وزارة التعليم العالي والبحث العلمي

BADJI MOKHTAR-ANNABA UNIVERSITY
UNIVERSITE BADJI MOKHTAR ANNABA

جامعة باجي مختار–عنابة

Faculty of Science

Department of Physics

University Session: 2023/2024

THESIS

In View of Requirements for the Degree
Doctorate in Physics

Matrix Gauge/Gravity Duality

Specialty
Physique Théorique
by
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In front of the jury

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الملخص

تقدم الهندسة الغير تبديلية (ونماذج المصفوفات الخاصة بها) حلاً مميزًا لمشكلة الجاذبية الكمومية، في حين أن الثنائية الثقالية /المعيارية هي حاليا أنجح اقتراح للجاذبية الكمومية. في هذه المذكرة نبدأ بالحديث عن الخصائص الأساسية لثنائية AdS2/CFT1 ونقدم دوافع تطوير النموذج الغير تبديلي لها.

تلعب الثنائية / الثقالية المعيارية دورا مهما في مفارقة ضياع المعلومات في بعدين حيث تقدم حلا لهذه المفارقة.

وعلى وجه الخصوص فإننا في هذه المذكرة سنناقش بالتفصيل استقرار الفضاءات الغير تبديليه، الكرات الغائمةS2، فضاء ديسيتر العكسي AdS2 و AdS2*S2 الغير تبديلي، تحت تأثير التقلبات الكمومية.

يتم تحديد فضاء الطور الذي وجد أنه يتميز بانبعاث الهندسة. التمديد للجاذبية الناشئة يكون ميسرا عن طريق إضافة مصفوفة حقل الديلاتون إلى مركبات الحقل المصفوفية .

Résumé

La géométrie non commutative (et ses modèles matriciels) présente une solution distincte au problème de la gravité quantique alors que la correspondance jauge/gravité est actuellement la proposition la plus réussie pour la gravité quantique. Dans cette étude, nous commençons à discuter des caractéristiques fondamentales de la correspondance AdS2/CFT 1 et fournir une motivation pour le développement d'un modèle non commutatif pour cette théorie. Nous présentons une déformation dans la quantification habituelle qui préserve les isométries de la théorie non déformée. Cette dualité jauge/gravité est essentielle pour le problème de perte d'informations en deux dimensions et fournit le cadre pour sa résolution. En particulier, dans la présente thèse, nous discutons en détail la stabilité des espaces non commutatifs *fuzzyS2*, **AdS2** non commutatif et **S2xAdS2** non commutatif sous fluctuation quantique. La structure des phases est déterminée et se révèle caractérisée par une géométrie émergente. Une extension à la gravité émergente, qui concerne les trous noirs AdS2 non commutatifs, est simple en incluant un champ matriciel de dilaton en plus des champs de coordonnées matricielles.

Abstract

Noncommutative geometry (and its matrix models) presents a distinct solution to the problem of quantum gravity, whereas gauge/gravity correspondence is currently the most successful proposal for quantum gravity. This thesis starts by talking about the basic features of the AdS^2/CFT_1 correspondence. It also motivates the development of a noncommutative model for this theory, we introduce a deformation in the usual quantization that preserves the isometries of the undeformed theory

In particular, in the current thesis, we discuss the stability of the noncommutative spaces fuzzy S^2, noncommutative AdS^2, and noncommutative $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$ under quantum fluctuation.

The phase structure is determined, which is characterized by emergent geometry. An extension to emergent gravity, relevant to noncommutative AdS^2_θ black holes,is straightforward by including a dilaton matrix field in addition to the matrix coordinate fields.

Key words:AdS/CFT correspondence, conformal quantum mechanics,quantum gravity,noncommutative, anti-de Sitter space, phase transition.

