وزارة التعليم العالي والبحث العلم

BADJI MOKHTAR-ANNABA
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THESIS

Presented to obtaining the Doctorate Degree

LIMIT CYCLES OF A CLASS OF 2-DIMENSIONAL DIFFERENTIAL SYSTEMS

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THÈSE

Présenté en vue de l'obtention du diplôme de Doctorat en mathématiques

Cycles Limites d'une Classe de Systèmes Différentiels de Dimension 2

Filière
Mathématiques Appliquées
Spécialité
Mathématiques Appliquées
Par
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To my dear mother for her encouragement me, and support during my whole life,
To the spirit of my dear father who taught me the importance of teaching and learning,
To my wife,my daughter Bissane and my son Mohamed Tamim,
To my brother and my sisters,
To people who helped me from near and far,
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Abstract

In this thesis,

Firstly, we study the maximum number of limit cycles that can bifurcate from the periodic orbits of the uniform isochronous center located at the origin of system $\dot{x} = -y + x^2 y,\ \dot{y} = x + xy^2$ when we perturb it inside the class of all continuous and discontinuous quartic polynomial systems separated by the straight line $y = 0$. The results are obtained by using the first order averaging method. Additional results are provided by using the averaging method of second order and perturbing the cubic uniform center inside the class of all quintic polynomial differential systems.

Secondly, we perturb the cubic degenerate center $\dot{x} = -y\left(3x^2 + y^2\right), \quad \dot{y} = x\left(x^2 - y^2\right)$ inside the class of all quintic polynomial differential systems. Interesting results are obtained by using the averaging method of second order.

Keywords: Averaging method, differential system, limit cycle, isochronous center, degenerate center.

Résumé

Dans cette thèse,

Premièrement, nous étudions le nombre maximum de cycles limites pouvant bifurquer à partir des orbites périodiques du centre isochrone uniforme situé à l'origine du système $\dot{x}=-y+x^2y,\ \dot{y}=x+xy^2$, lorsqu'on le perturbe par une classe de systèmes polynomiaux quartiques continus et discontinus séparés par la droite $y=0$. Les résultats sont obtenus en utilisant la méthode de moyennisation du premier ordre. D'autres résultats sont donnés en utilisant la méthode de moyennisation du deuxième ordre en perturbant le centre isochrone par une classe de systèmes différentiels polynomiaux quintiques.

Deuxièmement, nous perturbons le centre cubique dégénéré $\dot{x}=-y(3x^2+y^2)$, $\dot{y}=x(x^2-y^2)$ par une classe de systèmes différentiels polynomiaux quintiques. Des résultats très intéressants sont obtenus en utilisant la méthode de moyennisation du deuxième ordre.

Mots clés: Méthode de moyennisation, système differentiel, cycle limite, centre isochrone, centre dégénéré.

الملخص

في هذه الأطروحة،

أولاً، قمنا بدراسة العدد الأقصى لعدد الدورات الحدية التي يمكن أن تتشعب من المدارات الدورية للمركز المتزامن الموجود للنظام $\dot{x} = -y + x^2y$ ، $\dot{y} = x + xy^2$ عندما نقوم بتطبيق ارتياب بواسطة جميع أنظمة كثيرات الحدود من الدرجة الرابعة المستمرة وغير المستمرة المفصولة بخط مستقيم $y = 0$. تم الحصول على النتائج باستخدام طريقة المتوسط من الرتبة الأولى. بالإضافة الى ذلك، نتيجة أخرى تم الحصول عليها بتوظيف طريقة المتوسط من الرتبة الثانية بعد تطبيق ارتياب على المركز المتزامن المكعب بواسطة جميع الأنظمة التفاضلية لكثيرات الحدود من الدرجة الخامسة.

ثانيًا، نقوم بتطبيق ارتياب على المركز المفكك $\dot{x} = -y(3x^2 + y)$ ، $\dot{y} = x(x^2 - y^2)$ بواسطة جميع الأنظمة التفاضلية ذات كثيرات الحدود ذات الدرجة الخامسة. تم الحصول على نتائج مثيرة للاهتمام باستخدام طريقة المتوسط من الرتبة الثانية.

الكلمات المفتاحية: نظرية المتوسط، نظام تفاضلي، دورة حدية، مركز متزامن، مركز المفكك.

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General Introduction

Ever since humans became conscious of natural phenomena in their surroundings, they have tried to understand and comprehend these events. Furthermore, they also aims to identify a method for predicting them. The rain cycle might be considered as a fundamental problem that exemplifies these occurrences. To plant more and better, it would be helpful to determine when there is the most rainfall. Considering this perspective, it is indisputable that mathematics serves as the fundamental language for describing natural phenomena. There is a belief in the mathematics community that God created the universe and its laws using the language of mathematics. There is evidence that mathematics dates back to around 1900 BC, and there are currently no signs that the field is about to diminish. In the 17th century, Isaac Newton and Gottfried Leibniz introduced differential calculus. By proving to be a useful tool for modeling, in an abstract language, what happens in the real world over time, this novel approach has contributed to a better understanding of several natural phenomena. This lead the foundation for the study of ordinary differential equations.

An ordinary differential equation is an equation of the form

$$f(t, x, x', x'', \dots, x^{(n)}) = 0,$$

with $x^{(n)}$ denote the n^{th} derivative of x with respect to t. If x is a vector, then this equation becomes a differential system. If f does not depend explicitly on t, then the system is autonomous. Numerous situations, like the prey-predator problem (formulated by Alfred Lotka and Vito Volterra in 1925) and the n-bodies problem (formulated by Newton), can be modeled using ordinary differential equations.

Nearly two centuries later, the french mathematician Henry Poincaré, in his work "Mémoire sur les courbes définies par une équation différentielle", introduced a new approach and a fresh perspective to the study of differential systems. In this work, Poincaré presents a qualitative investigation of ordinary differential equations. By employing geometric and topological techniques, Poincaré successfully examined the qualitative characteristics of the solutions of a differential equation without the need for explicit determination of these solutions. Poincaré made important contributions to the field, including the introduction of the phase portrait idea and the development of essential concepts such as the return map and the Annular Region Theorem. These concepts are essential for classifying orbits based on their particular behaviors. These results would be the pillars of the Qualitative Theory of Differential Equations.

Poincaré also introduced the notion of a limit cycle, which is defined as a periodic orbit where at least one trajectory of the vector field approaches in either positive or negative time. Usually, an alternative definition is given when the vector field is of class C^1. A limit cycle is a closed orbit isolated from the other periodic orbits, where the term *"isolated"* means that the neighboring trajectories are not periodic.

In the early twentieth century, the Swedish mathematician Ivar Otto Bendixson published an important result proving that the principal solutions are called minimal or singular sets (periodic orbits, critical points and separatrix) defined a differential equation on a compact set has the property that the other solution goes to a singular solution. This is the well-known **Poincaré Bendixson Theorem**. This result inspired Lyapunov to investigate the dynamics of solutions in the neighborhood of an equilibrium position. Because of his work, Lyapunov has become recognized as the creator of the contemporary theory of motion stability, owing to his significant contributions.

A polynomial differential system in $\mathbb{R}^2$ or a planar polynomial differential system is a system of the form

$$\dot{x} = P(x(t), y(t)),$$
$$\dot{y} = Q(x(t), y(t)),$$

where the dot denotes the derivative with respect to time t (i.e. $\cdot = d/dt$), $x(t)$, $y(t)$, $P(x, y)$ and $Q(x, y)$ are real functions. In 1900, at the International Congress of Mathematicians in Paris, David Hilbert posed 23 open problems, which he believed would inspire progress in mathematics throughout the 20th century. The second part of the sixteenth Hilbert problem asks about the maximum number (called $H(n)$) and the position of the limit cycles with respect to the degree of a planar polynomial differential system. According to [59], the second part can be summarized as

> *"Proving that for any $n \geq 2$ there exists a finite number $H(n)$ such that any polynomial differential equation whose degree is lower or equal than n has less than $H(n)$ limit cycles."*

Until nowadays, even for the most simple case $n = 2$, the 16th Hilbert problem has not been solved. In 1923, Henri Dulac initiated the initial investigation of the 16th Hilbert problem. The aim of his study was to prove that the number of limit cycles for a planar polynomial differential system is finite, see [19]. In 1970, Yulij Ilyashenko verified the falsity of Dulac's proof. Several years later, Ilyashenko and Écalle independently presented a valid proof. While Dulac's proof was incorrect, his ideas were highly productive and generated important results, such as the classical Dulac Theorem and its generalization, known as the Bendixson-Dulac Theorem.

To provide the reader with an understanding of the complexity involved in solving the 16th Hilbert problem, a more limited version exists. This version focuses on determining the number $H(n)$ specifically for the Lienard family.

$$\begin{aligned} \dot{x} &= y - f(x), \\ \dot{y} &= -x, \end{aligned}$$

where f is a real function of degree n and $f(0) = 0$.
This weaker version is still unsolved.

In recent years, discontinuous planar differential systems have attracted significant attention because of the widespread of discontinuous phenomena in real life, the switching in electronic circuits, impact in mechanical devices, and stick-slip motion in oscillator with dry friction are discontinuous phenomena, see [2]. A planar discontinuous differential system is a system of the form

$$\dot{X} = Z(x,y) = \begin{cases} Z^{+}(x,y), & \text{when } f(x,y) \geq 0, \\ Z^{-}(x,y), & \text{when } f(x,y) < 0, \end{cases}$$

where $f:\mathbb{R}^2 \to \mathbb{R}$ is a C^1 function such that 0 is a regular value.
The discontinuity curve is given by $\Sigma = f^{-1}(0)$, and $Z^{\pm} = (X^{\pm}, Y^{\pm})$. For discontinuous differential systems, a limit cycle is said to be a crossing limit cycle if it intersects the discontinuity set only in the crossing region, which is a subset of the discontinuity set where the two vector fields are transverse to it with their normal components have the same sign (see [36]). Similar to continuous polynomial differential systems, the second part of the sixteenth Hilbert problem was extended to discontinuous piecewise polynomial differential systems. This problem asks for the maximum number of crossing limit cycles that a planar polynomial discontinuous system of a fixed degree can have.

In the aim to solve the second part of the sixteenth Hilbert problem, some theories and methods were created for detecting limit cycles for continuous systems and have been developed for the discontinuous systems such as the Averaging method. In a review of books on the method of averaging, J.A. Murdock [50] wrote: “The subject of averaging is vast, and it is possible to read four or five books entirely devoted to averaging and find very little overlap in the material which they cover.” The averaging method is a fruitful technique in perturbation theory. There are many different averaging theorems, whose main goal is to obtain information about a nonlinear differential system with a small parameter ε

$$\dot{x} = \varepsilon f(t,x,\varepsilon), \text{ where } f \text{ is } T-\text{periodic in} t,$$

from the averaged autonomous system

$$\dot{x} = \varepsilon F(x), \text{ where } F(x) = \frac{1}{T}\int_0^T f(t,x,0)dt.$$

The averaging theory for finding periodic solutions consists in providing sufficient conditions for the existence of periodic solutions in a vector field by studying the equilibrium points of its associated averaged system. This theory has become a classical tool for studying periodic solutions of nonlinear differential systems, see for instance [22, 31, 43, 48, 53, 58, 65]. For a brief historical review, the interested reader is referred to [60](Appendix A).

The contribution of this thesis is in this direction. We hope to determine a lower bound for $H(n)$, the maximum number of limit cycles, in some classes of planar polynomial differential systems. In fact, we present new results about this topic by using different theorems of the averaging theory for finding periodic solutions.

This thesis is divided into four chapters:

- **Chapter** 1 is an Overview on Dynamical Systems. This chapter presents some preliminary results and basic notions of the qualitative theory of dynamical systems. We begin by defining the mathematical tools that are necessary for the study of this thesis and basic definitions such as: planar differential systems, switching systems, the critical points and their local structure, linearization, periodic orbits, limit cycles, as well as theorems on the existence and non-existence of limit cycles.

- **Chapter** 2 is devoted to the averaging method for detecting periodic solutions, which is the main tool that we are going to use to prove the different results that are given in this thesis.

- **Chapter** 3 contains two parts. In the first part, we consider the cubic isochronous center located at the origin under the quartic continuous and discontinuous small perturbations and by applying the averaging method of first order for computing periodic solutions, we obtain at most three and at least nine limit cycles, respectively, can bifurcate from the periodic solutions of the unperturbed system. This study was published as follows:

  N. Rezaiki, A. Boulfoul, On the number of limit cycles coming from a uniform isochronous center with continuous and discontinuous quartic perturbations, Journal of Applied Analysis, 30(1),2024, 35-50, https://doi.org/10.1515/jaa-2023-0018.

  In the second part, we perturb the cubic isochronous center inside the class of all quintic differential systems with a small parameter. Using the second order averaging method, at most nineteen limit cycles can bifurcate from the periodic solution surrounding the origin. This study was published as follows:

  N. Rezaiki, A. Boulfoul, ON THE LIMIT CYCLES THAT CAN BIFURCATE FROM A UNIFORM ISOCHRONOUS CENTER VIA AVERAGING METHOD, Mem. Differential Equations Math. Phys, 2024

- **In chapter** 4, we study the maximum number of limit cycles that can bifurcate from a degenerate center of a cubic polynomial differential system. Using the second order averaging method and perturbing inside the class of all quintic polynomial differential system, we prove that at most five limit cycles can bifurcate from the degenerate center. This study was published as follows:

N. Rezaiki, A. Boulfoul, Limit cycles obtained by perturbing a degenerate center, Open Journal of Mathematical Analysis, 7(2023)1, 71-82.

Finally, the last chapter is devoted to the conclusion of this thesis and our future works.

We notice that all computations of the presented studies in this thesis have been verified with the help of the algebraic manipulators MAPLE and Mathematica.

CHAPTER 1

An overview on dynamical systems

1.1 Introduction

A dynamical system may be defined as a stable and predictable "rule" that governs the temporal evolution of a point in a geometric space. In the context of a dynamical system, a state is represented by a point, which is defined by a real vector. The "rule" is usually expressed as either differential equations or difference equations. A dynamical system always exhibits a certain state, and for each given time interval, there is only one future state that follows from the current state.

It is possible to get the precise formulation of the solutions for certain mathematical models that are represented by differential equations. In this instance, all of the solution's future dynamical behaviors can be predicted for any given starting situation. However, many of important problems, particularly nonlinear ones, are too difficult to resolve. How can we investigate the dynamical characteristics of these problems?

1.1.1 Planar differential systems

Consider the polynomial differential system

$$\begin{aligned} \dot{x} &= P_n(x,y), \\ \dot{y} &= Q_n(x,y), \end{aligned} \tag{1.1.1}$$

where $P_n(x,y)$ and $Q_n(x,y)$ real polynomial functions of degree n.
As far as we know, Hilbert's 16th problem [32] is a well-known one about the bifurcation of

limit cycles in planar differential systems (1.1.1). In 1900, Hilbert presented this problem at the International Congress of Mathematicians in Paris. The second part of the sixteenth Hilbert problem is about studying the distributions of limit cycles in (1.1.1) and determining the upper bound of the number $H(n)$, known as the Hilbert number, of limit cycles. In fact, This problem is being solved extremely slowly. More than a century, It is still unsolved even in the case of quadratic systems.

Early in the 1990s, it was independently demonstrated by Ilyashenko and Yakovenko [33] and Écalle [21] that the number of limit cycles for any given planar polynomial vector fields is finite. For any $n > 1$, it is still unknown if there is a finite uniform upper bound $H(n)$ for number of limit cycles of planar polynomial differential system of degree n (1.1.1). More than 30 years ago, it was demonstrated that $H(2) \geq 4$ for $n = 2$ [10, 61]. This result was recently also attained for quadratic systems that are almost integrable [66]. Many results on the lower bound of $H(n)$ have been obtained for cubic polynomial systems. Several results have been provided on $H(n)$'s lower bound. The obtained result in [39, 37] ,$H(3) \geq 13$, is the best among them . In addition, the distribution of the 13 limit cycles seen in [39, 37] are around a several singular points. See [38] for additional information on the study of Hilbert's sixteenth problem.

1.1.2 Switching differential equations

In recent years, a big interest has been focused on non-smooth dynamical systems whose the right hand functions are either non differentiable or discontinuous. For these kinds of systems, the fundamental techniques of qualitative theory can be found in [26, 35].

Consider the switching system of the form

$$(\dot{x}, \dot{y}) = \begin{cases} \left(f_1(x,y)^+, f_2(x,y)^+\right) = Z^+(x,y), \text{if } \ y > 0, \\ \left(f_1(x,y)^-, f_2(x,y)^-\right) = Z^-(x,y), \text{if } \ y < 0, \end{cases} \tag{1.1.2}$$

where $f_1(x,y)^\pm$ and $f_2(x,y)^\pm$ are analytic functions in x and y. The system defined for $y < 0$ ($y > 0$) is called the lower (upper) system. The plane is divided into two half-planes $\Sigma^\pm = \{(x,y) : \pm y > 0\}$ by the strait line, $\Sigma = y = 0$, and the trajectories are defined according the Filippov convention, see [26]. Three behaviors are possible on Σ when the two vector fields meet: crossing denoted Σ^C, sliding denoted Σ^S , and escaping denoted Σ^E. For $p \in \Sigma$, then

$p \in \Sigma^C$ if and only if $Z^+ f(p).Z^- f(p) > 0$ where $Z^{\pm} f(p) = \langle \nabla f(p), Z^{\pm}(p)\rangle$. As a result, we have $p \in \Sigma^S \cup \Sigma^E$ if, only if $Z^+ f(p).Z^- f(p) < 0$. Figure 1.1.1 shows the vector field in nearby of these three regions. A detailed discussion of the studies on the dynamics of switching systems can be found in a survey article [47].

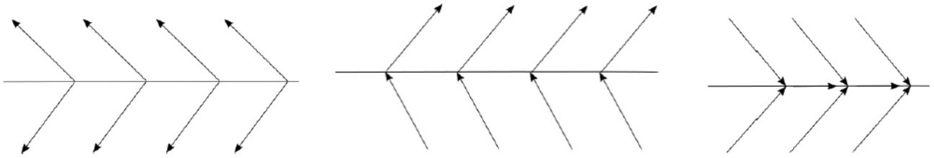


Figure 1.1.1 : Escaping, crossing and sliding segments

Many efforts have been made to extend the classical bifurcation theory and techniques for smooth systems to non-smooth ones, for example, the methods for Hopf bifurcation [14, 29, 41, 11, 42], Poincaré map [35, 41].

Switching systems can display more complex bifurcations, like border-collision bifurcation [51], grazing bifurcation [18, 6], and so on, that only non-smooth systems can have due to various forms of non-smoothness. In this thesis, these kinds of bifurcations will not be discussed.

1.2 Vector field

Before beginning with studying a differential system, it is convenient to graphically describe the vector field because it provides us with helpful information about the many forms of possible solutions and their asymptotic behavior. This geometrical interpretation was well known in the time of Poincaé which was the first to say that the differential equations are in fact a branch of geometry.

Definition 1.2.1 *([20]). A vector field of class C^r on $\Delta \subset \mathbb{R}^2$ an open subset is a map $\chi : \Delta \to \mathbb{R}^2$ where $\chi(x)$ is meant to represent the free part of a vector attached at the point $x \in \Delta$. The r of C^r denotes a positive integer, $+\infty$ or ω, where C^ω stands for an analytic function.*

The vector field associated to the differential system (1.1.1) *is usually represented by a differential operator*

$$\chi = P_n \frac{\partial}{\partial x} + Q_n \frac{\partial}{\partial y},$$

operating on function that are at least C^1.

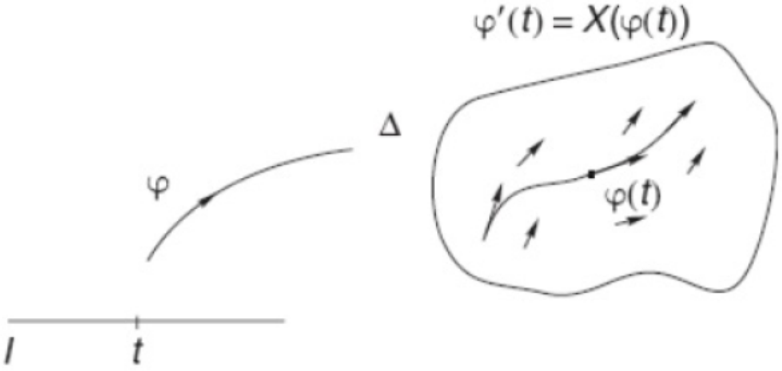


Figure 1.2.2 : Vector field

Remark 1.2.1 *A vector field is graphically represented on a plane by drawing a number of well chosen vectors* $(x, \chi(x))$, *as shown in Figure 1.2.2 ([20]).*

1.2.1 Flow

Let Consider a planar differential system of the form

$$\dot{x} = f(x), \tag{1.2.3}$$

where $f : I \subset \mathbb{R}^2 \to \mathbb{R}^2$ is a C^k function, $k \geq 1$.

For each point $x_0 \in I$, system (1.2.3) has a unique solution $\varphi(t, x_0)$ that satisfies the condition $\varphi(0, x_0) = x_0$.

Definition 1.2.2 *([52]) Let* $\varphi^t(x_0) = \varphi(t, x_0)$ *the family of the transformations* $\varphi : I \to \mathbb{R}^2$ *satisfies the following properties*

- *(a_1)* $\dfrac{d\varphi}{dt}(x) = \chi(\varphi_t(x)), \ \forall x \in I;$
- *(a_2)* $\varphi^0 = Id;$
- *(a_3)* $\varphi^{s+t} = \varphi^s \circ \varphi^t, \ \forall s, t \in \mathbb{R} \ \text{and} \ x \in I.$

The function φ *is called the flow generated by system* (1.2.3).

1.2.2 Phase portrait

Generically, it is very difficult to determine the explicit solution of a differential equation. It was already well known that for majority of the right hand sides, the explicit formulas for the solution simply can not be found. Poincaré in his works said that the differential equation should be studied by their right hand sides. He was the first to plot the phase portrait of differential equation near the singular points.

Definition 1.2.3 *([20]) A phase portrait of* (1.1.1) *is the set of (oriented) orbits that represent the solutions in phase plan.*

1.3 Local structure of Singular Points

Consider the autonomous differential equation

$$\dot{x} = f(x), \tag{1.3.4}$$

with $f : \mathscr{D} \to \mathbb{R}^n$ a C^1 function, $\mathscr{D}$ an open subset of $\mathbb{R}^n$. $\mathscr{D}$ is called the phase space of the differential equation (1.3.4).

1.3.1 Singular Point

Definition 1.3.1 *A point $x_0 \in \mathbb{R}^n$ is called singular point of* (1.3.4) *if $f(x_0) = 0$. If a singular point has a neighborhood that does not contain any other singular points, then that singular point is called an isolated equilibrium point, [24].*

Remark 1.3.1 *A singular point can also be called either a singularity, or critical point, or an equilibrium point.*

1.3.2 Stability of Singular Point

Definition 1.3.2 *Consider the differential system*

$$\dot{x} = f(t, x), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n, \tag{1.3.5}$$

let p be a singular point of system (1.3.5)*. Denote by $\psi(t)$ the solution of this system.*

We say that

- *p is stable if and only if*

$$\forall \varepsilon > 0, \exists \delta > 0, \| \psi(t_0) - p \| < \delta \Rightarrow \ (\forall t \geq t_0,\ \| \psi(t) - p \| < \varepsilon,).$$

- *p is asymptotically stable if and only if p is stable and if there exists a neighborhood V of p such that for all $x \in V$*

$$\lim_{t \to \infty} \psi(t) = p.$$

Remark 1.3.2 *Asymptotic stability implies that the equilibrium point is the limit of the trajectories when $t \to \infty$, however neutral stability (stable but not asymptotically stable) only states that the trajectories stay close to the equilibrium point without necessarily tending towards it.*

1.3.3 Linearization and Classification of singular points

Consider a planar differential system with a parameter ε of the form

$$\begin{aligned}\dot{x} =& f(x, y, \varepsilon),\\ \dot{y} =& g(x, y, \varepsilon).\end{aligned}$$

Let the vector field $\chi = (f, g)$ be a planar C^r with singular point p. The local behavior of the flow at p is generally quite complicated to study. The linear system display different classes even in cases of local topological equivalence.

Definition 1.3.3 *[20] we say that*

$$D\chi(p) = \begin{pmatrix} \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p), \\ \frac{\partial g}{\partial x}(p) & \frac{\partial g}{\partial y}(p), \end{pmatrix}$$

is the linear part of the vector field χ at the singular point p.

The linearized system of the vector field χ at the neighborhood of p is given by

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) \\ \frac{\partial g}{\partial x}(p) & \frac{\partial g}{\partial y}(p) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{1.3.6}$$

- *If* 0 *is not an eigenvalue of* $D\chi(p)$, *then the singular point* p *is called non-degenerate.*

- *A singular point* p *of system* (1.1.1) *is hyperbolic if the eigenvalues of the matrix* $D\chi(p)$ *have a non zero real part. Otherwise, the singular point is called non-hyperbolic. In this cases the singularities are also said to be elementary singular points.*

- *If the two eigenvalues are equal to* 0 *and* $D\chi(p) \not\equiv 0$, *then the singular point* p *is called nilpotent.*

- *The singular point is called linearly zero if* $D\chi(p) \equiv 0$.

In the plan, the linearized system at the origin is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \left.\frac{\partial(f,g)}{\partial(x,y)}\right|_{(0,0)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1.3.7}$$

where the characteristic polynomial is $P(\lambda) = \lambda^2 + a\lambda + b$, with $a = -(a_{1,1} + a_{2,2}) = -Tr(A)$, $b = a_{1,1}a_{2,2} - a_{2,1}a_{1,2} = \det(A)$. Denote by λ_1 and λ_2 its two eigenvalues. Then the origin of system (1.3.7) is a

- Node, if $\lambda_1\lambda_2 > 0$ and $Im(\lambda_1) = 0$.

- Saddle point, if $\lambda_1\lambda_2 < 0$.

- Center, if $Re(\lambda_1) = 0$ and $Im(\lambda_1) \neq 0$.

- Focus, if $Re(\lambda_1) \neq 0$ and $Im(\lambda_1) \neq 0$.

- Degenerate point, if $Re(\lambda_1\lambda_2) = 0$.

Therefore, we have a center point at the origin with a pair of purely imaginary eigenvalues, if $a = 0$ and $b > 0$. In this case, the origin in vector field could be either an elementary center or an elementary focus, and the center focus problem arises here. Under proper conditions of ε, small amplitude limit cycles can bifurcate from the origin.

Near the singular point p, the stability of the vector field χ can be studied according to the sign of λ_1 and λ_2 the eigenvalues of $D\chi(p)$.

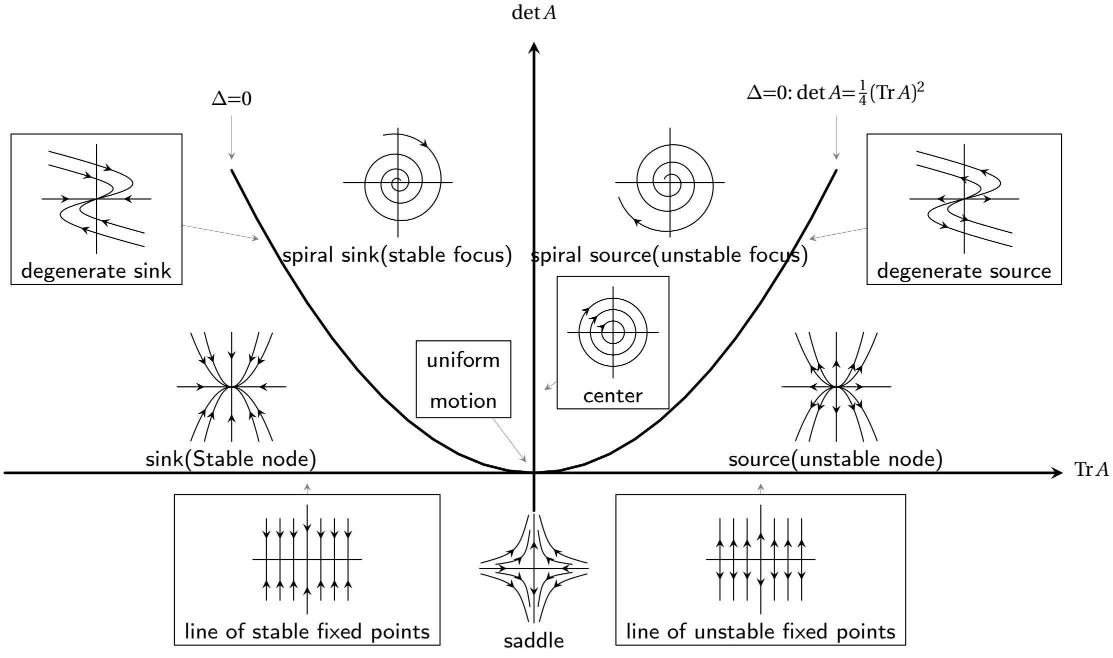
Theorem 1.3.1 *Let* p *be the singular point of the vector field* χ.

