho_k &= \mu_{k-1}\rho_{k-1}. \end{aligned} \tag{1.1.18}$$

From these we get

$$\begin{aligned} \rho_2 &= \frac{\mu_1}{\mu_2}\rho_1, \\ \rho_3 &= \frac{\mu_1}{\mu_3}\rho_1, \\ &\vdots \\ \rho_k &= \frac{\mu_1}{\mu_k}\rho_1, \end{aligned} \tag{1.1.19}$$

By setting $\rho_1 = 1$ and retaining generality. For $p_{n_1,n_2,\ldots,n_k}$, we get:

$$p_{n_1,n_2,\ldots,n_k} = \frac{1}{G(Q)}\frac{\mu_1^{Q-n_1}}{\mu_2^{n_2}\mu_3^{n_3}\ldots\mu_k^{n_k}}. \tag{1.1.20}$$

Using Buzen's algorithm, we obtain $G(Q)$ mentioned in (1.1.20).

1.2 Queuing system with feedback

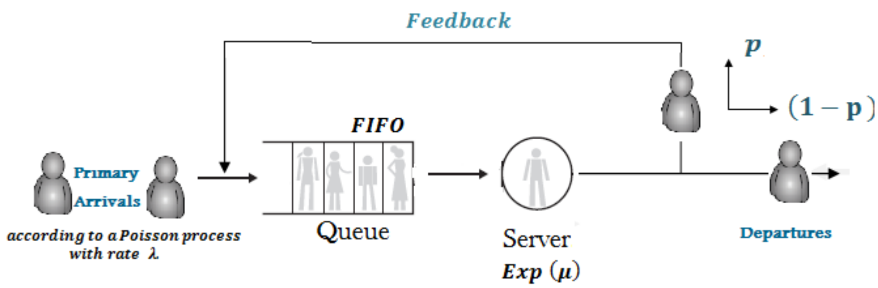
The exposition of the feedback notion can be found on the paper "A Queuing Model with Feedback" by Takas (1963), who notes that after each service a customer may return to the waiting room with probability p or may depart permanently with probability $q = 1 - p$. In the same context, a feedback takes the form of the return of certain calls that were handled for a new service, as it is mentioned on the paper of Melikov et al (2015).

We can highlight the reasons for feedback after completing service from the following papers:

- By de Vericourt & Zhou (2005), An example is given in a call centre, where customers will come back later if the initial service is not satisfactory.

- Another reason is exposed by Yom-Tov & Mandelbaum (2014), for the treatment of patients is monitored in stages by the doctor in the hospital, starting with an initial examination, then returning later to the check-up after requesting and carrying out tests.

Figure 1.2: A Single-Server Queue with Feedback.



From figure1.4, a Poisson process with rate λ is assumed for all incoming customers and gets served by a single server according to a First In First Out discipline (FIFO) infinite in capacity. Their service times are assumed to be i.i.d. random variables and have an exponential distribution with parameter μ.

An arriving customer starts his/her/its service instantaneously if he/she/it finds the server idle. After getting service, the customer may decide whether or not to provide feedback, which is supposed to occur instantly. Thus, a customer may either join the feedback flow with probability p if he/she/it provides feedback, by joining the end of the original queue (there is no difference between primary arrivals and feedback customers) or the departure process with probability $q = 1 - p$ and leaves the system forever if he/she/it does not provide feedback.

It is clear that feedback in a $M/M/1$ system is a special case of the general birth and death model with the following conditions

$$\begin{cases} \lambda_n = \lambda, & \forall n \geq 0\ ; \\ \mu_n = q\mu, & \forall n \geq 1 \text{ and } q = 1 - p. \end{cases}$$

Let $N(t)$ denote how many clients are in the system at an instant t.

The process $\{N(t), t \geq 0\}$ has an explicit expression for the stationary distribution of $p_n = \lim_{t \to +\infty} P(N(t) = n)$ given by

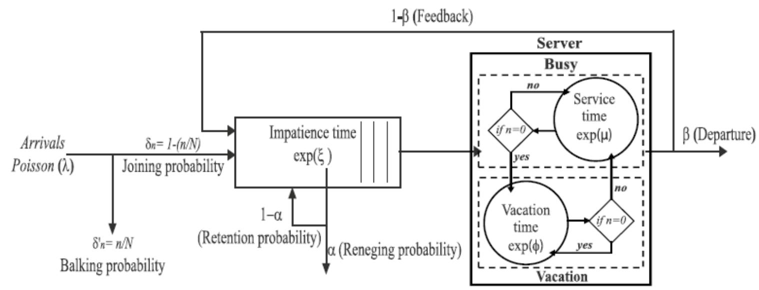
$$p_n = (1 - \frac{\lambda}{q\mu})(\frac{\lambda}{q\mu})^n,$$

for $\frac{\lambda}{q\mu} < 1, n \geq 0$, from the theorem 1 by Takas (1963).

To emphasize this idea, we mention a variety of papers dealing with a single server queueing system to model the customer's behaviour after the returns for a new service, see the paper of D'Avignon & Disney (1976),Santhakumaran & Thangaraj (2000), Liu & Whitt (2016), Bouchentouf & Guendouzi (2018), Shekhar et al (2019), Bouchentouf et al (2019) and Cherfaoui et al

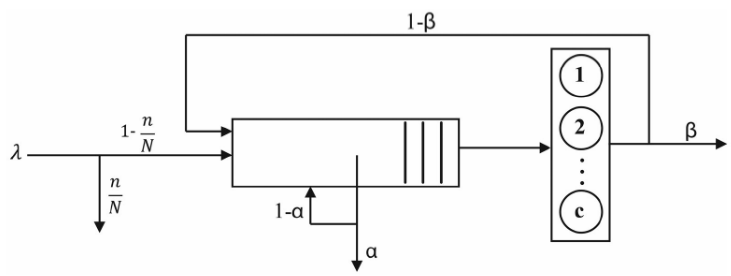
(2020).

Figure 1.3: $M/M/1/N$ feedback queueing system with multiple vacations, balking, reneging and retention of reneged customers. See Bouchentouf et al (2019).



The paper by Bouchentouf et al (2020), extended the investigation to a multiserver model. This feedback model is depicted in figure 1.4.

Figure 1.4: $M/M/c/N$ feedback queue with synchronous multiple vacation policy, balking, reneging, and retention. See Bouchentouf et al (2020)



The retrial queues that take into account the feedback were introduced too. As a selection of the related literature, we mention Lee (2005), Wang & Zhou (2010). It is possible to find in some models that the orbit (the source of retrial calls) is not formed by newly arrived calls but by serviced calls, like in the paper authored by Melikov et al (2015), or to find a combination of retrial phenomenon and classical queue with feedback, we refer to Kalyanaraman (2012), who deals with a feedback retrial queue with two types of clients, in which both types of clients arrive in batches of variable sizes.

1.3 Description of retrial queues

To describe models of a new branch of the queueing theory, known as retrial queues (or queues with returning customers, repeated attempts, etc.), we firstly introduce their characteristics by the following feature assumption: an arriving unit (as customer, call, message,...) that cannot get service (due to the finite capacity of the system when he/she/it finds all servers and waiting positions (if any) occupied, balking when a unit does not join the queue, reneging when a unit joins the queue and subsequently decides to leave, or because of his/her impatience, breakdowns or vacation among servers can also be considered, etc.) Exits area of the service but returns to the system after a random delay to repeat his/her/its request.

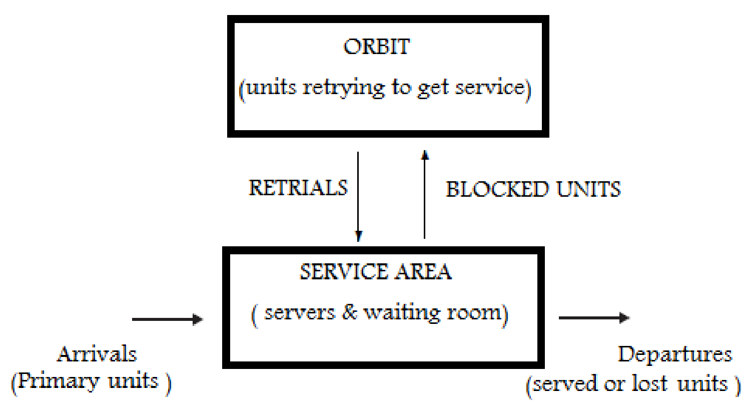
Those units that attempt service later are considered "in orbit". However, units in orbit can not notice the status of the servicing facility. Besides, they can only check the status of the server by 'returning' to the service facility and Retrial is related to such an action.

Units go back and forth from the orbit to the service facility until either service is received (in this case, each orbiting unit is treated the same as a primary unit (a new arriving unit)), or they abandon the system.

Above and beyond, it should be noted that the capacity O of the orbit can be either infinite or finite. In the case of finite O, if the orbit is full, any arriving unit to the orbit will be rejected (will be forced to leave the system forever).

As it's shown in Figure 1.5, which illustrates that Retrial queues stand for a type of networks with re-servicing after blocking. Whereat these networks contain 2 nodes: the main node, where blocking is possible, together with a delay node for repeated attempts.

Figure 1.5: General Structure of a Retrial Queue.



For a bibliography, we refer to the book by Artalejo & Gómez-Corral (2008) and to the book by Donald Gross et al (2008).

1.3.1 Examples

Retrial queues have been widely used to model many problems in telephone switching systems, telecommunication networks, computer networks and computer systems. The following are just a few examples which explain this general description in more details.

Telephone systems: Everybody knows from his/her own experience that a caller who receives a busy signal will keep repeating the call until it is connected. As a result, the flow of calls circulating in a telephone network consists of two parts: the flow of primary calls, which reflects the real wishes of the telephone subscribers, and the flow of repeated calls, which is the consequence of the lack of success of previous attempts.

Retail shopping queue: In a shop a customer who finds that a queue is too long may wish to do something else and return later on with the hope that the queue dissolves. Similar behaviour may demonstrate some impatient customers who entered the waiting line but then discovered that the residual waiting time is too long.

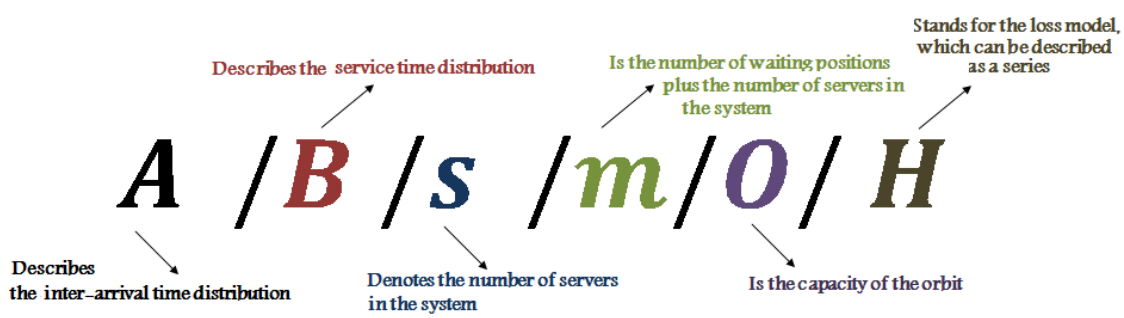
Random access protocols in digital communication networks: Consider a communication line with slotted time which is shared by several stations. Further, the duration of the slot equals the transmission time of a single packet of data. If two or more stations are transmitting packets simultaneously then a collision takes places, i.e. all packets are destroyed and must be retransmitted. If the stations involved in the conflict would try to retransmit destroyed packets in the nearest slot, then a collision occurs with certainty.

To avoid this, each station transmits, independently of other stations, the packet with probability p and delays actions until the next slot with probability $1-p$, or equivalently, each station introduces a random delay before next attempt to transmit the packet.

1.3.2 Notation

In Figure 1.6, we expressed the retrial queueing models described above using the extended notation of Kendall (1953).

Figure 1.6: Notation.



Note that the retrial time is not described in the notation.

$$H_k = \begin{cases} 1, & \text{In a no-loss system (NL)}, k \geq 0\ ; \\ \alpha < 1, & \text{In a geometric loss system (GL)}, k \geq 0. \end{cases}$$

When m, O, or H is omitted from the notation, we assume $m = s$, $O = \infty$ and $H = NL$ (No-Loss).

1.3.3 M/M/1 retrial queue

In this model, customers arrive at a single server queue according to a Poisson process with rate λ. Service times are exponential with mean $\frac{1}{\mu}$. Any arriving customer, upon finding the server busy, enters the orbit and remains there for an exponentially distributed period of time with mean $\frac{1}{\gamma}$. All inter arrival times (between primary arrivals), service times and orbit times are independent. Customers repeat service attempts until the server is available. In this model, we assume that no customers leave the system due to impatience.

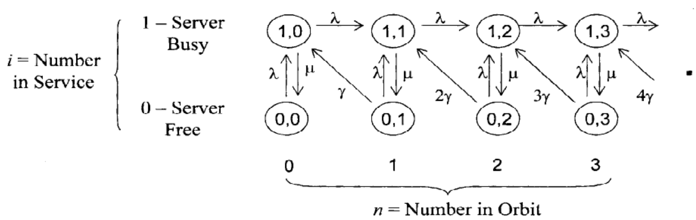


Figure 1.7: State Transition Rates For Retrial Queue.

Let $N_S(t)$ denote how many clients are in the system (once there is 1 server, $N_S(t) \in \{0, 1\}$) and $N_0(t)$ denotes how many clients are in orbit at an instant t. Then, $\{N_S(t), N_0(t)\}$ is a CTMC, with state space $\{i, n\}$, whereat $i \in \{0, 1\}$ and $n \in \{0, 1, 2, ...\}$. The total number of customers in the system is $N(t) = N_S(t) + N_0(t)$. Figure 4.2 shows the rate transitions between states. Let $P_{i,n}$ be the fraction of time that the system is in state $\{i, n\}$. Then, the rate balance equations are

$$(\lambda + n\gamma)p_{0,n} = \mu p_{1,n}; \tag{1.3.1}$$

$$(\lambda + \mu)p_{1,n} = \lambda p_{0,n} + (n + 1)\gamma p_{0,n+1} + \lambda p_{1,n-1}; \tag{1.3.2}$$

$$(\lambda + \mu)p_{1,0} = \lambda p_{0,0} + \gamma p_{0,1}. \tag{1.3.3}$$

We obtain steady-state solutions for this system using generating functions. Define the following partial generating functions:

$$P_0(z) = \sum_{n=0}^{\infty} z^n p_{0,n}, \quad P_1(z) = \sum_{n=0}^{\infty} z^n p_{1,n}.$$

Multiply (1.3.1) by z^n and sum over $n \geq 0$:

$$\sum_{n=0}^{\infty}(\lambda + n\gamma)p_{0,n}z^n = \sum_{n=0}^{\infty} \mu p_{1,n}z^n,$$

$$\lambda \sum_{n=0}^{\infty} p_{0,n}z^n + \gamma \sum_{n=0}^{\infty} np_{0,n}z^n = \mu \sum_{n=0}^{\infty} p_{1,n}z^n, \tag{1.3.4}$$

this can be rewritten as:

$$\lambda P_0(z) + \gamma z P_0'(z) = \mu P_1(z). \tag{1.3.5}$$

Similarly, multiply (1.3.2) by z^n, sum over $n \geq 1$, and add (1.3.4):

$$(\lambda + \mu) P_1(z) = \lambda P_0(z) + \gamma P_0'(z) + \lambda z P_1(z), \tag{1.3.6}$$

solving for $P_1(z)$ in (1.3.5) and substituting into (1.3.6) gives

$$P_0'(z) = \frac{\lambda\rho}{\gamma(1-\rho z)} P_0(z), \tag{1.3.7}$$

where $\rho = \frac{\lambda}{\mu}$. This is a separable differential equation, which we can write as follows:

$$\frac{P_0'(z)}{P_0(z)} = \frac{\lambda\rho}{\gamma(1-\rho z)}. \tag{1.3.8}$$

Integrating with respect to z gives

$$P_0(z) = C(1-\rho z)^{-\frac{\lambda}{\gamma}}. \tag{1.3.9}$$

Now, $P_1(z)$ can be found by plugging $P_0(z)$ into (1.3.5):

$$P_1(z) = C\rho(1-\rho z)^{-\frac{\lambda}{\gamma}-1}. \tag{1.3.10}$$

The constant C is found from the normalizing condition $P_0(1) + P_1(1) = 1$:

$$C = (1-\rho z)^{\frac{\lambda}{\gamma}+1}. \tag{1.3.11}$$

By substituting this into (1.3.9) and (1.3.10), we get:

$$\boxed{\begin{array}{lcl} P_0(z) & = & (1-\rho z)(\frac{1-\rho}{1-\rho z})^{\frac{\lambda}{\gamma}+1} \\ P_1(z) & = & \rho(\frac{1-\rho}{1-\rho z})^{\frac{\lambda}{\gamma}+1} \end{array}} \tag{1.3.12}$$

To obtain the steady-state probabilities, we expand $P_0(z)$ and $P_1(z)$ in a power series using the binomial formula

$$(1+z)^m = \sum_{n=0}^{\infty} \binom{m}{n} z^n = \sum_{n=0}^{\infty} \frac{z^n}{n!} \prod_{i=0}^{n-1} (m-i),$$

then, rearranging terms gives

$$\boxed{\begin{array}{lcl} p_{0,n} & = & (1-\rho)^{\frac{\lambda}{\gamma}+1} \frac{\rho^n}{n!\gamma^n} \prod_{i=0}^{n-1} (\lambda - i\gamma) \\ p_{1,n} & = & (1-\rho)^{\frac{\lambda}{\gamma}+1} \frac{\rho^{n+1}}{n!\gamma^n} \prod_{i=1}^{n} (\lambda - i\gamma) \end{array}} \tag{1.3.13}$$

We construct the generating function for the number of customers that are in the orbit:

$$P(z) = \sum_{n=0}^{\infty} z^n (p_{0,n} + p_{1,n}) = P_0(z) + P_1(z).$$

Let L_0 be the average number of orbiting clients. Then $L_o = P'(1)$. Hence it can be shown that

$$L_o = \frac{\rho^2}{1-\rho} \times \frac{\mu + \gamma}{\gamma}.$$

For a bibliography of retrial queues, see Falin & Templeton (1997) and Artalejo (1999).

1.4 Retrial queues with orbital search

Our objective in this section is to cover the description part for retrial queues with orbital search from the thesis of Shortle et al (2018). They investigated a single server queue with a linear retrial policy, where the server can go in search of customers immediately after each service completion with a known probability. They also obtained the necessary and sufficient condition for the ergodicity of $M/G/1$ and $M/M/1$ retrial queues with orbital search.

1.4.1 Description of the main model of $M/G/1$ type

Consider a single server queueing system in which customers arrive in a Poisson process with rate λ. These customers are identified as primary calls.

If the server is free at the time of a primary call arrival, the arriving call begins to be served immediately and leaves the system after service completion. Otherwise, if the server is busy, the arriving customer leaves the service area and joins the orbit. The interval between two successive repeated attempts is exponentially distributed with rate $\alpha(1-\delta_{0j}) + j\theta$ (the linear retrial policy), where δ_{0j} denotes Kronecker function and j is the number of customers in orbit. The service times are independent with distribution function $B(x)(B(0) = 0)$.

Let $\beta(s) = \int_0^\infty \exp^{-sx} dB(x)$ be the Laplace-Stieltjes transform of $B(x)$,$\beta_k = (-1)^k \beta^{(k)}(0)$ be the kth moment of the service time about the origin, $\rho = \lambda\beta$, the system load due to primary arrivals, $h(r) = \frac{B'(x)}{1-B(x)}$ be the instantaneous service intensity given that the elapsed service time is equal to x, $k(z) = (\beta(\lambda - \lambda z)$. It can be shown that $k(z) = \sum_{n=0}^{\infty} K_n z^n$, where $k_n = \int_0^\infty \frac{(\lambda x)^n}{n!} \exp^{-\lambda x} dB(x)$.

Let ζ_n be the time at which the nth service completion occurs. Immediately after this, the server goes for a search of customers in the orbit with a probability $p_i(p_o = 0)$ which depends on the number of customers j in orbit. Otherwise, the server remains free with probability $q_j = 1 - p_j$

. In the latter case the event to follow depends on the competition between a primary arrival of rate λ and the flow of repeated attempts of rate $\alpha(1-\delta_{0j})+j\theta$. The search time is assumed to be negligible.

The flow of primary arrivals, the intervals between repeated attempts, and service times are assumed to be mutually independent.

Let $N(t)$ denote how many clients are in orbit and $C(t)$ denote the state of the server at time t. We have $C(t)$ equal to 1 or 0 according to whether the server is busy or free. Note that the state space of the process $\chi(t)=\{C(t),N(t)\}$ is $S=\{0,1\}\times\mathbb{N}$. The transitions among states are shown in Illustration 1 for the case of exponential service times with rate μ.

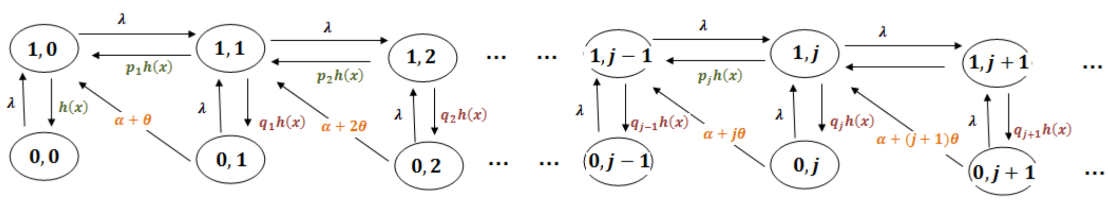


Figure 1.8: State space and transitions.

1.4.2 $M/M/1$ Retrial queues with orbital search

We consider $B(t)=1-\exp^{-\mu t}, t>0$, the process $\chi(t)$ becomes an irreducible continuous-time Markov chain and the principal characteristics can be easily expressed in hypergeometric functions.

The set of statistical equilibrium equations for the probabilities P_{0j} and P_{lj} is

$$\{\lambda+\alpha(1-\delta_{0j})+j\theta\}P_{0,j}=q_j\mu P_{1,j},\quad \forall j\geq 0; \tag{1.4.1}$$

$$(\lambda+\mu)P_{1j}=\lambda P_{1,j-1}+\lambda P_{0,j}+[\alpha+(j+1)\theta]P_{0,j+1}+\mu p_{j+1}P_{1,j+1},\quad \forall j\geq 0. \tag{1.4.2}$$

The equation (1.4.1) can be rewritten as

$$P_{1,j}=\frac{\lambda+\alpha(1-\delta_{0j})+j\theta}{q_j\mu}P_{0,j},\quad \forall j\geq 0. \tag{1.4.3}$$

Eliminating the probabilities P_{lj} from the equation (1.4.2), by substituting (1.4.3) into (1.4.2), we get:

$$(\lambda+\mu)\frac{\lambda+\alpha(1-\delta_{0j})+j\theta}{q_j\mu}P_{0,j}=\lambda\frac{\lambda+\alpha(1-\delta_{0,j-1})+(j-1)\theta}{q_{j-1}\mu}P_{0,j-1}+\lambda P_{0,j}+[\alpha+(j+1)\theta]P_{0,j+1}$$
$$+\mu p_{j+1}\frac{\lambda+\alpha(1-\delta_{0,j+1})+(j+1)\theta}{q_{j+1}\mu}P_{0,j+1};\quad \forall j\geq 1, \tag{1.4.4}$$

which implies that

$$\mu q_{j-1}q_j(\alpha+(j+1)\mu+\lambda p_{j+1})P_{0,j+1}-\lambda q_{j-1}q_{j+1}(\lambda+\alpha+j\mu)P_{0,j}=$$
$$\mu q_{j-1}q_{j+1}(\alpha+j\mu+\lambda p_j)P_{0,j}-\lambda q_j q_{j+1}(\lambda+\alpha(1-\delta_{0,j-1})+(j-1)\theta)P_{0,j-1},\quad \forall j\geq 1. \tag{1.4.5}$$

Now from (1.4.5), we have:

$$\mu q_{j-1}q_{j+1}(\alpha+j\mu+\lambda p_j)P_{0,j}-\lambda q_j q_{j+1}(\lambda+\alpha(1-\delta_{0,j-1})+(j-1)\theta)P_{0,j-1}=0,\quad \forall j\geq 1; \tag{1.4.6}$$

$$\mu q_{j-1}q_{j+1}(\alpha+j\mu+\lambda p_j)P_{0,j}=\lambda q_j q_{j+1}(\lambda+\alpha(1-\delta_{0,j-1})+(j-1)\theta)P_{0,j-1},\quad \forall j\geq 1;$$
$$P_{0,j}=\frac{\lambda q_j q_{j+1}(\lambda+\alpha(1-\delta_{0,j-1})+(j-1)\theta)}{\mu q_{j-1}q_{j+1}(\alpha+j\mu+\lambda p_j)}P_{0,j-1},\quad \forall j\geq 1.$$

Thus

$$P_{0,j}=\frac{\lambda q_j(\lambda+\alpha(1-\delta_{0,j-1})+(j-1)\theta)}{\mu q_{j-1}(\alpha+j\mu+\lambda p_j)}P_{0,j-1},\quad \forall j\geq 1. \tag{1.4.7}$$

Recursively, we find that $P_{0,j}$ depends on $P_{0,0}$ as follows:

$$P_{0,j}=P_{0,0}q_j\rho^j\prod_{k=0}^{j-1}\frac{\lambda+\alpha(1-\delta_{k0})+k\theta}{p_{k+1}\lambda+\alpha+(k+1)\mu},\quad \forall j\geq 1. \tag{1.4.8}$$

The probabilities $P_{1,j}$ can be also obtained directly from the equation (1.4.1) by

$$P_{1,j}=P_{0,0}\rho^{j+1}\prod_{k=1}^{j}\frac{\lambda+\alpha+k\theta}{p_k\lambda+\alpha+k\mu},\quad \forall j\geq 0 \tag{1.4.9}$$

and

$$P_{0,0}^{-1}=\sum_{j=0}^{\infty}\rho^{j+1}(1+\frac{q_j\mu}{\lambda+\alpha(1-\delta_{j0})+j\theta})\prod_{k=1}^{j}\frac{\lambda+\alpha+k\theta}{p_k\lambda+\alpha+k\mu}. \tag{1.4.10}$$

Then, in order to get closed-form expressions for the formulas (1.4.8), (1.4.9) and (1.4.10), they assumed the case of constant search, for $p_j=p, p\in[0,1], j\geq 1$.