Acknowledgements

I would like to begin by thanking my advisor Badis Ydri for his guidance, mentorship, positive energy, and neverending patience throughout my graduate career. I am grateful to him for taking such an active interest in my development as a researcher and for providing consistent encouragement throughout the past several years. I have also had the good fortune to work with a fantastic group of collaborators including Adel Bouchareb, Rafik Chemam,Reda Attallah. I am very grateful to each one of them for a series of educational and fun research experiences from which I learned a great deal. Of course, I cannot understate the importance of the constant support that I have received from my family during graduate school and, indeed, throughout my entire life. To my mother, my father , my sisters. I am grateful for a lifetime of love and encouragement

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Glossary

- **Fuzzy spaces** :Fuzzy spaces are finite-dimensional approximations to the algebra of functions on continuous manifolds which preserve the isometries and (super)symmetries of the underlying manifolds. Another very important motivation lies in the fact that string theory suggests that spacetime may be fuzzy and noncommutative at its fundamental level.
- **Quantum gravity**:Quantum gravity is a theory that can explain well-studied natural phenomena with high energies and areas of great gravity that cannot be neglected. In this theory, we try to combine the principles of quantum mechanics with general relativity. The two most important theories in quantum gravity are string theory and loop quantum gravity.
- **Gauge theory** :The most famous prototype is the Yang-Mills theory. The two main symmetries playing a major role here, besides local gauge invariance, are supersymmetry and conformal invariance.
- **The gauge/gravity duality**:It creates new links between quantum theory and gravity. It has led to new concepts in mathematics and physics and provided new tools for solving problems in many areas of theoretical physics.
- **M-(atrix) theory**:is a quantum mechanical model proposed by Tom Banks, Willy Fischler, Stephen Shenker, and Leonard Susskind (BFSS) in 1997.
- **the Killing vector**:A Killing vector field (often called a Killing field), named after Wilhelm Killing, is a vector field on a Riemannian manifold (or pseudo-Riemannian manifold) that preserves the metric. Killing fields are the infinitesimal generators of isometries.
- **the structure constants** or structure coefficients of an algebra over a field are used to explicitly specify the product of two basis vectors in the algebra as a linear combination.
- **primary field**:primary field is a local operaor in conformal field theory wich is anneleted by part of the conformal algebra .

  The conformal group: is the group consisting of globally defined, invertible, and finite conformal transformations (or, more concretely, conformal diffeomorphisms)

  the conformal algebra is the Lie algebra corresponding to the conformal group.

Introduction

Einstein's gravity (GR) and the world of elementary particles (QFT) have been incredibly successful in explaining most observed phenomena, this leads some to believe these theories might be pieces of a single grand unified theory explaining everything [1] . However, directly applying quantum mechanics to gravity seems to create a theory with mathematical problems, not renormalizable [2]. Conventional perturbative methods have not been able to quantify gravity. However, by abandoning the concept that particles are pointlike. The first person to try to describe such a quantum space carefully was von Neumann. He called his work pointless geometry because the Heisenberg uncertainty principle of quantum physics makes the idea of a point in a quantum phase space useless. It was this that led to the idea of Von Neumann algebras and the start of non-commutative geometry [3].In the absence of a complete theory of quantum gravitation, the non-commutative case is the quasi-classical regime of any quantum field theory [4] [5].

Quantum gravity can be incorporated into a consistent quantum theory by abandoning the concept that particles are pointlike and assuming that the fundamental objects in the theory are strings or one-dimensional extended objects [6]. String theory is the only physical theory that can explain stringed quantum phenomena, which occur at distances and dimensions close to Planck's length and are described by a quantum mechanic in which strings compensate primary particles. It can be closed or open, whose length is denoted by the symbol l s and is approximately 10^{-33} [141]. During the late 1960s, [8] string theory was developed as a means of organizing and explaining the observable spectrum of hadrons and their interactions. They identified a massless spin-two particle that does not exist in the hadronic cosmos. Scherk and Schwarz interpreted it as the graviton, or gravity's field quantum, in 1974. This stringy graviton interacts with the covariance laws of GR at low energies. String theory became a candidate for a quantum theory of gravity with this realization [86]. It is the most successful candidate for a quantum gravity theory. So far, the four essential books in string theory. [10], [11], [12] [8].

We can use string theory to quantize small oscillations in space-time around certain backgrounds. However, we have yet to determine the background itself through non-perturbative means. To tackle this problem, one possible approach is to study matrix models, which are suggested as non-perturbative versions of string theory. The BFSS matrix model [1](proposed in 1997) is a specific quantum approach using matrices to describe particles. A prominent example is the IKKT model [3] , inspired by the 10-dimensional super Yang-Mills theory to define type IIB superstring theory in ten dimensions in a non-perturbative manner [18].