- *If all the eigenvalues of the Jacobian matrix $D\chi(p)$ have negative real parts, then, the singular point p is asymptomatically stable.*

- *if there is at least one eigenvalue of $D\chi(p)$ with a positive real part, then p is said to be unstable.*

- *If $D\chi(p)$ has own values with negative real parts and others with zero real parts, then nothing can be said about the stability of the singular point p.*

Poincaré Diagram: Classification of Phase Portraits in the $(\det A, \operatorname{Tr} A)$-plane



1.3.4 Equivalence topological

Consider the following autonomous differential systems

$$(\mathbb{S}_1): \dot{x} = f(x), \;\; x \in \Omega_1, \qquad (\mathbb{S}_2): \dot{y} = g(y), \;\; y \in \Omega_2.$$

Definition 1.3.4 *A continuous map $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that $H^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ continuous and bijection is called homeomorphism of $\mathbb{R}^2$.*

Definition 1.3.5 *If there is a homeomorphism $H : \Omega_1 \rightarrow \Omega_2$ such that H transform the orbits of $(\mathbb{S}_1)$ into $(\mathbb{S}_2)$ and preserves their orientation. then two autonomous polynomial differential systems are said to be topologically equivalent.*

1.3.5 Hartman Grobman Theorem

Theorem 1.3.2 *([52]) Suppose that the Jacobian matrix associated to system (1.1.1) at the singular point p has two eigenvalues λ_1 and λ_2 such that the real part $\mathfrak{Re}(\lambda_i) \neq 0$ for $i = 1, 2$, then the solutions of system (1.1.1) can be approximated by the solutions of the linearized system (1.3.7) in the neighbourhood of the singular point p.*

1.4 Periodic solutions and Limit Cycles

Definition 1.4.1 *(Periodic solution [64]) A solution $\psi(t) = (x(t), y(t))$ of system (1.1.1) is said to be periodic if there exists $T > 0$ such that*

$$\psi(t+T) = \psi(t), \quad \forall t \in \mathbb{R}.$$

The minimal value of T is called the period.
In the phase plane, a periodic solution is represented by a closed curve.

Definition 1.4.2 *(Limit cycles [62]) A periodic orbits of system (1.1.1) is said to be limit cycles if there is no periodic orbits in a neighborhood of it. In other words, a limit cycle of system (1.1.1) is an isolated periodic solution in the set of all its periodic solutions. Isolated means that neighbooring trajectories are not periodic, they will spiral either away or towards the limit cycle.*

Remark 1.4.1 *Limit cycles appear only in differential systems,i.e. a linear vector field χ has no limit cycle.*

1.4.1 Stability of limit cycles

Theorem 1.4.1 *([52]) Let $\psi(t)$ be a periodic orbit of system (1.1.1) with the period T.*

- *If* $\int_0^T \nabla \cdot f(\psi(t))dt < 0$, *then the periodic solution* $\psi(t)$ *is a stable limit cycle.*
- *If* $\int_0^T \nabla \cdot f(\psi(t))dt > 0$, *then* $\psi(t)$ *is an unstable limit cycle.*
- *It may be a stable, unstable or semi-stable limit cycle or it may belong to a continuous band of cycles if* $\int_0^T \nabla \cdot f(\psi(t))dt = 0$.

Remark 1.4.2 *If* $\int_0^T div f(\psi(t))dt \neq 0$, *then the limit cycle* ψ *is said to be hyperbolic.*

Example 1.4.1 *Consider the system*

$$\begin{aligned} \dot{x} &= 2x - y - 2xr^2, \\ \dot{y} &= x + 2y - 2yr^2. \end{aligned} \tag{1.4.8}$$

with $r = \sqrt{x^2 + y^2}$. *In polar coordinates* $x = r\cos\theta$, $y = r\sin\theta$, *system* (1.4.8) *can be transformed into the following form*

$$\begin{aligned} \dot{r} &= 2r(1 - r^2), \\ \dot{\theta} &= 1. \end{aligned} \tag{1.4.9}$$

Taking θ as a new independent variable, we can write system (1.4.9) as follows

$$r' - 2r = 2r^3, \tag{1.4.10}$$

with $' = \frac{d}{d\theta}$, which is a *Bernoulli* equation. So by taking $z = r^{-2}$ and after some computations, equation (1.4.10) becomes

$$z' + 4z = 4.$$

The solution of the above differential equation is

$$z(\theta) = 1 + c\exp(-4\theta),$$

which implies that

$$r^2 = \frac{1}{1 + c\exp(-4\theta)}.$$

For $c = 0$, we have the periodic orbit $r^2 = 0$. In phase plane we get $x^2 + y^2 = 1$. Thus $\psi(t) = (\cos\theta, \sin\theta)$ is a periodic solution of system (1.4.8). In addition, we have $\nabla \cdot f(x, y) = 4 - 8x^2 - 8y^2$ then

$$\int_0^{2\pi} \nabla \cdot f(\psi(t)) = \int_0^{2\pi} \left(-8(\cos(t))^2 - 8(\sin(t))^2 + 4\right) = -8\pi < 0$$

So, system (1.4.8) has an hyperbolic limit cycles $\psi(t) = (\cos\theta, \sin\theta)$ which is a stable limit cycles, see Fig.1.4.3 .

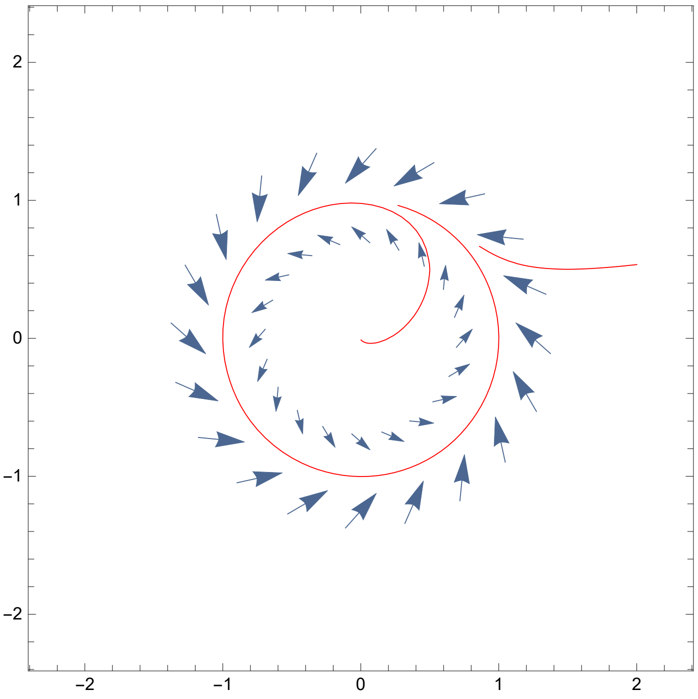


Figure 1.4.3 : Stable limit cycles for system (1.4.8)

1.4.2 Poincaré-Bendixson Theorem

Definition 1.4.3 *(ω-limit set of an orbit, [20]). Consider the C^r vector field with $r \geq 1$ $\chi : D \to \mathbb{R}^2$ where D is an open subset of $\mathbb{R}^2$. Let $\psi(t) = \psi(t, p) = \psi_p(t)$ be an integral curve of the vector field χ passing through the point p, defined on its maximal interval $I_p = (\omega_-(p), \omega_+(p))$. If $\omega_+(p) = \infty$ we define*

$$\omega(p) = \left\{ q \in D : \text{ there exist } \{t_n\} \text{ with } t_n \to \infty \text{ and } \psi(t_n) \to q \text{ when } n \to \infty \right\}.$$

The set $\omega(p)$ is called $\omega-$limit set or simply $\omega-$limit of p.
In the same way, if $\omega_{-}(p) = -\infty$, we define the set

$$\alpha(p) = \left\{ q \in D : \text{ there exist } \{t_n\} \text{ with } t_n \to -\infty \text{ and } \psi(t_n) \to q \text{ when } n \to \infty \right\}.$$

The set $\alpha(p)$ is called the $\alpha-$ limit set or $\alpha-$limit pf p.

A classical tools for studying the existence or the non-existence of limit cycles for the planar differential systems is the *Poincaré-Bendixson* theorem which is the following theorem.

Theorem 1.4.2 *([20]) Let $\gamma +_p = \{x \in D, x = \psi(t,p), t > 0\}$ be the positive half-orbit through the point p. Assume that $\gamma +_p$ is contained in a compact set $\mathbb{K} \in D$. Assume that the vector field χ has at most a finite number of singular points in K. Then one of the following statements holds*

- *If $\omega(p)$ contains only regular points, then $\omega(p)$ is a periodic orbits.*
- *if $\omega(p)$ contains both regular and singular points, then $\omega(p)$ is formed by a set of orbits, every one of which tends to one of the singular points in $\omega(p)$ as $t \to \pm\infty$.*
- *if $\omega(p)$ does not contain regular points, then $\omega(p)$ is a unique singular point.*

Remark 1.4.3 *The posed problem for applying Poincaré-Bendexon is the construction of the region $\mathbb{K}$.*

Example 1.4.2 *Consider the system*

$$\dot{x} = y + \frac{1}{4}x(1 - 2(x^2 + y^2)), \quad \dot{y} = -x + \frac{1}{2}y(1 - (x^2 + y^2)). \tag{1.4.11}$$

The origin is the unique equilibrium point of system (1.4.11). By using the polar coordinates, from system we can get

$$\dot{r} = \frac{1}{4}r(1 - 2r^2 + \sin^2\theta).$$

Note that $\dot{r} > 0$ for all θ as long as $1 - 2r^2 > 0$, i.e $r^2 < \dfrac{1}{2}$ and that $\dot{r} < 0$ for all θ as long as $1 - r^2 < 0$, i.e $r > 1$. Now consider the region $\mathbb{K} = \left\{(r,\theta) | \frac{1}{\sqrt{2}} \le r \le 1\right\}$ which is compact and does not contain any equilibrium point. because of $\dot{r} > 0$ for all $r < 1/\sqrt{2}$ and $\dot{r} < 0$ for all $r > 1$, then all trajectories remain in $\mathbb{K}$. The assumptions of the Poincaé-Bendixson Theorem are satisfied. As consequence, system (1.4.11) has at least one periodic orbit in $\mathbb{K}$.

Theorem 1.4.3 *[52](Bendixson criterion) Consider the system*

$$\begin{aligned}\dot{x} &= f(x,y),\\ \dot{y} &= g(x,y),\end{aligned} \tag{1.4.12}$$

and let $F=(f,g)^T \in C^1(D)$*, where* D *is a simply connected region in* $\mathbb{R}^2$*. If the divergence of the vector field* F *defined by*

$$divF = \nabla \cdot F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

is not identically zero and has a constant sign in D*, then* (1.4.12) *has no periodic orbit entirely contained in* D*.*

Example 1.4.3 *Consider the following differential system*

$$\begin{aligned}\dot{x} &= -x+y^2,\\ \dot{y} &= -y^3+x^2,\end{aligned}$$

It is clear that this system has two singular point, $(0,0)$ *which is degenerate and* $(1,1)$ *which is a saddle. We have*

$$\nabla \cdot F = \frac{\partial(-x+y^2)}{\partial x} + \frac{\partial(-y^3+x^2)}{\partial y} = -1-3y^2 < 0,$$

then the system has no periodic solutions.

Remark 1.4.4 *The theorem has been formulated for the system in* $\mathbb{R}^2$*. One might wonder whether the theorem can be generalised to systems with dimension larger than* 2*. Unfortunately this is not in this case.*

Theorem 1.4.4 *[52](Dulac's Criteria) Consider the differential system given in* (1.4.12)*, let* $\mathbb{K}$ *be a simply connected region in* $\mathbb{R}^2$*. If there exists a function* $B \in C^1(D)$ *such that* $\nabla \cdot (BF)$ *is not identically zero and does not change the sign in* $\mathbb{K}$*, then system* (1.4.12) *has no closed orbit lying entirely in* $\mathbb{K}$*.*

Example 1.4.4 *Consider the system*

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -ax-by+cx^2+dy^2,\end{aligned}$$

and consider the vector field $\chi = \left(y, -ax - by + cx^2 + dy^2\right)$ *also the Dulac function* $B(x,y) = b e^{-2dx}$.

Then we have $\nabla \cdot \chi = -b^2 \exp(-2dx) < 0$. *As consequence, there is no periodic solution for the system* (1.4.12).

CHAPTER

2 Averaging Theory

2.1 Introduction

The averaging method is a productive technique in perturbation theory. It goes back to the classical works on celestial mechanics by Clairaut, Lagrange, and Laplace in the XVIII$^{\text{th}}$ century. During the XIX$^{\text{th}}$ century, the field advanced under the expertise of well-known mathematicians like Jacobi and Poincaré. Until Fatou rigorously formalized it in 1928 and further developed and applied it to nonlinear mechanics in the 1930s by the Kiev school of mathematics, led by Bogoliubov, Krylov, and Mitropolsky, see [3, 25, 34, 60].

The averaging method for finding periodic solutions consists in providing sufficient conditions for the existence of periodic solutions in a vector field by studying the equilibrium points of its associated averaged system. In summary, the averaging method gives a quantitative relation between the solutions of the original system which is non-autonomous and the solutions of its autonomous averaged system. The averaging is with respect to the independent variable and the right hand sides of these systems are sufficiently small, depending on a small parameter ε.

The idea of the averaging method is to replace a differential system in standard form

$$\dot{x} = \varepsilon f(t, x, \varepsilon), \quad x \in U \subset \mathbb{R}^n,\ 0 < \varepsilon << 1, \tag{2.1.1}$$

where $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^+$ is C^r, $r \geq 1$, bounded on bounded sets, $T-$periodic in t, U an open and bounded subset. The associated averaged system is defined by

$$\dot{y} = \varepsilon \frac{1}{T} \int_0^T f(y,t,0)dt \equiv \varepsilon \bar{f}(y), \tag{2.1.2}$$

In other words, the idea is to replace a differential system in standard form (2.1.1) by some type of average as in (2.1.2) either over time or over an angular variable which is hopefully simpler. The main goal of this replacement is to obtain an asymptotic approximation to the solutions of the original system and to obtain a periodic solution by understanding of the dynamics of the averaged system.

In practice, a weakly nonlinear system is often represented as:

$$\dot{x} = Ax + \varepsilon f(t,x,\varepsilon)$$

which does not conform to the form (2.1.2). So how can the averaging method be used in these systems?

2.2 The Lagrange standard Form

Consider the initial value problem:

$$\dot{x} = A(t)x + \varepsilon g(t,x),\ x(0) = x_0, \tag{2.2.3}$$

where $A(t)$ is a continuous $n \times n$-matrix, $g(t,x)$ is a sufficiently smooth function of t and x.

Let $\Phi(t)$ be the fundamental matrix of the unperturbed system $(2.2.3)_{\varepsilon=0}$, and $y(t)$ be such that $y(0) = x_0$. Define the comoving coordinates as:

$$x = \Phi(t)y,$$

Then

$$\dot{x} = \dot{\Phi}(t)y(t) + \Phi(t)\dot{y}$$

Since that x(t) is the solution to equations (2.2.3), we get

$$\Phi(t)\dot{y} = \varepsilon g(\Phi(t)y(t),t),$$

i.e,

$$\dot{y} = \varepsilon\Phi^{-1}(t)g(\Phi(t)y(t), t). \tag{2.2.4}$$

The equation (2.2.4) is referred to as the Lagrange standard form and often written as:

$$\dot{y} = \varepsilon f(t, y). \tag{2.2.5}$$

Therefore, the weakly nonlinear differential equations in Lagrange standard form is exact in the form of (2.1.1).

2.3 Averaging Method for computing periodic solutions

Concerning the averaging theory for finding periodic solutions, two main hypotheses are usually assumed: (i) f is T-periodic in the first variable; and (ii) there exists a sub-manifold $\mathbb{W} \subset \mathbb{R}^n$ such that each solution of the unperturbed system $\dot{x} = f(t, x, 0)$ with initial condition in $\mathbb{W}$ is T-periodic. Under theses hypotheses, the averaging theory provides sufficient conditions for the existence of periodic solutions of $\dot{x} = f(t, x, \varepsilon)$.

2.3.1 Averaging method of first order for continuous differential systems

The following result was proved by applying topological methods based on Brouwer degree theory to solve operator equations equivalent to the problem of finding T-periodic solutions for a differential system. The regularity assumptions are weaker than in previously published results for the averaging method. The following theorem is the averaging method of first order that was developed in [8] for differential systems with continuous right hand.

Theorem 2.3.1 *Consider the differential system*

$$\dot{x} = \varepsilon F(t, x) + \varepsilon^2 G(t, x, \varepsilon), \tag{2.3.6}$$

with $F : \mathbb{R} \times D \to \mathbb{R}^n$, $G : \mathbb{R} \times D \times \left(-\varepsilon_f, \varepsilon_f\right) \to \mathbb{R}^n$ *are continuous functions,* T*-periodic in the first variable* t*, and* $D \subset \mathbb{R}^n$ *an open subset. The function is Defined* $F_{10} : D \to \mathbb{R}^n$ *by*

$$F_{10}(x) = \frac{1}{T}\int_0^T F(t, x)dt. \tag{2.3.7}$$

Assume that

(i) F *and* G *are locally Lipschitz with respect to* x.

(ii) *For* $a \in D$ *and* $F_{10}(a) = 0$, *there exists a neighborhood* V *of* a *such that* $F_{10}(z) \neq 0$ *for all* $z \in \bar{V} \setminus \{a\}$ *and* $d_B(F_{10}, V, 0) \neq 0$.

Then, for $|\varepsilon| > 0$ *sufficiently small, there exists a* $T-$*periodic solution* $x(., \varepsilon)$ *of system* (2.3.6) *such that* $x(., \varepsilon) \to a$ *as* ε *tends to 0.*

Example 2.3.1 ***(Van der Pol Equation)***

The ***Van Der Pol*** *oscillator has been extensively researched for its intriguing behaviors, which simulate a tunnel diode in electric circuits and is used in basic neuron models.*

The equation is given by

$$\ddot{u} - \varepsilon\left(1 - u^2\right)\dot{u} + u = 0 \tag{2.3.8}$$

where ε *is a non zero real parameter sufficiently small.*

This equation can be transformed into the following system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \varepsilon(1 - x^2)y. \end{aligned} \tag{2.3.9}$$

In order to apply the averaging method 2.3.1, *we should transform system* (2.3.9) *into the standard form* (2.3.6). *In polar coordinates,* $x = r\cos\theta$, $y = r\sin\theta$, *system* (2.3.9) *can be written as follows:*

$$\begin{aligned} \dot{r} &= \varepsilon r\left(1 - r^2\cos^2\theta\right)\sin^2\theta, \\ \dot{\theta} &= -1 + \varepsilon\left(1 - r^2\cos^2\theta\right)\sin\theta\cos\theta. \end{aligned} \tag{2.3.10}$$

Taking θ *as a new independent variable, system* (2.3.10) *can be transformed into the equivalent differential equation*

$$\frac{dr}{d\theta} = \varepsilon\left(r^2\cos^2\theta - 1\right)\sin^2\theta + O(\varepsilon^2) = \varepsilon f(\theta, r) + O(\varepsilon^2). \tag{2.3.11}$$

System (2.3.11) *is under the assumptions of Theorem* 2.3.1. *From* (2.3.7), *the averaged function is given by*

$$F_{10}(r) = \frac{1}{8}r\left(r^2 - 4\right). \tag{2.3.12}$$

It is obvious that $F_{10}(r)=0$ *has* $r=2$ *as the unique positive zero. It follows From the statement* (ii) *in Theorem* (2.3.1) *that system* (2.3.9) *has one limit cycles bifurcating from the periodic orbit of radius* 2 *of the unperturbed system* $(2.3.9)_{\varepsilon=0}$*, see figure 2.3.1 and 2.3.2 .*

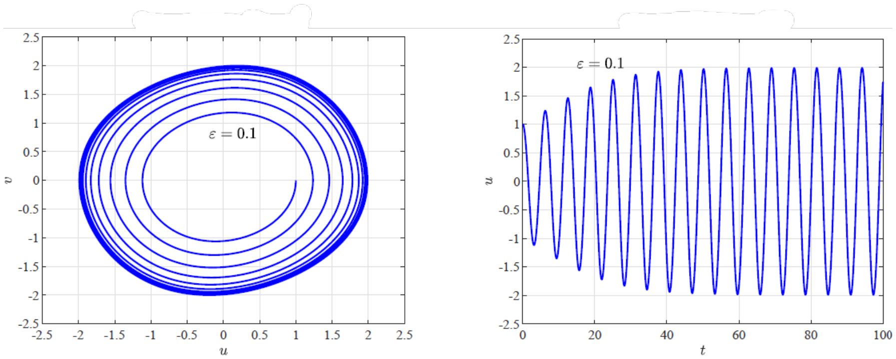


Figure 2.3.1 : Van der Pol oscillator for small ε

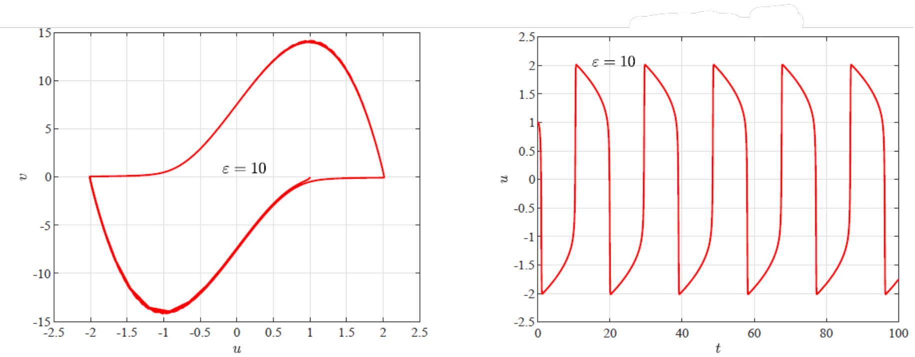


Figure 2.3.2 : Van der Pol oscillator for large ε

2.3.2 Averaging method of first order for discontinuous differential systems

The averaging method was extended, via Brouwer degree theory and regularization theory, for finding periodic solutions of certain discontinuous differential systems. The following theorem represents an insights in averaging, specifically its relation with non-smooth dynamical systems theory.

Theorem 2.3.2 *[43] Consider the following differential system*

$$\dot{x} = \varepsilon F(t,x) + \varepsilon^2 G(t,x,\varepsilon), \tag{2.3.13}$$

with

$$\begin{aligned} F(t,x) &= F_1(t,x) + sign(h(t,x))F_2(t,x), \\ G(t,x,\varepsilon) &= G_1(t,x,\varepsilon) + sign(h(t,x))G_2(t,x,\varepsilon), \end{aligned} \tag{2.3.14}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $G_1, G_2 : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ *are continuous functions,* $T-$*periodic in the first variable* t *and* D *is an open subset of* $\mathbb{R}^n$. *We also suppose that* h *is a* C^1 *function having* 0 *as a regular value. Denote by* $\mathcal{M} = h^{-1}(0)$, $\Sigma = 0 \times D \nsubseteq \mathcal{M}$, $\Sigma_0 = \Sigma \setminus \mathcal{M} \neq \emptyset$ *and its elements by* $z \equiv (0,z) \notin \mathcal{M}$.

Define the averaged function $F_{10}(x) : D \to \mathbb{R}^n$ *by*

$$F_{10}(x) = \frac{1}{T}\int_0^T F(t,x)\,dt. \tag{2.3.15}$$

Assume that

(i) F_1, F_2, G_1, G_2 *and* h *are locally* $L-$*Lipschitz with respect to* x.

(ii) For $a \in \Sigma_0$ *and* $F_{10}(a) = 0$, *there exists a neighborhood* V *of* a *such that* $F_{10}(z) \neq 0$ *for all* $z \in \bar{V} \setminus \{a\}$ *and* $d_B(F_{10}, V, 0) \neq 0$.

(iii) If $\partial_t h(t_0, z_0) = 0$ *for some* $(t_0, z_0) \in \mathcal{M}$, *then* $(\langle \bigtriangledown_x h.F_1\rangle^2 - \langle \bigtriangledown_x h.F_2\rangle^2)(t_0, z_0) > 0$.

Then for $|\varepsilon| > 0$ *sufficiently small, there exists a* $T-$*periodic solution* $x(.,\varepsilon)$ *of system* (2.3.13) *such that* $x(.,\varepsilon) \to a$ *as* $\varepsilon \to 0$.

Example 2.3.2 *Consider the following discontinuous differential system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y + \frac{16}{3}x^2 - \frac{4}{3}y^2 + \varepsilon(a_{10}x + a_{01}y) \\ x + \frac{8}{3}yx + \varepsilon(b_{10}x) \end{pmatrix}, & x > 0, \\ \begin{pmatrix} -y + \frac{16}{3}x^2 - \frac{4}{3}y^2 + \varepsilon(c_{20}x^2) \\ x + \frac{8}{3}yx \end{pmatrix}, & x < 0, \end{cases} \tag{2.3.16}$$

with $\varepsilon = 10^{-4}$, $a_{1,0} = -2/\pi$, $a_{0,1} = 1960.230699$, $b_{1,0} = -53.87877595$, $c_{2,0} = 963.2554059$.

Using the following change of variable

$$x = \frac{3r\sin\theta}{16(1 - r cos\theta)}, \quad y = \frac{3}{8}\left(-1 + \frac{1}{\sqrt{1 - r\cos\theta}}\right), \quad r \in (0,1),$$

system (2.3.16) *can be transformed into the following form:*

$$\frac{dr}{d\theta}=\begin{cases}\varepsilon f_1(\theta,r)+o(\varepsilon^2), & if\ \sin\theta>0,\\ \varepsilon f_2(\theta,r)+o(\varepsilon^2), & if\ \sin\theta<0,\end{cases}\tag{2.3.17}$$

where

$$\begin{aligned}f_1(\theta,r)=&-(1-r\cos\theta)\sin\theta\left(\frac{3r\sin\theta\,a_{1,0}}{16(1-r\cos\theta)}+\frac{3}{8}\left(-1+\frac{1}{\sqrt{1-r\cos\theta}}\right)a_{0,1}\right)\\&-\frac{3}{16}\frac{b_{1,0}\,r(-r+\cos\theta)\sin\theta}{\sqrt{1-r\cos\theta}},\\ f_2(\theta,r)=&\frac{9}{256}\frac{\sin^3\theta\,c_{2,0}r^2}{r\cos\theta-1}.\end{aligned}$$

Let

$$F_i(\theta,r)=\frac{1}{2}\Big(f_1(\theta,r)-(-1)^i f_2(\theta,r)\Big),\quad i=1,2,$$

Then system (2.3.17) *can be reduced to the standard form*

$$\frac{dr}{d\theta}=\varepsilon F(\theta,r)+o(\varepsilon^2),\tag{2.3.18}$$

where

$$F(\theta,r)=F_1(\theta,r)+sign(\sin\theta)F_2(\theta,r).$$

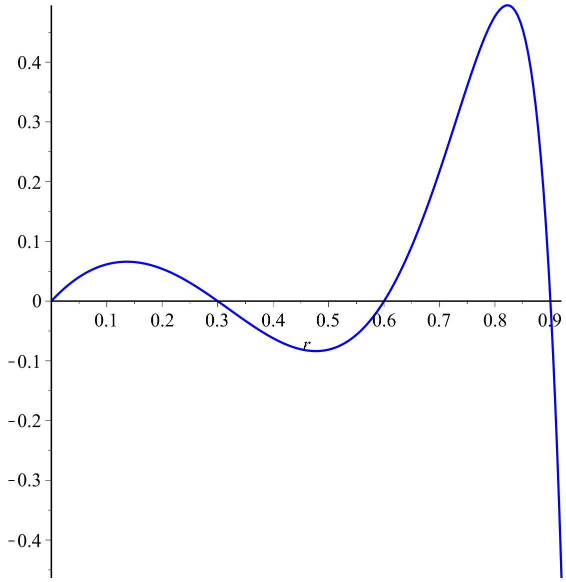


Figure 2.3.3 : The averaged function $F_{10}(r)$

The differential equation (2.3.18) *is under the condition of Theorem 2.3.2. From* (2.3.15)*, we can get*

$$\begin{aligned} F_{10}(r) &= \int_0^{2\pi} F(\theta, r)d\theta \\ &= \int_0^{\pi} f_1(\theta, t)d\theta + \int_0^{\pi} f_2(\theta, t)d\theta. \end{aligned} \tag{2.3.19}$$

By direct computation, the averaged function $F_{10}(r)$ *is given by*

$$F_{10}(r) = T_1 + 2541.802564T_2 - 107.7575519T_3 - 180.6103886T_4. \tag{2.3.20}$$

where

$$\begin{aligned} T_1 &= r, \\ T_2 &= \frac{3r + (1-r)^{3/2} - (1+r)^{3/2}}{r}, \\ T_3 &= -2 + (1-r)^{3/2} + (1+r)^{3/2}, \\ T_4 &= \frac{2r + (1-r^2)\ln\left(\frac{1-r}{1+r}\right)}{r}. \end{aligned}$$

The function $F_{10}(r)$ *has three positive zeros which are* $\frac{3j}{10}$ *with* $j = 1, 2, 3$, *see Fig 2.3.3 . In addition we have*

$$\frac{dF_{10}}{dr}(0.3) = -0.655961, \quad \frac{dF_{10}}{dr}(0.6) = 1.4791883, \quad \frac{dF_{10}}{dr}(0.9) = -17.5355,$$

then the conditions of Theorem 2.3.2 are verified. As a result, system (2.3.16) *has three limit cycles bifurcating from the periodic orbits of the center* $(2.3.16)_{\varepsilon=0}$.

2.3.3 Averaging theory of second order for computing periodic solutions

The averaging method for computing periodic solutions was extended to the second order. In [31], the authors developed it up to the fourth order.

