وزارة التعليم العالي والبحث العلمي

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THÈSE

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Existence, unicité et stabilité des solutions de certains systèmes paraboliques-hyperbolique avec un retard neutre

Filière
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Applied Mathematics

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In front of the jury

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DEDICATION

I am likewise indebted to my extended family members and to every single person who had a hand in the realization of this work.

To you, the readers of these lines.

Sara Labidi

ملخص

نهتم في هذه الأطروحة بدراسة بعض أنظمة المعادلات التفاضلية الجزئية. باستخدام نظرية أنصاف الزمر أو طريقة فاودو غالاركين، تم التوصل إلى نتائج وجود ووحدانية حلول بعض الأنظمة المرنة الحرارية المسامية بتأخير محايد حيث الإنتقال الحراري يتم وفق قانون لورد- شولمان. كما تم إثبات نتائج الإستقرار الأسي والجبري والعام للحل بالاعتماد على الطريقة الضربية والتي تتمثل في بناء دالة ليابونوف مكافئة لطاقة الأنظمة المدروسة.

الكلمات المفتاحية:

الاستقرار الأسي، الاستقرار الجبري، الاستقرار العام، نظام مسامي مرن، طريقة فاودو غالاركين، نظرية أنصاف الزمر، دالة ليابونوف، الطريقة الضربية، التأخير المحايد.

Abstract

In this thesis, we are interested in the study of some systems of partial derivative equations. Using the theory of semigroups and the Faedo-Galerkin method, we establish the existence and uniqueness results of the solutions of some thermos-elastic systems of porous types via Lord Shulman's law with a neutral delay.

Afterwards, an exponential, polynomial and general stability results of the solution were proven based on the multipliers technique which consists to construct a Lyapunov functional equivalent to the energy of the systems studied.

Keywords : Exponential stability, polynomial stability, general stability, porous-elastic system, Faedo-Galerkin method, semigroup theory, Lyapunov functional, multipliers technique, neutral delay.

Résumé

Dans cette thèse, nous nous intéressons à l'étude de certains systèmes d'équations dérivées partielles. En utilisant la théorie des semi-groupes et la méthode de Faedo Galerkin, on établit des résultats d'existence et d'unicité des solutions de certains systèmes thermo-élastiques de types poreux via la loi de Lord Shulman avec un retard neutre.

Puis, des résultats de stabilité exponentielle, polynomiale et générale de la solution ont été prouvés en se basant sur la méthode des multiplicateurs qui consiste à construire une fonctionnelle de Lyapunov équivalente à l'énergie des systèmes étudiés.

Mots-clés : Stabilité exponentielle, stabilité polynomiale, stabilité générale, système poreux-élastique, méthode de Faedo-Galerkin, théorie des semi-groupes, fonctionnelle de Lyapunov, méthode des multiplicateurs, retard neutre.

Contents

General introduction

Modeling a physical system is an essential phase in a scientific approach which aims to analyze its behavior and control it to improve its performance; it allows for a simplified representation of a system or a physical phenomenon. We take the example of vibrations which appear practically in all mechanical structures, several types of these vibrations are undesirable because they have a harmful influence on the operation and lifespan of these structures. They can cause fractures, breakage, malfunction, wear, or even destruction of structures. In addition, they can pose a certain danger to the user himself. The excitation dynamics causing these vibrations are numerous, they generally come either from the external environment, atmosphere, water, and contacts, or from an impact with other structures. The elimination or reduction of these vibrations is a major problem in the fields of engineering, for this reason, friction, thermal, or visco-elastic damping is developed and incorporated into the system, which explains the addition of certain dissipative terms in the equations or on the boundary.

In 1972, Goodman and Cowin [**?**] have given an extension of the classical elasticity theory to porous media by introducing the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition, Nunziato and Cowin [22] have presented a nonlinear theory for the behavior of porous solids in which the skeletal or matrix material is elastic and the interstices are void of material. In this theory the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field. Furthermore, this representation introduces an additional degree of kinematic freedom. The intended applications of the theory of elastic materials with voids are to geological materials like rocks and soils and manufactured porous materials. The basic evolution equations for one-dimensional porous materials theory are given by

$$\left\{ \rho u_{tt} = T_x, \ J\phi_{tt} = H_x + D, \right.$$

where T, H, and D represent respectively the stress tensor, the equilibrated stress vector, and the equilibrated body force. The parameter ρ designates the mass density and J equals to the product of the mass density by the equilibrated inertia. The functions u and ϕ represent respectively the longitudinal displacement and the volume fraction. The constitutive equations T, H, and D must take the following form

$$\begin{aligned} &T = \mu u_x + b\phi, \ H = \delta\phi_x, \\ &D = -bu_x - \xi\phi, \end{aligned}$$

where μ, δ, a are positive constants represent the constitutive parameters defining the coupling among the different components of the materials such that

$$\mu\xi > b^2,$$

and b is a real number different from zero.
By combining the two above constituted equations, we obtain the following one-dimensional porous elastic system

$$\left\{ \begin{array}{l} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi = 0. \end{array} \right.$$

The first investigation concerning the study of temporal asymptotic behavior of the solutions for a one-dimensional porous-elastic system was started by the work of Quintanilla [34] in which he considered a damping through porous-viscosity and he proved that the system does not decay exponentially with this complementary control. Casas and Quintanilla [32] showed that the porous-viscosity together with temperature are not strong enough to stabilize exponentially the system. The same authors [36] proved that the combination of porous-viscosity and thermal effects (temperature and micro-temperatures) provoke an exponential stability of solutions. Maganá and Quintanilla [29] showed that viscoelasticity damping and temperature produced slow decay in time and when the visco-elasticity is coupled with porous damping or with micro-temperatures, the system decays in an exponential way.

According to the thermo-elasticity theory, a body's deformation is connected to a change in enthalpy and, consequently, a change in temperature. In other words, thermo-elasticity is the study of the relationship between a material's thermal conductivity and pressure, as well as its elastic qualities and temperature. The Lord-Shulman theory [27] recovers the proposition of Maxwell-Cattaneo ([42],[**?**]) and combines it with the system describing the elastic vibrations of a material. This theory is based on the study of a system of four hyperbolic equations with heat dissipation. In this case, the heat equation is also hyperbolic unlike the one obtained for the Fourier's law which is parabolic. The basic evolution equations for one-dimensional porous materials theory are given by

$$\rho u_{tt} = T_x,\ J\phi_{tt} = H_x + D,\ \rho T_0 \eta_t = q_x,\ \rho e_t = P_x + q - Q,$$

where T_0 is the reference temperature at the equilibrium state (which we assume to be equal to one for simplicity), η is the entropy, q is the heat flux vector, P is the first heat flux moment, Q is the mean heat flux and e is the first moment of energy. The constitutive equations are given by

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x + \beta_0 \left(\tau\theta_{tx} + \theta_x\right), \\ J\varphi_{tt} = \delta\phi_{xx} - bu_x - \xi\phi - \beta_1 \left(\tau\theta_t + \theta\right), \\ a \left(\tau\theta_t + \theta\right)_t = -\beta_0 u_{tx} + \beta_1 \phi_t + \kappa\theta_{xx}, \end{cases}$$

where β_0, β_1 are positive constants.

In the context of partial differential equations (PDEs), a delay term refers to a mathematical expression that involves the value of a function at some previous time(s) or spatial location(s). The presence of delay may be a source of instability, as demonstrated in studies ([12],[13],[19],[30],[31],[39]), where they were proved that an arbitrarily small delay may destabilize a system, which is asymptotically stable in the absence of a delay. Among the delay kinds, we are interested here in the neutral delay (see ([14]-[16],[**?**],[38]) and this is because in some cases, we rely on it to achieve the stability of dynamical systems unlike

other types of delays. In fact, neutral delays are sometimes deliberately inserted into the systems to improve the performance of the structure.
This thesis is composed of four main chapters,

- The first chapter includes reminders of the classical preliminary notions and the mathematical tools which are necessary for the study of this thesis.

- In the second chapter, we study a Lord-Shulman porous-thermoelastic system with nonlinear damping term. First, an existence and uniqueness result of solution by using semi-group theory is established. Then, the general stability result of solution is proved irrespective on the wave speeds of the system or any other relationship between the coefficients of the system. This study was published in the journal "**Nonlinear Studies**" as follows "**Sara Labidi, Houssem Eddine Khochemane, Abdelhak Djebabla, General decay of a Lord-Shulman porous thermoelastic system with nonlinear damping term, Nonlinear Studies. 31(1) (2024), 165-182.**"

- The third chapter is devoted to the study of a porous-elastic system with nonlinear damping term and distributed delay of neutral type.A general stability result was proved based on the dissipation given by the nonlinear damping term despite of the destructive nature of delays in general. This has been the subject of the following international publication "**Sara Labidi, Houssem Eddine Khochemane and Abdelhak Djebabla, General stability for a neutral delayed porous-elastic system, DCDIS Series A: Mathematical Analysis. 31 (2024) 149-167.**"

- In the fourth chapter, we gave in detail the existence and uniqueness result of a neutral delayed porous-elastic system using the Faedo-Galerkin method. Then, a polynomial and exponential stability result was proved depending on the wave speeds propagation. The study was published as follows **"Houssem Eddine Khochemane, Sara Labidi, Sami Loucif, Abdelhak Djebabla, Global well-posedness and energy decay for a one dimensional porous-elastic system subject to a neutral delay, Mathematica Bohemica. DOI: 10.21136/MB.2024.0104-23."**

CHAPTER 1

Generalities

1.1 Introduction

In this chapter, we shall introduce a brief summary of dynamical systems theory and then state some necessary materials needed in the proof of our results, such as basic principles of functional analysis. Which relates to the Hilbert spaces, the L^p space, Sobolev spaces, some theorems on these last, existence and uniqueness theorem. Knowledge of all these notions and results is important for our study.

1.2 Functional Spaces

1.2.1 Normed space

Definition 1.2.1. A normed vector space is a vector space equipped with a norm. A semi-normed vector space is a vector space equipped with a semi-norm. A useful variation of the triangle inequality is

$$\|x - y\| \leq \|x\| - \|y\| .$$

1.2.2 Complete space

Definition 1.2.2. A space M is called complete (or a Cauchy) space if every Cauchy sequence of points in M has a limit that is also in M.

1.2.3 Banach space

Definition 1.2.3. A Banach space is a complete normed space $(X, \|.\|)$ .A normed space X is a pair $(X, \|.\|)$ consisting of a vector space over a scalar field $\Bbbk$ $(\mathbb{R}\text{or}\mathbb{C})$together with a distinguished norm $\|.\| : X \to \mathbb{R}$.

1.2.4 Inner product

Definition 1.2.4. An inner product in a complex linear space X is a map

$$(.,.) : X \times X \to C.$$

Such that for all $x, y, z \in X$ and $\lambda, \mu \in C$ we have the following properties

(a) $(x, \lambda y + \mu z) = \lambda (x, y) + \mu (x, z)$ (linear in the second argument).

(b) $(y, x) = \overline{(x, y)}$ (Hermitian symetric).

(c) $(x, x) \geq 0$ (non-negative).

(d) $(x, x) = 0$ if and only if $x = 0$ (positive definite).

We call a linear space with an inner product an inner product space or a pre-Hilbert space.

1.2.5 Hilbert space

Definition 1.2.5. A Hilbert space is a complete inner product space.

In particular, every Hilbert space is a Banach space with respect the norm

$$\|x\| = \sqrt{(x,x)}.$$

1.2.6 The $L^p(\Omega)$ spaces

Definition 1.2.6. ([7]) Let $1 \leq p \leq \infty$, and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$, by

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \text{ is measurable and } \int_\Omega |f(x)|^p \, dx < \infty \right\}.$$

For $p \in \mathbb{R}$ and $1 \leq p \leq \infty$, denote by

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

If $p = \infty$, we have

$$\begin{aligned} L^\infty(\Omega) &= \{ f : \Omega \to \mathbb{R} : f \text{ is measurable and there exists a constant } C \\ &\quad \text{such that, } |f(x)| \leq C \text{ a.e in } \Omega \}. \end{aligned}$$

Also, we denote by

$$\|f\|_\infty = \inf \{ C, \ |f(x)| \leq C \text{ a.e in } \Omega \}.$$

Remark 1.2.1. Let $1 \leq p \leq \infty$, we denote by q the conjugate of p i.e. $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Remark 1.2.2. In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x) g(x) \, dx,$$

is a Hilbert space.

1.2.7 The Sobolev spaces $\mathbf{W}^{m,p}(\Omega)$

Definition 1.2.7. Let $m \in \mathbb{N}$ and $p \in [1, \infty]$. The $W^{m,p}(\Omega)$ is the space of all $f \in L^p(\Omega)$, defined as

$$\begin{aligned} W^{m,p}(\Omega) &= \Big\{ f \in L^p(\Omega), \text{ such that } \partial^\alpha f \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^n \text{ such that} \\ &\qquad |\alpha| = \sum_{j=1}^{n} \alpha_j \leq m, \text{ where } \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n} \Big\}. \end{aligned}$$

Theorem 1.2.1. *([7]) $W^{m,p}(\Omega)$ is a Banach space with their usual norm*

$$\|f\|_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^p}, \ 1 \leq p \leq \infty, \textit{ for all } f \in L^p(\Omega).$$

Definition 1.2.8. When $p = 2$, we prefer to denote by $W^{m,2}(\Omega) = H^m(\Omega)$ and $W_0^{m,p}(\Omega) = H_0^m(\Omega)$, for $p \in [0, \infty[$ supplied with the norm

$$\|f\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} \left(\|\partial^\alpha f\|_{L^2} \right)^2 \right)^{\frac{1}{2}},$$

which do at $H^m(\Omega)$ a real Hilbert space with their usual scalar product

$$\langle u, v \rangle_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \partial^\alpha v dx.$$

The next result provides a basic characterization of functions in $W_0^{1,p}(\Omega)$.

Theorem 1.2.2. *([7]) Let $u \in W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$ if and only if $u = 0$ on $\partial\Omega$.*

Remark 1.2.3.

1. The theorem 1.2.2 explains the central role played by the space $W_0^{1,p}(\Omega)$. Differential equations (or partial differential equations) are often coupled with boundary conditions, i.e., the value of u is prescribed on $\partial\Omega$.

2. We have the following characterization of $H_0^m(\Omega)$

$$H_0^m(\Omega) = \{u \in H^m(\Omega),\ u = u' = ... = u^{(m-1)} = 0 \text{ on } \partial\Omega\}.$$

It is essential to notice the distinction between

$$H_0^2(\Omega) = \{u \in H^2(\Omega),\ u = u' = 0 \text{ on } \partial\Omega\},$$

and

$$H^2(\Omega) \cap H_0^1(\Omega) = \{u \in H^2(\Omega),\ u = 0 \text{ on } \partial\Omega\}.$$

1.3 Some functional inequalities

We give here some important functional inequalities.
These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

1.3.1 Hölder's Inequality

Lemma 1.3.1. *([7]) Let $1 \leq p, q \leq \infty$, assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ then, $fg \in L^1(\Omega)$ and*

$$\int_\Omega |fg|\, dx \leq \|f\|_p \|g\|_q \,. \tag{1.3.1}$$

Remark 1.3.1. For $p = q = 2$. We fall in the Cauchy-Schwarz inequality.i.e.,

$$\int_\Omega |fg|\, dx \leq \|f\|_2 \|g\|_2 \,. \tag{1.3.2}$$

1.3.2 Poincaré inequality

Lemma 1.3.2. *([7]) Let $h \in H_0^1(0, L)$. Then it holds*

$$\int_0^L |h|^2\, dx \leq \int_0^L |h_x|^2\, dx. \tag{1.3.3}$$

1.3.3 Young's Inequality

Lemma 1.3.3. *([7]) For all $a, b \in \mathbb{R}^+$, we have*

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}, \tag{1.3.4}$$

where ϵ is any positive constant. Moreover, for any $f, g \in L^2(\Omega)$ then, $fg \in L^1(\Omega)$. Therefore

$$\int_\Omega |fg|\, dx \leq \epsilon \int_\Omega f^2 dx + \frac{1}{4\epsilon} \int_\Omega g^2 dx.$$

Now, Let us denote by h^* the conjugate function in the sense of Young of a convex function h (see ([5]), p. 64), i.e.,

$$h^*(p) = \sup_{t \in \mathbb{R}_+} (pt - h(t)).$$

Assume that $h'' > 0$, then for $p \geq 0$ a given number, h^* is the Legendre transform of h (see Liu and Zuazua ([26])), which is given by

$$h^*(p) := p\left[h'\right]^{-1}(p) - h(\left[h'\right]^{-1}(p)), \tag{1.3.5}$$

and which satisfies the following inequality

Lemma 1.3.4. *([33])(Young's Inequality for the convex functions) Let h a convex function, h^* its conjugate in the sense of Young, we have*

$$px \leq h(x) + h^*(p) \ \ , \forall p, x \geq 0. \tag{1.3.6}$$

Remark 1.3.2. The relation (1.3.5) and the fact that $h(0) = 0$ and $(h')^{-1}$, h are increasing functions yield

$$h^*(p) \leq p\left[h'\right]^{-1}(p) \ \ , \forall p \geq 0. \tag{1.3.7}$$

1.4 Lax-Milgram Lemma

The existence and uniqueness of a solution to the weak formulation of the problem can be proved using the Lax-Milgram Lemma. This states that the weak formulation admits a unique solution.

Lemma 1.4.1. *([7]) (Lax-Milgram lemma) Let $a(\cdot,\cdot)$ be a bilinear form on a Hilbert space $\mathcal{H}$ equipped with norm $\|\cdot\|_{\mathcal{H}}$ and the following properties,*

i) $a(\cdot,\cdot)$ is continuous, that is

$$\exists\gamma_1 > 0 \textit{ such that } |a(w,v)| \leq \gamma_1 \|w\|_{\mathcal{H}} \|v\|_{\mathcal{H}}, \ \forall w,v \in \mathcal{H},$$

ii) $a(\cdot,\cdot)$ coercive (or $\mathcal{H}$-elliptic), that is

$$\exists\alpha > 0 \textit{ such that } |a(v,v)| \geq \alpha \|v\|_{\mathcal{H}}^2, \ \forall v \in \mathcal{H},$$

iii) L is a linear mapping on $\mathcal{H}$ (thus L is continuous), that is

$$\exists\gamma_2 > 0 \textit{ such that } |L(w)| \leq \gamma_2 \|w\|_{\mathcal{H}}, \ \forall w \in \mathcal{H}.$$

Then there exists a unique $u \in \mathcal{H}$, such that

$$a(w,u) = L(w), \ \forall w \in \mathcal{H}.$$

1.5 Uniformly Continuous Semigroups of Bounded Linear Operators

Definition 1.5.1. ([33]) Let X be a Banach space. A one parameter family $(T(t))_{t\geq 0}$ of bounded linear operators defined from X into X is a semi-group of bounded linear operators on X if

i) $T(0) = I$, (I is the identity operator on X).

ii) $T(t+s) = T(t)\,T(s)$,for every $t, s \geq 0$ (the semigroup property).

A semigroup of bounded linear operators, $T(t)$, is uniformly continuous if

$$\lim_{t\to 0} \|T(t) - I\| = 0.$$

The linear operator A defined by

$$D(A) = \left\{ x \in X : \lim_{t\to 0} \frac{T(t)\,x - x}{t} exists \right\},$$

and

$$Ax = \lim_{t\to 0} \frac{T(t)\,x - x}{t} = \left.\frac{d^{+}T(t)\,x}{dt}\right|_{t=0}, \text{ for } x \in D(A).$$

Is the infinitesimal generator of the semigroup $T(t)$, $D(A)$ is the domain of A.

This section is devoted to the study of uniformly continuous semigroup of bounded linear operators. From the definition, it is clear that if $T(t)$ is a uniformly continuous semigroup of bounded linear operators then

$$\lim_{s\to t} \|T(s) - T(t)\| = 0.$$

Theorem 1.5.1. *([33])A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

Theorem 1.5.2. *([33])Let $T(t)$ and $S(t)$ be uniformly continuous semigroups of bounded linear operators. If*

$$\lim_{t\to 0} \frac{T(t) - I}{t} = \lim_{t\to 0} \frac{S(t) - I}{t},$$

then $T(t) = S(t)$ for $t \geq 0$.

Corollary 1.5.1. *[33]Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators. Then*

(a) There exists a constant $\omega \geq 0$ such that $\|T(t)\| \leq e^{\omega t}$.

(b) There exists a unique bounded linear operator A such that $T(t) = e^{tA}$.

(c) The operator A in part (b) is the infinitesimal generator of $T(t)$.

(d) $t \to T(t)$ is differentiable in norm and

$$\frac{dT(t)}{dt} = AT(t) = T(t)A.$$

1.6 Strongly Continuous Semigroups $(C_0 - Semigroup)$of Bounded Linear Operators

Throughout this section X denotes a Banach space.

Definition 1.6.1. ([33]) A semigroup $T(t)$,$0 \leq t < \infty$,of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if

$$\lim_{t \to 0} T(t)x = x \text{ ,for every } x \in X.$$

A strongly continuous semigroup of bounded linear operators on X is called a semigroup of class C_0 or simply C_0-semigroup.

Theorem 1.6.1. *([33]) Let $T(t)$ be a C_0-semigroup. There exists constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t} \text{ ,for } 0 \leq t < \infty.$$

Corollary 1.6.1. *If $T(t)$ is a C_0-semigroup then for every $x \in X$, $t \to T(t)x$ is a continuous function from $\mathbb{R}_0^+$ (the nonnegative real line) into X.*

Theorem 1.6.2. *Let $T(t)$ be a C_0-semigroup and let A be its infinitesimal generator. Then*

(a) For $x \in X$,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x.$$

(b) For $x \in X, \int_0^t T(s)xds \in D(A)$ and

$$A\left(\int_0^t T(s)xds\right) = T(t)x - x.$$

(c) For $x \in D(A), T(t)x \in D(A)$. and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)xA.$$

(d) For $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau$$

Corollary 1.6.2. *If A is the infinitesimal generator of a C_0-semigroup $T(t)$,then $D(A)$ the domain of A is dense in X and A is a closed linear operator.*

Theorem 1.6.3. *Let $T(t)$ and $S(t)$ be C_0-semigroups of bounded linear operators with infinitesimal generators A and B respectively. If $A = B$, then $T(t) = S(t)$, for $t \geq 0$.*

Theorem 1.6.4. *Let A be the infinitesimal generator of the C_0-semigroups $T(t)$. If $D(A^n)$ is the domain of A^n,then $\cap_1^\infty D(A^n)$ is dense in X.*

Lemma 1.6.1. *Let A be the infinitesimal generator of a C_0-semigroup $T(t)$satisfying $\|T(t)\| \leq M$ for $t \geq 0$.If $x \in D(A^2)$, then*

$$\|Ax\|^2 \leq 4M^2 \left\|A^2x\right\| \|x\|.$$

Example 1.6.1. Let X be the Banach space of bounded uniformly continuous functions on $]-\infty, +\infty[$ with the supremum norm. For $f \in X$, we define

$$(T(t)f)(s) = f(t+s).$$

It is easy to check that $T(t)$ is a C_0-semigroup satisfying $\|T(t)\| \leq 1$, for $t \geq 0$. The infinitesimal generator of $T(t)$ is defined on $D(A) = \{f : f \in$X$, f'$ exists $f' \in$

$X, and\,(Af)\,(s) = f'\,(s)\,, forf \in D(A).$

From Lemma 1.6 we obtain Landau's inequality

$$\left(\sup\left|f'\,(s)\right|\right)^2 \leq 4\left(\sup\left|f''\,(s)\right|\right)\left(\sup\left|f\,(s)\right|\right),$$

where the sup are taken over $]-\infty,,+\infty[$ .Example 1.6.1 can be easily modified to the case where $X = L^p\,(-\infty,+\infty)\,, 1 < p < +\infty.$

1.7 The m-dissipatives operators

1.7.1 Definitions and preliminary notions

Definition 1.7.1. ([33]) An unbounded linear operator in X is a couple $(A, D(A))$ where $D(A)$ is a vector subspace of X that represents the domain of A and A is a linear application of $D(A)$ in X. Similarly, an unbounded linear operator of X in Y is a couple $(A, D(A))$ such that $D(A)$ a vector subspace of X and A is a linear application of $D(A)$ of X in Y .

Definition 1.7.2. ([33]) An unbounded linear operator $(A, D(A))$ in X, is closed if its graph

$$G\,(A) = \{(x, Ax)\ \mid x \in D(A)\}\ \text{is closed in}\ (X \times X).$$

Definition 1.7.3. ([33]) Let $(A, D(A))$ be an unbounded linear operator in X where $D(A)$ is dense in X, we said that $(A, D(A))$ is a dense domain in X.

Definition 1.7.4. ([33])Let $(A, D(A))$ be an unbounded linear operator in X where $D(A)$ is dense in X.we call assistant of A the operator $(A^*, D(A)^*)$ defined by

$$D(A)^* = \left\{y \in X' \mid \exists c \geq 0 \text{ such as } \langle Ax, y\rangle_{X,X'} \leq c\,\|x\|\,,\ \text{for all } x \in D(A)\right\},$$

and

$$\langle x, A^*y\rangle_{X,X'} = \langle Ax, y\rangle_{X,X'}\,,\text{for all } x \in D(A) \text{ and } y \in D(A^*).$$

Theorem 1.7.1. *Let $(A, D(A))$ be an unbounded linear operator in X where $D(A)$ is dense in X. If X is a reflexive space and A is closed then $D(A^*)$ is dense in X'.*

1.7.2 The m-dissipatives operators

Definition 1.7.5. ([33]) An unbounded linear operator $(A, D(A))$ in X is dissipative if

$$\forall x \in D(A), \forall \lambda > 0, \ \|\lambda x - Ax\| \ \geq \lambda \|x\|.$$

Definition 1.7.6. ([33]) An unbounded linear operator $(A, D(A))$ in X is m-dissipative if A is dissipative and

$$\forall f \in X, \forall \lambda > 0, \quad \exists x \in D(A) \quad \textit{as that } \lambda x - Ax = f.$$

Theorem 1.7.2. *([33]) If A is m-dissipative, then for all $\lambda > 0$, the operator $(\lambda I - A)$ admits an inverse, $(\lambda I - A)^{-1} f$ belongs to $D(A)$ for all $f \in X$, and $(\lambda I - A)^{-1}$is a bounded verifying linear operator*

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}.$$

Theorem 1.7.3. *([33])Let $(A, D(A))$ be an unbounded dissipative linear operator in X. The operator A is m-dissipative if and only if*

$$\exists \ \lambda_0 \textit{ such that } \forall \ f \in Y, \ \exists \ x \in D(A) \textit{ verifies } \lambda_0 x - Ax = f.$$

Theorem 1.7.4. *([33]) Let $(A, D(A))$ be an unbounded linear operator in X. If it exists $\lambda_0 > 0$ for which the operator $\lambda_0 I - A$ is a bijection of $D(A)$ on X and if $(\lambda_0 I - A)^{-1}$ is a bounded operator on X,then A is closed.*

Particularly, if A is m-dissipative, then A is closed.

Theorem 1.7.5. *([33]) Let A be a dissipative operator and $R(I - A) = X$. If X is reflexive,* *then*

$$\overline{D(A)} = X.$$

Theorem 1.7.6. *Let A be a maximal monotone operator. Then, given any $u_0 \in D(A)$, there exists a unique function u such that*

$$u \in C([0,\infty[, D\,(A)) \cap C^1([0,\infty[, X),$$

satisfying

$$\begin{cases} u' + A(t)u = 0 \quad \textit{on } [0,\infty[, \\ u\,(0) = u_0. \end{cases}$$

Moreover,

$$|u(t)| \leq |u_0|, |\frac{du}{dt}(t)| = |Au(t)| \leq |Au_0|, \, \forall t \geq 0$$

Remark 1.7.1.

1. The main interest of Theorem 1.7.6 lies in the fact that we reduce the study of an "evolution problem" to the study of the "stationary equation" $u' + Au = f$.
2. The space $D(A)$ is equipped with the graph norm $|u| + |Au|$ or with the equivalent Hilbert norm $\sqrt{|u|^2 + |Au|^2}$.
3. We refer the interested readers to [3, **?**] and references therein for details discussion on existence and uniqueness of local or global solutions of nonlinear evolution equations.