Let F be the hyper-geometric series given by

$$F(a,b,;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k z^k}{(c)_k k!},$$

where $(x)_k$ is the Pochhammer symbol defined by

$$(x)_k=\begin{cases} 1, & \text{if } k=0;\\ x(x+1)...(x+k-1), & \text{if } k\geq 1.\end{cases}$$

Proposition 1.1 *Let us assume that $\{\chi(t); t \geq 0\}$ is positive recurrent, then the limiting probabilities $\{P_{i,j}\}_{(i,j)\in S}$ are given by*

$$
\begin{aligned}
P_{0,j} &= P_{0,0}\frac{(1-p)\lambda}{\lambda+\alpha}\rho^j \frac{(\frac{\lambda+\alpha}{\theta})_j}{(\frac{p\lambda+\alpha}{\theta}+1)_j}, \quad \forall j \geq 1;\\
P_{1,j} &= P_{0,0}\rho^{j+1}\frac{(\frac{\lambda+\alpha}{\theta}+1)_j}{(\frac{p\lambda+\alpha}{\theta}+1)_j}, \quad \forall j \geq 0;\\
P_{0,0}^{-1} &= F(1, \frac{\lambda+\alpha}{\theta}+1; \frac{p\lambda+\alpha}{\theta}+1; \rho).
\end{aligned}
$$

We also introduce the partial generating functions

$$
P_i(z) = \sum_{j=0}^{\infty} z^j P_{i,j}, i \in \{0,1\}, |z| \leq 1,
$$

Proposition 1.2 *The partial generating functions $P_i(z), 0 \leq i \leq 1$, are given by*

$$
\begin{aligned}
P_0(z) &= P_{0,0}(1-\rho z)F(1, \frac{\lambda+\alpha}{\theta}+1; \frac{p\lambda+\alpha}{\theta}+1; \rho z);\\
P_1(z) &= P_{0,0}\rho F(1, \frac{\lambda+\alpha}{\theta}+1; \frac{p\lambda+\alpha}{\theta}+1; \rho z).
\end{aligned}
$$

Note also that

$$
M_0^i = \sum_{j=0}^{\infty} P_{i,j} \ and \ M_k^i = \sum_{j=k}^{\infty} j(j-1)...(j-k+1)P_{i,j}, for\ i \in \{0,1\}, k \geq 1,
$$

where M_k^i is the partial factorial moments.

Proposition 1.3 *The partial factorial moments M_k^i, for $i \in \{0,1\}$ and $k \geq 0$, are given by*

$$
\begin{aligned}
M_0^0 &= 1-\rho;\\
M_k^0 &= P_{0,0}k!\frac{(1-p)\lambda}{\lambda+\alpha}\rho^k\frac{(\frac{\lambda+\alpha}{\theta})_k}{(\frac{p\lambda+\alpha}{\theta}+1)_k}F(k+1, \frac{\lambda+\alpha}{\theta}+k; \frac{p\lambda+\alpha}{\theta}+k+1; \rho), k \geq 1;\\
M_0^1 &= \rho;\\
M_k^1 &= P_{0,0}k!\rho^{k+1}\frac{(\frac{\lambda+\alpha}{\theta}+1)_k}{(\frac{p\lambda+\alpha}{\theta}+1)_k}F(k+1, \frac{\lambda+\alpha}{\theta}+k+1; \frac{p\lambda+\alpha}{\theta}+k+1; \rho), k \geq 1.
\end{aligned}
$$

In particular, the expressions corresponding to the classical retrial policy (for $\alpha = 0$ and $\theta > 0$) are given by

Corollary 1.1 *The limiting probabilities are given by*

$$
\begin{aligned}
P_{0,j} &= P_{0,0}(1-\rho)\rho^j \frac{(\frac{\lambda}{\theta})_j}{(\frac{p\lambda}{\theta}+1)_j}, \forall j \geq 1;\\
P_{1,j} &= P_{0,0}\rho^{j+1}\frac{(\frac{\lambda}{\theta}+1)_j}{(\frac{p\lambda}{\theta}+1)_j}, \forall j \geq 0;\\
P_{0,0}^{-1} &= F(1, \frac{\lambda}{\theta}+1; \frac{p\lambda}{\theta}+1; \rho).
\end{aligned}
$$

Corollary 1.2 *The partial generating functions are given by*

$$
\begin{aligned}
P_0(z) &= P_{0,0}(1-\rho z)F(1, \frac{\lambda}{\theta}+1; \frac{p\lambda}{\theta}+1; \rho z);\\
P_1(z) &= P_{0,0}\rho F(1, \frac{\lambda}{\theta}+1; \frac{p\lambda}{\theta}+1; \rho z).
\end{aligned}
$$

Corollary 1.3 *The partial factorial moments are given by*

$$
\begin{aligned}
M_0^0 &= 1-\rho;\\
M_k^0 &= P_{0,0}k!(1-p)\rho^k \frac{(\frac{\lambda}{\theta})_k}{(\frac{p\lambda}{\theta}+1)_k} F(k+1, \frac{\lambda}{\theta}+k; \frac{p\lambda}{\theta}+k+1; \rho), k \geq 1;\\
M_0^1 &= \rho;\\
M_k^1 &= P_{0,0}k!\rho^{k+1}\frac{(\frac{\lambda}{\theta}+1)_k}{(\frac{p\lambda}{\theta}+1)_k} F(k+1, \frac{\lambda}{\theta}+k+1; \frac{p\lambda}{\theta}+k+1; \rho), k \geq 1.
\end{aligned}
$$

Moreover, the expressions corresponding to the constant retrial policy (for $\theta = 0$ and $\alpha > 0$) are given by

Corollary 1.4 *The limiting probabilities are given by*

$$
\begin{aligned}
P_{0,j} &= P_{0,0}\frac{(1-p)\lambda}{\lambda+\alpha}\beta^j, \forall j \geq 1;\\
P_{1,j} &= P_{0,0}\rho\beta^j, \forall j \geq 0;\\
P_{0,0}^{-1} &= 1-\beta.
\end{aligned}
$$

Corollary 1.5 *The partial generating functions are given by*

$$
\begin{aligned}
P_0(z) &= \frac{(1-\rho z)(1-\beta)}{1-\beta z};\\
P_1(z) &= \frac{\rho(1-\beta)}{1-\beta z}.
\end{aligned}
$$

Corollary 1.6 *The partial factorial moments are given by*

$$
\begin{aligned}
&M_0^0 = 1-\rho;\\
&M_k^0 = k!\frac{(1-p)\lambda}{\lambda+\alpha}(\frac{\beta}{1-\beta})^k, k \geq 1;\\
&M_0^1 = \rho;\\
&M_k^1 = k!\rho(\frac{\beta}{1-\beta})^k, k \geq 1.
\end{aligned}
$$

Remark 1.1 *The performance characteristics of the standard $M/M/1$ queue and the $M/M/1$ retrial queue can be deduced by fixing the value of the parameter p_j, for both choices $p_j = 1$ and $p_j = 0, \forall j \geq 1$.*

Chapter 2

$M/M/1$ Queue With Retrials After Service Interruption Option Selected By Customer

In this chapter, we present the framework we will use for modelling a $M/M/1$ queue system with interrupted service by the customer being served, where the access to the orbit is done after a first pass through the server. We consider a single server, whose orbit and queue have infinite capacity , according to the constant retrial policy. Our interest is in the customers that they can decide between leave the system forever or join the orbit for coming back again to the server after a random time in order to get another service.

Such model is applicable to many practical situations where the customer can make a decision to join orbit with probability p_1 if he interrupted his first service, waiting for completing; or to left the system immediately after complaining with probability $(1 - p_1)$. We assume that the server after the completion of customers' service and being idle, there is a competition between primary and orbital customers (which are waiting in the queue or in the orbit) for getting into the server for the next service.

The results of our model can be applied to improve the management of several systems in many fields, so to help take an optimal control policy to minimize the expected discounted cost. In concrete terms, we were inspired by the following situations:

- An employee who gets temporary leave because of maternity/paternity leave, medical leave, or ... When at the end of it, he/she has to resume his/her duties under the contract. Thus, he/she returns to the server. There is also the case, in which he/she goes away forever (due to retirement, or after a lay-off, ...).
- The sale and purchase in instalments guarantee the return of the customer to complete the remaining instalments.
- A site for student assignments that is only accessible once for students, but teachers have

unlimited access, so they can view and return as many times as they want.

2.1 Model description

In this model, we consider a $M/M/1$ queue with retrials after the service interruption option selected by the client under a constant retrial policy. We consider a single server retrial queueing system; whose orbit and queue have infinite capacity which primary clients arrive according to a Poisson process with rate $\lambda > 0$. The service times are independent and exponentially distributed with parameter μ.

The following rules govern the dynamic of the customers:

- If an arriving client finds the server idle, he immediately begins his service. Otherwise, an arriving client who finds the server busy joins the queue line in the service area according to FCFS discipline (first come, first served).
- We assume that the client can interrupt the service and go on vacations or take a break, where the break choice can be only applied to the primary customer who started his service and decided to leave the service space before completing its. Thus, the client may leave the system forever with probability $(1 - p_1)$ after completing his first service, or joins the orbit with probability p_1 (in case that the client wants to take a break). After a period of time, the orbiting client coming back to the server.
- We assume that the clients only have access to the orbit after an initial service.
- Customers in service can join the orbit and spend an amount of time. If and only if they decided to come back to the service and the queue line was empty, an orbiting customer attempts to access the server directly at random intervals of time (without rejoining the queue line in a service area), according to the constant retrial policy with rate θ, where the inter-retrials time are exponentially distributed with rate $\theta > 0$. However, these orbiting customers upon the completion of their break can regain access to the server to resume service; as the service is exponentially distributed and orbiting customers resume their service from the beginning. Such retrial policy is called constant retrial; see Wang et al (2017).
- It should be denoted that the server after the completion of customer's service and being idle, there is competition between the primary and orbital customers for getting into the server for the next service if and only if the queue line is empty.
- All the random variables defined above are mutually independent.

2.2 Stochastic analysis

We have come to analyse a $M/M/1$ retrial queue with customers' break choice and constant retrial policy. We consider a single server retrial queueing system; whose orbit and queue have infinite capacity which primary customers arrive according to a Poisson process with rate $\lambda > 0$. The service times are independent and exponentially distributed with parameter μ. We have also the global traffic intensity given by

$$\rho = \rho_q + \rho_o = \frac{\lambda}{\mu(1-p_1)} + \frac{\lambda p_1}{\mu(1-p_1)} = \frac{\lambda(1+p_1)}{\mu(1-p_1)}.$$

The system state at time t can be described by the process $X(t) = \{C(t), N_q(t), N_o(t); t \geq 0\}$, where $C(t)$ is the state of the server (0 or 1 according as the server is idle or busy) and $N_o(t)$ denotes how many clients are in orbit and $N_q(t)$ denotes how many clients are in the queue line (excluding any clients that may be in service) at time t.

Let $N(t)$ denote how many clients are in the system at an instant t (i.e. in orbit, in queue line and service). Where $N(t) = N_q(t) + N_o(t) + C(t)$.

So that the continuous-time stochastic process $X(t) = \{C(t), N_q(t), N_o(t); t \geq 0\}$, describes the state of the system with state space $\{c, i, j\}$, where $c \in \{0, 1\}$; $i \in \mathbb{N}$ and $j \in \mathbb{N}$.

Its infinitesimal transition rates $q_{(0,i,j)(c,m,n)}$ and $q_{(1,i,j)(c,m,n)}$ are given by

$$\begin{aligned}
q_{(0.0.0)(1.0.0)} &= \lambda; \\
q_{(0.0.j)(1.0.j)} &= \lambda \; ; q_{(0.0.j)(1.0.j-1)} = \theta, \forall j \geq 1; \\
q_{(1.0.j)(0.0.j+1)} &= \lambda p_1, \forall j \geq 0; \\
q_{(1.0.0)(0.0.0)} &= \mu(1-p_1); \\
q_{(1.0.j)(0.0.j)} &= \mu(1-p_1), \forall j \geq 1; \\
q_{(0.i.0)(1.i-1.0)} &= \lambda \; ; q_{(1.i.0)(1.i+1.0)} = \lambda, \forall i \geq 1; \\
q_{(1.0.0)(1.1.0)} &= \lambda \; ; q_{(1.0.0)(0.0.1)} = \lambda p_1; \\
q_{(1.i.0)(0.i.1)} &= \lambda p_1 \; ; q_{(1.i.0)(0.i.0)} = \mu(1-p-1), \forall i \geq 1; \\
q_{(0.i.j)(1.i-1.j)} &= \lambda \; ; q_{(1.i.j)(1.i+1.j)} = \lambda, \forall i \geq 1, \forall j \geq 1; \\
q_{(1.i.j)(0.i.j+1)} &= \lambda p_1 \; ; q_{(1.i.j)(0.i.j)} = \mu(1-p_1), \forall i \geq 1, \forall j \geq 1. \qquad (2.1)
\end{aligned}$$

Remark 2.1 *It is more convenient to split our analysis in two, by considering a special case, i.e. we limit ourselves to analyse the $M/M/1$ model with no queue line in the service area (the case where the queue is empty). Then, considering the main model with the queue line with the orbit.*

Special case. When the queue is empty the model becomes without waiting space. Then if an arriving customer finds the server idle, he immediately begins his service. Otherwise, an arriving customer who finds the server busy leaves the system without any effect on the system.

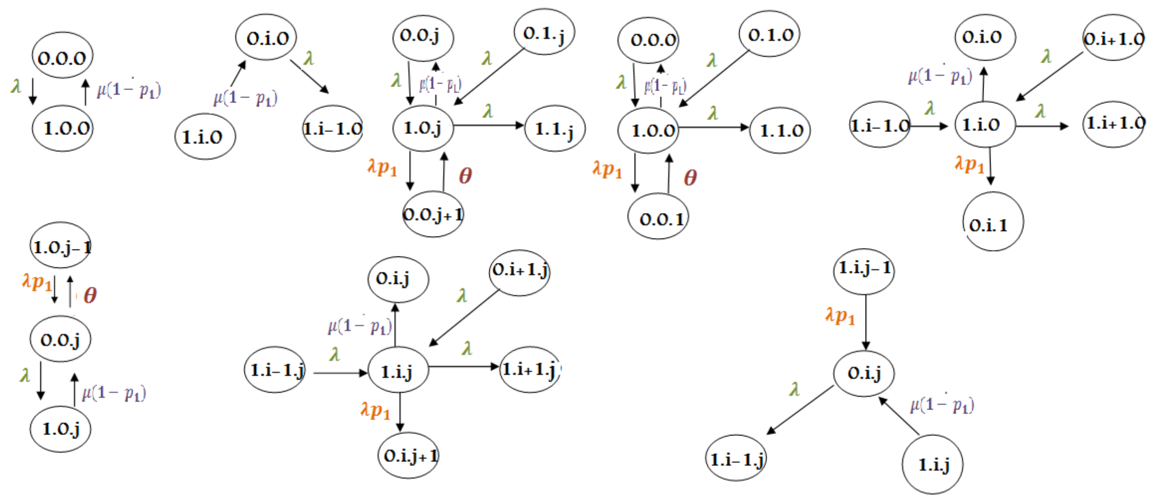


Figure 2.1: State space and transitions.

Let $N(t)$ denote how many clients are in the system at time t. Where $N(t) = N_o(t) + C(t)$. In the case of exponentially distributed service times process $\chi = \{C(t), N_o(t); t \geq 0\}$ is a Markov process with the state space $\{0; 1\} \times \mathbb{N}$, we define the limiting probabilities that the system is in an idle or busy period respectively:

$$\pi_{0,n} = \lim_{t \to +\infty} P(C(t) = 0, No(t) = n), n \geq 0;$$

$$\pi_{1,n} = \lim_{t \to +\infty} P(C(t) = 1, No(t) = n), n \geq 0.$$

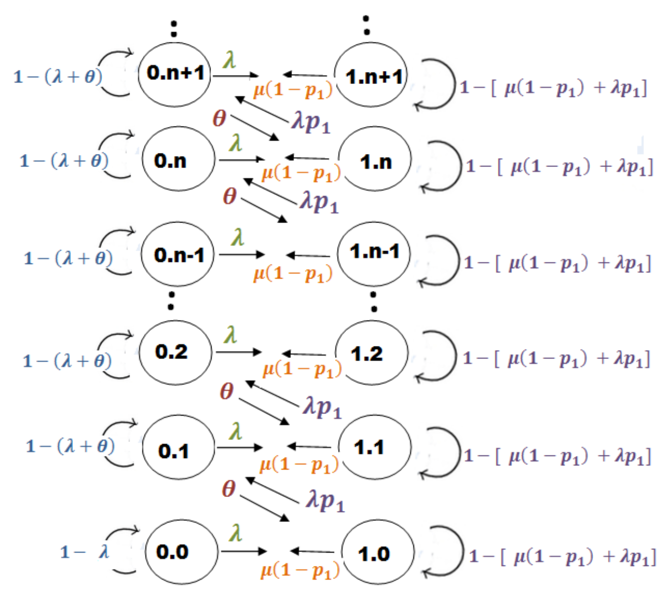


Figure 2.2: Dynamic diagram queueing system.

From a state $(0, n)$ only transitions into the following states are possible:

- $(1, n)$ with rate λ, due to arrival of a primary customer;

- $(1, n-1)$ with rate θ, due to arrival of an orbital customer.

Reaching state $(0, n)$ is possible only from states:

- $(1, n)$ with rate $\mu(1-p_1)$, due to a service completion in case that the customer leaves the system forever;
- $(1, n-1)$ with rate λp_1, due to an interrupted service in case that the customer wants to take a break and joins to the orbit.

From a state $(1, n)$ only transitions into the following states are possible:

- $(0, n)$ with rate $\mu(1-p_1)$, due to a service completion in case that the customer leaves the system forever;
- $(0, n+1)$ with rate λp_1, due to an interrupted service in case that the customer wants to take a break and joins to the orbit.

Reaching state $(1, n)$ is possible from only from states:

- $(0, n)$ with rate λ, due to arrival of a primary customer;
- $(0, n+1)$ with rate θ, due to arrival of an orbital customer.

The set of statistical equilibrium equations for the probabilities $\{\pi_{0,n}, \pi_{1,n}; \forall n \geq 0\}$ is

$$\lambda\pi_{0,0} = \mu(1-p_1)\pi_{1,0}; \tag{2.2}$$

$$[\theta+\lambda]\pi_{0,n} = \lambda p_1 \pi_{1,n-1} + \mu(1-p_1)\pi_{1,n}, \forall n \geq 1; \tag{2.3}$$

$$[\lambda p_1 + \mu(1-p_1)]\pi_{1,n} = \theta\pi_{0,n+1} + \lambda\pi_{0,n}, \forall n \geq 0. \tag{2.4}$$

We have then the following result:

Theorem 2.1 *For a $M/M/1$ retrial queue in the steady state, the joint distribution of server state $C(t)$ and queue length $N_o(t)$, $\pi_{i,n} = P\{C(t) = i, N_o(t) = n\}$ is given by*

$$\pi_{0,n} = (\frac{\lambda}{\theta}\rho_o)^j \times \frac{\theta - \lambda\rho_o}{\theta(1+\rho_q)}$$

and

$$\pi_{1,n} = (\frac{\lambda}{\theta}\rho_o)^j \times \frac{\theta - \lambda\rho_o}{\theta(1+\rho_q)} \times \rho_q.$$

Proof 2.1 *The way of solving the equations* (2.3) *and* (2.4) *consists of the following.*

With the help of equation (2.3)*, we eliminate probabilities* $\pi_{0,n}$ *from equation* (2.4) *and rewrite the resulting equation as*

$$[\lambda p_1+\mu(1-p_1)]\pi_{1,n} = \frac{\theta}{\theta+\lambda}\{\lambda p_1\pi_{1,n}+\mu(1-p_1)\pi_{1,n+1}\}+\frac{\lambda}{\theta+\lambda}\{\lambda p_1\pi_{1,n-1}+\mu(1-p_1)\pi_{1,n}\},\ \forall n \geq 1;$$

$$[\lambda p_1+\mu(1-p_1)]\times[\theta+\lambda]\pi_{1,n} = \{\theta\lambda p_1\pi_{1,n}+\theta\mu(1-p_1)\pi_{1,n+1}\}+\{\lambda^2 p_1\pi_{1,n-1}+\lambda\mu(1-p_1)\pi_{1,n}\},\ \forall n \geq 1;$$

$$\theta\mu(1-p_1)\pi_{1,n} - \lambda^2 p_1 \pi_{1,n-1} = \theta\mu(1-p_1)\pi_{1,n+1} - \lambda^2 p_1 \pi_{1,n}, \forall n \geq 1;$$

which implies that

$$\theta\mu(1-p_1)\pi_{1,n} - \lambda^2 p_1 \pi_{1,n-1} = 0, \forall n \geq 1;$$

i.e.

$$\theta\mu(1-p_1)\pi_{1,n} = \lambda^2 p_1 \pi_{1,n-1}, \forall n \geq 1;$$

$$\pi_{1,n} = \frac{\lambda^2 p_1}{\theta\mu(1-p_1)} \times \pi_{1,n-1}, \forall n \geq 1;$$

$$\pi_{1,n} = \{\frac{\lambda^2 p_1}{\theta\mu(1-p_1)}\}^j \ \frac{\lambda}{\mu(1-p_1)} \pi_{0,0}, \forall n \geq 0;$$

$$\pi_{1,n} = (\frac{\lambda}{\theta}\rho_o)^j \times \rho_q \times \pi_{0,0}, \forall n \geq 0. \tag{2.5}$$

By substituting (2.5) *into* (2.3) *and after some rearrangement we get:*

$$\pi_{0,n} = \{\frac{\lambda^2 p_1}{\theta\mu(1-p_1)}\}^j \ \pi_{0,0}, \forall n \geq 0;$$

$$\pi_{0,n} = (\frac{\lambda}{\theta}\rho_o)^j \ \pi_{0,0}, \forall n \geq 0. \tag{2.6}$$

The probability $\pi_{0,0}$ *may be found with the help of the normalizing condition* $\sum_{n\geq 0}(\pi_{0,n}+\pi_{1,n}) = 1$, *then we get:*

$$\pi_{0,0} = \frac{\theta\mu(1-p_1) - \lambda^2 p_1}{\theta[\mu(1-p_1)+\lambda]} = \frac{\theta - \lambda\rho_o}{\theta(1+\rho_q)}.$$

Introducing $\Pi_0(z)$ and $\Pi_1(z)$ the generating functions corresponding to the orbit size defined by

$$\Pi_0(z) = \sum_{n=0}^{+\infty} \pi_{0,n} \ z^n, |z| \leq 1;$$

$$\Pi_1(z) = \sum_{n=0}^{+\infty} \pi_{1,n} \ z^n, |z| \leq 1;$$

and under the necessary and sufficient condition $\mid \frac{\lambda^2 p_1 z}{\theta\mu(1-p_1)} \mid = \mid \frac{\lambda}{\theta}\rho_o z \mid < 1$ which allows the stability of the system, we have the following results:

Theorem 2.2 *If* $\mid \frac{\lambda^2 p_1 z}{\theta\mu(1-p_1)} \mid < 1$, *the generating functions corresponding to the orbit size* $\Pi_0(z)$ *and* $\Pi_1(z)$ *have the following expressions*

$$\Pi_0(z) = \frac{1}{1+\rho_q} \times \frac{\theta - \lambda\rho_o}{\theta - \lambda\rho_o z};$$

$$\Pi_1(z) = \frac{\rho_q}{1+\rho_q} \times \frac{\theta - \lambda\rho_o}{\theta - \lambda\rho_o z}.$$

Proposition 2.1 • *The fraction of time the server is busy is* $\mathbf{Pr}$*[the server is busy]*$= M_0^1 = \sum_{n\geq 0}\pi_{1,n}$, *this can be found from the generation function, since*

$$\Pi_1(1) = \sum_{n\geq 0}\pi_{1,n} = \frac{\lambda}{\mu(1-p_1)+\lambda} = \frac{\rho_q}{1+\rho_q}.$$

- *The fraction of time the server is idle is* $\mathbf{Pr}$*[the server is idle]*$= M_0^0 = \sum_{j\geq 0}\pi_{0,0,j}$, *this can be found from the generation function, since*

$$\Pi_0(1) = \sum_{n\geq 0}\pi_{0,n} = \frac{\mu(1-p_1)}{\mu(1-p_1)+\lambda} = \frac{1}{1+\rho_q}.$$

- *The generating function for the number of customers in orbit is*

$$\Pi(z) = \sum_{n\geq 0} z^n(\pi_{0,n}+\pi_{1,n}) = \Pi_0(z)+\Pi_1(z) = \frac{\lambda+\mu(1-p_1)}{\theta\mu(1-p_1)-\lambda^2p_1z} \times \frac{\theta\mu(1-p_1)-\lambda^2p_1}{\mu(1-p_1)+\lambda};$$
$$= \frac{\theta-\lambda\rho_o}{\theta-\lambda\rho_o z}.$$

- *The generating function for the number of customers in the system is*

$$Q(z) = \Pi_0(z)+z\Pi_1(z) = \frac{\lambda z+\mu(1-p_1)}{\theta\mu(1-p_1)-\lambda^2p_1z} \times \frac{\theta\mu(1-p_1)-\lambda^2p_1}{\mu(1-p_1)+\lambda};$$
$$= \frac{\theta-\lambda\rho_o}{\theta-\lambda\rho_o z} \times \frac{1+\rho_q z}{1+\rho_q}.$$

2.3 Performance measures of the special case

- The mean number of customers in the system $\bar{n}$ can be derived from

$$\bar{n} = Q(1)' = \frac{\lambda(\theta+\rho_o)}{(1+\rho_q)(\theta-\lambda\rho_o)}. \qquad (*)$$

- The mean number $\bar{n}_o$ of customers in orbit is,

$$\bar{n}_o = \Pi(1)' = \frac{\lambda^2p_1}{(1+\rho_q)(\theta\mu(1-p_1)-\lambda^2p_1};$$
$$= \frac{\lambda\rho_o}{(1+\rho_q)(\theta-\lambda\rho_o}.$$

- The mean time spend in orbit W_o (i.e. , the mean in orbit until finding the server idle and beginning service) can be derived from (*) using Litles's law:

$$\bar{W} = \frac{\bar{n}}{\lambda} = \frac{(\theta + \rho_o)}{(1 + \rho_q)(\theta - \lambda\rho_o)}.$$

- The average time in system $\bar{W}_s$ can be similarly derived

$$\bar{W}_s = \bar{W} + \frac{1}{\mu(1 - p_1)} = \frac{\mu(1 - p_1)(\theta + \rho_o) + (1 + \rho_q)(\theta - \lambda\rho_o)}{\mu(1 - p_1)(1 + \rho_q)(\theta - \lambda\rho_o)}.$$

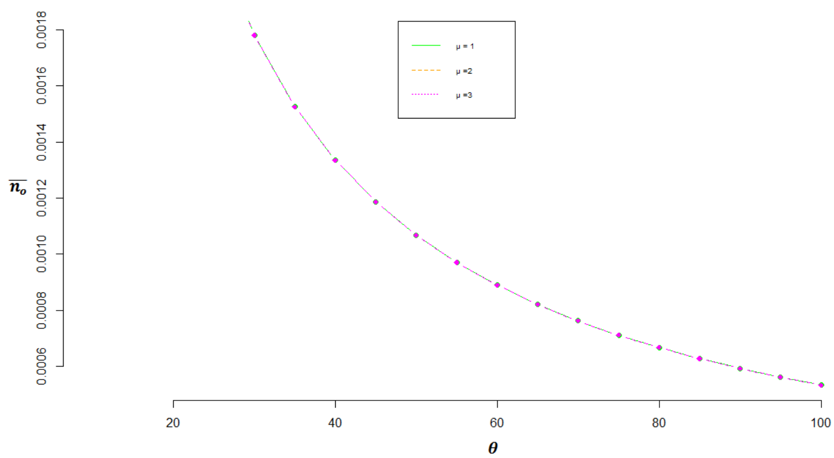


Figure 2.3: $\bar{n_o}$ by varying θ & $(\mu, p) \in \{(1, 0.25); (2, 0.25); (3, 0.25)\}$.