Consider the analytic differential equation:

$$\frac{dr}{d\theta} = F_0(\theta, r) + \varepsilon F_1(\theta, r) + \varepsilon^2 F_2(\theta, r) + \varepsilon^3 R(\theta, r, \varepsilon), \tag{2.3.21}$$

where $r \in \mathbb{R}$, $\theta \in \mathbb{S}^1$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with ε_0 a small positive real number, and the functions $F_k(\theta, r)$ with $k \in \{0,1,2\}$ are $2\pi-$periodic functions in the variable θ. Note that for $\varepsilon = 0$, system (2.3.21) is unperturbed with the solution $r_s(\theta, r_0)$ satisfying $r_s(0, r_0) = r_0 \in \mathscr{I} \subset \mathbb{R}$ ($\mathscr{I}$ is a real open interval), and $r_s(\theta, r_0)$ is 2π-periodic function in variable θ. We are interested in the limit cycles of (2.3.21) which bifurcate from the periodic orbits of the unperturbed system with initial value $r_0 \in \mathscr{I}$.

So, we define by $r_\varepsilon(\theta, r_0)$ the solution of equation (2.3.21) satisfying $r_\varepsilon(0, r_0) = r_0$.

In what follows, we denote by $v = v(\theta, r_0)$ the solution of the variational equation:

$$\frac{\partial v}{\partial \theta} = \frac{\partial F_0}{\partial r}(\theta, r_s(\theta, r_0)).v. \tag{2.3.22}$$

satisfying $v(0, r_0) = 1$.

Now, the functions $F_{j0}(r_0)$ with $j \in \{1,2\}$ are defined by:

$$v_1(\theta, r_0) = \int_0^\theta \frac{F_1\left(\phi, r_s\left(\phi, r_0\right)\right)}{v\left(\phi, r_0\right)} d\phi, \tag{2.3.23}$$

$$F_{10}(r_0) = \int_0^{2\pi} \frac{F_1(\theta, r_s(\theta, r_0))}{v(\theta, r_0)} d\theta, \tag{2.3.24}$$

$$\begin{aligned} F_{20}(r_0) = & \int_0^{2\pi} \left(\frac{F_2(\theta, r_s(\theta, r_0))}{v(\theta, r_0)} + \frac{\partial F_1}{\partial r}(\theta, r_s(\theta, r_0)).v_1(\theta, r_0) \right) d\theta, \\ & + \frac{1}{2} \int_0^{2\pi} \frac{\partial^2 F_0}{\partial r^2}(\theta, r_s(\theta, r_0)) v_1(\theta, r_0)^2 d\theta. \end{aligned} \tag{2.3.25}$$

Theorem 2.3.3 *Assume that the solution $r_s(\theta, r_0)$ of the unperturbed equation (2.3.21) such that $r_s(0, r_0) = r_0$ is 2π-periodic for $r_0 \in \mathscr{I}$ with $\mathscr{I}$ a real open interval. If $F_{10}(r_0)$ is identically zero in $\mathscr{I}$ and $F_{20}(r_0)$ is not identically zero in $\mathscr{I}$. Then for each simple zero r^* of $F_{20}(r_0)$, there exists a periodic solution $r_\epsilon(\theta, r_0)$ of (2.3.21) such that $r_\epsilon(\theta, r_0) \to r^*$ when $\epsilon \to 0$.*

Another second order averaging method given in [60, 63] deals with the bifurcation of limit cycles from the constant solutions of the unperturbed system.

Theorem 2.3.4 *Consider the two initial value problems*

$$\dot{x} = \varepsilon f(t, x) + \varepsilon^2 g(t, x) + \varepsilon^3 h(t, x, \varepsilon), \qquad x(0) = x_0, \tag{2.3.26}$$

and

$$\dot{y} = \varepsilon f^0(y) + \varepsilon^2\big(f^{10}(y) + g^0(y)\big), \qquad y(0) = x_0, \tag{2.3.27}$$

with $f, g : [0, +\infty) \times D \to \mathbb{R}$, $h : (0, \infty) \times D \times (0, \varepsilon) \to \mathbb{R}$

D *an open subset of* $\mathbb{R}$, f, g *and* h *periodic of period* T *in* t, *and*

$$f^1(t,x) = \frac{\partial f}{\partial x} y^1(t,x) - \frac{\partial y^1}{\partial x} f^0(t,x), \tag{2.3.28}$$

where

$$y^1(t,x) = \int_0^t \big(f(s,x) - f^0(x)ds\big) + z(x), \tag{2.3.29}$$

with $z(x) \in C^1$ *function such that the averaging of* $y^1(\theta, r)$ *is zero.*

f^0, f^{10}, g^0 *denote the averaged functions of* f, f^1 *and* g, *respectively, defined as in* (2.3.7).

Suppose that

1. $\frac{\partial f}{\partial x}$ *is Lipschitz in* x *and all these functions are continuous on their domain of definition.*
2. $|h(t,x,\varepsilon)|$ *is bounded by a constant uniformly in* $[0, L/\varepsilon) \times D \times (0, \varepsilon_0]$.
3. T *is a constant independent of* ε, *and* $y(t)$ *belongs to* D *on the time scale* $1/\varepsilon$.

Then the following statements hold:

1. *On the timescale* $1/\varepsilon$, *we have that* $x(t) = y(t) + \varepsilon y^1(t, y(t)) + O(\varepsilon^2)$, *as* $\varepsilon \to 0$.
   If, in addition, $f^0(y) \equiv 0$, *then the following statements hold.*
2. ***If*** p ***is an equilibrium point*** *of the averaged system* (2.3.27) *such that*

$$\frac{\partial}{\partial y}\big(f^{10}(y) + g^0(y)\big)\Big|_{y=p} \neq 0, \tag{2.3.30}$$

   then there exists a T*-periodic solution* $\Phi(t,\varepsilon)$ *of* (4.4.8) *which is close to* p *such that* $\Phi(t,\varepsilon) \to p$ *as* $\varepsilon \to 0$.

3. *If* (2.3.30) *is negative, then the corresponding periodic solution* $\Phi(t,\varepsilon)$ *of* (4.4.8) *in the space* (t,x) *is asymptotically stable for* ε *sufficiently small. If* ε *is positive, then it is unstable.*

CHAPTER 3

Bifurcation of limit cycles from continuous and discontinuous perturbations of a cubic uniform isochronous center

3.1 Introduction

The study of the isochronous system goes back to the XVII century by Christian Huygens. With his works on the cycloidal pendulum, C.Huygens, a mathematician and physicist, was a pioneer in studying isochronicity and inspired interest in this field of research. He studied the cycloidal pendulum, which has isochronous oscillations, in opposition to the monotonicity of the period of the usual pendulum. It is probably the first example of a nonlinear isochronous center. For more details, see [27].

Isochronocity occurs in a wide range of physical processes. Isochronicity is crucial for solving bifurcation, perturbation, and boundary value problems, beyond its practical applications. The importance of isochronicity also appears in stability theory. For example a periodic solution of the central region is Lyapunov stable if and only if the neighbouring periodic solutions have the same period. For more details, see the reference [30].

Recall that if $p \in \mathbb{R}$ is a *uniform isochronous center* or *rigid center*, the angular velocity of the vector $\overrightarrow{pq}$ is the same for all periodic orbits in $U \setminus p$ with $q \in U \setminus p$.

In the last 30 years, there has been a significant interest in the study of the bifurcation of limit cycles of planar polynomial differential systems with an equilibrium point of center type that satisfy this isochronicity property, see [40, 12, 49, 9]. Because of the wide range of discontinuous phenomena in real life, see for instance [2], many researchers are interested in studying limit cycles of discontinuous differential systems, see for example [43].

3.2 Uniform isochronous center of degree 3

The following result characterizes planar cubic polynomial differential systems with a uniform isochronous center. It was obtained by Collins [13] in 1997, Devlin et al. [17] in 1998, and Gasull et al. [28] in 2005.

Theorem 3.2.1 *A cubic planar differential system has a uniform isochronous center located at the origin if and only if it can be represented as*

$$\begin{aligned} \dot{x} &= -y + x\left(a_1 x + a_2 y + a_3 x^2 + a_4 x y - a_3 y^2\right), \\ \dot{y} &= x + y\left(a_1 x + a_2 y + a_3 x^2 + a_4 x y - -a_3 y^2\right), \end{aligned} \tag{3.2.1}$$

and satisfy $a_1^2 a_3 - a_2^2 a_3 + a_1 a_2 a_4 = 0$, $a_i \in \mathbb{R}$, $i = 1, \ldots, 4$.

In [13], Collins proved that the cubic isochronous center (3.2.1) can be reduced into two forms with at most one parameter. The following result was proved by Collins in [13]

Proposition 3.2.0 *If* $a_1^2 a_3 - a_2^2 a_3 + a_1 a_2 a_4 = 0$ *with* $a_i \in \mathbb{R}$, *then system* (3.2.1) *can be reduced to one of the following forms:*

$$\dot{x} = -y + x^2 y, \qquad \dot{y} = x + x y^2, \tag{3.2.2}$$

$$\dot{x} = -y + x^2 + A x^2 y, \quad \dot{y} = x + x y + A x y^2, \tag{3.2.3}$$

with $A \in \mathbb{R}$.

3.3 Results on the number of limit cycles bifurcation from continuous and discontinuous perturbations of cubic uniform isochronous center

In this section, firstly, we apply the averaging method of first order for finding the periodic solutions of piecewise differential systems of the following form:

$$\dot{r}=\begin{cases} \varepsilon f^{+}(\theta,r) & \text{if } 0\leq\theta\leq\alpha,\\ \varepsilon f^{-}(\theta,r) & \text{if } \alpha\leq\theta\leq 2\pi,\end{cases}$$

with $|\varepsilon|\neq 0$ a real parameter small enough, $r\in D$, with $D\subset\mathbb{R}^{+}$ an open interval and $\theta\in\mathbb{S}^{1}$.

We apply this method to investigate the bifurcation of limit cycles in planar polynomial differential systems with a cubic uniform isochronous center located at the origin under two different types of perturbation, either inside the class of all quartic and quintic continuous polynomial perturbations or inside the class of all discontinuous quartic polynomial perturbations separated by the straight line $y=0$.

3.3.1 Main results

Due to Collins [13], if the origin is a cubic uniform isochronous center, then the system can be reduced to the following form

$$\begin{aligned}\dot{x}&=-y+x^{2}y=P(x,y),\\ \dot{y}&=x+xy^{2}=Q(x,y).\end{aligned}\tag{3.3.4}$$

We recall that system (3.3.4) has a first integral $H(x,y)=\dfrac{x^{2}+y^{2}}{1-x^{2}}$ and an integrating factor $\mu(x,y)=\dfrac{2}{(1-x^{2})^{2}}$. There are two invariant straight lines $x=-1$ and $x=1$. The period annulus can be expressed by $\Gamma_h:\{H(x,y)=h, h\in(0,+\infty)\}$. It is clear that $H=0$ corresponds to the isochronous center $(0,0)$.

Now, we consider system (3.3.4) under a small perturbations as follows:

$$\dot{X}=\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}-y+x^{2}y+\varepsilon\sum\limits_{i+j=0}^{4}a_{i,j}x^{i}y^{j},\\ x+xy^{2}+\varepsilon\sum\limits_{i+j=0}^{4}b_{i,j}x^{i}y^{j},\end{pmatrix}=\begin{pmatrix}P(x,y)+\varepsilon p_{1}(x,y),\\ Q(x,y)+\varepsilon q_{1}(x,y),\end{pmatrix},\tag{3.3.5}$$

$$\dot{X}=\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=Z(x,y)=\begin{cases}Y_1 & \text{if } y>0,\\ Y_2 & \text{if } y<0,\end{cases}\tag{3.3.6}$$

where

$$Y_1=\begin{pmatrix}-y+x^2y+\varepsilon\sum\limits_{i+j=0}^{4}a_{i,j}x^iy^j,\\ x+xy^2+\varepsilon\sum\limits_{i+j=0}^{4}b_{i,j}x^iy^j,\end{pmatrix}=\begin{pmatrix}P(x,y)+\varepsilon p_1(x,y),\\ Q(x,y)+\varepsilon q_1(x,y),\end{pmatrix},$$

$$Y_2=\begin{pmatrix}-y+x^2y+\varepsilon\sum\limits_{i+j=0}^{4}c_{i,j}x^iy^j,\\ x+xy^2+\varepsilon\sum\limits_{i+j=0}^{4}d_{i,j}x^iy^j,\end{pmatrix}=\begin{pmatrix}P(x,y)+\varepsilon p_2(x,y),\\ Q(x,y)+\varepsilon q_2(x,y),\end{pmatrix},$$

and

$$\dot{X}=\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}-y+x^2y+\varepsilon\sum\limits_{i+j=0}^{5}a_{i,j}x^iy^j,\\ x+xy^2+\varepsilon\sum\limits_{i+j=0}^{5}b_{i,j}x^iy^j,\end{pmatrix}=\begin{pmatrix}P(x,y)+\varepsilon p_3(x,y),\\ Q(x,y)+\varepsilon q_3(x,y),\end{pmatrix},\tag{3.3.7}$$

with $|\varepsilon|\neq 0$ a positive real number sufficiently small. In what follows, we state the main results.

Theorem 3.3.1 *[56] For $|\varepsilon|\neq 0$ real number sufficiently small, using the averaging method of first order, the maximum number of limit cycles that can bifurcate from the isochronous center $(3.3.5)_{\varepsilon=0}$ is three. In addition, there are perturbations with only $0,1,2$ and 3 limit cycles bifurcate from the isochronous center $(3.3.5)_{\varepsilon=0}$.*

Theorem 3.3.2 *[56] For $|\varepsilon|\neq 0$ sufficiently small, the discontinuous system (3.3.6) can have at least 9 limit cycles bifurcating from the isochronous center $(3.3.6)_{\varepsilon=0}$ using the first order averaging method. Moreover, there are perturbations with at least $0,1,2,\ldots,9$ limit cycles bifurcate from the isochronous center $(3.3.6)_{\varepsilon=0}$.*

Other remarkable results can be obtained from Theorems 3.3.1 and 3.3.2.

Corollary 3.3.1 *[56] Let $p_1(x,y)$, $p_2(x,y)$, $q_1(x,y)$ and $q_2(x,y)$ be real homogeneous polynomials of degree d with $1\leq d\leq 4$ (i.e. Polynomials with only the terms of degree d).*
For system *(3.3.5), by using the averaging method of first order,*

- *If* $d=4$ *or* $d=2$*, then* $F_{10}(r)$ *is identically zero for all* $r\in(0,+\infty)$*, then no information can be provided about limit cycles of system* (3.3.5).
- *If* $d=3$*, then system* (3.3.5) *has at most two limit cycles .*
- *If* $d=1$*, then system* (3.3.5) *has at most one limit cycles.*

For system (3.3.6)*, by using averaging method of first order we have*

- *For* $d=4$ *system* (3.3.6) *at least five limit cycles.*
- *For* $d=3$ *system* (3.3.6) *at most two limit cycles.*
- *For* $d=2$ *system* (3.3.6) *at least three limit cycles.*
- *For* $d=1$ *system* (3.3.6) *at most one limit cycle.*

Corollary 3.3.2 *[56] Let* $p_1(x,y)$*,* $p_2(x,y)$*,* $q_1(x,y)$ *and* $q_2(x,y)$ *be real polynomials. Using the averaging method of first order, we have*

- *If system* (3.3.5) *is of degree* 3 *and system* (3.3.6) *is of degree* 4 *then system* (3.3.6) *has at least six more limit cycles than system* (3.3.5).
- *If system* (3.3.5) *is of degree* 4 *and system* (3.3.6) *is of degree* 3 *then system* (3.3.6) *has at least four more limit cycles than system* (3.3.5).
- *If* $p_1(x,y), q_1(x,y)$ *are of degree* 3 *and* $p_2(x,y), q_2(x,y)$ *are of degree* 4 *or* $p_1(x,y), q_1(x,y)$ *are of degree* 4 *and* $p_2(x,y), q_2(x,y)$ *are of degree* 3*, then system* (3.3.6) *can have at least* 8 *limit cycles.*

Theorem 3.3.3 *[54] Let* $\varepsilon\neq 0$ *be a real parameter sufficiently small,*

(a) *using the averaging method of first order, system* (3.3.7) *can have up to* 5 *limit cycles that bifurcate from the isochronous center* $(3.3.7)_{\varepsilon=0}$.

(b) *by using the averaging method of second order, the polynomial differential system* (3.3.7) *has at most* 19 *limit cycles bifurcating from the isochronous center* $(3.3.7)_{\varepsilon=0}$.

In order to prove our results, the following theorem provides a way to transform a perturbed differential system into the standard form for applying the averaging method.

Theorem 3.3.4 *(see[8]) Consider the following planar differential system*

$$\begin{aligned} \dot{x} &= P(x,y)+\varepsilon p(x,y), \\ \dot{y} &= Q(x,y),+\varepsilon q(x,y). \end{aligned} \tag{3.3.8}$$

Assume that

(a_1) *System* $(3.3.8)_{\varepsilon=0}$ *has a continuous family of ovals*

$$\{\Gamma_h\} \subset \left\{(x,y) : H(x,y) = h, h_1 < h < h_2\right\},$$

where H *is the first integral of* $(3.3.8)_{\varepsilon=0}$.

(a_2) $xQ(x,y) - yP(x,y) \neq 0$ *for all* (x,y) *in the period annulus formed by the ovals* $\{\Gamma_h\}$. *Let* $\rho : \left(\sqrt{h_1}, \sqrt{h_2}\right) \times [0,2\pi) \to [0,+\infty)$ *be a continuous function such that*

$$H(\rho(R,\phi)\cos\phi, \rho(R,\phi)\sin\phi) = R^2, \tag{3.3.9}$$

for all $R \in (\sqrt{h_1}, \sqrt{h_2})$ *and all* $\phi \in [0,2\pi)$.

Then the differential equation which describes the dependence between the square root of energy, $R = \sqrt{h}$*, and the angle* ϕ *for system* (3.3.8) *is*

$$\frac{dR}{d\phi} = \frac{\varepsilon}{2R} \frac{\mu(x^2+y^2)(Qp - Pq)}{xQ - yP + \varepsilon(xq - yp)}, \tag{3.3.10}$$

with $x = \rho(R,\phi)\cos\phi$ *and* $y = \rho(R,\phi)\sin\phi$.

T determine the number of the zeros of the averaged functions (2.3.6) and (2.3.15), the following results will be used.

Lemma 3.3.1 *(Descartes Theorem [1]) Consider the polynomial* $p\left(y\right) = a_{i_1} y^{i_1} + a_{i_2} y^{i_2} + \cdots + a_{i_n} y^{i_n}$ *with* $0 \leq i_1 < i_2 < \cdots < i_n$ *and the real constant* $a_{i_k} \neq 0 \ \forall k \in \{1,2,\ldots,n\}$. *When* $a_{i_k} a_{i_{k+1}} < 0$, *we say that* a_{i_k} *and* $a_{i_{k+1}}$ *have a variation of sign. If the number of the variations are* $m \in \mathbb{N}$, *then* $p\left(y\right)$ *has at most* m *positive real roots. Moreover, it is always possible to choose the coefficients of* $p\left(y\right)$ *in such a way that* $p\left(y\right)$ *has exactly* $n-1$ *positive real roots.*

Lemma 3.3.2 *(see [45]) Let A be a set and let $f_1, f_2, \ldots, f_n : A \to \mathbb{R}$ be linearly independent functions, then there exist $a_1, a_2, \ldots, a_{n-1} \in A$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ such that $\forall i \in \{1, 2, \ldots, n-1\}$*

$$\sum_{k=1}^{n} \alpha_k f_k(a_i) = 0.$$

3.4 Proofs of Results

3.4.1 Proof of Theorem 3.3.1

By Lemma 3.3.4, $\rho(\theta, r) = \dfrac{r}{\sqrt{1 + r^2\cos^2\theta}}$ with $r \in (0, +\infty)$ is the function ρ satisfying the assumptions (a_1) and (a_2). Taking $x = \dfrac{r\cos\theta}{\sqrt{1 + r^2\cos^2\theta}}$ and $y = \dfrac{r\sin\theta}{\sqrt{1 + r^2\cos^2\theta}}$, we can transform system (3.3.5) into the form

$$\frac{dr}{d\theta} = \varepsilon F(\theta, r) + O(\varepsilon^2), \tag{3.4.11}$$

where

$$F(\theta, r) = \sqrt{1 + r^2\cos^2\theta}\left((1 + r^2)\cos\theta\,\bar{p}(\theta, r) + \sin\theta\,\bar{q}(\theta, r)\right),$$

with

$$\bar{p}(\theta, r) = p\Big(\frac{r\cos\theta}{\sqrt{1 + r^2\cos^2\theta}}, \frac{r\sin\theta}{\sqrt{1 + r^2\cos^2\theta}}\Big) \text{ and } \bar{q}(\theta, r) = q\Big(\frac{r\cos\theta}{\sqrt{1 + r^2\cos^2\theta}}, \frac{r\sin\theta}{\sqrt{1 + r^2\cos^2\theta}}\Big).$$

The assumptions of Theorem 2.3.1 are verified for the equation (3.4.30). Then by direct calculations, $F_{10}(r)$ is given by

$$F_{10}(r) = \frac{1}{2r}\left(A_1\left(\left(r^2 + 1\right)^{3/2} - 1\right) + A_2\left(\sqrt{r^2 + 1} - 1\right) + \left(A_3\left(r^2 + 1\right) + A_4\right) r^2\right), \tag{3.4.12}$$

where

$$\begin{aligned} A_1 &= 2\,b_{0,3} - 2\,a_{1,2}, \\ A_2 &= 2\,a_{3,0} - 2\,b_{2,1}, \\ A_3 &= a_{1,0} + a_{3,0} + a_{1,2}, \\ A_4 &= -2\,a_{3,0} + b_{0,1} - 3\,b_{0,3} + 2\,a_{1,2} + b_{2,1}. \end{aligned} \tag{3.4.13}$$

In order to determine the number of limit cycles of the system (3.3.5), we study the number of zeros of the function $F_{10}(r)$.

Let $t=\sqrt{1+r^2}$, note that $t\in(1,\infty)$, then $F_{10}(r)=0$ is equivalent to

$$\begin{aligned} f(t) &= \left(a_{3,0}+a_{1,2}+a_{1,0}\right)t^4+\left(2\,b_{0,3}-2\,a_{1,2}\right)t^3+\left(+a_{1,2}-a_{1,0}-3\,a_{3,0}-3\,b_{0,3}\right. \\ &\quad \left.+b_{2,1}+b_{0,1}\right)t^2-2\left(b_{2,1}-a_{3,0}\right)t+b_{0,3}+b_{2,1}-b_{0,1}=0. \end{aligned} \tag{3.4.14}$$

From the above expression of the function $f(t)$, it is clear that $f(t)$ is generated by a linear combination of the set $\mathscr{S}=\left\{1,t,t^2,t^3,t^4\right\}$. Using Lemma 3.3.1 (Descartes Theorem), it follows that $f(t)$ can have at most four simple zeros and because 1 is a zero of $f(t)$, then the function $f(t)$ has at most 3 zeros in $(1,+\infty)$. Hence the first order averaged function $F_{10}(r)$ has at most 3 zeros in $(0,+\infty)$. As a consequence, we conclude that there are three limit cycles bifurcating from the period annulus around the isochronous center for (3.3.5) with $|\varepsilon|\neq 0$ a real parameter sufficiently small.

In this part, we provide conditions on the coefficients $a_{i,j}$ and $b_{i,j}$ in a way that the system (3.3.5) can have exactly 3 limit cycles.

In the first step, we suppose that the function (3.4.12) has three zeros $R_i=i,\ i=1,2,3$. By substituting all R_i in (3.4.12), we obtain 3 linear equations for variables $A_j,\ j=1,2,3,4$.

$$\begin{cases} \left(2\sqrt{2}-1\right)A_1+\left(\sqrt{2}-1\right)A_2+2\,A_3+A_4=0, \\ \left(5\sqrt{5}-1\right)A_1+\left(\sqrt{5}-1\right)A_2+20\,A_3+4\,A_4=0, \\ \left(10\sqrt{10}-1\right)A_1+\left(\sqrt{10}-1\right)A_2+90\,A_3+9\,A_4=0. \end{cases} \tag{3.4.15}$$

Solving this system of equations we get

$$\begin{aligned} A_1 &= 1, \\ A_2 &= \sqrt{10}, \\ A_3 &= -\frac{1}{4}(\sqrt{5}-1)(\sqrt{2}-1), \\ A_4 &= \tfrac{1}{2}\Big((3\sqrt{5}-5)\sqrt{2}-5\sqrt{5}+3\Big). \end{aligned} \tag{3.4.16}$$

Substituting (3.4.16) in (3.4.13) we get

$$\begin{aligned} a_{1,2} &= -1/2+b_{0,3}, \\ a_{3,0} &= \sqrt{5/2}+b_{2,1}, \\ a_{1,0} &= 1/4\Big(\left(-3\sqrt{5}+1\right)\sqrt{2}+\sqrt{5}+1\Big)-b_{2,1}-b_{0,3}, \\ b_{0,1} &= 5/2\left(\sqrt{5}-1\right)\left(\sqrt{2}-1\right)+b_{0,3}+b_{2,1}. \end{aligned} \tag{3.4.17}$$

For a numerical application, we can get $a_{1,2}=-1/2,\ a_{3,0}=\sqrt{5/2},\ a_{1,0}=1/4\big((-3\sqrt{5}+1)\sqrt{2}+\sqrt{5}+1\big),\ b_{0,1}=5/2\big(\big(\sqrt{5}-1\big)(\sqrt{2}-1)\big)$ and all other coefficients can take any real numbers. In this case the averaged function of system (3.3.5) is given by

$$\begin{aligned} F_{10}(r) \ &= \frac{1}{2r}\Big(\big(r^2+1\big)^{3/2}-1+\sqrt{10}\big(\sqrt{r^2+1}-1\big)+\big(-1/4\left(\sqrt{5}-1\right)\big(\sqrt{2}-1\big)\big(r^2+1\big)\\ &+1/2(\sqrt{2}\left(3\sqrt{5}-5\right)-5\sqrt{5}+3)\big)\,r^2\Big). \end{aligned} \tag{3.4.18}$$

The function $F_{10}(r)$ defined in (3.4.18) has exactly 3 zeros $R_1=1,\ R_2=2,\ R_3=3$. $F_{10}(r)$ is shown in Fig.4.5.1 .

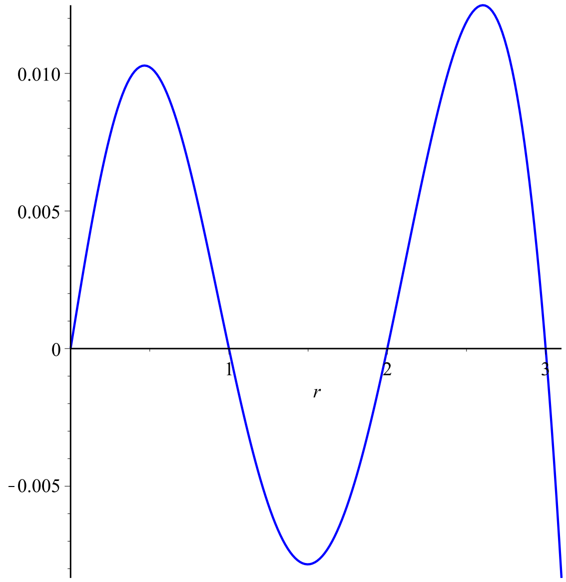


Figure 3.4.1 : Function $F_{10}(r)$ of system (3.3.5) with the conditions (3.4.17).

Moreover, we have

$$\frac{dF_{10}(R_1)}{dr}=-0.269301180,\quad \frac{dF_{10}(R_2)}{dr}=0.0269301179,\quad \frac{dF_{10}(R_3)}{dr}=-0.708524383.$$

As a consequence, by applying the first order averaging method for continuous differential systems, system (3.3.5) under conditions (3.4.17) has exactly three limit cycles (shown in Fig. 3.4.2) bifurcating from the periodic annulus of the isochronous center $(3.3.5)_{\varepsilon=0}$.

This completes the proof of Theorem 3.3.1.

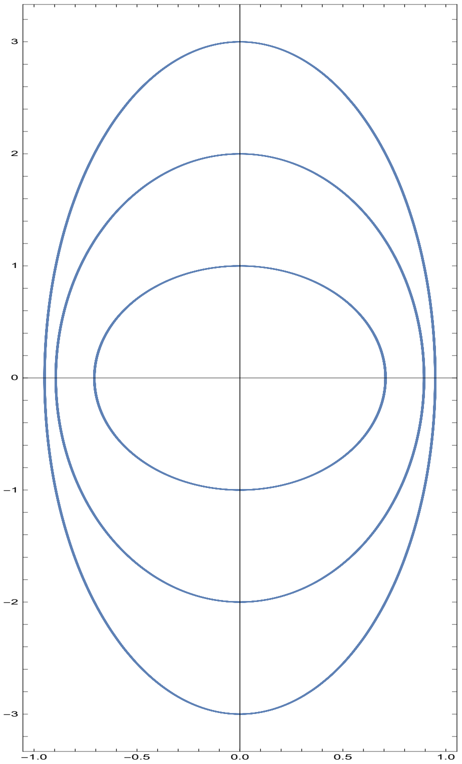


Figure 3.4.2 : For $\varepsilon = 10^{-5}$, system (3.3.5) under the conditions (3.4.17) has exactly 3 limit cycles.