Theorem 1.7.7. *([33]) Let $(S(t)_{t\geq 0})$ a strongly continued semigroup on Y and $(A, D(A))$ are infinitesimal generator, the following properties are verified*

i) For all $x \in X$, we have

$$\lim_{t \longrightarrow 0} \frac{1}{h} \int_t^{t+h} S(s)xds = S(t)x.$$

ii) For all $x \in X$, and all $t > 0$,

$$\int_0^t S(s)xds,$$

belongs to $D(A)$ and

$$A(\int_0^t S(s)xds) = S(t)x - x.$$

iii) If $x \in D(A)$, then $S(t)x \in D(A)$ and

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax.$$

iv) If $x \in D(A)$, so

$$S(t)x - S(s)x = \int_s^t S(\tau)Axd\tau = \int_s^t As(\tau)xd\tau.$$

1.7.3 The m-dissipative (m-monotone) operators in a Hilbert space

Definition 1.7.7. ([7]) Let $\mathcal{H}$ be a Hilbert space ,an unbounded linear operator $A : D\left(A\right) \subset \mathcal{H} \ \rightarrow \mathcal{H}$ is said to be monotone if it satisfies

$$\left(Au, u\right) \geq 0,\ \forall u \in D\left(A\right),$$

it is called maximal monotone if, in addition

$$R\left(I + A\right) = \mathcal{H}, \text{i.m. } \forall f \in \mathcal{H}, \exists u \in D\left(A\right) \text{ such that } u + Au = f.$$

Proposition 1.7.1. *([7]) Let $\mathcal{A}$ be a maximal monotone operator. Then $D\left(A\right)$ is dense in $\mathcal{H}$.*

Theorem 1.7.8. *([7]) An unbounded linear operator $\left(A, D(A)\right)$ in $\mathcal{H}$, is dissipative if and only if*

$$\forall x \in D(A),\ \left(Ax, x\right) \leq 0.$$

In the case of a complex Hilbert space, the previous condition is replaced by

$$\forall x \in D(A),\ Re\left(Ax, x\right) \leq 0.$$

Theorem 1.7.9. *([7]) If A is m-dissipative,then $D(A)$ dense in $\mathcal{H}$.*

1.7.4 Hille–Yosida theorem

Let $T(t)$ be a C_0-semigroup. From Theorem 2.5 it follows that there are constants $\omega \geq 0$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$ (for $0 \leq t < \infty$).If $\omega \geq 0$, $T(t)$ is called uniformly bounded and if moreover $M = 1$ it is called a C_0-semigroup of contractions. This section is devoted to the characterization of the infinitesimal generators of C_0-semigroup of contractions. Conditions on the behavior of the resolvent of an operator A, which are necessary and sufficient for A to be the infinitesimal generator of a C_0-semigroup of contractions, are given.

Recall that if A is a linear, not necessarily bounded operator in X, the resolvent set $\rho(A) of\ A$ is the set of all complex numbers λ for which $\lambda I - A$ is invertible, i.e., $(\lambda I - A)^{-1}$ is a bounded linear operator in X. The family $R(\lambda : A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ of bounded linear operators is called the resolvent of A.

Theorem 1.7.10. *([33])(Hille-Yosida) A linear (unbounded) operator A is the infinitesimal generator of a C_0-semigroup of contraction $T(t), t \geq 0$ if and only if the following conditions are satisfied*

(i) *A is closed and $\overline{D(A)} = X$.*

(ii) *The resolvent set $\rho(A)$of A contains $\mathbb{R}^+$and for all $\lambda > 0$,*

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}.$$

Lemma 1.7.1. *([33])Let A satisfy the conditions (i) and (ii) of Theorem 1.7.10 and let $\|R(\lambda : A)\| \leq \frac{1}{\lambda}$.Then*

$$\lim_{\lambda \to \infty} \lambda R(\lambda : A)x = x, \quad for \quad x \in X$$

Lemma 1.7.2. *([33])Let A satisfy the conditions (i) and (ii) of Theorem 1.7.10 and if A_λ is the Yosida approximation of A,then*

$$\lim_{\lambda \to \infty} A_\lambda x = Ax, \quad for \quad x \in D(A).$$

Lemma 1.7.3. *([33])Let A satisfy the conditions (i) and (ii) of Theorem 1.7.10 and if A_λ is the Yosida approximation of A, then A_λ is the infinitesimal generator of a uniformly continuous semigroup of contractions e^{tA_λ}. Furthermore, for every $x \in X, \lambda, \mu > 0$ we have*

$$\left\| e^{tA_\lambda} x - e^{tA_\mu} x \right\| \leq t \left\| A_\lambda x - A_\mu x \right\| .$$

1.7.5 Lumer-Phillips theorem

In this section, we see a different characterization of such infinitesimal generators. In order to state and prove the result we need some preliminaries.Let X be a Banach space and let X^* be its dual. We denote the value of $x^* \in X^*$ at $x \in X$ by $\langle x, x^* \rangle$ or $\langle x^*, x \rangle$ .For every $x \in X$ we define the duality set $F(x) \subseteq X^*$ by

$$F(x) = \left\{ x^* : x^* \in X^* \text{ and } \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}.$$

From the Hahn-Banach theorem it follows that $F(x) \neq \phi$ for every $x \in X$.

Definition 1.7.8. ([33])A linear operator A is dissipative if for every $x \in D(A)$, there is a $x^* \in F(x)$ such that $Re \langle Ax, x^* \rangle \leq 0$.

A useful characterization of dissipative operators is given next.

Theorem 1.7.11. *([33])A linear operator A is dissipative if and only if*

$$\|(\lambda I - A) x\| \geq \lambda \|x\| \quad \text{,for all} \quad x \in D(A) \text{ and } \lambda > 0.$$

Theorem 1.7.12. *([33])(Lumer-Phillips) Let A an unbounded linear operator of $D(A)$ with dense domain $D(A)$ in X.*

a) *If A is dissipative and if there exists $\lambda_0 > 0$ such that the range $R(\lambda_0 I - A)$* is *X. Therefore, A is an infinitesimal generator of a strongly continued C_0-semigroup of contraction on X.*

b) *If A is an infinitesimal generator of a strongly continued contraction* C_0-semigroup on *X,then* $R(\lambda I - A) = X, \ \forall \lambda > 0$ *and* A is *dissipative*.Moreover, for every $x \in D(A)$ and every $x^* \in F(x)$, $Re\langle Ax, x^* \rangle \leq 0$.

1.8 Concept of stability

Definition 1.8.1. ([23])The semigroup $T(t) = e^{At}$ is said to be exponentially stable if there exists two constants $\alpha > 0$ and $M \geq 1$ as that

$$\|T(t)\| \leq Me^{\alpha t}, \quad \forall t \geq 0.$$

1.8.1 Stability in the Lyapunov sense

Definition 1.8.2. (Internal stability) An equilibrium point is stable if the state trajectories of the system converge to an initial state different from the equilibrium state.

Definition 1.8.3. (Balance state) x_e is a state of balance if $x(t_0) = x_e \Longleftrightarrow x(t) = x_e$ with $t \geq t_0$, in the absence of control and disturbances.

Definition 1.8.4. (Asymptotic stability) The state of equilibrium point x_e is said to be stable if

$$\forall t \geq 0, \forall \epsilon > 0, \ \exists \alpha > 0 \mid \quad \|x(0) - x_e\| < \alpha \Longrightarrow \|x(t) - x_e\| < \epsilon.$$

Otherwise, x_e is said to be unstable.

Definition 1.8.5. (Lyapunov's stability) An equilibrium point is asymptotically stable if it is stable and if

$$\exists \alpha > 0, \mid \quad \|x(0) - x_e\| < \alpha \Longrightarrow \lim_{t \to +\infty} x(t) = x_e.$$

CHAPTER

2

General decay of a Lord-Shulman porous thermoelastic system with nonlinear damping term

2.1 Introduction

In [?], Goodman and Cowin have extented of the classical elasticity theory to porous media by introducing the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition, Nunziato and Cowin [32] have presented a nonlinear theory for the behavior of porous solids in which the skeletal or matrix material is elastic and the interstices are void of material. In this theory, the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field.

The best known constitutive equation to model thermal conduction is Fourier's law. However, this law has a major drawback. Therefore, many researchers have been interested in the formulation of alternative constitutive relations to overcome this paradox. It is worth recalling the theories of Green and Lindsay [18] or Lord and

Shulman [27] which are based on the heat conduction equation of Cattaneo and Maxwell [8] or those proposed by Gurtin et al. [20] which take into account the acceleration of the heat flow. Among the works that have been recently released concerning the study of the asymptotic behavior of the solution of different type of problem based on Cattaneo law for the heat conduction, we cite the following references ([1],[6],[36],[41],[23],[22]). The Lord-Shulman theory [27] recovers the proposition of Maxwell and Cattaneo and combines it with the system describing the elastic vibrations of a material. Lord-Shulman thermoelasticity has received a lot of attention among scientists in the last few years and the amount of contributions to explain this theory is enormous. This theory consists of studying a system of four hyperbolic equations with heat dissipation. In this case, the heat equation is also hyperbolic unlike the one obtained for the Fourier law which is parabolic. Lord-Shulman thermoelasticity has attracted many mathematicians over the years and a lot of work has gone into understanding it.

Bazarra, Fernandez and Quintanilla [6] study the Lord-Shulman porous thermoelastic system with microtemperature. They used the semigroup theory together with the method developed by Liu and Zheng [25] to establish an exponential stability.

In this work, we consider the one-dimensional Lord-Shulman porous thermoelastic system by adding a nonlinear damping term to the equilibrated body force in the second equation

$$\begin{cases} \rho u_{tt} = \mu^* u_{xx} + \mu_0 \phi_x - \beta_0 \left(\tau \theta_{tx} + \theta_x\right), & \text{in } (0,1) \times (0,\infty) \\ J\phi_{tt} = a_0 \phi_{xx} - \mu_0 u_x - \xi \phi - \beta_1 \left(\tau \theta_t + \theta\right) - \alpha(t) g(\phi_t), & \text{in } (0,1) \times (0,\infty) \\ a\left(\tau \theta_t + \theta\right)_t = -\beta_0 u_{tx} + \beta_1 \phi_t + \kappa \theta_{xx}, & \text{in } (0,1) \times (0,\infty). \end{cases} \tag{2.1.1}$$

Now, we supplement system (2.1.1) by the following boundary and initial conditions

$$\begin{cases} u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), \phi_t(x,0) = \phi^1(x), & x \in (0,\,1) \\ \phi(x,0) = \phi^0(x),\ \theta(x,0) = \theta^0(x), & x \in (0,\,1) \end{cases} \tag{2.1.2}$$

here u^0, u^1, ϕ^0, ϕ^1, w^0,θ^0 and θ^1 are given functions, and the boundary conditions

$$\begin{cases} u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = 0, & t > 0 \\ \theta(0,t) = \theta(1,t) = 0, & t > 0. \end{cases} \tag{2.1.3}$$

The variables u, ϕ and θ represent, respectively, the displacement of the solid elastic material, the volume fraction and the temperature are. The parameters ρ, J, a and τ are, respectively, the mass density, product of the mass density by the equilibrated inertia, the thermal capacity and the relaxation parameter which is assumed to be small but strictly positive. The coefficients μ^*, μ_0, ξ, β_0, β_1, κ, a_0 are positive constants in which their physical meaning is well known such that

$$\chi = \xi - \frac{\mu_0^2}{\mu^*} > 0, \tag{2.1.4}$$

and the term $\alpha(t)\, g(\phi_t)$ is the nonlinear damping term where the functions α and g are specified later.

The main goal of this work is to prove that the system (2.1.1)-(2.1.3) is well-posed in the sense of semi-group on one hand and on the other hand we show that the dissipation given only by the nonlinear damping term guarantees the general stability of solution without imposing any restrictive growth assumption near the origin on the damping term. Furthermore, our result does not depend on the wave speeds of the system or any other relationship between the coefficients of the system. Meanwhile, from the first equation of (2.1.1) and the boundary conditions (2.1.3), we get

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx = 0, \ \forall\, t \geq 0, \tag{2.1.5}$$

and therefore

$$\int_0^1 u(x,t)dx = t\int_0^1 u^1(x)dx + \int_0^1 u^0(x)dx, \ \forall\, t \geq 0.$$

Consequently, if we set

$$\overline{u}(x,t) = u(x,t) - t\int_0^1 u^1 dx + \int_0^1 u^0 dx,\ t \geq 0,\ x \in [0,1],$$

we find

$$\int_0^1 \overline{u}(x,t)dx = 0,\ t \geq 0.$$

and $(\overline{u},\, \phi, \theta)$ satisfies the same equations in (2.1.1)-(2.1.3). In what follows we will work with $\overline{u}$ but, for convenience, we write u instead of $\overline{u}$.

The spirit of this manuscript is as follows.

In Section 2, we introduce some assumptions and transformations are needed in the next sections to prove the main result.

In Section 3, we study the existence and uniqueness of solution for the system (2.1.1)-(2.1.3) using semigroup techniques.

In Section 4, we prove the general stability of the solution.

2.2 Preliminaries

In this section, we present the background mathematics needed later to prove our main result. We shall use the following hypothesis

(H1) $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing differentiable function.

(H2) $g : \mathbb{R} \to \mathbb{R}$ is a non-decreasing C^0-function such that there exists a positive constants $\nu_1, \nu_{2,}\, \epsilon$ and a strictly increasing function $G \in C^1\left([0,\infty)\right)$, with $G(0) = 0$, and G is linear or strictly convex C^2-function on $(0,\epsilon]$ such that

$$\begin{cases} s^2 + g^2(s) \leq G^{-1}\left(sg\left(s\right)\right),\ \forall \left|s\right| \leq \epsilon, \\ \nu_1 \left|s\right| \leq \left|g(s)\right| \leq \nu_2 \left|s\right|,\ \forall \left|s\right| \geq \epsilon, \end{cases} \tag{2.2.1}$$

which implies that $sg(s) > 0$, for all $s \neq 0$.

(H3) The function g satisfies the following property

$$|g(\phi_2) - g(\phi_1)| \leq k_0(|\phi_1|^{\varrho} + |\phi_2|^{\varrho})|\phi_1 - \phi_2|, \phi_1, \phi_2 \in [-\epsilon, \epsilon], \tag{2.2.2}$$

where $k_0 > 0$, $\varrho > 0$.

Note that the hypothesis (H2) was first introduced by (Lasiecka and Tataru [24]) in 1993.

2.3 Existence and uniqueness

In this section, we give the existence and uniqueness result for the system (2.1.1)-(2.1.3) using the semigroup theory ([25]). So, if we denote $U = (u, v, \phi, \psi, \theta, \vartheta)^T$, where $v = u_t$, $\psi = \phi_t$ and $\vartheta = \theta_t$. Then, system (2.1.1)-(2.1.3) can be rewritten as follows:

$$\begin{cases} U_t\,(t) = \mathbb{A}U\,(t) + \Gamma\,(U)\,, \ \ t > 0 \\ U\,(0) = U_0 = (u_0,\ u_1,\ \phi_0,\ \phi_1,\ \theta_0,\ w_0)^T\,, \end{cases}$$

where the operator $\mathbb{A} : \mathbb{D}(\mathbb{A}) \subset \mathbb{H} \longrightarrow \mathbb{H}$ is defined by

$$\mathbb{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ \dfrac{\mu^*}{\rho}\partial_x^2\,(.) & 0 & \dfrac{\mu_0}{\rho}\partial_x\,(.) & 0 & -\dfrac{\beta_0}{\rho}\partial_x\,(.) & -\dfrac{\beta_0\tau}{\rho}\partial_x\,(.) \\ 0 & 0 & 0 & I & 0 & 0 \\ -\dfrac{\mu_0}{J}\partial_x\,(.) & 0 & \dfrac{a_0}{J}\partial_x^2\,(.) - \dfrac{\xi}{J}I & 0 & -\dfrac{\beta_1}{J}I & -\dfrac{\beta_1\tau}{J}I \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & -\dfrac{\beta_0}{a\tau}\partial_x\,(.) & 0 & \dfrac{\beta_1}{a\tau}I & \dfrac{\kappa}{a\tau}\partial_x^2\,(.) & -\dfrac{1}{\tau}I \end{pmatrix}, \tag{2.3.1}$$

and $\Gamma : D(\Gamma) = \mathbb{H} \longrightarrow \mathbb{H}$ is the nonlinear operator defined by

$$\Gamma(U) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{\alpha(t)}{J} g(\psi) \\ 0 \\ 0 \end{pmatrix}.$$

We consider the following spaces

$$\begin{aligned} &L_*^2(0,1) = \left\{ \Psi \in L^2(0,1),\ \int_0^1 \Psi(x)\, dx = 0 \right\}, \\ &H_*^2(0,1) = \left\{ \Psi \in H^2(0,1),\ \Psi_x(0) = \Psi_x(1) = 0 \right\}, \\ &H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1), \end{aligned} \tag{2.3.2}$$

and let

$$\mathbb{H} = H_*^1(0,1) \times L_*^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1), \tag{2.3.3}$$

be the Hilbert space equipped with the following inner product

$$\begin{aligned} \langle U, \tilde{U} \rangle_{\mathbb{H}} = &\rho \int_0^1 v\tilde{v}dx + J \int_0^1 \psi\tilde{\psi}dx + \mu_0 \int_0^1 (u_x\tilde{\phi} + \phi\tilde{u}_x)\, dx \\ &+ \xi \int_0^1 \phi\widetilde{\phi}dx + a_0 \int_0^1 \phi_x\widetilde{\phi}_x dx + a \int_0^1 (\theta + \tau\vartheta)\left(\widetilde{\theta} + \tau\widetilde{\vartheta}\right) dx \\ &+ \mu^* \int_0^1 u_x\widetilde{u}_x dx + \tau\kappa \int_0^1 \theta_x\widetilde{\theta}_x dx, \end{aligned} \tag{2.3.4}$$

for $U = (u, v, \varphi, \psi, \theta, \vartheta)^T \in \mathbb{H}$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\varphi}, \tilde{\psi}, \tilde{\theta}, \tilde{\vartheta})^T \in \mathbb{H}$. The domain of $\mathbb{A}$ is given by

$$D(\mathbb{A}) = \begin{pmatrix} U \in \mathbb{H} \mid u \in H_*^2(0,1) \cap H_*^1(0,1),\ v \in H_*^1(0,1), \\ \phi \in H^2(0,1) \cap H_0^1(0,1),\ \psi \in H_0^1(0,1), \\ \theta \in H^2(0,1) \cap H_0^1(0,1),\ \vartheta \in H_0^1(0,1). \end{pmatrix} \tag{2.3.5}$$

Clearly, the domain $\mathbb{D}(\mathbb{A})$ is dense in $\mathbb{H}$.

Remark 2.3.1. Under the assumption (2.1.4), it is clear that (2.1.3) defines an inner product. Precisely, from (2.1.3), we have

$$\begin{aligned}\langle U, U\rangle_{\mathbb{H}} = \frac{1}{2}\int_0^1 \Big(&\rho v^2 + J\psi^2 + \mu^* u_x^2 + a\left(\tau\theta_t + \theta\right)^2 + a_0\phi_x^2 + \xi\phi^2 \\ &+ 2\mu_0 u_x \phi + \kappa\tau\theta_x^2\Big)\, dx.\end{aligned}$$

Using the fact that

$$\begin{aligned}\mu^* u_x^2 + 2\mu_0 u_x\phi + \xi\phi^2 = \frac{1}{2}\Bigg[&\mu^*\left(u_x + \frac{\mu_0}{\mu^*}\phi\right)^2 + \xi\left(\phi + \frac{\mu_0}{\xi}u_x\right)^2 \\ &+ \left(\mu^* - \frac{\mu_0^2}{\xi}\right)u_x^2 + \left(\xi - \frac{\mu_0^2}{\mu^*}\right)\phi^2\Bigg],\end{aligned}$$

since $\mu_0^2 < \mu^*\xi$, we deduce that

$$\mu^* u_x^2 + 2\mu_0 u_x\phi + \xi\phi^2 > \frac{1}{2}\left(\mu^* - \frac{\mu_0^2}{\xi}\right)u_x^2 + \frac{1}{2}\left(\xi - \frac{\mu_0^2}{\mu^*}\right)\phi^2,$$

which yields $(U, U)_{\mathbb{H}} > 0$, by using the relation $\mu_0^2 < \mu^*\xi$.

Consequently, we conclude that $\left\langle U, \widetilde{U}\right\rangle_{\mathbb{H}}$ defines an inner product on $\mathbb{H}$ and the associated norm $\|.\|_{\mathbb{H}}$ is equivalent to the usual one.

Now, we can give the following well-posedness result.

Theorem 2.3.1. *Let $U_0 \in \mathbb{H}$ and assume that $(H1) - (H3)$ hold. Then, there exists a unique solution $U \in C\left(\mathbb{R}_+, \mathbb{H}\right)$. Moreover, if $U_0 \in D(\mathbb{A})$, then*

$$U \in C\left(\mathbb{R}_+, \mathbb{D}(\mathbb{A})\right) \cap C^1\left(\mathbb{R}_+, \mathbb{H}\right).$$

Proof. First, we prove that $\mathcal{A}$ is a maximal dissipative operator. For any $U \in \mathbb{D}(\mathbb{A})$, using (2.3.4) and integrating by parts, we have

$$\langle \mathbb{A}U, U\rangle_{\mathbb{H}} = -\kappa \int_0^1 \theta_x^2 dx \leq 0.$$

Thus, $\mathbb{A}$ is dissipative. Next, we prove that the operator $I-\ \mathbb{A}$ is surjective. Given $K = (k_1, k_2, k_3, k_4, k_5, k_6)^T \in \mathbb{H}$, we prove that there exists $U \in D(\mathbb{A})$ such that

$$(I - \mathbb{A})\, U = K. \tag{2.3.6}$$

That is

$$\begin{cases} u - v = k_1 \in H_*^1(0,1), \\ \rho v - \mu^* u_{xx} - \mu_0 \phi_x + \beta_0 \theta_x + \beta_0 \vartheta_x = \rho k_2 \in L_*^2(0,1), \\ \phi - \psi = k_3 \in H_0^1(0,1), \\ J\psi + \mu_0 u_x - a_0 \phi_{xx} + \xi\phi + \beta_1 \theta + \beta_1 \tau \vartheta = J k_4 \in L^2(0,1), \\ \theta - \vartheta = k_5 \in H_0^1(0,1), \\ a\tau\vartheta + \beta_0 v_x - \beta_1 \psi - \kappa\theta_{xx} + a\vartheta = a\tau k_6 \in L^2(0,1). \end{cases} \tag{2.3.7}$$

Inserting $u - v = k_1$, $\phi - \psi = k_3$ and $\theta - \vartheta = k_5$ in $(2.3.7)_2$, $(2.3.7)_4$ and $(2.3.7)_6$, we obtain

$$\begin{cases} \rho u - \mu^* u_{xx} - \mu_0 \phi_x + \beta_0 (1+\tau)\, \theta_x = \mu_1, \\ (J + \xi)\, \phi + \mu_0 u_x - a_0 \phi_{xx} + \beta_1 (1+\tau)\, \theta = \mu_2, \\ a(\tau+1)\, \theta + \beta_0 u_x - \beta_1 \phi - \kappa \theta_{xx} = \mu_3, \end{cases} \tag{2.3.8}$$

where

$$\begin{cases} \mu_1 = \rho k_2 + \rho k_1 + \beta_0 \tau k_{5x}, \\ \mu_2 = J k_4 + J k_3 + \beta_1 \tau k_5, \\ \mu_3 = a\tau k_6 + a(\tau+1)\, k_5 + \beta_0 k_{1x} - \beta_1 k_3. \end{cases}$$

To solve (2.3.8), we consider the following variational formulation

$$B\left((u,\phi,\theta), (\tilde{u},\tilde{\phi},\tilde{\theta})\right) = \mathcal{L}\left(\tilde{u},\tilde{\phi},\tilde{\theta}\right), \ \forall \left(\tilde{u},\tilde{\phi},\tilde{\theta}\right) \in V, \tag{2.3.9}$$

where $B : V \times V \longmapsto \mathbb{R}$ with

$$V = H_*^1(0,1) \times H_0^1(0,1) \in H_0^1(0,1),$$

is the bilinear form defined by

$$\begin{aligned}
B\left((u,\phi,\theta),\left(\tilde{u},\tilde{\phi},\tilde{\theta}\right)\right) &= \rho\int_0^1 u\tilde{u}dx + \mu^*\int_0^1 u_x\tilde{u}_xdx - \mu_0\int_0^1 \phi_x\tilde{u}dx \\
&+ \beta_0\left(1+\tau\right)\int_0^1 \theta_x\tilde{u}dx + \left(J+\xi\right)\int_0^1 \phi\tilde{\phi}dx \\
&+ \mu_0\int_0^1 u_x\tilde{\phi}dx + a_0\int_0^1 \phi_x\tilde{\phi}_xdx + \beta_1\left(1+\tau\right)\int_0^1 \theta\tilde{\phi}dx \\
&+ a\left(1+\tau\right)^2\int_0^1 \theta\tilde{\theta}dx + \beta_0\left(1+\tau\right)\int_0^1 u_x\tilde{\theta}dx \\
&- \beta_1\left(1+\tau\right)\int_0^1 \phi\tilde{\theta}dx + \kappa\left(1+\tau\right)\int_0^1 \theta_x\tilde{\theta}_xdx,
\end{aligned}$$

and $\mathcal{L}: V \longmapsto \mathbb{R}$ is the linear form given by

$$\mathcal{L}\left(\tilde{u},\tilde{\phi},\tilde{\theta}\right) = \int_0^1 \mu_1\tilde{u}dx + \int_0^1 \mu_2\tilde{\phi}dx + \int_0^1 \mu_3\tilde{\theta}dx.$$

Now, if we equipped the space V with the norm

$$\begin{aligned}
\|(u,\phi,\theta)\|_V^2 &= \|u\|_2^2 + \|u_x\|_2^2 + \|\phi\|_2^2 + \|u\|_2^2 + \|\phi_x\|_2^2 \\
&+ \|\theta\|_2^2 + \|\theta_x\|_2^2,
\end{aligned}$$

we have

$$\begin{aligned}
&\|B\left((u,\phi,\theta),(u,\phi,\theta)\right)\|_V^2 \\
&= \rho\int_0^1 u^2dx + \mu^*\int_0^1 u_x^2dx + 2\mu_0\int_0^1 \phi u_xdx + \left(J+\xi\right)\int_0^1 \phi^2dx \\
&+ a_0\int_0^1 \phi_x^2dx + a\left(1+\tau\right)^2\int_0^1 \theta^2dx + \kappa\left(1+\tau\right)\int_0^1 \theta_x^2dx.
\end{aligned}$$

On the other hand, we can write

$$\begin{aligned}
&\mu^*\int_0^1 u_x^2dx + 2\mu_0\int_0^1 \phi u_xdx + \left(J+\xi\right)\int_0^1 \phi^2dx \\
&> \frac{1}{2}\left(\mu^* - \frac{\mu_0^2}{J+\xi}\right)u_x^2 + \frac{1}{2}\left(J+\xi-\frac{\mu_0^2}{\mu^*}\right)\phi^2,
\end{aligned}$$

and since $\xi - \dfrac{\mu_0^2}{\mu^*} > 0$, we deduce that

$$\begin{aligned}
&\left|B\left(\left(u,\phi,\theta\right),\left(u,\phi,\theta\right)\right)\right| \\
&\geq \rho\int_0^1 u^2dx + \frac{1}{2}\left(\mu^* - \frac{\mu_0^2}{J+\xi}\right)\int_0^1 u_x^2 + \frac{1}{2}\left(J+\xi-\frac{\mu_0^2}{\mu^*}\right)\int_0^1 \phi^2 \\
&+ a_0\int_0^1 \phi_x^2dx + a\left(1+\tau\right)^2\int_0^1 \theta^2dx + \kappa\left(1+\tau\right)\int_0^1 \theta_x^2dx \\
&\geq M_0\left\|\left(u,\phi,\theta\right)\right\|_V^2 .
\end{aligned}$$

Thus, B is coercive.