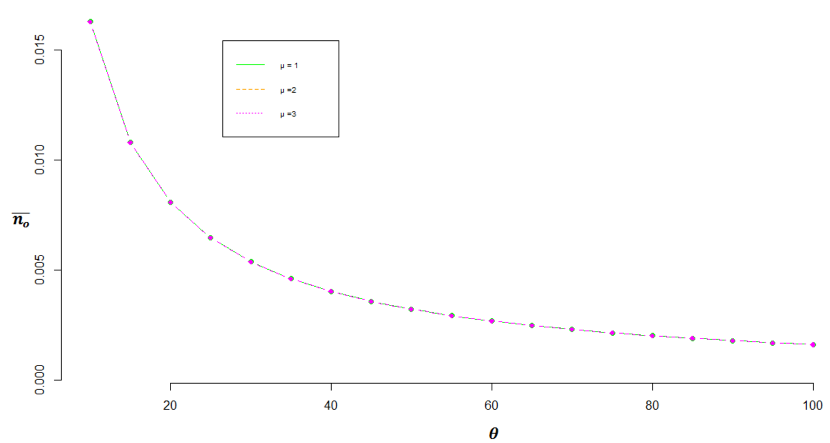


Figure 2.4: $\bar{n_o}$ by varying θ & $(\mu, p) \in \{(1, 0.50); (2, 0.50); (3, 0.50)\}$.

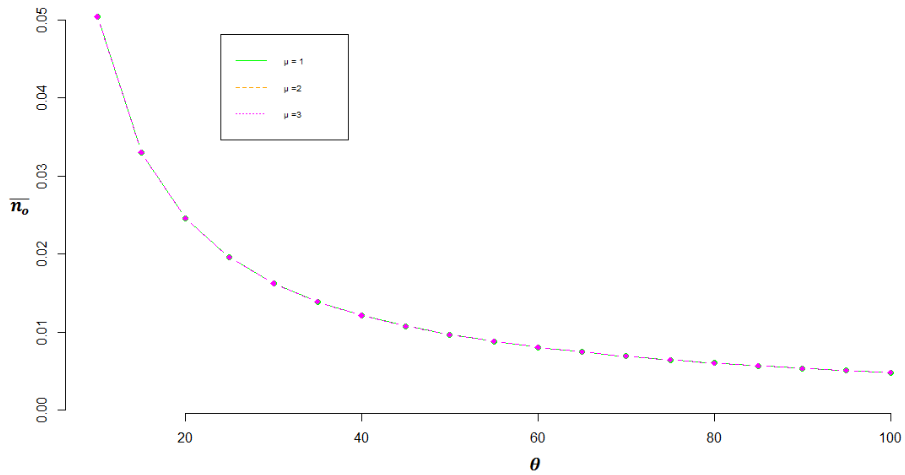


Figure 2.5: $\bar{n_o}$ by varying θ & $(\mu, p) \in \{(1, 0.75); (2, 0.75); (3, 0.75)\}$.

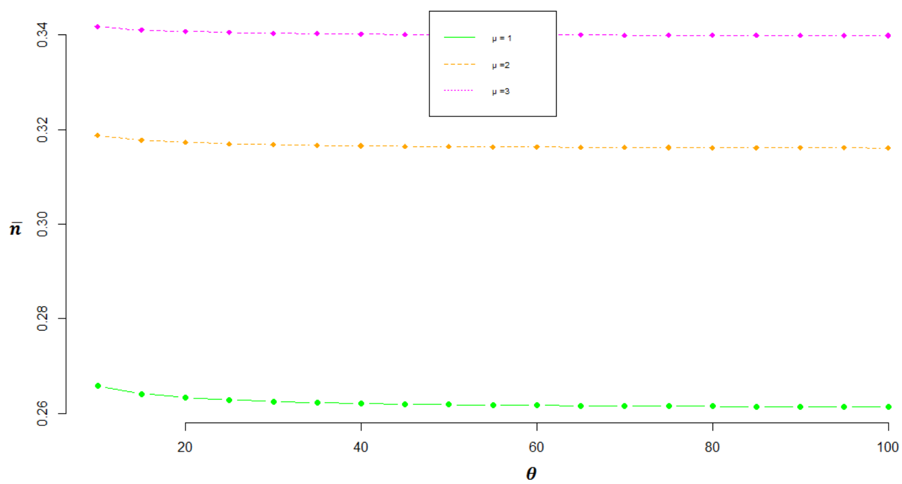


Figure 2.6: $\bar{n}$ by varying θ & $(\mu, p) \in \{(1, 0.25); (2, 0.25); (3, 0.25)\}$.

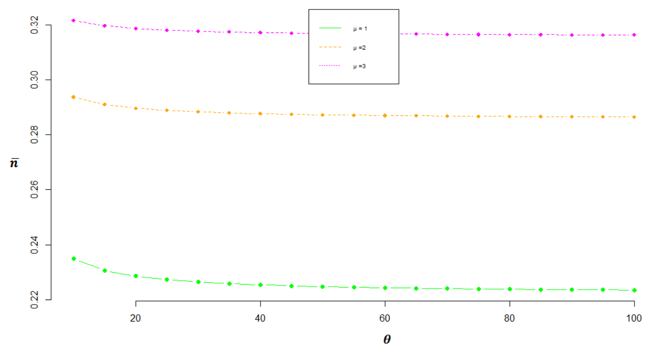


Figure 2.7: $\bar{n}$ by varying θ & $(\mu, p) \in \{(1, 0.50); (2, 0.50); (3, 0.50)\}$

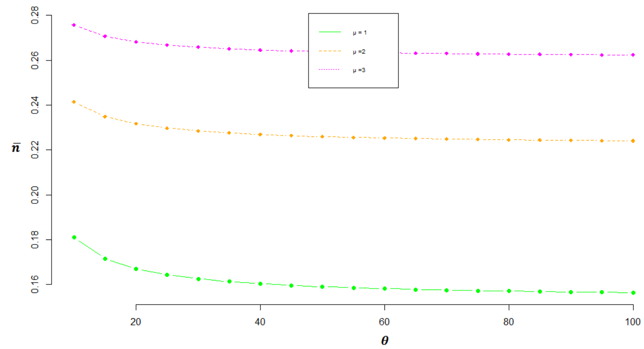


Figure 2.8: $\bar{n}$ by varying θ & $(\mu, p) \in \{(1, 0.75); (2, 0.75); (3, 0.75)\}$.

The main model. The set of statistical equilibrium equations for the probabilities $\{\pi_{0,i,j}, \pi_{1,i,j}; \forall i \geq 0; \forall j \geq 0\}$, under the ergodicity condition $\rho = \frac{\lambda(1+p_1)}{\mu(1-p_1)} < 1$, have the following expressions

$$\lambda\pi_{0,0,0} = \mu(1-p_1)\pi_{1,0,0}; \tag{2.7}$$
$$\lambda\pi_{0,i,0} = \mu(1-p_1)\pi_{1,i,0}, \forall i \geq 1; \tag{2.8}$$
$$\{\lambda + \lambda p_1 + \mu(1-p_1)\}\pi_{1,i,0} = \lambda\pi_{1,i-1,0} + \lambda\pi_{0,i+1,0}, \forall i \geq 1; \tag{2.9}$$
$$[\lambda + \theta]\pi_{0,0,j} = \lambda p_1\pi_{1,0,j-1} + \mu(1-p_1)\pi_{1,0,j}, \forall j \geq 1; \tag{2.10}$$
$$\{\lambda + \lambda p_1 + \mu(1-p_1)\}\pi_{1,0,j} = \lambda\pi_{0,0,j} + \lambda\pi_{0,1,j} + \theta\pi_{0,0,j+1}, \forall j \geq 0; \tag{2.11}$$
$$\lambda\pi_{0,i,j} = \lambda p_1\pi_{1,i,j-1} + \mu(1-p_1)\pi_{1,i,j}, \forall i \geq 1, \forall j \geq 1; \tag{2.12}$$
$$\{\lambda + \lambda p_1 + \mu(1-p_1)\}\pi_{1,i,j} = \lambda\pi_{0,i+1,j} + \lambda\pi_{1,i-1,j}, \forall i \geq 1, \forall j \geq 1; \tag{2.13}$$

with the normalization equation

$$\sum_{i\geq 0}\sum_{j\geq 0}\pi_{0,i,j} + \sum_{i\geq 0}\sum_{j\geq 0}\pi_{1,i,j} = 1.$$

Theorem 2.3 *The generating functions of the Markov chain $\{C(t), N_q(t), N_o(t); t \geq 0\}$ have the* *following expressions*

$$\Pi_{0,0}(z) = \frac{\theta\mu(1-p_1)}{\theta\mu(1-p_1) - \lambda^2 p_1 z}\pi_{0,0,0} = \frac{\theta}{\theta - \lambda\rho_o z}\pi_{0,0,0};$$
$$\Pi_{1,0}(z) = \frac{\theta\lambda)}{\theta\mu(1-p_1) - \lambda^2 p_1 z}\pi_{0,0,0} = \frac{\theta\rho_q}{\theta - \lambda\rho_o z}\pi_{0,0,0};$$
$$\Pi_1(x,z) = \frac{\alpha(x,z) - \frac{\theta}{\mu(1-p_1)}}{\rho_o(1-z) + \rho_q(1-x)}(\frac{\theta}{\theta - \lambda\rho_o z})\pi_{0,0,0};$$

where $\alpha(x,z) = \frac{x(\lambda z+\theta)(1+\rho_o z) - \rho_q z}{z(1+\rho_o z)}$ *and*

$$\Pi_0(x,z) = (\frac{\beta(x,z)}{\rho_o(1-z) + \rho_q(1-x)} - \frac{\theta}{\lambda})(\frac{\theta}{\theta - \lambda\rho_o z})\pi_{0,0,0},$$

where $\beta(x,z) = \frac{1+\rho_o z}{\rho_q}(\alpha(x,z) - \frac{\theta}{\mu(1-p_1)})$.

Proof 2.2 *We introduce the following partial generating functions:*

$$\Pi_{0,0}(z) = \sum_{j=0}^{+\infty} \pi_{0,0,j}\ z^j\ ; \Pi_{1,0}(z) = \sum_{j=0}^{+\infty} \pi_{1,0,j}\ z^j,\ |z| \leq 1;$$
$$\Pi_{0,i}(z) = \sum_{j=0}^{+\infty} \pi_{0,i,j}\ z^j\ ; \Pi_{1,i}(z) = \sum_{j=0}^{+\infty} \pi_{1,i,j}\ z^j,\ |z| \leq 1,\ \forall i \geq 0.$$

Further, we define the generating functions for $\Pi_{0,i}(z)$ *and* $\Pi_{1,i}(z)$*, respectively, with respect to the queue line size.*

$$\Pi_0(x,z) = \sum_{i=0}^{+\infty} \Pi_{0,i}(z)\ x^i;\ \Pi_1(x,z) = \sum_{i=0}^{+\infty} \Pi_{1,i}(z)\ x^i, |x| \leq 1,\ |z| \leq 1.$$

From (2.7) *and* (2.8)*, we obtain:*

$$\sum_{i=0}^{+\infty} \lambda\pi_{0,i,0}\ x^i = \sum_{i=0}^{+\infty} \mu(1-p_1)\pi_{1,i,0}\ x^i;$$
$$\lambda\sum_{i=0}^{+\infty} \pi_{0,i,0}\ x^i = \mu(1-p_1)\sum_{i=0}^{+\infty} \pi_{1,i,0}\ x^i;$$
$$\lambda\Pi_{0,0}(x) = \mu(1-p_1)\Pi_{1,0}(x);$$
$$\Pi_{1,0}(x) = \frac{\lambda}{\mu(1-p_1)}\Pi_{0,0}(x);$$
$$\Pi_{1,0}(x) = \rho_q\Pi_{0,0}(x).$$

By substituting $\pi_{1,i,0}$ *from* (2.8) *into equation* (2.9)*, we obtain:*

$$
\begin{aligned}
\{\lambda+\lambda p_1+\mu(1-p_1)\}\pi_{1,i,0} &= \lambda\pi_{1,i-1,0}+\lambda\pi_{0,i+1,0}, \forall i\geq 1;\\
\{\lambda+\lambda p_1+\mu(1-p_1)\}\frac{\lambda}{\mu(1-p_1)}\pi_{0,i,0} &= \lambda\frac{\lambda}{\mu(1-p_1)}\pi_{0,i-1,0}+\lambda\pi_{0,i+1,0}, \forall i\geq 1;\\
\{\lambda^2+\lambda^2 p_1+\lambda\mu(1-p_1)\}\pi_{0,i,0} &= \lambda^2\pi_{0,i-1,0}+\lambda\mu(1-p_1)\pi_{0,i+1,0}, \forall i\geq 1; \qquad (2.14)
\end{aligned}
$$

multiplying (2.14) *by* x^i *and summing over* i*, we get:*

$$
\begin{aligned}
\sum_{i\geq 1}\{\lambda^2+\lambda^2 p_1+\lambda\mu(1-p_1)\}\pi_{0,i,0}x^i &= \sum_{i\geq 1}\lambda^2\pi_{0,i-1,0}x^i+\sum_{i\geq 1}\lambda\mu(1-p_1)\pi_{0,i+1,0}x^i, \forall i\geq 1;\\
\{\lambda^2+\lambda^2 p_1+\lambda\mu(1-p_1)\}[\Pi_{0,0}(x)-\pi_{0,0,0}] &= \lambda^2 x\Pi_{0,0}(x)+\frac{\lambda\mu(1-p_1)}{x}[\Pi_{0,0}(x)-\pi_{0,0,0}-x\pi_{0,1,0}];\\
\{\lambda^2 x+\lambda^2 p_1 x+\lambda x\mu(1-p_1)\}[\Pi_{0,0}(x)-\pi_{0,0,0}] &= \lambda^2 x^2\Pi_{0,0}(x)+\lambda\mu(1-p_1)[\Pi_{0,0}(x)-\pi_{0,0,0}-x\pi_{0,1,0}];\\
\{\lambda^2 x+\lambda^2 p_1 x+\lambda x\mu(1-p_1)\}\Pi_{0,0}(x)-\{\lambda^2 x+\lambda^2 p_1 x+\lambda x\mu(1-p_1)\}\pi_{0,0,0} &= \lambda^2 x^2\Pi_{0,0}(x)\\
&+\lambda\mu(1-p_1)\Pi_{0,0}(x)-\lambda\mu(1-p_1)\pi_{0,0,0}-\lambda x\mu(1-p_1)\pi_{0,1,0};\\
\{\lambda^2 x+\lambda^2 p_1 x+\lambda x\mu(1-p_1)-\lambda^2x^2-\lambda\mu(1-p_1))\}\Pi_{0,0}(x) &= -\lambda x\mu(1-p_1)\frac{\lambda}{\mu(1-p_1)}\pi_{0,0,0}\\
&+\{\lambda^2 x+\lambda^2 p_1 x-\lambda\mu(1-p_1)(1-x)\}\pi_{0,0,0};\\
\{(1-x)[\lambda^2 x-\lambda\mu(1-p_1)]+\lambda^2 p_1 x\}\Pi_{0,0}(x) &= \{\lambda^2 p_1 x-\lambda\mu(1-p_1)(1-x)\}\pi_{0,0,0};\\
\Pi_{0,0}(x) &= \frac{\lambda p_1 x-\mu(1-p_1)(1-x)}{(1-x)[\lambda x-\mu(1-p_1)]+\lambda p_1 x}\pi_{0,0,0};\\
\Pi_{0,0}(x) &= \frac{\rho_o x-(1-x)}{\rho_o x-(1-x)(1-\rho_q x)}\pi_{0,0,0}.
\end{aligned}
$$

So,

$$
\begin{aligned}
\Pi_{1,0}(x) =& \rho_q\Pi_{0,0}(x);\\
\Pi_{1,0}(x) =& \frac{\rho_q[\rho_o x-(1-x)]}{\rho_o x-(1-x)(1-\rho_q x)}\pi_{0,0,0}. \qquad (2.15)
\end{aligned}
$$

Multiply equation (2.10) *by* z^j *and sum over* j*, we get:*

$$
\begin{aligned}
\sum_{j=1}^{+\infty}[\lambda+\theta]\pi_{0,0,j}z^j &= \sum_{j=1}^{+\infty}\lambda p_1\pi_{1,0,j-1}z^j+\sum_{j=1}^{+\infty}\mu(1-p_1)\pi_{1,0,j}z^j;\\
[\lambda+\theta][\Pi_{0,0}(z)-\pi_{0,0,0}] &= \lambda p_1 z\Pi_{1,0}(z)+\mu(1-p_1)[\Pi_{1,0}(z)-\pi_{1,0,0}];\\
[\lambda+\theta]\Pi_{0,0}(z) &= [\lambda p_1 z+\mu(1-p_1)]\Pi_{1,0}(z);\\
\Pi_{0,0}(z) &= \frac{\lambda p_1 z+\mu(1-p_1)}{\lambda+\theta}\Pi_{1,0}(z);\\
\Pi_{1,0}(z) &= \frac{\lambda+\theta}{\lambda p_1 z+\mu(1-p_1)}\Pi_{0,0}(z) = \frac{\rho_q+\frac{\theta}{\mu(1-p_1)}}{\rho_o z+1}\Pi_{0,0}(z). \qquad (2.16)
\end{aligned}
$$

Multiply equation (2.11) *by* z^j *and sum over* j*, we get:*

$$\sum_{j=0}^{+\infty}\{\lambda+\lambda p_1+\mu(1-p_1)\}\pi_{1,0,j}z^j = \sum_{j=0}^{+\infty}\lambda\pi_{0,0,j}z^j + \sum_{j=0}^{+\infty}\lambda\pi_{0,1,j}z^j + \sum_{j=0}^{+\infty}\theta\pi_{0,0,j+1}z^j;$$
$$\{\lambda+\lambda p_1+\mu(1-p_1)\}\sum_{j=0}^{+\infty}\pi_{1,0,j}z^j = \lambda\sum_{j=0}^{+\infty}\pi_{0,0,j}z^j + \lambda\sum_{j=0}^{+\infty}\pi_{0,1,j}z^j + \theta\sum_{j=0}^{+\infty}\pi_{0,0,j+1}z^j;$$
$$\{\lambda+\lambda p_1+\mu(1-p_1)\}\pi_{1,0}(z) = \lambda\pi_{0,0}(z)+\lambda\pi_{0,1}(z)+\frac{\theta}{z}\pi_{0,0}(z);$$
$$\{\lambda z+\lambda p_1 z+z\mu(1-p_1)\}\Pi_{1,0}(z) = [\lambda z+\theta]\Pi_{0,0}(z)+\lambda z\Pi_{0,1}(z)-\theta\pi_{0,0,0};$$
$$\{\lambda z+\lambda p_1 z+z\mu(1-p_1)\}\Pi_{1,0}(z) = [\lambda z+\theta][\frac{\lambda p_1 z+\mu(1-p_1)}{\lambda+\theta}\Pi_{1,0}(z)]+\lambda z\Pi_{0,1}(z);$$
$$\{\lambda z+\lambda p_1 z+z\mu(1-p_1)\}\{\lambda+\theta\}\Pi_{1,0}(z) = [\lambda z+\theta][\lambda p_1 z+\mu(1-p_1)]\Pi_{1,0}(z)+\lambda z\{\lambda+\theta\}\Pi_{0,1}(z);\Pi_{1,0}(z)$$
$$\Pi_{0,1}(z) = \frac{\lambda z(\lambda+\theta)+(1-z)[\lambda^2 p_1 z-\theta\mu(1-p_1)]}{\lambda z\{\lambda+\theta\}}\Pi_{1,0}(z);$$

$$\Pi_{0,1}(z) = \frac{\lambda z(\lambda+\theta)+(1-z)[\lambda^2 p_1 z-\theta\mu(1-p_1)]}{\lambda z[\lambda p_1 z-\mu(1-p_1)]}\Pi_{0,0}(z);$$
$$\Pi_{0,1}(z) = \frac{(\theta-\lambda\rho_o z)(1-z)-(\lambda+\theta)\rho_q z}{(1-\rho_o z)\lambda z}\Pi_{0,0}(z). \tag{2.17}$$

Now, multiply equation (2.12) *by* z^j *and sum over* j*, we get:*

$$\sum_{j=1}^{+\infty}\lambda\pi_{0,i,j}z^j = \sum_{j=1}^{+\infty}\lambda p_1\pi_{1,i,j-1}z^j + \sum_{j=1}^{+\infty}\mu(1-p_1)\pi_{1,i,j}z^j, \forall i\geq 1;$$
$$\lambda[\Pi_{0,i}(z)-\pi_{0,i,0}] = \lambda p_1 z\Pi_{1,i}(z)+\mu(1-p_1)[\Pi_{1,i}(z)-\pi_{1,i,0}], \forall i\geq 1;$$

after using equation (2.8)*, we get:*

$$\lambda\Pi_{0,i}(z) = \lambda p_1 z\Pi_{1,i}(z)+\mu(1-p_1)\Pi_{1,i}(z), \forall i\geq 1;$$
$$\rho_q\Pi_{0,i}(z) = (\rho_o z+1)\Pi_{1,i}(z), \forall i\geq 1; \tag{2.18}$$

multiply equation (2.18) *by* x^i *and sum over* i*, we get :*

$$\sum_{i=1}^{+\infty}\rho_q\Pi_{0,i}(z)x^i = \sum_{i=1}^{+\infty}(\rho_o z+1)\Pi_{1,i}(z)x^i;$$
$$\rho_q[\Pi_0(x,z)-\Pi_{0,0}(z)] = (\rho_o z+1)[\Pi_1(x,z)-\Pi_{1,0}(z)];$$
$$\rho_q\Pi_0(x,z) = (\rho_o z+1)\Pi_1(x,z)+\rho_q\Pi_{0,0}(z)-(\rho_o z+1)\Pi_{1,0}(z);$$

by substituting (2.16) *and after some rearrangement we get:*

$$\rho_q\Pi_0(x,z) = (\rho_o z+1)\Pi_1(x,z) + \rho_q\Pi_{0,0}(z) - (\rho_o z+1)\frac{\rho_q + \frac{\theta}{\mu(1-p_1)}}{\rho_o z+1}\Pi_{0,0}(z);$$
$$\rho_q\Pi_0(x,z) = (\rho_o z+1)\Pi_1(x,z) - \frac{\theta}{\mu(1-p_1)}\Pi_{0,0}(z). \tag{2.19}$$

Multiply equation (2.13) *by* z^j *and sum over* j*, we get:*

$$\sum_{i=1}^{+\infty}\{\lambda+\lambda p_1+\mu(1-p_1)\}\pi_{1,i,j}z^j = \sum_{i=1}^{+\infty}\lambda\pi_{0,i+1,j}z^j + \sum_{i=1}^{+\infty}\lambda\pi_{1,i-1,j}z^j, \forall i\geq 1;$$
$$\{\lambda+\lambda p_1+\mu(1-p_1)\}[\Pi_{1,i}(z)-\pi_{1,i,0}] = \lambda[\Pi_{0,i+1}(z)-\pi_{0,i+1,0}] + \lambda[\Pi_{1,i-1}(z)-\pi_{1,i-1,0}], \forall i\geq 1;$$

using equation (2.9)*, we get:*

$$\{\lambda+\lambda p_1+\mu(1-p_1)\}\Pi_{1,i}(z) = \lambda\Pi_{0,i+1}(z) + \lambda\Pi_{1,i-1}(z), \forall i\geq 1;$$
$$\{\rho+1\}\Pi_{1,i}(z) = \rho_q[\Pi_{0,i+1}(z)+\Pi_{1,i-1}(z)], \forall i\geq 1; \tag{2.20}$$

multiply equation (2.20) *by* x^i *and sum over* i*, we get:*

$$\sum_{i=1}^{+\infty}\{\rho+1\}\Pi_{1,i}(z)x^i = \sum_{i=1}^{+\infty}\rho_q\Pi_{0,i+1}(z)x^i + \sum_{i=1}^{+\infty}\rho_q\Pi_{1,i-1}(z)x^i;$$
$$\{\rho+1\}[\Pi_1(x,z)-\Pi_{1,0}(z)] = \rho_q[\Pi_0(x,z)-\Pi_{0,0}(z)] + x\rho_q\Pi_1(x,z);$$
$$\{\rho+1-x\rho_q\}\Pi_1(x,z) = \rho_q\Pi_0(x,z) - \rho_q\Pi_{0,0}(z) + \{\rho+1\}\Pi_{1,0}(z); \tag{2.21}$$

where

$$(\rho+1)\Pi_{1,0}(z) - \rho_q\Pi_{0,0}(z) = (\rho+1)\frac{(\lambda+\theta)}{\lambda p_1 z+\mu(1-p_1)}\Pi_{0,0}(z) - \rho_q\Pi_{0,0}(z);$$

by substituting (2.16) *and after some rearrangement we get:*

$$= [\frac{(\lambda+\theta)(\rho+1)}{\lambda p_1 z+\mu(1-p_1)} - \rho_q]\Pi_{0,0}(z);$$
$$= \frac{(\lambda+\theta)(\rho+1) - \rho_q[\lambda p_1 z+\mu(1-p_1)]}{\lambda p_1 z+\mu(1-p_1)}\Pi_{0,0}(z);$$

then,

$$[(\rho+1)-\rho_q x]\Pi_1(x,z) = \rho_q\Pi_0(x,z) + \alpha(x,z)\Pi_{0,0}(z);$$

where $\alpha(x,z) = \frac{(\lambda+\theta)(\rho+1)-\rho_q[\lambda p_1 z+\mu(1-p_1)]}{\lambda p_1 z+\mu(1-p_1)}$.