Yang-Mills matrix models are quantum field theories that offer a non-perturbative approach to studying gauge theories. They represent gauge fields as matrices and provide insights into strong coupling regimes that are challenging to analyze using traditional perturbative methods. Many models that exhibit emergent phenomena can be formulated as matrix models [3].

String theory demonstrates several instances of dualities, where a physical theory may have multiple equivalent formulations known as being "dual" to each other. This concept was significantly advanced in 1997 by J. Maldacena's articulation of his conjecture [134], is often referred to as the AdS/CFT correspondence.

The AdS/CFT correspondence, or more generally, the gauge/gravity duality, i.e., the the emergence of gravity from gauge theory refers to the duality between string theory in anti-de Sitter space-time (AdS) and a conformal field theory (CFT) on the boundary of this space. It demonstrates the connection between gravity and gauge theory; moreover, this conjectured correspondence is the direct manifestation of the holographic principle [41], [84], was first proposed in A$dS^5 \times S^5$ and $\mathcal{N} = 4$ Super Yang-Mills, has been used to study black hole, QFT, and QG.

In this thesis, we study the case $d = 1$ and construct a consistent AdS^2/CFT_1 correspondence; this is a duality/correspondence between the dAFF conformal quantum mechanics (QM) on the boundary (which is only quasi-conformal in the sense that there is neither an $SO(1,2)$-invariant vacuum state nor there are strictly speaking primary operators), and between the non-commutative geometry ofAdS^2_θ. In the bulk (which is only quasi-AdS) in the sense of being only asymptotically AdS^2. $\mathrm{AdS}^2/\mathrm{CFT}_1$ [72]correspondence it is the most mysterious case among all examples of the AdS/CFT. On the CFT side, this is given by dAFF conformal quantum mechanics. However, this conformal quantum mechanics is only quasi-conformal because there is neither an $SO(1,2)$ invariant quantum vacuum state nor primary operators in the strict sense. However, the bulk correlators are correctly reproduced by appropriately defined boundary quantum fields, as argued in [22]

The main argument underlying this QM/NCG duality consists of the following main observations: Both non-commutative AdS^2_θ and the dAFF conformal quantum mechanics enjoy the same symmetry structure given by the group $SO(1,2)$ and the asymptotically AdS^2_θ is an AdS^2 space-time, i.e., it has the same boundary, and the boundary correlation functions computed using the quasi-primary operators reproduce the bulk AdS^2 correlation functions [91].

One of the most important motivations for studying this case is its appearance on the event horizon of black holess [73], [91] , where quantum mechanics meets relativity [8], [20]. We begin by quantizing these spaces to obtain the non-commutative space $\mathbb{S}^2_N$ [87] and the non-commutative pseudo-sphere AdS^2_θ [84] , then We write its Yang-Mills matrix models corresponding To know the quantum gravitational fluctuations Who controls it from cubic Myers terms. Finally, let us discuss Their phase transition and emergent geometry, which the matrix model characterizes.

The introductory chapter provides an overview of gauge/gravity duality, focusing on the AdS/CFT correspondence. Maximally symmetric spaces in two dimensions, including the sphere S^2 and the pseudo-sphere (such as AdS^2) are explored in Chapter 2, along with the symmetry groups $su(2)$ and $su(1,1)$ and their representations. We establish a solid theoretical framework for our subsequent analysis. Chapter 3 delves into Conformal Field Theory (CFT) fundamentals, emphasizing the conformal group, conformal transformations, conserved currents, correlation functions, and the state-operator correspondence. The chapter also covers AdS space-time and its scalar fields; in chapter 4, We discussed the non-commutative $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$ and presented the phase structure of the corresponding Yang-Mills matrix models. Chapter 5, we apply this theory to the information loss of black holes in two dimensions and provide the framework for its resolution; this is done by constructing a non-commutative black hole in the two-dimensional Jackiw-Teitelboim dilaton gravity, this is done by adding the dilation field that distinguishes them. Which presents a proposal to solve the dilemma of information loss in the hole.