Because B is bilinear form and we can easily show that it is bounded on $\left(V \times V\right)$, thus B is continuous. By the same way, we show that $\mathcal{L}$ is continuous . Consequently by applying the Lax-Milgram theorem, we conclude that the variational problem (2.3.9) has a unique solution $u \in H_*^1\left(0,1\right), \phi \in H_0^1\left(0,1\right), \theta \in H_0^1\left(0,1\right)$. The substitution of u, ϕ and θ, respectively, in $(2.3.7)_1$, $(2.3.7)_3$ and $(2.3.7)_5$ yields

$$\left(v,\psi,\vartheta\right) \in H_*^1\left(0,1\right) \times H_0^1\left(0,1\right) \times H_0^1\left(0,1\right).$$

Moreover, if we take $\left(\tilde{\phi},\tilde{\theta}\right) \equiv \left(0,0\right) \in H_0^1\left(0,1\right) \times H_0^1\left(0,1\right)$ in (2.3.9), we get

$$\mu^*\int_0^1 u_x\tilde{u}_xdx = \int_0^1\left(\mu_1 + \mu_0\phi_x - \beta_0\left(1+\tau\right)\theta_x - \rho u\right)\tilde{u}dx,\ \forall\tilde{u} \in H_*^1\left(0,1\right). \quad (2.3.10)$$

Here, we cannot use the regularity theorem, because $\tilde{u} \in H_*^1\left(0,1\right)$. So, we take $\bar{\Psi} \in H_0^1\left(0,1\right)$ and set

$$\tilde{u}(x) = \bar{\Psi}\left(x\right) - \int_0^1 \bar{\Psi}\left(x\right)dx.$$

It is clear that $\tilde{u} \in H_*^1\left(0,1\right)$. Then, a substitution in (2.3.10) leads to

$$\mu^*\int_0^1 u_x\bar{\Psi}_xdx = \int_0^1 r\bar{\Psi}dx,\ \forall\bar{\Psi} \in H_0^1\left(0,1\right),$$

where

$$r = \mu_1 + \mu_0 \phi_x - \beta_0 (1+\tau) \theta_x - \rho u \in L^2_* (0,1) .$$

So

$$u \in H^2 (0,1) ,$$

and

$$-\mu^* u_{xx} = \mu_1 + \mu_0 \phi_x - \beta_0 (1+\tau) \theta_x - \rho u. \tag{2.3.11}$$

We replace (2.3.7)$_1$, (2.3.7)$_5$ and $\mu_1 = \rho k_2 + \rho k_1 + \beta_0 \tau k_{5x}$ in (2.3.11), we obtain

$$\rho v - \mu^* u_{xx} - \mu_0 \phi_x + \beta_0 \theta_x + \beta_0 \vartheta_x = \rho k_2.$$

This gives (2.3.7)$_2$. Since $-\mu^* u_{xx} = r(x)$, then

$$-\mu^* \int_0^1 u_{xx} \Phi dx = \int_0^1 r \Phi dx, \ \forall \Phi \in H^1 (0,1) .$$

Namely,

$$-\mu^* u_x \Phi \mid_0^1 + \mu^* \int_0^1 u_x \Phi_x dx = \int_0^1 r \Phi dx, \ \forall \Phi \in H^1 (0,1) .$$

Because $H^1_* (0,1) \subset H^1 (0,1)$, we can write

$$-\mu^* u_x \tilde{u} \mid_0^1 dx + \mu^* \int_0^1 u_x \tilde{u}_x dx = \int_0^1 r \tilde{u} dx, \ \forall \tilde{u} \in H^1_* (0,1) ,$$

and on the other hand, we have (2.3.10). Thus,

$$u_x (1) \tilde{u} (1) - u_x (0) \tilde{u} (0) = 0.$$

Since $\tilde{u} \in H^1_* (0,1)$ is arbitrary. Then,

$$u_x (1) = u_x (0) = 0.$$

Consequently,

$$u \in H^2_* (0,1) \cap H^1_* (0,1) .$$

If we choose $\left(\tilde{u},\tilde{\theta}\right) \equiv (0,0) \in H_*^1(0,1) \times H_0^1(0,1)$ in (2.3.9), we get

$$a_0 \int_0^1 \phi_x \tilde{\phi}_x dx = \int_0^1 \left(\mu_2 - (J+\xi)\phi - \mu_0 u_x - \beta_1(1+\tau)\theta\right)\tilde{\phi}dx, \ \forall \tilde{\phi} \in H_0^1(0,1).$$

That is

$$-a_0\phi_{xx} = \mu_2 - (J+\xi)\phi - \mu_0 u_x - \beta_1(1+\tau)\theta \in L^2(0,1),$$

and by applying the regularity theory for the linear elliptic equations, we conclude that

$$\phi \in H^2(0,1) \cap H_0^1(0,1).$$

If we take $(\tilde{u},\tilde{\phi}) \equiv (0,0) \in H_*^1(0,1) \times H_0^1(0,1)$ in (2.3.9), we get

$$\begin{aligned}&\kappa(1+\tau)\int_0^1 \theta_x \tilde{\theta}_x dx \\ &= \int_0^1 \left(\mu_3 - a(1+\tau)^2\theta - \beta_0(1+\tau)u_x + \beta_1(1+\tau)\phi\right)\tilde{\theta}dx, \ \forall \tilde{\theta} \in H_0^1(0,1).\end{aligned}$$

That is

$$-\kappa(1+\tau)\theta_{xx} = \mu_3 - a(1+\tau)^2\theta - \beta_0(1+\tau)u_x + \beta_1(1+\tau)\varphi \in L^2(0,1),$$

by the regularity theory for the linear elliptic equations, we deduce that

$$\theta \in H^2(0,1) \cap H_0^1(0,1).$$

Now, we prove that the operator Γ is locally Lipschitz in $\mathbb{H}$.
Let $U = (u,v,\phi,\psi,\theta,\vartheta)^T \in \mathbb{H}$ and $U_1 = (u_1,v_1,\phi_1,\psi_1,\theta_1,\vartheta_1)^T \in \mathbb{H}$ with $\|U\|_{\mathbb{H}} \leq \epsilon$, $\|U_1\|_{\mathbb{H}} \leq \epsilon$, then we have

$$\|\Gamma(U) - \Gamma(U_1)\|_{\mathbb{H}} \leq \eta_1 \|g(\psi) - g(\psi_1)\|_2.$$

By using (2.2.2) and Hölder's inequalities, we can get

$$\|g(\psi) - g(\psi_1)\|_2 \leq k_0(\|\psi\|_{2\varrho}^{\varrho} + \|\psi_1\|_{2\varrho}^{\varrho})\|\psi - \psi_1\|_2 \leq \eta_2\|\psi - \psi_1\|_2.$$

Which gives us

$$\|\Gamma(U)-\Gamma(U_1)\|_{\mathbb{H}} \leq \eta_3 \|\psi-\psi_1\|_{\mathbb{H}}.$$

Then the operator Γ is locally Lipschitz in $\mathbb{H}$.

Hence, there exists a unique $U \in D(\mathbb{A})$ such that (2.3.6) is satisfied. Consequently, the operator $\mathbb{A}$ is maximal. With this, we conclude that $\mathbb{A}$ is a maximal dissipative operator. Then, $\mathbb{A}$ is the infinitesimal generator of a linear contraction C_0-semigroup on $\mathbb{H}$. Therefore, the well-posedness result follows from the Lumer–Phillips theorem. □

2.4 General stability

In this section, we use the energy method to prove a general stability result of the system (2.1.1)-(2.1.3). First, we state and prove some technical lemmas needed in the proof of our result.

Lemma 2.4.1. *Let* (u,ϕ,θ) *be the solution of system* (2.1.1)-(2.1.3). *Then, the energy functional* $E(t)$*, defined by*

$$E(t)=\frac{1}{2}\int_0^1\left(\rho u_t^2+\mu^* u_x^2+J\phi_t^2+a_0\phi_x^2+\xi\phi^2+2\mu_0\phi u_x+a(\tau\theta_t+\theta)^2+\kappa\tau\theta_x^2\right)dx, \tag{2.4.1}$$

satisfies

$$E'(t)=-\kappa\int_0^1\theta_x^2dx-\alpha(t)\int_0^1\phi_t g(\phi_t)\,dx\leq 0, \tag{2.4.2}$$

and

$$E'(t)\leq-\kappa\int_0^1\theta_x^2dx+\frac{\alpha(0)}{2}\int_0^1\phi_t^2dx+\frac{\alpha(0)}{2}\int_0^1 g^2(\phi_t)\,dx. \tag{2.4.3}$$

Proof. Equation (2.4.1) can be obtained by multiplying the first three equations of (2.1.1) by u_t, ϕ_t, and $(\tau\theta_t+\theta)$ respectively, integrating by parts over $(0,1)$ and using the boundary conditions (2.1.3). The estimate (2.4.3) is produced by using Young's inequality to the last term of (2.4.2). □

Lemma 2.4.2. *Let* (u, ϕ, θ) *be the solution of* (2.1.1)-(2.1.3). *Then, the functional*

$$I_1(t) = \rho \int_0^1 u_t u dx - \beta_0 \tau \int_0^1 \theta u_x dx,\ t \geq 0, \tag{2.4.4}$$

satisfies

$$I_1'(t) \leq -\frac{\mu^*}{2} \int_0^1 u_x^2 dx + \left(\frac{\beta_0 \tau}{2} + \frac{\beta_0^2}{\mu^*} \right) \int_0^1 \theta_x^2 dx + \frac{\beta_0 \tau}{2} \int_0^1 u_t^2 dx + \frac{\mu_0^2}{\mu^*} \int_0^1 \phi^2 dx,\ t \geq 0. \tag{2.4.5}$$

Proof. Direct computation, by using equation $(2.1.1)_1$ and integrating by parts, we get

$$\begin{aligned} I_1'(t) = &- \mu^* \int_0^1 u_x^2 dx + \rho \int_0^1 u_t^2 dx - \mu_0 \int_0^1 \phi u_x dx - \beta_0 \int_0^1 \theta_x u dx \\ &+ \beta_0 \tau \int_0^1 \theta_x u_t dx. \end{aligned} \tag{2.4.6}$$

Using Young's inequality, we get

$$-\mu_0 \int_0^1 \phi u_x dx \leq \frac{\mu_0^2}{\mu^*} \int_0^1 \phi^2 dx + \frac{\mu^*}{4} \int_0^1 u_x^2 dx, \tag{2.4.7}$$

$$\beta_0 \tau \int_0^1 \theta_x u_t dx \leq \frac{\beta_0 \tau}{2} \int_0^1 \theta_x^2 dx + \frac{\beta_0 \tau}{2} \int_0^1 u_t^2 dx. \tag{2.4.8}$$

Young's and Poincaré inequalities lead to

$$-\beta_0 \int_0^1 \theta_x u dx \leq \frac{\beta_0^2}{\mu^*} \int_0^1 \theta_x^2 dx + \frac{\mu^*}{4} \int_0^1 u_x^2 dx. \tag{2.4.9}$$

Inserting (2.4.7)-(2.4.9) in (2.4.6), we obtain (2.4.5). □

Lemma 2.4.3. *Let* (u, ϕ, θ) *be the solution of* (2.1.1)-(2.1.3). *Then, the functional*

$$\begin{aligned} I_2(t) := J \int_0^1 \phi_t \phi dx - \frac{\mu_0 \rho}{\mu^*} \int_0^1 u_t \left(\int_0^x \phi(y)\, dy \right) dx \\ + \left(\frac{\beta_0 \mu_0 \tau}{\mu^*} + \beta_1 \tau \right) \int_0^1 \theta \phi dx, \quad t \geq 0\ , \end{aligned}$$

satisfies, for any $\varepsilon_1 > 0$*, the following estimate*

$$\begin{aligned} I_2'(t) \leq & - a_0 \int_0^1 \phi_x^2 dx - \frac{1}{2}\chi \int_0^1 \phi^2 dx + \varepsilon_1 \int_0^1 u_t^2 dx \\ & + \left(\varsigma_2 + \frac{\mu_0^2 \rho^2}{4{\mu^*}^2 \varepsilon_1} \right) \int_0^1 \phi_t^2 dx + (\varsigma_1 + \varsigma_2) \int_0^1 \theta_x^2 dx \\ & + \frac{\alpha(0)^2}{\chi} \int_0^1 g^2(\phi_t)\, dx, \ t > 0, \end{aligned} \tag{2.4.10}$$

where $\varsigma_1 = \frac{1}{\chi}\left(\beta_1 + \frac{\beta_0 \mu_0}{\mu^*}\right)^2 > 0$ *and* $\varsigma_2 = \frac{\tau}{2}\left(\beta_1 + \frac{\beta_0 \mu_0}{\mu^*}\right) > 0.$

Proof. By taking a derivative of $I_2(t)$, using $(2.1.1)_2$ and then integrating by parts together with the boundary conditions, we obtain

$$\begin{aligned} I_2'(t) = & -a_0 \int_0^1 \phi_x^2 dx - \left(\xi - \frac{\mu_0^2}{\mu^*}\right) \int_0^1 \varphi^2 dx + J \int_0^1 \phi_t^2 dx \\ & - \left(\beta_1 + \frac{\beta_0 \mu_0}{\mu^*}\right) \int_0^1 \theta \phi dx - \frac{\mu_0 \rho}{\mu^*} \int_0^1 u_t \left(\int_0^x \phi_t(y)\, dy \right) dx \\ & + \left(\frac{\beta_0 \mu_0 \tau}{\mu^*} + \beta_1 \tau \right) \int_0^1 \theta \phi_t dx - \alpha(t) \int_0^1 \phi g(\phi_t)\, dx. \end{aligned} \tag{2.4.11}$$

By using Young's inequality, we get

$$-\alpha(t) \int_0^1 \phi g(\phi_t)\, dx \leq \frac{1}{4}\chi \int_0^1 \phi^2 dx + \frac{\alpha(0)^2}{\chi} \int_0^1 g^2(\phi_t)\, dx. \tag{2.4.12}$$

By using Young's and Poincaré inequalities, we have

$$-\left(\beta_1 + \frac{\beta_0 \mu_0}{\mu^*}\right) \int_0^1 \theta \phi dx \leq \frac{1}{4}\chi \int_0^1 \phi^2 dx + \varsigma_1 \int_0^1 \theta_x^2 dx, \tag{2.4.13}$$

$$\tau \left(\beta_1 + \frac{\beta_0 \mu_0}{\mu^*}\right) \int_0^1 \theta \phi_t dx \leq \varsigma_2 \int_0^1 \left(\theta_x^2 + \phi_t^2\right) dx, \tag{2.4.14}$$

where $\varsigma_1 = \frac{1}{\chi}\left(\beta_1 + \frac{\beta_0 \mu_0}{\mu^*}\right)^2$ and $\varsigma_2 = \frac{\tau}{2}\left(\beta_1 + \frac{\beta_0 \mu_0}{\mu^*}\right)$.

Young's and Cauchy Schwarz inequalities, give us

$$-\frac{\mu_0\rho}{\mu^*}\int_0^1 u_t\left(\int_0^x \phi_t\left(y\right)dy\right)dx \le \varepsilon_1\int_0^1 u_t^2dx+\frac{\mu_0^2\rho^2}{4{\mu^*}^2\varepsilon_1}\int_0^1\phi_t^2dx. \tag{2.4.15}$$

Inserting (2.4.12)-(2.4.15) in (2.4.11), we get (2.4.10). □

Lemma 2.4.4. *Let* (u,ϕ,θ) *be the solution of* (2.1.1)-(2.1.3). *Then, the functional*

$$I_3(t)=-a\tau^2\int_0^1\theta_t\theta dx-\frac{a\tau}{2}\int_0^1\theta^2dx,$$

satisfies, for any $\varepsilon_2>0$,

$$\begin{aligned} I_3'\left(t\right) &\le -\frac{a}{2}\int_0^1\left(\tau\theta_t+\theta\right)^2dx+\varepsilon_2\int_0^1 u_t^2dx+\frac{\beta_1\tau}{2}\int_0^1\phi_t^2dx \\ &\quad+\left(\frac{\beta_1\tau}{2}+\frac{\left(\beta_0\tau\right)^2}{4\varepsilon_2}+\frac{a}{2}\right)\int_0^1\theta_x^2dx. \end{aligned} \tag{2.4.16}$$

Proof. By differentiating $I_3(t)$, using $(2.1.1)_3$ and integrating by parts, we obtain

$$\begin{aligned} I_3'\left(t\right) &= -a\tau^2\int_0^1\theta_t^2dx-\beta_0\tau\int_0^1 u_t\theta_xdx-\beta_1\tau\int_0^1\phi_t\theta dx \\ &\quad+\kappa\tau\int_0^1\theta_x^2dx. \end{aligned} \tag{2.4.17}$$

Using the fact that

$$\int_0^1\left(\tau\theta_t+\theta\right)^2dx\le 2\int_0^1\left(\tau\theta_t\right)^2dx+2\int_0^1\theta^2dx,$$

which yields,

$$-\int_0^1\left(\tau\theta_t\right)^2dx\le-\frac{1}{2}\int_0^1\left(\tau\theta_t+\theta\right)^2dx+\frac{1}{2}\int_0^1\theta_x^2dx. \tag{2.4.18}$$

By using Young's and Poincaré inequalities, we obtain

$$-\beta_1\tau\int_0^1 \phi_t\theta dx \leq \frac{\beta_1\tau}{2}\int_0^1 \phi_t^2 dx + \frac{\beta_1\tau}{2}\int_0^1 \theta_x^2 dx. \tag{2.4.19}$$

By using Young's inequality, we get

$$-\beta_0\tau\int_0^1 u_t\theta_x dx \leq \varepsilon_2\int_0^1 u_t^2 dx + \frac{(\beta_0\tau)^2}{4\varepsilon_2}\int_0^1 \theta_x^2 dx. \tag{2.4.20}$$

By substituting (2.4.18)-(2.4.20) in (2.4.17), we have (2.4.16). □

Lemma 2.4.5. *Let* (u,ϕ,θ) *be the solution of* (2.1.1)-(2.1.3). *Then, the functional*

$$I_4(t) = -\rho a\int_0^1 \left(\int_0^x u_t(y)\,dy\right)(\tau\theta_t+\theta)\,dx, \tag{2.4.21}$$

satisfies, for any $\varepsilon_3,\varepsilon_4>0$, *the following estimate*

$$\begin{aligned} I_4'(t) \leq & -\frac{\beta_0\rho}{2}\int_0^1 u_t^2 dx + \varepsilon_3\int_0^1 u_x^2 dx + \frac{\rho\beta_1^2}{\beta_0}\int_0^1 \phi_t^2 dx + \varepsilon_4\int_0^1 \phi^2 dx \\ & + \frac{\rho\kappa^2}{\beta_0}\int_0^1 \theta_x^2 dx + \left(\beta_0 a + \frac{(\mu_0 a)^2}{4\varepsilon_4} + \frac{(a\mu^*)^2}{4\varepsilon_3}\right)\int_0^1 (\tau\theta_t+\theta)^2\,dx. \end{aligned} \tag{2.4.22}$$

Proof. By differentiating $I_4(t)$, we obtain

$$\begin{aligned} I_4'(t) = & -a\mu^*\int_0^1 u_x(\tau\theta_t+\theta)\,dx - \mu_0 a\int_0^1 \phi(\tau\theta_t+\theta)\,dx + \beta_0 a\int_0^1 (\tau\theta_t+\theta)^2\,dx \\ & + \beta_0\rho\int_0^1\left(\int_0^x u_t(y)\,dy\right)u_{tx}dx - \beta_1\rho\int_0^1\left(\int_0^x u_t(y)\,dy\right)\phi_t dx \\ & - \kappa\rho\int_0^1\left(\int_0^x u_t(y)\,dy\right)\theta_{xx}dx. \end{aligned}$$

By integrating by parts and using the fact that $\int_0^1 u dx = 0$, we arrive at

$$I_4'(t) = -a\mu^* \int_0^1 u_x \left(\tau\theta_t + \theta\right) dx - \mu_0 a \int_0^1 \phi \left(\tau\theta_t + \theta\right) dx + \beta_0 a \int_0^1 \left(\tau\theta_t + \theta\right)^2 dx$$
$$- \beta_0 \rho \int_0^1 u_t^2 dx - \beta_1 \rho \int_0^1 \left(\int_0^x u_t \left(y\right) dy \right) \phi_t dx + \kappa\rho \int_0^1 u_t \theta_x dx. \tag{2.4.23}$$

Young's inequality, leads to

$$-a\mu^* \int_0^1 u_x \left(\tau\theta_t + \theta\right) dx \leq \varepsilon_3 \int_0^1 u_x^2 dx + \frac{\left(a\mu^*\right)^2}{4\varepsilon_3} \int_0^1 \left(\tau\theta_t + \theta\right)^2 dx, \tag{2.4.24}$$
$$-\mu_0 a \int_0^1 \phi \left(\tau\theta_t + \theta\right) dx \leq \varepsilon_4 \int_0^1 \phi^2 dx + \frac{\left(\mu_0 a\right)^2}{4\varepsilon_4} \int_0^1 \left(\tau\theta_t + \theta\right)^2 dx, \tag{2.4.25}$$
$$\kappa\rho \int_0^1 u_t \theta_x dx \leq \frac{\beta_0 \rho}{4} \int_0^1 u_t^2 dx + \frac{\rho\kappa^2}{\beta_0} \int_0^1 \theta_x^2 dx. \tag{2.4.26}$$

Applying Young's and Cauchy Schwarz inequalities, we arrive at

$$-\beta_1 \rho \int_0^1 \left(\int_0^x u_t \left(y\right) dy \right) \phi_t dx \leq \frac{\beta_0 \rho}{4} \int_0^1 u_t^2 dx + \frac{\rho \beta_1^2}{\beta_0} \int_0^1 \phi_t^2 dx. \tag{2.4.27}$$

By substituting (2.4.24)-(2.4.27) in (2.4.23), we have (2.4.22). □

Now, we define the Lyapunov functional $L(t)$ by

$$L(t) = NE(t) + N_1 I_1\left(t\right) + N_2 I_2\left(t\right) + N_3 I_3\left(t\right) + N_4 I_4\left(t\right), \tag{2.4.28}$$

where N, N_1, N_2, N_3, and N_4 are positive constants.

Theorem 2.4.1. *Let* (u, φ, θ) *be the solution of* (2.1.1)-(2.1.3). *Then, there exists two positive constants* κ_1 *and* κ_2 *such that the Lyapunov functional* (2.4.28) *satisfies*

$$\kappa_1 E\left(t\right) \leq L(t) \leq \kappa_2 E\left(t\right), \ \forall t \geq 0, \tag{2.4.29}$$

and

$$L'(t) \leq -\beta_2 E(t) + \beta_3 \int_0^1 \left(\varphi_t^2 + g^2(\varphi_t) \right) dx, \ \beta_2, \beta_3 > 0. \tag{2.4.30}$$

Proof. From (2.4.28), we get

$$\begin{aligned}|L(t)-NE(t)| &\leq \rho N_1 \int_0^1 |u_t u|\,dx + \beta_0 \tau N_1 \int_0^1 |\theta u_x|\,dx \\ &+ JN_2 \int_0^1 |\phi_t \phi|\,dx + N_2 \frac{\mu_0 \rho}{\mu^*} \int_0^1 \left| u_t \left(\int_0^x \phi(y)\,dy \right) \right| dx \\ &+ \left(\frac{\beta_0 \mu_0 \tau}{\mu^*} + \beta_1 \tau \right) \int_0^1 |\theta \phi|\,dx \\ &+ aN_3 \int_0^1 \tau^2 |\theta_t \theta|\,dx + N_4 \rho a \int_0^1 \left| \left(\int_0^x u_t(y)\,dy \right) (\tau \theta_t + \theta) \right| dx.\end{aligned}$$

By using Young's, Cauchy-Schwarz and Poincaré inequalities, we have

$$|L(t)-NE(t)| \leq cE(t),\ c>0,$$

that is

$$(N-c)\,E(t) \leq L(t) \leq (N+c)\,E(t).$$

Now, by choosing N (depending on N_1, N_2, N_3, and N_4) sufficiently large, we obtain (2.4.29).

By differentiating equation (2.4.28), then, recalling the estimations (2.4.2), (2.4.5), (2.4.10), (2.4.16) and (2.4.22), we get

$$\begin{aligned}
L'(t) \leq & -\left[N\kappa - N_1\left(\frac{\beta_0\tau}{2}+\frac{\beta_0^2}{\mu^*}\right) - N_2\left(\varsigma_1+\varsigma_2\right) - N_3\left(\frac{\beta_1\tau}{2}+\frac{(\beta_0\tau)^2}{4\varepsilon_2}+\frac{a}{2}\right)\right. \\
& \left. -N_4\frac{\rho\kappa^2}{\beta_0}\right]\int_0^1 \theta_x^2 dx + \left[\frac{\alpha^2(0)}{\chi}N_2 + \frac{\alpha(0)}{2}N\right]\int_0^1 g^2\left(\phi_t\right)dx \\
& + \left[N_2\left(\varsigma_2 + \frac{\mu_0^2\rho^2}{4\mu^{*2}\varepsilon_1}\right) + N_3\frac{\beta_1\tau}{2} + N_4\frac{\beta_1^2\rho}{\beta_0} + \frac{\alpha(0)}{2}N\right]\int_0^1 \varphi_t^2 dx \\
& - \left[N_1\frac{\mu^*}{2} - N_4\varepsilon_3\right]\int_0^1 u_x^2 dx - a_0 N_2\int_0^1 \phi_x^2 dx \\
& - \left[\frac{1}{2}\chi N_2 - N_1\frac{\mu_0^2}{\mu^*} - N_4\varepsilon_4\right]\int_0^1 \phi^2 dx \\
& - \left[N_3\frac{a}{2} - N_4\left(\beta_0 a + \frac{(\mu_0 a)^2}{4\varepsilon_4} + \frac{(a\mu^*)^2}{4\varepsilon_3}\right)\right]\int_0^1 \left(\tau\theta_t+\theta\right)^2 dx \\
& - \left[N_4\frac{\beta_0\rho}{2} - N_2\varepsilon_1 - N_3\varepsilon_2 - N_1\frac{\beta_0\tau}{2}\right]\int_0^1 u_t^2 dx,
\end{aligned}$$

if we choose $\varepsilon_1 = \frac{1}{N_2}$, $\varepsilon_2 = \frac{1}{N_3}$ and $\varepsilon_3 = \varepsilon_4 = \frac{1}{N_4}$, we obtain

$$\begin{aligned}
L'(t) \leq & -\left[N\kappa - N_1\left(\frac{\beta_0\tau}{2}+\frac{\beta_0^2}{\mu^*}\right) - N_2\left(\varsigma_1+\varsigma_2\right) - N_3\left(\frac{\beta_1\tau}{2}+\frac{N_3(\beta_0\tau)^2}{4}+\frac{a}{2}\right)\right. \\
& \left. -N_4\frac{\rho\kappa^2}{\beta_0}\right]\int_0^1 \theta_x^2 dx + \left[\frac{\alpha^2(0)}{\chi}N_2 + \frac{\alpha(0)}{2}N\right]\int_0^1 g^2\left(\phi_t\right)dx \\
& + \left[N_2\left(\varsigma_2 + N_2\frac{\mu_0^2\rho^2}{4\mu^{*2}}\right) + N_3\frac{\beta_1\tau}{2} + N_4\frac{\beta_1^2\rho}{\beta_0} + \frac{\alpha(0)}{2}N\right]\int_0^1 \phi_t^2 dx \\
& - \left[N_1\frac{\mu^*}{2} - 1\right]\int_0^1 u_x^2 dx - a_0 N_2\int_0^1 \phi_x^2 dx \\
& - \left[\frac{1}{2}\chi N_2 - N_1\frac{\mu_0^2}{\mu^*} - 1\right]\int_0^1 \phi^2 dx \\
& - \left[N_3\frac{a}{2} - N_4\left(\beta_0 a + \frac{N_4}{4}\left((\mu_0 a)^2 + (a\mu^*)^2\right)\right)\right]\int_0^1 \left(\tau\theta_t+\theta\right)^2 dx \\
& - \left[\frac{\beta_0\rho}{2}N_4 - 2 - N_1\frac{\beta_0\tau}{2}\right]\int_0^1 u_t^2 dx.
\end{aligned}$$

We select our parameters appropriately as follows:

First, we choose N_1 large enough such that

$$\frac{\mu^*}{2}N_1 - 1 > 0.$$

We take N_2 large such that

$$\frac{1}{2}\chi N_2 - N_1\frac{\mu_0^2}{\mu^*} - 1 > 0.$$

We pick N_4 large such that

$$\frac{\beta_0\rho}{2}N_4 - N_1\frac{\beta_0\tau}{2} - 2 > 0.$$

Then, we choose N_3 large enough such that

$$N_3\frac{a}{2} - N_4\left(\beta_0 a + \frac{N_4}{4}\left((\mu_0 a)^2 + (a\mu^*)^2\right)\right) > 0.$$

Finally, we take N large enough (even larger so that (2.4.29) remains valid) such that

$$N\kappa - N_1\left(\frac{\beta_0\tau}{2} + \frac{\beta_0^2}{\mu^*}\right) - N_2\left(\varsigma_1 + \varsigma_2\right) - N_3\left(\frac{\beta_1\tau}{2} + \frac{N_3\left(\beta_0\tau\right)^2}{4} + \frac{a}{2}\right) - N_4\frac{\rho\kappa^2}{\beta_0} > 0.$$

All these choices lead to (2.4.30). □

In this section, we state and prove our stability result.