By substituting (2.19) *into* (2.21), *we get:*

$$
\begin{aligned}
[(\rho+1)-\rho_q x]\Pi_1(x,z) &= \rho_q\Pi_0(x,z)+\alpha(x,z)\Pi_{0,0}(z);\\
[(\rho+1)-\rho_q x]\Pi_1(x,z) &= (\rho_o z+1)\Pi_1(x,z)-\frac{\theta}{\mu(1-p_1)}\Pi_{0,0}(z)+\alpha(x,z)\Pi_{0,0}(z);\\
([(\rho+1)-\rho_q x]-(\rho_o z+1))\Pi_1(x,z) &= [\alpha(x,z)-\frac{\theta}{\mu(1-p_1)}]\Pi_{0,0}(z);\\
[(1-x)\rho_o+(1-z)\rho_q]\Pi_1(x,z) &= [\alpha(x,z)-\frac{\theta}{\mu(1-p_1)}]\Pi_{0,0}(z);\\
\Pi_1(x,z) &= \frac{\alpha(x,z)-\frac{\theta}{\mu(1-p_1)}}{(1-x)\rho_o+(1-z)\rho_q}\Pi_{0,0}(z);
\end{aligned}
$$

also,

$$
\Pi_1(x,z)=\frac{\alpha(x,z)-\frac{\theta}{\mu(1-p_1)}}{(1-x)\rho_o+(1-z)\rho_q}\times\frac{\theta}{\theta-\lambda\rho_o z}\Pi_{0,0,0}.
$$

By substituting $\Pi_1(x,z)$ *into* (2.19), *we get:*

$$
\begin{aligned}
\rho_q\Pi_0(x,z) &= (\rho_o z+1)\Pi_1(x,z)-\frac{\theta}{\mu(1-p_1)}\Pi_{0,0}(z);\\
\rho_q\Pi_0(x,z) &= (\rho_o z+1)\times\frac{\alpha(x,z)-\frac{\theta}{\mu(1-p_1)}}{(1-x)\rho_o+(1-z)\rho_q}\times\frac{\theta}{\theta-\lambda\rho_o z}\Pi_{0,0,0}-\frac{\theta}{\mu(1-p_1)}\Pi_{0,0}(z);
\end{aligned}
$$

$$
\begin{aligned}
\rho_q\Pi_0(x,z) &= (\rho_o z+1)\times\frac{\alpha(x,z)-\frac{\theta}{\mu(1-p_1)}}{(1-x)\rho_o+(1-z)\rho_q}\times\frac{\theta}{\theta-\lambda\rho_o z}\Pi_{0,0,0}-\frac{\theta}{\mu(1-p_1)}\times\frac{\theta}{\theta-\lambda\rho_o z}\Pi_{0,0,0};\\
\rho_q\Pi_0(x,z) &= \{\frac{(\rho_o z+1)\times[\alpha(x,z)-\frac{\theta}{\mu(1-p_1)}]}{(1-x)\rho_o+(1-z)\rho_q}-\frac{\theta}{\mu(1-p_1)}\}\times\frac{\theta}{\theta-\lambda\rho_o z}\Pi_{0,0,0};\\
\Pi_0(x,z) &= \{\frac{\frac{\rho_o z+1}{\rho_q}\times[\alpha(x,z)-\frac{\theta}{\mu(1-p_1)}]}{(1-x)\rho_o+(1-z)\rho_q}-\frac{\theta}{\lambda}\}\times\frac{\theta}{\theta-\lambda\rho_o z}\Pi_{0,0,0};\\
\Pi_0(x,z) &= \{\frac{\beta(x,z)}{(1-x)\rho_o+(1-z)\rho_q}-\frac{\theta}{\lambda}\}\times\frac{\theta}{\theta-\lambda\rho_o z}\Pi_{0,0,0};
\end{aligned}
$$

where $\beta(x,z)=\frac{\rho_o z+1}{\rho_q}\times[\alpha(x,z)-\frac{\theta}{\mu(1-p_1)}]$.

Remark There is no obvious guess for the limiting distribution, by do not provide a closed-form symbolic solution, in terms of λ's, μ's, θ's, and p_1's, but rather we can only solve an instance of the chain (where the rates are all numbers) by using The matrix analytic methods as approximate numerical methods for solving Markov chain that are quite complex.

2.4 The Matrix-Analytic Method

The matrix analytic methods are developed by Neuts & Rao (1990) and Latouche & Ramaswami (1999) as approximate numerical methods for solving Markov chains that are quite complex. There is no obvious guess for the stationary distribution, by do not provide a closed-form symbolic solution, but rather we can only solve an instance of the chain (where the rates are all numbers).

To illustrate the method, it is useful to start by rewriting the balance equations in terms of a "generator matrix", **Q**. This is a matrix such that

$$\vec{\pi}\ .\ \mathbf{Q} = \vec{0} \quad \text{where } \vec{\pi}\ .\ \vec{1} = 1. \tag{**}$$

Here $\vec{\pi}$ is a $2 \times (b_1+1) \times (b_2+1)$ row vector of all the limiting distribution probabilities

$$\vec{\pi} = (\pi_{000}, \pi_{100}, \pi_{001}, \pi_{101}, ..., \pi_{00j}, \pi_{10j}, \pi_{010}, \pi_{110}, \pi_{011}, \pi_{111}, ..., \pi_{01j}, \pi_{11j}, ..., \pi_{0ij}, \pi_{1ij}),$$

$\forall 0 \le i \le b_1, 0 \le j \le b_2$ and $\vec{1}$ is an appropriately sized vector of 1s, and $\vec{0}$ denotes a vector with an infinite number of null entries.

Partitioning the limiting probability vector $\vec{\pi}$ as $\vec{\pi} = (\vec{\pi}_0, \vec{\pi}_1, ...\vec{\pi}_i)$, for $0 \le i \le b_1$, where $\vec{\pi}_i = (\pi_{0i0}, \pi_{1i0}, \pi_{0i1}, \pi_{1i1}, ..., \pi_{0ij}, \pi_{1ij})$, for $0 \le j \le b_2$.

By ordering the states as $S = \{(0,0,0), (1,0,0), ..., (0,0,j), (1,0,j), (0,1,0), (1,1,0), ..., (0,1,j), (1,1,j), ..., (0,i,0), (1,i,0), ..., (0,i,j), (1,i,j)\}$, we can express the infinitesimal generator **Q** of the process $\{C(t), N_q(t), N_o(t); t \ge 0\}$ in the following matrix block form:

$$\mathbf{Q} = \begin{pmatrix} L_0 & F & & & \\ B & L & F & & \\ & B & L & F & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where

$$L_0 = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \mu(1-p_1) & C & \lambda p_1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \theta & -(\lambda+\theta) & \lambda & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \mu(1-p_1) & C & \lambda p_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \theta & -(\lambda+\theta) & \lambda & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & & \vdots \end{pmatrix}$$

$$L = \begin{pmatrix} -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \mu(1-p_1) & C & \lambda p_1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -(\lambda+\theta) & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \mu(1-p_1) & C & \lambda p_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & -(\lambda+\theta) & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & & \vdots \end{pmatrix}$$

where $C = -[\lambda + \lambda p_1 + \mu(1 - p_1)]$.

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & \dots & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & \lambda & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \end{pmatrix}$$

By using the notation $\vec{\pi}_i = (\pi_{0i0}, \pi_{1i0}, \pi_{0i1}, \pi_{1i1}, ..., \pi_{0ij}, \pi_{1ij})$; for $0 \leq j \leq b_2$, we rewrite the $\vec{\pi} \ . \ \mathbf{Q} = \vec{0}$ for the probabilities as matrix equations:

$$\begin{aligned} \vec{\pi}_0 \ . \ L_0 + \vec{\pi}_1 \ . \ B = & \quad \vec{0}, \\ \vec{\pi}_0 \ . \ F + \vec{\pi}_1 \ . \ L + \vec{\pi}_2 \ . \ B = & \quad \vec{0}, \\ \vec{\pi}_1 \ . \ F + \vec{\pi}_2 \ . \ L + \vec{\pi}_3 \ . \ B = & \quad \vec{0}, \\ \vdots \qquad & \quad \vdots \\ \vec{\pi}_{i-1} \ . \ F + \vec{\pi}_i \ . \ L + \vec{\pi}_{i+1} \ . \ B = & \quad \vec{0}, \quad \forall i \geq 1. \end{aligned}$$

The idea behind matrix-analytic methods is that we recursively express $\vec{\pi}_i$ in terms of $\vec{\pi}_0$. However, rather than being related by a constant, ρ, as in $M/M/1$ queue, they are instead related by a matrix $\mathbf{R}$, such that

$$\vec{\pi}_i = \vec{\pi}_{i-1} \ . \ \mathbf{R}, \quad \forall i > 0$$

which, when expanded, yields

$$\vec{\pi}_i = \vec{\pi}_o \ . \ \mathbf{R}^i, \quad \forall i > 0.$$

By substituting this guess into the matrix equations yields the following:

$$\begin{array}{ccccc} \vec{\pi}_0 \ . \ L_0 + \vec{\pi}_1 \ . \ B & = \vec{0} & \Rightarrow & \vec{\pi}_0 \ . \ L_0 + \vec{\pi}_1 \ . \ B & = \vec{0}, \\ \vec{\pi}_0 \ . \ F + \vec{\pi}_o \ . \ \mathbf{R} \ . \ L + \vec{\pi}_o \ . \ \mathbf{R}^2 \ . \ B & = \vec{0} & \Rightarrow & \vec{\pi}_0 \ . \ (F + \mathbf{R} \ . \ L + \mathbf{R}^2 \ . \ B) & = \vec{0}, \\ \vec{\pi}_1 \ . \ F + \vec{\pi}_1 \ . \ \mathbf{R} \ . \ L + \vec{\pi}_1 \ . \ \mathbf{R}^2 \ . \ B & = \vec{0} & \Rightarrow & \vec{\pi}_1 \ . \ (F + \mathbf{R} \ . \ L + \mathbf{R}^2 \ . \ B) & = \vec{0}, \\ \vdots & \vdots & & \vdots & \vdots \\ \vec{\pi}_{i-1} \ . \ F + \vec{\pi}_{i-1} \ . \ \mathbf{R} \ . \ L + \vec{\pi}_{i-1} \ . \ \mathbf{R}^2 \ . \ B & = \vec{0} & \Rightarrow & \vec{\pi}_{i-1} \ . \ (F + \mathbf{R} \ . \ L + \mathbf{R}^2 \ . \ B) & = \vec{0}, \quad \forall i \geq 1. \end{array}$$

We observe that the common portion is $(F + \mathbf{R} \ . \ L + \mathbf{R}^2 \ . \ B) = 0$. Then, we use this common portion to determine $\mathbf{R}$ as follows:

$$\begin{array}{rcl} (F + \mathbf{R} \ . \ L + \mathbf{R}^2 \ . \ B) & = & 0, \\ \Rightarrow \qquad \mathbf{R} \ . \ L & = & -(\mathbf{R}^2 \ . \ B + F), \\ \Rightarrow \qquad \mathbf{R} & = & -(\mathbf{R}^2 \ . \ B + F)L^{-1}, \end{array}$$

then, we solve for $\mathbf{R}$ by iterating (here $\mathbf{R}_n$ denotes the nth iteration of $\mathbf{R}$):

- Let $\mathbf{R}_0 = 0$ (or a better guess, if available).
- While $\| \mathbf{R}_{n+1} - \mathbf{R}_n \| > \epsilon$ (The typical definition is the maximum of all the elementsnis the matrix $\mathbf{R}_{n+1} - \mathbf{R}_n$), set $\mathbf{R}_{n+1} = -(\mathbf{R}^2 \ . \ B + F)L^{-1}$.

This process keeps iterating until it determines that $\mathbf{R}$ has converged. Once $\mathbf{R}$ converges, we set $\vec{\pi}_i = \vec{\pi}_o \ . \ \mathbf{R}^i$.

- Then, we have two equations involving $\vec{\pi}_o$: $\vec{\pi}_0 \ . \ (L_0 + \mathbf{R} \ . \ B) = \vec{0}$ and the normalizing equation $\vec{\pi} \ . \ \vec{1} = 1$. We rewrite the normalizing equation in terms of $\vec{\pi}_0$:

$$\begin{aligned} \sum_{i=0}^{+\infty} \vec{\pi}_i \ . \ \vec{1} &= 1, \\ \sum_{i=0}^{+\infty} \vec{\pi}_o \ . \ \mathbf{R}^i \ . \ \vec{1} &= 1, \\ \vec{\pi}_o (\sum_{i=0}^{+\infty} \mathbf{R}^i) \ . \ \vec{1} &= 1, \\ \vec{\pi}_o (\mathbf{I} - \mathbf{R})^{-1} \ . \ \vec{1} &= 1. \end{aligned}$$

- By using the notation $\phi = L_0 + \mathbf{R} \ . \ B$ and $\psi = (\mathbf{I} - \mathbf{R})^{-1}\vec{1}$. Thus, $\vec{\pi}_0(L_0 + \mathbf{R} \ . \ B) = \vec{0}$ becomes $\vec{\pi}_0 \phi = \vec{0}$ and $\vec{\pi}_o(\mathbf{I} - \mathbf{R})^{-1} \ . \ \vec{1} = 1$ becomes $\vec{\pi}_0 \psi = 1$.
- After replacing one equation of ϕ (the first column) with the normalizing equation ψ and the first element of the zero with 1, the system of equations has a unique solution, and we solve this system for $\vec{\pi}_0$.
- Using $\vec{\pi}_i = \vec{\pi}_o \ . \ \mathbf{R}^i$, we get all the $\vec{\pi}_i$.

2.5 Numerical examples

We present a numerical example to determine the steady state probabilities $\{(\pi_{0ij}, \pi_{1ij})$ for $0 \leq i \leq 2 \, ; 0 \leq j \leq 2\}$. Therefore, note that

$$
\begin{aligned}
\vec{\pi}_0 &= (\pi_{000}, \pi_{100}, \pi_{001}, \pi_{101}, \pi_{002}, \pi_{102}), \\
\vec{\pi}_1 &= (\pi_{010}, \pi_{110}, \pi_{011}, \pi_{111}, \pi_{012}, \pi_{112}), \\
\vec{\pi}_2 &= (\pi_{020}, \pi_{120}, \pi_{021}, \pi_{121}, \pi_{022}, \pi_{122}),
\end{aligned}
$$

also satisfies $\vec{\pi} \ . \ \vec{1} = 1$, $\sum_{i=0}^{2} \vec{\pi}_i \ . \ \vec{1} = 1$ (where $\vec{1}$ is a 6×1 column vector of ones) and $\sum_{i=0}^{2} \sum_{j=0}^{2} (\pi_{0ij} + \pi_{1ij}) = 1$

$$
\left[\begin{array}{cccccc|cccccc|cccccc}
-\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
H & C & \lambda p_1 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \theta & D & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & H & C & \lambda p_1 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \theta & D & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & H & C & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & \lambda & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & H & C & \lambda p_1 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H & C & \lambda p_1 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H & C & 0 & 0 & 0 & 0 & 0 & \lambda \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H & C & \lambda p_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H & C & \lambda p_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & -\lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H & C
\end{array}\right]
$$

Where $H = \mu(1 - p_1)$; $D = -[\lambda + \theta\ ]$
and $C = -[\lambda + \lambda p_1 + \mu(1 - p_1)\ ]$

Figure 2.9: $\mathbf{Q}$ for $0 \leq i \leq 2\ ; 0 \leq j \leq 2\}$.

In Figure 2.9, the matrix obtained here is similar than these in section 4. Then, based on the matrix analytic method proposed, we briefly provide some numerical examples in some cases that examine the sensitivity and the impact of the system parameters: customers' arrival rate λ, service rate μ, retrial rate θ and p_1 on the stationary distribution $\vec{\pi} = (\vec{\pi}_0, \vec{\pi}_1, \vec{\pi}_2)$. The values of these parameters are chosen so that they satisfy the stability condition.

Tables 2.1, 2.2 and 2.3 list values of $\vec{\pi}$ for different values p_1, λ and μ. The results exhibit the expected behaviour, that is for each fixed value of θ, μ, λ and p_1.

In Tables 2.4, 2.5 and 2.6 we calculate some performance measures of the system as the mean number of customers in the orbit $\bar{n}_o$, in the queue $\bar{n}_q$ and in the system $\bar{n}$.

Table 2.1: Stationary distributions for $\mu = 1$.

$\vec{\pi}$	$\theta = 0,05$ The stationary distribution	$p_1 = 0,25$ $\lambda = 0,18$ $\rho = 0,30$	$p_1 = 0,5$ $\lambda = 0,2$ $\rho = 0,60$	$p_1 = 0,75$ $\lambda = 0,1$ $\rho = 0,70$
$\vec{\pi}_0$	π_{000}	0, 251361	0, 241127	0, 293949
	π_{100}	0, 094564	0, 086836	0, 101765
	π_{001}	0, 140953	0, 155591	0, 146214
	π_{101}	0, 056322	0, 060428	0, 057199
	π_{002}	0, 067175	0, 086198	0, 067801
	π_{102}	0, 025716	0, 031013	0, 023521
$\vec{\pi}_1$	π_{010}	0, 081337	0, 067320	0, 070536
	π_{110}	0, 032535	0, 026928	0, 028214
	π_{011}	0, 059996	0, 064573	0, 065840
	π_{111}	0, 020745	0, 020443	0, 017872
	π_{012}	0, 029689	0, 037855	0, 033340
	π_{112}	0, 009801	0, 011053	0, 007974
$\vec{\pi}_2$	π_{020}	0, 027984	0, 020876	0, 019556
	π_{120}	0, 011194	0, 008350	0, 007822
	π_{021}	0, 021818	0, 021346	0, 019650
	π_{121}	0, 007608	0, 006868	0, 005513
	π_{022}	0, 011202	0, 013200	0, 010769
	π_{122}	0, 003720	0, 003906	0, 002654
$\sum_{i=0}^{2}\sum_{j=0}^{2}(\pi_{0ij}+\pi_{1ij})$		0, 953721	0, 963912	0, 980188

Table 2.2: Stationary distributions for $\mu = 2$.

$\vec{\pi}$	$\theta = 0,05$ The stationary distribution	$p_1 = 0,25$ $\lambda = 0,36$ $\rho = 0,30$	$p_1 = 0,5$ $\lambda = 0,4$ $\rho = 0,60$	$p_1 = 0,75$ $\lambda = 0,2$ $\rho = 0,70$
$\vec{\pi}_0$	π_{000}	0, 381733	0, 211534	0, 236644
	π_{100}	0, 090449	0, 072130	0, 078785
	π_{001}	0, 174477	0, 168535	0, 170511
	π_{101}	0, 042263	0, 061415	0, 061620
	π_{002}	0, 063818	0, 118307	0, 108420
	π_{102}	0, 014908	0, 040955	0, 035724
$\vec{\pi}_1$	π_{010}	0, 083967	0, 055919	0, 055565
	π_{110}	0, 020152	0, 022368	0, 022226
	π_{011}	0, 045586	0, 062335	0, 064269
	π_{111}	0, 009732	0, 020461	0, 019040
	π_{012}	0, 016933	0, 045514	0, 043407
	π_{112}	0, 003480	0, 014113	0, 011651
$\vec{\pi}_2$	π_{020}	0, 018708	0, 017341	0, 015675
	π_{120}	0, 004490	0, 006936	0, 006270
	π_{021}	0, 010449	0, 020428	0, 019299
	π_{121}	0, 002238	0, 006784	0, 005838
	π_{022}	0, 003942	0, 015498	0, 013792
	π_{122}	0, 000812	0, 004843	0, 003765
$\sum_{i=0}^{2}\sum_{j=0}^{2}(\pi_{0ij}+\pi_{1ij})$		0, 988138	0, 965415	0, 972502

Table 2.3: Stationary distributions for $\mu = 3$.

$\vec{\pi}$	$\theta = 0,05$ The stationary distribution	$p_1 = 0,25$ $\lambda = 0,54$ $\rho = 0,30$	$p_1 = 0,5$ $\lambda = 0,5$ $\rho = 0,50$	$p_1 = 0,75$ $\lambda = 0,3$ $\rho = 0,70$
$\vec{\pi}_0$	π_{000}	0, 339849	0, 230564	0, 228960
	π_{100}	0, 079551	0, 067690	0, 072467
	π_{001}	0, 195791	0, 189494	0, 176521
	π_{101}	0, 046568	0, 058200	0, 060636
	π_{002}	0, 088553	0, 132544	0, 124222
	π_{102}	0, 020426	0, 038900	0, 039779
$\vec{\pi}_1$	π_{010}	0, 073658	0, 055090	0, 050811
	π_{110}	0, 017678	0, 018363	0, 020324
	π_{011}	0, 048683	0, 059148	0, 061491
	π_{111}	0, 010623	0, 016656	0, 018499
	π_{012}	0, 022280	0, 042502	0, 045503
	π_{112}	0, 004710	0, 011392	0, 012652
$\vec{\pi}_2$	π_{020}	0, 016368	0, 014945	0, 014251
	π_{120}	0, 003928	0, 004982	0, 005700
	π_{021}	0, 011073	0, 016752	0, 018293
	π_{121}	0, 002422	0, 004754	0, 005607
	π_{022}	0, 005128	0, 012364	0, 014201
	π_{122}	0, 001085	0, 003329	0, 003998
$\sum_{i=0}^{2}\sum_{j=0}^{2}(\pi_{0ij}+\pi_{1ij})$		0, 988372	0, 977666	0, 973916

Table 2.4: Performance measures for $\mu = 1$.

p_1	ρ	$\overline{n}_0$	$\overline{n}_q$	$\overline{n}$
$0,25$	$0,3$	$0,602050$	$0,401155$	$1,265409$
$0,25$	$0,7$	$0,719818$	$0,484046$	$1,504296$
$0,50$	$0,6$	$0,695701$	$0,377266$	$1,328794$
$0,50$	$0,9$	$0,802063$	$0,457317$	$1,559500$
$0,75$	$0,7$	$0,604406$	$0,355702$	$1,212643$

Table 2.5: Performance measures for $\mu = 2$.

p_1	ρ	$\overline{n}_0$	$\overline{n}_q$	$\overline{n}$
$0,25$	$0,1$	$0,393624$	$0,094697$	$0,570672$
$0,25$	$0,2$	$0,400394$	$0,114475$	$0,604274$
$0,25$	$0,3$	$0,492531$	$0,261127$	$0,942183$
$0,25$	$0,4$	$0,579220$	$0,317857$	$1,121311$
$0,25$	$0,5$	$0,773235$	$0,397808$	$1,430926$
$0,25$	$0,6$	$0,810404$	$0,445972$	$1,538947$
$0,25$	$0,7$	$0,838871$	$0,487261$	$1,627525$
$0,50$	$0,2$	$0,207289$	$0,138650$	$0,459690$
$0,50$	$0,3$	$0,457421$	$0,213101$	$0,832645$
$0,50$	$0,5$	$0,738598$	$0,317964$	$1,281377$
$0,50$	$0,6$	$0,818418$	$0,364370$	$1,432792$
$0,50$	$0,8$	$0,845337$	$0,421438$	$1,549216$
$0,50$	$0,9$	$0,854948$	$0,446507$	$1,597139$
$0,75$	$0,4$	$0,168929$	$0,032056$	$0,358454$
$0,75$	$0,7$	$0,774095$	$0,345438$	$1,364452$

Table 2.6: Performance measures for $\mu = 3$.

p_1	ρ	$\overline{n}_0$	$\overline{n}_q$	$\overline{n}$
$0,25$	$0,2$	$0,343849$	$0,174650$	$0,655122$
$0,25$	$0,3$	$0,599522$	$0,257641$	$1,044153$
$0,25$	$0,4$	$0,765932$	$0,332529$	$1,325862$
$0,25$	$0,5$	$0,818513$	$0,392652$	$1,469175$
$0,50$	$0,3$	$0,578726$	$0,210169$	$0,949430$
$0,50$	$0,5$	$0,827064$	$0,317399$	$1,368726$
$0,50$	$0,7$	$0,879828$	$0,393043$	$1,539275$
$0,75$	$0,2$	$0,180412$	$0,120561$	$0,402036$
$0,75$	$0,7$	$0,821759$	$0,333380$	$1,394802$
$0,75$	$0,9$	$1,336944$	$0,575866$	$2,166579$

We also visualize these results graphically, and we obtain the following curves for different performance measures with respect to ρ and for each μ.

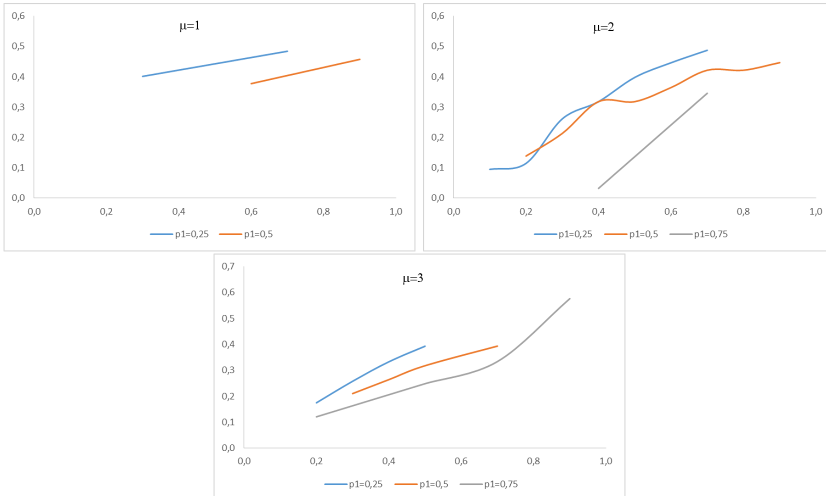


Figure 2.10: $\bar{n_q}$ with respect to ρ for $\mu = \{1, 2, 3\}$ and $p_1 = \{0.25, 0.50, 0.75\}$.

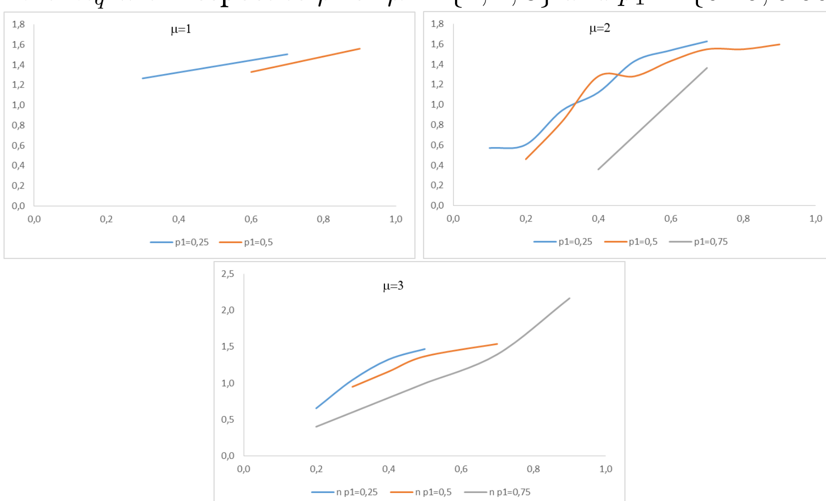


Figure 2.11: $\bar{n}$ with respect to ρ for $\mu = \{1, 2, 3\}$ and $p_1 = \{0.25, 0.50, 0.75\}$.

According to the curves obtained from Tables 2.4, 2.5 and 2.6 and for the different values of $\mu = \{1, 2, 3\}$, we observe that $\bar{n}$, $\bar{n_o}$ or $\bar{n_q}$ increase with respect to the values of the traffic intensity ρ, decrease with respect to p_1 the probability of service interruption and joining the orbit. We note that the mean number of customers reaches the maximum values for the smallest probability $p_1 = 0.25$. And that the curves are almost identical for $p_1 = 0.25$ and $p_1 = 0.5$. But in all cases, $\bar{n_o}$ is higher than $\bar{n_q}$.

For $\mu = 1$, the curves are linear because we did not obtain much data since there are very few solutions of the stationary distributions of the studied system. Otherwise, when $\mu = 2$, there are more values.

Chapter 3

$M/M/1$ Queue With Retrials After Service Interruption Option Selected By Customer And Orbital Search

In the study of retrial queues, several papers are interested in obtaining the limiting distribution of the system state, the performance measures and emphasizes the impact of specific descriptors defined in the model under study. Sometimes it is possible to derive closed-form expressions, we refer to Arrar et al. (2012), Arrar et al. (2017), Krishnamoorthy et al. (2005), Wang (2004), Wang & Zhao (2007) as a selection of the related literature, but very often the absence of explicit formulas and recursive schemes for the computation of the limiting probabilities or even impossible to find the limiting distribution of the system state. This difficulty motivates the implementation of approximations numerically.

The literature on approximations of retrial queues is various. The paper by Gómez-Corral (2006) includes a bibliographical guide to the use of the matrix analytic methods in retrial queuing systems, and for a related approximation, we refer to Baumann & Sandmann (2012), Harchol-Balter (2013), Neuts (1981), Neuts & Rao (1990).

Our aim in this chapter is to describe the main model first, then provide a simple and efficient procedure for the computation of the limiting probabilities $\{\pi_{c,i,j}, 0 \leq c \leq 1, i \geq 0, j \geq 0\}$ by using the matrix analytic methods and calculating the performance measures of the main model under study. Finally, the influence of some parameters on the performance measures of the system has been examined numerically and illustrated.