3.4.2 Proof of Theorem 3.3.2

Now, using Theorem 2.3.2, we will prove Theorem 3.3.2. From Lemma 3.3.4, we take $x = \dfrac{r\cos\theta}{\sqrt{1+r^2\cos^2\theta}}$ and $y = \dfrac{r\sin\theta}{\sqrt{1+r^2\cos^2\theta}}$. Then system (3.3.6) can be written as

$$\frac{dr}{d\theta} = \varepsilon\Big(F_1(\theta,r) + sign(y)F_2(\theta,r)\Big), \tag{3.4.19}$$

where

$$F_1(\theta,r) = \frac{1}{2}\left(Y_1^1 + Y_2^1\right) \text{ and } F_2(\theta,r) = \frac{1}{2}\left(Y_1^1 - Y_2^1\right),$$

with

$$Y_1^1 = \sqrt{1+r^2\cos^2\theta}\Big((1+r^2)\cos\theta\,\bar{p}_1(\theta,r) + \sin\theta\,\bar{q}_1(\theta,r)\Big),$$

$$Y_2^1 = \sqrt{1+r^2\cos^2\theta}\Big((1+r^2)\cos\theta\,\bar{p}_2(\theta,r) + \sin\theta\,\bar{q}_2(\theta,r)\Big),$$

and the polynomials $\bar{p}_i(\theta,r)$ and $\bar{q}_i(\theta,r)$ with $i = 1,2$ are expressed as

$$\bar{p}_i(\theta,r) = p_i\Big(\frac{r\cos\theta}{\sqrt{1+r^2\cos^2\theta}}, \frac{r\sin\theta}{\sqrt{1+r^2\cos^2\theta}}\Big),$$

and

$$\bar{q}_i(\theta,r)=q_i\Big(\frac{r\cos\theta}{\sqrt{1+r^2\cos^2\theta}},\frac{r\sin\theta}{\sqrt{1+r^2\cos^2\theta}}\Big).$$

The assumptions of Theorem 2.3.2 are verified for System (3.4.19). As a result, we can compute the first order averaged function $F_{10}(r)$.

Some integrals which appear in $F_{10}(r)$ are listed in the following lemma.

Lemma 3.4.1 *The following integrals hold*

$$I_0=\int \sin\theta\sqrt{1+r^2\cos^2\theta}\,d\theta=-\frac{1}{2}\cos\theta\sqrt{1+r^2\cos^2\theta}-\frac{1}{2r}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big),$$

$$I_1=\int \frac{\sin\theta}{\sqrt{1+r^2\cos^2\theta}}d\theta=-\frac{1}{r}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big),$$

$$I_2=\int \frac{\sin\theta\cos^2\theta}{\sqrt{1+r^2\cos^2\theta}}d\theta=\frac{1}{2r^3}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)-\frac{1}{2r^2}\cos\theta\sqrt{1+r^2\cos^2\theta},$$

$$I_3=\int \frac{\sin\theta\cos^4\theta}{(1+r^2\cos^2\theta)^{3/2}}d\theta=\frac{3}{2r^5}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)-\frac{1}{2r^4}\cos\theta\sqrt{1+r^2\cos^2\theta}$$
$$-\frac{\cos\theta}{r^4\sqrt{1+r^2\cos^2\theta}}.$$

Proof. We can easily compute these integrals. For I_2, it is clear that $I_2=\frac{1}{r^2}(I_0-I_1)$.
Now we compute the integral I_3. We first take the derivative of I_2 with respect to r. We obtain $I_3(r)=-\frac{1}{r}I_2'$.
The rest of the integrals that appear in the expression of $F_{10}(r)$ can be computed in a similar way. ■

Now the first order averaged function of system (3.3.6) is

$$
\begin{aligned}
F_{10}(r) \;&= \frac{1}{2r\sqrt{1+r^2}}\Big(4\left(a_{1,3}-c_{1,3}\right)g_1(r)+4\left(b_{2,2}-d_{2,2}\right)g_2(r)+2\left(-4\,b_{0,4}-2\,d_{0,2}\right.\\
&\left.+4\,d_{0,4}-a_{1,1}-3\,a_{3,1}+c_{1,1}+3\,c_{3,1}+2\,b_{0,2}\right)g_3(r)+2\left(3\,d_{4,0}-d_{0,0}+b_{0,0}\right.\\
&\left.-b_{0,2}+b_{0,4}-b_{2,0}-3\,b_{4,0}-d_{0,4}+d_{0,2}+d_{2,0}\right)g_4(r)+\pi\left(c_{1,0}+c_{3,0}+a_{1,2}\right.\\
&\left.+c_{1,2}+a_{3,0}+a_{1,0}\right)g_5(r)+\pi\left(-2\,c_{3,0}+d_{0,1}-3\,d_{0,3}+2\,a_{1,2}+b_{2,1}+2\,c_{1,2}\right.\\
&\left.+d_{2,1}-2\,a_{3,0}+b_{0,1}-3\,b_{0,3}\right)g_6(r)-2\,\pi\left(-c_{3,0}-d_{0,3}+a_{1,2}+b_{2,1}+c_{1,2}\right.\\
&\left.+d_{2,1}-a_{3,0}-b_{0,3}\right)g_7(r)+2\,\pi\left(d_{0,3}-a_{1,2}-c_{1,2}+b_{0,3}\right)g_8(r)+2\left(-3\,a_{1,3}\right.\\
&\left.+a_{1,1}+a_{3,1}-c_{1,1}+3\,c_{1,3}-c_{3,1}+2\,b_{0,4}-2\,d_{0,4}\right)g_9(r)+2\left(-3\,a_{1,3}+3\,a_{3,1}\right.\\
&-d_{2,0}-d_{4,0}+a_{1,1}-3\,b_{2,2}-c_{1,1}+3\,c_{1,3}-3\,c_{3,1}+3\,d_{2,2}+b_{0,0}-d_{0,0}-b_{0,2}\\
&\left.+3\,b_{0,4}+b_{2,0}+b_{4,0}+d_{0,2}-3\,d_{0,4}\right)g_{10}(r)+2\left(-2\,d_{4,0}+2\,b_{4,0}\right)g_{11}(r)\Big),
\end{aligned}
\tag{3.4.20}
$$

where

$$
\begin{aligned}
g_1(r) &= \left(r^2+1\right)^{3/2}\left(r^2+3/2\right)\ln\left(r+\sqrt{r^2+1}\right),\\
g_2(r) &= \sqrt{r^2+1}\left(r^2+3/2\right)\ln\left(r+\sqrt{r^2+1}\right),\\
g_3(r) &= \left(r^2+1\right)^{3/2}\ln\left(r+\sqrt{r^2+1}\right),\\
g_4(r) &= \sqrt{r^2+1}\ln\left(r+\sqrt{r^2+1}\right),\\
g_5(r) &= r^2\left(r^2+1\right)^{3/2},\\
g_6(r) &= \left(r^2+1-\sqrt{r^2+1}\right),\\
g_7(r) &= r^2\sqrt{r^2+1},\\
g_8(r) &= r^2\left(r^2+1\right),\\
g_9(r) &= r^3\left(r^2+1\right),\\
g_{10}(r) &= r\left(r^2+1\right),\\
g_{11}(r) &= r.
\end{aligned}
$$

Due to the equality

$$g_4(r) = 2(g_2(r)-g_3(r)),$$

we can rewrite the function $F_{10}(r)$ as

$$
\begin{aligned}
F_{10}(r) = \frac{1}{2r\sqrt{1+r^2}}\Big(\; & \alpha_1 g_1(r)+\alpha_2 g_2(r)+\alpha_3 g_3(r)+\alpha_5 g_5(r)+\alpha_6 g_6(r)+\alpha_7 g_7(r)\\
& +\alpha_8 g_8(r)+\alpha_9 g_9(r)+\alpha_{10} g_{10}(r)+\alpha_{11} g_{11}(r)\Big),
\end{aligned}
\tag{3.4.21}
$$

where

$$
\begin{aligned}
\alpha_1 = \;& 4\left(a_{1,3}-c_{1,3}\right),\\
\alpha_2 = \;& 4\left(b_{2,2}-d_{2,2}+3\,d_{4,0}+b_{0,0}-d_{0,0}-b_{0,2}+b_{0,4}-b_{2,0}-3\,b_{4,0}+d_{0,2}\right.\\
& \quad \left.-d_{0,4}+d_{2,0}\right),\\
\alpha_3 = \;& 2\left(-6\,b_{0,4}-4\,d_{0,2}+6\,d_{0,4}-a_{1,1}-3\,a_{3,1}+c_{1,1}+3\,c_{3,1}+4\,b_{0,2}-6\,d_{4,0}\right.\\
& \left.-2\,b_{0,0}+2\,d_{0,0}+2\,b_{2,0}+6\,b_{4,0}-2\,d_{2,0}\right),\\
\alpha_5 = \;& \pi\left(c_{1,0}+c_{3,0}+a_{1,2}+c_{1,2}+a_{1,0}+a_{3,0}\right),\\
\alpha_6 = \;& -2\,\pi\left(-c_{3,0}-d_{0,3}+a_{1,2}+b_{2,1}+c_{1,2}+d_{2,1}-a_{3,0}-b_{0,3}\right),\\
\alpha_7 = \;& \pi\left(-2\,c_{3,0}+d_{0,1}-3\,d_{0,3}+2\,a_{1,2}+b_{2,1}+2\,c_{1,2}+d_{2,1}-2\,a_{3,0}+b_{0,1}-3\,b_{0,3}\right),\\
\alpha_8 = \;& 2\,\pi\left(d_{0,3}-a_{1,2}-c_{1,2}+b_{0,3}\right),\\
\alpha_9 = \;& 2\left(a_{1,1}-3\,a_{1,3}+a_{3,1}-c_{1,1}+3\,c_{1,3}-c_{3,1}+2\,b_{0,4}-2\,d_{0,4}\right),\\
\alpha_{10} = \;& 2\left(-d_{2,0}-d_{4,0}+a_{1,1}-3\,a_{1,3}+3\,a_{3,1}-3\,b_{2,2}-c_{1,1}+3\,c_{1,3}-3\,c_{3,1}+3\,d_{2,2}\right.\\
& \left.+b_{0,0}-d_{0,0}-b_{0,2}+3\,b_{0,4}+b_{2,0}+b_{4,0}+d_{0,2}-3\,d_{0,4}\right),\\
\alpha_{11} = \;& 2\left(-2\,d_{4,0}+2\,b_{4,0}\right).
\end{aligned}
\tag{3.4.22}
$$

Note that the coefficients $a_{i,j},\ b_{i,j}$ in the function $F_{10}(r)$ can be selected freely. Based on (3.4.21), the averaged function $F_{10}(r)$ is generated by ten linearly independent functions $g_i(r)$, $i = 1,2,3,5,6,7,8,9,10,11$. According to Lemma 3.3.2, there exist α_i such that $F_{10}(r)$ has at least 9 zeros, $r_j \in (0,\infty),\ j \in \{1,2,\dots,9\}$. Applying Theorem 2.3.2, we conclude from this analysis that we can choose the coefficients $a_{i,j},\ b_{i,j},\ c_{i,j}$ and $d_{i,j}$ such that the discontinuous differential system (3.3.6) has at least *nine* limit cycles.

In order to verify this result, we will provide conditions on coefficients $a_{i,j},\ b_{i,j},\ c_{i,j}$ and $d_{i,j}$ in such a way that the averaged function $F_{10}(r)$ of the discontinuous differential system (3.3.6) can have at least *nine* zeros in $(0,+\infty)$.

We have

$$F_{10}(r) = \frac{1}{2r\sqrt{1+r^2}}f(r),$$

with

$$
\begin{aligned}
f(r) = \;& \alpha_1 g_1(r)+\alpha_2 g_2(r)+\alpha_3 g_3(r)+\alpha_5 g_5(r)+\alpha_6 g_6(r)+\alpha_7 g_7(r)\\
& +\alpha_8 g_8(r)+\alpha_9 g_9(r)+\alpha_{10} g_{10}(r)+\alpha_{11} g_{11}(r).
\end{aligned}
$$

Suppose that $f(r)$ has at least 9 zeros which are $r_j = \frac{j}{5}$ with $j = 1, 2, \ldots, 9$. By substituting all r_j in (3.4.21) we obtain 9 linear equations with 10 variables $\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}$.

$$
\begin{cases}
0.3245241206\,\alpha_1 + 0.3120424235\,\alpha_2 + 0.2107299485\,\alpha_3 + 0.04242384236\,\alpha_5 \\
+0.020196097\,\alpha_6 + 0.04079215612\,\alpha_7 + 0.0416\,\alpha_8 + 0.00832\,\alpha_9 + 0.208\,\alpha_{10} + 0.2\,\alpha_{11} = 0, \\
0.8089077709\,\alpha_1 + 0.6973342857\,\alpha_2 + 0.4872938380\,\alpha_3 + 0.1998973176\,\alpha_5 \\
+0.082967039\,\alpha_6 + 0.1723252738\,\alpha_7 + 0.1856\,\alpha_8 + 0.07424\,\alpha_9 + 0.464\,\alpha_{10} + 0.4\,\alpha_{11} = 0, \\
1.678030712\,\alpha_1 + 1.233846111\,\alpha_2 + 0.9021670492\,\alpha_3 + 0.5709668096\,\alpha_5 \\
+0.193809621\,\alpha_6 + 0.4198285364\,\alpha_7 + 0.4896\,\alpha_8 + 0.29376\,\alpha_9 + 0.816\,\alpha_{10} + 0.6\,\alpha_{11} = 0, \\
3.292963525\,\alpha_1 + 2.007904590\,\alpha_2 + 1.538768002\,\alpha_3 + 1.344143839\,\alpha_5 \\
+0.359375153\,\alpha_6 + 0.8195999021\,\alpha_7 + 1.0496\,\alpha_8 + 0.83968\,\alpha_9 + 1.312\,\alpha_{10} + 0.8\,\alpha_{11} = 0, \\
6.232252400\,\alpha_1 + 3.116126200\,\alpha_2 + 2.492900960\,\alpha_3 + 2.828427124\,\alpha_5 \\
+0.585786438\,\alpha_6 + 1.414213562\,\alpha_7 + 2.0\,\alpha_8 + 2.0\,\alpha_9 + 2.0\,\alpha_{10} + \alpha_{11} = 0, \\
11.38450871\,\alpha_1 + 4.665782258\,\alpha_2 + 3.872281873\,\alpha_3 + 5.488418652\,\alpha_5 \\
+0.877950065\,\alpha_6 + 2.249351906\,\alpha_7 + 3.5136\,\alpha_8 + 4.21632\,\alpha_9 + 2.928\,\alpha_{10} + 1.2\,\alpha_{11} = 0, \\
20.05160198\,\alpha_1 + 6.774189860\,\alpha_2 + 5.795260686\,\alpha_3 + 9.981450051\,\alpha_5 \\
+1.239534947\,\alpha_6 + 3.372111504\,\alpha_7 + 5.8016\,\alpha_8 + 8.12224\,\alpha_9 + 4.144\,\alpha_{10} + 1.4\,\alpha_{11} = 0, \\
34.06102174\,\alpha_1 + 9.567702741\,\alpha_2 + 8.389414222\,\alpha_3 + 17.19550609\,\alpha_5 \\
+1.673203774\,\alpha_6 + 4.830198339\,\alpha_7 + 9.1136\,\alpha_8 + 14.58176\,\alpha_9 + 5.696\,\alpha_{10} + 1.6\,\alpha_{11} = 0, \\
55.88595255\,\alpha_1 + 13.18064919\,\alpha_2 + 11.79028535\,\alpha_3 + 28.28744972\,\alpha_5 \\
+2.180873972\,\alpha_6 + 6.671568331\,\alpha_7 + 13.7376\,\alpha_8 + 24.72768\,\alpha_9 + 7.632\,\alpha_{10} + 1.8\,\alpha_{11} = 0.
\end{cases}
\tag{3.4.23}
$$

Solving these equations we get

$$\begin{aligned}
\alpha_1 &= 0.1035131340,\\
\alpha_2 &= -4.659739831,\\
\alpha_3 &= 4.950983302,\\
\alpha_5 &= 0.4763226687,\\
\alpha_6 &= 6.536587951,\\
\alpha_7 &= -4.729222571,\\
\alpha_8 &= 0.9923827304,\\
\alpha_9 &= -0.8528463864,\\
\alpha_{10} &= 0.8828379259,\\
\alpha_{11} &= 1.
\end{aligned} \tag{3.4.24}$$

By substituting (3.4.24) in (3.4.22), we get

$$\begin{aligned}
b_{4,0} &= 1/4 + d_{4,0},\\
a_{1,3} &= 0.02587828350 + c_{1,3},\\
b_{0,0} &= -0.0001297048 + d_{0,0},\\
d_{0,1} &= 0.007776804\,\pi^{-1} - c_{1,0} - a_{1,0} - b_{0,1},\\
c_{3,0} &= 2.772102611\,\pi^{-1} - a_{3,0} + b_{2,1} + d_{2,1},\\
a_{3,1} &= 0.5163885572 + c_{3,1} - 2\,d_{2,2} - b_{2,0} + d_{2,0} + 2\,b_{2,2},\\
a_{1,1} &= -0.0355663934 - b_{2,0} + d_{2,0} + c_{1,1} + 2\,d_{0,2} - 2\,b_{0,2},\\
d_{0,3} &= -1.799588577\,\pi^{-1} - c_{1,0} - a_{1,0} - b_{2,1} - d_{2,1} - b_{0,3},\\
a_{1,2} &= -2.295779942\,\pi^{-1} - c_{1,0} - c_{1,2} - a_{1,0} - b_{2,1} - d_{2,1},\\
d_{0,4} &= 0.4148052532 + b_{0,4} + d_{0,2} - b_{2,0} + d_{2,0} - b_{0,2} - d_{2,2} + b_{2,2}.
\end{aligned} \tag{3.4.25}$$

Let consider the discontinuous system (3.3.6) with conditions (3.4.25), then the function

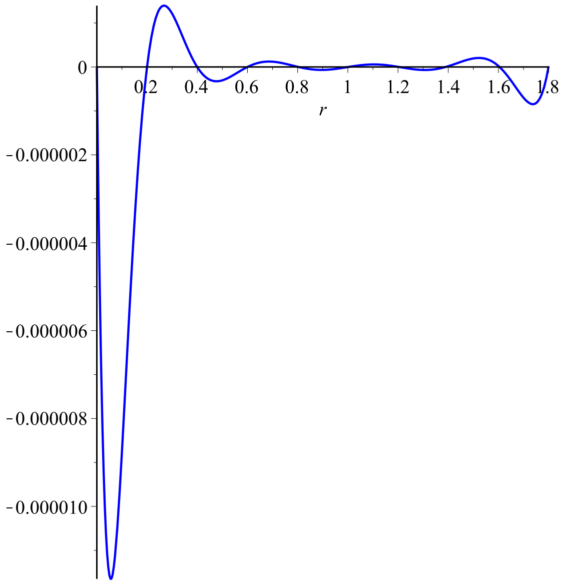


Figure 3.4.3 : Function $f(r)$ of system (3.3.6) with conditions (3.4.25).

$f(r)$ is

$$\begin{aligned} f(r) \;&= \frac{1}{2r\sqrt{1+r^2}}\Big(0.1035131340\left(r^2+1\right)^{3/2}\left(r^2+3/2\right)\ln\!\left(r+\sqrt{r^2+1}\right) \\ &-4.659739831\sqrt{r^2+1}(r^2+3/2)\ln\!\left(r+\sqrt{r^2+1}\right) \\ &+4.950983302(r^2+1)^{3/2}\ln\!\left(r+\sqrt{r^2+1}\right) \\ &+0.4763226687r^2(r^2+1)^{3/2}+0.9923827304r^2(r^2+1) \\ &-4.729222571r^2\sqrt{r^2+1}+6.536587951(r^2+1-\sqrt{r^2+1}) \\ &-0.8528463864r^3(r^2+1)+0.8828379259r(r^2+1)+r\Big). \end{aligned} \tag{3.4.26}$$

The function $f(r)$ defined in (3.4.26) has at least 9 zeros which are $r_j = j/5$ with $j = 1, 2, \ldots, 9$. $f(r)$ is shown in Fig. 4.4.12.

In addition we have

$$\frac{df(0.2)}{dr} = 0.00004685331,\ \frac{df(0.4)}{dr} = -0.00000895994,\ \frac{df(0.6)}{dr} = 0.000002830879,$$

$$\frac{df(0.8)}{dr} = -0.0000013685,\ \frac{df(1)}{dr} = 0.0000009124,\ \frac{df(1.2)}{dr} = -0.0000009714,$$

$$\frac{df(1.4)}{dr} = 0.000001740,\ \frac{df(1.6)}{dr} = -0.000005055,\ \frac{df(1.8)}{dr} = 0.00003348.$$

The assumptions of Theorem 2.3.2 are satisfied. Thus, from Theorem 2.3.2, system (3.3.6) has at least nine limit cycles surrounding the origin with $|\varepsilon| \neq 0$ is a real parameter sufficiently small. This completes the proof of 3.3.2.

3.4.3 Proof of Corollaries 3.3.1 and 3.3.2

In this part, we introduce the proof of corollaries 3.3.1 and 3.3.2 respectively.

Proof of Corollary 3.3.1

Here, only the cases $d = 4$ for system (3.3.6) and $d = 3$ for system (3.3.5) and (3.3.6) will be proved. The other cases can be proved in a similar way.

For $d = 4$: In this case, we consider the polynomials $p_1(x,y), p_2(x,y), q_1(x,y)$ and $q_2(x,y)$ (3.3.6) with only the terms of degree 4. By (3.4.21), the averaged function $F_{10}(r)$ of system (3.3.6) is

$$\begin{aligned} F_{10}(r) = \; & \frac{1}{2r\sqrt{1+r^2}}\Big(4\left(a_{1,3}-c_{1,3}\right)g_1(r)+4\left(b_{2,2}-d_{2,2}+3\,d_{4,0}+b_{0,4}-3\,b_{4,0}-d_{0,4}\right)g_2(r) \\ & +6\left(-2\,b_{0,4}+2\,d_{0,4}-a_{3,1}+c_{3,1}-2\,d_{4,0}+2\,b_{4,0}\right)g_3(r)+2\left(-3\,a_{1,3}+a_{3,1}+3\,c_{1,3}\right. \\ & \left.-c_{3,1}+2\,b_{0,4}-2\,d_{0,4}\right)g_9(r)+2\left(-d_{4,0}-3\,a_{1,3}+3\,a_{3,1}-3\,b_{2,2}+3\,c_{1,3}-3\,c_{3,1}\right. \\ & \left.+3\,d_{2,2}+3\,b_{0,4}+b_{4,0}-3\,d_{0,4}\right)g_{10}(r)+4\left(-d_{4,0}+b_{4,0}\right)g_{11}(r)\Big). \end{aligned} \tag{3.4.27}$$

The function $F_{10}(r)$ is generated by 6 linearly independent functions $g_i(r),\ i = 1,2,3,9,10,11$. By Lemma 3.3.2, there exist α_i such that $F_{10}(r)$ has at least 5 zeros $r_j \in (0,\infty),\ j = 1,2,3,4,5$. Applying Theorem 2.3.2, we conclude that the discontinuous differential system (3.3.6) can have at least 5 limit cycles.

For $d = 3$: $p_1(x,y), q_1(x,y), p_2(x,y)$ and $q_2(x,y)$ contain only the terms of degree 3.

- From (3.4.12), the averaged function $F_{10}(r)$ of system (3.3.5) can be written as

$$\begin{aligned} F_{10}(r) = \; & \frac{1}{2r}\Big((2\,b_{0,3}-2\,a_{1,2})\Big(\left(r^2+1\right)^{3/2}-1\Big)+(2\,a_{3,0}-2\,b_{2,1})\left(\sqrt{r^2+1}-1\right) \\ & +\left((a_{3,0}+a_{1,2})\left(r^2+1\right)+(-2\,a_{3,0}-3\,b_{0,3}+2\,a_{1,2}+b_{2,1})\right)r^2\Big), \end{aligned}$$

Taking $r = \sqrt{t^2-1}$, note that $t \in (1,+\infty)$, then the averaged function $F_{10}(r) = 0$ is equivalent with

$$g(t)=(t-1)^2\big((a_{1,2}+a_{3,0})\,t^2+\big(2\,a_{3,0}+2\,b_{0,3}\big)\,t+b_{0,3}+b_{2,1}\big)=0.$$

- In the case of system (3.3.6), from (3.4.21), we have

$$\begin{aligned} F_{10}(r)= \; & \frac{\pi}{2r\sqrt{1+r^2}}\Big(\big(c_{3,0}+a_{1,2}+c_{1,2}+a_{3,0}\big)g_5(r)-2\big(-c_{3,0}-d_{0,3}+a_{1,2}+b_{2,1}+c_{1,2}+d_{2,1}\\ & -a_{3,0}-b_{0,3}\big)g_6(r)+\big(2(-c_{3,0}+a_{1,2}+c_{1,2}-a_{3,0})-3(d_{0,3}\,b_{0,3})+b_{2,1}+d_{2,1}\big)g_7(r)\\ & +2\big(d_{0,3}-a_{1,2}-c_{1,2}+b_{0,3}\big)g_8(r)\Big). \end{aligned}$$

Taking $r=\sqrt{t^2-1}$ with $t\in(1,+\infty)$, then the averaged function $F_{10}(r)=0$ is equivalent with

$$\begin{aligned} h(t)= \; & t\,(t-1)^2\Big(\big(c_{3,0}+a_{1,2}+c_{1,2}+a_{3,0}\big)\,t^2+2\big(a_{3,0}+b_{0,3}+c_{3,0}+d_{0,3}\big)\,t+d_{2,1}+b_{0,3}\\ & +d_{0,3}+b_{2,1}\Big)=0. \end{aligned}$$

Using Lemma 3.3.1 (Descartes Theorem), it is obvious that the functions $g(t)$ and $h(t)$ can have at most 2 zeros in $(1,+\infty)$. Hence, the first order averaged functions $F_{10}(r)$ of (3.3.5) and (3.3.6) can have at most 2 zeros in $(0,+\infty)$. By Theorems 2.3.1 and 2.3.2 respectively, we conclude that systems (3.3.5) and (3.3.6) can have at most 2 limit cycles bifurcating from the period annulus around the isochronous center (3.3.4) with $|\varepsilon|\neq 0$ a real parameter sufficiently small.

Proof of Corollary 3.3.2

Case 1. If system (3.3.5) is of degree 3, then the averaged function is given by formula (3.4.12). From the analysis in the proof of Theorem 3.3.1, system (3.3.5) has at most 3 limit cycles bifurcating from the periodic orbits of the center (3.3.4). Comparing this result with the one in Theorem 3.3.2, we conclude that system (3.3.6) has at least 6 limit cycles more than system (3.3.5).

Case 2. In this case, system (3.3.6) has degree 3. From (3.4.21), the averaged function $F_{10}(r)$ is given by

$$\begin{aligned}F_{10}(r)=&\frac{1}{2r\sqrt{1+r^2}}\Big(4\left(b_{0,0}-d_{0,0}-b_{0,2}-b_{2,0}+d_{0,2}+d_{2,0}\right)g_2(r)+2\left(-4\,d_{0,2}-a_{1,1}+c_{1,1}+4\,b_{0,2}\right.\\&\left.-2\,b_{0,0}+2\,d_{0,0}+2\,b_{2,0}-2\,d_{2,0}\right)g_3(r)+\pi\left(c_{1,0}+c_{3,0}+a_{1,2}+c_{1,2}+a_{1,0}+a_{3,0}\right)g_5(r)\\&-2\pi\left(-c_{3,0}-d_{0,3}+a_{1,2}+b_{2,1}+c_{1,2}+d_{2,1}-a_{3,0}-b_{0,3}\right)g_6(r)+\pi\left(-2\,c_{3,0}+d_{0,1}\right.\\&\left.-3\,d_{0,3}+2\,a_{1,2}+b_{2,1}+2\,c_{1,2}+d_{2,1}-2\,a_{3,0}+b_{0,1}-3\,b_{0,3}\right)g_7(r)+2\pi\left(d_{0,3}-a_{1,2}-c_{1,2}\right.\\&\left.+b_{0,3}\right)g_8(r)+2\left(a_{1,1}-c_{1,1}\right)g_9(r)+2\left(-d_{2,0}+a_{1,1}-c_{1,1}+b_{0,0}-d_{0,0}-b_{0,2}+b_{2,0}\right.\\&\left.+d_{0,2}\right)g_{10}(r)\Big).\end{aligned}$$

The function $F_{10}(r)$ is generated by a linear combination of 8 linearly independent functions. By Lemma 3.3.2, $F_{10}(r)$ has at least 7 zeros. From Theorem 2.3.2, we conclude that the discontinuous differential system (3.3.6) can have at least 7 limit cycles. In this case, system (3.3.6) can have at least 4 limit cycles more than system (3.3.5).

Case 3.

- If $p_1(x,y), q_1(x,y)$ are of degree 3 and $p_2(x,y), q_2(x,y)$ are of degree 4, then from (3.4.21), the averaged function $F_{10}(r)$ is given by

  $$\begin{aligned}F_{10}(r)=\frac{1}{2r\sqrt{1+r^2}}\Big(\;&\alpha_2 g_2(r)+(\alpha_3-6c_{3,1}+6a_{3,1})g_3(r)+\alpha_5 g_5(r)+\alpha_6 g_6(r)+\alpha_7 g_7(r)\\&+\alpha_8 g_8(r)+(\alpha_9+6a_{1,3}-6c_{1,3}-2a_{3,1}+2c_{3,1})g_9(r)+(\alpha_{10}+6a_{1,3}\\&-6c_{1,3}-6a_{3,1}+6c_{3,1})g_{10}(r)+\alpha_{11}g_{11}(r)\Big),\end{aligned}\tag{3.4.28}$$

  with $\alpha_i,\ i=2,3,5,6,7,8,9,10,11$ are defined in (3.4.22).