Theorem 2.4.2. *Assume that (H1)- (H3) hold. Then, for any* $U_0 \in D(\mathbb{A})$, *there exists the positive constants* a_1, a_2, *such that*

$$E(t) \leq a_1 G_0^{-1}\left(\frac{a_2}{\int_0^t \alpha(s)ds}\right), \tag{2.4.31}$$

where

$$G_0\left(t\right) = tG'\left(\varepsilon_0 t\right),\ \forall\varepsilon_0 \geq 0.$$

Proof. Multiplying (2.4.30) by $\alpha(t)$, we get

$$\alpha(t)\,Ł'(t) \leq -\beta_2\alpha(t)\,E(t) + \beta_3\alpha(t)\int_0^1 \left(\phi_t^2 + g^2(\phi_t)\right)dx. \tag{2.4.32}$$

We distinguish two cases

1) G **is linear on** $[0,\epsilon]$. By using (2.4.32) and the hypothesis $(H2)$, we have

$$\begin{aligned}\alpha(t)\,L'(t) &\leq -\beta_2\alpha(t)\,E(t) + \beta_3\alpha(t)\int_0^1 \phi_t g(\phi_t)dx \\ &\leq -\beta_2\alpha(t)\,E(t) - \beta_3\alpha(t)\,E'(t).\end{aligned} \tag{2.4.33}$$

Since $\alpha'(t) \leq 0$, then (2.4.33) is equivalent to

$$L_1'(t) \leq -\beta_2\alpha(t)\,E(t), \tag{2.4.34}$$

where

$$L_1(t) = \alpha(t)\,L(t) + (\beta_3\alpha(t) + \varsigma)\,E(t)\sim E(t),\, \varsigma > 0. \tag{2.4.35}$$

Because $E(t)$ is a non-increasing function, for all $T \in \mathbb{R}_+$, by using (2.4.34), we have

$$E(T)\int_0^T \alpha(t)\,dt \leq \frac{L_1(0)}{\beta_2}. \tag{2.4.36}$$

Using the fact that $G_0^{-1}(t)$ is linear, then (2.4.36) can be rewritten as follows

$$E(T) \leq \lambda G_0^{-1}\left(\frac{L_1(0)}{\beta_2\int_0^T \alpha(t)\,dt}\right),\ \lambda > 0,$$

which gives (2.4.31) with $a_1 = \lambda$ and $a_2 = \dfrac{L_1(0)}{\beta_2}$. The proof is complete.

2) G **is nonlinear on** $[0,\epsilon]$. To estimate the last term of (2.4.32), we first choose $0 \leq \epsilon_1 \leq \epsilon$, such that $sg(s) \leq \min(\epsilon, G(\epsilon)), \forall |s| \leq \epsilon_1$ and by using $(H2)$, for $s \neq 0$, it follows that

$$\begin{cases} s^2 + g^2(s) \leq G^{-1}(sg(s)),\ \forall |s| \leq \epsilon_1, \\ \nu_1 |s| \leq |g(s)| \leq \nu_2 |s|,\ \forall |s| \geq \epsilon_1, \end{cases}$$

and we consider the following two sets

$$I_1 = \{x \in (0,1) : |\phi_t| \leq \epsilon_1\},\ I_2 = \{x \in (0,1) : |\phi_t| > \epsilon_1\}.$$

Now, we define $I(t)$ by

$$I(t) = \int_{I_1} \phi_t g(\phi_t)dx,$$

using Jensen's inequality and the hypothesis $(H2)$, we have

$$\beta_3\alpha(t)\int_0^1 \left(\phi_t^2 + g^2(\phi_t)\right)dx \leq \beta_2'\alpha(t)\,G^{-1}(I(t)) - \beta_2'\alpha(t)\,E'(t). \tag{2.4.37}$$

Inserting (2.4.37) in (2.4.32), we obtain

$$L_2'(t) \leq -\beta_2\alpha(t)\,E(t) + \beta_2'\alpha(t)\,G^{-1}(I(t)), \tag{2.4.38}$$

where

$$L_2(t) = \alpha(t)\,L(t) + \left(\beta_2'\alpha(t) + \tau_1\right)E(t) \sim E(t),\ \tau_1 > 0.$$

Now, for $\varepsilon_0 < \epsilon_1$ and using the fact that $E'(t) \leq 0$, $G' > 0$, $G'' > 0$ on $(0,\epsilon]$, we find that the functional $L_3(t)$, defined by

$$L_3(t) = G'(\varepsilon_0 E(t))\,L_2(t) + \tau_2 E(t) \sim E(t),\ \tau_2 > 0,$$

satisfies

$$\begin{aligned} L_3'(t) &= E'(t)\left(\varepsilon_0 G''(\varepsilon_0 E(t))_2(t) + \tau_2\right) +_2'(t)G'(\varepsilon_0 E(t)) \\ &\leq -\beta_2\alpha(t)\,G_0(E(t)) + \beta_2'\alpha(t)\,G'(\varepsilon_0 E(t))\,G^{-1}(I(t)). \end{aligned} \tag{2.4.39}$$

Note that, the equivalence between $L_3(t)$ and $E(t)$ is due to the fact that $G'(\varepsilon_0 E(t))$ is positive non-increasing function and $L_2(t) \sim E(t)$. Indeed, we have for all $t \geq 0$,

$$m_1 E(t) \leq L_2(t) \leq m_2 E(t),$$

and

$$0 < G'\left(\varepsilon_0 E(t)\right) \leq G'\left(\varepsilon_0 E(0)\right),$$

then

$$\tau_2 E\left(t\right) \leq L_3(t) \leq \left(G'\left(\varepsilon_0 E(0)\right) m_2 + \tau_2\right) E(t).$$

Therefore, there exists $\sigma_1, \sigma_2 > 0$, satisfying

$$\sigma_1 E\left(t\right) \leq L_3(t) \leq \sigma_2 E(t),$$

with $\sigma_1 = \tau_2, \sigma_2 = G'\left(\varepsilon_0 E(0)\right) m_2 + \tau_2$.

To estimate the last term of (2.4.39), we apply the following general Young's inequality

$$AB \leq G^*\left(A\right) + G\left(B\right), \text{ if } A \in \left(0, G'\left(\epsilon\right)\right], \ B \in \left(0, \epsilon\right],$$

where

$$G^*\left(s\right) = s\left(G'\right)^{-1}\left(s\right) - G\left(\left(G'\right)^{-1}\left(s\right)\right), \text{ if } s \in \left(0, G'\left(\epsilon\right)\right],$$

we deduce that

$$\beta_2'\alpha\left(t\right) G'\left(\varepsilon_0 E(t)\right) G^{-1}\left(I\left(t\right)\right) \leq \beta_2'\varepsilon_0\alpha\left(t\right) G_0\left(E(t)\right) - \beta_2'\alpha\left(t\right) E'(t). \tag{2.4.40}$$

Substituting (2.4.40) in (2.4.39) and letting $\varepsilon_0 = \dfrac{\beta_2}{2\beta_2'}$, we have

$$L_3'(t) + \beta_2'\alpha\left(t\right) E'(t) \leq -k\alpha\left(t\right) G_0\left(E(t)\right), \tag{2.4.41}$$

which can be rewritten as

$$\left(L_3(t) + \beta_2'\alpha\left(t\right) E(t)\right)' - \beta_2'\alpha'(t)E(t) \leq -k\alpha\left(t\right) G_0\left(E(t)\right) \tag{2.4.42}$$

since $\alpha'(t) \leq 0$, then (2.4.42) is equivalent to

$$L_4'(t) \leq -k\alpha\left(t\right) G_0\left(E(t)\right), \tag{2.4.43}$$

where

$$L_4(t) = L_3(t) + \beta_2' \alpha\,(t)\,E(t) \sim E(t),$$

this last relation is checked from the fact that $\alpha\,(t)$ is a positive non-increasing function and $L_3(t) \sim E(t)$. Indeed, for every $t \geq 0$, we have already

$$\sigma_1 E\,(t) \leq L_3(t) \leq \sigma_2 E(t),$$

then

$$\sigma_1 E(t) \leq L_4(t) \leq \sigma_3 E(t),$$

with $\sigma_3 = \sigma_2 + \beta_2' h(0)$.

By using (2.4.43), because $G_0\,(E(t))$ and $G'\,(\varepsilon_0 E(t))$ are non-increasing functions, then for all $T \in \mathbb{R}_+$, we have

$$kG_0\,(E(T)) \int_0^T \alpha\,(t)\,dt \leq L_4(0),$$

that can be rewritten as follows

$$E(T) \leq G_0^{-1}\left(\frac{L_4(0)}{k\int_0^T \alpha\,(t)\,dt}\right),$$

which gives (2.4.31) with $a_1 = 1$ and $a_2 = \dfrac{L_4(0)}{k}$. The proof is complete. $\square$

Conclusion In this work, we proved the existence and uniqueness of the solution of heat porous elastic system where the heat conduction is given by Lord-Shulman law and we gave a general decay of the solution by using the multipliers technique in which the exponential and polynomial decay are only special cases.

CHAPTER 3

General stability for a neutral delayed porous-elastic system

3.1 Introduction

In the present work, we consider the following porous-elastic system with nonlinear damping term and subject to a distributed delay of neutral type

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \ x \in (0,1)\,, \ t > 0, \\ J\left(\phi_t + \int_0^t k\,(t-s)\,\phi_t\,(s)\,ds\right)_t - \delta\phi_{xx} + bu_x + \xi\phi \\ +\alpha\,(t)\,g\,(\phi_t) = 0, \ x \in (0,1)\,, \ t > 0, \\ u\,(x,0) = u_0\,(x)\,, \ u_t\,(x,0) = u_1\,(x)\,, \ x \in (0,1)\,, \\ \phi\,(x,0) = \phi_0\,(x)\,, \ \phi_t\,(x,0) = \phi_1\,(x)\,, \ x \in (0,1)\,, \\ u_x\,(0,t) = u_x\,(1,t) = \phi\,(0,t) = \phi\,(1,t) = 0, \ t > 0, \end{cases} \tag{3.1.1}$$

where the functions u and ϕ represent, respectively, the displacement of the solid elastic material and the volume fraction. The parameter ρ designates the mass density and J equals to the product of the mass density by the equilibrated inertia. The term $\alpha(t) g(\phi_t)$ is the nonlinear damping term where α is a positive non-increasing differentiable function and g is specified in the preliminaries. The coefficients μ, δ, ξ are positive constants represent the constitutive parameters defining the coupling among the different components of the materials such that

$$\mu\xi > b^2. \tag{3.1.2}$$

where b is a real number different from zero. The initial data u_0, u_1, ϕ_0, ϕ_1 belongs to the suitable functional space and the integral represents the neutral delay term where k is the relaxation function that specified in the preliminaries.

The presence of delay may be a source of instability of systems which are asymptotically stable in the absence of delay unless additional control terms have been used (see [12],[13],[19],[30],[31],[39]). Also, the introducing of delay may lead to ill-posedness as shown in many works such as ([13],[35]). In addition to the well-known discrete delays, there are several others and we are interested here in the neutral delay where the delay is occurring in the second (highest) derivative, for more details, see previous studies ([14]-[16],[**?**],[38]) and the references therein. It is also worth mentioning that besides the fact that systems are very reactive to small delays, on the contrary, they can be stabilized by 'large' neutral delays of course under well-chosen assumptions and conditions. In fact, neutral delays are sometimes deliberately inserted into the systems to improve the performance of the structure.

Many investigations have been realized concerning the study of asymptotic behavior of the solutions for a one-dimensional porous-elastic system for different damping mechanisms such as ([20],[8],[32],[41],[29],[34]). Also, among the works concerned with the study of stability in the presence of neutral delay, we cite the work of Seghour et al. [37] where they considered the following thermo-elastic laminated system subject to a neutral delay

$$\begin{cases} \rho w_{tt} + G\left(\psi - w_x\right)_x + Aw_t = 0,\ x \in (0,1)\,,\ t > 0, \\ I_\rho\left(3s_{tt} - \psi_{tt}\right) - G\left(\psi - w_x\right) - \left(3s - \psi\right) + \mu\theta_x = 0,\ x \in (0,1)\,,\ t > 0, \\ 3I_\rho\left(s_t + \displaystyle\int_0^t h\left(t-r\right) s_t\left(r\right) dr\right)_t + 3G\left(\psi - w_x\right) + 4\gamma s - 3s_{xx} = 0,\ x \in (0,1)\,,\ t > 0, \\ \theta_t - \kappa\theta_{xx} + \mu\left(3s - \psi\right)_{tx} = 0,\ x \in (0,1)\,,\ t > 0, \end{cases}$$

with boundary conditions

$$\begin{cases} \psi\left(0,t\right) = s\left(0,t\right) = \theta_x\left(0,t\right) = w_x\left(0,t\right) = 0,\ t \geq 0, \\ \theta\left(1,t\right) = w\left(1,t\right) = s_x\left(1,t\right) = \psi_x\left(1,t\right) = 0,\ t \geq 0, \end{cases}$$

and initial data

$$\begin{cases} \left(w,\psi,s,\theta\right)\left(x,0\right) = \left(w_0,\psi_0,s_0,\theta_0\right),\ x \in (0,1)\,, \\ \left(w_t,\psi_t,s_t\right)\left(x,0\right) = \left(w_1,\psi_1,s_1\right),\ x \in (0,1)\,. \end{cases}$$

The authors showed, under some appropriate assumptions, that the dissipation produced by the heat equation with the frictional damping stabilize exponentially the system even in the presence of neutral delay for the case of equal wave speeds. In the opposite one, and with an additional assumption on the kernel, they proved a polynomial stability.

The main goal of this paper is to give a global well-posedness result for the problem (3.1.1) by using the Faedo-Galerkin method. Moreover, based on the multipliers method with some assumptions on the kernel of neutral delay which already exist in the literature alongside with some properties of the convex functions, we construct a suitable Lyapunov functional and we show that the dissipation given by the nonlinear damping term is strong enough to guarantee a general decay of the solutions, despite of the destructive nature of delays in general, for the case of equal speeds of wave propagation, that is

$$\chi = \frac{\mu}{\rho} - \frac{\delta}{J} = 0. \tag{3.1.3}$$

Introducing a neutral delay makes our problem different from those considered so far in the literature. Moreover, the study of the asymptotic behavior becomes different and more complicated than in the case of other types of delay that has appeared in the recent literature such as in ([2]-[3], [10]-[11], [17], [?]-[?]). In other words, a neutral-type delayed dynamical system is a more general class than delayed systems, in the sense where it's described by a model in which the highest derivative of the state at the present time is a function not only of the values of the passed state, but also of the highest derivative of the passed state and this strengthens the challenges.

This paper is organized as follows. In Section 2, we introduce some assumptions and transformations needed in the next sections to prove the main result. In Section 3, we give the existence and uniqueness result of the solution. In Section 4, we use the energy method to prove the general stability result.

3.2 Preliminaries

In this section, we present some assumptions to achieve our goal. For that, we use the standard Lebesgue space $L^2(0,1)$ and the Sobolev space $H_0^1(0,1)$ with their usual scalar products and norms. To simplify the calculations, we use the following Lemma which help us later to make some estimates.

Lemma 3.2.1 ([37]). *For any function $\psi \in C^1\left([0,\infty); L^2(0,1)\right)$ and any $k \in C^1([0,\infty))$, we have the following identity*

$$\begin{aligned}
&\int_0^1 \psi(t)\left(\int_0^t k(t-s)\,\psi_t(s)ds\right)dx \\
&= -\frac{1}{2}\left(k'\Box\psi\right)(t) + \frac{1}{2}\frac{d}{dt}\int_0^1\left(\int_0^t k(t-s)\,\psi^2(s)ds\right)dx \\
&+ \frac{k(t)}{2}\int_0^1 \psi^2 dx - k(t)\int_0^1 \psi(0)\psi(t)\,dx,
\end{aligned}$$

where

$$(k\Box\psi) = \int_0^t k(t-s)\left(\int_0^1 (\psi(t)-\psi(s))^2\,dx\right)ds,\ t \geq 0.$$

Also, we need the following hypothesis to reach our aim

(H1) The kernel k is a nonnegative continuously differentiable and summable function satisfying

$$k'(t) \leq 0, \; \forall t \geq 0, \; \bar{k} = \int_0^\infty k(s)ds.$$

(H2) $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing differentiable function.

(H3) $\exp(\varsigma t)\, k(t) \in L^1(\mathbb{R}_+)$ for some $\varsigma > 0$.

Note that if $\int_0^{+\infty} e^{\varsigma s} k(s)ds < \infty$ and $\lim\limits_{t \longrightarrow \infty} \exp(\varsigma t)\, k(t) < \infty$, then

$$\int_0^{+\infty} e^{\varsigma s} \left|k'(s)\right| ds = -\int_0^{+\infty} e^{\varsigma s} k'(s)ds = -e^{\varsigma s} k(s) \mid_0^\infty + \varsigma \int_0^{+\infty} e^{\varsigma s} k(s)ds < \infty.$$

(H4) $g : \mathbb{R} \to \mathbb{R}$ is a non-decreasing C^0-function such that there exists a positive constants $\nu_1, \nu_{2,}\ \epsilon$ and a strictly increasing function $G \in C^1([0, \infty))$, with $G(0) = 0$, and G is linear or strictly convex C^2-function on $(0, \epsilon]$ such that

$$\begin{cases} s^2 + g^2(s) \leq G^{-1}(sg(s)), \; \forall |s| \leq \epsilon, \\ \nu_1 |s| \leq |g(s)| \leq \nu_2 |s|, \; \forall |s| \geq \epsilon, \end{cases} \tag{3.2.1}$$

which implies that $sg(s) > 0$, for all $s \neq 0$.

Also, we need to use the following transformation to calculate the energy of the system and for other necessary estimations

$$\left(\int_0^t k(t-s)\, \phi_t(s)ds \right)_t = k(t)\, \phi_t(0) + \int_0^t k(t-s)\, \phi_{tt}(s)ds. \tag{3.2.2}$$

In view of the boundary conditions, our system can have solutions (uniform in the variable x), which do not decay. In other words, it is known that for the problem determined by (3.1.1) we can always take solutions where u is constant, for this reason, we impose that

$$\int_0^1 u_0 dx = \int_0^1 u_1 dx = 0. \tag{3.2.3}$$

It is worth noting that condition (3.2.3) is imposed to guarantee that the solution decays. Thus, if we want to avoid this behavior we need to impose condition (3.2.3).

3.3 Global well-posedness

In this section, we prove the global existence and the uniqueness of the solution of the problem (3.1.1) by using the classical Faedo-Galerkin approximations. First, let us define the following spaces

$$H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1),$$

with

$$L_*^2(0,1) = \left\{ g \in L^2(0,1) : \int_0^1 g(x)\,dx = 0 \right\}.$$

Also

$$H_*^2(0,1) = \left\{ \psi \in H^2(0,1) : \psi_x(0) = \psi_x(1) = 0 \right\}.$$

For completeness, we state without proof the following global existence and regularity result which can be proved by using the standard Faedo-Galerkin method, for which we refer the reader to [28].

Theorem 3.3.1. *Assume that* ***(H1)-(H4),*** (3.1.2) *hold, and the initial data*

$$\begin{aligned} (u_0, u_1) &\in H_*^1(0,1) \times L_*^2(0,1), \\ (\phi_0, \phi_1) &\in H_0^1(0,1) \times L^2(0,1), \end{aligned} \tag{3.3.1}$$

problem (3.1.1) *has a unique global weak solution*

$$\begin{aligned} u &\in C\left(\mathbb{R}_+, H_*^2(0,1) \cap H_*^1(0,1)\right) \cap C^1\left(\mathbb{R}_+, H_*^1(0,1)\right) \cap C^2\left(\mathbb{R}_+, L_*^2(0,1)\right), \\ \phi &\in C\left(\mathbb{R}_+, H^2(0,1) \cap H_0^1(0,1)\right) \cap C^1\left(\mathbb{R}_+, H_0^1(0,1)\right) \cap C^2\left(\mathbb{R}_+, L^2(0,1)\right). \end{aligned} \tag{3.3.2}$$

In addition, the solution (u, ϕ) depends continuously on the initial data.

3.4 Stability result

In this section, we use the energy method to study the asymptotic behavior of solutions of the system (3.1.1) and we establish a general decay result of solutions of the problem (3.1.1) in the case when (3.1.3) holds. So, we need the following lemmas.

Lemma 3.4.1. *The energy $E(t)$ of the system* (3.1.1) *given by*

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \left(\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + 2bu_x\phi + \xi\phi^2 + \delta\phi_x^2\right) dx \\ &+ \frac{J}{2}\int_0^1 \left(\int_0^t k\left(t-s\right)\phi_t^2(s)ds\right) dx,\ t \geq 0. \end{aligned} \tag{3.4.1}$$

satisfies

$$E'(t) \leq -\alpha\left(t\right)\int_0^1 \phi_t g\left(\phi_t\right) dx. \tag{3.4.2}$$

Proof. Multiplying $(3.1.1)_1, (3.1.1)_2$ by u_t, ϕ_t and integrating over $(0,1)$ and summing them up, we obtain

$$\begin{aligned} &\frac{d}{2dt}\int_0^1 \left(\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + 2bu_x\phi + \xi\phi^2 + \delta\phi_x^2\right) dx \\ &+ J\int_0^1 \left[\phi_t\left(\int_0^t k\left(t-s\right)\phi_t(s)ds\right)_t\right] dx \ = \ -\alpha\left(t\right)\int_0^1 \phi_t g\left(\phi_t\right) dx. \end{aligned} \tag{3.4.3}$$

By exploiting (3.2.2) and applying the result in Lemma 1, we obtain

$$\begin{aligned} &J\int_0^1 \left[\phi_t\left(\int_0^t k\left(t-s\right)\phi_t(s)ds\right)_t\right] dx \\ &= -\frac{J}{2}\left(k'\Box\phi_t\right)(t) + J\frac{k(t)}{2}\int_0^1 \phi_t^2 dx \\ &+ \frac{J}{2}\frac{d}{dt}\int_0^1 \left(\int_0^t k\left(t-s\right)\phi_t^2(s)ds\right) dx. \end{aligned} \tag{3.4.4}$$

Inserting (3.4.4) in (3.4.3) and taking into account the positivity of $k(t)$, we have (3.4.2). □

Remark 3.4.1. Note that

$$\mu u_x^2 + 2bu_x\phi + \xi\phi^2 = \mu\left(u_x + \frac{b}{\mu}\phi\right)^2 + \left(\xi - \frac{b^2}{\mu}\right)(\phi)^2$$

and because $\mu\xi > b^2$, we deduce that the energy $E(t)$ defined by (3.4.1) is non negative.

Lemma 3.4.2. *Let* (u,ϕ) *be the solution of system* (3.1.1). *Then, the functional*

$$F_1(t) = J\int_0^1 \phi\left(\phi_t + \int_0^t k(t-s)\,\phi_t(s)ds\right)dx + \frac{b\rho}{\mu}\int_0^1 \phi\left(\int_0^x u_t(y)\,dy\right)dx,$$

satisfies for any $\varepsilon_0 > 0$,

$$\begin{aligned} F_1'(t) \le & -\delta\int_0^1 \phi_x^2 dx - \frac{\xi_1}{2}\int_0^1 \phi^2 dx + \left[\frac{3J}{2} + \frac{b^2\rho^2}{4\mu^2\varepsilon_0}\right]\int_0^1 \phi_t^2 dx \\ & + \varepsilon_0\int_0^1 u_t^2 dx + \frac{J\bar{k}}{2}\int_0^1\left(\int_0^t k(t-s)\,\phi_t^2(s)ds\right)dx \\ & + \frac{\alpha^2(0)}{2\xi_1}\int_0^1 g^2(\phi_t)\,dx. \end{aligned} \tag{3.4.5}$$

Proof. Multiplying $(3.1.1)_2$ by ϕ and integrating by parts, we obtain

$$\begin{aligned} & J\int_0^1 \phi\left(\phi_t + \int_0^t k(t-s)\,\phi_t(s)ds\right)_t dx + J\int_0^1 \phi_t\left(\phi_t + \int_0^t k(t-s)\,\phi_t(s)ds\right)dx \\ & = -\delta\int_0^1 \phi_x^2 dx - b\int_0^1 u_x\phi dx - \xi\int_0^1 \phi^2 dx + J\int_0^1 \phi_t\left(\phi_t + \int_0^t k(t-s)\,\phi_t(s)ds\right)dx \\ & - \alpha(t)\int_0^1 \phi g(\phi_t)\,dx, \end{aligned}$$

that is

$$\begin{aligned} & J\frac{d}{dt}\int_0^1 \phi\left(\phi_t + \int_0^t k(t-s)\,\phi_t(s)ds\right)dx \\ & = -\delta\int_0^1 \phi_x^2 dx - b\int_0^1 u_x\phi dx - \xi\int_0^1 \phi^2 dx \\ & + J\int_0^1 \phi_t^2 dx + J\int_0^1 \phi_t\left(\int_0^t k(t-s)\,\phi_t(s)ds\right)dx. \end{aligned} \tag{3.4.6}$$

Integrating over $(0,x)$ the first equation of (3.1.1) and multiplying it by $\dfrac{b}{\mu}$, we get

$$\frac{b\rho}{\mu}\int_0^x u_{tt}\left(y\right)dy = bu_x + \frac{b^2}{\mu}\phi.$$

Now, multiplying this last equation by ϕ and integrating over $\left(0,1\right)$, we have

$$\frac{b\rho}{\mu}\int_0^1 \phi\left(\int_0^x u_{tt}\left(y\right)dy\right)dx = b\int_0^1 \phi u_x dx + \frac{b^2}{\mu}\int_0^1 \phi^2 dx,$$

which is equivalent to

$$\begin{aligned}&\frac{b\rho}{\mu}\frac{d}{dt}\int_0^1 \phi\left(\int_0^x u_t\left(y\right)dy\right)dx \\ &= b\int_0^1 \phi u_x dx + \frac{b^2}{\mu}\int_0^1 \phi^2 dx + \frac{b\rho}{\mu}\int_0^1 \phi_t\left(\int_0^x u_t\left(y\right)dy\right)dx. \qquad (3.4.7)\end{aligned}$$

The combination of (3.4.6) and (3.4.7), leads to

$$\begin{aligned}F_1'\left(t\right) &= J\int_0^1 \phi_t^2 dx + J\int_0^1 \phi_t\left(\int_0^t k\left(t-s\right)\phi_t(s)ds\right)dx \\ &\quad - \delta\int_0^1 \phi_x^2 dx - \xi_1\int_0^1 \phi^2 dx \\ &\quad + \frac{b\rho}{\mu}\int_0^1 \phi_t\left(\int_0^x u_t\left(y\right)dy\right)dx - \alpha\left(t\right)\int_0^1 \phi g\left(\phi_t\right)dx. \qquad (3.4.8)\end{aligned}$$

Using Young's and Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned}&J\int_0^1 \phi_t\left(\int_0^t k\left(t-s\right)\phi_t(s)ds\right)dx \\ &\leq \frac{J}{2}\int_0^1 \phi_t^2 dx + \frac{J\bar{k}}{2}\int_0^1\left(\int_0^t k\left(t-s\right)\phi_t^2(s)ds\right)dx. \qquad (3.4.9)\end{aligned}$$

Using Young's inequality, we get

$$-\alpha\left(t\right)\int_0^1 \phi g\left(\phi_t\right)dx \leq \frac{\xi_1}{2}\int_0^1 \phi^2 dx + \frac{\alpha^2\left(0\right)}{2\xi_1}\int_0^1 g^2\left(\phi_t\right)dx, \qquad (3.4.10)$$

$$\frac{b\rho}{\mu}\int_0^1 \phi_t\left(\int_0^x u_t\left(y\right)dy\right)dx \leq \frac{b^2\rho^2}{4\mu^2\varepsilon_0}\int_0^1 \phi_t^2 dx + \varepsilon_0\int_0^1 u_t^2 dx. \tag{3.4.11}$$

Inserting (3.4.9)-(3.4.11) into (3.4.8), we obtain (3.4.5). □

Lemma 3.4.3. *Let* (u, ϕ) *be the solution of system* (3.1.1). *Then, the functional*

$$F_2\left(t\right) = \frac{\delta\rho b}{\mu J}\int_0^1 \phi_x u_t dx + b\int_0^1\left(\phi_t + \int_0^t k\left(t-s\right)\phi_t(s)ds\right)u_x dx,$$

satisfies, for any $\varepsilon_1 > 0$,

$$\begin{aligned} F_2'\left(t\right) \leq & -\frac{b^2}{2J}\int_0^1 u_x^2 dx + C_{\varepsilon_1}\int_0^1 \phi_x^2 dx + \varepsilon_1\left(2+k\left(0\right)\right)\int_0^1 u_t^2 dx \\ & + \frac{b^2k\left(t\right)}{4\varepsilon_1}\int_0^1 \phi_{0x}^2 dx + \frac{b^2k\left(0\right)}{4\varepsilon_1}\int_0^1\left(\int_0^t\left|k'\left(t-s\right)\right|\phi_x^2(s)ds\right)dx \\ & + \frac{J\alpha^2\left(0\right)}{b^2}\int_0^1 g^2\left(\phi_t\right)dx + \frac{\rho b}{\mu}\chi\int_0^1 \phi_t u_{tx} dx, \end{aligned} \tag{3.4.12}$$

where $C_{\varepsilon_1} = \dfrac{\delta b^2}{\mu J} + \dfrac{b^2k^2\left(0\right)}{4\varepsilon_1} + \dfrac{\xi^2}{2J}$.