3.1 Model description

In this model, we analyze a $M/M/1$ retrial queue with customers' break choice and constant retrial policy. We consider a single server retrial queueing system; whose orbit and queue have infinite capacity. We suppose that primary customers arrive according to a Poisson process with rate $\lambda > 0$. The service time is exponentially distributed with parameter μ. The following rules govern the dynamic of the customers:

- If an arriving customer finds the server idle, he immediately begins his service. Otherwise, an incoming customer that finds the server occupied, will join the line of the queue in the service area according to FCFS discipline.
- We assume that a customer who has started his service, may decide to interrupt it and go on vacation or take a break. For this fact, he has to leave the service area and enter the orbit before returning for another service. Thus, the customer can leave the system permanently with probability $(1 - p_1)$, after finishing his service, or join the orbit with probability p_1 and return to the server after a period of time.
- We assume that the customers have first access to the orbit after an initial service with rate λp_1, Takas (1963).
- An orbiting customer attempts to access to the server directly at random intervals time (without rejoin the queue line in service area), where the inter-retrials times are exponentially distributed with rate $\theta > 0$, according to the linear retrial policy $\alpha(1 - \delta_{0j}) + j\theta$, given that α is a constant rate, δ_{0j} denotes Kronecker function and the rate $j\theta$ is the so-called classical retrial policy rate depending on how many customers j are on orbit.
- An orbiting customer can access the server for another service only if the queue is empty.
- The server can go in search of customers immediately after each service completion, by picking up an orbital customer with probability p. The search time is assumed to be negligible. The probability for not going for the search of customers is $q = 1 - p$.
- All the random variables defined above are mutually independent.

Adding to the previous parameters, we define the global traffic intensity given by

$$\rho = \frac{\lambda + \lambda p_1}{\mu(1 - p_1)},$$

it is the ratio of the arrival rate $\lambda + \lambda p_1$ to the departure rate $\mu(1 - p_1)$, Artalejo et al. (2002). We can write also, $\rho = \rho_q + \rho_o$, where $\rho_q = \frac{\lambda}{\mu(1-p_1)}$ is the traffic intensity of primary customers and $\rho_o = \frac{\lambda p_1}{\mu(1-p_1)}$ is the traffic intensity of orbiting customers.

3.2 Stochastic analysis

We denote by $N_q(t)$ how many customers queued at given time t. excluding any customer who may be in service. $N_o(t)$ is the number of orbital customers at the instant t. And let $C(t)$ be equal to 0 or 1 depending on the state of the server if it is idle or busy at time t.

And, depending on the state of the server, whether it is idle or busy at time t, let $C(t)$ be 0 or 1. Let $N(t)$ denotes the total number of customers in the system at time t (i.e. in orbit, in queue line and in service), where $N(t) = N_q(t) + N_o(t) + C(t)$.

So that the continuous-time stochastic process $\chi = \{C(t), N_q(t), N_o(t); t \geq 0\}$, describes the state of the system with state space $(c, i, j) \in \{0, 1\} \times \mathbb{N} \times \mathbb{N}$.

Its infinitesimal transition rates $q_{(0,i,j)(c,m,n)}$ and $q_{(1,i,j)(c,m,n)}$ are given by

- For $i = 0$ and $j = 0$:

$$q_{(0,i,j)(c,m,n)} = \begin{cases} \lambda, & \text{if } (c,m,n) = (1,0,0); \\ -\lambda, & \text{if } (c,m,n) = (0,i,j); \\ 0, & \text{otherwise} \end{cases}$$

  and

$$q_{(1,i,j)(c,m,n)} = \begin{cases} \lambda, & \text{if } (c,m,n) = (1,1,0); \\ \lambda p_1, & \text{if } (c,m,n) = (0,0,1); \\ \mu(1-p_1), & \text{if } (c,m,n) = (0,0,0); \\ -[\lambda + \lambda p_1 + \mu(1-p_1)], & \text{if } (c,m,n) = (1,i,j); \\ 0, & \text{otherwise.} \end{cases}$$

- For $i = 0$ and $j \geq 1$:

$$q_{(0,i,j)(c,m,n)} = \begin{cases} \lambda, & \text{if } (c,m,n) = (1,0,j); \\ \alpha(1-\delta_{0j}) + j\theta, & \text{if } (c,m,n) = (1,0,j-1); \\ -[\lambda + \alpha(1-\delta_{0j}) + j\theta], & \text{if } (c,m,n) = (0,i,j); \\ 0, & \text{otherwise} \end{cases}$$

  and

$$q_{(1,i,j)(c,m,n)} = \begin{cases} \lambda, & \text{if } (c,m,n) = (1,i,j); \\ \lambda p_1, & \text{if } (c,m,n) = (0,0,j+1); \\ \mu(1-p_1) + q\mu, & \text{if } (c,m,n) = (0,0,j); \\ p\mu, & \text{if } (c,m,n) = (1,0,j-1); \\ -[\lambda + \lambda p_1 + \mu(1-p_1) + q\mu + p\mu], & \text{if } (c,m,n) = (1,i,j); \\ 0, & \text{otherwise.} \end{cases}$$

- For $j = 0$ and $i \geq 1$:

$$q_{(0,i,j)(c,m,n)} = \begin{cases} \lambda, & \text{if } (c,m,n) = (1,i-1,0); \\ -\lambda, & \text{if } (c,m,n) = (0,i,j); \\ 0, & \text{otherwise} \end{cases}$$

  and

$$q_{(1,i,j)(c,m,n)} = \begin{cases} \lambda, & \text{if } (c,m,n) = (1, i+1, 0); \\ \lambda p_1, & \text{if } (c,m,n) = (0, i, 1); \\ \mu(1-p_1), & \text{if } (c,m,n) = (0, i, 0); \\ -[\lambda + \lambda p_1 + \mu(1-p_1)], & \text{if } (c,m,n) = (1, i, j); \\ 0, & \text{otherwise.} \end{cases}$$

- For $i \geq 1$ and $j \geq 1$:

$$q_{(0,i,j)(c,m,n)} = \begin{cases} \lambda, & \text{if } (c,m,n) = (1, i-1, j); \\ -\lambda, & \text{if } (c,m,n) = (0, i, j); \\ 0, & \text{otherwise} \end{cases}$$

and

$$q_{(1,i,j)(c,m,n)} = \begin{cases} \lambda, & \text{if } (c,m,n) = (1, i+1, j); \\ \lambda p_1, & \text{if } (c,m,n) = (0, i, j+1); \\ \mu(1-p_1) + q\mu, & \text{if } (c,m,n) = (0, i, j); \\ p\mu, & \text{if } (c,m,n) = (1, i, j-1); \\ -[\lambda + \lambda p_1 + \mu(1-p_1) + q\mu + p\mu], & \text{if } (c,m,n) = (1, i, j); \\ 0, & \text{otherwise.} \end{cases}$$

The stochastic behaviour of the process χ can be represented with the help of the graphical transitions shown in Figure 3.1.

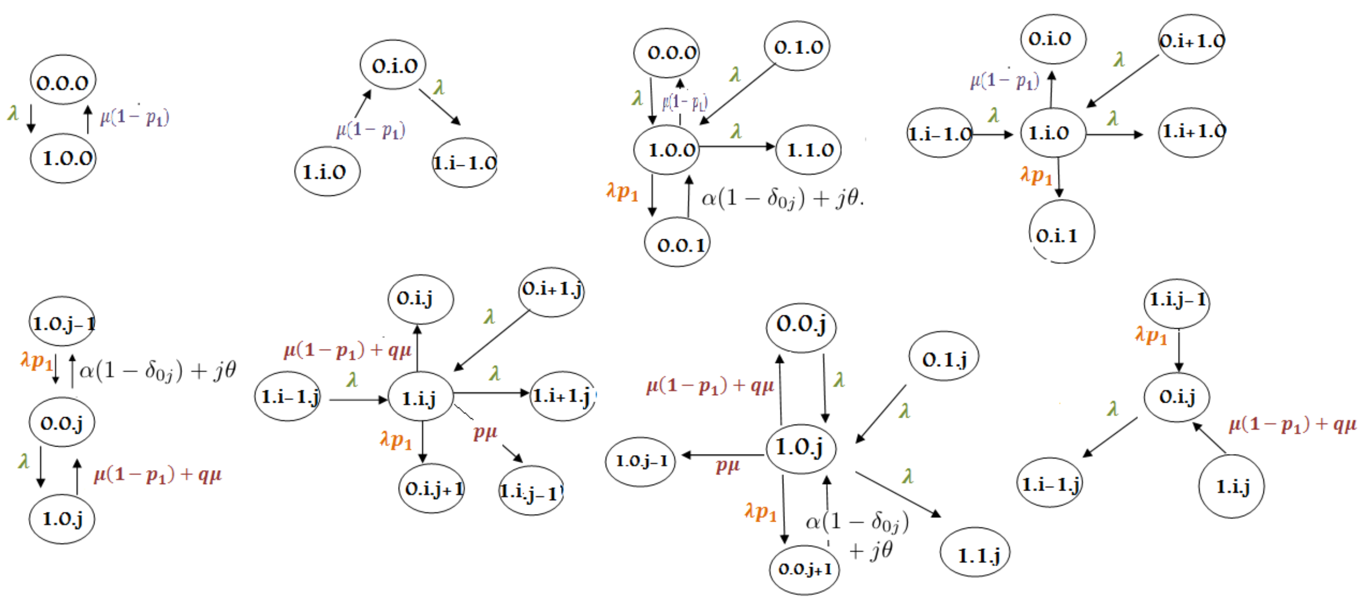


Figure 3.1: Graphical transitions.

Particular cases

In Figure 3.2, we present some special cases of our model by setting appropriate parameters as follows:

- The main model behaves like a $M/M/1$ queue with retrials after interruption service and orbital search according to a constant retrial policy if $\theta = 0$;
- The main model behaves like a $M/M/1$ queue with retrials after interruption service and orbital search according to a classical retrial policy if $\alpha = 0$;
- The main model behaves like a $M/M/1$ standard queue according to first come, first served (FCFS) discipline if $p_1 = 0$;
- The main model behaves like a $M/M/1$ queue with retrials after interruption service if $p = 0$ (where there is no orbital search). In this case, we can get three cases, depending on the retrial policy that is selected (it can be according to a linear retrial policy or either according to a classical retrial policy when $\alpha = 0$ or constant retrial policy when $\theta = 0$);
- The main model can be without waiting space. Then, if an arriving customer finds the server idle, he immediately begins his service. Otherwise, an arriving customer who finds the server busy, leaves the system without any effect on the system. Its infinitesimal transition rates $q_{(0,n)(c,m)}$ and $q_{(1,n)(c,m)}$ are given by

$$\begin{aligned}
q_{(0,n)(1,n)} &= \lambda, \forall n \geq 0; \\
q_{(0,n)(1,n-1)} &= \alpha(1-\delta_{0n}) + n\theta, \forall n \geq 1; \\
q_{(1,0)(0,0)} &= \mu(1-p_1); \\
q_{(1,n)(0,n)} &= \mu(1-p_1) + q\mu, \forall n \geq 1; \\
q_{(1,n)(1,n-1)} &= p\mu, \forall n \geq 1; \\
q_{(1,n)(0,n+1)} &= \lambda p_1, \forall n \geq 0.
\end{aligned}$$

The set of statistical equilibrium equations for the probabilities $\{\pi_{0,n}, \pi_{1,n}; \forall n \geq 0\}$ have the following expressions

$$\lambda\pi_{0,0} = \mu(1-p_1)\pi_{1,0}; \tag{3.1}$$

$$[\lambda + \alpha(1-\delta_{0j}) + j\theta]\pi_{0,n} = \lambda p_1 \pi_{1,n-1} + [\mu(1-p_1) + q\mu]\pi_{1,n}, \quad \forall n \geq 1; \tag{3.2}$$

$$\{\lambda p_1 + \mu(1-p_1)\}\pi_{1,0} = \lambda\pi_{0,0} + [\alpha(1-\delta_{0j}) + j\theta]\pi_{0,1} + p\mu\pi_{1,1}; \tag{3.3}$$

$$\{\lambda p_1 + \mu(1-p_1) + q\mu + p\mu\}\pi_{1,n} = \lambda\pi_{0,n} + [\alpha(1-\delta_{0j}) + j\theta]\pi_{0,n+1} + p\mu\pi_{1,n+1}, \quad \forall n \geq 1; \tag{3.4}$$

and the normalization equation $\sum_{n\geq 0} \pi_{0,n} + \sum_{n\geq 0} \pi_{1,n} = 1$.

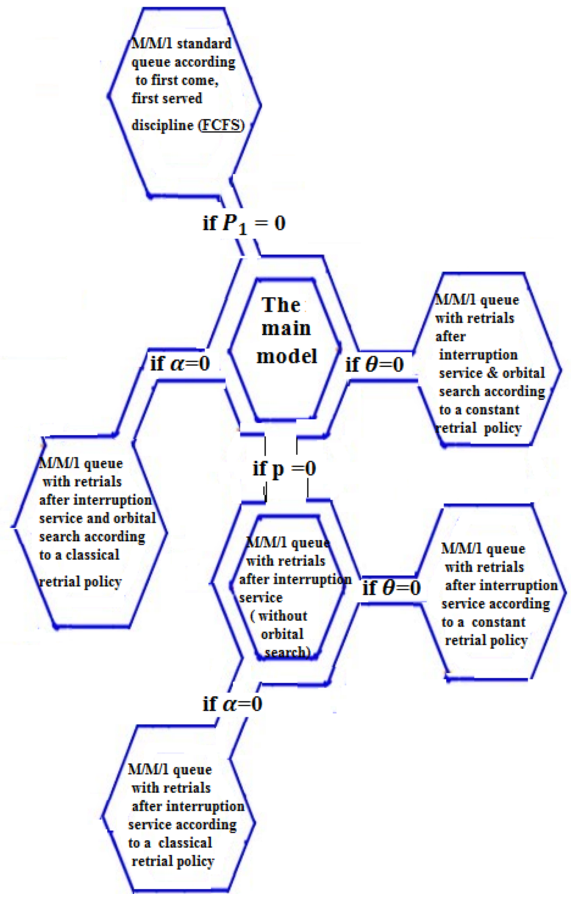


Figure 3.2: Particular cases.

In this paper, the retrial models operating under the classical retrial policy or the linear policy have transitions between states $(0, 0, j)$ that depend on the third coordinate j. The main analytical difficulties are related to this fact. Since we cannot obtain the steady state distributions of the model in an explicit form. We can solve only one instance of the chain, when the rates are all numbers, by using the Matrix analytic methods, which are approximate numerical methods for solving complex Markov chains.

3.3 Matrix-Analytic Method

To illustrate the method, it is useful to start by rewriting the balance equations in terms of a "generator matrix", **Q**. This is a matrix such that

$$\vec{\pi}\ .\ \mathbf{Q} = \vec{0}, \text{ where } \vec{\pi}\ .\ \vec{1} = 1. \tag{3.5}$$

Here, $\vec{\pi}$ is a $2 \times (b_1 + 1) \times (b_2 + 1)$ row vector of all the limiting distribution probabilities

$$\vec{\pi} = (\pi_{000}, \pi_{100}, \pi_{001}, \pi_{101}, ..., \pi_{00j}, \pi_{10j}, \pi_{010}, \pi_{110}, \pi_{011}, \pi_{111}, ..., \pi_{01j}, \pi_{11j}, ..., \pi_{0ij}, \pi_{1ij}), \tag{3.6}$$

$\forall 0 \leq i \leq b_1, 0 \leq j \leq b_2$, and $\vec{1}$ is an appropriately sized vector of 1s, and $\vec{0}$ denotes a vector with an infinite number of null entries.

Partitioning the limiting probability vector $\vec{\pi}$ as

$$\vec{\pi} = (\vec{\pi}_0, \vec{\pi}_1, ...\vec{\pi}_i), \text{ for } 0 \leq i \leq b_1, \tag{3.7}$$

where

$$\vec{\pi}_i = (\pi_{0i0}, \pi_{1i0}, \pi_{0i1}, \pi_{1i1}, ..., \pi_{0ij}, \pi_{1ij}), \text{ for } 0 \leq j \leq b_2. \tag{3.8}$$

By ordering the states as $S = \{(0,0,0), (1,0,0), ..., (0,0,j), (1,0,j), (0,1,0), (1,1,0), ..., (0,1,j), (1,1,j), ..., (0,i,0), (1,i,0), ..., (0,i,j), (1,i,j)\}$, we can express the infinitesimal generator $\mathbf{Q}$ of the process $\{C(t), N_q(t), N_o(t); t \geq 0\}$ in the following matrix block form:

$$\mathbf{Q} = \begin{pmatrix} L_0 & F & & & \\ B & L & F & & \\ & B & L & F & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where

$$L_0 = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu(1-p_1) & A_0 & \lambda p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & S & T & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p\mu & V & A_1 & \lambda p_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S & T & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p\mu & V & A_1 & \lambda p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & S & T & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p\mu & V & A_1 & \lambda p_1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$L = \begin{pmatrix} -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu(1-p_1) & A_0 & \lambda p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p\mu & V & A_1 & \lambda p_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p\mu & V & A_1 & \lambda p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p\mu & V & A_1 & \lambda p_1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

with $A_0 = -[\lambda + \lambda p_1 + \mu(1-p_1)]$, $A_1 = -[\lambda + \lambda p_1 + \mu(1-p_1) + q\mu + p\mu]$, $S = \alpha(1-\delta_{0j}) + j\theta$, $T = -[\lambda + \alpha(1-\delta_{0j}) + j\theta]$ and $V = \mu(1-p_1) + q\mu$.

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & \dots & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & \lambda & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \end{pmatrix}$$

3.4 Numerical examples

We present a numerical example to determine the steady state probabilities $\{(\pi_{0ij}, \pi_{1ij})$, for $0 \leq i \leq 2$ and $0 \leq j \leq 3\}$.

Therefore, note that

$$\vec{\pi}_0 = (\pi_{000} \quad \pi_{100} \quad \pi_{001} \quad \pi_{101} \quad \pi_{002} \quad \pi_{102} \quad \pi_{003} \quad \pi_{103}),$$

$$\vec{\pi}_1 = (\pi_{010} \quad \pi_{110} \quad \pi_{011} \quad \pi_{111} \quad \pi_{012} \quad \pi_{112} \quad \pi_{013} \quad \pi_{113}),$$

$$\vec{\pi}_2 = (\pi_{020} \quad \pi_{120} \quad \pi_{021} \quad \pi_{121} \quad \pi_{022} \quad \pi_{122} \quad \pi_{023} \quad \pi_{123}),$$

also satisfies

$$\vec{\pi} \, . \, \vec{1} = 1;$$

$$\sum_{i=0}^{2} \vec{\pi}_i \, . \, \vec{1} = 1; \qquad \text{(where } \vec{1} \text{ is a } 8 \times 1 \text{ column vector of ones)}$$

$$\sum_{i=0}^{2} \sum_{j=0}^{3} (\pi_{0ij} + \pi_{1ij}) = 1. \tag{3.9}$$

And

$$\mathbf{Q} = \begin{pmatrix} L_0 & F & 0 \\ B & L & F \\ 0 & B & L \end{pmatrix}$$

where

$$L_0 = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{V_0} & \mathbf{A_0} & \lambda p_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{S} & \mathbf{T} & \lambda & 0 & 0 & 0 & 0 \\ 0 & p\mu & \mathbf{V} & \mathbf{A_1} & \lambda p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{S} & \mathbf{T} & \lambda & 0 & 0 \\ 0 & 0 & 0 & p\mu & \mathbf{V} & \mathbf{A_1} & \lambda p_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{S} & \mathbf{T} & \lambda \\ 0 & 0 & 0 & 0 & 0 & p\mu & \mathbf{V} & \mathbf{A_2} \end{pmatrix},$$

$$L = \begin{pmatrix} -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{V_0} & \mathbf{A_0} & \lambda p_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & p\mu & \mathbf{V} & \mathbf{A_1} & \lambda p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & p\mu & \mathbf{V} & \mathbf{A_1} & \lambda p_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & p\mu & \mathbf{V} & \mathbf{A_2} \end{pmatrix},$$

with $\mathbf{S} = \alpha + j\theta$, $\mathbf{T} = -\lambda - \alpha - j\theta$, $\mathbf{V} = \mu(1-p_1) + q\mu$, $\mathbf{V_0} = \mu(1-p_1)$, $\mathbf{A_0} = -[\mu(1-p_1) + \lambda + \lambda p_1]$, $\mathbf{A_1} = -[\mu(1-p_1) + \mu + \lambda + \lambda p_1]$ and $\mathbf{A_2} = -[\mu(1-p_1) + \mu + \lambda]$.

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Then, based on the matrix analytic method proposed, we briefly provide some numerical examples in some cases that examine the sensitivity and the impact of the system parameters: customers' arrival rate λ, service rate μ, retrial rate θ, α, p and p_1 on the limiting distribution $\vec{\pi} = (\vec{\pi}_0, \vec{\pi}_1, \vec{\pi}_2)$. The values of all the parameters were chosen, so that they satisfy the stability condition $\rho < 1$, Sumitha & Udaya Chandrika (2012) and the normalizing condition $\sum_{i=0}^{2}\sum_{j=0}^{3}(\pi_{0ij}+\pi_{1ij})=1$.

After the above conditions have been verified, we study the behaviour of the following performance measures according to the retrial rate θ, the orbital search rate p and the traffic intensity ρ:

- The mean number of customers in the system: $\bar{n}=\sum_{j=0}^{3}\sum_{i=0}^{2}[(i+j)\pi_{0ij}+(i+j+1)\pi_{1ij}]$;
- The mean number of customers in the queue: $\bar{n}_q=\sum_{i=0}^{2} i\sum_{j=0}^{3}(\pi_{0ij}+\pi_{1ij})$;
- The mean number of customers in the orbit: $\bar{n}_o=\sum_{j=0}^{3} j\sum_{i=0}^{2}(\pi_{0ij}+\pi_{1ij})$.

For different values of (p_1,λ) $((0.25,0.18),(0.50,1.2),(0.75,0.04285714))$ and for a fixed value of $\rho=0.3$, $\mu=1$, $p=0.4$ and $\theta=0.1$, the Table 3.1 presents the values of $\vec{\pi}$ in case of the linear retrial policy, for $\alpha=0.05$, Table 3.3 has the values of $\vec{\pi}$ in case of the classical retrial policy and Table 3.5 has the values of $\vec{\pi}$ in case of the constant retrial policy for $\alpha=0.05$.

In a similar way, for another different values of (p_1,λ) $((0.25,0.3),(0.50,0.1666667),(0.75,0.07142857))$ and for a fixed value of $\rho=0.5$, $\mu=1$, $p=0.6$ and $\theta=0.2$, the Tables 3.2, 3.4 and 3.6 present the values of $\vec{\pi}$, respectively, in case of: the linear retrial policy, the classical retrial policy and the constant retrial policy, with $\alpha=0.1$.

Table 3.1: Values of $\vec{\pi}$ for the linear retrial policy with $\alpha=0.05$.

The limiting distribution	$\rho=0.3,\mu=1,\epsilon=10^{-7},p=0.4\ ,\theta=0.1$		
	$p_1=0.25\ \lambda=0.18$	$p_1=0.50,\lambda=0.1$	$p_1=0.75,\lambda=0.04285714$
π_{000}	0.590148	0.645851	0.7008437
π_{100}	0.1390074	0.1250751	0.1150974
π_{001}	0.03858691	0.0402677	0.02506241
π_{101}	0.004798773	0.00346652	0.001334005
π_{002}	0.0008211582	0.0006755377	0.0001726865
π_{102}	0.000101595	5.737472×10^{-5}	9.051445×10^{-6}
π_{003}	1.285912×10^{-5}	8.154333×10^{-6}	8.443465×10^{-7}
π_{103}	1.661895×10^{-6}	7.279214×10^{-7}	4.796267×10^{-8}
π_{010}	0.1307079	0.1130031	0.1010489
π_{110}	0.03136989	0.02260062	0.01732267
π_{011}	0.01292814	0.01534931	0.01529023
π_{111}	0.0006780884	0.000368091	0.0001158773
π_{012}	0.0002684396	0.0002437705	0.000100796
π_{112}	1.3189×10^{-5}	5.429549×10^{-6}	7.002361×10^{-7}
π_{013}	4.960086×10^{-6}	3.492411×10^{-6}	6.025273×10^{-7}
π_{113}	2.217114×10^{-7}	7.069421×10^{-8}	3.900015×10^{-9}
π_{020}	0.02940598	0.0203566	0.01518462
π_{120}	0.007057436	0.00407132	0.002603077
π_{021}	0.002612059	0.002585267	0.002242
π_{121}	0.0001130267	4.996426×10^{-5}	1.460633×10^{-5}
π_{022}	4.262527×10^{5}	3.193033×10^{-5}	1.256121×10^{-5}
π_{122}	1.915812×10^{-6}	6.316543×10^{-7}	8.09982×10^{-8}
π_{023}	7.153594×10^{-7}	4.032019×10^{-7}	6.969096×10^{-8}
π_{123}	3.152084×10^{-8}	7.943157×10^{-9}	4.508726×10^{-10}
$\sum_{i=0}^{2}\sum_{j=0}^{3}(\pi_{0ij}+\pi_{1ij})$	0.988683	0.994072	0.996457

Table 3.2: Values of $\vec{\pi}$ for the linear retrial policy with $\alpha = 0.1$.

The limiting distribution	$\rho = 0.5, \mu = 1, \epsilon = 10^{-7}, p = 0.6, \theta = 0.2.$		
	$p_1 = 0.25\ \lambda = 0.3$	$p_1 = 0.50, \lambda = 0.1666667$	$p_1 = 0.75, \lambda = 0.07142857$
π_{000}	0.4222393	0.4985866	0.5696219
π_{100}	0.1636481	0.1575284	0.1516412
π_{001}	0.03797358	0.04210704	0.02741862
π_{101}	0.009139597	0.007247313	0.003169886
π_{002}	0.001345373	0.001201436	0.0003476965
π_{102}	0.0003398508	0.0002189052	4.441288×10^{-5}
π_{003}	3.755425×10^{-5}	2.678745×10^{-5}	3.546893×10^{-6}
π_{103}	1.049169×10^{-5}	5.526326×10^{-6}	5.490973×10^{-7}
π_{010}	0.1478462	0.1336726	0.123095
π_{110}	0.0591385	0.04455754	0.03516999
π_{011}	0.0242917	0.03031824	0.03128152
π_{111}	0.002480107	0.001488791	0.0005389042
π_{012}	0.0009658119	0.0009807389	0.0004728451
π_{112}	9.020479×10^{-5}	4.376732×10^{-5}	7.545818×10^{-6}
π_{013}	3.413895×10^{-5}	2.84758×10^{-5}	6.611407×10^{-6}
π_{113}	3.022892×10^{-6}	1.220767×10^{-6}	1.046202×10^{-7}
π_{020}	0.05316112	0.03762091	0.02847453
π_{120}	0.02126445	0.0125403	0.008135581
π_{021}	0.008247441	0.008227445	0.007140646
π_{121}	0.0007646947	0.0003624618	0.0001141715
π_{022}	0.0002930559	0.0002362586	9.996592×10^{-5}
π_{122}	2.657797×10^{-5}	1.019032×10^{-5}	1.575531×10^{-6}
π_{023}	1.016491×10^{-5}	6.681495×10^{-6}	1.386409×10^{-6}
π_{123}	9.183687×10^{-7}	2.93766×10^{-7}	2.250118×10^{-8}
$\sum_{i=0}^{2}\sum_{j=0}^{3}(\pi_{0ij} + \pi_{1ij})$	0.953352	0.977018	0.9867881

Table 3.3: Values of $\vec{\pi}$ for the classical retrial policy $(\alpha = 0)$.