- If $p_1(x,y), q_1(x,y)$ are of degree 4 and $p_2(x,y), q_2(x,y)$ are of degree 3, then from (3.4.21), the function $F_{10}(r)$ can be written as

  $$\begin{aligned}F_{10}(r)=\frac{1}{2r\sqrt{1+r^2}}\Big(\;&\alpha_1 g_1(r)+(\alpha_2-4b_{2,2}+4d_{2,2}-12d_{4,0}-4b_{0,4}+12b_{4,0}+4d_{0,4})g_2(r)\\&+(\alpha_3+12b_{0,4}-12d_{0,4}+12d_{4,0}-12b_{4,0})g_3(r)+\alpha_5 g_5(r)+\alpha_6 g_6(r)\\&+\alpha_7 g_7(r)+\alpha_8 g_8(r)+(\alpha_9-4b_{0,4}+4d_{0,4})g_9(r)+(\alpha_{10}+2d_{4,0}+6b_{2,2}\\&-6d_{2,2}-6b_{0,4}-2b_{4,0}+6d_{0,4})g_{10}(r)\Big),\end{aligned}\tag{3.4.29}$$

  with $\alpha_i,\ i=1,2,3,5,6,7,8,9,10$ defined in (3.4.22).

  From (3.4.28) and (3.4.29) respectively, the averaged functions $F_{10}(r)$ are generated by a linear combination of 9 linearly independent functions. By Lemma 3.3.2, $F_{10}(r)$ has at least 8 zeros. From Theorem 2.3.2, we conclude that the discontinuous differential system (3.3.6) can have at least 8 limit cycles.

3.4.4 Proof of Theorem 3.3.3

Proof of statement (a) of Theorem 3.3.3

In order to apply the averaging method, system (3.3.7) should be transformed into the standard form (2.3.6). Using the ideas of Lemma 3.3.4, we take

$$x = \frac{r\cos\theta}{\sqrt{1+r^2\cos^2\theta}}, \quad y = \frac{r\sin\theta}{\sqrt{1+r^2\cos^2\theta}}.$$

Then system (3.3.7) can be written as

$$\frac{dr}{d\theta} = \varepsilon F_1(\theta, r) + O(\varepsilon^2), \tag{3.4.30}$$

where

$$F_1(\theta, r) = \sqrt{1+r^2\cos^2\theta}\,\big((1+r^2)\cos\theta\,\bar{p}(\theta, r) + \sin\theta\,\bar{q}(\theta, r)\big),$$

with

$$\bar{p}(\theta, r) = p\Big(\frac{r\cos\theta}{\sqrt{1+r^2\cos^2\theta}}, \frac{r\sin\theta}{\sqrt{1+r^2\cos^2\theta}}\Big)$$

and

$$\bar{q}(\theta, r) = q\Big(\frac{r\cos\theta}{\sqrt{1+r^2\cos^2\theta}}, \frac{r\sin\theta}{\sqrt{1+r^2\cos^2\theta}}\Big).$$

The assumptions of Theorem 2.3.1 are verified for the equation (3.4.30). Then by direct computations, the expression of $f^{\circ}(r)$ is given by

$$
\begin{aligned}
f^{\circ}(r) =& \frac{1}{2\pi}\int_{0}^{2\pi} F_1(\theta, r)d\theta \\
=& \frac{1}{2r\sqrt{1+r^2}}\Big(\big(\big(a_{3,2}+a_{1,0}+a_{3,0}+a_{5,0}+a_{1,2}-3\,a_{1,4}\big)r^4+\big(5\,a_{3,2} \\
& -3\,b_{2,3}+b_{2,1}+b_{4,1}+a_{1,0}-a_{3,0}-3\,a_{5,0}+b_{0,1}-3\,b_{0,3}+5\,b_{0,5}+3\,a_{1,2} \\
& -7\,a_{1,4}\big)r^2+4\,a_{3,2}+2\,b_{2,1}-4\,b_{2,3}+4\,b_{4,1}-2\,a_{3,0}-4\,a_{5,0}-2\,b_{0,3} \\
& +4\,b_{0,5}+2\,a_{1,2}-4\,a_{1,4}\big)\sqrt{r^2+1}+\big(b_{0,5}+a_{1,4}\big)r^6+\big(-3\,a_{3,2}+b_{2,3} \\
& +2\,b_{0,3}-2\,b_{0,5}-2\,a_{1,2}+6\,a_{1,4}\big)r^4+\big(-7\,a_{3,2}-2\,b_{2,1}+5\,b_{2,3}-3\,b_{4,1} \\
& +2\,a_{3,0}+5\,a_{5,0}+4\,b_{0,3}-7\,b_{0,5}-4\,a_{1,2}+9\,a_{1,4}\big)r^2-4\,a_{3,2}-2\,b_{2,1} \\
& +4\,b_{2,3}-4\,b_{4,1}+2\,a_{3,0}+4\,a_{5,0}+2\,b_{0,3}-4\,b_{0,5}-2\,a_{1,2}+4\,a_{1,4}\Big).
\end{aligned}
\tag{3.4.31}
$$

Taking $s=\sqrt{1+r^2}$ with $s\in(1,+\infty)$, we have

$$
\begin{aligned}
f^{\circ}(s) =& \frac{1}{2s\sqrt{s^2-1}}\Big(\big(a_{1,4}+b_{0,5}\big)s^6+\big(a_{1,0}+a_{1,2}-3\,a_{1,4}+a_{3,0}+a_{3,2}+a_{5,0}\big)s^5 \\
& +\big(-2\,a_{1,2}+3\,a_{1,4}-3\,a_{3,2}+2\,b_{0,3}-5\,b_{0,5}+b_{2,3}\big)s^4+\big(-3\,a_{3,0}+3\,a_{3,2} \\
& -a_{1,0}+a_{1,2}-a_{1,4}-5\,a_{5,0}+b_{0,1}-3\,b_{0,3}+5\,b_{0,5}+b_{2,1}-3\,b_{2,3}+b_{4,1}\big)s^3 \\
& +\big(2\,a_{3,0}-a_{3,2}+5\,a_{5,0}-2\,b_{2,1}+3\,b_{2,3}-3\,b_{4,1}\big)s^2+\big(-b_{0,1}+b_{0,3}-b_{0,5} \\
& +b_{2,1}-b_{2,3}+3\,b_{4,1}\big)s-a_{5,0}-b_{4,1}\Big) \\
=& \frac{1}{2s\sqrt{s^2-1}}\big(A_6s^6+A_5s^5+A_4s^4+A_3s^3+A_2s^2+A_1s+A_0\big),
\end{aligned}
\tag{3.4.32}
$$

where

$$
\begin{aligned}
A_0 &= -a_{5,0}-b_{4,1}, \\
A_1 &= -b_{0,1}+b_{0,3}-b_{0,5}+b_{2,1}-b_{2,3}+3\,b_{4,1}, \\
A_2 &= 2\,a_{3,0}-a_{3,2}+5\,a_{5,0}-2\,b_{2,1}+3\,b_{2,3}-3\,b_{4,1}, \\
A_3 &= -3\,a_{3,0}+3\,a_{3,2}-a_{1,0}+a_{1,2}-a_{1,4}-5\,a_{5,0}+b_{0,1}-3\,b_{0,3}+5\,b_{0,5}+b_{2,1}-3\,b_{2,3}+b_{4,1}, \\
A_4 &= -2\,a_{1,2}+3\,a_{1,4}-3\,a_{3,2}+2\,b_{0,3}-5\,b_{0,5}+b_{2,3}, \\
A_5 &= a_{1,0}+a_{1,2}-3\,a_{1,4}+a_{3,0}+a_{3,2}+a_{5,0}, \\
A_6 &= a_{1,4}+b_{0,5}.
\end{aligned}
$$

Using Theorem 3.3.1, the function $f^0(s)$ can have at most 6 real zeros. Because $\sum_0^6 A_i = 0$, then $f^0(s)$ can have at most 5 zeros $s \in (1, \infty)$. This allows us to conclude that the system (3.3.7) has at most 5 limit cycles bifurcating from the isochronous center $(3.3.7)_{\varepsilon=0}$ using the averaging method of first order and there are perturbations with only $0, 1, 2, \ldots, 5$ limit cycles.

Proof of statement (b) of Theorem 3.3.3

To apply the second order averaging method, we use the same changes of variables as in the proof of statement (a) of Theorem 3.3.3. Then the system (3.3.7) can be written as follows

$$\frac{dr}{d\theta} = \varepsilon F_1(\theta, r) + \varepsilon^2 F_2(\theta, r), \tag{3.4.33}$$

with

$$F_1(\theta, r) = \sqrt{1 + r^2 \cos^2\theta}\left((1 + r^2)\cos\theta\, \bar{p}(\theta, r) + \sin\theta\, \bar{q}(\theta, r)\right), \tag{3.4.34}$$

and

$$\begin{aligned} F_2(\theta, r) = \frac{1 + r^2\cos^2\theta}{r} & \left(\sin(\theta)\bar{p}(\theta, r) - \cos(\theta)\bar{q}(\theta, r)\right) \times \\ & \left(\left(r^2 + 1\right)\cos(\theta)\bar{p}(\theta, r) + \sin(\theta)\bar{q}(\theta, r)\right). \end{aligned} \tag{3.4.35}$$

To compute the second order averaged function, the first order averaged function $f^\circ(r)$ must be vanished.

Lemma 3.4.2 *For* $a_{1,4} = -b_{0,5}$, $a_{5,0} = -b_{4,1}$, $b_{0,1} = -a_{1,0}$, $b_{2,1} = -3\,a_{3,0} - 4\,a_{1,0} - a_{1,2} - 3\,b_{0,3}$, $a_{3,2} = -a_{1,0} - a_{3,0} + b_{4,1} - a_{1,2} - 3\,b_{0,5}$, $b_{2,3} = -2\,b_{0,3} - b_{0,5} - 3\,a_{1,0} - 3\,a_{3,0} + 3\,b_{4,1} - a_{1,2}$, *then the function* $f^0(r)$ *defined in* (3.4.31) *is identically zero.*

Proof. From (3.4.32), it is easy to verify that $f^0(r) = 0$ if and only if $a_{1,4} = -b_{0,5}$, $a_{5,0} = -b_{4,1}$, $b_{0,1} = -a_{1,0}$, $b_{2,1} = -3\,a_{3,0} - 4\,a_{1,0} - a_{1,2} - 3\,b_{0,3}$, $a_{3,2} = -a_{1,0} - a_{3,0} + b_{4,1} - a_{1,2} - 3\,b_{0,5}$, $b_{2,3} = -2\,b_{0,3} - b_{0,5} - 3\,a_{1,0} - 3\,a_{3,0} + 3\,b_{4,1} - a_{1,2}$. ■

In order to determine the explicit expression of $y^1(\theta, r)$, Lemma 3.4.1 presents some useful integrals. Because $f^0(r) \equiv 0$, we have $y^1(\theta, r) = \int_0^\theta f(\phi, r)d\phi + z(r)$.

The explicit expression of $y^1(\theta,r)$ is:

$$\begin{aligned}
y^1(\theta,r)= \quad & \tilde{A}_1 \ln\left(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\right)+\tilde{A}_2 \arctan\left(\frac{r\sin\theta}{\sqrt{1+r^2\cos^2\theta}}\right)\\
&+\tilde{A}_3 \sin\theta\sqrt{1+r^2\cos^2\theta}+\tilde{A}_4\cos\theta\sqrt{1+r^2\cos^2\theta}\\
&+\tilde{A}_5\left(\ln\left(1+r^2\cos^2\theta\right)+2\ln(2)-2\ln\left(1+\sqrt{r^2+1}\right)\right)\\
&+\tilde{A}_6\frac{\sin\theta}{\sqrt{1+r^2\cos^2\theta}}+\tilde{A}_7\frac{\cos\theta}{\sqrt{1+r^2\cos^2\theta}}+\tilde{A}_8\left(\frac{1}{1+r^2\cos^2\theta}\right.\\
&\left.-\frac{1}{\sqrt{r^2+1}}\right)+\tilde{A}_9\frac{\sin\theta\cos\theta}{1+r^2\cos^2\theta}+\tilde{A}_{10}\frac{\sin\theta\cos^3\theta}{1+r^2\cos^2\theta}\\
&+\tilde{A}_{11}\left(\cos^2\theta-\frac{1}{2}\right)+\tilde{A}_{12}\sin\theta\cos\theta.
\end{aligned} \tag{3.4.36}$$

The coefficients $\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_{14}$ are given in the Appendix A. (The computations of $y^1(\theta,r)$ can be revised by using Maple or Mathematica).

If the conditions in Lemma 3.4.2 are satisfied, then we must compute $F_{20}(r)=f^{10}(r)+g^0(r)$ the second order averaged function, where $f^{10}(r)=\displaystyle\int_0^{2\pi}\frac{dF_1(\theta,r)}{dr}y^1(\theta,r)d\theta$ and $g^{\circ}(r)=\displaystyle\int_0^{2\pi}F_2(\theta,r)$.

The following lemma lists some integrals which appear in $f^{10}(r)$.

Lemma 3.4.3 *The following integrals hold*

$$M_0(r)=\int_0^{2\pi}\cos\theta\sqrt{1+r^2\cos^2\theta}\,\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)\,d\theta=\frac{\pi}{2r}((r^2+1)\ln\big(1+r^2\big)+r^2),$$

$$M_1(r)=\int_0^{2\pi}\frac{\cos\theta}{\sqrt{1+r^2\cos^2\theta}}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)\,d\theta=\frac{\pi}{r}\ln\big(1+r^2\big).$$

$$M_2(r)=\int_0^{2\pi}\frac{\cos^3\theta}{(1+r^2\cos^2\theta)^{3/2}}\arctan\left(\frac{\sin\theta}{\sqrt{1+r^2\cos^2\theta}}\right)=-\frac{\pi}{r^3}\Big(\ln\big(1+r^2\big)-2\Big(r^2+1-\sqrt{r^2+1}\Big)\Big),$$

$$M_3(r)=\int_0^{2\pi}\frac{\cos^3\theta}{(1+r^2\cos^2\theta)^{3/2}}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)d\theta=\frac{\pi}{r^3}\left(\ln\big(1+r^2\big)+\frac{2(1-\sqrt{(1+r^2)})}{1+r^2}\right),$$

$$M_4(r)=\int_0^{2\pi}\frac{\cos^5\theta}{(1+r^2\cos^2\theta)^{3/2}}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)d\theta=\frac{\pi}{3r^5}\left(3\ln\big(1+r^2\big)+\frac{-11r^4-19r^2-8}{(r^2+1)^{5/2}}\right.$$
$$\left.+\frac{12r^2+8}{(r^2+1)^2}\right),$$

$$M_5(r)=\int_0^{2\pi}\frac{\cos^4\theta}{1+r^2\cos^2\theta}\ln\big(1+r^2\cos^2\theta\big)d\theta=\frac{\pi}{r^4}\left(\left(r^2-2-\frac{2}{\sqrt{r^2+1}}\right)\ln\Big(1+\sqrt{r^2+1}\Big)+\frac{2\ln\big(r^2+1\big)}{\sqrt{r^2+1}}\right.$$
$$\left.+\ln(2)\left(\frac{-2r^4+2r^2+8}{\sqrt{r^2+1}\big(1+\sqrt{r^2+1}\big)}+\frac{-2r^2+8}{1+\sqrt{r^2+1}}\right)+\frac{r^2\big(\sqrt{r^2+1}-1\big)}{1+\sqrt{r^2+1}}\right).$$

Proof. We just compute the integrals $M_0(r), M_1(r), M_3(r)$ and the others can be computed by the same way.

Consider the integrals

$$\tilde{M}_0(r)=r\,M_0(r)=\int_0^{2\pi}\frac{r\cos\theta}{\sqrt{1+r^2\cos^2\theta}}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)\,d\theta,$$

$$\tilde{M}_1(r)=r\,M_1(r)=\int_0^{2\pi}\frac{r\cos\theta}{\sqrt{1+r^2\cos^2\theta}}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)\,d\theta.$$

Taking the derivative with respect to r, we get

$$r\tilde{M}_0^{'}(r) = r\frac{d}{dr}\tilde{M}_0(r) = 2\int_0^{2\pi} r\cos\theta\sqrt{1+r^2\cos^2\theta}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)d\theta$$

$$-\int_0^{2\pi}\frac{r\cos\theta}{\sqrt{1+r^2\cos^2\theta}}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)d\theta+\int_0^{2\pi} r^2\cos^2\theta\, d\theta.$$

Hence,

$$r\tilde{M}_0^{'}(r) = 2\tilde{M}_0^{'}(r) - \tilde{M}_1(r) + \pi r^2. \tag{3.4.37}$$

On the other hand, we perform integration by parts:

$$\tilde{M}_0(r) = \int_0^{2\pi} r\sqrt{1+r^2\cos^2\theta}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)d\sin\theta.$$

Using the following derivatives:

$$\frac{d}{d\theta}\sqrt{1+r^2\cos^2\theta} = -\frac{r^2\cos\theta\sin\theta}{\sqrt{1+r^2\cos^2\theta}},$$

$$\frac{d}{d\theta}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big) = -\frac{r\sin\theta}{\sqrt{1+r^2\cos^2\theta}},$$

we get

$$\begin{aligned}
\tilde{M}_0(r) &= \int_0^{2\pi} r\sin^2\theta\frac{r^2\cos\theta}{\sqrt{1+r^2\cos^2\theta}}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)d\theta+\int_0^{2\pi} r^2\sin^2\theta\, d\theta\\
&= r^2\int_0^{2\pi}\frac{r\cos\theta}{\sqrt{1+r^2\cos^2\theta}}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)d\theta\\
&+\int_0^{2\pi}\frac{r\cos\theta}{\sqrt{1+r^2\cos^2\theta}}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)d\theta\\
&-\int_0^{2\pi}(1+r^2\cos^2\theta)\frac{r\cos\theta}{\sqrt{1+r^2\cos^2\theta}}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)d\theta+\pi r^2\\
&= (r^2+1)\int_0^{2\pi}\frac{r\cos\theta}{\sqrt{1+r^2\cos^2\theta}}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)d\theta\\
&-\int_0^{2\pi} r\cos\theta\sqrt{1+r^2\cos^2\theta}\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)d\theta+\pi r^2.
\end{aligned}$$

Then,

$$2\tilde{M}_0(r) = (r^2+1)\tilde{M}_1(r) + \pi r^2, \tag{3.4.38}$$

(3.4.37) $\times (r^2+1) +$ (3.4.38) $\Rightarrow$

$$r(r^2+1)\tilde{M}_0'(r) + 2\tilde{M}_0(r) = 2(r^2+1)\tilde{M}_0(r) + \pi(r^2+1) + \pi r^2,$$

yields

$$\tilde{M}_0'(r) - \frac{2r}{r^2+1}\tilde{M}_0(r) = \frac{\pi r}{r^2+1} + \pi r. \tag{3.4.39}$$

Equation (3.4.39) is a linear differential equation of first order of the form:

$$\tilde{M}_0'(r) + P(r)\tilde{M}_0(r) = Q(r),$$

with

$$P(r) = -\frac{2r}{r^2+1}, \quad Q(r) = \frac{\pi r}{r^2+1} + \pi r,$$

its solution is given by:

$$\tilde{M}_0(r) = e^{R(r)}\left(\int Q(r)e^{R(r)dr+C}\right),$$

where

$$R(r) = \int P(r)dr = -\int \frac{2r}{r^2+1}dr = -\ln\left(r^2+1\right),$$

and C is a constant of integration. So we have

$$\tilde{M}_0(r)\frac{\pi}{2}(r^2+1)\ln\left(r^2+1\right) + C(r^2+1) - \frac{\pi}{2},$$

and due to $\tilde{M}_0(0) = 0$, we get $C = \frac{\pi}{2}$.
As a consequence

$$M_0(r) = \frac{\pi}{2r}\left((r^2+1)\ln\left(r^2+1\right) + r^2\right).$$

From equation (3.4.38), we obtain

$$M_1(r) = \frac{1}{r}\tilde{M}_1(r) = \frac{1}{r(r^2+1)}\left(2\tilde{M}_0(r) - \pi r^2\right) = \frac{\pi}{r}\ln\left(r^2+1\right).$$

Firstly, this integral hold [40]

$$J_1 = \int_0^{2\pi} \frac{\cos\theta}{\sqrt{1+r^2\cos^2\theta}}\ln\left(r\cos\theta + \sqrt{1+r^2\cos^2\theta}\right)d\theta = \frac{\pi}{r}\ln\left(1+r^2\right).$$

Now, taking the derivative of the both sides of $M_1(r)$ with respect to r, we get

$$\frac{dM_1(r)}{dr}=-rM_3(r)+\frac{\cos^2\theta}{1+r^2\cos^2\theta}=-\frac{\pi\ln\left(r^2+1\right)}{r^2}+\frac{2\pi}{r^2+1},$$

yields:

$$M_3(r)=\frac{\pi}{r^3}\left(\ln\left(1+r^2\right)+\frac{2(1-\sqrt{(1+r^2)})}{1+r^2}\right).$$

The other integrals can be computed by applying the above ideas. ■

Now, using the ideas in the proof of Lemma 3.4.3, we compute $f^{10}(r)$.

Taking $s=\sqrt{1+r^2}$ with $s\in(0,\infty)$, the second order averaged function $F_{20}(s)$ is given by:

$$\begin{aligned}
F_{20}(s)=&\frac{1}{2\sqrt{s^2-1}}\left(B_1+B_2s^2+B_3s^4\right)\ln(s)+\frac{1}{8s^5(s+1)\sqrt{s^2-1}}\left(B_4s^2+B_5s^3+B_6s^4\right.\\
&\left.+B_7s^5+B_8s^6+B_9s^7+B_{10}s^8+B_{11}s^9+B_{12}s^{10}+B_{13}s^{11}+B_{14}s^{12}+B_{15}s^{13}\right)\\
=&\frac{1}{8s^5\sqrt{s^2-1}(s+1)}\left(4\left(\left(s^5+s^6\right)B_1+\left(s^7+s^8\right)B_2+\left(s^9+s^{10}\right)B_3\right)\ln(s)\right.\\
&+B_4s^2+B_5s^3+B_6s^4+B_7s^5+B_8s^6+B_9s^7+B_{10}s^8+B_{11}s^9+B_{12}s^{10}\\
&\left.+B_{13}s^{11}+B_{14}s^{12}+B_{15}s^{13}\right),\\
=&\frac{1}{8s^5\sqrt{s^2-1}(s+1)}F(s),
\end{aligned}\tag{3.4.40}$$

where

$$\begin{aligned}
F(s)=\quad&4\left(\left(s^5+s^6\right)B_1+\left(s^7+s^8\right)B_2+\left(s^9+s^{10}\right)B_3\right)\ln(s)\\
&+B_4s^2+B_5s^3+B_6s^4+B_7s^5+B_8s^6+B_9s^7+B_{10}s^8\\
&+B_{11}s^9+B_{12}s^{10}+B_{13}s^{11}+B_{14}s^{12}+B_{15}s^{13},
\end{aligned}$$

with the coefficients $B_1,B_2,\ldots,B_{15}$ are given in Appendix A.

Taking the 11^{th} order derivative of $F(s)$ with respect to s, we obtain:

$$\begin{aligned}
F^{(11)}(s)\quad=&\frac{5760}{s^6}\left(540540\,B_{15}s^8+83160\,B_{14}s^7+6930\,B_3s^6+2520\,B_3s^5-252\,B_3s^4\right.\\
&\left.+56\,B_2s^3-21\,B_2s^2+12\,B_1s-10\,B_1\right).
\end{aligned}$$

Using Theorem 3.3.1, we conclude that $F^{(11)}(s)$ can have at most 8 zeros for $s\in(1,+\infty)$. By applying the Rolle's rule, the function $F(s)$ can have at most 19 zeros, taking into account their multiplicities in $(1,+\infty)$. Hence, the second order averaged function $F_{20}(r)$ has at most 19 zeros for $r\in(0,+\infty)$. This allow us to conclude that the system (3.3.7) has at most 19 limit cycles that can bifurcate from the center $(3.3.7)_{\varepsilon=0}$.

CHAPTER

4 Bifurcation of limit cycles from a cubic degenerate center

4.1 Introduction

In the qualitative theory of ordinary differential equations, the degenerate-center problem, concerning degenerate singular points of planar systems, is a challenging problem. It is a poorly-understood because of the two problem: the stability problem, to decide whether it is a focus, and the monodromy problem (the singular point p is termed monodromic if there are no orbits tending to or leaving p with a certain angle), to decide if the singular point is of focus-center type or a center.

4.2 Degenerate center

Consider the polynomial differential system:

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{4.2.1}$$

As usual, we denote $\dot{} = d/dt$. Assume that system (4.2.1) has a center located at the origin. After a linear change of variables and a possible scaling of time, if system (4.2.1) can be written in the following form:

$$\dot{x} = F_1(x,y), \dot{y} = F_2(x,y),$$

where F_1 and F_2 are nonlinear real polynomials (i.e. without linear terms and constant), then the center is said to be of degenerate type.

4.3 Results on the number of limit cycles bifurcating from continuous perturbations of cubic degenerate center

4.3.1 Main result

Consider the planar differential system

$$\begin{aligned} \dot{x} &= -y\left(3x^2+y^2\right), \\ \dot{y} &= x\left(x^2-y^2\right). \end{aligned} \tag{4.3.2}$$

It is clear that $(0,0)$ is the equilibrium point of system (4.3.2). Moreover, it is a global center, i.e., all the orbits contained in $\mathbb{R}^2 \setminus \{(0,0)\}$ are periodic. Note that system (4.3.2) has the non-rational first integral $H(x,y)=(x^2+y^2)\exp\left(-\frac{2x^2}{x^2+y^2}\right)$.

Several references have dealt with the problem of the bifurcation of limit cycles from the periodic solutions of the global centre (4.3.2). For example, the following result is one of the first examples in which the limit cycles bifurcating from the periodic orbits of a 2—dimensional polynomial differential system with center having a non-rational first integral have been studied.

Proposition 4.3.1 *([46]) Consider the following system*

$$\begin{aligned} \dot{x} &= -y\left(3x^2+y^2\right)+\varepsilon\sum_{i+j=0}^{n} a_{i,j}x^i y^j, \\ \dot{y} &= x\left(x^2-y^2\right)+\varepsilon\sum_{i+j=0}^{n} a_{i,j}x^i y^j. \end{aligned} \tag{4.3.3}$$

Then for $|\varepsilon| \neq 0$ *sufficiently small, system* (4.3.3) *can have up to* $[(n-1)/2]$ *limit cycles bifurcating from the periodic orbits of the global center* (4.3.2)*, where* $[\cdot]$ *denotes the integer part function.*

Another result, given in [7], discusses the bifurcation of limit cycles from the degenerate center (4.3.2) under small cubic polynomials perturbations.

Proposition 4.3.2 *Let* $P_1(x,y), P_2(x,y), Q_1(x,y)$ *and* $Q_2(x,y)$ *be polynomials of degree at most* *three. Consider the perturbed system*

$$\begin{aligned}\dot{x} =& -y(3x^2+y^2)+\varepsilon P_1(x,y)+\varepsilon^2 P_2(x,y),\\ \dot{y} =& x(x^2+y^2)+\varepsilon Q_1(x,y)+\varepsilon^2 Q_2(x,y).\end{aligned} \tag{4.3.4}$$

For $|\varepsilon|>0$ *sufficiently small, using the averaging method of first and second order, system* (4.3.6) *has one and two limit cycles ,respectively, bifurcating from the periodic solutions of the center* (4.3.2).

A remarkable result was given in [44] by LLibre and Pantazi. As far as we know, it was the first time that a complete study up to second order in the small parameter of the perturbation was done for studying limit cycles that can bifurcate from the periodic orbits surrounding a degenerate center having neither a rational first integral nor a Hamiltonian one.

Proposition 4.3.3 *The perturbation of the degenerate center* (4.3.2) *as follows*

$$\begin{aligned}\dot{x} =& -y(3x^2+y^2)+\varepsilon\sum_{i+j=0}^{3} a_{i,j}x^i y^j+\varepsilon^2\sum_{i+j=0}^{3} b_{i,j}x^i y^j),\\ \dot{y} =& x(x^2+y^2)+\varepsilon\sum_{i+j=0}^{3} c_{i,j}x^i y^j+\varepsilon^2\sum_{i+j=0}^{3} d_{i,j}x^i y^j).\end{aligned} \tag{4.3.5}$$

can have up to three limit cycles bifurcating from the periodic orbits of the center of system (4.3.2) *using averaging theory of second order, and there are perturbations with exactly* 0, 1, 2 *and* 3 *limit cycles.*

Another result in this direction was given in [55] but by perturbing the degenerate center (4.3.2) inside the class of all quintic polynomial differential systems. This result is the following theorem.