Proof. Multiplying $(3.1.1)_2$ by $\frac{b}{J}u_x$ and integrating by parts, we obtain

$$\begin{aligned} & b\int_0^1 u_x\left(\phi_t + \int_0^t k\left(t-s\right)\phi_t(s)ds\right)_t dx + b\int_0^1 u_{tx}\left(\phi_t + \int_0^t k\left(t-s\right)\phi_t(s)ds\right)dx \\ & = -\frac{b^2}{J}\int_0^1 u_x^2 dx + \frac{\delta b}{J}\int_0^1 \phi_{xx}u_x dx - \frac{b\xi}{J}\int_0^1 \phi u_x dx \\ & + b\int_0^1 u_{tx}\left(\phi_t + \int_0^t k\left(t-s\right)\phi_t(s)ds\right)dx - \alpha\left(t\right)\int_0^1 u_x g\left(\phi_t\right)dx, \end{aligned}$$

that is

$$\begin{aligned} & b\frac{d}{dt}\int_0^1 u_x\left(\phi_t + \int_0^t k\left(t-s\right)\phi_t(s)ds\right)dx \\ & = -\frac{b^2}{J}\int_0^1 u_x^2 dx - \frac{\delta b}{J}\int_0^1 \phi_x u_{xx} dx - \frac{b\xi}{J}\int_0^1 \phi u_x dx \\ & + b\int_0^1 u_{tx}\left(\phi_t + \int_0^t k\left(t-s\right)\phi_t(s)ds\right)dx - \alpha\left(t\right)\int_0^1 u_x g\left(\phi_t\right)dx. \end{aligned} \tag{3.4.13}$$

Multiplying $(3.1.1)_1$ by $\frac{\delta b}{\mu J}\phi_x$ and integrating by parts, we obtain

$$\frac{\delta b\rho}{\mu J}\int_0^1 \phi_x u_{tt}dx = \frac{\delta b}{J}\int_0^1 \phi_x u_{xx}dx + \frac{\delta b^2}{\mu J}\int_0^1 \phi_x^2 dx,$$

which is equivalent

$$\frac{\delta b\rho}{\mu J}\frac{d}{dt}\int_0^1 \phi_x u_t dx = \frac{\delta b}{J}\int_0^1 \phi_x u_{xx}dx + \frac{\delta b^2}{\mu J}\int_0^1 \phi_x^2 dx + \frac{\delta \rho b}{\mu J}\int_0^1 \phi_{tx}u_t dx. \tag{3.4.14}$$

By combining (3.4.13), (3.4.14) and integrating by parts, we obtain

$$\begin{aligned} F_2'(t) = {} & \frac{\rho b}{\mu}\chi\int_0^1 \phi_t u_{tx}dx - \frac{b^2}{J}\int_0^1 u_x^2 dx + \frac{\delta b^2}{\mu J}\int_0^1 \phi_x^2 dx - \frac{b\xi}{J}\int_0^1 \phi u_x dx \\ & - \alpha(t)\int_0^1 u_x g(\phi_t)\,dx + b\int_0^1 u_{tx}\left(\int_0^t k(t-s)\,\phi_t(s)ds\right)dx. \end{aligned} \tag{3.4.15}$$

Integrating by parts with respect to t the last term of (3.4.15), we have

$$\begin{aligned} & b\int_0^1 u_{tx}\left(\int_0^t k(t-s)\,\phi_t(s)ds\right)dx \\ & = b\int_0^1 u_{tx}\left[k(0)\,\phi(t) - k(t)\,\phi(0) + \int_0^t k'(t-s)\,\phi(s)ds\right]dx \\ & = -bk(0)\int_0^1 u_t\phi_x dx + bk(t)\int_0^1 u_t\phi_x(0)\,dx \\ & - b\int_0^1 u_t\left(\int_0^t k'(t-s)\,\phi_x(s)ds\right)dx. \end{aligned}$$

Then, (3.4.15) becomes

$$\begin{aligned} F_2'(t) = {} & \frac{\rho b}{\mu}\chi\int_0^1 \phi_t u_{tx}dx - \frac{b^2}{J}\int_0^1 u_x^2 dx + \frac{\delta b^2}{\mu J}\int_0^1 \phi_x^2 dx \\ & - bk(0)\int_0^1 u_t\phi_x dx + bk(t)\int_0^1 u_t\phi_x(0)\,dx - \frac{b\xi}{J}\int_0^1 \phi u_x dx \\ & - \alpha(t)\int_0^1 u_x g(\phi_t)\,dx - b\int_0^1 u_t\left(\int_0^t k'(t-s)\,\phi_x(s)ds\right)dx. \end{aligned} \tag{3.4.16}$$

By using Young's inequality, we arrive at

$$-bk(0)\int_0^1 u_t\phi_x dx \le \varepsilon_1 \int_0^1 u_t^2 dx + \frac{b^2k^2(0)}{4\varepsilon_1}\int_0^1 \phi_x^2 dx, \tag{3.4.17}$$

$$-\alpha(t)\int_0^1 u_x g(\phi_t)\,dx \le \frac{b^2}{4J}\int_0^1 u_x^2 dx + \frac{J\alpha^2(0)}{b^2}\int_0^1 g^2(\phi_t)\,dx, \tag{3.4.18}$$

and

$$\begin{aligned}+bk(t)\int_0^1 u_t\phi_x(0)\,dx &\le \varepsilon_1 k(t)\int_0^1 u_t^2 dx + \frac{b^2k(t)}{4\varepsilon_1}\int_0^1 \phi_{0x}^2 dx\\ &\le \varepsilon_1 k(0)\int_0^1 u_t^2 dx + \frac{b^2k(t)}{4\varepsilon_1}\int_0^1 \phi_{0x}^2 dx,\end{aligned} \tag{3.4.19}$$

Young's and Cauchy-Schwarz inequalities leads to

$$\begin{aligned}&-b\int_0^1 u_t\left(\int_0^t k'(t-s)\,\phi_x(s)ds\right)dx\\ &\le \varepsilon_1\int_0^1 u_t^2 dx + \frac{b^2k(0)}{4\varepsilon_1}\int_0^1\left(\int_0^t \left|k'(t-s)\right|\phi_x^2(s)ds\right)dx.\end{aligned} \tag{3.4.20}$$

By using Young's and Poincaré inequalities, we have

$$-\frac{b\xi}{J}\int_0^1 \phi u_x dx \le \frac{b^2}{4J}\int_0^1 u_x^2 dx + \frac{\xi^2}{J}\int_0^1 \phi_x^2 dx. \tag{3.4.21}$$

By substituting (3.4.17)-(3.4.21) in (3.4.16), we get (3.4.12). □

Lemma 3.4.4. *Let* (u,ϕ) *be the solution of system* (3.1.1). *Then, the functional*

$$F_3(t) = -\rho\int_0^1 uu_t dx,$$

satisfies,

$$F_3'(t) \le -\rho\int_0^1 u_t^2 dx + \frac{b^2}{2\mu}\int_0^1 \phi_x^2 dx + \frac{3\mu}{2}\int_0^1 u_x^2 dx. \tag{3.4.22}$$

Proof. Multiplying $(3.1.1)_1$ by $-u$ and integrating by parts, we obtain

$$-\rho\int_0^1 u_{tt}u dx = \mu\int_0^1 u_x^2 dx - b\int_0^1 u\phi_x dx,$$

which is equivalent to

$$-\rho\int_0^1 u_{tt}udx-\rho\int_0^1 u_t^2dx=-\rho\int_0^1 u_t^2dx+\mu\int_0^1 u_x^2dx-b\int_0^1 u\phi_xdx.$$

Therefore

$$F_3'(t)=-\rho\int_0^1 u_t^2dx+\mu\int_0^1 u_x^2dx-b\int_0^1 u\phi_xdx,$$

Young's and Poincaré inequalities give (3.4.22). □

Lemma 3.4.5. *([37])Let (u,ϕ) be the solution of system* (3.1.1)*. Then, the functionals*

$$F_4(t)=e^{-\varsigma t}\int_0^1\left(\int_0^t e^{\varsigma s}\tilde{H}_1(t-s)\phi_t^2(s)ds\right)dx,$$
$$F_5(t)=e^{-\tau t}\int_0^1\left(\int_0^t e^{\tau s}\tilde{H}_2(t-s)\phi_x^2(s)ds\right)dx,$$

satisfy, $\forall t\geq 0$,

$$F_4'(t)=-\varsigma F_4(t)+\tilde{H}_1(0)\int_0^1\phi_t^2dx-\int_0^1\left(\int_0^t k(t-s)\phi_t^2(s)ds\right)dx,\tag{3.4.23}$$

$$F_5'(t)=-\tau F_5(t)+\tilde{H}_2(0)\int_0^1\phi_x^2dx-\int_0^1\left(\int_0^t\left|k'(t-s)\right|\phi_x^2(s)ds\right)dx,\tag{3.4.24}$$

where $\tilde{H}_1(t)=\int_t^\infty e^{\varsigma s}\left|k(s)\right|ds$ *and* $\tilde{H}_2(t)=\int_t^\infty e^{\tau s}\left|k'(s)\right|ds.$

Now, we define the Lyapunov functional $L(t)$

$$L(t)=NE(t)+N_1F_1(t)+N_2F_2(t)+F_3(t)+N_3F_4(t)+N_4F_5(t),\tag{3.4.25}$$

where N, N_1, N_2, N_3 and N_4 are positive constants.

Lemma 3.4.6. *Let (u,ϕ) be the solution of* (3.1.1)*. Then, there exists two positive constants κ_1 and κ_2 such that the Lyapunov functional* (3.4.25) *satisfies*

$$\kappa_1\left(E(t)+F_4(t)+F_5(t)\right)\leq L(t)\leq\kappa_2\left(E(t)+F_4(t)+F_5(t)\right),\ \forall t\geq 0,\tag{3.4.26}$$

and

$$L'(t) \leq -\beta_1 \left(E(t) + F_4(t) + F_5(t)\right) + C_2 k(t) + C_3 \int_0^1 (\phi_t^2 + g^2(\phi_t))dx \\ + N_2 \frac{\rho b}{\mu} \chi \int_0^1 \phi_t u_{tx} dx,\ \beta_1 > 0. \tag{3.4.27}$$

Proof. From (3.4.25), we have

$$\begin{aligned} &|L(t) - NE(t) - N_3 F_4(t) - N_4 F_5(t)| \\ &\leq N_1 J \int_0^1 |\phi| \cdot \left|\phi_t + \int_0^t k(t-s)\phi_t(s)ds\right| dx + N_1 \frac{\mu_1}{2} \int_0^1 \phi^2 dx \\ &+ N_1 \frac{|b|\rho}{\mu} \int_0^1 |\phi| \left(\int_0^x |u_t(y)| dy\right) dx + N_2 \frac{\delta\rho |b|}{\mu J} \int_0^1 |\phi_x| |u_t| dx \\ &+ N_2 |b| \int_0^1 |u_x| \left|\phi_t + \int_0^t k(t-s)\phi_t(s)ds\right| dx + \rho \int_0^1 |u| |u_t| dx. \end{aligned}$$

By using Young's, Cauchy-Schwarz and Poincaré inequalities, we obtain

$$|L(t) - NE(t) - N_3 F_4(t) - N_4 F_5(t)| \leq \lambda_1 E(t).$$

Therefore,

$$(N - \lambda_1) E(t) + N_3 F_4(t) + N_4 F_5(t) \leq L(t) \leq (N + \lambda_1) E(t) + N_3 F_4(t) + N_4 F_5(t).$$

Now, by choosing N (depending on N_1, N_2, N_3 and N_4) sufficiently large, we obtain (3.4.26) with

$$\begin{aligned} \kappa_1 &= \min\{N - \lambda_1, N_3, N_4\}, \\ \kappa_2 &= \max\{N + \lambda_1, N_3, N_4\}. \end{aligned}$$

Now, by differentiating $L(t)$, exploiting (3.4.2), (3.4.5), (3.4.12), (3.4.22), (3.4.23), (3.4.24), using Young's inequality and setting $\varepsilon_0 = \frac{\rho}{4N_1}$, $\varepsilon_1 = \frac{\rho}{4N_2(2+k(0))}$, we get

$$\begin{aligned}
L'(t) \leq & \left[\left(\frac{3J}{2}+\frac{b^2\rho}{\mu^2}N_1\right)+N_3\tilde{H}_1(0)+\frac{N}{2}\alpha(0)\right]\int_0^1 \phi_t^2 dx \\
& -\frac{3\rho}{4}\int_0^1 u_t^2 dx - N_1\frac{\xi_1}{2}\int_0^1 \phi^2 dx \\
& -\left[\delta N_1 - N_2 C_{\varepsilon_1} - \frac{b^2}{2\mu} - N_4\tilde{H}_2(0)\right]\int_0^1 \phi_x^2 dx \\
& -\left(\frac{b^2}{2J}N_2 - \frac{3\mu}{2}\right)\int_0^1 u_x^2 dx - \varsigma N_3 F_4(t) - \tau N_4 F_5(t) \\
& -\left(N_3 - \frac{J\bar{k}}{2}N_1\right)\int_0^1\left(\int_0^t k(t-s)\,\phi_t^2(s)ds\right)dx \\
& -\left[N_4 - \frac{N_2^2 b^2 k(0)\,(2+k(0))}{\rho}\right]\int_0^1\left(\int_0^t \left|k'(t-s)\right|\phi_x^2(s)ds\right)dx \\
& +\frac{b^2N_2^2 k(t)\,(2+k(0))}{\rho}\int_0^1 \phi_{0x}^2 dx + N_2\frac{\rho b}{\mu}\chi\int_0^1 \phi_t u_{tx} dx \\
& +\alpha(0)\left(\alpha(0)\left(\frac{1}{2\xi_1}+\frac{J}{b^2}\right)+\frac{N}{2}\right)\int_0^1 g^2(\phi_t)\,dx.
\end{aligned}$$

We select our parameters appropriately as follows

First, we choose N_2 large enough such that

$$\frac{b^2}{2J}N_2 - \frac{3\mu}{2} > 0.$$

We pick N_4 large such that

$$N_4 - \frac{N_2^2 b^2 k(0)\,(2+k(0))}{\rho} > 0.$$

We select N_1 large enough such that

$$\delta N_1 - N_2 C_{\varepsilon_1} - \frac{b^2}{2\mu} - N_4\tilde{H}_2(0) > 0.$$

Finally, we choose N_3 large such that

$$N_3 - \frac{J\bar{k}}{2} N_1 > 0.$$

All these choices leads to

$$\begin{aligned} L'(t) &\leq -\alpha_1 \int_0^1 \left(\phi_t^2 + \phi_x^2 + u_t^2 + u_x^2 + \phi^2\right) dx - \int_0^1 \left(\int_0^t k\left(t-s\right) \phi_t^2(s) ds\right) dx \\ &+ \alpha_2 k(t) \int_0^1 \phi_{0x}^2 dx - \varsigma N_3 F_4\left(t\right) - N_4 \tau F_5\left(t\right) + N_2 \frac{\rho b}{\mu} \chi \int_0^1 \phi_t u_{tx} dx \\ &+ C_3 \int_0^1 (\phi_t^2 + g^2\left(\phi_t\right)) dx, \end{aligned} \tag{3.4.28}$$

where $\alpha_1, \alpha_2 > 0$.

On the other hand, from Eq. (3.4.1) and by using Young's inequality, we obtain

$$\begin{aligned} E\left(t\right) &\leq \frac{1}{2} \int_0^1 \left(\rho u_t^2 + J\phi_t^2 + \left(\mu + |b|\right) u_x^2 + \delta \phi_x^2 + \left(\xi + |b|\right) \phi^2\right) dx \\ &+ \frac{J}{2} \int_0^1 \left(\int_0^t k\left(t-s\right) \phi_t^2(s) ds\right) dx \\ &\leq \varrho_1 \left(\int_0^1 \left(u_t^2 + \phi_t^2 + u_x^2 + \phi_x^2 + \phi^2\right) dx\right. \\ &+ \left.\int_0^1 \left(\int_0^t k\left(t-s\right) \phi_t^2(s) ds\right) dx\right), \ \varrho_1 > 0, \end{aligned}$$

which implies that

$$- \int_0^1 \left(u_t^2 + \phi_t^2 + u_x^2 + \phi_x^2 + \phi^2\right) dx - \int_0^1 \left(\int_0^t k\left(t-s\right) \phi_t^2(s) ds\right) dx \leq -\varrho_2 E\left(t\right), \tag{3.4.29}$$

where $\varrho_2 > 0$.

The combination of (3.4.28) and (3.4.29) gives (3.4.27) with $C_2 = \alpha_2 \int_0^1 \phi_{0x}^2 dx$. □

We are now ready to state and prove the following exponential stability result

Theorem 3.4.1. *Let (u, ϕ) be the solution of* (3.1.1) *and assume that* (3.1.2), (3.1.3), ***(H1)-***

(H4) *hold. Then, there exists two positive constants* a_1 *and* a_2 *such that*

$$E(t) \leq a_1 G_0^{-1}\left(\frac{a_2}{\int_0^t \alpha(s)\,ds}\right). \tag{3.4.30}$$

Proof. By using (3.4.26) *and* (3.1.3), *then,* (3.4.27) becomes

$$L'(t) \leq -\beta_1 L(t) + C_2 k(t) + C_3 \int_0^1 (\phi_t^2 + g^2(\phi_t))dx. \tag{3.4.31}$$

Multiplying (3.4.31) by $\alpha(t)$, we get

$$\alpha(t)\,L'(t) \leq -\beta_1 \alpha(t)\,L(t) + C_2 \alpha(t)\,k(t) + C_3 \alpha(t) \int_0^1 \left(\phi_t^2 + g^2(\phi_t)\right) dx. \tag{3.4.32}$$

We distinguish two cases

1) G **is linear on** $[0,\epsilon]$. By using (3.4.32) and the hypothesis **(H4)**, we have

$$\begin{aligned}\alpha(t)\,L'(t) &\leq -\beta_1 \alpha(t)\,L(t) + C_2 \alpha(t)\,k(t) + C_3 \alpha(t) \int_0^1 \phi_t g(\phi_t)dx \\ &\leq -\beta_1 \alpha(t)\,L(t) + C_2 \alpha(t)\,k(t) - C_3 E'(t).\end{aligned} \tag{3.4.33}$$

Since $\alpha'(t) \leq 0$, then (3.4.33) is equivalent to

$$L_1'(t) \leq -\beta_1 \alpha(t)\,L(t) + C_2 \alpha(t)\,k(t), \tag{3.4.34}$$

where

$$L_1(t) = \alpha(t)\,L(t) + C_3 E(t), \tag{3.4.35}$$

and $L_1(t)$ satisfies

$$\iota_2 E(t) \leq L_1(t) \leq \iota_1 L(t),\ \iota_1, \iota_2 > 0. \tag{3.4.36}$$

Indeed, because $\alpha(t)$ is positive, $\alpha'(t) \leq 0$ and by using (3.4.35), (3.4.26), then, we get (3.4.36).

So, (2.4.34) is equivalent to

$$L_1'(t) \leq -\beta_2 \alpha(t) L_1(t) + C_2 \alpha(t) k(t),\ \beta_2 = \frac{\beta_1}{\iota_1}. \tag{3.4.37}$$

Integrating (3.4.37) over $(0,T)$, exploiting (3.4.36), using the fact that $E(T), \alpha(t)$ are decreasing and $L_1(t)$ is positive, we arrive at

$$\begin{aligned} \beta_3 E(T) \int_0^T \alpha(t)\,dt &\leq L_1(0) + C_4 \int_0^T k(t)dt \\ &\leq L_1(0) + C_4 \int_0^\infty k(t)dt. \end{aligned} \tag{3.4.38}$$

Because G_0^{-1} is linear, then, the estimation (3.4.38) is equivalent to

$$E(T) \leq \lambda G_0^{-1}\left(\frac{L_1(0) + C_4\bar{k}}{\beta_3 \int_0^T \alpha(t)\,dt}\right),$$

which gives (2.4.31) with $a_1 = \lambda$ and $a_2 = \dfrac{L_1(0) + C_4\bar{k}}{\beta_3}$. The proof is complete.

2) G **is nonlinear on** $[0,\epsilon]$. To estimate the last term of (2.4.32), we first choose $0 \leq \epsilon_1 \leq \epsilon$, such that

$$sg(s) \leq \min(\epsilon, G(\epsilon)), \forall |s| \leq \epsilon_1.$$

Using (**H4**) along with fact that g is continuous and $|g(s)| > 0$, for $s \neq 0$, it follows that

$$\begin{cases} s^2 + g^2(s) \leq G^{-1}(sg(s)),\ \forall |s| \leq \epsilon_1, \\ \nu_1 |s| \leq |g(s)| \leq \nu_2 |s|,\ \forall |s| \geq \epsilon_1, \end{cases} \tag{3.4.39}$$

and we consider the following two sets

$$I_1 = \{x \in (0,1) : |\phi_t| \leq \epsilon_1\},\ I_2 = \{x \in (0,1) : |\phi_t| > \epsilon_1\}.$$

Now, we define $I(t)$ by

$$I(t) = \int_{I_1} \phi_t g(\phi_t) dx,$$

using Jensen's inequality, we have

$$G^{-1}\left(I\left(t\right)\right) \geq C_5 \int_{I_1} G^{-1}\left(\phi_t g(\phi_t)\right) dx. \tag{3.4.40}$$

Then, the combination of (3.4.39) and (3.4.40) gives

$$C_3 \alpha\left(t\right) \int_0^1 \left(\phi_t^2 + g^2(\phi_t)\right) dx \leq C_6 \alpha\left(t\right) G^{-1}\left(I\left(t\right)\right) - C_3' E'(t). \tag{3.4.41}$$

Inserting (3.4.41) in (3.4.32), we obtain

$$L_2'(t) \leq -\beta_4 \alpha\left(t\right) L_2(t) + C_2 \alpha\left(t\right) k(t) + C_6 \alpha\left(t\right) G^{-1}\left(I\left(t\right)\right), \tag{3.4.42}$$

where

$$L_2(t) = \alpha\left(t\right) L(t) + C_3' E\left(t\right),$$

satisfies

$$m_2 E(t) \leq L_2(t) \leq m_1 L(t),\ m_1, m_2 > 0.$$

Now, for $\varepsilon_0 < \epsilon_1$ and using the fact that $E' \leq 0$, $G' > 0$, $G'' > 0$ on $(0, \epsilon]$, we find that the functional $L_3\left(t\right)$, defined by

$$L_3\left(t\right) = G'\left(\varepsilon_0 E(t)\right) L_2\left(t\right) + m_3 E\left(t\right),$$

satisfies

$$\begin{aligned} L_3'\left(t\right) &= \varepsilon_0 E'(t) G''\left(\varepsilon_0 E(t)\right) L_2\left(t\right) + L_2'\left(t\right) G'\left(\varepsilon_0 E(t)\right) + C_3' E'\left(t\right) \\ &\leq -\beta_5 \alpha\left(t\right) G_0\left(E(t)\right) + C_2' \alpha\left(t\right) k(t) + C_6 \alpha\left(t\right) G'\left(\varepsilon_0 E(t)\right) G^{-1}\left(I\left(t\right)\right), \end{aligned} \tag{3.4.43}$$

and

$$m_2' E(t) \leq L_3\left(t\right) \leq m_1' L(t),\ m_1', m_2' > 0.$$

To estimate the last term of (3.4.43), we apply the following general Young's inequality

$$AB \leq G^*\left(A\right) + G\left(B\right), \text{ if } A \in \left(0, G'\left(\epsilon\right)\right],\ B \in \left(0, \epsilon\right],$$

where

$$G^{*}(s)=s\left(G'\right)^{-1}(s)-G\left(\left(G'\right)^{-1}(s)\right),\text{ if } s\in\left(0,G'(\epsilon)\right],$$

we deduce that

$$C_6\alpha(t)\,G'(\varepsilon_0 E(t))\,G^{-1}(I(t))\leq C_6\alpha(t)\,\varepsilon_0 G_0(E(t))-C_6E'(t). \tag{3.4.44}$$

Substituting (2.4.40) in (2.4.39) and letting $\varepsilon_0=\frac{\beta_5}{2C_6}$, we have

$$L_4'(t)\leq-\beta_6\alpha(t)\,G_0(E(t))+C_4'k(t), \tag{3.4.45}$$

with $L_4(t)=L_3(t)+C_6E(t)$ verifies

$$0\leq m_2E(t)\leq L_4(t).$$

Integrating (3.4.45) over $(0,T)$,

$$\begin{aligned}\beta_6G_0(E(T))\int_0^T\alpha(t)dt&\leq L_4(0)+C_4'\int_0^T k(t)dt\\&\leq L_4(0)+C_4'\int_0^\infty k(t)dt.\end{aligned}$$

the estimation (3.4.38) is equivalent to

$$E(T)\leq G_0^{-1}\left(\frac{L_4(0)+C_4'\bar{k}}{\beta_6\int_0^T\alpha(t)\,dt}\right),$$

which gives (3.4.30) with $a_1=1$ and $a_2=\frac{L_4(0)+C_4'\bar{k}}{\beta_6}$. The proof is complete. □

Conclusion and open problem

In this article, we established a general stability of the solution of porous-elastic system in the presence of neutral delay and nonlinear damping term for the case of equal speeds of wave propagation. Introducing a nonlinear damping mechanism to control the side effects given by this type of delay makes our problem different from those considered so far in the literature and under some assumptions imposed on the kernel of delay along with some hypotheses on the nonlinear damping term, we have been able to prove an explicit energy decay rate that depends on the wave speeds of propagation. As an open problem, we propose to study the same problem when the wave speeds are not equal.