The limiting distribution	$\rho = 0.3, \mu = 1, \epsilon = 10^{-7}, p = 0.4, \theta = 0.1$		
	$p_1 = 0.25\ \lambda = 0.18$	$p_1 = 0.5, \lambda = 1.2$	$p_1 = 0.75, \lambda = 0.04285714$
π_{000}	0.5754542	0.6264096	0.6833294
π_{100}	0.138109	0.1252819	0.1171422
π_{001}	0.0518391	0.05630338	0.03674832
π_{101}	0.00614818	0.004542346	0.001746442
π_{002}	0.001259196	0.001072448	0.000276671
π_{102}	0.000149501	0.00008601555	0.0000130069
π_{003}	0.00002151897	0.00001399097	1.39423×10^{-6}
π_{103}	2.667823×10^{-6}	1.17783×10^{-6}	7.052077×10^{-8}
π_{010}	0.1301741	0.1134501	0.1029526
π_{110}	0.03124178	0.02269002	0.01764901
π_{011}	0.01388877	0.01615636	0.01583333
π_{111}	0.000810443	0.000437396	0.0001309193
π_{012}	0.000339369	0.0003001244	0.0001150413
π_{112}	1.823443×10^{-5}	7.402401×10^{-6}	8.496686×10^{-7}
π_{013}	$7.086021 \times \times 10^{-6}$	4.854298×10^{-6}	7.331469×10^{-7}
π_{113}	3.369883×10^{-7}	1.04827×10^{-7}	4.835063×10^{-9}
π_{020}	0.02931635	0.02045363	0.01547427
π_{120}	0.007035924	0.004090726	0.002652733
π_{021}	0.002703665	0.002645083	0.00229322
π_{121}	0.0001259578	0.00005451999	0.0000153111
π_{022}	0.00004982266	0.00003570503	0.00001321699
π_{122}	2.444428×10^{-6}	7.677305×10^{-7}	8.741172×10^{-8}
π_{023}	9.454552×10^{-7}	4.994191×10^{-7}	7.53402×10^{-8}
π_{123}	4.457977×10^{-8}	1.050489×10^{-8}	4.931804×10^{-10}
$\sum_{i=0}^{2}\sum_{j=0}^{3}(\pi_{0ij} + \pi_{1ij})$	0.9886985	0.9940382	0.9963889

Table 3.4: Values of $\vec{\pi}$ for the classical retrial policy ($\alpha = 0$).

The limiting distribution	$\rho = 0.5$, $\mu = 1$, $\epsilon = 10^{-7}$, $p = 0.6$, $\theta = 0.2$		
	$p_1 = 0.25$ $\lambda = 0.3$	$p_1 = 0.50$, $\lambda = 0.1666667$	$p_1 = 0.75$, $\lambda = 0.07142857$
π_{000}	0.4066091	0.4748591	0.5445161
π_{100}	0.1626436	0.1582863	0.155576
π_{001}	0.05005936	0.05822721	0.0401744
π_{101}	0.01115775	0.009066058	0.003953922
π_{002}	0.00193731	0.001802785	0.0005279602
π_{102}	0.0004515529	0.0002956372	0.00005704345
π_{003}	0.00005594639	0.00004099495	5.210992×10^{-6}
π_{103}	0.00001433503	7.547805×10^{-6}	6.814002×10^{-7}
π_{010}	0.1476159	0.1349189	0.1265567
π_{110}	0.05904637	0.04497297	0.03615906
π_{011}	0.0254885	0.03157541	0.03250287
π_{111}	0.002798325	0.001683134	0.0005916009
π_{012}	0.001120627	0.001126652	0.0005206301
π_{112}	0.0001098381	0.00005279341	8.453775×10^{-6}
π_{013}	0.00004200964	0.00003448311	7.394912×10^{-6}
π_{113}	3.795683×10^{-6}	1.497482×10^{-6}	1.15888×10^{-7}
π_{020}	0.05318374	0.03803282	0.02928964
π_{120}	0.02127349	0.01267761	0.00836847
π_{021}	0.008444061	0.008416814	0.007363395
π_{121}	0.0008153968	0.0003848167	0.0001194553
π_{022}	0.0003188773	0.0002533044	0.0001047189
π_{122}	0.00003000734	0.00001127704	1.662357×10^{-6}
π_{023}	0.00001160231	7.427171×10^{-6}	1.462561×10^{-6}
π_{123}	1.069688×10^{-6}	3.312314×10^{-7}	2.371352×10^{-8}
$\sum_{i=0}^{2}\sum_{j=0}^{3}(\pi_{0ij} + \pi_{1ij})$	0.9532326	0.9767359	0.986407

Table 3.5: Values of $\vec{\pi}$ for the constant retrial policy ($\theta = 0$).

The limiting distribution	$\rho = 0.3$, $\mu = 1$, $\epsilon = 10^{-7}$, $p = 0.4$, $\alpha = 0.05$		
	$p_1 = 0.25$ $\lambda = 0.18$	$p_1 = 0.50$, $\lambda = 0.1$	$p_1 = 0.75$, $\lambda = 0.04285714$
π_{000}	0.5540986	0.5988841	0.6620737
π_{100}	0.1329836	0.1197768	0.1134984
π_{001}	0.07636424	0.08839677	0.0634247
π_{101}	0.008577415	0.006609705	0.002636794
π_{002}	0.004504865	0.004522928	0.001381255
π_{102}	0.0004815817	0.0003163217	5.118274×10^{-5}
π_{003}	0.0002493836	0.0002137289	2.644291×10^{-5}
π_{103}	2.643485×10^{-5}	1.476659×10^{-5}	9.532396×10^{-7}
π_{010}	0.1259563	0.109028	0.1000165
π_{110}	0.03022951	0.0218056	0.01714569
π_{011}	0.01542753	0.0171366	0.01600702
π_{111}	0.001049354	0.0005667095	0.00015871
π_{012}	0.0006511389	0.0005304503	0.0001613974
π_{112}	5.184005×10^{-5}	2.246323×10^{-5}	2.136046×10^{-6}
π_{013}	3.405675×10^{-5}	2.253655×10^{-5}	2.313148×10^{-6}
π_{113}	2.812898×10^{-6}	1.027721×10^{-6}	3.585448×10^{-8}
π_{020}	0.02842768	0.01969275	0.01504186
π_{120}	0.006822644	0.00393855	0.002578605
π_{021}	0.002821136	0.002651154	0.002250057
π_{121}	0.00014873	6.198894×10^{-5}	1.593798×10^{-5}
π_{022}	8.096822×10^{-5}	5.021091×10^{-5}	1.452213×10^{-5}
π_{122}	5.838095×10^{-6}	1.746949×10^{-6}	1.295113×10^{-7}
π_{023}	3.725683×10^{-6}	1.676963×10^{-6}	1.283736×10^{-7}
π_{123}	3.021545×10^{-7}	7.304443×10^{-8}	1.575133×10^{-9}
$\sum_{i=0}^{2}\sum_{j=0}^{3}(\pi_{0ij} + \pi_{1ij})$	0.9889997	0.9942467	0.9964885

Table 3.6: Values of $\vec{\pi}$ for the constant retrial policy ($\theta = 0$).

The limiting distribution	$\rho = 0.5, \mu = 1, \epsilon = 10^{-7}, p = 0.4 , \alpha = 0.1$		
	$p_1 = 0.25$ $\lambda = 0.3$	$p_1 = 0.50, \lambda = 0.1666667$	$p_1 = 0.75, \lambda = 0.07142857$
π_{000}	0.3906255	0.4520127	0.5256363
π_{100}	0.1562502	0.1506709	0.1501818
π_{001}	0.07058136	0.08794192	0.06748801
π_{101}	0.01435981	0.01210586	0.005421417
π_{002}	0.005877452	0.00653187	0.002294034
π_{102}	0.001107822	0.000814456	0.0001582
π_{003}	0.0004408081	0.00042967	6.565423×10^{-5}
π_{103}	8.107532×10^{-5}	5.189704×10^{-5}	4.276941×10^{-6}
π_{010}	0.1430658	0.1296599	0.1227949
π_{110}	0.05722632	0.04321998	0.03508427
π_{011}	0.02695917	0.03231844	0.03233373
π_{111}	0.003300677	0.001983047	0.0006615969
π_{012}	0.001660531	0.001575287	0.0006234733
π_{112}	0.0002179206	0.0001081043	1.398633×10^{-5}
π_{013}	0.0001132066	8.930036×10^{-5}	1.346917×10^{-5}
π_{113}	1.531995×10^{-5}	6.527447×10^{-6}	3.274094×10^{-7}
π_{020}	0.05174727	0.03668014	0.02845325
π_{120}	0.02069891	0.01222671	0.0081295
π_{021}	0.008584162	0.008326975	0.007196849
π_{121}	0.0008894178	0.0004099293	0.0001208488
π_{022}	0.0004051442	0.0002971419	0.0001082873
π_{122}	4.768429×10^{-5}	1.706986×10^{-5}	1.939634×10^{-6}
π_{023}	2.361115×10^{-5}	1.337758×10^{-5}	1.780166×10^{-6}
π_{123}	3.049585×10^{-6}	8.967875×10^{-7}	3.57627×10^{-8}
$\sum_{i=0}^{2} \sum_{j=0}^{3} (\pi_{0ij} + \pi_{1ij})$	0.9542822	0.9774921	0.986788

On the other hand, it is worthwhile to note that the matrix analytic method proposed in this paper works and is numerically stable one. Moreover, it can be applied on models which satisfy all the previously mentioned conditions.

The impact of the retrial rate θ.

Table 3.7: Performance measures for $\rho = 0.3, \mu = 1, p_1 = 0.25, \alpha = 0.05$ and $p = 0.4$.

θ	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.06227618	1.586191	1.83161
0,5	0.03062629	1.575898	1.794099
1	0.02344094	1.573528	1.785564
5	0.01666917	1.571285	1.777516
10	0.01574638	1.570979	1.776419
50	0.0149947	1.57073	1.775525
100	0.01489988	1.570698	1.775412

Table 3.8: Performance measures for $\rho = 0.3, \mu = 1, p_1 = 0.50, \alpha = 0.05$ and $p = 0.4$.

θ	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.06415477	1.6579	1.877751
0,5	0.03302917	1.645738	1.839925
1	0.02636358	1.643106	1.83181
5	0.02021255	1.640671	1.824317
10	0.01938403	1.640342	1.823308
50	0.01871086	1.640075	1.822487
100	0.01862606	1.640042	1.822384

Table 3.9: Performance measures for $\rho = 0.3, \mu = 1, p_1 = 0.75, \alpha = 0.05$ and $p = 0.4$.

θ	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.04465559	1.705097	1.886251
0,5	0.02655224	1.694198	1.86227
1	0.02291502	1.692	1.857448
5	0.01963069	1.690014	1.853092
10	0.01919352	1.689749	1.852512
50	0.01883922	1.689534	1.852042
100	0.01879464	1.689507	1.851983

The influence of the retrial rate θ is illustrated in Figure 3.3, from the numerical results listed in Tables 3.7, 3.8 and 3.9. We plot the performance measures by taking $p = 0.4, \rho = 0.3, \mu = 1, \alpha = 0.05$, for the values of $p_1 = 0.25, 0.50$ and 0.75.

We observe that $\overline{n}_q$, $\overline{n}_o$ and $\overline{n}$ decrease when θ increases, with α fixed for several choices of the probability of service interruption and joining the orbit p_1.

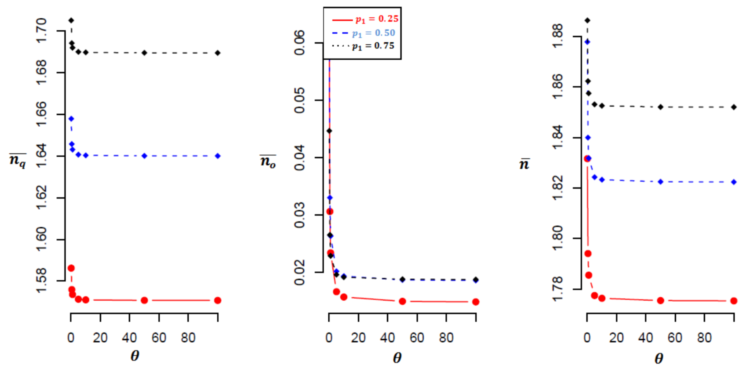


Figure 3.3: $\overline{n}_0$, $\overline{n}_q$ and $\overline{n}$ by varying the θ.

The impact of the orbital search rate p.

Table 3.10: Performance measures for $\rho = 0.3, \mu = 1, p_1 = 0.25, \alpha = 0.05$ and $\theta = 0.1$.

p	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.07616402	1.590922	1.848426
0, 2	0.07092634	1.589142	1.84209
0.3	0.06633377	1.587577	1.836528
0.4	0.06227618	1.586191	1.83161
0.5	0.058667	1.584954	1.827232
0.6	0.05543715	1.583844	1.82331
0.7	0.05253092	1.582843	1.819778
0.8	0.04990288	1.581936	1.816582
0.9	0.04751564	1.58111	1.813676

Table 3.11: Performance measures for $\rho = 0.3, \mu = 1, p_1 = 0.50, \alpha = 0.05$ and $\theta = 0.1$.

p	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.07341219	1.661547	1.88917
0, 2	0.07006187	1.660228	1.88504
0.3	0.06698661	1.659017	1.881246
0.4	0.06415477	1.6579	1.877751
0.5	0.06153932	1.656868	1.874521
0.6	0.05911702	1.655911	1.871528
0.7	0.05686783	1.655022	1.868748
0.8	0.05477432	1.654193	1.866158
0.9	0.0528213	1.653419	1.863741

Table 3.12: Performance measures for $\rho = 0.3, \mu = 1, p_1 = 0.75, \alpha = 0.05$ and $\theta = 0.1$.

p	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.04835619	1.707146	1.891142
0, 2	0.04706388	1.70643	1.889434
0.3	0.04583162	1.705748	1.887805
0.4	0.04465559	1.705097	1.886251
0.5	0.04353225	1.704477	1.884766
0.6	0.04245836	1.703885	1.883346
0.7	0.04143092	1.703319	1.881988
0.8	0.04044717	1.702777	1.880687
0.9	0.03950454	1.702259	1.879441

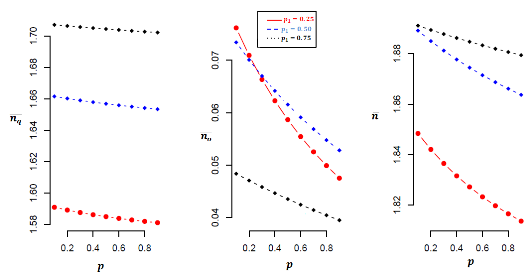


Figure 3.4: $\overline{n}_o$, $\overline{n}_q$ and $\overline{n}$ by varying the p.

The effect of the orbital search rate p is shown in Figure 3.4, from the numerical results listed in Tables 3.10, 3.11 and 3.12, where we have plotted the three performance measures, with respect to p, for $p_1 = 0.25, 0.50$ and 0.75.

We observe that for several choices of the probability of service interruption and joining the orbit p_1, $\overline{n}$, $\overline{n}_o$ and $\overline{n}_q$ always decrease.

The impact of the traffic intensity ρ.

Table 3.13: Performance measures for $p = 0.4, \mu = 1, p_1 = 0.25, \alpha = 0.05$ and $\theta = 0.1$.

ρ	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.008594285	1.84759	1.929467
0, 2	0.03080721	1.710121	1.874878
0.3	0.06227618	1.586191	1.83161
0.4	0.09979338	1.47379	1.795509
0.5	0.1410306	1.370917	1.763275
0.6	0.184278	1.275856	1.732519
0.7	0.2282345	1.187254	1.701635
0.8	0.271846	1.104098	1.669598
0.9	0.3141844	1.025631	1.635729

Table 3.14: Performance measures for $p = 0.4, \mu = 1, p_1 = 0.50, \alpha = 0.05$ and $\theta = 0.1$.

ρ	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.00873337	1.872918	1.943186
0, 2	0.03157344	1.758953	1.903693
0.3	0.06415477	1.6579	1.877751
0.4	0.1031173	1.568676	1.861936
0.5	0.1460473	1.489864	1.853523
0.6	0.1913093	1.420023	1.850527
0.7	0.2378539	1.357835	1.851594
0.8	0.2850423	1.302153	1.855845
0.9	0.3325048	1.252001	1.862714

Table 3.15: Performance measures for $p = 0.4, \mu = 1, p_1 = 0.75, \alpha = 0.05$ and $\theta = 0.1$.

ρ	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.005983884	1.890856	1.949946
0, 2	0.02182542	1.792536	1.912772
0.3	0.04465559	1.705097	1.886251
0.4	0.07212007	1.627936	1.868175
0.5	0.1024134	1.560115	1.856631
0.6	0.1342283	1.500577	1.850082
0.7	0.1666678	1.448272	1.847361
0.8	0.1991511	1.402225	1.847609
0.9	0.2313297	1.361565	1.85021

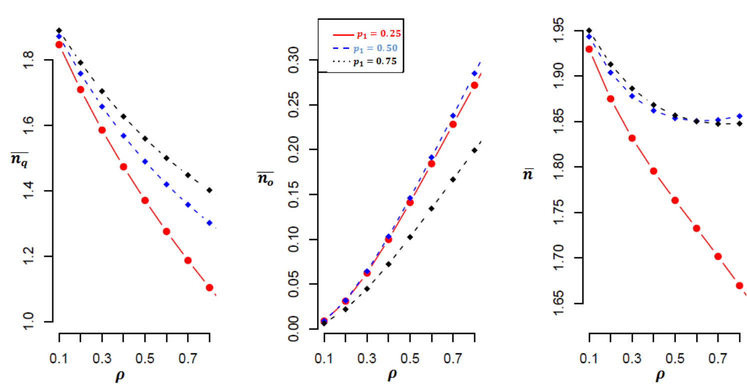


Figure 3.5: $\overline{n}_o$, $\overline{n}_q$ and $\overline{n}$ by varying the ρ.

In Figure 3.5, we choose the values $p_1 = 0.25, 0.50$ and 0.75 to represent the performance measures $\overline{n}_q$, $\overline{n}_o$ and $\overline{n}$, from the numerical results listed in Tables 3.13, 3.14 and 3.15, as functions of ρ.

We observe that for several choices of p_1, the probability of service interruption and joining the orbit, $\overline{n}_q$ and $\overline{n}$ respectively, have a decreasing shape with increasing values of ρ, but $\overline{n_o}$ is strictly an increasing function of ρ.

Chapter 4

Approximation Of Models With Retrials After Service Interruption Option, Through The Matrix-Analytic Method

Numerical analysis deals with the study of methods, techniques, or algorithms for obtaining approximations for solutions to mathematical problems, which has played a tremendous role in the evaluation and advancement of science and technology. Sometimes these methods involve the development of an algorithm for the solution of problems where an analytical solution does not exist. In this chapter, we were interested in the estimation of the steady-state distribution of some particular cases under consideration and their performance measures by using the matrix analytic method. Furthermore, we illustrated graphically the impact of Some parameters on the distribution and the performance measures of each case.

4.1 Numerical analysis for a $M/M/1$ queue with interruption service, retrials and orbital search according to a constant retrial policy

The model presented in this section can be seen as a particular case of the model already treated in chapter 3. We analyse a $M/M/1$ retrial queue according to the constant retrial policy with rate θ. Where we consider a single server and non-conventional retrial queueing system with a new form of access to the orbit, we assume that there is no waiting space but its orbit has an infinite capacity at which primary customers arrive according to a Poisson process with a rate $\lambda > 0$. The service times are independent and exponentially distributed with parameter μ, as it is shown in Figure 4.1.

We denote by N_o how many clients are in the orbit and let $C(t)$ be 0 or 1 according to the server

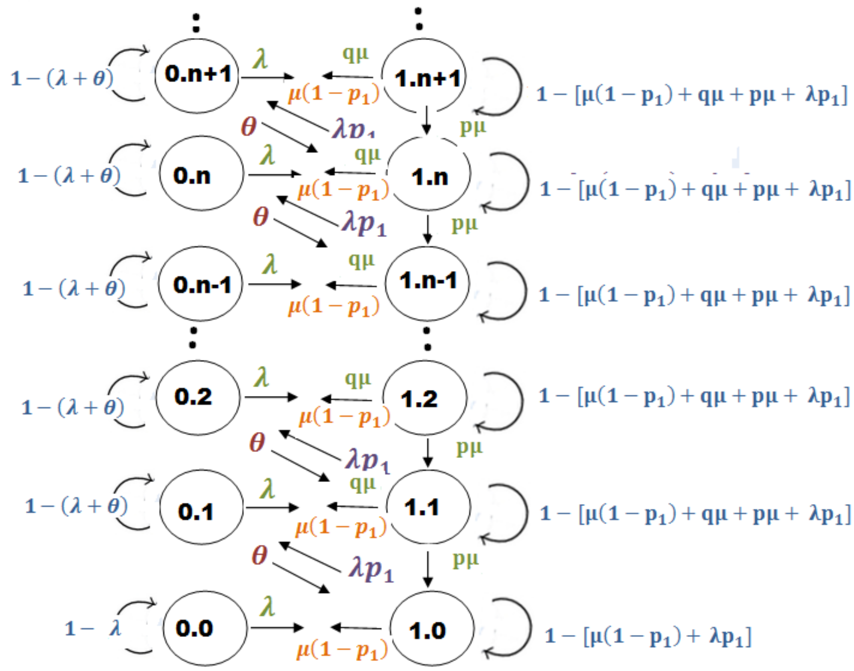


Figure 4.1: State Space And Transitions.

being idle or busy at a given instant t.

Let $N(t)$ denote how many clients are in the system at an instant t (i.e. in the orbit and service). Where $N(t) = N_o(t) + C(t)$.

So that continuous-time stochastic process $\chi = \{C(t), N_o(t); t \geq 0\}$, describes the state of the system with state space $\{c, n\}$, where $c \in \{0, 1\}$ and $n \in \mathbb{N}$.

Its infinitesimal transition rates $q_{(0,n)(c,m)}$ and $q_{(1,n)(c,m)}$ are given by

- For $n \geq 1$:

$$q_{(0,n)(c,m)} = \begin{cases} \lambda & \text{if } (c,m) = (1,n); \\ \theta & \text{if } (c,m) = (0,n-1); \\ -(\lambda+\theta) & \text{if } (c,m) = (0,n); \\ 0 & \text{otherwise.} \end{cases}$$

$$q_{(1,n)(c,m)} = \begin{cases} \mu(1-p_1) + q\mu & \text{if } (c,m) = (0,n); \\ p\mu & \text{if } (c,m) = (1,n-1); \\ \lambda p_1 & \text{if } (c,m) = (1,n+1); \\ -(\lambda p_1 + p\mu + q\mu + \mu(1-p_1)) & \text{if } (c,m) = (1,n); \\ 0 & \text{otherwise.} \end{cases}$$

and

$$q_{(0,0)(1,0)} = \lambda; \quad q_{(1,0)(0,0)} = \mu(1-p_1); \quad q_{(1,0)(0,1)} = \lambda p_1.$$

Where, from a state $(0, n)$ only transitions into the following states are possible:

- $(1, n)$ with rate λ, due to arrival of a primary customer;
- $(1, n-1)$ with rate θ, due to arrival of an orbital customer.

Reaching state $(0, n)$ is possible only from states:

- $(1, n)$ with rate $\mu(1-p_1)$ (for $n = 0$) due to a service completion in case that the customer leaves the system forever;
- $(1, n)$ with rate $\mu(1-p_1) + q\mu$, where $\mu(1-p_1)$ due to a service completion in case that the customer leaves the system forever and $q\mu$ if no orbital search is made on a service completion epoch;
- $(1, n-1)$ with rate λp_1 due to an interrupted service in case that the customer wants to take a break and joins to the orbit.

From a state $(1, n)$ only transitions into the following states are possible:

- $(0, n)$ wit rate $\mu(1-p_1)$ (for $n = 0$) due to a service completion in case that the customer leaves the system forever;
- $(0, n)$ with rate $\mu(1-p_1) + q\mu$, where $\mu(1-p_1)$ due to a service completion in case that the customer leaves the system forever and $q\mu$ if no orbital search is made on a service completion epoch;
- $(0, n+1)$ with rate λp_1 due to an interrupted service in case that the customer wants to take a break and joins to the orbit;
- $(1, n-1)$ with rate $p\mu$ if an orbital search is made on a service completion epoch.

Reaching state $(1, n)$ is possible from only from states:

- $(0, n)$ with rate λ, due to arrival of a primary customer;
- $(0, n+1)$ with rate θ, due to arrival of orbital customer;
- $(1, n+1)$ with rate $p\mu$, if an orbital search is made on a service completion epoch.

Then, we define the limiting probabilities that the system is in an idle or busy period respectively:

$$\pi_{0,n} = \lim_{t \to +\infty} P(C(t) = 0; N_o(t) = n), n \geq 0;$$
$$\pi_{1,n} = \lim_{t \to +\infty} P(C(t) = 1; N_o(t) = n), n \geq 0.$$

The set of statistical equilibrium equations for the probabilities $\{\pi_{0,n}, \pi_{1,n}; \forall n \geq 0\}$ have the following expressions

$$\lambda\pi_{0,0} = \mu(1-p_1)\pi_{1,0}; \tag{4.1}$$
$$[\lambda+\theta]\pi_{0,n} = \lambda p_1\pi_{1,n-1} + [\mu(1-p_1)+q\mu]\pi_{1,n}, \quad \forall n \geq 1; \tag{4.2}$$
$$\{\lambda p_1 + \mu(1-p_1)\}\pi_{1,0} = \lambda\pi_{0,0} + \theta\pi_{0,1} + p\mu\pi_{1,1}; \tag{4.3}$$
$$\{\lambda p_1 + \mu(1-p_1) + p\mu + q\mu\}\pi_{1,n} = \lambda\pi_{0,n} + \theta\pi_{0,n+1} + p\mu\pi_{1,n+1}, \quad \forall n \geq 1; \tag{4.4}$$

and the normalization equation

$$\sum_{n\geq 0}\pi_{0,n} + \sum_{n\geq 0}\pi_{1,n} = 1.$$

We present a numerical example to determine the steady state probabilities $\{(\pi_{0i}, \pi_{1i})$ for $0 \leq i \leq 7\}$ through the matrix-analytic method.