Theorem 4.3.1 *Consider the perturbed system*

$$\begin{aligned}\dot{x} =& -y(3x^2+y^2)+\varepsilon P_1(x,y)+\varepsilon^2 P_2(x,y),\\ \dot{y} =& x(x^2+y^2)+\varepsilon Q_1(x,y)+\varepsilon^2 Q_2(x,y),\end{aligned} \tag{4.3.6}$$

with

$$P_1(x,y)=\sum_{i+j=0}^{5} a_{i,j}x^i y^j, P_2(x,y)=\sum_{i+j=0}^{5} b_{i,j}x^i y^j, Q_1(x,y)=\sum_{i+j=0}^{3} d_{i,j}x^i y^j, Q_2(x,y)=\sum_{i+j=0}^{5} e_{i,j}x^i y^j.$$

For $\varepsilon\neq 0$ *sufficiently small, system* (4.3.6) *can have up to five limit cycles bifurcating from the periodic orbits of the global center* (4.3.2) *using the second order averaging method.*

The following results can be obtained from Theorem 4.3.1.

Corollary 4.3.1 *Let $P_1(x, y), P_2(x, y), Q_1(x, y), Q_2(x, y)$ be real polynomials of degree four. Then system* (4.3.6) *has at most three limit cycles bifurcating from the periodic orbits surrounding the origin by using the second order averaging method.*

Corollary 4.3.2 *Let $P_1(x, y), P_2(x, y), Q_1(x, y), Q_2(x, y)$ be real polynomials of degree five without constants. Then by using the second order averaging method, system* (4.3.6) *has at most four limit cycles bifurcating from the periodic orbits surrounding the origin.*

4.4 Proofs of Results

In this section we only introduce the proof of Theorem 4.3.1, the two corollaries 4.3.1 and 4.3.2.

4.4.1 Proof of Theorem 4.3.1

Consider the system

$$\begin{aligned} \dot{x} &= -y(3x^2 + y^2), \\ \dot{y} &= x(x^2 - y^2). \end{aligned} \tag{4.4.7}$$

In polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$, system (4.4.7) can be transformed into the following form:

$$\frac{dr}{d\theta} = -r\sin(2\theta). \tag{4.4.8}$$

The solution $r_u(\theta, r_0) = r_0 e^{-\sin^2\theta}$ of (4.4.8) is 2π–periodic in the first variable θ and satisfies $r_u(0, r_0) = r_0$.

Consider the variational equation:

$$\frac{\partial v}{\partial \theta} = \frac{\partial F_0}{\partial r}(\theta, r_u(\theta, r_0)).v,$$

with its solution being $v(\theta, r_0) = e^{\sin^2\theta}$ satisfying $v(0, r_0) = 1$.

Now, we shall perturb the degenerate center (4.4.7) inside the class of all polynomial systems of degree 5 as in (4.3.6). With the goal of applying the averaging method, the system (4.3.6) should be transformed to the standard form (2.3.21). By taking θ as an independent variable, system (4.3.6) could be written in the as follows:

$$\frac{dr}{d\theta} = F_0\big(\theta, y\big) + \varepsilon F_1\big(\theta, y\big) + \varepsilon^2 F_2\big(\theta, y\big) + \varepsilon^3 R(\theta, r, \varepsilon), \tag{4.4.9}$$

where

$$\begin{aligned}
F_0(\theta, r) &= -r\sin(2\theta), && (4.4.10)\\
F_1(\theta, r) &= \frac{1}{r^2} f_2(\theta) + \frac{1}{r} f_1(\theta) + f_0(\theta) + r\tilde{f}_1(\theta) + r^2\tilde{f}_2(\theta) + r^3\tilde{f}_3(\theta), && (4.4.11)\\
F_2(\theta, r) &= \frac{1}{r^5} g_5(\theta) + \frac{1}{r^4} g_4(\theta) + \frac{1}{r^3} g_3(\theta) + \frac{1}{r^2} g_2(\theta) + \frac{1}{r} g_1(\theta) + g_0(\theta) && (4.4.12)\\
&\quad + r\tilde{g}_1(\theta) + r^2\tilde{g}_2(\theta) + r^3\tilde{g}_3(\theta) + r^4\tilde{g}_4(\theta) + r^5\tilde{g}_5(\theta),
\end{aligned}$$

with

$$\begin{aligned}
f_2(\theta) &= \frac{1}{2}\Bigg(d_{0,0}(\sin(3\theta) + 3\sin\theta) + a_{0,0}(\cos\theta + \cos(3\theta)) \Bigg),\\
f_1(\theta) &= \frac{1}{4}\Bigg(\big(a_{0,1} + d_{1,0}\big)\sin(4\theta) - \big(d_{0,1} - a_{1,0}\big)(\cos(4\theta) + 2\cos(2\theta)) + 4d_{1,0}\sin(2\theta) + 3d_{0,1} + a_{1,0} \Bigg),\\
f_0(\theta) &= \frac{1}{8}\big(a_{1,1} - d_{0,2} + d_{2,0}\big)\sin(5\theta) + \frac{1}{8}\big(a_{1,1} - d_{0,2} + 5d_{2,0}\big)\sin(3\theta) + (\frac{1}{2}d_{2,0} + d_{0,2})\sin\theta\\
&\quad + \frac{1}{8}\big(-a_{0,2} + a_{2,0} - d_{1,1}\big)\cos(5\theta) + \frac{1}{8}\big(a_{0,2} + 3a_{2,0} - 3d_{1,1}\big)\cos(3\theta) + \frac{1}{2}\big(a_{2,0} + d_{1,1}\big)\cos\theta,\\
\tilde{f}_1(\theta) &= \frac{1}{16}\big(a_{2,1} + d_{3,0} - a_{0,3} - d_{1,2}\big)\sin(6\theta) + \frac{1}{8}\big(a_{0,3} + a_{2,1} + 3d_{3,0} - d_{1,2}\big)\sin(4\theta)\\
&\quad + \frac{1}{16}\big(a_{2,1} - a_{0,3} + 7d_{1,2} + 9d_{3,0}\big)\sin(2\theta) + \frac{1}{16}\big(a_{3,0} - a_{1,2} + d_{0,3} - d_{2,1}\big)\cos(6\theta)\\
&\quad + \frac{1}{4}\big(a_{3,0} - d_{2,1}\big)\cos(4\theta) + \frac{1}{16}\big(+7a_{3,0} + a_{1,2} - 9d_{0,3} + d_{2,1}\big)\cos(2\theta) + \frac{1}{4}\big(a_{3,0} + d_{2,1} + 2d_{0,3}\big),\\
\tilde{f}_2(\theta) &= \frac{1}{32}\Bigg(\big(-a_{1,3} + a_{3,1} + d_{4,0} + d_{0,4} - d_{2,2}\big)\sin(7\theta) + (3a_{3,1} + a_{1,3} - 3d_{2,2} + 7d_{4,0} - d_{0,4})\sin(5\theta)\\
&\quad + \big(3a_{3,1} + a_{1,3} + 15d_{4,0} - 9d_{0,4} + 5d_{2,2}\big)\sin(3\theta) + (-a_{1,3} + a_{3,1} + 7d_{2,2} + 9d_{4,0} + 25d_{0,4})\sin\theta\\
&\quad + \big(-a_{2,2} + a_{0,4} + a_{4,0} + d_{1,3} - d_{3,1}\big)\cos(7\theta) + \big(5a_{4,0} - 3a_{0,4} - a_{2,2} + d_{1,3} - 5d_{3,1}\big)\cos(5\theta)\\
&\quad + (a_{2,2} + 3a_{0,4} + 11a_{4,0} - 9d_{1,3} - 3d_{3,1})\cos(3\theta) + (+a_{2,2} - a_{0,4} + 15a_{4,0} + 9d_{3,1} + 7d_{1,3})\cos\theta \Bigg),
\end{aligned}$$

$$\begin{aligned}\tilde{f}3(\theta)=\frac{1}{64}\Big(&\big(a_{4,1}+a_{0,5}-a_{2,3}+d_{5,0}+d_{1,4}-d_{3,2}\big)\sin(8\theta)+4(2d_{5,0}-d_{3,2}-a_{0,5}+a_{4,1})\sin(6\theta)\\&+2(11d_{5,0}+d_{3,2}-5d_{1,4}+a_{2,3}+3a_{4,1}+3a_{0,5})\sin\big(4\theta\big)+4(6d_{5,0}+4d_{1,4}+a_{4,1}+3d_{3,2}\\&-a_{0,5}\big)\sin(2\theta)+(-a_{3,2}+a_{5,0}+a_{1,4}-d_{4,1}-d_{0,5}+d_{2,3})\cos(8\theta)+2\big(d_{2,3}+d_{0,5}-3d_{4,1}\\&+3a_{5,0}-a_{1,4}-a_{3,2}\big)\cos(6\theta)+8\big(2a_{5,0}+d_{0,5}-d_{4,1}-d_{2,3}\big)\cos(4\theta)+2(13a_{5,0}+a_{1,4}+a_{3,2}\\&-17d_{0,5}-d_{2,3}+3d_{4,1})\cos(2\theta)+\big(a_{3,2}+15a_{5,0}-a_{1,4}+9d_{4,1}+7d_{2,3}+25d_{0,5}\big)\Big).\end{aligned}$$

It is clear that $F_k(\theta, r_0)$ with $k \in \{0,1,2\}$ are $2\pi-$periodic functions in the first variable θ. The assumptions of the second order averaging method are verified, so, we should determine the expression of the averaged function $F_{10}(r_0)$.

The following lemma lists some helpful integrals for computing the first order averaged function $F_{10}(r_0)$.

Lemma 4.4.1 *The following integrals hold*

$$J_1=\int_0^{2\pi} e^{2\sin^2\theta}\,d\theta=21.62373221\ldots,\qquad J_5=\int_0^{2\pi}\cos(2\theta)\,e^{-2\sin^2\theta}\,d\theta=1.306339667\ldots,$$

$$J_2=\int_0^{2\pi}\cos(2\theta)\,e^{2\sin^2\theta}\,d\theta=-9.652617083\ldots,\qquad J_6=\int_0^{2\pi}\cos(4\theta)\,e^{-2\sin^2\theta}\,d\theta=0.313774589\ldots,$$

$$J_3=\int_0^{2\pi}\cos(4\theta)\,e^{2\sin^2\theta}\,d\theta=2.318498042\ldots,\qquad J_7=\int_0^{2\pi}\cos(6\theta)\,e^{-2\sin^2\theta}\,d\theta=0.051241309\ldots,$$

$$J_4=\int_0^{2\pi} e^{-2\sin^2\theta}\,d\theta=2.926453923\ldots,\qquad J_8=\int_0^{2\pi}\cos(8\theta)\,e^{-2\sin^2\theta}\,d\theta=0.0063267292\ldots.$$

Proof. The integrals J_i with $i = 1,2,\ldots,8$ are computed by using the algebraic manipulator Maple, because they give some terms that cannot be expressed by means of elementary functions or in terms of the error function $erf(z)$, where the error function is the integral of the Gaussian distribution given by $erf(z)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$. ■

Form (2.3.24), we have that

$$
\begin{aligned}
F_{10}(r_0) &= \int_0^{2\pi} \frac{F_1(\theta, r_u(\theta, r_0))}{v(\theta, r_0)} d\theta \\
&= \frac{1}{r_0^2} \int_0^{2\pi} f_2(\theta) e^{3\sin^2\theta} d\theta + \frac{1}{r_0} \int_0^{2\pi} f_1(\theta) e^{2\sin^2\theta} d\theta + \int_0^{2\pi} f_0(\theta) e^{\sin^2\theta} d\theta \\
&+ r_0 \int_0^{2\pi} \tilde{f}_1(\theta) d\theta + r_0^2 \int_0^{2\pi} \tilde{f}_2(\theta) e^{-\sin^2\theta} d\theta + r_0^3 \int_0^{2\pi} \tilde{f}_3(\theta) e^{-2\sin^2\theta} d\theta,
\end{aligned}
$$

in order to simplify the expression of $F_{10}(r_0)$, we use the trigonometric formulas

$$
\cos(x + 2k\pi) = \cos(x) \text{ with } k \in \mathbb{Z}\,,\ \cos(x + \pi) = -\cos(x)\,,\ \sin(x + \pi) = -\sin(x), \tag{4.4.13}
$$

so, we have

$$
\begin{aligned}
F_{10}(r_0) &= \frac{1}{r_0} \int_0^{2\pi} f_1(\theta)\, e^{2\sin^2\theta} d\theta + r_0 \int_0^{2\pi} \tilde{f}_1(\theta) d\theta + r_0^3 \int_0^{2\pi} \tilde{f}_3(\theta) e^{-2\sin^2\theta} d\theta \\
&= (\frac{1}{4}(-J_3 - 2J_2 + 3J_1) d_{0,1} + \frac{1}{4}(J_3 + 2J_2 + J_1) a_{1,0}) \times \frac{1}{r_0} + 2\pi(\frac{1}{2} d_{0,3} \\
&+ \frac{1}{4} d_{2,1} + \frac{1}{4} a_{3,0}) r_0 + ((-J_8 + 6J_5 - 6J_7 + 9J_4 - 8J_6) d_{4,1} + (J_8 - 2J_5 \\
&+ 2J_7 + 7J_4 - 8J_6) d_{2,3} + (-J_8 - 34J_5 + 2J_7 + 25J_4 + 8J_6) d_{0,5} \\
&+ (J_8 + 26J_5 + 6J_7 + 15J_4 + 16J_6) a_{5,0} + (-J_8 + 2J_5 - 2J_7 + J_4) a_{3,2} \\
&+ (J_8 + 2J_5 - 2J_7 - J_4) a_{1,4})) r_0^3.
\end{aligned}
$$

A necessary condition for applying the second order averaging method is $F_{10}(r)$ must be vanished.

So, $F_{10}(r_0) \equiv 0$ if and only if the following conditions satisfied.

$$
a_{1,0} = \frac{-3J_1 + 2J_2 + J_3}{J_1 + 2J_2 + J_3} d_{0,1}, \quad a_{0,3} = -d_{1,2} - 2d_{0,3},
$$

$$
\begin{aligned}
a_{5,0} = & \frac{-9\,J_4-6\,J_5+8\,J_6+6\,J_7+J_8}{J_8+26\,J_5+6\,J_7+15\,J_4+16\,J_6}d_{4,1}+\frac{-7\,J_4+2\,J_5+8\,J_6-2\,J_7-J_8}{J_8+26\,J_5+6\,J_7+15\,J_4+16\,J_6}d_{2,3}\\
& +\frac{-25\,J_4+34\,J_5-8\,J_6-2\,J_7+J_8}{J_8+26\,J_5+6\,J_7+15\,J_4+16\,J_6}d_{0,5}+\frac{-J_4-2\,J_5+2\,J_7+J_8}{J_8+26\,J_5+6\,J_7+15\,J_4+16\,J_6}a_{3,2}\\
& +\frac{J_4-2\,J_5+2\,J_7-J_8}{J_8+26\,J_5+6\,J_7+15\,J_4+16\,J_6}a_{1,4}.
\end{aligned}
$$

In other words, we have

$$
\begin{aligned}
a_{1,0} = & -17.65322449 d_{0,1}, \quad a_{0,3} = -d_{1,2}-2d_{0,3}\\
a_{5,0} = & 0.004927298284 a_{1,4}-0.3768477366 d_{1,4}-0.06527160474 a_{3,2}-0.3768477367 d_{0,5}\\
& -0.1859602196 d_{2,3}.
\end{aligned}
$$

On the other hand, we have $\frac{\partial^2 F_0}{\partial r^2}(\theta, r_u(\theta, r_0))=0$, then (2.3.25) can be written as

$$
F_{20}(r_0)=F_{20}^{(1)}(r_0)+F_{20}^{(2)}(r_0), \tag{4.4.14}
$$

where

$$
\begin{aligned}
F_{20}^{(1)}(r_0) = & \int_0^{2\pi} \left(\frac{F_2(\theta, r_u(\theta, r_0))}{u(\theta, r_0)}\right) d\theta\\
= & \frac{1}{r_0^5}\int_0^{2\pi} g_5(\theta) e^{6\sin^2\theta} d\theta+\frac{1}{r_0^4}\int_0^{2\pi} g_4(\theta) e^{5\sin^2\theta} d\theta+\frac{1}{r_0^3}\int_0^{2\pi} g_3(\theta) e^{4\sin^2\theta} d\theta\\
& +\frac{1}{r_0^2}\int_0^{2\pi} g_2(\theta) e^{3\sin^2\theta} d\theta+\frac{1}{r_0}\int_0^{2\pi} g_1(\theta) e^{2\sin^2\theta} d\theta+\int_0^{2\pi} g_0(\theta) e^{\sin^2\theta} d\theta\\
& +r_0\int_0^{2\pi} \tilde{g}_1(\theta) d\theta+r_0^2\int_0^{2\pi} \tilde{g}_2(\theta) e^{-\sin^2\theta} d\theta+r_0^3\int_0^{2\pi} \tilde{g}_3(\theta) e^{-2\sin^2\theta} d\theta\\
& +r_0^4\int_0^{2\pi} \tilde{g}_4(\theta) e^{-3\sin^2\theta} d\theta+r_0^5\int_0^{2\pi} \tilde{g}_5(\theta) e^{-4\sin^2\theta} d\theta,
\end{aligned}
$$

and

$$F^{(2)}_{20}(r_0)=\int_0^{2\pi}\left(\frac{\partial F_1}{\partial r}(r_u(\theta,r_0)).v_1(\theta,r_u(\theta,r_0))\right)d\theta$$

$$=\frac{1}{r_0^5}\int_0^{2\pi}C_5(\theta)d\theta+\frac{1}{r_0^4}\int_0^{2\pi}C_4(\theta)d\theta+\frac{1}{r_0^3}\int_0^{2\pi}C_3(\theta)d\theta+\frac{1}{r_0}\int_0^{2\pi}C_1(\theta)d\theta+\int_0^{2\pi}C_0(\theta)d\theta$$

$$+r_0\int_0^{2\pi}\tilde{C}_1(\theta)d\theta+r_0^2\int_0^{2\pi}\tilde{C}_2(\theta)d\theta+r_0^3\int_0^{2\pi}\tilde{C}_3(\theta)d\theta+r_0^4\int_0^{2\pi}\tilde{C}_4(\theta)d\theta+r_0^5\int_0^{2\pi}\tilde{C}_5(\theta)d\theta,$$

with

$$\frac{\partial F_1}{\partial r}(\theta,r_u(\theta,r_0))=-\frac{2}{r_0^3}f_2(\theta)e^{3\sin^2\theta}-\frac{1}{r_0^2}f_1(\theta)e^{2\sin^2\theta}+\tilde{f}_1(\theta)$$
$$+2r_0\tilde{f}_2(\theta)e^{-\sin^2\theta}+3r_0^2\tilde{f}_3(\theta)e^{-2\sin^2\theta},$$

and

$$v_1(\theta,y)=\int_0^{\theta}\frac{F_1(\phi,r_u(\phi,r_0))}{v(\phi,r_0)}d\theta.$$

Now, by using formulas (4.4.13), we can simplify $F^{(1)}_{20}(r_0)$ as

$$F^{(1)}_{20}(r_0)=\frac{1}{r_0^5}\int_0^{2\pi}g_5(\theta)e^{6\sin^2\theta}d\theta+\frac{1}{r_0^3}\int_0^{2\pi}g_3(\theta)e^{4\sin^2\theta}d\theta+\frac{1}{r_0}\int_0^{2\pi}g_1(\theta)e^{2\sin^2\theta}d\theta$$
$$+r_0\int_0^{2\pi}\tilde{g}_1(\theta)d\theta+r_0^3\int_0^{2\pi}\tilde{g}_3(\theta)e^{-2\sin^2\theta}d\theta+r_0^5\int_0^{2\pi}\tilde{g}_5(\theta)e^{-4\sin^2\theta}d\theta.$$

Again, by using (4.4.13), $F^2_{20}(r_0)$ becomes

$$F^{(2)}_{20}(r_0)=\frac{1}{r_0^5}\int_0^{2\pi}C_5(\theta)d\theta+\frac{1}{r_0^3}\int_0^{2\pi}C_3(\theta)d\theta+\frac{1}{r_0}\int_0^{2\pi}C_1(\theta)d\theta+r_0\int_0^{2\pi}\tilde{C}_1(\theta)d\theta$$
$$+r_0^3\int_0^{2\pi}\tilde{C}_3(\theta)d\theta+r_0^5\int_0^{2\pi}\tilde{C}_5(\theta)d\theta,$$

yields

$$F_{20}(r_0)=\omega_0\frac{1}{r_0^5}+\omega_2\frac{1}{r3_0}+\omega_4\frac{1}{r_0}+\omega_6 r_0+\omega_8 r_0^3+\omega_{10}r_0^5, \tag{4.4.15}$$

where

$$
\begin{aligned}
w_0 &= 665.2264929\ldots a_{0,0}d_{0,0},\\
w_2 &= 16.68739735\ldots a_{0,0}d_{2,0} - 7.649140221\ldots a_{0,0}a_{1,1} + 95.95703341\ldots a_{0,0}d_{0,2}\\
&\quad - 31.79348852\ldots d_{0,0}a_{2,0} - 110.9164315\ldots d_{0,0}d_{1,1} + 105.7377762\ldots d_{0,0}a_{0,2}\\
&\quad - 244.7413301\ldots d_{1,0}d_{0,1} + 251.5279874\ldots a_{0,1}d_{0,1} - 0.0000010\ldots d_{0,1}^2,\\
w_4 &= -38.88726631\ldots d_{0,1}d_{3,0} + 38.88726638\ldots d_{0,1}a_{0,3} + 2.223841814\ldots d_{2,1}a_{0,1}\\
&\quad + 3.507120048\ldots a_{0,0}d_{2,2} - 74.40755738\ldots d_{0,0}a_{4,0} + 1.738873532\ldots d_{1,0}a_{1,2}\\
&\quad - 68.30745734\ldots d_{0,1}d_{1,2} + 7.629774867\ldots d_{1,0}d_{0,3} - 1.179588309\ldots a_{0,0}a_{1,3}\\
&\quad + 1.159249020\ldots b_{1,0} + 20.46448319\ldots e_{0,1} + 2.318498043\ldots a_{0,2}d_{2,0}\\
&\quad - 4.731652314\ldots d_{0,2}d_{1,1} + 2.318498043\ldots d_{0,2}a_{2,0} - 1.253905250\ldots a_{1,2}a_{0,1}\\
&\quad + 14.28961316\ldots d_{0,3}a_{0,1} + 14.47892563\ldots a_{0,2}d_{0,2} - 1.253905250\ldots a_{0,2}a_{1,1}\\
&\quad + 0.09465622855\ldots a_{1,1}a_{2,0} - 3.761715750\ldots d_{1,1}d_{2,0} + 0.1893124564\ldots a_{2,0}d_{2,0}\\
&\quad + 2.318498043\ldots a_{1,1}d_{1,1} + 17.33550510\ldots d_{0,0}a_{0,4} - 2.740323153\ldots d_{0,0}a_{2,2}\\
&\quad - 42.35725611\ldots d_{0,0}d_{3,1} - 1.443217705\ldots d_{1,0}d_{2,1} - 3.155691809\ldots d_{0,1}a_{2,1}\\
&\quad + 2.728264900\ldots a_{0,0}d_{4,0} + 14.19972526\ldots a_{0,0}d_{0,4} + 0.5199940616\ldots a_{0,0}a_{3,1}\\
&\quad - 31.00730858\ldots d_{0,0}d_{1,3},\\
w_6 &= 0.2137640471\ldots a_{1,4}d_{1,0} + 1.960592990\ldots d_{0,5}a_{0,1} + 3.184149598\ldots d_{0,5}d_{1,0}\\
&\quad + 2.018194236\ldots d_{0,4}a_{0,2} + 0.01176643430\ldots a_{2,2}d_{2,0} + 6.395609425\ldots a_{0,5}d_{0,1}\\
&\quad + 0.4226848073\ldots d_{2,3}d_{1,0} + 0.3951177630\ldots d_{2,3}a_{0,1} + 0.2601849590\ldots a_{3,2}d_{1,0}\\
&\quad + 1.063795942\ldots d_{0,4}d_{1,1} - 0.004353626539\ldots a_{3,2}a_{0,1} - 3.534291735\ldots d_{3,1}d_{0,2}\\
&\quad - 0.9905725301\ldots d_{1,3}d_{2,0} - 0.1441784055\ldots d_{2,2}d_{1,1} - 2.748893572\ldots d_{1,3}d_{0,2}\\
&\quad + 0.3727810141\ldots a_{3,1}a_{2,0} - 0.01486268923\ldots a_{1,3}a_{2,0} + 3.141592654\ldots e_{0,3}\\
&\quad + 1.570796327\ldots e_{2,1} - 0.2051743665\ldots a_{0,4}a_{1,1} + 0.2684387459\ldots d_{4,0}a_{0,2}\\
&\quad + 0.1934471228\ldots d_{4,1}a_{0,1} - 0.1889181963\ldots a_{1,3}a_{0,2} + 1.570796327\ldots b_{3,0}\\
&\quad + 0.4864614390\ldots d_{1,3}a_{1,1} + 0.3056713282\ldots a_{1,3}d_{1,1} - 0.1886097603\ldots a_{1,4}a_{0,1}\\
&\quad + 0.7158366500\ldots d_{2,2}a_{2,0} + 0.009959033800\ldots a_{3,1}a_{0,2} - 13.73654973\ldots d_{5,0}d_{0,1}\\
&\quad - 5.497787143\ldots a_{4,0}d_{0,2} - 0.1537925955\ldots d_{4,1}d_{1,0} - 0.3847809687\ldots a_{2,3}d_{0,1}
\end{aligned}
$$

$$
\begin{aligned}
&+0.2112880916\ldots\ a_{3,1}d_{1,1}+0.2989367237\ldots\ d_{3,1}a_{1,1}-0.3926990818\ldots\ d_{2,1}d_{3,0}\\
&-3.540174952\ldots\ a_{4,0}d_{2,0}-15.30734609\ldots\ d_{1,4}d_{0,1}+0.1934079318\ldots\ a_{4,0}a_{1,1}\\
&-13.20767224\ldots\ d_{3,2}d_{0,1}+0.4274798378\ldots\ d_{2,2}a_{0,2}-2.151020123\ldots\ d_{3,1}d_{2,0}\\
&+2.159844950\ldots\ d_{0,3}a_{0,3}+0.005883217058\ldots\ a_{2,2}a_{1,1}+1.461398678\ldots\ d_{0,4}a_{2,0}\\
&-0.1963495409\ldots\ a_{1,2}a_{0,3}+0.3926990818\ldots\ d_{1,2}a_{1,2}+0.9817477044\ldots\ d_{3,0}d_{0,3}\\
&+0.1963495409\ldots\ a_{1,2}d_{3,0}+0.3926990818\ldots\ d_{2,1}a_{0,3}-0.7853981635\ldots\ d_{2,1}d_{1,2}\\
&-8.767640129\ldots\ a_{4,1}d_{0,1}+0.2485206765\ldots\ d_{4,0}a_{2,0}+0.3750494305\ldots\ a_{0,4}d_{2,0}\\
&+2.356194491\ldots\ a_{0,4}d_{0,2}-1.178097245\ldots\ d_{1,2}d_{0,3}-0.7754391284\ldots\ d_{4,0}d_{1,1},\\
w_8={}&-0.006405163750\ldots\ b_{1,4}+0.4898773753\ldots\ e_{0,5}+0.4898773752\ldots\ e_{4,1}\\
&+0.08484881107\ldots\ b_{3,2}+0.2417361057\ldots\ e_{2,3}+1.299934503\ldots\ b_{5,0}\\
&+0.03710327108\ldots\ d_{1,2}a_{3,2}+0.06642416099\ldots\ a_{0,4}d_{2,2}-0.001186261761\ldots\ a_{0,4}a_{3,1}\\
&+0.06631353951\ldots\ a_{0,3}d_{2,3}+0.2926356580\ldots\ a_{0,3}d_{0,5}+0.1442724248\ldots\ d_{1,2}d_{0,5}\\
&-0.001698899757\ldots\ a_{0,3}a_{3,2}+0.05232791382\ldots\ d_{1,2}a_{1,4}-0.03217562874\ldots\ d_{1,2}d_{2,3}\\
&+0.03808922399\ldots\ a_{0,3}d_{4,1}+0.01360116867\ldots\ a_{4,0}d_{2,2}-0.8756904479\ldots\ d_{4,0}a_{4,0}\\
&+0.02040175301\ldots\ a_{1,3}a_{4,0}+0.01136518658\ldots\ a_{2,1}d_{4,1}+0.01704777983\ldots\ a_{2,1}a_{3,2}\\
&+0.01672245248\ldots\ d_{3,0}a_{1,4}+0.1754225661\ldots\ d_{3,0}d_{2,3}+0.6871190930\ldots\ d_{3,0}d_{0,5}\\
&-0.1893085336\ldots\ d_{2,1}d_{5,0}+0.06602874065\ldots\ d_{2,1}a_{0,5}-0.5192937428\ldots\ d_{0,3}d_{1,4}\\
&+0.02040175301\ldots\ a_{2,2}a_{3,1}+0.03843098248\ldots\ d_{1,3}a_{3,1}-0.1572827152\ldots\ d_{1,3}d_{4,0}\\
&+0.03843098248\ldots\ d_{3,1}a_{1,3}-0.09726371801\ldots\ d_{3,1}d_{0,4}-0.1572827153\ldots\ d_{3,1}d_{2,2}\\
&+0.06642416099\ldots\ a_{2,2}d_{0,4}-0.1102740074\ldots\ d_{1,2}d_{4,1}-0.03808922386\ldots\ d_{3,0}d_{4,1}\\
&+0.08654771082\ldots\ d_{3,0}a_{3,2}-0.02953278001\ldots\ a_{0,3}a_{1,4}-0.001186261762\ldots\ a_{2,2}a_{1,3}\\
&+0.04902727950\ldots\ a_{1,2}d_{5,0}-0.5192937424\ldots\ d_{0,3}d_{3,2}+0.03843098248\ldots\ d_{0,4}a_{4,0}\\
&+0.06642416098\ldots\ d_{1,3}a_{1,3}-0.0001977103018\ldots\ a_{1,2}a_{2,3}-0.1316620603\ldots\ d_{1,3}d_{0,4}\\
&-0.09726371795\ldots\ d_{1,3}d_{2,2}-0.01300803776\ldots\ d_{0,3}a_{2,3}-0.0003954205650\ldots\ d_{2,1}a_{2,3}\\
&-0.5679256019\ldots\ d_{3,1}d_{4,0}-0.2941636775\ldots\ d_{2,1}d_{3,2}+0.3592038668\ldots\ d_{0,3}a_{0,5}\\
&+0.1300803779\ldots\ a_{3,1}a_{4,0}+0.3250032343\ldots\ d_{0,4}a_{0,4}+0.05543244326\ldots\ a_{1,2}d_{3,2}\\
&-0.03222352905\ldots\ a_{1,2}a_{0,5}-0.2349355213\ldots\ d_{2,1}a_{4,1}+0.03843098248\ldots\ a_{0,4}d_{4,0}
\end{aligned}
$$

$$
\begin{aligned}
&-0.2021188616\ldots d_{2,1}d_{1,4}-0.4964802492\ldots d_{0,3}a_{4,1}+0.3363903734\ldots d_{0,3}d_{5,0}\\
&-0.03083955696\ldots a_{0,4}a_{1,3}+0.02621378586\ldots a_{1,2}a_{4,1}+0.08184393943\ldots a_{1,2}d_{1,4}\\
&+0.04709488526\ldots a_{2,1}d_{2,3}+0.01360116867\ldots a_{2,2}d_{4,0}+0.03843098248\ldots a_{2,2}d_{2,2}\\
&-0.002024227119\ldots a_{2,1}a_{1,4}+0.08340366999\ldots a_{2,1}d_{0,5}+0.01360116867\ldots d_{3,1}a_{3,1},\\
w_{10}=&\,0.001592179550\ldots d_{5,0}a_{1,4}+0.2947572926\ldots d_{5,0}d_{0,5}-0.004852826544\ldots a_{1,4}a_{0,5}\\
&+0.04840928673\ldots d_{0,5}a_{0,5}+0.006676120637\ldots a_{2,3}d_{0,5}-0.04618970216\ldots a_{4,1}d_{4,1}\\
&+0.006010711593\ldots d_{4,1}a_{0,5}-0.02589608392\ldots d_{4,1}d_{1,4}-0.02640370046\ldots a_{4,1}d_{0,5}\\
&+0.04233997947\ldots a_{3,2}d_{5,0}-0.0004746561370\ldots a_{3,2}a_{0,5}+0.006900020517\ldots d_{3,2}a_{1,4}\\
&-0.01162168283\ldots d_{3,2}d_{0,5}+0.006587474736\ldots a_{3,2}d_{1,4}+0.006434781110\ldots a_{4,1}a_{3,2}\\
&+0.01080894622\ldots d_{2,3}a_{0,5}+0.01130732797\ldots d_{1,4}a_{1,4}-0.02441752356\ldots d_{1,4}d_{0,5}\\
&+0.004691871440\ldots d_{2,3}a_{2,3}+0.09426342039\ldots d_{2,3}d_{5,0}+0.001730289429\ldots a_{3,2}a_{2,3}\\
&+0.002963794680\ldots a_{4,1}a_{1,4}-0.01070894437\ldots a_{4,1}d_{2,3}-0.0002494984216\ldots a_{2,3}a_{1,4}\\
&-0.02134227821\ldots d_{3,2}d_{2,3}+0.001153526288\ldots d_{4,1}a_{2,3}-0.02225290313\ldots d_{4,1}d_{5,0}\\
&-0.01752438166\ldots d_{1,4}d_{2,3}+0.006593334965\ldots d_{3,2}a_{3,2}-0.05806429194\ldots d_{3,2}d_{4,1}.
\end{aligned}
$$