CHAPTER 4

Global well-posedness and energy decay for a one dimensional porous-elastic system subject to a neutral delay

4.1 Introduction

In 1972, Goodman and Cowin [23] have given an extension of the classical elasticity theory to porous media by introducing the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition, Nunziato and Cowin [22] have presented a nonlinear theory for the behavior of porous solids in which the skeletal or matrix material is elastic and the interstices are void of material. In this theory, the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field. Furthermore, this representation introduces an additional degree of kinematic freedom. The intended applications of the theory of elastic materials with voids are to geological materials like rocks and soils, and manufactured porous materials.

In [34], Quintanilla gave the first investigation concerning the study of asymptotic behavior of the solutions for a one-dimensional porous-elastic system where he proved that the damping through porous-viscosity is not strong enough to provoke an exponential decay. In ([20],[8]), Apalara showed that the same system considered in [34] is exponentially stable in the case of equal speeds of wave propagation. In [32], Casas and Quintanilla studied the one-dimensional porous-elastic system in the presence of the usual thermal effect with micro-temperature damping and they used the semi-group approach to prove the exponential stability of the solutions irrespective of the speeds of wave propagations. In [41], Casas and Quintanilla proved that the combination of porous-viscosity and thermal effects provokes an exponential stability of the solutions. In [29], Magańa and Quintanilla showed that visco-elasticity damping and temperature produced slow decay in time and when the visco-elasticity is coupled with porous damping or with micro-temperatures, the system decays in an exponential way.

Delay effect arises in many applications depending not only on the present state but also on some past occurrences and it has attracted lots of attention from researchers in diverse fields of human endeavor such as mathematics, engineering, science, and economics. The presence of delay may be a source of instability of systems which are uniformly asymptotically stable in the absence of delay unless additional control terms have been used (see [12],[13],[19],[30],[31],[39]). Also, the introduction of this complementary control may lead to ill-posedness as shown in many works such as ([13],[35]) and the references therein. In addition to the well-known discrete delays, there are several others and we are interested here in the neutral delay where the delay is occurring in the second (highest) derivative, for more details, see the previous studies ([14]-[16],[**?**],[38]) and the references therein.

Among the investigations that have been realized concerning the asymptotic behavior with neutral delay, we cite the work of Seghour et al. [37] where they considered the following thermo-elastic laminated system subject to a neutral delay

$$\begin{cases} \rho w_{tt} + G\left(\psi - w_x\right)_x + A w_t = 0,\ x \in (0,1)\,,\ t > 0, \\ I_\rho\left(3s_{tt} - \psi_{tt}\right) - G\left(\psi - w_x\right) - \left(3s - \psi\right) + \mu\theta_x = 0,\ x \in (0,1)\,,\ t > 0, \\ 3I_\rho\left(s_t + \displaystyle\int_0^t h\left(t-r\right) s_t\left(r\right) dr\right)_t + 3G\left(\psi - w_x\right) + 4\gamma s - 3s_{xx} = 0,\ x \in (0,1)\,,\ t > 0, \\ \theta_t - \kappa\theta_{xx} + \mu\left(3s - \psi\right)_{tx} = 0,\ x \in (0,1)\,,\ t > 0, \end{cases}$$

with boundary conditions

$$\begin{cases} \psi\left(0,t\right) = s\left(0,t\right) = \theta_x\left(0,t\right) = w_x\left(0,t\right) = 0,\ t \geq 0, \\ \theta\left(1,t\right) = w\left(1,t\right) = s_x\left(1,t\right) = \psi_x\left(1,t\right) = 0,\ t \geq 0, \end{cases}$$

and initial data

$$\begin{cases} \left(w, \psi, s, \theta\right)\left(x,0\right) = \left(w_0, \psi_0, s_0, \theta_0\right),\ x \in (0,1)\,, \\ \left(w_t, \psi_t, s_t\right)\left(x,0\right) = \left(w_1, \psi_1, s_1\right),\ x \in (0,1)\,. \end{cases}$$

The authors showed, under some appropriate assumptions, that the dissipation produced by the heat equation with the frictional damping stabilize exponentially the system even in the presence of neutral delay for the case of equal wave speeds. In the opposite one, and with an additional assumption on the kernel, they proved a polynomial stability.

In this paper, we consider the following porous-elastic system with porous-viscosity subject to a distributed delay of neutral type

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0,\ x \in (0,1)\,,\ t > 0, \\ J\left(\phi_t + \displaystyle\int_0^t k\left(t-s\right)\phi_t\left(s\right) ds\right)_t - \delta\phi_{xx} + b u_x + \xi\phi + \mu_1\phi_t = 0,\ x \in (0,1)\,,\ t > 0, \\ u\left(x,0\right) = u_0\left(x\right),\ u_t\left(x,0\right) = u_1\left(x\right),\ x \in (0,1)\,, \\ \phi\left(x,0\right) = \phi_0\left(x\right),\ \phi_t\left(x,0\right) = \phi_1\left(x\right),\ x \in (0,1)\,, \\ u_x\left(0,t\right) = u_x\left(1,t\right) = \phi\left(0,t\right) = \phi\left(1,t\right) = 0,\ t > 0, \end{cases} \tag{4.1.1}$$

where the functions u and ϕ represent respectively the displacement of the solid elastic material and the volume fraction. The parameter ρ designates the mass density and J equals to the product of the mass density by the equilibrated inertia. The coefficients μ, δ, ξ, μ_1 are positive constants represent the constitutive parameters defining the coupling among the different components of the materials such that

$$\mu\xi > b^2. \tag{4.1.2}$$

Where b is a real number different from zero. The initial data u_0, u_1, ϕ_0, ϕ_1 belongs to the suitable functional space and the integral represents the neutral delay term where k is the relaxation function that specified in the preliminaries. The system (4.1.1) was constructed by considering the following basic evolution equations of the one-dimensional porous materials theory

$$\rho u_{tt} = T_x,\ J\left(\phi_t + \int_0^t k(t-s)\,\phi_t(s)\,ds\right)_t = H_x + D, \tag{4.1.3}$$

where T, H and D represent respectively the stress tensor, the equilibrated stress vector and the equilibrated body force. Consequently, to get the system (4.1.1) we take the constitutive equations T, H and D at this form

$$\begin{aligned} &T = \mu u_x + b\phi,\ H = \delta\phi_x, \\ &D = -bu_x - \xi\phi - \mu_1\phi_t, \end{aligned} \tag{4.1.4}$$

and by combining (4.1.4) in (4.1.3), we obtain (4.1.1).

The main goal of this paper is to prove a global well-posedness of the problem (4.1.1) by using the Faedo-Galerkin method with some a priori estimates. Moreover, based on the multipliers method along side with some assumptions on the kernel of neutral delay, we construct a suitable Lyapunov functional and we show that the dissipation given by the porous-viscosity is strong enough to guarantee an exponential decay in spite of the existence of the neutral delay for the case of equal speeds of wave propagation, that is

$$\chi = \frac{\mu}{\rho} - \frac{\delta}{J} = 0. \tag{4.1.5}$$

In the opposite one, we establish a polynomial stability result. Furthermore, in our case and compared to the work of Seghour et al. in [37], we were able to dispense the thermal effect depends only on the damping mechanism to control the neutral delay term.

Introducing a neutral delay makes our problem different from those considered so far in the literature. Moreover, the study of the asymptotic behavior becomes different and more complicated than in the case of other types of delay that has appeared in the recent literature such as in ([2]-[3], [10]-[11], [17], [**?**]-[**?**]). In other words, a neutral-type delayed dynamical system is a more general class than delayed systems, in the sense where it's described by a model in which the highest derivative of the state at the present time is a function not only of the values of the passed state, but also of the highest derivative of the passed state and this strengthens the challenges. It is also worth mentioning that besides the fact that systems are very reactive to small delays, on the contrary, they can be stabilized by 'large' neutral delays. In fact, neutral delays are sometimes deliberately inserted into the systems to improve the performance of the structure and this has been proven by some works, such as [**?**] in which, the authors showed that the dissipation given only by the neutral delay, without damping or other dissipation terms, provokes exponential stability of the solution.

This paper is organized as follows. In Section 2, we introduce some assumptions and transformations needed in the next sections to prove the main result. In Section 3, we prove the existence and uniqueness of the solution. In Section 4, we show the decay of the energy. In Sections (5 and 6), we use the energy method to prove the exponential and polynomial stability results.

4.2 Preliminaries

In this section, we present our assumptions on both kernels and introduce the energy functional and some other functionals.

We use the standard Lebesgue space $L^2(0,1)$ and the Sobolev space $H_0^1(0,1)$ with their usual scalar products and norms. Let's define the space $\mathcal{H}$ as

$$\mathcal{H} = H_*^1(0,1) \times L_*^2(0,1) \times H_0^1(0,1) \times L^2(0,1),$$

where $H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1)$ and

$$L_*^2(0,1) = \left\{ f \in L^2(0,1) : \int_0^1 f(x)\,dx = 0 \right\}.$$

Moreover, we define the following space

$$H_*^2(0,1) = \left\{ \psi \in H^2(0,1) : \psi_x(0) = \psi_x(1) = 0 \right\}.$$

To simplify the calculations, we are obliged to announce this Lemma which is usable in the following sections.

Lemma 4.2.1 ([37]). *For any function $\psi \in C^1\left([0,\infty); L^2(0,1)\right)$ and any $k \in C^1([0,\infty))$, we have the following identity*

$$\begin{aligned}
&\int_0^1 \psi(t)\left(\int_0^t k(t-s)\,\psi_t(s)ds\right)dx \\
&= -\frac{1}{2}\left(k'\Box\psi\right)(t) + \frac{1}{2}\frac{d}{dt}\int_0^1\left(\int_0^t k(t-s)\,\psi^2(s)ds\right)dx \\
&+ \frac{k(t)}{2}\int_0^1 \psi^2 dx - k(t)\int_0^1 \psi(0)\psi(t)\,dx,
\end{aligned}$$

where

$$(k\Box\psi) = \int_0^t k(t-s)\left(\int_0^1 (\psi(t)-\psi(s))^2\,dx\right)ds, \; t \geq 0.$$

Also, we need the following hypothesis to reach our aim

(H1) The kernel k is a nonnegative continuously differentiable and summable function satisfying

$$k'(t) \leq 0, \; \forall t \geq 0, \; \bar{k} = \int_0^\infty k(s)ds < 1.$$

(H2) $\exp(\varsigma t)\,k(t) \in L^1(\mathbb{R}_+)$ for some $\varsigma > 0$.

Note that if $\int_0^{+\infty} e^{\varsigma s}k(s)ds < \infty$ and $\lim\limits_{t\longrightarrow\infty} \exp(\varsigma t)\, k(t) < \infty$, then

$$\int_0^{+\infty} e^{\varsigma s}\left|k'(s)\right| ds = -\int_0^{+\infty} e^{\varsigma s}k'(s)ds = -e^{\varsigma s}k(s)\mid_0^{\infty} +\varsigma\int_0^{+\infty} e^{\varsigma s}k(s)ds < \infty.$$

Theorem 4.2.1 ([40])**.** *Let $B_0 \subset B_1 \subset B_2$ be three Banach spaces. We assume that the embedding of B_1 in B_2 is continuous and that the embedding of B_0 in B_1 is compact. Let p, r such that $1 \le p, r \le +\infty$. For $T > 0$, we define*

$$E_{p,r} = \left\{ v \in L^p\left(0, T; B_0\right),\ \frac{dv}{dt} \in L^r\left(0, T; B_2\right) \right\}.$$

i) If $p < +\infty$, the embedding of $E_{p,r}$ in $L^p(0, T; B_1)$ is compact.

ii) If $p = +\infty$ and $r > 1$, the embedding of $E_{p,r}$ in $C^0(0, T; B_1)$ is compact.

Also, we need to use the following transformation in order to calculate the energy of the system and for other necessary estimations

$$\left(\int_0^t k(t-s)\,\phi_t(s)ds\right)_t = k(t)\,\phi_t(0) + \int_0^t k(t-s)\,\phi_{tt}(s)ds. \tag{4.2.1}$$

We shall consider the classical energy defined by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\int_0^1 \left(\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + 2bu_x\phi + \xi\phi^2 + \delta\phi_x^2\right) dx \\ & + \frac{J}{2}\int_0^1 \left(\int_0^t k(t-s)\,\phi_t^2(s)ds\right) dx,\ t \ge 0. \end{aligned} \tag{4.2.2}$$

Lemma 4.2.2. *The energy $E(t)$ given by* (4.2.2) *satisfies*

$$E'(t) \le \frac{J}{2}\left(k'\Box\phi_t\right)(t) - \mu_1\int_0^1 \phi_t^2 dx. \tag{4.2.3}$$

Proof. Multiplying $(4.1.1)_1, (4.1.1)_2$ by u_t, ϕ_t and integrating over $(0, 1)$ and summing them up, we obtain

$$\begin{aligned}
&\frac{d}{2dt}\int_0^1 \left(\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + 2bu_x\phi + \xi\phi^2 + \delta\phi_x^2\right) dx \\
&+ J\int_0^1 \left[\phi_t \left(\int_0^t k\left(t-s\right)\phi_t(s)ds\right)_t\right] dx \;=\; -\mu_1 \int_0^1 \phi_t^2 dx. \qquad (4.2.4)
\end{aligned}$$

By exploiting (4.2.1) and applying the result in Lemma 1, we obtain

$$\begin{aligned}
&J\int_0^1 \left[\phi_t \left(\int_0^t k\left(t-s\right)\phi_t(s)ds\right)_t\right] dx \\
&= -\frac{J}{2}\left(k'\square\phi_t\right)(t) + J\frac{k(t)}{2}\int_0^1 \phi_t^2 dx \\
&+ \frac{J}{2}\frac{d}{dt}\int_0^1 \left(\int_0^t k\left(t-s\right)\phi_t^2(s)ds\right) dx. \qquad (4.2.5)
\end{aligned}$$

Inserting (4.2.5) in (4.2.4) and taking into account the positivity of $k(t)$, we have (4.2.3). □

Remark 4.2.1. Note that

$$\mu u_x^2 + 2bu_x\phi + \xi\phi^2 = \mu\left(u_x + \frac{b}{\mu}\phi\right)^2 + \left(\xi - \frac{b^2}{\mu}\right)(\phi)^2,$$

and because $\mu\xi > b^2$, we deduce that the energy $E(t)$ defined by (4.2.2) is non negative.

In view of the boundary conditions, our system can have solutions (uniform in the variable x), which do not decay. In other words, it is known that for the problem determined by (4.1.1) we can always take solutions where u is constant, for this reason, we impose that

$$\int_0^1 u_0 dx = \int_0^1 u_1 dx = 0. \qquad (4.2.6)$$

It is worth noting that condition (4.2.6) is imposed to guarantee that the solution decays. Thus, if we want to avoid this behavior we need to impose condition (4.2.6).

In addition as in [3], to be able to use Poincaré inequality for u , we perform the following transformation

From $(4.1.1)_1$, we observe that

$$\int_0^1 u_{tt} dx = 0.$$

If we take $v(t) = \int_0^1 u dx$, we observe that $v(0) = \int_0^1 u_0 dx = 0$ and $v'(0) = \int_0^1 u_1 dx = 0$. Moreover, v is a solution of the following initial value problem

$$\begin{cases} v''(t) = 0, \ \forall t \geq 0, \\ v(0) = \int_0^1 u_0(x) \, dx = 0, \ \ v'(0) = \int_0^1 u_1(x) \, dx = 0. \end{cases}$$

The solution of the problem is given by

$$v(t) = \int_0^1 u(x,t) \, dx = t \int_0^1 u_1(x) \, dx + \int_0^1 u_0(x) \, dx = 0.$$

Consequently

$$\int_0^1 u(x,t) \, dx = 0, \ \forall t \geq 0.$$

4.3 Global well-posedness

In this section, we prove the global existence and the uniqueness of the solution of problem (4.1.1) by using the classical Faedo-Galerkin approximations along with some priori estimates. The well-posedness of (4.1.1) is given by the following theorem

Theorem 4.3.1. *Assume that* ***(H1)-(H2),*** (4.1.2) *hold, and the initial data*

$$\begin{aligned} (u_0, u_1) &\in H_*^1(0,1) \times L_*^2(0,1), \\ (\phi_0, \phi_1) &\in H_0^1(0,1) \times L^2(0,1), \end{aligned} \tag{4.3.1}$$

problem (4.1.1) *has a unique global weak solution*

$$\begin{aligned} &u \in C\left(\mathbb{R}_+, H_*^2(0,1) \cap H_*^1(0,1)\right) \cap C^1\left(\mathbb{R}_+, H_*^1(0,1)\right) \cap C^2\left(\mathbb{R}_+, L_*^2(0,1)\right), \\ &\phi \in C\left(\mathbb{R}_+, H^2(0,1) \cap H_0^1(0,1)\right) \cap C^1\left(\mathbb{R}_+, H_0^1(0,1)\right) \cap C^2\left(\mathbb{R}_+, L^2(0,1)\right). \end{aligned} \tag{4.3.2}$$

In addition, the solution (u, ϕ) *depends continuously on the initial data.*

Proof. We divide the proof into three steps: we first construct Faedo–Galerkin approximations, then thanks to a priori estimates we look to prove that $t_n = T$ for $n \in \mathbb{N}^*$. Finally, we pass to the limit.

Step 1: Faedo–Galerkin approximations.

We construct an approximations of the solution (u, ϕ) by the Faedo-Galerkin method as follows (see [15] and [**?**]): For every $n \geq 1$, let $W_n = span\{e_1, e_2, ..., e_n\}$ be a Hilbert basis (orthonormal basis) of $H_*^2(0,1) \cap H_*^1(0,1)$ and $L_*^2(0,1)$. Also, we denote by $\Gamma_n = span\{\sigma_1, \sigma_2, ..., \sigma_n\}$ a Hilbertian basis of $H^2(0,1) \cap H_0^1(0,1)$ and $L^2(0,1)$. For given initial data

$$\begin{aligned} (u_0, u_1) &\in H_*^1(0,1) \times L_*^2(0,1), \\ (\phi_0, \phi_1) &\in H_0^1(0,1) \times L^2(0,1), \end{aligned}$$

we seek functions $y_j^n, h_j^n \in C^2([0,T])$, such that the approximations

$$\begin{cases} u^n(x,t) = \sum\limits_{j=1}^{j=n} y_j^n(t)\, e_j(x), \\ \phi^n(x,t) = \sum\limits_{j=1}^{j=n} h_j^n(t)\, \sigma_j(x), \end{cases} \tag{4.3.3}$$

check the following approximate problem

$$\begin{cases} \rho u_{tt}^n, -\mu u_{xx}^n - b\phi_x^n = 0, \\ J\phi_{tt}^n + J\left(\int_0^t k(t-s)\,\phi_t^n(s)ds\right)_t - \delta\phi_{xx}^n + bu_x^n + \xi\phi^n + \mu_1\phi_t^n = 0, \end{cases} \tag{4.3.4}$$

with the initial data

$$\begin{cases} u^n(x,0) = u_0^n(x),\ u_t^n(x,0) = u_1^n(x), \\ \phi^n(x,0) = \phi_0^n(x),\ \phi_t^n(x,0) = \phi_1^n(x), \end{cases} \tag{4.3.5}$$

which satisfies

$$\begin{cases} u_0^n = \sum_{j=1}^{n} \left\{ \int_0^1 u_0 e_j dx \right\} e_j \underset{n\to\infty}{\longrightarrow} u_0 \text{ strongly in } H_*^1(0,1), \\ u_1^n = \sum_{j=1}^{n} \left\{ \int_0^1 u_1 e_j dx \right\} e_j \underset{n\to\infty}{\longrightarrow} u_1 \text{ strongly in } L_*^2(0,1), \\ \phi_0^n = \sum_{j=1}^{n} \left\{ \int_0^1 \phi_0 \sigma_j dx \right\} \sigma_j \underset{n\to\infty}{\longrightarrow} \phi_0 \text{ strongly in } H_0^1(0,1), \\ \phi_1^n = \sum_{j=1}^{n} \left\{ \int_0^1 \phi_1 \sigma_j dx \right\} \sigma_j \underset{n\to\infty}{\longrightarrow} \phi_1 \text{ strongly in } L^2(0,1). \end{cases} \tag{4.3.6}$$

Through (4.3.4), we get

$$\begin{cases} \rho \langle u_{tt}^n, e_k \rangle_{L^2(0,1)} - \mu \langle u_{xx}^n, e_k \rangle_{L^2(0,1)} - b \langle \phi_x^n, e_k \rangle_{L^2(0,1)} = 0, \\ J \langle \phi_{tt}^n, \sigma_k \rangle_{L^2(0,1)} + J \left\langle \left(\int_0^t k(t-s)\, \phi_t^n(s) ds \right)_t, \sigma_k \right\rangle_{L^2(0,1)} \\ -\delta \langle \phi_{xx}^n, \sigma_k \rangle_{L^2(0,1)} + b \langle u_x^n, \sigma_k \rangle_{L^2(0,1)} + \xi \langle \phi^n, \sigma_k \rangle_{L^2(0,1)} + \mu_1 \langle \phi_t^n, \sigma_k \rangle_{L^2(0,1)} = 0, \end{cases} \tag{4.3.7}$$

with (u_0^n, u_1^n) and (ϕ_0^n, ϕ_1^n), respectively, in W_n and Γ_n. According to the standard ordinary differential equations theory, the finite dimensional problem (4.3.7) has a solution $\left(y_j^n, h_j^n\right)_{j=1,..,n} \in C^2([0,t_n])^2$. Then, the a priori estimates that follow imply that in fact $t_n = T, \forall T > 0$.

Step 2: Energy estimates

A priori estimate I.

For every $n \geq 1$, by integrating by parts in (4.3.7), we get

$$\begin{cases} \rho \displaystyle\int_0^1 u_{tt}^n e_k dx + \mu \int_0^1 u_x^n e_{kx} dx - b \int_0^1 \phi_x^n e_k dx = 0, \\ J \displaystyle\int_0^1 \phi_{tt}^n \sigma_k dx + J \int_0^1 \sigma_k \left(\int_0^t k\left(t-s\right) \phi_t^n(s) ds \right)_t dx \\ +\delta \displaystyle\int_0^1 \phi_x^n \sigma_{kx} dx + b \int_0^1 u_x^n \sigma_k dx + \xi \int_0^1 \phi^n \sigma_k dx \\ +\mu_1 \displaystyle\int_0^1 \phi_t^n \sigma_k dx = 0, \qquad \forall k = 1, .., n. \end{cases} \tag{4.3.8}$$

Multiplying $(4.3.8)_1$ and $(4.3.8)_2$, respectively, by $(y_k^n)_t$ and $(h_k^n)_t$, then, by using integration by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left\{ \rho \left(u_t^n\right)^2 + \mu \left(u_x^n\right)^2 + J \left(\phi_t^n\right)^2 + 2bu_x^n \phi^n + \xi \left(\phi^n\right)^2 + \delta \left(\phi_x^n\right)^2 \right\} dx \\ &+ J \int_0^1 \phi_t^n \left(\int_0^t k\left(t-s\right) \phi_t^n(s) ds \right)_t dx + \mu_1 \int_0^1 \left(\phi_t^n\right)^2 dx = 0. \end{aligned} \tag{4.3.9}$$

We use the same technique in the proof of Lemma 2, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left\{ \rho \left(u_t^n\right)^2 + \mu \left(u_x^n\right)^2 + J \left(\phi_t^n\right)^2 + 2bu_x^n \phi^n + \xi \left(\phi^n\right)^2 + \delta \left(\phi_x^n\right)^2 \right\} dx \\ &+ \frac{J}{2}\frac{d}{dt}\int_0^1 \left(\int_0^t k\left(t-s\right) \left(\phi_t^n\right)^2(s) ds \right) dx = \frac{J}{2} \left(k' \Box \phi_t^n\right)(t) \\ &- \left(J\frac{k(t)}{2} + \mu_1 \right) \int_0^1 \left(\phi_t^n\right)^2 dx \ \leq \ 0. \end{aligned} \tag{4.3.10}$$

Now integrating (4.3.10), we obtain

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left\{ \rho \left(u_t^n\right)^2 + \mu \left(u_x^n\right)^2 + J \left(\phi_t^n\right)^2 + 2bu_x^n \phi^n + \xi \left(\phi^n\right)^2 + \delta \left(\phi_x^n\right)^2 \right\} dx \\ &+ \frac{J}{2}\int_0^1 \left(\int_0^t k\left(t-s\right) \left(\phi_t^n\right)^2(s) ds \right) dx \\ &\leq \frac{1}{2}\int_0^1 \left\{ \rho \left(u_1^n\right)^2 + J \left(\phi_1^n\right)^2 + \left[\delta \left(\phi_x^n\right)^2 + \mu \left(u_x^n\right)^2 + 2bu_x^n \phi^n + \xi \left(\phi^n\right)^2 \right] (x,0) \right\} dx. \end{aligned}$$

Hence, the previous inequality takes the following form

$$E^n\left(t\right) \leq E^n\left(0\right),$$

where

$$E^n(t) = \frac{1}{2}\int_0^1 \left\{\rho\left(u_t^n\right)^2 + \mu\left(u_x^n\right)^2 + J\left(\phi_t^n\right)^2 + 2bu_x^n\phi^n + \xi\left(\phi^n\right)^2 + \delta\left(\phi_x^n\right)^2\right\} dx$$
$$+\frac{J}{2}\int_0^1 \left(\int_0^t k(t-s)\left(\phi_t^n\right)^2(s)ds\right) dx. \tag{4.3.11}$$

In view of the hypotheses on the function k and as in the Remark 1, we deduce

$$0 \leq E^n(t) \leq E^n(0).$$

Now, since the sequences $(u_0^n)_{n\in\mathbb{N}}, (u_1^n)_{n\in\mathbb{N}}, (\phi_0^n)_{n\in\mathbb{N}}, (\phi_1^n)_{n\in\mathbb{N}}$, converge (see (4.3.6)), using **(H1)** and **(H2),** we can find a positive constant C independent of n such that

$$E^n(t) \leq C. \tag{4.3.12}$$

Then $t_n = T$, for all $T > 0$.

A priori estimate II

Throught (4.3.3), also as $\left(y_j^n, h_j^n\right)_{j=1,..,n} \in \left(C^2[0,T]\right)^2$ and

$$\left(e_j\right)_{j\geq 1} \subset H_*^2(0,1) \cap H_*^1(0,1) \subset H^1(0,1) \hookrightarrow C(0,1),$$
$$\left(\sigma_j\right)_{j\geq 1} \subset H^2(0,1) \cap H_0^1(0,1) \subset H^1(0,1) \hookrightarrow C(0,1),$$

we have

$$\begin{cases} u^n \in C^2\left(0,T;\ H_*^2(0,1) \cap H_*^1(0,1)\right), \\ \phi^n \in C^2\left(0,T;\ H^2(0,1) \cap H_0^1(0,1)\right). \end{cases} \tag{4.3.13}$$

Because $E = C^2\left(0,T;\ H_*^2(0,1) \cap H_*^1(0,1)\right)$ is a Banach space equipped with the norm

$$\|u^n\|_E = \sup_{t\in[0,T]} \|u^n(.,t)\|_{H^2(0,1)} + \sup_{t\in[0,T]} \|u_t^n(.,t)\|_{H^2(0,1)}$$
$$+\sup_{t\in[0,T]} \|u_{tt}^n(.,t)\|_{H^2(0,1)},\ \forall n \in \mathbb{N}^*,$$

also $F = C^2\left(0, T;\ H^2(0,1) \cap H_0^1(0,1)\right)$ is a Banach space equipped with the norm

$$\begin{aligned}\|\phi^n\|_F = &\sup_{t\in[0,T]} \|\phi^n(.,t)\|_{H^2(0,1)} + \sup_{t\in[0,T]} \|\phi_t^n(.,t)\|_{H^2(0,1)} \\ &+ \sup_{t\in[0,T]} \|\phi_{tt}^n(.,t)\|_{H^2(0,1)}, \ \forall n \in \mathbb{N}^*.\end{aligned}$$

Taking into account (4.3.13), we get $\|u^n\|_E < \infty$, $\|\phi^n\|_E < \infty$, and using the fact that

$$\begin{aligned}\|u_{xx}^n\|_{L^2(0,1)} &\leq \|u^n\|_E < \infty, \ \forall n \in \mathbb{N}^* \text{ and } \forall t \in [0,T], \\ \|\phi_{xx}^n\|_{L^2(0,1)} &\leq \|\phi^n\|_F < \infty, \ \forall n \in \mathbb{N}^* \text{ and } \forall t \in [0,T],\end{aligned}$$

we get

$$\int_0^1 (u_{xx}^n)^2 + (\phi_{xx}^n)^2\, dx < \infty, \ \forall t \in [0,T]. \tag{4.3.14}$$

Step 3 : The limit process.