	π_{00}	π_{10}	π_{01}	π_{11}	π_{02}	π_{12}	π_{03}	π_{13}	π_{04}	π_{14}	π_{05}	π_{15}	π_{06}	π_{16}	π_{07}	π_{17}
π_{00}	$-\lambda$	$-\lambda$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
π_{10}	$\mu(1-p_1)$	$-\mu(1-p_1)-\lambda p_1$	λp_1	0	0	0	0	0	0	0	0	0	0	0	0	0
π_{01}	0	θ	$-\lambda-\theta$	λ	0	0	0	0	0	0	0	0	0	0	0	0
π_{11}	0	μp	$q\mu+\mu(1-p_1)$	$-\mu(1-p_1)-\lambda p_1-\mu$	λp_1	0	0	0	0	0	0	0	0	0	0	0
π_{02}	0	0	0	θ	$-\lambda-\theta$	λ	0	0	0	0	0	0	0	0	0	0
π_{12}	0	0	0	μp	$q\mu+\mu(1-p_1)$	$-\mu(1-p_1)-\lambda p_1-\mu$	λp_1	0	0	0	0	0	0	0	0	0
π_{03}	0	0	0	0	0	θ	$-\lambda-\theta$	λ	0	0	0	0	0	0	0	0
π_{13}	0	0	0	0	0	μp	$q\mu+\mu(1-p_1)$	$-\mu(1-p_1)-\lambda p_1-\mu$	λp_1	0	0	0	0	0	0	0
π_{04}	0	0	0	0	0	0	0	θ	$-\lambda-\theta$	λ	0	0	0	0	0	0
π_{14}	0	0	0	0	0	0	0	μp	$q\mu+\mu(1-p_1)$	$-\mu(1-p_1)-\lambda p_1-\mu$	λp_1	0	0	0	0	0
π_{05}	0	0	0	0	0	0	0	0	0	θ	$-\lambda-\theta$	λ	0	0	0	0
π_{15}	0	0	0	0	0	0	0	0	0	μp	$q\mu+\mu(1-p_1)$	$-\mu(1-p_1)-\lambda p_1-\mu$	λp_1	0	0	0
π_{06}	0	0	0	0	0	0	0	0	0	0	0	0	$-\lambda-\theta$	λ	0	0
π_{16}	0	0	0	0	0	0	0	0	0	0	0	μp	$q\mu+\mu(1-p_1)$	$-\mu(1-p_1)-\lambda p_1-\mu$	λp_1	0
π_{07}	0	0	0	0	0	0	0	0	0	0	0	0	0	θ	$-\lambda-\theta$	λ
π_{17}	0	0	0	0	0	0	0	0	0	0	0	0	0	μp	$q\mu+\mu(1-p_1)$	$-\mu(1-p_1)-\lambda p_1-\mu$

Figure 4.2: The generator matrix $\mathbf{Q}$ for $0 \leq i \leq 7$.

Therefore, note that

$$\vec{\pi}_0 = (\pi_{00} \quad \pi_{10}),$$
$$\vec{\pi}_1 = (\pi_{01} \quad \pi_{11}),$$
$$\vec{\pi}_2 = (\pi_{02} \quad \pi_{12}),$$
$$\vec{\pi}_3 = (\pi_{03} \quad \pi_{13}),$$
$$\vec{\pi}_4 = (\pi_{04} \quad \pi_{14}),$$
$$\vec{\pi}_5 = (\pi_{05} \quad \pi_{15}),$$
$$\vec{\pi}_6 = (\pi_{06} \quad \pi_{16}),$$
$$\vec{\pi}_7 = (\pi_{07} \quad \pi_{17}),$$

also satisfies

$$\vec{\pi}\ .\ \vec{1} = 1,$$

$$\sum_{i=0}^{7} \vec{\pi}_i\ .\ \vec{1} = 1, \qquad \text{(where } \vec{1} \text{ is a } 2\times 1 \text{ column vector of ones)}$$

$$\sum_{i=0}^{7} (\pi_{0i} + \pi_{1i}) = 1.$$

From Figure 4.2, we can express the infinitesimal generator $\mathbf{Q}$ of the process $\{C(t), N_o(t); t \geq 0\}$ in the following matrix block form:

$$\mathbf{Q} = \begin{pmatrix} L_0 & F & & & \\ B & L & F & & \\ & B & L & F & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where

$$L_0 = \begin{pmatrix} -\lambda & \lambda \\ \mu(1-p_1) & -[\mu(1-p_1) + \lambda p_1] \end{pmatrix}$$

$$L = \begin{pmatrix} -[\lambda+\theta] & \lambda \\ [q\mu + \mu(1-p_1)] & -[\mu(1-p_1) + \lambda p_1 + \mu] \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & 0 \\ \lambda p_1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & \theta \\ 0 & p\mu \end{pmatrix}.$$

Then, based on the matrix analytic method proposed, we briefly provide some numerical examples in some cases that examine the sensitivity and the impact of the system parameters: customers' arrival rate λ, service rate μ, retrial rate θ, orbital search rate p and p_1 on $\vec{\pi} = (\vec{\pi}_0, \vec{\pi}_1, \vec{\pi}_2, \vec{\pi}_3, \vec{\pi}_4, \vec{\pi}_5, \vec{\pi}_6, \vec{\pi}_7)$. The values of these parameters are chosen to satisfy the condition of stability $\rho < 1$.

As it is shown in the Tables 4.1 and 4.2, we can make the following observations:

Tables 4.1 and 4.2 lists values of $\vec{\pi}$ where the rates are all numbers.

The results in Tables 4.1 and 4.2 exhibit the predicted results, that is for each fixed value of p_1, p, λ, θ and μ. It's easy to notice that the normalizing condition is satisfied and closed to 1 for a small fixed error $\epsilon = 10^{-7}$ during all the simulations.

Also, some performance measures of the system were calculated like the mean number of clients in the orbit $\bar{n_o}$, and in the system $\bar{n}$.

Table 4.1: The limiting distribution $\rho = 0.2$, $\epsilon = 10^{-7}$, $p_1 = p = 0.25$ and $\theta = 0.05$.

$\vec{\pi}$	The limiting distribution	$\rho = 0.2$, $\epsilon = 10^{-7}$, $p_1 = p = 0.25$, $\theta = 0.05$		
		$\mu = 1, \lambda = 0.6$	$\mu = 2, \lambda = 1.2$	$\mu = 3, \lambda = 1.8$
$\vec{\pi}_0$	π_{00}	0.2278067	0.2213336	0.201507
	π_{10}	0.2277973	0.1770669	0.1612055
$\vec{\pi}_1$	π_{01}	0.2517757	0.2398968	0.2361846
	π_{11}	0.08632308	0.08225033	0.08097755
$\vec{\pi}_2$	π_{02}	0.09540963	0.1114358	0.1186414
	π_{12}	0.03271187	0.03820656	0.04067704
$\vec{\pi}_3$	π_{03}	0.03615519	0.05176367	0.05959652
	π_{13}	0.01239606	0.01774754	0.02043309
$\vec{\pi}_4$	π_{04}	0.0137009	0.02404503	0.02993681
	π_{14}	0.00469745	0.008244008	0.01026405
$\vec{\pi}_5$	π_{05}	0.005191914	0.01116929	0.01503801
	π_{15}	0.001780084	0.00382947	0.005155887
$\vec{\pi}_6$	π_{06}	0.00196746	0.005188309	0.007553964
	π_{16}	0.0006745575	0.001778848	0.00258993
$\vec{\pi}_7$	π_{07}	0.0007455629	0.00241005	0.003794544
	π_{17}	0.0002556215	0.0008263028	0.001300986
$\sum_{n=0}^{7}(\pi_{0n} + \pi_{1n}) =$		0.999389	0.9971926	0.9948569
$\bar{n}_o = \sum_{n=0}^{7} n \times (\pi_{0n} + \pi_{1n}) =$		0.8713093	1.098573	1.234193
$\bar{n} = \sum_{n=0}^{7} n \times \pi_{0n} + \sum_{n=0}^{7}(n+1) \times \pi_{1n} =$		1.237945	1.428523	1.556797

Table 4.2: The limiting distribution for $\rho = 0.6$, $\epsilon = 10^{-7}$, $p_1 = p = 0.5$ and $\theta = 0.05$.

$\vec{\pi}$	The limiting distribution	$\rho = 0.6$, $\epsilon = 10^{-7}$, $p_1 = p = 0.5$, $\theta = 0.05$		
		$\mu = 1$, $\lambda = 0.6$	$\mu = 2$, $\lambda = 1.2$	$\mu = 3$, $\lambda = 1.8$
$\vec{\pi}_0$	π_{00}	0.1645574	0.1428577	0.1348843
	π_{10}	0.1974687	0.1714291	0.161861
$\vec{\pi}_1$	π_{01}	0.2369622	0.2285718	0.2241149
	π_{11}	0.0947848	0.09142862	0.08964587
$\vec{\pi}_2$	π_{02}	0.1137416	0.1219046	0.1241249
	π_{12}	0.04549662	0.04876181	0.0496499
$\vec{\pi}_3$	π_{03}	0.05459588	0.06501565	0.06874591
	π_{13}	0.02183834	0.02600624	0.02749834
$\vec{\pi}_4$	π_{04}	0.02620598	0.03467493	0.03807456
	π_{14}	0.01048238	0.01386996	0.01522981
$\vec{\pi}_5$	π_{05}	0.01257885	0.01849325	0.0210874
	π_{15}	0.005031535	0.007397295	0.008434952
$\vec{\pi}_6$	π_{06}	0.006037835	0.009863044	0.01167915
	π_{16}	0.002415132	0.003945214	0.004671654
$\vec{\pi}_7$	π_{07}	0.002898156	0.005260278	0.006468434
	π_{17}	0.001159261	0.002104109	0.002587371
$\sum_{n=0}^{7}(\pi_{0n} + \pi_{1n}) =$		0.9962546	0.9915836	0.9887584
$\bar{n_o} = \sum_{n=0}^{7} n \times (\pi_{0n} + \pi_{1n}) =$		1.193451	1.392432	1.472368
$\bar{n} = \sum_{n=0}^{7} n \times \pi_{0n}$ $+ \sum_{n=0}^{7}(n+1) \times \pi_{1n} =$		1.572128	1.757374	1.831947

The impact of the traffic intensity ρ.

The effect of the ρ rate on the performance measures is shown in Figures 4.3, 4.4 and 4.5 from the values that are listed respectively on the Tables 4.3, 4.4 and 4.5.

Where we have plotted the two performance measures $\overline{n_o}$ and $\overline{n}$ by varying ρ. Firstly, for $\mu = 1, 2$ and 3, while we fix $\theta = 0.05, p_1 = 0.5$ and $p = 0.4$ as it is shown in Figure 4.3. Then, for $p = 0.3, 0.6$ and 0.9, while we fix $\theta = 0.05, \mu = 1$ and $p_1 = 0.25$ as it is shown shown in Figure 4.4. Finally, for $p_1 = 0.25, 0.50$ and 0.75, while we fix $\theta = 0.05, \mu = 1$ and $p = 0.4$ as it is shown in Figure 4.5.

Table 4.3: Performance measures for $\theta = 0.05, p = 0.4$ and $p_1 = 0.5$.

	$\mu = 1$		$\mu = 2$		$\mu = 3$	
ρ	$\overline{n_o}$	$\overline{n}$	$\overline{n_o}$	$\overline{n}$	$\overline{n_o}$	$\overline{n}$
0.1	0.01798412	0.07975569	0.009204181	0.0412671	0.006185808	0.02783613
0, 2	0.05948657	0.1727642	0.03113004	0.09237502	0.02109934	0.063067
0.3	0.1130221	0.2684275	0.06024504	0.1476975	0.04116091	0.1020075
0.4	0.1730014	0.3627735	0.09346982	0.204341	0.06431509	0.1426268
0.5	0.2369006	0.4548493	0.1290658	0.2608487	0.08933933	0.1837838
0.6	0.3037888	0.5450339	0.1660529	0.3165383	0.1154906	0.224838
0.7	0.3735843	0.6042774	0.2038927	0.3711508	0.1423129	0.2654407
0.8	0.4466815	0.7237702	0.2423096	0.4246601	0.1695279	0.3054169
0.9	0.5237638	0.8148063	0.281188	0.4771679	0.1969697	0.3446969

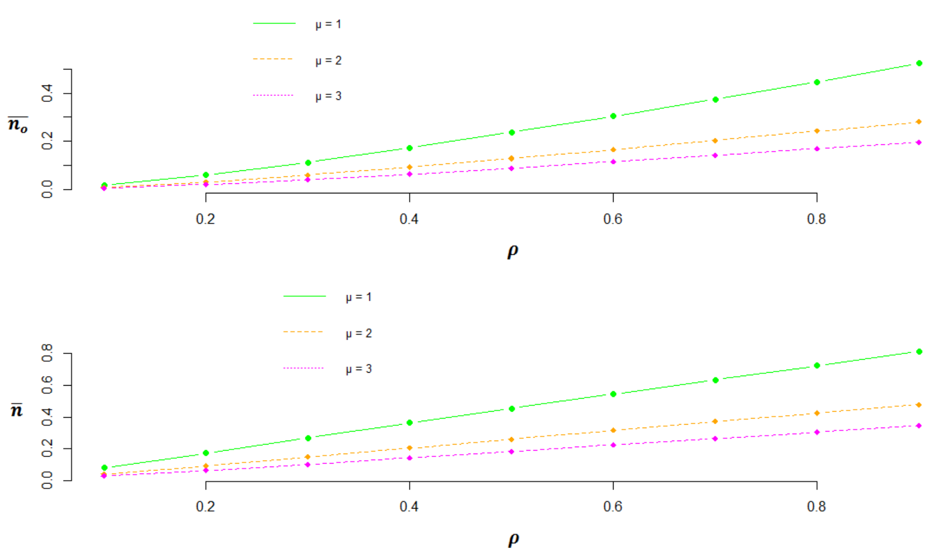


Figure 4.3: $\overline{n_o}$ and $\overline{n}$ by varying (ρ, μ). ($\theta = 0.05, p = 0.4$ and $p_1 = 0.5$)

Table 4.4: Performance measures for $\theta = 0.05, \mu = 1$ and $p_1 = 0.5$.

	$p = 0.3$		$p = 0.6$		$p = 0.9$	
ρ	$\overline{n}_o$	$\overline{n}$	$\overline{n}_o$	$\overline{n}$	$\overline{n}_o$	$\overline{n}$
0.1	0.01868548	0.08042882	0.01672834	0.07855047	0.01514233	0.07702838
0, 2	0.06359531	0.1765785	0.05268052	0.166448	0.04496505	0.1592911
0.3	0.1235294	0.2779412	0.09660106	0.2535778	0.07932692	0.2379808
0.4	0.1925391	0.3801666	0.1438671	0.3369133	0.1149034	0.3112945
0.5	0.2678567	0.4821424	0.1925896	0.4159938	0.1505352	0.3793488
0.6	0.3485816	0.5843894	0.242081	0.4912822	0.185862	0.4427889
0.7	0.434935	0.6882624	0.2922048	0.5635389	0.2208431	0.5023577
0.8	0.52786	0.7955802	0.3430896	0.6335865	0.2555697	0.558758
0.9	0.6288068	0.9084593	0.3949942	0.7022228	0.2901855	0.6126154

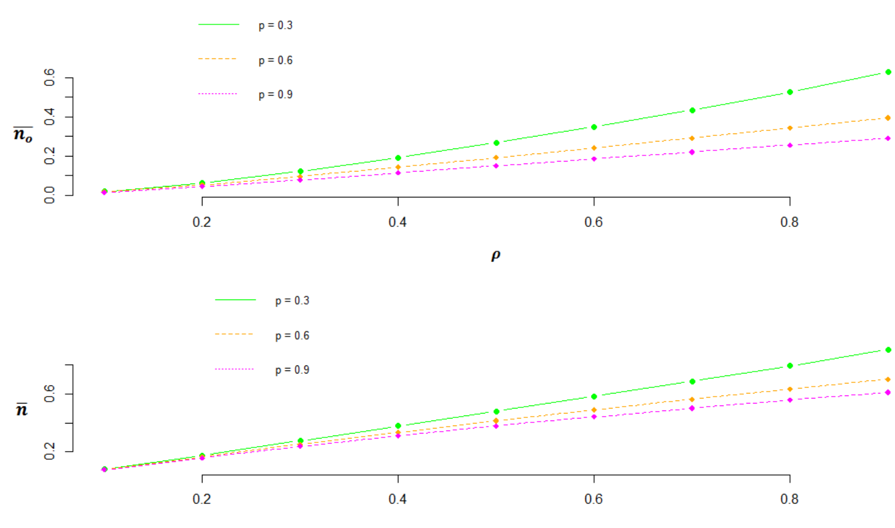


Figure 4.4: $\overline{n_o}$ and $\overline{n}$ by varying (ρ, p). ($\theta = 0.05, \mu = 1$ and $p_1 = 0.5$)

Table 4.5: Performance measures for $\theta = 0.05, \mu = 1$ and $p = 0.4$.

$p_1 = 0.25$			$p_1 = 0.50$		$p_1 = 0.75$	
ρ	$\overline{n}_o$	$\overline{n}$	$\overline{n}_o$	$\overline{n}$	$\overline{n}_o$	$\overline{n}$
0.1	0.01800423	0.09134743	0.01798412	0.07975569	0.01066757	0.06426655
0, 2	0.05692669	0.1907704	0.05948657	0.1727642	0.03722117	0.1368242
0.3	0.1052793	0.2888061	0.1130221	0.2684275	0.07336888	0.2115492
0.4	0.1586017	0.3832211	0.1730014	0.3627735	0.1149705	0.2851227
0.5	0.2152248	0.4741951	0.2369006	0.4548493	0.1594666	0.3559984
0.6	0.2746994	0.5627093	0.3037888	0.5450339	0.2053645	0.4236635
0.7	0.337157	0.649982	0.3735843	0.6342774	0.251858	0.4881692
0.8	0.4030375	0.7372791	0.4466815	0.7237702	0.2985692	0.549852
0.9	0.472965	0.825867	0.5237638	0.8148063	0.3453838	0.6091776

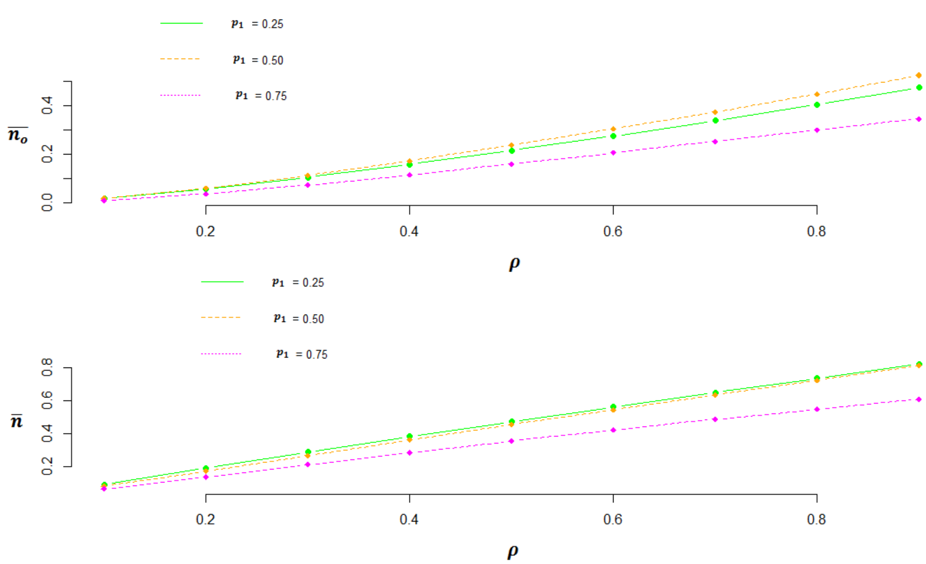


Figure 4.5: The effect of ρ and p_1 on $\overline{n_o}$ and $\overline{n}$ by varying (ρ, p_1). ($\theta = 0.05, \mu = 1$ and $p = 0.4$)

We observe that, for several choices of μ, p and p_1, the main number of customers in orbit $\overline{n_o}$ and the system $\overline{n}$ are strictly increasing function of ρ.

The impact of the constant retrial rate θ.

The influence of the constant retrial rate θ on $\overline{n_o}$ and $\overline{n}$ is illustrated in Figures 4.6, 4.7 and 4.8 respectively from Tables 4.6, 4.7 and 4.8.

We keep $\rho = 0.3$, then we fix $(p_1, p) = (0.5, 0.4)$ and we plot $\overline{n_o}$ and $\overline{n}$ First, for $\mu = 1, 2$ and 3 as it is shown in Figure 4.6. Then, we fix $(\mu, p_1) = (1, 0.25)$ and we plot $\overline{n_o}$ and $\overline{n}$ for $p = 0.3, 0.6$ and 0.9 as it is shown in Figure 4.7. Finally, we fix $(\mu, p) = (1, 0.4)$ for $p_1 = 0.25, 0.50$ and 0.75 as it is shown in Figure 4.8.

Table 4.6: Performance measures for $\rho = 0.3, p_1 = 0.5$ and $p = 0.4$.

	$\mu = 1$		$\mu = 2$		$\mu = 3$	
θ	$\overline{n_o}$	$\overline{n}$	$\overline{n_o}$	$\overline{n}$	$\overline{n_o}$	$\overline{n}$
0.1	0.04485799	0.1590084	0.02328744	0.08478656	0.01573813	0.05782496
0, 5	0.01154696	0.1282807	0.005974831	0.06821615	0.004032137	0.04646526
1	0.005987809	0.1231601	0.003096727	0.06545249	0.002089375	0.0445803
5	0.001234189	0.1187832	0.000638014	0.06311033	0.0004303909	0.04297075
10	0.0006194623	0.1182173	0.0003202132	0.06280632	0.0002160035	0.04276275
50	0.0001242739	0.1177615	0.00006423658	0.06256145	0.00004333097	0.04259523
100	0.00006215831	0.1177043	0.00003213048	0.06253074	0.00002167365	0.04257422

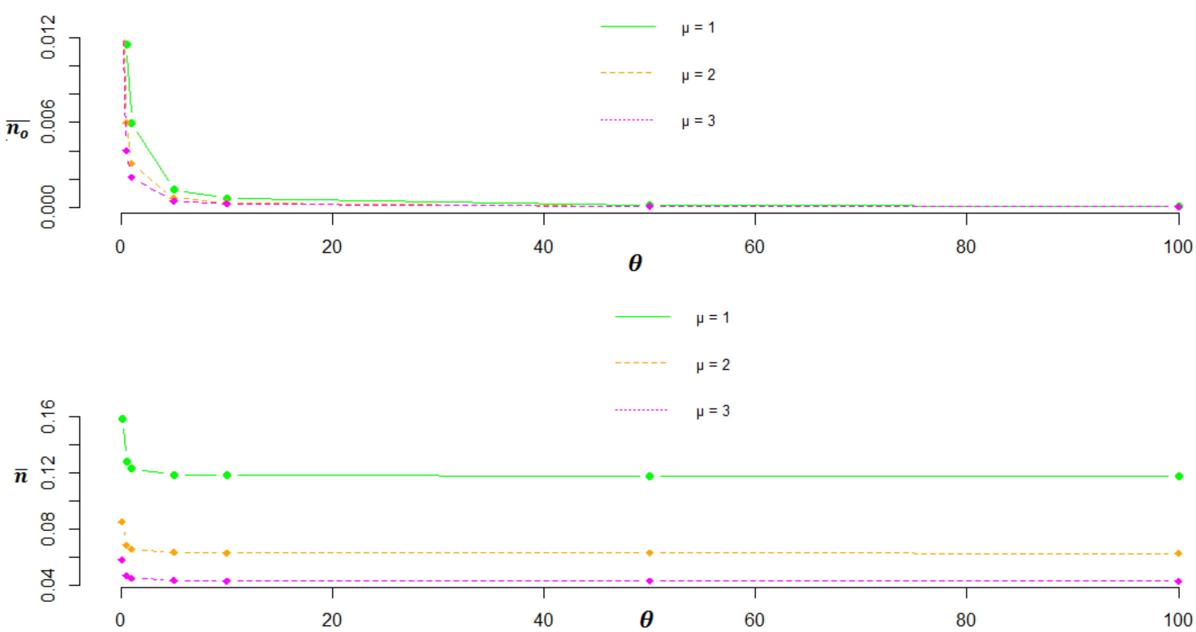


Figure 4.6: $\overline{n_o}$ and $\overline{n}$ by varying (θ, μ). ($\rho = 0.3, p_1 = 0.5$ and $p = 0.4$)

Table 4.7: Performance measures for $\rho = 0.3, \mu = 1$ and $p_1 = 0.25$.

	$p = 0.3$		$p = 0.6$		$p = 0.9$	
θ	$\overline{n}_o$	$\overline{n}$	$\overline{n}_o$	$\overline{n}$	$\overline{n}_o$	$\overline{n}$
0.1	0.07330814	0.2597649	0.05932261	0.2470922	0.04982115	0.2384934
0,5	0.01808398	0.2098346	0.01709102	0.2089396	0.01620144	0.2081378
1	0.009315055	0.2019335	0.009044405	0.2016897	0.008789038	0.2014598
5	0.001909188	0.1952663	0.00189755	0.1952558	0.001886054	0.1952455
10	0.000957633	0.19441	0.0009546268	0.1944073	0.0009517083	0.1944047
50	0.0001919905	0.1937211	0.0001918721	0.193721	0.000191754	0.1937209
100	0.00009602519	0.1936348	0.00009599558	0.1936348	0.00009596598	0.1936347

Table 4.8: Performance measures for $\rho = 0.3, \mu = 1$ and $p = 0.4$.

	$p_1 = 0.25$		$p_1 = 0.50$		$p_1 = 0.75$	
θ	$\overline{n}_o$	$\overline{n}$	$\overline{n}_o$	$\overline{n}$	$\overline{n}_o$	$\overline{n}$
0.1	0.0679662	0.2549222	0.06904383	0.2287077	0.04181842	0.1834722
0.5	0.01774042	0.2095249	0.01680783	0.1817348	0.009421796	0.1546991
1	0.009223056	0.2018506	0.008639458	0.1744091	0.004786736	0.150587
5	0.001905293	0.1952628	0.001767561	0.1682502	0.0009698446	0.1472016
10	0.0009565825	0.1944091	0.0008863264	0.1674607	0.0004857151	0.1467722
50	0.000191951	0.1937211	0.0001776747	0.1668258	0.00009726821	0.1464277
100	0.00009601532	0.1936348	0.00008886303	0.1667463	0.00004864208	0.1463846

We observe that the three curves of $\overline{n_o}$ seem to be undistinguished. However, the tail of the performance measures decrease and become heavier as far as θ increases.