The coefficients ω_0, ω_2, ω_4, ω_6, ω_8 and ω_{10} are independent because $a_{0,0}d0,0$ only appears in ω_0, $d_{0,1}^2$ only appears in ω_2, $e_{0,1}$ only appears in ω_4, $e_{0,3}$ only appears in ω_6, $b5,0$ only appears in ω_8 and $d_{5,0}d0,5$ only appears in ω_{10}. Using lemma 3.3.1 (Descartes Theorem), we can choose the coefficients ω_0, ω_2, ω_4, ω_6, ω_8 and ω_{10} such that $F_{20}(r_0)$ has 5, 4, 3, 2, 1 or 0 simple positive real zeros. As a consequence, system (4.3.6) can have at most five limit cycles bifurcating from the periodic orbits surrounding the origin by using the second order averaging method (Theorem 2.3.3) and this number can be reached.

4.4.2 Proof of Corollary 4.3.1

System (4.3.6) with the assumptions that $P_1(x,y), P_2(x,y), Q_1(x,y), Q_2(x,y)$ are real polynomials of degree four is given by:

$$
\begin{aligned}
\dot{x} =& -y(3x^2+y^2)+\varepsilon\sum_{i+j=0}^{4} a_{i,j}x^i y^j+\varepsilon^2\sum_{i+j=0}^{4} b_{i,j}x^i y^j,\\
\dot{y} =& \,x(x^2+y^2)+\varepsilon\sum_{i+j=0}^{4} d_{i,j}x^i y^j+\varepsilon^2\sum_{i+j=0}^{4} e_{i,j}x^i y^j.
\end{aligned}
\tag{4.4.16}
$$

By following the same steps as in the proof of Theorem 4.3.1, the second order averaged function associated to system (4.4.16) can be obtained immediately from (4.4.15) by taking $a_{i,j} = b_{i,j} = d_{i,j} = e_{i,j} = 0$ for all $i + j = 5$. The explicit expression of $F_{20}(r_0)$ is:

$$F_{20}(r_0) = \frac{1}{r_0^5}\Big(\vartheta_0 + \vartheta_2 r^2 + \vartheta_4 r_0^4 + \vartheta_6 r_0^6 + \vartheta_8 r_0^8\Big), \tag{4.4.17}$$

with $\vartheta_0, \vartheta_2, \vartheta_4, \vartheta_6, \vartheta_8$ being exactly $\omega_0, \omega_2, \omega_4, \omega_6, \omega_8$ respectively, with $a_{i,j} = b_{i,j} = d_{i,j} = e_{i,j} = 0$ for all $i + j = 5$. Then $F_{20}(r_0) = 0$ equivalent to

$$f(r_0) = \vartheta_0 + \vartheta_2 r^2 + \vartheta_4 r_0^4 + \vartheta_6 r_0^6 + \vartheta_8 r_0^8 = 0.$$

Because the coefficients ϑ_l with $l = 0, 2, 4, 6, 8$ are independent, it is obvious that $f(r_0)$ has at most four positive zeros. In addition, the coefficients $\vartheta_0, \vartheta_2, \vartheta_4, \vartheta_6, \vartheta_8$ can be chosen in a way such that $f(r_0)$ has exactly four zeros. From Theorem 2.3.3, system (4.4.16) can have at most three limit cycles bifurcating from the periodic orbits of the degenerate center (4.3.2).

4.4.3 Proof of Corollary 4.3.2

Let $P_1(x, y), P_2(x, y), Q_1(x, y), Q_2(x, y)$ be real polynomials of degree five without constants, then system (4.3.6) becomes:

$$\begin{aligned} \dot{x} &= -y(3x^2 + y^2) + \varepsilon \sum_{i+j=1}^{5} a_{i,j} x^i y^j + \varepsilon^2 \sum_{i+j=1}^{5} b_{i,j} x^i y^j, \\ \dot{y} &= x(x^2 + y^2) + \varepsilon \sum_{i+j=1}^{5} d_{i,j} x^i y^j + \varepsilon^2 \sum_{i+j=1}^{5} e_{i,j} x^i y^j. \end{aligned} \tag{4.4.18}$$

The second order averaged function of system (4.4.18) can be obtained from (4.4.15). So $F_{20}(r)$ is given by

$$F_{20}(r_0) = \upsilon_2 \frac{1}{r_0^3} + \upsilon_4 \frac{1}{r_0} + \upsilon_6 r_0 + \upsilon_8 r_0^3 + \upsilon_{10} r_0^5, \tag{4.4.19}$$

where $\upsilon_2, \upsilon_4, \upsilon_6, \upsilon_8, \upsilon_{10}$ are exactly $\omega_2, \omega_4, \omega_6, \omega_8, \omega_{10}$ respectively with $a_{0,0} = 0$ and $d_{0,0} = 0$. The coefficients $\upsilon_2, \upsilon_4, \upsilon_6, \upsilon_8, \upsilon_{10}$ are independent.

$F_{20}(r_0) = 0$ is equivalent to $\upsilon_2 + \upsilon_4 r_0^2 + \upsilon_6 r_0^4 + \upsilon_8 r_0^6 + \upsilon_{10} r_0^8 = 0$. Using lemma 3.3.1 (Descartes Theorem), we can choose the coefficients υ_2, υ_4, υ_6, υ_8 and υ_{10} such that $F_{20}(r_0)$ has at most four simple positive real zeros. A a result, from Theorem 2.3.3 system (4.4.18) can have at most four limit cycles bifurcating from the periodic orbits of the degenerate center (4.3.2).

4.5 Examples

To ensure the results of Theorem 4.3.1, we present the following numerical examples.

Example 4.5.1 ***(Example with*** 5 ***limit cycles)***

Consider the following system

$$\begin{cases} \dot{x} &= -y(3x^2+y^2)+\varepsilon(y^5+813.8487421x^5)+\varepsilon^2(4874.474235x \\ & -1925.496694x^3+500.3665045x^6), \\ \dot{y} &= x(x^2-y^2)+\varepsilon(\frac{1}{2}-20.29310103y^2-4376.46688x^2y^3). \end{cases} \tag{4.5.20}$$

For $\varepsilon = 0.00001$*, the expression of* $F_{20}(r_0)$ *of* (4.5.20) *is given by*

$$F_{20}(r_0) = 665.2264929\frac{1}{r_0^5} - 3894531546\frac{1}{r_0^3} + 5650.729487\frac{1}{r_0} - 3024.563121r_0 \\ +650.4436819r_0^3 - 47.30499505r_0^5.$$

$F_{20}(r_0)$ *has five positive zeros near to:* $r_0 = 0.5\,, 1\,, 1.5\,, 2\,, 2.5$. *This means that the system* (4.5.20) *has* 5 *limit cycles bifurcating from the degenerate center of the unperturbed system of* (4.5.20), *see Figure 4.5.1* .

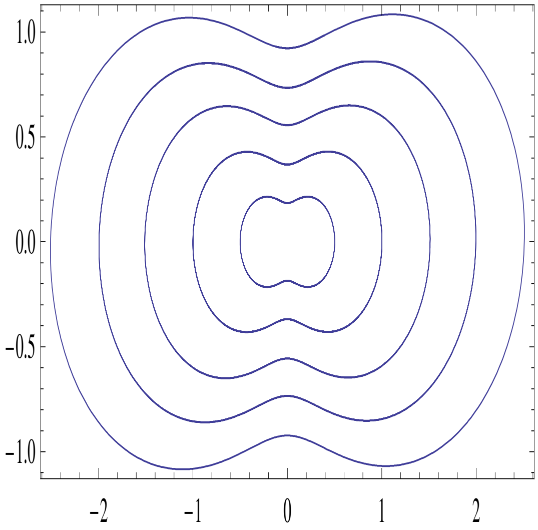


Figure 4.5.1 : Five limit cycles for system (4.5.20) with $\varepsilon = 0.00001$

Example 4.5.2 *(**Example with** 4 **limit cycles**)*

We consider the system

$$\left\{\begin{array}{ll} \dot{x} & =-y(3x^2+y^2)+\varepsilon(1-0.3768477367x^5)-0.8626274275x\varepsilon^2, \\ \dot{y} & =x(x^2-y^2)+\varepsilon(3-7.558400223y^2+y^5+3.91816296x^5) \\ & +\varepsilon^2(67.27415794y^3-63.65366812x^4y). \end{array}\right. \tag{4.5.21}$$

The function $F_{20}(r_0)$ *of the previous system with* $\varepsilon=0.001$ *is given by*

$$\begin{aligned} F_{20}(r_0) = {} & 1995.679479\frac{1}{r_0^5}-2175.844988\frac{1}{r_0^3}-1.154907016\frac{1}{r_0}+211.3480004r_0 \\ & -31.18249186r_0^3+1.154907106r_0^5. \end{aligned}$$

The equation $F_{20}(r_0)=0$ *has exactly four positive solutions:*$r_0=1\,;\,2\,;\,3\,;\,4$. *As consequence of it, there is* 4 *limit cycles bifurcating from the global center of the unperturbed* (4.5.21)*, see Figure 4.5.2 .*

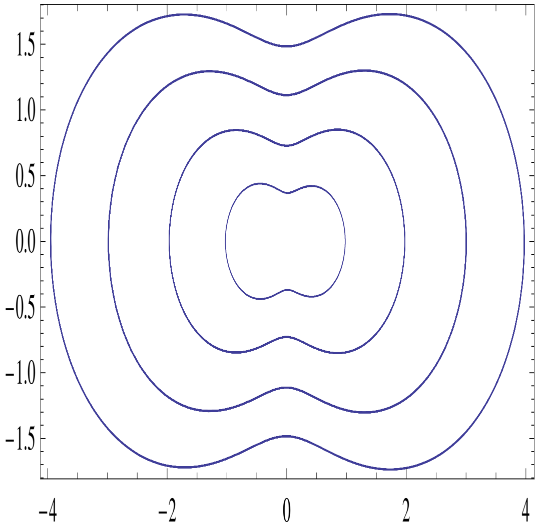


Figure 4.5.2 : Four limit cycles for system (4.5.21) with $\varepsilon=0.001$

Example 4.5.3 *(**Example with** 3 **limit cycles**)*

For $\varepsilon=0.0001$*, the system*

$$\left\{\begin{array}{ll} \dot{x} & =-y(3x^2+y^2)+\varepsilon(4-0.376877367x^5)+508.1571728x^3y^2\varepsilon^2, \\ \dot{y} & =x(x^2-y^2)+\varepsilon(\tfrac{1}{2}-1.179494576y^2+y^5-10.44843456x^5) \\ & +\varepsilon^2(11.73841899y-101.9528222x^2y). \end{array}\right. \tag{4.5.22}$$

has the second averaged function

$$F_{20}(r_0) = 332.6132465\frac{1}{r_0^5} - 452.7235855\frac{1}{r_0^3} + 240.220678\frac{1}{r_0} - 160.1471187r_0 + 43.11653195r_0^3 - 3.079752282r_0^5.$$

If we suppose that $F_{20}(r_0) = 0$, *we obtain three positive values of* r_0 *which are:* $r_0 = 1$, $r_0 = 2$ *and* $r_0 = 3$. *Thus the maximum number of limit cycles bifurcates from the global center of the unperturbed* (4.5.21) *is three, see Figure* 4.5.3 .

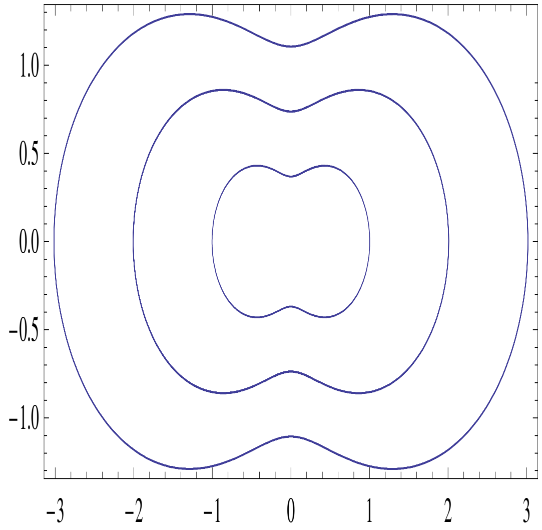


Figure 4.5.3 : Three limit cycles for system (4.5.22) with $\varepsilon = 0.0001$

Example 4.5.4 ***(Example with** 2 **limit cycles)***

The system

$$\begin{cases} \dot{x} &= -y(3x^2+y^2) + \varepsilon(4 - 0.376877367x^5) + 508.1571728x^3y^2\varepsilon^2, \\ \dot{y} &= x(x^2-y^2) + \varepsilon(\frac{1}{2} - 1.179494576y^2 + y^5 - 10.44843456x^5) \\ & +\varepsilon^2(11.73841899y - 101.9528222x^2y). \end{cases} \tag{4.5.23}$$

with $\varepsilon = 0.001$ *has*

$$F_{20}(r_0) = 332.6132465\frac{1}{r_0^5} - 452.7235855\frac{1}{r_0^3} + 240.220678\frac{1}{r_0} - 160.1471187r_0 + 43.11653195r_0^3 - 3.079752282r_0^5.$$

The equation $F_{20}(r_0) = 0$ *has two positive real solutions* $r_0 = 1$ *and* $r_0 = 3$. *Hence two limit cycles bifurcated from the global center of the unperturbed system* (4.5.23), *see Figure*4.5.4 .

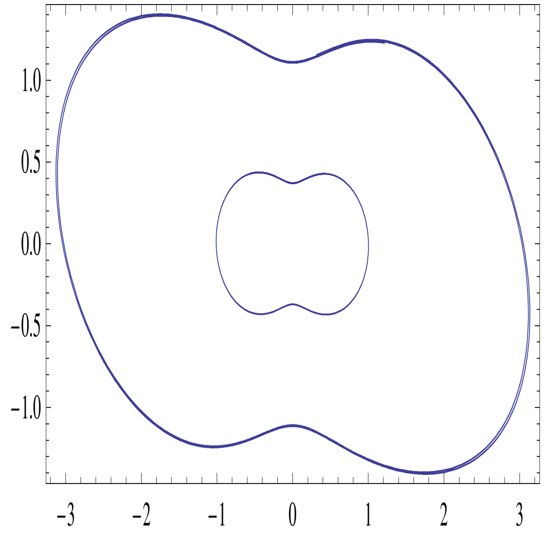


Figure 4.5.4 : Two limit cycles for system (4.5.23) with $\varepsilon = 0.001$

Example 4.5.5 ***(Example with one limit cycles)***

For $\varepsilon = 0.0001$, *we consider the system*

$$\begin{cases} \dot{x} & = -y(3x^2+y^2)+\varepsilon(1+152.1930999xy-0.376877367x^5) \\ & \quad +\varepsilon^2(5355.864437x), \\ \dot{y} & = x(x^2-y^2)+\varepsilon(7+y^5-1316.502754x^5)+(-988.1581229x^2y \\ & \quad +6421.031503x^2y^3)\varepsilon^2. \end{cases} \tag{4.5.24}$$

The function $F_{20}(r_0)$ *of* (4.5.24) *is*

$$\begin{aligned} F_{20}(r_0) = +4656.585450\frac{1}{r_0^5} - 1164.146362\frac{1}{r_0^3} + 6208.780600\frac{1}{r_0} \\ + 1552.195150r_0^3 - 1558.19150r_0 - 388.0487875r_0^5. \end{aligned}$$

One limit cycle can bifurcate from the degenerate center of the unperturbed system of (4.5.24) *because* $r_0 = 2$ *is the only positive root of* $F_{20}(r_0)$, *see Figure* 4.5.5 .

Example 4.5.6 ***(Example with zero limit cycle)***

Consider the system

$$\begin{cases} \dot{x} & = -y(3x^2+y^2)+\varepsilon(1-0.376877367x^5)+2613.379742x^3y^2\varepsilon^2, \\ \dot{y} & = x(x^2-y^2)+\varepsilon(1+3.466272714y^2+y^5+376.1436442x^5) \\ & \quad +\varepsilon^2(43.34185473y+282.3308924x^2y), \end{cases} \tag{4.5.25}$$

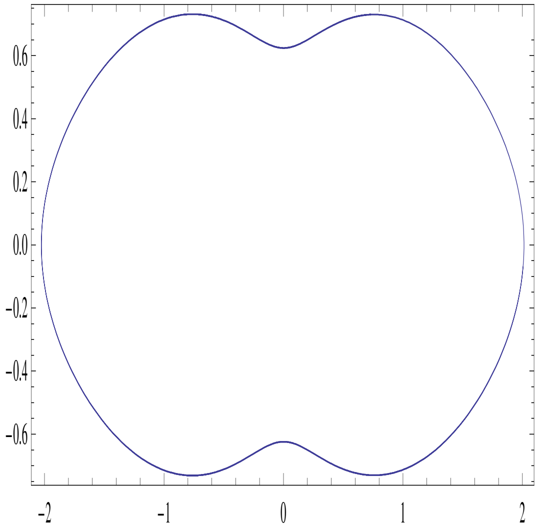


Figure 4.5.5 : One limit cycle for system (4.5.24) with $\varepsilon = 0.0001$

with $\varepsilon = 0.001$. *We have that*

$$F_{20}(r_0) = 665.2264932\frac{1}{r_0^5} + 332.6132466\frac{1}{r_0^3} + 886.9686576\frac{1}{r_0} + 443.4843288\, r_0 + 221.7421644\, r_0^3 + 110.8710822\, r_0^5.$$

The equation $F_{20}(r_0) = 0$ *does not have a positive zero, which implies that there is no limit cycle bifurcating from the global center of the unperturbed system* (4.5.25), *see Figure* 4.5.6 .

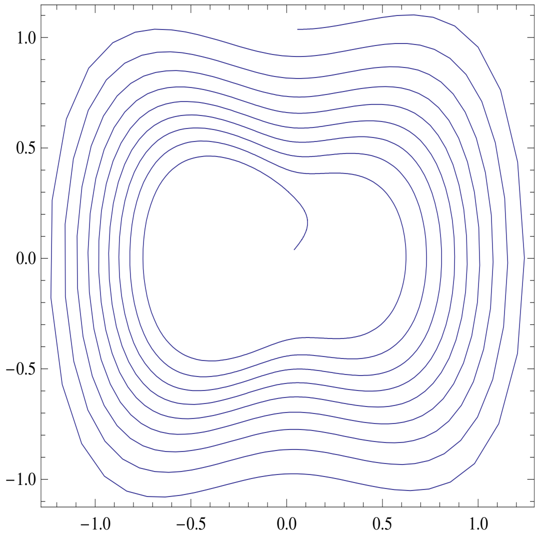


Figure 4.5.6 : No limit cycle for system (4.5.25) with $\varepsilon = 0.001$

CHAPTER

5 Conclusion and future work

5.1 Conclusion

This thesis investigates the bifurcation of limit cycles in smooth and non-smooth systems using *Averaging theory*. In particular, we provide new lower bounds for the maximum number of limit cycles bifurcating from a center in planar polynomial vector fields and switching systems.

In Chapter 3, it is shown that the differential systems (3.3.5), (3.3.6) and (3.3.7) can have at most 3, at least 9, at most 19 limit cycles, respectively, by using the averaging theory. Several examples are presented to ensure the obtained results. The algebraic manipulators Maple and Mathematica were used to verify the computations.

In Chapter 4, we investigate the limit cycles that can bifurcate from a degenerate center. The problem of the degenerate center is poorly understood. Under quintic perturbations up to second order and using the averaging method of second order, we prove that system (4.3.6) can have at most five limit cycles. As far as we know, this study is one of the first complete studies up to second order in the small parameter of perturbations conducted for studying the limit cycles which bifurcate from the periodic orbits surrounding a degenerate center.

5.2 Future works

For future works, there are many interesting but also challenging problems that are worth exploring. In Chapter 4, the cubic degenerate center is perturbed by quintic polynomial differential system. Our interest is in the maximum number of limit cycles when we perturb the degenerate center inside the class of all polynomial systems of degree n by using the second order averaging method. Thus, our first question is: can we find a way to overcome the difficulty and obtain more limit cycles? On the other hand, the limit cycles bifurcating from the origin should be finite. The next question is: can we provide an upper bound for the number of limit cycles?

For switching systems, we apply the averaging method for computing the crossing limit cycles. We may complete this study by including the Pseudo-Hopf bifurcation for studying the existence of limit cycles in the sliding region.

We also want to extend our investigation to the study of the existence of limit cycles for biological and chemical models, see for example [23], using the averaging method and considering these models in cases of discontinuity.

APPENDIX A

Appendix A

In this appendix, we present the coefficients $A_1, A_2, \dots, A_{13}$ and $B_1, B_2, \dots, B_{15}$, which appear in $y^1(\theta, r)$ and in $F_{20}(r)$, respectively.

$$\begin{aligned}
\tilde{A}_1 &= -\frac{\left(r^2+1\right)\left(2\left(r^2+1\right)a_{1,3}+2\,b_{2,2}-4\,b_{0,4}+2\,b_{0,2}-3\,a_{3,1}+a_{1,3}-a_{1,1}\right)+b_{0,0}-b_{2,0}-3\,b_{4,0}+b_{2,2}+b_{0,4}-b_{0,2}}{2r},\\
\tilde{A}_2 &= \frac{\left(r^2+1\right)\left(b_{3,1}-3\,b_{1,3}+b_{1,1}-4\,a_{4,0}+\left(r^2+1\right)\left(a_{4,0}+a_{2,2}+a_{2,0}+a_{0,2}+a_{0,0}-3\,a_{0,4}\right)+2\,a_{2,2}-2\,a_{2,0}\right)+2\,b_{3,1}}{2r},\\
\tilde{A}_3 &= \frac{\left(r^2+1\right)\left(a_{2,0}+a_{0,4}-a_{0,2}+a_{0,0}-a_{2,2}+a_{4,0}\right)-1/2\,b_{1,1}+1/2\,b_{1,3}-1/2\,b_{3,1}}{2},\\
\tilde{A}_4 &= \frac{\left(r^2+1\right)\left(-a_{3,1}+a_{1,3}-a_{1,1}\right)-1/2\,b_{0,0}+1/2\,b_{0,2}-1/2\,b_{0,4}-1/2\,b_{2,0}+1/2\,b_{2,2}-1/2\,b_{4,0}}{2},\\
\tilde{A}_5 &= -\frac{\left(r^2+1\right)\left(\left(r^2+1\right)\left(a_{2,3}-2\,a_{0,5}+a_{0,3}\right)+b_{3,2}-2\,b_{1,4}+b_{1,2}-2\,a_{4,1}+a_{2,3}-a_{2,1}\right)-b_{3,0}-2\,b_{5,0}+b_{3,2}}{2r},\\
\tilde{A}_6 &= \frac{\left(r^2+1\right)\left(2\left(r^2+1\right)a_{0,4}+2\,b_{1,3}-2\,a_{2,2}\right)-b_{3,1}+a_{4,0}}{2},\\
\tilde{A}_7 &= \left(r^2+1\right)\left(\left(r^2+1\right)\left(a_{1,3}-b_{0,4}\right)+b_{2,2}-a_{3,1}\right)-b_{4,0},\\
\tilde{A}_8 &= \frac{\left(r^2+1\right)\left(\left(r^2+1\right)\left(\left(r^2+1\right)a_{0,5}+b_{1,4}-a_{2,3}\right)-b_{3,2}+a_{4,1}\right)+b_{5,0}}{2r},\\
\tilde{A}_9 &= -\frac{{,}r\left(\left(2\,r^2+4\right)b_{0,3}+\left(r^2+4\right)a_{3,0}+\left(r^2+4\right)a_{1,0}+r^2\left(2\,r^2\,b_{0,5}-a_{1,2}\right)\right)}{2},\\
\tilde{A}_{10} &= \frac{r^3\left(\left(r^2-2\right)\left(a_{3,0}+a_{1,0}\right)+r^2\left(a_{1,2}+2\,b_{0,5}-2\,b_{4,1}\right)-2\,b_{0,3}\right)}{2},\\
\tilde{A}_{11} &= -\frac{r\left(\left(r^2+1\right)\left(a_{4,1}+a_{2,1}+a_{0,5}-a_{0,3}+a_{0,1}-a_{2,3}\right)+b_{1,0}-b_{1,2}+b_{1,4}+b_{3,0}-b_{3,2}+b_{5,0}\right)}{2},\\
\tilde{A}_{12} &= \frac{r\left(\left(r^2+1\right)\left(a_{3,0}+a_{1,0}-a_{1,2}\right)+5\,a_{1,0}+3\,a_{3,0}+a_{1,2}+4\,b_{0,3}\right)}{2}.
\end{aligned}$$