From (4.3.12) and (4.3.14), we conclude that

$$\begin{aligned}&(u^n)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty\left(0,T; H_*^2(0,1) \cap H_*^1(0,1)\right), \\ &(u_t^n)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty\left(0,T; L_*^2(0,1)\right), \\ &(\phi^n)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty\left(0,T; H^2(0,1) \cap H_0^1(0,1)\right), \\ &(\phi_t^n)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty\left(0,T; L^2(0,1)\right).\end{aligned} \tag{4.3.15}$$

Note that the boundedness in $L^\infty\left(0,T; H_*^2(0,1) \cap H_*^1(0,1)\right)$ is not a consequence of (4.3.14) only, but we exploit it as follows

Throughout (4.3.12), we conclude

$$\int_0^1 (u_x^n)^2\, dx < \infty, \ \forall n \geq 1, \ \forall t \in [0,T], \tag{4.3.16}$$

and by using Poincaré inequality with (4.3.14), we have

$$\sup_{t\in[0,T]}\left(\int_0^1 (u^n)^2\,dx+\int_0^1 (u_x^n)^2\,dx+\int_0^1 (u_{xx}^n)^2\,dx\right)<\infty,\ \forall n\geq 1.$$

Then

$$u^n \text{ is bounded in } L^\infty\left(0,T;H_*^2(0,1)\cap H_*^1(0,1)\right),\ \forall n\geq 1.$$

By using Theorem (4.2.1), Since

The embedding of $H_*^1(0,1)$ in $L_*^2(0,1)$ is continuous.
The embedding of $H_*^2(0,1)\cap H_*^1(0,1)$ in $H_*^1(0,1)$ is compact.
The embedding of $H_0^1(0,1)$ in $L^2(0,1)$ is continuous.
The embedding of $H^2(0,1)\cap H_0^1(0,1)$ in $H_0^1(0,1)$ is compact.

Then, the embedding of $E_{\infty,\infty}$ in $C(0,T;H_*^1(0,1))$ is compact where

$$\begin{aligned}E_{\infty,\infty}=\Big\{u^n \mid u^n &\in L^\infty\left(0,T;H_*^2(0,1)\cap H_*^1(0,1)\right),\\ u_t^n&=\frac{du^n}{dt}\in L^\infty\left(0,T;L_*^2(0,1)\right)\Big\},\end{aligned}$$

and the embedding of $\tilde{E}_{\infty,\infty}$ in $C([0,T],H_0^1(0,1))$ is compact with

$$\begin{aligned}\tilde{E}_{\infty,\infty}=\Big\{\phi^n \mid \phi^n &\in L^\infty\left(0,T;H^2(0,1)\cap H_0^1(0,1)\right),\\ \phi_t^n&=\frac{d\phi^n}{dt}\in L^\infty\left(0,T;L^2(0,1)\right)\Big\}.\end{aligned}$$

On the other hand, from (4.3.15), we have $(u^n)_{n\in\mathbb{N}^*}$ and $(\phi^n)_{n\in\mathbb{N}^*}$ bounded in $E_{\infty,\infty}$ and $\tilde{E}_{\infty,\infty}$ respectively. So, there exists two sub-sequences $(u^m)_{m\geq 1}$ of $(u^n)_{n\geq 1}$ and $(\phi^m)_{m\geq 1}$ of $(\phi^n)_{n\geq 1}$ such that

$$u^m \overset{m\to\infty}{\longrightarrow} u \text{ strongly in } C(0,T;H_*^1(0,1)), \tag{4.3.17}$$

$$\phi^m \overset{m\to\infty}{\longrightarrow} \phi \text{ strongly in } C(0,T;H_0^1(0,1)). \tag{4.3.18}$$

Which implies that

$$\{u^m\}_{m\geq 1} \text{ simply converges to } u,\ \forall t\in\ [0,T]. \tag{4.3.19}$$

By using (4.3.13), we have $u_t^m \in C^1\left(0,T;H_*^2(0,1)\cap H_*^1(0,1)\right)$, $\forall m \geq 1$, and by (4.3.19) with the **dominated convergence theorem**, we obtain for any $t\in[0,T]$ and $k\in\mathbb{N}^*$

$$\begin{aligned}\lim_{m\to\infty}\left\|u_t^m(.,t)-u_t^{m+k}(.,t)\right\|_{L^2(0,1)}^2 &= \lim_{m\to\infty}\int_0^1\left|u_t^m(x,t)-u_t^{m+k}(x,t)\right|^2dx\\ &=\int_0^1\lim_{m\to\infty}\left|u_t^m(x,t)-u_t^{m+k}(x,t)\right|^2dx=0,\end{aligned} \tag{4.3.20}$$

also, by the same way, we can write

$$\begin{aligned}\lim_{m\to\infty}\left\|u_{tx}^m(.,t)-u_{tx}^{m+k}(.,t)\right\|_{L^2(0,1)}^2 &= \lim_{m\to\infty}\int_0^1\left|u_{tx}^m(x,t)-u_{tx}^{m+k}(x,t)\right|^2dx\\ &=\int_0^1\lim_{m\to\infty}\left|u_{tx}^m(x,t)-u_{tx}^{m+k}(x,t)\right|^2dx=0.\end{aligned} \tag{4.3.21}$$

Combining (4.3.20)-(4.3.21), we get

$$\lim_{m\to\infty}\sup_{t\in[0,T]}\left\|u_t^m(.,t)-u_t^{m+k}(.,t)\right\|_{H^1(0,1)}^2=0,$$

it means that $(u_t^m)_{m\geq 1}$ is a Cauchy sequence in $X=C\left(0,T;H_*^1(0,1)\right)$ equipped with the norm

$$\|u\|_X=\sup_{t\in[0,T]}\|u(.,t)\|_{H^1(0,1)}.$$

Since $X=\left(C\left(0,T;\ H_*^1(0,1)\right);\|.\|_X\right)$ is a Banach space, then there exists a unique $g\in C\left(0,T;H_*^1(0,1)\right)$ such that

$$u_t^m \overset{m\to\infty}{\longrightarrow} g \text{ strongly in } C\left(0,T;\ H_*^1(0,1)\right). \tag{4.3.22}$$

Now, it's left to prove that $g = u_t$. Since the operator A is define as follows

$$\begin{cases} A : D(A) = C^1\left(0, T;\ H^1_*(0,1)\right) \subset C\left(0, T;\ H^1_*(0,1)\right) \longrightarrow C\left(0, T;\ H^1_*(0,1)\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad u \longrightarrow u_t \end{cases}$$

is closed. That is to say if $(u^m)_{m\geq 1} \subset D(A)$ converges strongly to $u \in C\left(0, T;\ H^1_*(0,1)\right)$ and $u^m_t = Au^m$, $m \geq 1$ converges strongly to $g \in C\left(0, T;\ H^1_*(0,1)\right)$, then, we get $u \in C^1\left(0, T;\ H^1_*(0,1)\right)$ and $g = Au = u_t$. Using (4.3.17) and (4.3.22), we obtain

$$u^m_t \overset{m\to\infty}{\longrightarrow} g = u_t \text{ strongly in } X = C\left(0, T;\ H^1_*(0,1)\right), \tag{4.3.23}$$

Similarly, by using (4.3.13) and (4.3.18), we can easily prove that

$$\phi^m_t \overset{m\to\infty}{\longrightarrow} \phi_t \text{ strongly in } Y = C\left(0, T;\ H^1_0(0,1)\right). \tag{4.3.24}$$

Also, by using (4.3.13), we have $u^m_{tt} \in C\left(0, T;\ H^2_*(0,1) \cap H^1_*(0,1)\right)$, $\forall m \geq 1$, and by (4.3.19) with the **dominated convergence theorem,** we obtain for any $t \in [0, T]$ and $k \in \mathbb{N}^*$

$$\begin{aligned} \lim_{m\to\infty} \left\| u^m_{tt}(.,t) - u^{m+k}_{tt}(.,t) \right\|^2_{L^2(0,1)} &= \lim_{m\to\infty} \int_0^1 \left| u^m_{tt}(x,t) - u^{m+k}_{tt}(x,t) \right|^2 dx \\ &= \int_0^1 \lim_{m\to\infty} \left| u^m_{tt}(x,t) - u^{m+k}_{tt}(x,t) \right|^2 dx = 0, \end{aligned} \tag{4.3.25}$$

this last formula implies that

$$\lim_{m\to\infty} \sup_{t\in[0,T]} \left\| u^m_{tt}(.,t) - u^{m+k}_{tt}(.,t) \right\|_{L^2(0,1)} = 0,$$

it means that $(u^m_{tt})_{m\geq 1}$ is a Cauchy sequence in $Z = C\left(0, T;\ L^2(0,1)\right)$ equipped with the norm

$$\|u\|_Z = \sup_{t\in[0,T]} \|u(.,t)\|_{L^2(0,1)}.$$

Since $Z = \left(C\left(0, T,\ L^2(0,1)\right);\ \|.\|_Z\right)$ is a Banach space, then there exists a unique $f \in C\left(0, T,\ L^2(0,1)\right)$ such that

$$u_{tt}^m \overset{m\to\infty}{\longrightarrow} f \text{ strongly in } C\left(0, T;\ L^2(0,1)\right). \tag{4.3.26}$$

Now, it's left to prove that $f = u_{tt}$.

By using (4.3.17) and (4.3.23), we get

$$u^m \overset{m\to\infty}{\longrightarrow} u \text{ strongly in } C^1\left(0, T;\ H_*^1(0,1)\right). \tag{4.3.27}$$

Since the operator B is defined as follows

$$\begin{cases} B : D(B) = C^2\left(0, T;\ L^2(0,1)\right) \subset C^1\left(0, T;\ L^2(0,1)\right) \longrightarrow C\left(0, T;\ L^2(0,1)\right) \\ \qquad\qquad\qquad\qquad\qquad u \longrightarrow u_{tt} \end{cases}$$

is closed. Now, by using (4.3.26) and (4.3.27), we obtain

$$f = Bu = u_{tt}, \tag{4.3.28}$$

which implies

$$u_{tt}^m \overset{m\to\infty}{\longrightarrow} f = u_{tt} \text{ strongly in } C\left(0, T;\ L^2(0,1)\right).$$

Similarly, by using (4.3.13), (4.3.18) and (4.3.24), we can easily prove that

$$\phi_{tt}^m \overset{m\to\infty}{\longrightarrow} \phi_{tt} \text{ strongly in } C\left(0, T;\ L^2(0,1)\right).$$

By passing to the limit in (4.3.5) and (4.3.8), the problem (4.1.1) admits a global weak solution satisfies (4.3.2).

The proof now can be completed arguing as in [**?**, Theorem 3.1]

Continuous dependence and uniqueness

For uniqueness: Let us assume that $\left(\Lambda^1, \Upsilon^1\right)$ and $\left(\Lambda^2, \Upsilon^2\right)$ are two global solutions of (4.1.1). Then, $(\chi, \Xi) = \left(\Lambda^1 - \Lambda^2, \Upsilon^1 - \Upsilon^2\right)$ satisfies $(4.1.1)_1$ and $(4.1.1)_2$ with

$$\begin{cases} \chi(x,0) = \chi_t(x,0) = \Xi(x,0) = \Xi_t(x,0) = 0, \ x \in (0,1), \\ \chi_x(0,t) = \chi_x(1,t) = \Xi(0,t) = \Xi(1,t) = 0, \ t > 0. \end{cases} \tag{4.3.29}$$

From the linearity of the equations and the fact that the energy $E(t)$ is decreasing, so that, for (χ, Ξ), we have $0 \leq E(t) \leq E(0) = 0$, for any $t \geq 0$, where

$$\begin{aligned} E(t) = &\frac{1}{2}\int_0^1 \left(\rho\chi_t^2 + \mu\chi_x^2 + J\Xi_t^2 + 2b\chi_x\Xi + \xi\Xi^2 + \delta\Xi_x^2\right) dx \\ &+ \frac{J}{2}\int_0^1 \left(\int_0^t k(t-s)\,\Xi_t^2(s)ds\right) dx, \end{aligned}$$

satisfies

$$\frac{d}{dt}E(t) = \frac{J}{2}\left(k'\Box\Xi_t\right)(t) - \left(J\frac{k(t)}{2} + \mu_1\right)\int_0^1 \Xi_t^2 dx \leq 0. \tag{4.3.30}$$

Hence, $(\chi, \Xi)(t) = (0,0)$, identically. So, the problem (4.1.1) has a unique global solution.

The continuous dependence on initial data

Let (Θ, Φ) be a global solution of (4.1.1). A simple integration, and by using the Young's inequality and the positivity of energy, we get

$$\begin{aligned} E(t) \leq &E(0) + \frac{1}{2}\int_0^t \left[\int_0^1 \left(\rho\Theta_t^2 + \mu\Theta_x^2 + J\Phi_t^2 + 2b\Theta_x\Phi + \xi\Phi^2 + \delta\Phi_x^2\right) dx \right.\\ &\left. + \frac{J}{2}\int_0^1 \left(\int_0^t k(t-s)\,\Phi_t^2(s)ds\right) dx\right] d\tau \\ \leq &E(0) + \frac{1}{2}\int_0^t \left[\int_0^1 \left(\rho\Theta_t^2 + (\mu + |b|)\,\Theta_x^2 + J\Phi_t^2 + (\xi + |b|)\,\Phi^2 + \delta\Phi_x^2\right) dx \right.\\ &\left. + \frac{J}{2}\int_0^1 \left(\int_0^t k(t-s)\,\Phi_t^2(s)ds\right) dx\right] d\tau \\ \leq &E(0) + \varsigma_1\int_0^t \left[\int_0^1 \left(\Theta_t^2 + \Theta_x^2 + \Phi_t^2 + \Phi^2 + \Phi_x^2\right) dx \right.\\ &\left. + \int_0^t k(t-s)\,\Phi_t^2(s)ds\right] d\tau, \end{aligned} \tag{4.3.31}$$

where $\varsigma_1 = \max\left\{\frac{\rho}{2}, \frac{1}{2}(\mu + |b|), \frac{J}{2}, \frac{1}{2}(\xi + |b|), \frac{\delta}{2}, \frac{J}{4}\right\}$.
On the other hand, we have

$$E(t) > \varsigma_2 \int_0^1 \left(\Theta_t^2 + \Theta_x^2 + \Xi_t^2 + \Xi^2 + \Xi_x^2 + \int_0^t k(t-s)\, \Xi_t^2(s) ds\right) dx,$$

with $\varsigma_2 = \min\left\{\frac{\rho}{2}, \frac{J}{2}, \frac{\delta}{2}, \frac{1}{2}\left(\mu - \frac{b^2}{\xi}\right), \frac{1}{2}\left(\xi - \frac{b^2}{\mu}\right), \frac{J}{4}\right\}$.
Applying Gronwall's inequality to (4.3.31), we obtain

$$\int_0^1 \left(\Theta_t^2 + \Theta_x^2 + \Phi_t^2 + \Phi^2 + \Phi_x^2 + \int_0^t k(t-s)\, \Phi_t^2(s) ds\right) dx \leq e^{\varsigma_1 t} E(0).$$

This shows that solution of problem (4.1.1) depends continuously on the initial data. This ends the proof of Theorem **2**. □

4.4 Stability result

In this section, we use the energy method to study the asymptotic behavior of solutions of the system (4.1.1).

4.4.1 Exponential stability

In this subsection, we establish an exponential decay result of solutions of the problem (4.1.1) in the case when (4.1.5) holds. The same result is obtained in [**?**] where the authors considered the one dimensional porous-elastic system subject to a distributed delay and by some assumptions on the weight of delay they proved an exponential decay of the solution. Also, in [3] the author established an explicit and general decay rate of solution of the same system damped via a nonlinear damping term under some properties of convex functions. In our case the situation is completely different and this due to the nature and form of the neutral delay. So, we need the following lemmas

Lemma 4.4.1.

Let (u, ϕ) be the solution of system (4.1.1). Then, the functional

$$
\begin{aligned}
F_1(t) = {} & J \int_0^1 \phi \left(\phi_t + \int_0^t k(t-s)\, \phi_t(s) ds \right) dx + \frac{b\rho}{\mu} \int_0^1 \phi \left(\int_0^x u_t(y)\, dy \right) dx \\
& + \frac{\mu_1}{2} \int_0^1 \phi^2 dx,
\end{aligned}
$$

satisfies for any $\varepsilon_0 > 0$,

$$
\begin{aligned}
F_1'(t) \leq {} & -\delta \int_0^1 \phi_x^2 dx - 2\xi_1 \int_0^1 \phi^2 dx + \left[\frac{3J}{2} + \frac{b^2\rho^2}{4\mu^2\varepsilon_0} \right] \int_0^1 \phi_t^2 dx \\
& + \varepsilon_0 \int_0^1 u_t^2 dx + \frac{J\bar{k}}{2} \int_0^1 \left(\int_0^t k(t-s)\, \phi_t^2(s) ds \right) dx. \qquad (4.4.1)
\end{aligned}
$$

Proof. By differentiating $F_1(t)$ and integrating by parts, we obtain

$$
\begin{aligned}
F_1'(t) = {} & J \int_0^1 \phi_t^2 dx + J \int_0^1 \phi_t \left(\int_0^t k(t-s)\, \phi_t(s) ds \right) dx \\
& - \delta \int_0^1 \phi_x^2 dx - b \int_0^1 u_x \phi dx - 2\xi_1 \int_0^1 \phi^2 dx + b \int_0^1 u_x \phi dx \\
& + \frac{b\rho}{\mu} \int_0^1 \phi_t \left(\int_0^x u_t(y)\, dy \right) dx. \qquad (4.4.2)
\end{aligned}
$$

Using Young's and Cauchy-Schwarz inequalities, we obtain

$$
\begin{aligned}
& J \int_0^1 \phi_t \left(\int_0^t k(t-s)\, \phi_t(s) ds \right) dx \\
& \leq \frac{J}{2} \int_0^1 \phi_t^2 dx + \frac{J\bar{k}}{2} \int_0^1 \left(\int_0^t k(t-s)\, \phi_t^2(s) ds \right) dx. \qquad (4.4.3)
\end{aligned}
$$

Using Young's inequality, we get

$$
\frac{b\rho}{\mu} \int_0^1 \phi_t \left(\int_0^x u_t(y)\, dy \right) dx \leq \frac{b^2\rho^2}{4\mu^2\varepsilon_0} \int_0^1 \phi_t^2 dx + \varepsilon_0 \int_0^1 u_t^2 dx. \qquad (4.4.4)
$$

Inserting (4.4.3) and (4.4.4) into (4.4.2), we obtain (4.4.6). □

Lemma 4.4.2. *Let* (u,ϕ) *be the solution of system* (4.1.1). *Then, the functional*

$$F_2(t)=\frac{\delta\rho b}{\mu J}\int_0^1\phi_x u_t dx+b\int_0^1\left(\phi_t+\int_0^t k(t-s)\,\phi_t(s)ds\right)u_x dx,$$

satisfies, for any $\varepsilon_1>0$,

$$\begin{aligned}F_2'(t)\leq&-\frac{b^2}{4J}\int_0^1 u_x^2dx+C_{\varepsilon_1}\int_0^1\phi_x^2dx+\varepsilon_1(2+k(0))\int_0^1 u_t^2dx\\&+\frac{b^2k(t)}{4\varepsilon_1}\int_0^1\phi_{0x}^2dx+\frac{b^2k(0)}{4\varepsilon_1}\int_0^1\left(\int_0^t\left|k'(t-s)\right|\phi_x^2(s)ds\right)dx\\&+\frac{\mu_1^2}{J}\int_0^1\phi_t^2dx+\frac{\rho b}{\mu}\chi\int_0^1\phi_t u_{tx}dx,\end{aligned}\tag{4.4.5}$$

where $C_{\varepsilon_1}=\dfrac{\delta b^2}{\mu J}+\dfrac{b^2k^2(0)}{4\varepsilon_1}+\dfrac{\xi^2}{2J}$.

Proof. By differentiating $F_2(t)$, and integrating by parts, we obtain

$$\begin{aligned}F_2'(t)=&\frac{\rho b}{\mu}\chi\int_0^1\phi_t u_{tx}dx-\frac{b^2}{J}\int_0^1 u_x^2dx+\frac{\delta b^2}{\mu J}\int_0^1\phi_x^2dx-\frac{b\xi}{J}\int_0^1\phi u_x dx\\&+b\int_0^1 u_{tx}\left(\int_0^t k(t-s)\,\phi_t(s)ds\right)dx-\frac{b\mu_1}{J}\int_0^1\phi_t u_x dx.\end{aligned}\tag{4.4.6}$$

Integrating by parts with respect to t the last term of (4.4.6),we have

$$\begin{aligned}&b\int_0^1 u_{tx}\left(\int_0^t k(t-s)\,\phi_t(s)ds\right)dx\\&=b\int_0^1 u_{tx}\left[k(0)\,\phi(t)-k(t)\,\phi(0)+\int_0^t k'(t-s)\,\phi(s)ds\right]dx\\&=-bk(0)\int_0^1 u_t\phi_x dx+bk(t)\int_0^1 u_t\phi_x(0)\,dx\\&-b\int_0^1 u_t\left(\int_0^t k'(t-s)\,\phi_x(s)ds\right)dx.\end{aligned}$$

Then, (4.4.6) becomes

$$\begin{aligned}F_2'(t) = &\frac{\rho b}{\mu}\chi\int_0^1 \phi_t u_{tx}dx - \frac{b^2}{J}\int_0^1 u_x^2 dx + \frac{\delta b^2}{\mu J}\int_0^1 \phi_x^2 dx\\ &- bk(0)\int_0^1 u_t\phi_x dx + bk(t)\int_0^1 u_t\phi_x(0)\,dx - \frac{b\xi}{J}\int_0^1 \phi u_x dx\\ &- b\int_0^1 u_t\left(\int_0^t k'(t-s)\,\phi_x(s)ds\right)dx - \frac{b\mu_1}{J}\int_0^1 \phi_t u_x dx. \qquad (4.4.7)\end{aligned}$$

By using Young's inequality, we arrive at

$$-bk(0)\int_0^1 u_t\phi_x dx \le \varepsilon_1\int_0^1 u_t^2 dx + \frac{b^2k^2(0)}{4\varepsilon_1}\int_0^1 \phi_x^2 dx, \qquad (4.4.8)$$

$$-\frac{b\mu_1}{J}\int_0^1 \phi_t u_x dx \le \frac{b^2}{4J}\int_0^1 u_x^2 dx + \frac{\mu_1^2}{J}\int_0^1 \phi_t^2 dx \qquad (4.4.9)$$

and

$$\begin{aligned}+bk(t)\int_0^1 u_t\phi_x(0)\,dx &\le \varepsilon_1 k(t)\int_0^1 u_t^2 dx + \frac{b^2k(t)}{4\varepsilon_1}\int_0^1 \phi_{0x}^2 dx\\ &\le \varepsilon_1 k(0)\int_0^1 u_t^2 dx + \frac{b^2k(t)}{4\varepsilon_1}\int_0^1 \phi_{0x}^2 dx, \qquad (4.4.10)\end{aligned}$$

Young's and Cauchy-Schwarz inequalities leads to

$$\begin{aligned}&- b\int_0^1 u_t\left(\int_0^t k'(t-s)\,\phi_x(s)ds\right)dx\\ &\le \varepsilon_1\int_0^1 u_t^2 dx + \frac{b^2k(0)}{4\varepsilon_1}\int_0^1\left(\int_0^t \left|k'(t-s)\right|\phi_x^2(s)ds\right)dx. \qquad (4.4.11)\end{aligned}$$

By using Young's and Poincaré inequalities, we have

$$-\frac{b\xi}{J}\int_0^1 \phi u_x dx \le \frac{b^2}{4J}\int_0^1 u_x^2 dx + \frac{\xi^2}{J}\int_0^1 \phi_x^2 dx. \qquad (4.4.12)$$

By substituting (4.4.8)-(4.4.12) in (4.4.7) and taking into account that $\chi = 0$, we get (4.4.5).

□

Lemma 4.4.3. *Let* (u, ϕ) *be the solution of system* (4.1.1)*. Then, the functional*

$$F_3(t) = -\int_0^1 u u_t dx,$$

satisfies,

$$F_3'(t) \leq -\rho \int_0^1 u_t^2 dx + \frac{b^2}{2\mu} \int_0^1 \phi_x^2 dx + \frac{3\mu}{2} \int_0^1 u_x^2 dx. \tag{4.4.13}$$

Proof. Differentiating $F_3(t)$ and integrating by parts, we obtain

$$F_3'(t) = -\rho \int_0^1 u_t^2 dx + \mu \int_0^1 u_x^2 dx - b \int_0^1 u \phi_x dx,$$

Young's and Poincaré inequalities give (4.4.13). □

Lemma 4.4.4. *([37])Let* (u, ϕ) *be the solution of system* (4.1.1)*. Then, the functionals*

$$F_4(t) = e^{-\varsigma t} \int_0^1 \left(\int_0^t e^{\varsigma s} \tilde{H}_1(t-s) \phi_t^2(s) ds \right) dx,$$
$$F_5(t) = e^{-\tau t} \int_0^1 \left(\int_0^t e^{\tau s} \tilde{H}_2(t-s) \phi_x^2(s) ds \right) dx,$$

satisfy, $\forall t \geq 0$,

$$F_4'(t) = -\varsigma F_4(t) + \tilde{H}_1(0) \int_0^1 \phi_t^2 dx - \int_0^1 \left(\int_0^t k(t-s) \phi_t^2(s) ds \right) dx, \tag{4.4.14}$$

$$F_5'(t) = -\tau F_5(t) + \tilde{H}_2(0) \int_0^1 \phi_x^2 dx - \int_0^1 \left(\int_0^t \left| k'(t-s) \right| \phi_x^2(s) ds \right) dx, \tag{4.4.15}$$

where $\tilde{H}_1(t) = \int_t^\infty e^{\varsigma s} |k(s)| ds$ *and* $\tilde{H}_2(t) = \int_t^\infty e^{\tau s} |k'(s)| ds$.

Now, we define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = NE(t) + N_1 F_1(t) + N_2 F_2(t) + F_3(t) + N_3 F_4(t) + N_4 F_5(t), \tag{4.4.16}$$

where N, N_1, N_2, N_3, and N_4 are positive constants.