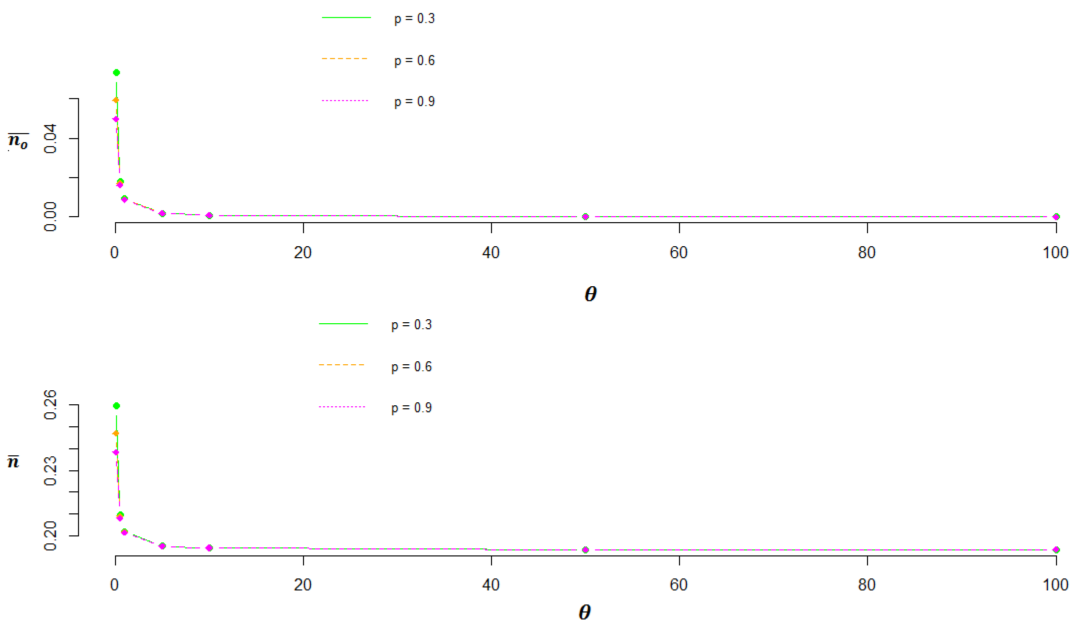


Figure 4.7: $\overline{n_o}$ and $\overline{n}$ by varying (θ, p). ($\rho = 0.3, \mu = 1$ and $p_1 = 0.25$)

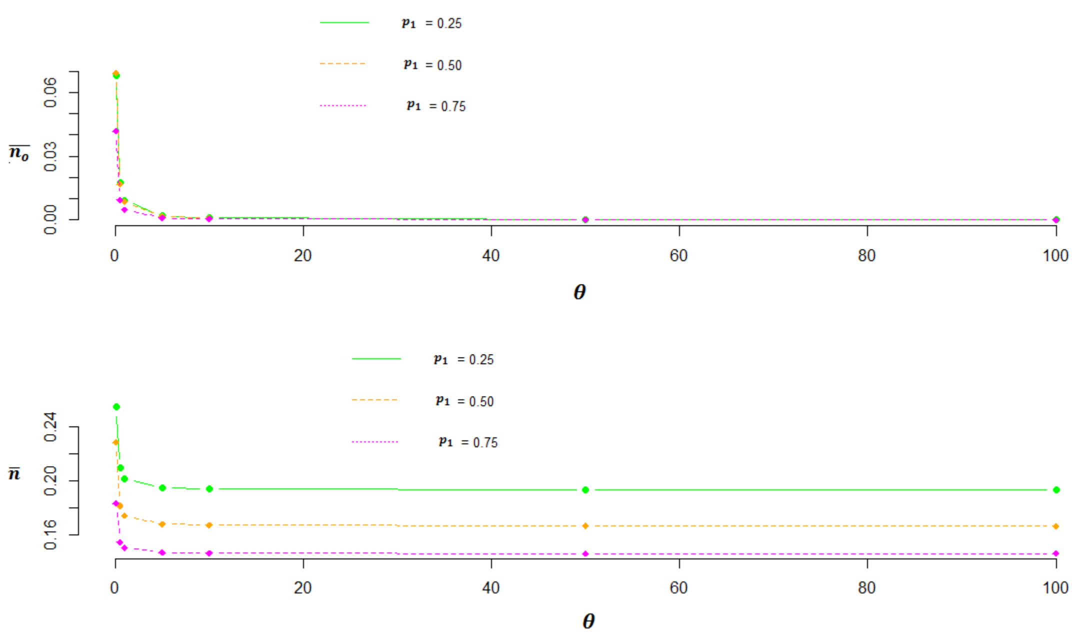


Figure 4.8: The effect of θ and p_1 on $\overline{n_o}$ and $\overline{n}$, for $\rho = 0.3, \mu = 1$ and $p = 0.4$.

4.2 Numerical analysis for a $M/M/1$ queue with interruption service and retrials according to a linear retrial policy

In this section, the present model includes a generalisation of the model treated in chapter 2, for a retrial queuing system with a single server and infinite capacity of the orbit and the queue, but according to a linear retrial policy $\alpha(1-\delta_{0j})+j\theta$, instead of a constant retrial policy, as it is shown in Figure 4.9.

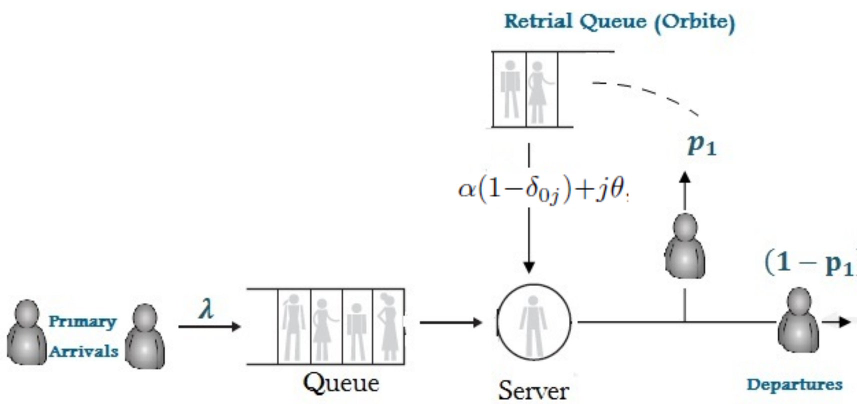


Figure 4.9: A typical queuing system.

The system state at time t can be described by the process $\chi(t) = \{C(t), N_q(t), N_o(t); t \geq 0\}$, where $C(t)$ is the state of the server and $N_o(t)$ denotes how many clients are in orbit and $N_q(t)$ denotes how many clients are in the queue at time t.

Under the above assumptions the process $\chi(t) = \{C(t), N_q(t), N_o(t); t \geq 0\}$ is Markovian with the lattice semi-strip $S = \{O, 1\} \times \mathbb{N} \times \mathbb{N}$ as the state space.

The set of statistical equilibrium equations for the probabilities $\pi_{c,i,j}$ ($c \in \{0,1\}$, $i \geq 0$ and $j \geq 0$), have the following expressions:

$$\lambda\pi_{0,0,0} = \mu(1-p_1)\pi_{1,0,0};$$

$$\lambda\pi_{0,i,0} = \mu(1-p_1)\pi_{1,i,0}, \forall i \geq 1; \tag{4.5}$$

$$\{\lambda + \lambda p_1 + \mu(1-p_1)\}\pi_{1,i,0} = \lambda\pi_{1,i-1,0} + \lambda\pi_{0,i+1,0}, \forall i \geq 1; \tag{4.6}$$

$$[\lambda + \alpha(1-\delta_{0j}) + j\theta]\pi_{0,0,j} = \lambda p_1\pi_{1,0,j-1} + \mu(1-p_1)\pi_{1,0,j}, \forall j \geq 1; \tag{4.7}$$

$$\{\lambda + \lambda p_1 + \mu(1-p_1)\}\pi_{1,0,j} = \lambda\pi_{0,0,j} + \lambda\pi_{0,1,j} + [\alpha(1-\delta_{0j}) + j\theta]\pi_{0,0,j+1}, \forall j \geq 0; \tag{4.8}$$

$$\lambda\pi_{0,i,j} = \lambda p_1\pi_{1,i,j-1} + \mu(1-p_1)\pi_{1,i,j}, \forall i \geq 1, \forall j \geq 1; \tag{4.9}$$

$$\{\lambda + \lambda p_1 + \mu(1-p_1)\}\pi_{1,i,j} = \lambda\pi_{0,i+1,j} + \lambda\pi_{1,i-1,j}, \forall i \geq 1, \forall j \geq 1; \tag{4.10}$$

with the normalization equation $\sum_{i\geq 0}\sum_{j\geq 0}\pi_{0,i,j} + \sum_{i\geq 0}\sum_{j\geq 0}\pi_{1,i,j} = 1$ and the global traffic intensity given by $\rho = \rho_q + \rho_o = \frac{\lambda(1+p_1)}{\mu(1-p_1)}$.

To illustrate the method, it is useful to start by rewriting the equations in terms of a "generator matrix", $\mathbf{Q}$. By ordering the states as

$$\begin{aligned} S = &\{(0,0,0),(1,0,0),...,(0,0,j),(1,0,j),(0,1,0),(1,1,0),...,(0,1,j),(1,1,j),...,\\ &(0,i,0),(1,i,0),...,(0,i,j),(1,i,j)\}. \end{aligned}$$

We expressed the infinitesimal generator $\mathbf{Q}$ of the process $X(t) = \{C(t), N_q(t), N_o(t); t \geq 0\}$ in the following matrix block form:

$$\mathbf{Q} = \begin{pmatrix} L_0 & F & & & \\ B & L & F & & \\ & B & L & F & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

$$L_0 = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \mu(1-p_1) & C & \lambda p_1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & R & A & \lambda & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \mu(1-p_1) & C & \lambda p_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & R & A & \lambda & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & & \vdots \end{pmatrix}$$

$$L = \begin{pmatrix} -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \mu(1-p_1) & C & \lambda p_1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & A & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \mu(1-p_1) & C & \lambda p_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & A & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & & \vdots \end{pmatrix}$$

Where $A = -(\lambda + [\alpha(1-\delta_{0j}) + j\theta])$, $C = -[\lambda + \lambda p_1 + \mu(1-p_1)]$ and $R = \alpha(1-\delta_{0j}) + j\theta$.

$$B = \begin{pmatrix} 0 & \lambda & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & \dots & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \end{pmatrix}$$

Then, we concentrate on the computation of the limiting distribution $\vec{\pi} = \{\pi_{c,i,j}, 0 \leq c \leq 1, 0 \leq i \leq 2, 0 \leq j \leq 3\}$, by using the Matrix-analytic Method and fixing the values of all the rates as it is listed in the Table 4.9, where we take $\rho = 0.35, \mu = 1, \lambda = 0.21, \theta = 0.50, \alpha = 0.05, p_1 = 0.25$ and $\epsilon = 10^{-7}$.

Table 4.9: The limiting distribution of the system.

The limiting distribution	$\rho = 0.35$, $\mu = 1$, $\epsilon = 10^{-7}$, $\theta = 0.5$, $\alpha = 0.05$, $p_1 = 0.25$ $\lambda = 0.21$
π_{000}	0.3572064
π_{100}	0.1428824
π_{001}	0.1144135
π_{101}	0.02172484
π_{002}	0.01508984
π_{102}	0.002705245
π_{003}	0.00182629
π_{103}	0.0003231911
π_{010}	0.1305525
π_{110}	0.05222101
π_{011}	0.03334282
π_{111}	0.004508349
π_{012}	0.003336225
π_{112}	0.0004909195
π_{013}	0.0003820957
π_{113}	5.763684×10^{-5}
π_{020}	0.04694554
π_{120}	0.01877822
π_{021}	0.009558205
π_{121}	0.001080811
π_{022}	0.0006956546
π_{122}	9.454485×10^{-5}
π_{023}	7.070254×10^{-5}
π_{123}	1.045918×10^{-5}
$\sum_{i=0}^{2} \sum_{j=0}^{3} (\pi_{0ij} + \pi_{1ij})$	0.9582975

Next, we present some numerical examples to visualize the performance of $\bar{n}$ and the effect of some rates on $\bar{n}$, first by varying the pair (ρ, p_1), then by varying the pair (θ, p_1) and finally by varying the pair (α, p_1).

θ	α	μ	ρ	p_1	$\sum_{i=0}^{2}\sum_{j=0}^{3}\pi_{0ij}+\pi_{1ij}$	$\bar{n}$
0.1	0.05	1	0.30	0.25	0.9705365	0.9860876
				0.50	0.9828092	0.6865405
				0.75	0.9894097	0.8091561
			0.45	0.25	0.9306556	1.001155
				0.50	0.5853329	0.3737411
				0.75	0.9771188	0.6953235
			0.60	0.25	0.8739677	0.9823651
				0.50	0.8969764	0.3808211
				0.75	0.9597357	0.5428684
			0.75	0.25	0.80271	0.9305052
				0.50	0.9703247	0.8872514
				0.75	0.9355151	0.333578
			0.90	0.25	0.7196339	0.8469858
				0.50	0.8783594	0.5760105
				0.75	0.8997168	0.02688644

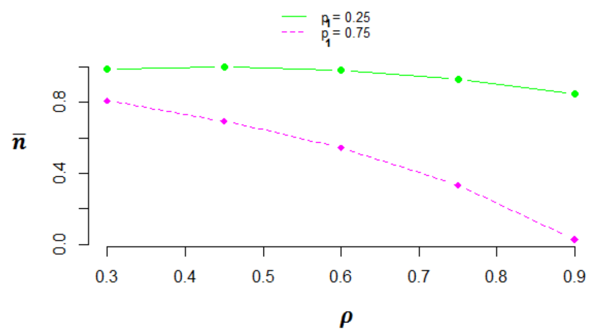


Figure 4.10: $\bar{n}$ by varying the (ρ, p_1).

ρ	α	μ	θ	p_1	$\sum_{i=0}^{2}\sum_{j=0}^{3}\pi_{0ij}+\pi_{1ij}$	$\bar{n}$
0.45	0.05	1	0.5	0.25	0.9344017	1.091594
			0.6		0.9338431	1.00707
			0.75		0.9338367	0.9397666
			1		0.9342661	0.85822
			0.5	0.50	0.963817	0.656832
			0.6		0.9642875	0.5917391
			0.75		0.9650411	0.497881
			1		0.9663478	0.3458103
			0.1	0.75	0.9771188	0.6953235
			0.25		0.9779121	0.4488867
			0.50		0.9802112	0.1313738
			0.60		0.9810887	0.01772057

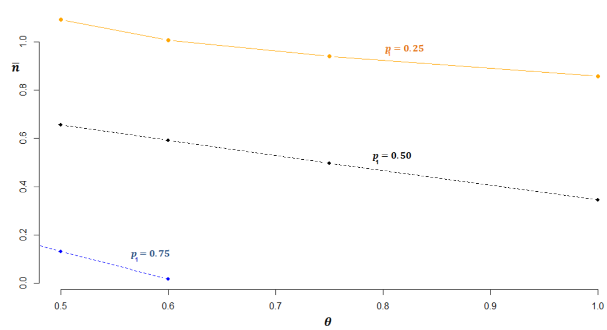


Figure 4.11: $\bar{n}$ by varying the (θ, p_1).

ρ	θ	μ	α	p_1	$\sum_{i=0}^{2}\sum_{j=0}^{3}\pi_{0ij}+\pi_{1ij}$	$\bar{n}$
0.45	0.50	1	5.00	0.25	0.9668304	1.807303
			10		0.9549928	1.47293
			50		0.948623	1.205207
			100		0.9455462	0.9920366
			5.00	0.50	0.9742141	1.340501
			10		0.9731097	1.238351
			50		0.9714967	1.0509
			100		0.9700448	0.8627342
			5.00	0.75	0.9833611	1.158131
			10		0.9832609	1.098081
			50		0.9828496	0.9864828
			100		0.9823433	0.8840413

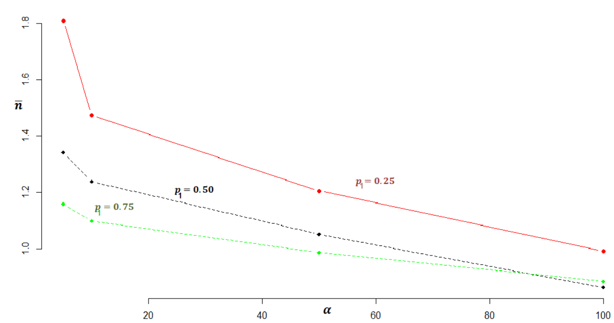


Figure 4.12: $\bar{n}$ by varying the (α, p_1).

From the above results that are listed on the last three tables and illustrated on the last three figures 4.10, 4.11 and 4.12, we can reveal the following observations:

1. $\bar{n}$ decreases as ρ increases, even when θ and α increases too.

2. $\bar{n}$ appears to be small for $p_1 = 0.75$ as compared to the rest values of p_1 when $p_1 = 0.50$

and $p_1 = 0.25$.

General conclusion

In this work, we have presented a detailed approximation of the stationary distribution for a single-server Markovian queueing model with several parameters, by using the matrix-analytic method.

The present investigation includes many features simultaneously such as (1) retrials according to retrial linear policy; (2) Interruption service; (3) Orbital search. We note that all these realistic assumptions have not been gathered together in the existing literature.

Our study has two main objectives. The first one is to link the corresponding retrial queues with interruption service under several retrial policies. Our analysis applies to the different retrial policies such as a constant retrial policy, classical retrial policy or linear retrial policy. Furthermore, includes the classical queue. That is why our model can be considered as a generalized version of many existing queuing models associated with many practical situations. The second objective is to introduce orbital search in retrial queueing models which allows minimizing the idle time of the server. If the holding costs and cost of using the search of clients are introduced, the obtained results can be used for the optimal tuning of the parameters of the search mechanism.

The analytical results have been obtained by using the Q-matrix (infinitesimal generating matrix) technique. We have obtained approximated values of the steady-state distribution and some performance measures of the model. Moreover, some numerical results are presented to demonstrate how the different parameters of the considered models influence the behaviour of the system. Some special cases are illustrated

The work carried out during this thesis and the results obtained open up a range of prospects. For our future work, we plan to direct our research in the following directions:

- This investigation can be further extended for systems with set-up times, server vacations (breakdowns) or by incorporating the batch arrival of primary clients.
- More broadly, we are optimistic that techniques similar to the matrix-analytic method could help analyze an extremely broad class of multiserver systems. These results are only treated for single-server systems. Nothing is known for the $M/M/k$ retrial models with interruption service and orbital search in the case of $k \geq 2$ servers.

Appendix
Some numerical analysis programs in R

In Chapter 2, for $\mu = 2$, $\epsilon = 10^{-7}$, $p_1 = 0.25$, $\rho = 0.2$, $\lambda = 1.2$, $\theta = 0.05$, we have:

```
l0 <- c(-1.2,1.5,1.2,-1.8 )
L0 <- matrix(l0,2,2)
f <- c(0,0.3,0,0)
F <- matrix(f,2,2)
b <- c(0,0,0.05,0.5 )
B <- matrix(b,2,2)
l <- c(-1.25,3,1.2,-3.8 )
L <-matrix(l,2,2)
INV <- solve(L)
R1 <- - F* INV
R2 <- -(F+R1 * R1* B)* INV
e1 <- R2-R1
R3<- -(F+R2 * R2* B)* INV
e2<- R3-R2
R4<- -(F+R3 * R3* B)* INV
e3<- R4-R3
```

```
R5<- -(F+R4 * R4 * B)* INV
e4<- R5-R4
R6<- -(F+R5 * R5 * B)* INV
e5<- R6-R5
R7<- -(F+R6 * R6 * B)* INV
e6<- R7-R6
R8<- -(F+R7 * R7* B)* INV
e7<- R8-R7
R9<- -(F+R8 * R8* B)*INV
e8<- R9-R8
R10<- -(F+R9 * R9* B)* INV
e9<- R10-R9
R11<- -(F+R10 * R10* B)* INV
e10<- R11-R10
R12<- -(F+R11 * R11* B)* INV
e11<- R12-R11
R13<- -(F+R12 * R12* B) INV
e12<- R13-R12
R14<- -(F+R13* R13* B)* INV
e13<- R14-R13
R15<- -(F+R14 * R14* B)* INV
e14<- R15-R14
R16<- -(F+R15 * R15* B)* INV
e15<- R16-R15
R17<- -(F+R16 * R16* B)* INV
e16<- R17-R16
R18<- -(F+R17 * R17* B)* INV
e17<- R18-R17
```

```
R19<- -(F+R18 * R18* B)* INV
e18<- R19-R18
R20<- -(F+R19 * R19* B)* INV
e19<- R20-R19
R21<- -(F+R20 * R20* B)* INV
e20<- R21-R20
R22<- -(F+R21 * R21* B)* INV
e21<- R22-R21
R23<- -(F+R22 * R22* B)* INV
e22<- R23-R22
R24<- -(F+R23 * R23* B)* INV
e23<- R24-R23
R25<- -(F+R24 * R24* B)* INV
e24<- R25-R24
R26<- -(F+R25 * R25* B)* INV
e25<- R25-R24
R27<- -(F+R26 * R26* B)* INV
e26<- R27-R26
R28<- -(F+R27 * R27* B)* INV
e27<- R28-R27
PHI<- L0+R28* B
d<- diag(1,2,2)
o<- c(1,1,1,1)
O<- matrix(o,2,1)
i<- d- R28
I<- solve(i)
psi<- I * O
a<- c(1,0,0,0)
```

```
A<- matrix(a,1,2)
Z<- c(1,4.397583,1.2,-1.5)
M<- matrix(Z,2,2)
T<- solve(M)
p0<- A * T
p1<- p0 * R28
p2<- p0 * R28 * R28
p3<- p0 * R28 * R28 * R28
p4<- p0 * R28 * R28* R28 * R28
p5<- p0 * R28 * R28* R28* R28* R28
p6<- p0 * R28 * R28* R28* R28* R28 * R28
p7<- p0 * R28 * R28* R28* R28* R28 * R28* R28
```

In Chapter 3, for $\mu = 1$, $\epsilon = 10^{-7}$, $p_1 = 0.25$, $p = 0.4$, $\rho = 0.3$, $\theta = 0.05$, we have:

```
rho<- 0.3
mu<- 1
p<- 0.4
theta<- 0.05
pf<- 0.25
a<- mu *(1-pf) + (1-p) * mu
lambda<- rho*(1-pf)/(1+pf)
b<- (-lambda - theta)
q<- (1-pf)
x<- p*mu
y<- lambda*pf
z<- -lambda
t<- mu * q
```

```
A<- (-t +z - y)

B<- (- t +z -y - mu)

C<- (- t +z - mu )

l0<- c(-lambda,mu *(1-pf),0,0,0,0,0,0, lambda,A,theta,p*mu,0,0,0,0, 0,lambda*pf,b,a,0,0,0,0, 0,0,lambda,B,theta,
p*mu,0,0, 0,0,0,lambda*pf,b,a,0,0, 0,0,0,0,lambda,B,theta,p*mu, 0,0,0,0,0,lambda*pf,b,a, 0,0,0,0,0,0,lambda,C)

L0<- matrix(l0,8,8)

f<- c(0,0,0,0,0,0,0,0, 0,lambda,0,0,0,0,0,0, 0,0,0,0,0,0,0,0, 0,0,0,lambda,0,0,0,0, 0,0,0,0,0,0,0,0, 0,0,0,0,0,lambda,0,0,
0,0,0,0,0,0,0,0, 0,0,0,0,0,0,0,lambda)

F<- matrix(f,8,8)

b<- c(0,0,0,0,0,0,0,0, lambda,0,0,0,0,0,0,0, 0,0,0,0,0,0,0,0, 0,0,lambda,0,0,0,0,0, 0,0,0,0,0,0,0,0, 0,0,0,0,lambda,0,0,0,
0,0,0,0,0,0,0,0, 0,0,0,0,0,0,lambda,0)

Bo<- matrix(b,8,8)

a<- mu *(1-pf) + (1-p) * mu

b<- (-lambda - theta)

q<- (1-pf)

x<- p*mu

y<- lambda*pf

z<- -lambda

t<- mu * q

A<- (-t +z - y)

B<- (- t +z -y - mu)

C<- (- t +z - mu )

l<- c(-lambda,mu*q,0,0,0,0,0,0, 0,A,0,p*mu,0,0,0,0, 0,lambda*pf,-lambda,a,0,0,0,0, 0,0,0,B,0,x,0,0, 0,0,0,y,z,a,0,0,
0,0,0,0,0,B,0,x, 0,0,0,0,0,lambda*pf,-lambda,a, 0,0,0,0,0,0,0,C )

L<- matrix(l,8,8)

INV<- solve(L)

R1<- - F* INV

R2<- -(F+R1 * R1* Bo)* INV

e1<- R2-R1

R3<- -(F+R2 * R2* Bo)* INV
```

```
e2<- R3-R2
R4<- -(F+R3 * R3* Bo)* INV
e3<- R4-R3
R5<- -(F+R4 * R4* Bo)* INV
e4<- R5-R4
R6<- -(F+R5 * R5* Bo)* INV
e5<- R6-R5
R7<- -(F+R6 * R6* Bo)* INV
e6<- R7-R6
R8<- -(F+R7 * R7* Bo)* INV
e7<- R8-R7
R9<- -(F+R8 * R8* Bo)* INV
e8<- R9-R8
R10<- -(F+R9 * R9* Bo)* INV
e9<- R10-R9
R11<- -(F+R10 * R10* Bo)* INV
e10<- R11-R10
R12<- -(F+R11 * R11* Bo)* INV
e11<- R12-R11
R13<- -(F+R12 * R12* Bo)* INV
e12<- R13-R12
R14<- -(F+R13 * R13* Bo)* INV
e13<- R14-R13
R15<- -(F+R14 * R14* Bo)*INV
e14<- R15-R14
R16<- -(F+R15 * R15* Bo)* INV
e15<- R16-R15
R17<- -(F+R16 * R16* Bo)* INV
```

```
e16<- R17-R16
R18<- -(F+R17 * R17* Bo)* INV
e17<- R18-R17
R19<- -(F+R18 * R18* Bo)* INV
e18<- R19-R18
R20<- -(F+R19 * R19* Bo)* INV
e19<- R20-R19
R21<- -(F+R20 * R20* Bo)* INV
e20<- R21-R20
R22<- -(F+R21 * R21* Bo)* INV
e21<- R22-R21
R23<- -(F+R22 * R22* Bo)* INV
e22<- R23-R22
R24<- -(F+R23 * R23* Bo)* INV
e23<- R24-R23
R25<- -(F+R24 * R24* Bo)* INV
e24<- R25-R24
R26<- -(F+R25 * R25* Bo)* INV
e25<- R26-R25
R27<- -(F+R26 * R26* Bo)* INV
e26<- R27-R26
R27<- -(F+R26 * R26* Bo)* INV
e26<- R27-R26
R28<- -(F+R27 * R27* Bo)* INV
e27<- R28-R27
R29<- -(F+R28 * R28* Bo)* INV
e28<- R29-R28
R30<- -(F+R29 * R29* Bo)* INV
```

```
e29<- R30-R29
R31<- -(F+R30 * R30* Bo)* INV
e30<- R31-R30
R32<- -(F+R30 * R30* Bo)* INV
e31<- R31-R30
R33<- -(F+R32 * R32* Bo)* INV
e32<- R33-R32
R34<- -(F+R33 * R33* Bo)* INV
e33<- R34-R33
R35<- -(F+R34 * R34* Bo)* INV
e34<- R35-R34
R36<- -(F+R35 * R35* Bo)* INV
e35<- R36-R35
R37<- -(F+R36 * R36* Bo)* INV
e36<- R37-R36
R38<- -(F+R37 * R37* Bo)* INV
e37<- R38-R37
PHI<- L0+R14 * Bo
d<- diag(1,8,8)
o<- c(1,1,1,1,1,1,1,1)
O<- matrix(o,8,1)
i<- d- R14
I<- solve(i)
psi<- I * O
a<- c(1,0,0,0,0,0,0,0)
A<- matrix(a,1,8)
PHI[1,1]<- psi[1,1]
PHI[2,1]<- psi[2,1]
```

```
PHI[3,1]<-psi[3,1]
PHI[4,1]<-psi[4,1]
PHI[5,1]<-psi[5,1]
PHI[6,1]<-psi[6,1]
PHI[7,1]<-psi[7,1]
PHI[8,1]<-psi[8,1]
T<- solve(PHI)
p0<- A * T
p1<- p0 * R14
p2<- p0 * R14* R14
```

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