$$
\begin{aligned}
B_1 = {} & (-b_{0,0} - 6\,a_{3,1} + 3\,b_{2,2} + 5\,b_{0,2} - 9\,b_{0,4} + b_{2,0} + 3\,b_{4,0} - 2\,a_{1,1} + 2\,a_{1,3})\,b_{3,1} \\
& -(2\,a_{2,2} + b_{1,1} - 3\,b_{1,3} - 2\,a_{2,0} - 4\,a_{4,0})(b_{0,0} + b_{2,2} - b_{0,2} + b_{0,4} - b_{2,0} - 3\,b_{4,0}), \\
B_2 = {} & (6\,b_{4,0} - 2\,b_{2,2} - 2\,b_{0,0} + 2\,b_{0,2} - 2\,b_{0,4} + 2\,b_{2,0})\,a_{2,2} + (-2\,a_{0,0} - 2\,a_{0,2} + 6\,a_{0,4} \\
& -2\,a_{2,0} - 2\,a_{4,0})\,b_{2,2} + (6\,b_{4,0} - 2\,b_{0,0} + 2\,b_{0,2} - 2\,b_{0,4} + 2\,b_{2,0})\,a_{0,2} + (-18\,b_{4,0} \\
& +6\,b_{0,0} - 6\,b_{0,2} + 6\,b_{0,4} - 6\,b_{2,0})\,a_{0,4} + (6\,b_{4,0} - 2\,b_{0,0} + 2\,b_{0,2} - 2\,b_{0,4} + 2\,b_{2,0})\,a_{2,0} \\
& +(6\,b_{4,0} - 2\,b_{0,0} + 2\,b_{0,2} - 2\,b_{0,4} + 2\,b_{2,0})\,a_{4,0} + 6\,a_{0,0}b_{4,0} - 2\,a_{0,0}b_{0,0} + 2\,a_{0,0}b_{0,2} \\
& -2\,a_{0,0}b_{0,4} + 2\,a_{0,0}b_{2,0} + 8\,a_{1,3}b_{3,1}, \\
B_3 = {} & (-a_{0,0} + 3\,a_{2,2} + 2\,b_{1,1} - 6\,b_{1,3} + 2\,b_{3,1} - a_{0,2} + 3\,a_{0,4} - 5\,a_{2,0} - 9\,a_{4,0})\,a_{1,3} \\
& +(a_{0,0} + a_{2,2} + a_{0,2} - 3\,a_{0,4} + a_{2,0} + a_{4,0})(3\,a_{3,1} - 2\,b_{2,2} - 2\,b_{0,2} + 4\,b_{0,4} + a_{1,1}), \\
B_4 = {} & (4\,a_{1,0} + 4\,a_{3,0} + 3\,b_{0,3} + a_{1,2} - 4\,b_{4,1})\,b_{5,0} - 4\,b_{3,1}b_{4,0} + 2\,b_{3,2}b_{4,1}, \\
B_5 = {} & (-4\,b_{0,0} - 4\,b_{2,2} + 4\,b_{0,2} - 4\,b_{0,4} + 4\,b_{2,0} + 8\,b_{4,0})\,b_{3,1} + (4\,a_{1,0} + 4\,a_{3,0} + 3\,b_{0,3} \\
& +a_{1,2} - 4\,b_{4,1})\,b_{5,0} + 2\,b_{3,2}b_{4,1}, \\
B_6 = {} & (16\,b_{3,0} + 56\,b_{5,0} + 8\,a_{2,1} - 6\,a_{2,3} + 12\,a_{4,1} - 8\,b_{1,2} + 12\,b_{1,4} - 34\,b_{3,2})\,b_{4,1} \\
& +(-4\,b_{0,0} - 4\,b_{2,0} + 8\,b_{4,0} - 8\,a_{3,1} + 16\,b_{2,2} + 4\,b_{0,2} - 4\,b_{0,4})\,b_{3,1} + (3\,a_{1,2} + 6\,a_{1,0} \\
& +14\,a_{3,0} + 3\,b_{0,3} + 6\,b_{0,5})\,b_{3,2} + (8\,a_{2,2} + 8\,b_{1,1} - 12\,b_{1,3} - 16\,a_{2,0} - 32\,a_{4,0})\,b_{4,0} \\
& +(-10\,a_{1,2} - 56\,a_{1,0} - 56\,a_{3,0} - 30\,b_{0,3} - 12\,b_{0,5})\,b_{5,0} + (a_{1,2} + 4\,a_{1,0} + 4\,a_{3,0} \\
& +3\,b_{0,3})\,a_{4,1} + (-4\,a_{1,2} - 16\,a_{1,0} - 16\,a_{3,0} - 12\,b_{0,3})\,b_{3,0} - 4(a_{3,1} - 2\,b_{2,2})\,a_{4,0}, \\
B_7 = {} & (-12\,a_{0,5} - 44\,b_{3,0} + 28\,b_{1,2} - 12\,b_{1,0} - 84\,b_{5,0} - 28\,a_{2,1} + 30\,a_{2,3} - 48\,b_{1,4} - 12\,a_{0,1} \\
& +66\,b_{3,2} + 12\,a_{0,3} - 48\,a_{4,1})\,b_{4,1} + (44\,b_{3,0} - 8\,a_{4,1} + 4\,a_{2,3} - 30\,b_{3,2} - 4\,b_{1,4} - 4\,a_{2,1} \\
& +12\,b_{1,0} - 4\,a_{0,3} - 4\,b_{1,2} + 84\,b_{5,0} + 4\,a_{0,5} + 4\,a_{0,1})\,a_{1,0} + (-4\,a_{2,1} - 4\,a_{0,3} + 30\,b_{5,0} \\
& -9\,b_{3,2} + 4\,a_{0,5} + 4\,a_{2,3} - 4\,b_{1,4} + 24\,b_{3,0} - 4\,b_{1,2} - 9\,a_{4,1} + 4\,a_{0,1} + 12\,b_{1,0})\,b_{0,3} \\
& +(-4\,a_{0,5} + 4\,a_{2,1} - 4\,a_{2,3} + 12\,a_{4,1} - 4\,b_{1,2} + 12\,b_{1,4} - 30\,b_{3,2} + 48\,b_{5,0} - 4\,a_{0,1} \\
& +12\,b_{3,0} - 4\,b_{1,0} + 4\,a_{0,3})\,b_{0,5} + (2\,a_{0,0} + 2\,a_{4,0} - 14\,a_{2,2} - 6\,b_{1,1} + 22\,b_{1,3} + 2\,b_{3,1} \\
& -2\,a_{0,2} + 2\,a_{0,4} + 6\,a_{2,0})\,b_{2,2} + (6\,b_{0,2} - 18\,b_{0,4} - 2\,b_{2,0} - 6\,b_{4,0} + 6\,b_{0,0} - 12\,a_{1,1} \\
& +12\,a_{1,3} - 20\,a_{3,1})\,b_{3,1} + (-14\,b_{0,2} + 30\,b_{0,4} - 10\,b_{2,0} - 2\,b_{4,0} - 2\,b_{0,0} + 8\,a_{1,1} - 8\,a_{1,3} \\
& +20\,a_{3,1})\,a_{4,0} + (-6\,a_{0,0} + 10\,a_{2,2} + 2\,b_{1,1} - 10\,b_{1,3} + 6\,a_{0,2} - 6\,a_{0,4} - 2\,a_{2,0})\,b_{0,2} \\
& +(6\,a_{0,0} - 10\,a_{2,2} - 6\,b_{1,1} + 14\,b_{1,3} - 6\,a_{0,2} + 6\,a_{0,4} + 2\,a_{2,0})\,b_{0,4} + (-2\,a_{0,0} + 14\,a_{2,2} \\
& +2\,b_{1,1} - 18\,b_{1,3} + 2\,a_{0,2} - 2\,a_{0,4} - 6\,a_{2,0})\,b_{2,0} + (6\,a_{0,0} + 14\,a_{2,2} + 2\,b_{1,1} - 30\,b_{1,3}
\end{aligned}
$$

$$
\begin{aligned}
& -6\,a_{0,2}+6\,a_{0,4}+2\,a_{2,0})\,b_{4,0}+(6\,a_{0,0}-10\,a_{2,2}+2\,b_{1,1}+6\,b_{1,3}-6\,a_{0,2}+6\,a_{0,4} \\
& +2\,a_{2,0})\,b_{0,0}+(4\,b_{1,0}+8\,b_{3,0}+10\,b_{5,0}+a_{4,1}-4\,b_{1,2}+4\,b_{1,4}-9\,b_{3,2})\,a_{1,2}+(12\,b_{1,0} \\
& +44\,b_{3,0}+84\,b_{5,0}+4\,a_{4,1}-12\,b_{1,2}+12\,b_{1,4}-46\,b_{3,2})\,a_{3,0}, \\
B_8 = \; & (-4\,a_{0,5}-20\,a_{2,1}+26\,a_{2,3}-48\,a_{4,1}+20\,b_{1,2}-48\,b_{1,4}+54\,b_{3,2}-60\,b_{5,0}-4\,a_{0,1} \\
& -36\,b_{3,0}-4\,b_{1,0}+4\,a_{0,3})\,b_{0,5}+(48\,a_{0,5}+20\,b_{3,0}-28\,b_{1,2}+20\,b_{1,0}+28\,a_{2,1}-54\,a_{2,3} \\
& +60\,b_{1,4}+20\,a_{0,1}-30\,b_{3,2}-36\,a_{0,3}+60\,a_{4,1})\,b_{4,1}+(-20\,b_{3,0}-20\,a_{4,1}-14\,a_{2,3} \\
& +40\,b_{3,2}-4\,b_{1,4}+4\,a_{2,1}-20\,b_{1,0}-4\,a_{0,3}+20\,b_{1,2}+4\,a_{0,5}+4\,a_{0,1})\,a_{1,0}+(2\,a_{0,0} \\
& +2\,a_{4,0}+2\,a_{2,2}+2\,b_{1,1}-14\,b_{1,3}-14\,b_{3,1}-2\,a_{0,2}+2\,a_{0,4}+6\,a_{2,0})\,b_{2,2}+(-2\,b_{0,2} \\
& -22\,b_{0,4}+6\,b_{2,0}-2\,b_{4,0}+6\,b_{0,0}+4\,a_{1,1}+12\,a_{1,3}+12\,a_{3,1})\,b_{3,1}+(8\,a_{2,1}-4\,a_{0,3} \\
& +15\,b_{5,0}+6\,b_{3,2}+4\,a_{0,5}-17\,a_{2,3}+11\,b_{1,4}+8\,b_{1,2}+6\,a_{4,1}+4\,a_{0,1}-12\,b_{1,0})\,b_{0,3} \\
& +(2\,b_{0,2}-18\,b_{0,4}-10\,b_{2,0}+30\,b_{4,0}-2\,b_{0,0}-16\,a_{1,1}+28\,a_{1,3}-28\,a_{3,1})\,a_{4,0} \\
& +(-6\,a_{0,0}+10\,a_{2,2}+2\,b_{1,1}-10\,b_{1,3}+6\,a_{0,2}-6\,a_{0,4}-2\,a_{2,0})\,b_{0,2}+(6\,a_{0,0}-10\,a_{2,2} \\
& -6\,b_{1,1}+14\,b_{1,3}-6\,a_{0,2}+6\,a_{0,4}+2\,a_{2,0})\,b_{0,4}+(-2\,a_{0,0}-2\,a_{2,2}+2\,b_{1,1}+6\,b_{1,3} \\
& +2\,a_{0,2}-2\,a_{0,4}-6\,a_{2,0})\,b_{2,0}+(6\,a_{0,0}+14\,a_{2,2}-6\,b_{1,1}+18\,b_{1,3}+26\,a_{0,2}-42\,a_{0,4} \\
& +18\,a_{2,0})\,b_{4,0}+(6\,a_{0,0}-10\,a_{2,2}+2\,b_{1,1}+6\,b_{1,3}-6\,a_{0,2}+6\,a_{0,4}+2\,a_{2,0})\,b_{0,0} \\
& +(-4\,b_{1,0}+5\,b_{5,0}-4\,a_{2,1}+3\,a_{2,3}-10\,a_{4,1}+8\,b_{1,2}-11\,b_{1,4}+6\,b_{3,2})\,a_{1,2} \\
& +(-20\,b_{1,0}-20\,b_{3,0}-8\,a_{2,1}+6\,a_{2,3}-40\,a_{4,1}+28\,b_{1,2}-36\,b_{1,4}+40\,b_{3,2})\,a_{3,0} \\
& +4\,a_{3,1}\,(a_{2,2}-2\,a_{2,0}), \\
B_9 = \; & (6\,a_{2,3}+4\,b_{3,2}-5\,a_{0,1}-8\,b_{1,0}-40\,b_{3,0}-84\,b_{5,0}+16\,a_{2,1}+48\,a_{4,1}-16\,b_{1,2}+24\,b_{1,4} \\
& +24\,a_{0,3}-40\,a_{0,5})\,a_{1,0}+(-11\,b_{5,0}-9\,a_{2,3}+10\,a_{4,1}+5\,b_{1,4}-4\,b_{1,0}+4\,a_{0,1}+8\,a_{2,1} \\
& -4\,a_{0,3}+4\,a_{0,5}+6\,b_{3,2}-8\,b_{3,0})\,a_{1,2}+(60\,b_{1,4}-58\,a_{2,3}-32\,b_{1,2}+24\,a_{0,1}+60\,a_{4,1} \\
& -30\,b_{3,2}+56\,a_{0,5}+32\,a_{2,1}-40\,a_{0,3}+24\,b_{3,0}+24\,b_{1,0})\,b_{0,5}+(8\,a_{0,1}+8\,a_{2,1}+84\,b_{5,0} \\
& -60\,a_{0,5}+30\,a_{2,3}+24\,a_{0,3}-42\,b_{3,2}+40\,b_{3,0}-8\,b_{1,2}+8\,b_{1,0})\,b_{4,1}+(-33\,b_{5,0}+23\,a_{2,3} \\
& +6\,b_{3,2}-5\,b_{1,4}-44\,a_{0,5}-12\,b_{1,0}+28\,a_{0,3}+6\,a_{4,1}-12\,a_{0,1}-24\,b_{3,0})\,b_{0,3}+(-16\,b_{0,2} \\
& +20\,b_{0,4}-12\,b_{2,0}-12\,b_{4,0}-2\,a_{1,1}+2\,a_{1,3}-14\,a_{3,1}+12\,b_{0,0}+16\,b_{2,2})\,a_{2,2}+(-16\,a_{2,0} \\
& -12\,a_{4,0}-12\,a_{0,0}+6\,b_{1,1}-18\,b_{1,3}+2\,b_{3,1}+4\,a_{0,2}+20\,a_{0,4})\,b_{2,2}+(-8\,b_{1,0}-40\,b_{3,0} \\
& -84\,b_{5,0}+16\,a_{2,1}-18\,a_{2,3}+40\,a_{4,1}-8\,b_{1,2}+24\,b_{1,4}+20\,b_{3,2})\,a_{3,0}+(12\,b_{0,2}-48\,b_{0,4} \\
& +16\,b_{2,0}+8\,b_{4,0}-2\,a_{1,1}-18\,a_{1,3}-6\,a_{3,1}+8\,b_{0,0})\,a_{4,0}+(-4\,a_{2,0}-8\,a_{0,0}+22\,b_{1,1} \\
& -46\,b_{1,3}+50\,b_{3,1}+16\,a_{0,2}-8\,a_{0,4})\,b_{0,4}+(-2\,a_{2,0}+2\,a_{0,0}-4\,b_{1,1}+4\,b_{1,3}-2\,a_{0,2}
\end{aligned}
$$

$$
\begin{aligned}
& +2\,a_{0,4})\,a_{1,3}+(-10\,b_{0,2}-2\,b_{2,0}-2\,b_{4,0}+12\,a_{1,1}+20\,a_{3,1}-2\,b_{0,0})\,b_{3,1}+(-12\,b_{0,2}\\
& +8\,b_{4,0}+2\,a_{1,1}-6\,a_{3,1}+8\,b_{0,0})\,a_{0,2}+(4\,b_{0,2}-24\,b_{2,0}-48\,b_{4,0}-2\,a_{1,1}+6\,a_{3,1})\,a_{0,4}\\
& +(12\,b_{2,0}+4\,b_{4,0}+2\,a_{1,1}+10\,a_{3,1}+4\,b_{0,0})\,a_{2,0}+(4\,a_{0,0}-2\,b_{1,1}+18\,b_{1,3})\,b_{0,2}+(8\,a_{0,0}\\
& -2\,b_{1,1}+18\,b_{1,3})\,b_{2,0}+(-2\,b_{1,1}+30\,b_{1,3})\,b_{4,0}-2(b_{1,1}+3\,b_{1,3})\;b_{0,0}+2(-a_{1,1}+3a_{3,1})\,a_{0,0},\\
B_{10}= \;& (20\,a_{2,3}-16\,b_{1,2}-42\,b_{3,2}-5\,a_{0,1}+24\,b_{1,0}+40\,b_{3,0}+56\,b_{5,0}-16\,a_{2,1}-4\,a_{4,1}-8\,b_{1,4}\\
& -8\,a_{0,3}+40\,a_{0,5})\,a_{1,0}+(12\,b_{5,0}-2\,a_{2,3}-9\,b_{3,2}-10\,b_{1,4}+51\,a_{0,5}+12\,b_{1,0}-16\,a_{0,3}-8\,a_{2,1}\\
& -8\,b_{1,2}-9\,a_{4,1}-4\,a_{0,1}+12\,b_{3,0})\,b_{0,3}+(4\,b_{5,0}+6\,a_{2,3}+5\,a_{4,1}+10\,b_{1,4}+4\,b_{1,0}-4\,a_{0,1}\\
& -8\,b_{1,2}+8\,a_{0,3}-11\,a_{0,5}-9\,b_{3,2}+4\,b_{3,0})\,a_{1,2}+(-24\,a_{0,1}-32\,a_{2,1}-56\,b_{5,0}+30\,a_{2,3}\\
& +24\,a_{0,3}-60\,b_{1,4}-60\,a_{4,1}+58\,b_{3,2}-40\,b_{3,0}+32\,b_{1,2}-24\,b_{1,0})\,b_{4,1}+(42\,a_{2,3}+8\,b_{1,2}\\
& -8\,a_{0,1}-30\,b_{3,2}+60\,b_{5,0}-84\,a_{0,5}-8\,a_{2,1}+40\,a_{0,3}+24\,b_{3,0}-8\,b_{1,0})\,b_{0,5}+(-16\,b_{0,2}\\
& +12\,b_{0,4}+4\,b_{2,0}-20\,b_{4,0}+6\,a_{1,1}-2\,a_{1,3}+18\,a_{3,1}+12\,b_{0,0}-16\,b_{2,2})\,a_{2,2}+(-16\,a_{2,0}\\
& -20\,a_{4,0}-12\,a_{0,0}-2\,b_{1,1}+14\,b_{1,3}-2\,b_{3,1}-12\,a_{0,2}+12\,a_{0,4})\,b_{2,2}+(-4\,b_{0,2}+48\,b_{0,4}\\
& +16\,b_{2,0}+8\,b_{4,0}+22\,a_{1,1}-50\,a_{1,3}+46\,a_{3,1}+8\,b_{0,0})\,a_{4,0}+(-10\,a_{2,0}+2\,a_{0,0}+12\,b_{1,1}\\
& -20\,b_{1,3}-2\,a_{0,2}+2\,a_{0,4})\,a_{1,3}+(12\,a_{2,0}-8\,a_{0,0}-2\,b_{1,1}+6\,b_{1,3}+18\,b_{3,1}+16\,a_{0,2}\\
& -8\,a_{0,4})\,b_{0,4}+(-2\,b_{0,2}-2\,b_{2,0}-2\,b_{4,0}-4\,a_{1,1}-4\,a_{3,1}-2\,b_{0,0})\,b_{3,1}+(4\,b_{0,2}+8\,b_{2,0}\\
& +48\,b_{4,0}-2\,a_{1,1}-30\,a_{3,1})\,a_{0,4}+(-12\,b_{0,2}-24\,b_{4,0}+2\,a_{1,1}+18\,a_{3,1}+8\,b_{0,0})\,a_{0,2}\\
& +(24\,b_{1,0}+40\,b_{3,0}+56\,b_{5,0}+12\,a_{2,3}+20\,a_{4,1}-24\,b_{1,2}+24\,b_{1,4}-50\,b_{3,2})\,a_{3,0}\\
& +(12\,b_{2,0}+4\,b_{4,0}+2\,a_{1,1}+18\,a_{3,1}+4\,b_{0,0})\,a_{2,0}+(4\,a_{0,0}-2\,b_{1,1}+10\,b_{1,3})\,b_{0,2}\\
& +(8\,a_{0,0}-2\,b_{1,1}-6\,b_{1,3})\,b_{2,0}+(-2\,b_{1,1}-6\,b_{1,3})\,b_{4,0}-2(-a_{1,1}+3a_{3,1})a_{0,0}\\
& -2\,b_{0,0}(b_{1,1}+3\,b_{1,3}),\\
B_{11}= \;& (-12\,a_{2,1}-16\,a_{2,3}-40\,a_{4,1}+20\,b_{1,2}-20\,b_{1,4}+26\,b_{3,2}+a_{0,1}-4\,b_{1,0}-4\,b_{3,0}\\
& -4\,b_{5,0}-20\,a_{0,3}+20\,a_{0,5})\,a_{1,0}+(4\,a_{0,1}+20\,a_{2,1}+4\,b_{5,0}+60\,a_{0,5}-54\,a_{2,3}\\
& -36\,a_{0,3}+48\,b_{1,4}+48\,a_{4,1}-26\,b_{3,2}+4\,b_{3,0}-20\,b_{1,2}+4\,b_{1,0})\,b_{4,1}+(-60\,b_{1,4}\\
& +30\,a_{2,3}+28\,b_{1,2}-20\,a_{0,1}-60\,a_{4,1}+54\,b_{3,2}-48\,b_{5,0}-28\,a_{2,1}+20\,a_{0,3}-36\,b_{3,0}\\
& -20\,b_{1,0})\,b_{0,5}+(-12\,b_{3,1}-2\,a_{2,0}+2\,a_{0,4}-6\,a_{0,0}+22\,a_{4,0}-12\,b_{1,3}+6\,a_{0,2}+14\,a_{2,2}\\
& +4\,b_{1,1})\,a_{1,3}+(2\,a_{1,1}+14\,a_{3,1}-2\,b_{4,0}-2\,b_{2,0}+6\,b_{0,2}-2\,b_{0,4}-2\,b_{2,2}-2\,b_{0,0})\,a_{2,2}\\
& +(-2\,b_{2,0}-2\,b_{2,2}-2\,b_{4,0}+6\,b_{0,2}-10\,b_{0,4}-2\,a_{1,1}+6\,a_{3,1}-2\,b_{0,0})\,a_{0,2}+(-6\,b_{0,0}\\
& +26\,b_{2,0}+42\,b_{4,0}-6\,a_{1,1}-18\,a_{3,1}-30\,b_{0,4}+18\,b_{0,2}-14\,b_{2,2})\,a_{0,4}+(2\,b_{0,2}-2\,a_{1,1}
\end{aligned}
$$

$$
\begin{aligned}
& -6\,b_{0,0}-6\,b_{4,0}-10\,a_{3,1}-6\,b_{2,0}+10\,b_{2,2}+2\,b_{0,4}\big)\,a_{2,0}+\big(-4\,b_{1,0}-4\,b_{3,0}-4\,b_{5,0}\\
& -16\,a_{2,1}+12\,a_{2,3}-44\,a_{4,1}+20\,b_{1,2}-36\,b_{1,4}+26\,b_{3,2}\big)\,a_{3,0}+\big(2\,b_{0,2}-6\,b_{2,0}+18\,b_{0,4}\\
& -14\,a_{3,1}+10\,b_{2,2}-6\,a_{1,1}-6\,b_{4,0}-6\,b_{0,0}\big)\,a_{4,0}+\big(4\,a_{2,1}-22\,a_{2,3}+3\,a_{4,1}+4\,b_{1,2}\\
& +10\,b_{1,4}+3\,b_{3,2}+8\,a_{0,1}-20\,a_{0,3}+15\,a_{0,5}\big)\,b_{0,3}+\big(2\,b_{0,2}-6\,b_{2,0}-6\,b_{4,0}+2\,a_{1,1}\\
& -6\,a_{3,1}-6\,b_{0,0}+10\,b_{2,2}\big)\,a_{0,0}+\big(2\,a_{0,0}-16\,b_{1,1}+28\,b_{1,3}-28\,b_{3,1}\big)\,b_{0,4}+\big(-8\,a_{2,1}\\
& +6\,a_{2,3}-11\,a_{4,1}+4\,b_{1,2}-10\,b_{1,4}+3\,b_{3,2}-4\,a_{0,1}+5\,a_{0,5}\big)\,a_{1,2}-\big(8\,b_{0,2}+4\,b_{2,2}\big)\,b_{1,3},\\
B_{12}= \quad & \big(-6\,a_{2,3}+12\,b_{1,4}+12\,a_{2,1}+20\,a_{4,1}+a_{0,1}-4\,b_{1,0}-4\,b_{3,0}-4\,b_{5,0}-4\,b_{1,2}-4\,b_{3,2}\\
& +12\,a_{0,3}-44\,a_{0,5}\big)\,a_{1,0}+\big(4\,a_{0,1}-4\,a_{2,1}+4\,b_{5,0}-48\,a_{0,5}+30\,a_{2,3}+12\,a_{0,3}-12\,b_{1,4}\\
& -12\,a_{4,1}+4\,b_{3,2}+4\,b_{3,0}+4\,b_{1,2}+4\,b_{1,0}\big)\,b_{4,1}+\big(48\,b_{1,4}-66\,a_{2,3}-28\,b_{1,2}+12\,a_{0,1}\\
& +48\,a_{4,1}-30\,b_{3,2}+12\,b_{5,0}+84\,a_{0,5}+28\,a_{2,1}-44\,a_{0,3}+12\,b_{3,0}+12\,b_{1,0}\big)\,b_{0,5}\\
& +\big(-12\,b_{3,1}+6\,a_{2,0}+6\,a_{0,4}-6\,a_{0,0}+18\,a_{4,0}+20\,b_{1,3}-2\,a_{0,2}-2\,a_{2,2}-12\,b_{1,1}\big)\,a_{1,3}\\
& +\big(-6\,a_{1,1}-22\,a_{3,1}-2\,b_{4,0}-2\,b_{2,0}+6\,b_{0,2}-2\,b_{0,4}+14\,b_{2,2}-2\,b_{0,0}\big)\,a_{2,2}+\big(-2\,b_{2,0}\\
& +14\,b_{2,2}-2\,b_{4,0}+6\,b_{0,2}-10\,b_{0,4}-2\,a_{1,1}-18\,a_{3,1}-2\,b_{0,0}\big)\,a_{0,2}+\big(-6\,b_{0,0}-6\,b_{2,0}\\
& -6\,b_{4,0}+2\,a_{1,1}+30\,a_{3,1}+2\,b_{0,4}+2\,b_{0,2}-14\,b_{2,2}\big)\,a_{0,4}+\big(2\,b_{0,2}-2\,a_{1,1}-6\,b_{0,0}-6\,b_{4,0}\\
& -10\,a_{3,1}-6\,b_{2,0}+10\,b_{2,2}-14\,b_{0,4}\big)\,a_{2,0}+\big(-4\,b_{1,0}-4\,b_{3,0}-4\,b_{5,0}+8\,a_{2,1}-18\,a_{2,3}\\
& +16\,a_{4,1}-4\,b_{1,2}+12\,b_{1,4}-4\,b_{3,2}\big)\,a_{3,0}+\big(2\,b_{0,2}-6\,b_{2,0}-30\,b_{0,4}-14\,a_{3,1}+10\,b_{2,2}\\
& -6\,a_{1,1}-6\,b_{4,0}-6\,b_{0,0}\big)\,a_{4,0}+\big(2\,a_{0,0}+8\,b_{1,1}-20\,b_{1,3}+8\,b_{3,1}\big)\,b_{0,4}+\big(2\,b_{0,2}-6\,b_{2,0}\\
& -6\,b_{4,0}+2\,a_{1,1}-6\,a_{3,1}-6\,b_{0,0}+10\,b_{2,2}\big)\,a_{0,0}+\big(4\,a_{2,1}-9\,a_{2,3}+4\,a_{4,1}+b_{1,4}+4\,a_{0,1}\\
& -8\,a_{0,3}+10\,a_{0,5}\big)\,a_{1,2}+\big(19\,a_{2,3}-b_{1,4}+20\,a_{0,3}-54\,a_{0,5}\big)\,b_{0,3},\\
B_{13}= \quad & \big(-8\,a_{2,1}+34\,a_{2,3}-12\,a_{4,1}+8\,b_{1,2}-12\,b_{1,4}+6\,b_{3,2}+16\,a_{0,3}-56\,a_{0,5}\big)\,b_{0,5}+\big(4\,a_{0,0}-16\,a_{2,2}\\
& -4\,a_{0,2}+8\,b_{1,3}-8\,a_{0,4}+4\,a_{2,0}+4\,a_{4,0}\big)\,a_{1,3}+\big(-5\,b_{0,3}+3\,a_{1,2}-6\,b_{4,1}+6\,a_{1,0}+6\,a_{3,0}\big)\,a_{2,3}\\
& +\big(-16\,b_{0,2}+32\,b_{0,4}+8\,a_{1,1}+12\,a_{3,1}-8\,b_{2,2}\big)\,a_{0,4}+\big(26\,b_{0,3}-10\,a_{1,2}+12\,b_{4,1}+16\,a_{1,0}\big)\,a_{0,5}\\
& +\big(b_{1,4}+4\,a_{0,3}\big)\,a_{1,2}+\big(-b_{1,4}-4\,a_{0,3}\big)\,b_{0,3}-\big(8\,a_{2,2}-4\,b_{1,3}\big)\,b_{0,4},\\
B_{14}= \quad & \big(4\,a_{0,0}+4\,a_{2,2}+4\,a_{0,2}-8\,a_{0,4}+4\,a_{2,0}+4\,a_{4,0}\big)\,a_{1,3}+\big(-b_{0,3}+4\,b_{0,5}+a_{1,2}\big)\,a_{0,5}-2\,a_{2,3}\,b_{0,5},\\
B_{15}= \quad & \big(-b_{0,3}+4\,b_{0,5}+a_{1,2}\big)\,a_{0,5}+4\,a_{0,4}a_{1,3}-2\,a_{2,3}\,b_{0,5}.
\end{aligned}
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