Lemma 4.4.5. *Let* (u, ϕ) *be the solution of* (4.1.1)*. Then, there exists two positive constants* κ_1

and κ_2, such that the Lyapunov functional (4.4.16) *satisfies*

$$\kappa_1 \left(E\left(t\right) +F_4\left(t\right) +F_5\left(t\right) \right) \leq \mathcal{L}(t)\leq \kappa_2\left(E\left(t\right) +F_4\left(t\right) +F_5\left(t\right) \right) ,\ \forall t\geq 0, \tag{4.4.17}$$

and

$$\mathcal{L}'(t)\leq -\beta_1\left(E(t)+F_4\left(t\right) +F_5\left(t\right) \right) +C_2k(t)+N_2\frac{\rho b}{\mu }\chi \int_0^1\phi_t u_{tx}dx,\ \beta_1>0. \tag{4.4.18}$$

Proof. From (4.4.16), we have

$$\begin{aligned}
&\left\vert \mathcal{L}(t)-NE\left(t\right) -N_3F_4\left(t\right) -N_4F_5\left(t\right) \right\vert \\
&\leq N_1J\int_0^1\left\vert \phi \right\vert .\left\vert \phi_t+\int_0^t k\left(t-s\right) \phi_t(s)ds\right\vert dx+N_1\frac{\mu_1}{2}\int_0^1\phi^2dx \\
&+N_1\frac{\left\vert b\right\vert \rho }{\mu }\int_0^1\left\vert \phi \right\vert \left(\int_0^x\left\vert u_t\left(y\right) \right\vert dy\right) dx+N_2\frac{\delta \rho \left\vert b\right\vert }{\mu J}\int_0^1\left\vert \phi_x\right\vert \left\vert u_t\right\vert dx \\
&+N_2\left\vert b\right\vert \int_0^1\left\vert u_x\right\vert \left\vert \phi_t+\int_0^t k\left(t-s\right) \phi_t(s)ds\right\vert dx+\rho \int_0^1\left\vert u\right\vert \left\vert u_t\right\vert dx.
\end{aligned}$$

By using Young's, Cauchy-Schwarz and Poincaré inequalities, we obtain

$$\left\vert \mathcal{L}(t)-NE\left(t\right) -N_3F_4\left(t\right) -N_4F_5\left(t\right) \right\vert \leq \lambda_1E(t).$$

Therefore,

$$\left(N-\lambda_1\right) E\left(t\right) +N_3F_4\left(t\right) +N_4F_5\left(t\right) \leq \mathcal{L}(t)\leq \left(N+\lambda_1\right) E\left(t\right) +N_3F_4\left(t\right) +N_4F_5\left(t\right) ,$$

by choosing N (depending on N_1, N_2, N_3, N_4) sufficiently large we obtain (4.4.17) with

$$\begin{aligned}
\kappa_1 &=\min \left\{ N-\lambda_1,N_3,N_4\right\} , \\
\kappa_2 &=\max \left\{ N+\lambda_1,N_3,N_4\right\} .
\end{aligned}$$

Now, by differentiating $\mathcal{L}\left(t\right)$, exploiting (4.2.3), (4.4.1), (4.4.5), (4.4.13), (4.4.14), (4.4.15) and setting $\varepsilon_0 = \dfrac{\rho}{4N_1}$, $\varepsilon_1 = \dfrac{\rho}{4N_2\left(2+k(0)\right)}$, we get

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[N\mu_1 - N_1\left(\frac{3J}{2} + \frac{b^2\rho^2}{4\mu^2\varepsilon_0}\right) - N_3\tilde{H}_1\left(0\right) - N_2\frac{\mu_1^2}{J}\right]\int_0^1 \phi_t^2 dx \\
& + \frac{NJ}{2}\left(k'\Box\phi_t\right)\left(t\right) - \frac{\rho}{2}\int_0^1 u_t^2 dx - 2N_1\xi_1\int_0^1 \phi^2 dx \\
& - \left[\delta N_1 - N_2 C_{\varepsilon_1} - \frac{b^2}{2\mu} - N_4\tilde{H}_2\left(0\right)\right]\int_0^1 \phi_x^2 dx \\
& - \left(\frac{b^2}{4J}N_2 - \frac{3\mu}{2}\right)\int_0^1 u_x^2 dx - \varsigma N_3 F_4\left(t\right) - \tau N_4 F_5\left(t\right) \\
& - \left(N_3 - \frac{J\bar{k}}{2}N_1\right)\int_0^1\left(\int_0^t k\left(t-s\right)\phi_t^2(s)ds\right)dx \\
& - \left[N_4 - \frac{N_2^2 b^2 k(0)\left(2+k(0)\right)}{\rho}\right]\int_0^1\left(\int_0^t \left|k'\left(t-s\right)\right|\phi_x^2(s)ds\right)dx \\
& + \frac{b^2 N_2^2 k(t)\left(2+k(0)\right)}{\rho}\int_0^1 \phi_{0x}^2 dx + N_2\frac{\rho b}{\mu}\chi\int_0^1 \phi_t u_{tx} dx.
\end{aligned}$$

We select our parameters appropriately as follows.
First, we choose N_2 large enough such that

$$\frac{b^2}{4J}N_2 - \frac{3\mu}{2} > 0.$$

We pick N_4 large such that

$$N_4 - \frac{N_2^2 b^2 k(0)\left(2+k(0)\right)}{\rho} > 0.$$

We select N_1 large enough such that

$$\delta N_1 - N_2 C_{\varepsilon_1} - \frac{b^2}{2\mu} - N_4\tilde{H}_2\left(0\right) > 0.$$

We choose N_3 large such that

$$N_3 - \frac{J\bar{k}}{2}N_1 > 0.$$

Finally, we take N large enough (even larger so that (4.4.17) remains valid) such that

$$N\mu_1 - N_1\left(\frac{3J}{2} + \frac{b^2\rho^2}{4\mu^2\varepsilon_0}\right) - N_3\tilde{H}_1(0) - N_2\frac{\mu_1^2}{J} > 0.$$

All these choices leads to

$$\begin{aligned}\mathcal{L}'(t) \leq & -\alpha_1\int_0^1\left(\phi_t^2+\phi_x^2+u_t^2+u_x^2+\phi^2\right)dx - \int_0^1\left(\int_0^t k(t-s)\,\phi_t^2(s)ds\right)dx \\ & +\alpha_2 k(t)\int_0^1 \phi_{0x}^2 dx - \varsigma N_3 F_4(t) - N_4\tau F_5(t) + N_2\frac{\rho b}{\mu}\chi\int_0^1 \phi_t u_{tx}dx, \end{aligned} \tag{4.4.19}$$

where $\alpha_1, \alpha_2 > 0$.

On the other hand, from Eq. (4.2.2) and by using Young's inequality, we obtain

$$\begin{aligned}E(t) \leq & \frac{1}{2}\int_0^1\left(\rho u_t^2 + J\phi_t^2 + (\mu+|b|)\,u_x^2 + \delta\phi_x^2 + (\xi+|b|)\,\phi^2\right)dx \\ & +\frac{J}{2}\int_0^1\left(\int_0^t k(t-s)\,\phi_t^2(s)ds\right)dx \\ \leq & \varrho_1\left(\int_0^1\left(u_t^2+\phi_t^2+u_x^2+\phi_x^2+\phi^2\right)dx \right.\\ & \left. +\int_0^1\left(\int_0^t k(t-s)\,\phi_t^2(s)ds\right)dx\right),\ \varrho_1 > 0,\end{aligned}$$

which implies that

$$-\int_0^1\left(u_t^2+\phi_t^2+u_x^2+\phi_x^2+\phi^2\right)dx - \int_0^1\left(\int_0^t k(t-s)\,\phi_t^2(s)ds\right)dx \leq -\varrho_2 E(t), \tag{4.4.20}$$

where $\varrho_2 > 0$.

The combination of (4.4.19) and (4.4.20) gives (4.4.18) with $C_2 = \alpha_2\int_0^1 \phi_{0x}^2 dx$. □

We are now ready to state and prove the following exponential stability result

Theorem 4.4.1. *Let (u,ϕ) be the solution of (4.1.1) and assume that (4.1.2),* ***(H1)-(H2)*** *hold and $\chi = 0$. Then, there exist two positive constants τ_1 and τ_2 such that*

$$E(t) \leq \tau_2 e^{-\tau_1 t},\ \forall t \geq 0. \tag{4.4.21}$$

Proof. By using (4.4.18) and the right side of (4.4.19), we get

$$\mathcal{L}'(t) \leq -C_1\mathcal{L}(t) + C_2 k(t), \tag{4.4.22}$$

where $C_1 = \dfrac{\beta_1}{\kappa_2} > 0$.
Multiplying (4.4.22) by $\exp\left(C_1 t\right)$, we obtain

$$\frac{d}{dt}\left(\mathcal{L}(t)\exp\left(C_1 t\right)\right) \leq C_2 \exp\left(C_1 t\right) k(t). \tag{4.4.23}$$

Integrating over $(0, T)$ the inequation (4.4.23) and choosing C_1 smaller than ς, we have

$$\begin{aligned}\mathcal{L}(T)\exp\left(C_1 T\right) &\leq \mathcal{L}(0) + C_2 \int_0^T \exp\left(\varsigma t\right) k(t)dt \\ &\leq \mathcal{L}(0) + C_2 \int_0^\infty \exp\left(\varsigma t\right) k(t)dt.\end{aligned}$$

Thanks to the hypothesis **(H2)**, we can write

$$\mathcal{L}(T) \leq C_3 \exp\left(-C_1 T\right),\ C_3 > 0,$$

which yields the serial result (4.4.21), using the fact that $F_4\left(t\right), F_5\left(t\right)$ are positive and the other side of the equivalence relation (4.4.17) again. The proof is complete. □

4.4.2 Polynomial stability

Here, similarly to [37], we prove a polynomial decay result of solutions of the problem (4.1.1) when (4.1.5) does not hold by assuming that the function k verifies the same hypotheses **(H1)-(H2)** and the following additional assumption

- **(H3)** $-\omega k\left(t\right) \leq k'\left(t\right) \leq 0$, where ω is a positive constant.

In order to establish this result, we need to introduce the second-order energy $E_2\left(t\right)$ by using the multipliers technique as in the case of $E\left(t\right)$. For that, by differentiating $(4.1.1)_1$

and $(4.1.1)_2$ with respect to time, we arrive at

$$\begin{cases} \rho u_{ttt} = \mu u_{xxt} + b\phi_{xt},\ x \in (0,1)\,,\ t > 0, \\ J\phi_{ttt} + J\left(\int_0^t k\,(t-s)\,\phi_t\,(s)\,ds\right)_{tt} = \delta\phi_{xxt} - bu_{xt} - \xi\phi_t + \mu_1\phi_{tt},\ x \in (0,1)\,,\ t > 0, \end{cases} \tag{4.4.24}$$

with boundary conditions

$$u_{xt}\,(0,t) = u_{xt}\,(1,t) = \phi_t\,(0,t) = \phi_t\,(1,t) = 0,\ t \geq 0,$$

and initial data

$$\begin{cases} u\,(x,0) = u_0\,(x)\,,\ u_t\,(x,0) = u_1\,(x)\,,\ u_{tt}\,(x,0) = u_2\,(x)\,,\ x \in (0,1)\,, \\ \phi\,(x,0) = \phi_0\,(x)\,,\ \phi_t\,(x,0) = \phi_1\,(x)\,,\ \phi_{tt}\,(x,0) = \phi_2\,(x)\,,\ x \in (0,1)\,. \end{cases}$$

Note that

$$\begin{aligned} &\left(\int_0^t k\,(t-s)\,\phi_t\,(s)\,ds\right)_{tt} \\ &= \left(\int_0^t k\,(s)\,\phi_t\,(t-s)\,ds\right)_{tt} \\ &= \left(\int_0^t k\,(s)\,\phi_{tt}\,(t-s)\,ds + k(t)\phi_t\,(0)\right)_t \\ &= \int_0^t k\,(t-s)\,\phi_{ttt}\,(s)\,ds + k(t)\phi_{tt}\,(0) + k'(t)\phi_t\,(0)\,. \end{aligned}$$

Then, the system (4.4.24) can be rewritten as follows

$$\begin{cases} \rho u_{ttt} = \mu u_{xxt} + b\phi_{xt},\ x \in (0,1)\,,\ t > 0, \\ J\phi_{ttt} + J\int_0^t k\,(t-s)\,\phi_{ttt}\,(s)\,ds + Jk(t)\phi_2 + Jk'(t)\phi_1 \\ = \delta\phi_{xxt} - bu_{xt} - \xi\phi_t - \mu_1\phi_{tt},\ x \in (0,1)\,,\ t > 0, \end{cases} \tag{4.4.25}$$

where $\phi_2 = \phi_{tt}\,(0)$ and $\phi_1 = \phi_t\,(0)$ depend on x.

Theorem 4.4.2. *The second-order energy* $E_2\left(t\right)$ *associated to the system* (3.1.1) *defined by*

$$\begin{aligned} E_2\left(t\right) &= \frac{1}{2}\int_0^1 \left(\rho u_{tt}^2 + J\phi_{tt}^2 + \xi\phi^2 + \delta\phi_{xt}^2 + \mu u_{xt}^2 + 2b\phi_t u_{tx}\right) dx \\ &+ \frac{J}{2}\int_0^1 \left(\int_0^t k\left(t-s\right)\phi_{tt}^2\left(s\right) ds\right) dx, \end{aligned} \tag{4.4.26}$$

satisfies

$$E_2^{'}\left(t\right) \leq -Jk'(t)\int_0^1 \phi_1\phi_{tt}dx - \mu_1\int_0^1 \phi_{tt}^2 dx + \frac{J}{2}\left(k'\Box\phi_{tt}\right)\left(t\right), \tag{4.4.27}$$

and

$$E_2\left(t\right) \leq l,\ \forall t \geq 0. \tag{4.4.28}$$

Proof. By multiplying $(4.4.25)_1$ by u_{tt}, $(4.4.25)_2$ by ϕ_{tt}, integrating over $\left(0,1\right)$ and summing up, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\rho u_{tt}^2 + J\phi_{tt}^2 + \xi\phi^2 + \delta\phi_{xt}^2 + \mu u_{xt}^2 + 2b\phi_t u_{tx}\right) dx \\ &+ Jk(t)\int_0^1 \phi_{tt}\phi_2 dx + Jk'(t)\int_0^1 \phi_{tt}\phi_1 dx \\ &+ J\int_0^1 \phi_{tt}\left(\int_0^t k\left(t-s\right)\phi_{ttt}(s)ds\right) dx = -\mu_1\int_0^1 \phi_{tt}^2 dx. \end{aligned} \tag{4.4.29}$$

By using again the result in Lemma **1** to estimate the last term of (4.4.29), we get

$$\begin{aligned} &J\int_0^1 \phi_{tt}\left(\int\int_0^t k\left(t-s\right)\phi_{ttt}(s)ds\right) dx \\ &= \frac{J}{2}\frac{d}{dt}\int_0^1 \left(\int_0^t k\left(t-s\right)\phi_{tt}^2\left(s\right) ds\right) dx - Jk(t)\int_0^1 \phi_2\phi_{tt}dx \\ &+ \frac{Jk(t)}{2}\int_0^1 \phi_{tt}^2 dx - \frac{J}{2}\left(k'\Box\phi_{tt}\right)\left(t\right). \end{aligned} \tag{4.4.30}$$

By using the positivity of $k(t)$ and the combination of (4.4.29) with (4.4.30), we have (4.4.26) and (4.4.27).

Now, by using the hypothesis **(H3)** and Young's inequality, we can write

$$-Jk'(t)\int_0^1 \phi_1\phi_{tt}dx \leq J\delta_1\omega k(t)\int_0^1 \phi_{tt}^2 dx + \frac{J\omega k(t)}{4\delta_1}\int_0^1 \phi_1^2 dx, \tag{4.4.31}$$

letting $\delta_1 = \dfrac{1}{2\omega}$ and because $k'(t) \leq 0$, then, (4.4.27) becomes

$$E_2^{'}(t) \leq \frac{J\omega^2 k(t)}{2} \int_0^1 \phi_1^2 dx = \zeta k(t),$$

where $\zeta = \dfrac{J\omega^2}{2} \displaystyle\int_0^1 \phi_1^2 dx > 0$. A simple integration over $(0, T)$ and by the hypothesis **(H1)**, we obtain (4.4.28). $\square$

We introduce the following functional

$$\tilde{F}_2(t) = -\frac{\rho b}{\mu} \chi \int_0^1 \phi_t u_x dx,$$

that satisfies

$$\tilde{F}_2'(t) = -\frac{\rho b}{\mu} \chi \int_0^1 u_{tx} \phi_t dx - \frac{\rho b}{\mu} \chi \int_0^1 \phi_{tt} u_x dx.$$

By using Young's inequality, we get

$$-\frac{\rho b}{\mu} \chi \int_0^1 \phi_{tt} u_x dx \leq \frac{b^2}{8J} \int_0^1 u_x^2 dx + C_0 \int_0^1 \phi_{tt}^2 dx.$$

Then,

$$\tilde{F}_2'(t) \leq \frac{b^2}{8J} \int_0^1 u_x^2 dx + C_0 \int_0^1 \phi_{tt}^2 dx - \frac{\rho b}{\mu} \chi \int_0^1 u_{tx} \phi_t dx.$$

We define the following Lyapunov functional as follows

$$\begin{aligned} \mathcal{L}_1(t) = {} & N\left(E(t) + E_2(t)\right) + N_1 F_1(t) + N_2 \left(F_2(t) + \tilde{F}_2(t)\right) + F_3(t) \\ & + N_3 F_4(t) + N_4 F_5(t). \end{aligned} \tag{4.4.32}$$

The Lyapunov functional $\mathcal{L}_1$ defined by (4.4.32) is not equivalent to the energy functional E, but it is equivalent to $E + E_2 + F_4(t) + F_5(t)$. Indeed by using (4.4.32), Young's, Poincaré and Cauchy-Schwarz inequalities, we have

$$\begin{aligned}
&\left|\mathcal{L}_1(t) - N\left(E\left(t\right) + E_2(t)\right) - N_3F_4\left(t\right) - N_4F_5\left(t\right)\right| \\
&\le \lambda_1 E\left(t\right) + \lambda_2 E_2(t) \\
&\le \beta\left(E\left(t\right) + E_2(t)\right),\ \beta = \max\left(\lambda_1, \lambda_2\right),
\end{aligned}$$

so

$$\begin{aligned}
&\left(N-\beta\right)\left(E\left(t\right) + E_2(t)\right) + N_3F_4\left(t\right) + N_4F_5\left(t\right) \\
&\le \mathcal{L}_1(t) \le \left(N+\beta\right)\left(E\left(t\right) + E_2(t)\right) + N_3F_4\left(t\right) + N_4F_5\left(t\right).
\end{aligned}$$

Now by choosing N sufficiently large, we obtain

$$\rho_1\left(E\left(t\right) + E_2(t) + F_4\left(t\right) + F_5\left(t\right)\right) \le \mathcal{L}_1(t) \le \rho_2\left(E\left(t\right) + E_2(t) + F_4\left(t\right) + F_5\left(t\right)\right),$$

where

$$\rho_1 = \min\left\{N-\beta, N_3, N_4\right\},\ \rho_2 = \max\left\{N+\beta, N_3, N_4\right\}.$$

Therefore,

$$\mathcal{L}_1(t) \sim E + E_2 + F_4 + F_5.$$

Now, we are ready to state and prove the polynomial stability result

Theorem 4.4.3. *Let* (u, ϕ) *be the solution of* (4.1.1) *and assume that* (4.1.2), ***(H1)-(H3)*** *hold and* $\chi \neq 0$. *Then, there exits a positive constant* C_3 *such that*

$$E(t) \le \frac{C_3}{t},\ t > 0.$$

Proof. First, note that when $\chi \neq 0$, we have

$$\begin{aligned} F_2'(t) + \tilde{F}_2'(t) \leq & -\frac{b^2}{8J}\int_0^1 u_x^2 dx + \left(\frac{\delta b^2}{\mu J} + \frac{b^2 k^2(0)}{4\varepsilon_1} + \frac{\xi^2}{2J}\right)\int_0^1 \phi_x^2 dx \\ & + \varepsilon_1 (2 + k(0)) \int_0^1 u_t^2 dx + \frac{b^2 k(t)}{4\varepsilon_1}\int_0^1 \phi_{0x}^2 dx + \frac{\mu_1^2}{J}\int_0^1 \phi_t^2 dx \\ & + \frac{b^2 k(0)}{4\varepsilon_1}\int_0^1 \left(\int_0^t k'(t-s)\,\phi_x^2(s) ds\right) dx + C_0 \int_0^1 \phi_{tt}^2 dx. \end{aligned} \tag{4.4.33}$$

By differentiating $\mathcal{L}_1$ and using (4.2.3), (4.4.1), (4.4.33), (4.4.13), (4.4.14) and (4.4.15), we get

$$\begin{aligned} \mathcal{L}_1'(t) \leq & -\delta_1 \int_0^1 \phi_t^2 dx + \delta_2 \left(k' \Box \phi_t\right)(t) - \delta_3 \int_0^1 u_t^2 dx - \delta_4 \int_0^1 \phi^2 dx \\ & - \delta_5 \int_0^1 \phi_x^2 dx - \delta_6 \int_0^1 u_x^2 dx - \delta_7 F_4(t) - \delta_8 F_5(t) \\ & - \delta_9 \int_0^1 \left(\int_0^t k(t-s)\,\phi_t^2(s) ds\right) dx - \delta_{10} \int_0^1 \left(\int_0^t \left|k'(t-s)\right| \phi_x^2(s) ds\right) dx \\ & + (\delta_{11} + \delta_{14}) k(t) - \delta_{12} \int_0^1 \phi_{tt}^2 dx + \delta_{13} \left(k' \Box \phi_{tt}\right)(t), \end{aligned}$$

where

$$\begin{aligned} \delta_1 &= \left[N\mu_1 - N_1\left(\frac{3J}{2} + \frac{b^2\rho^2}{4\mu^2\varepsilon_0}\right) - N_3 \tilde{H}_1(0) - N_2 \frac{\mu_1^2}{J}\right], \\ \delta_2 &= \frac{NJ}{2}, \delta_3 = \frac{\rho}{2}, \ \delta_4 = 2N_1 \xi_1, \\ \delta_5 &= \left[\delta N_1 - N_2 C_{\varepsilon_1} - \frac{b^2}{2\mu} - N_4 \tilde{H}_2(0)\right], \\ \delta_6 &= \left(\frac{b^2}{8J} N_2 - \frac{3\mu}{2}\right), \ \delta_7 = \varsigma N_3, \ \delta_8 = \tau N_4, \\ \delta_9 &= \left(N_3 - \frac{J\bar{k}}{2} N_1\right), \ \delta_{10} = \left[N_4 - \frac{N_2^2 b^2 k(0)\,(2 + k(0))}{\rho}\right], \\ \delta_{11} &= \frac{b^2 N_2^2\,(2 + k(0))}{\rho}\int_0^1 \phi_{0x}^2 dx, \\ \delta_{12} &= (N\mu_1 - N_2 C_0), \ \delta_{13} = \frac{JN}{2}, \ \delta_{14} = N\zeta. \end{aligned}$$

We select our parameters as follows. First, we choose N_2 large enough such that

$$\delta_6 = \frac{b^2}{8J}N_2 - \frac{3\mu}{2} > 0.$$

We pick N_4 large such that

$$\delta_{10} = N_4 - \frac{N_2^2 b^2 k(0)\,(2 + k(0))}{\rho} > 0.$$

We select N_1 large enough such that

$$\delta_5 = \delta N_1 - N_2 C_{\varepsilon_1} - \frac{b^2}{2\mu} - N_4 \tilde{H}_2\,(0) > 0.$$

We choose N_3 large such that

$$\delta_9 = N_3 - \frac{J\bar{k}}{2}N_1 > 0.$$

Finally, we take N large enough (even larger so that (4.4.17) remains valid) such that

$$\begin{cases} \delta_1 = N\mu_1 - N_1\left(\dfrac{3J}{2} + \dfrac{b^2\rho^2}{4\mu^2\varepsilon_0}\right) - N_3\tilde{H}_1\,(0) - N_2\dfrac{\mu_1^2}{J} > 0, \\ \text{and} \\ \delta_{12} = N\mu_1 - N_2 C_0 > 0. \end{cases}$$

Because $k'\,(t) \leq 0$ and $\delta_2, \delta_7, \delta_8, \delta_{10}, \delta_{11}, \delta_{12}, \delta_{13} > 0$, then

$$\begin{aligned} \mathcal{L}_1'(t) &\leq -\delta_1 \int_0^1 \phi_t^2 dx - \delta_3 \int_0^1 u_t^2 dx - \delta_4 \int_0^1 \phi^2 dx - \delta_5 \int_0^1 \phi_x^2 dx \\ &\quad - \delta_6 \int_0^1 u_x^2 dx - \delta_9 \int_0^1 \left(\int_0^t k\,(t-s)\,\phi_t^2(s) ds\right) dx + (\delta_{11} + \delta_{14})\,k(t) \\ &\leq -v_1 \int_0^1 \left(u_t^2 + \phi_t^2 + u_x^2 + \phi_x^2 + \phi^2 + \int_0^t k\,(t-s)\,\phi_t^2(s) ds\right) dx \\ &\quad + v_2 k(t), \end{aligned}$$

where $v_1 = \min\{\delta_1, \delta_3, \delta_4, \delta_5, \delta_6, \delta_9\}$, $v_2 = \delta_{11} + \delta_{14}$.

On the other hand, from the energy formula and by using Young's inequality, we obtain

$$\begin{aligned}
E(t) &\leq \frac{1}{2}\int_0^1 \left(\rho u_t^2 + J\phi_t^2 + (\mu + |b|)\, u_x^2 + \delta\phi_x^2 + (\xi + |b|)\, \phi^2\right) dx \\
&\quad + \frac{J}{2}\int_0^1 \left(\int_0^t k(t-s)\, \phi_t^2(s) ds\right) dx \\
&\leq \varrho_1 \left(\int_0^1 \left(u_t^2 + \phi_t^2 + u_x^2 + \phi_x^2 + \phi^2\right) dx \right.\\
&\quad \left. + \int_0^1 \left(\int_0^t k(t-s)\, \phi_t^2(s) ds\right) dx\right),\ \varrho_1 > 0,
\end{aligned}$$

which implies that

$$-\int_0^1 \left(u_t^2 + \phi_t^2 + u_x^2 + \phi_x^2 + \phi^2 + \int_0^t k(t-s)\, \phi_t^2(s) ds\right) dx \leq -\varrho_2 E(t),$$

where $\varrho_2 > 0$. Then

$$\mathcal{L}_1'(t) \leq -\omega_0 E(t) + \omega_1 k(t),$$

with $\omega_0 = v_1 \varrho_2, \omega_1 = v_2$.
By integrating over $(0, T)$, we get

$$\begin{aligned}
\omega_0 E(T) T &\leq -\mathcal{L}_1(T) + \mathcal{L}_1(0) + \omega_1 \int_0^T k(t) dt, \\
&\leq \mathcal{L}_1(0) + \omega_1 \int_0^\infty k(t) dt = l.
\end{aligned}$$

So

$$E(T) \leq \frac{C_3}{T},$$

with

$$C_3 = \frac{l}{\omega_0}.$$

The proof is complete. □

Remark 4.4.1. We note that the results obtained hold even for $\mu\xi = b^2$. In this case, we have to redefine the energy as in [45] as follows

$$\begin{aligned} E(t) = {} & \frac{1}{2}\int_0^1 \left(\rho u_t^2 + J\phi_t^2 + \delta\phi_x^2 + \mu\left(u_x + \frac{b}{\mu}\phi\right)^2 + \left(\xi - \frac{b^2}{\mu}\right)(\phi)^2\right)dx \\ & + \frac{J}{2}\int_0^1 \left(\int_0^t k\,(t-s)\,\phi_t^2(s)ds\right)dx, \end{aligned}$$

and adjust our calculations accordingly.

Conclusion

In this paper, we studied the asymptotic behavior of the solution of porous-elastic system in the presence of neutral delay. Introducing a single damping mechanism given by this type of delay makes our problem different from those considered so far in the literature and under some assumptions imposed on the kernel of delay, we have been able to prove an explicit energy decay rate that depends of the wave speeds of propagation.

Conclusion

In this thesis, a qualitative study and analysis have been presented of some real problems arising from physics and mechanics corresponding to the one-dimensional porous-elastic system in the presence of both the thermal effect and neural delay. We gave a global well-posedness using the Faedo-Galerkin approximations along with some energy estimates and the semigroup theory. Then, based on the energy method, we established an exponential, polynomial, and general stability results of solutions of the studied problems.

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