Finally, we summarize the essential findings and contributions of this thesis. We discuss the implications of our results for understanding matrix gauge/gravity duality and the phase structure of non-commutative $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$. Moreover, we identify potential avenues for future research in this intriguing field. Overall, this thesis contributes to the advancement of knowledge in gauge/gravity duality and sheds light on the phase structure of the non-commutative $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$, offering valuable insights into the interplay between quantum field theorie and gravity in two-dimensional space-time. The future question is whether this can be how to generalize this information loss paradox.

Chapter 1

Introduction to the gauge/gravity duality

1.1 AdS/CFT –The Overview

AdS/CFT correspondence [6] is based on the idea that a :"type IIB superstring theory where the strings propagate on an $AdS_5 \times S^5$ background is dual to a highly symmetric N = 4 super Yang-Mills theory in the large N limit". AdS stands for anti-de Sitter space, and CFT for conformal field theory. In this conjecture, quantum gravity can be equivalent to quantum field theory in fewer dimensions; more precisely, local CFT in d dimensions can be equivalent to gravity in Anti-de-Sitter space in d+1 dimensions. The key features of the AdS/CFT duality are the following:

- There is a mapping between a quantum theory of gravity (string theory) and an ordinary (non-gravitational) quantum field theory.
- Sam Hilbert space $\mathcal{H}_{CFT} = \mathcal{H}_{AdS-QG}$ [43]
- Same symmetry group $SO(1,2)$, a natural relation between observation
- The two theories have the same amount of degrees of freedom per unit volume [159], Any theory of quantum gravity in (d+1)-dimensional spacetime must be holographic; microscopic degrees of freedom are d-dimensional. [39]

  In a more formal way, the $\text{AdS}^{d+1}/\text{CFT}_d$ correspondence states that "the CFT_d generating functional with source $J = \phi_0$ is equal to the path integral on the gravity side evaluated over a bulk field, which has the value ϕ_0 at the boundary of AdS^{d+1}"

$$Z_{str}\left[\phi|_{\partial AdS} = J\right] = Z_{CFT}[J] \tag{1.1.1}$$

This duality relates certain gauge theories to gravitational theories in higher dimensions, providing a powerful tool for investigating the behavior of strongly coupled gauge theories; this allows, for example, the holographic description of a quantum black hole and the calculation of the corresponding Hawking radiation. This duality allows us to describe one side (the more difficult), gravity or the standard field, in terms of the other (the more accessible). The gauge theory is a theory that is wholly defined in quantum mechanics, and even more so, it is non-perturbatively defined by a lattice. So the gauge/gravity duality gives a complete non-perturbative quantum definition of quantum gravity, which is required of any theory of quantum gravity.

This is a general view; in this thesis, we specialized in studyingAdS^2/CFT_1,because, according to the proposal put forward in [91], the near-horizon classical geometry of a Reissner-Nordstrom black hole is given by the noncommutative $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$ where the CFT_1 theory at the boundary of the noncommutative AdS^2_θ is postulated to be given by the dAFF conformal quantum mechanics, i.e. $\mathrm{CFT}_1 \equiv \mathrm{QM}$.This is then a correspondence or duality between quantum mechanics (QM) on the boundary and noncommutative geometry (NCG) in the bulk which provides a concrete model for the $\mathrm{AdS}^{d+1}/\mathrm{CFT}_d$ correspondence [134] in one dimension (see figure 1). The main argument underlying this QM/NCG duality consists in the following main observations:

1. Both noncommutative AdS^2_θ and the dAFF conformal quantum mechanics enjoy the same symmetry structure given by the group $SO(1,2)$. However, dAFF conformal quantum mechanics is only quasi-conformal in the sense that there is neither an invariant vacuum state nor strictly speaking primary operators [130], [35]. Analogously, noncommutative AdS^2_θ is only quasi-AdS as it approaches AdS^2 only at large distances (commutative limit).

2. Asymptotically AdS^2_θ is an AdS^2 spacetime, i.e. it has the same boundary [84]. And furthermore the algebra of quasi-primary operators on the boundary (which defines in the same time the geometry of the boundary and the dAFF quantum mechanics) is in some sense a subalgebra of the operator algebra of noncommutative AdS^2_θ [91].

3. Metrically the Laplacian operator on the noncommutative AdS^2_θ shares the same spectrum as the Laplacian operator on the commutative AdS^2 spacetime [62]. The boundary correlation functions computed using the quasi-primary operators reproduces the bulk AdS^2 correlation functions [35].

Thus here, noncommutative geometry provides the fundamental mathematical structure for the eluding $\mathrm{AdS}^2/\mathrm{CFT}_1$ correspondence [72] [38]

However in general, noncommutative geometry [36] provides a description for classical gravity while the corresponding IKKT-type Yang-Mills matrix models [160, 161] provide a proper description for a quantum theory gravity.

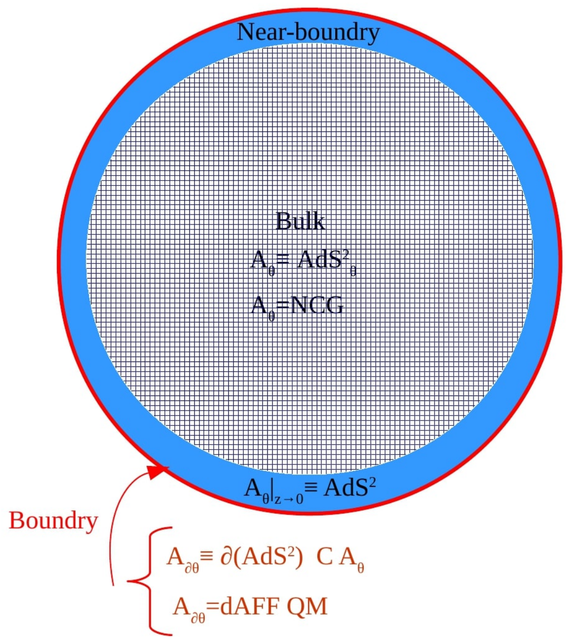

Figure 1.1: The QM/NCG correspondence: The bulk is given by the algebra associated with noncommutative AdS^2_θ while the boundary is given by a subalgebra thereof. The behavior near-boundary is precisely that of commutative AdS^2.

1.2 the Holography principle

One of the most crucial aspects of the AdS/CFT duality is that the two theories have an equal number of degrees of freedom per unit volume [159] This, according to 't Hooft (1994) [136], involves a concept known as holography , which implies that the number of fundamental degrees of freedom is related to the area of surfaces in spacetime [146], which allows for the boundary projection of all physical degrees of freedom. [133].

The holographic principle, which is at the core of the famous AdS/CFT correspondence, provides a partial explanation of how a higher-dimensional gravity theory can possess an equivalent number of degrees of freedom of information, and entropy as a lower-dimensional quantum field theory [134].

For a clearer understanding of this idea, we consider the example of AdS_{d+1}/CFT_d.the metric of AdS_{d+1} in poincar coordinates given by

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + dx_\mu dx^\mu\right) \tag{1.2.1}$$

As previously explained, the conformal boundary $z = 0$ is a Minkowski spacetime. We usually regularize the boundary as [145]:

1. **UV regulator**:By putting in a small epsilon $z = \epsilon$ (lattice spacing)
2. **IR regulator**: By placing it inside a spatial box with dimensions of R

We need to know the total number of degrees of freedom contained in the box. First, we have a lattice with lattice spacing ϵ. The dimension of the lattice is d, but we fixed the timing,i.e., the dimension is $d-1$. The formula for the box's number of cells is $\left(\frac{R}{\epsilon}\right)^{d-1}$ (we raise it for the dimension calculation).

By definition [140], the number of degrees of freedom per lattice site is equal to the central charge, denoted by c_{QFT}, and therefore the formula for the box's total degree of freedom is provided by [139]

$$N_{\text{QFT}} = \left(\frac{R}{\epsilon}\right)^{d-1} c_{\text{QFT}}. \tag{1.2.2}$$

this is on the one hand, from CFT_d, Let us calculate the degrees of freedom of the AdS_{d+1}T. AdS_{d+1}The amount of degrees of freedom within a particular region equals the maximal entropy, determined via the Bekenstein-Hawking formula cit :

$$N_{\text{AdS}} = \frac{A_H}{4G_N}. \tag{1.2.3}$$

- G_N is Newton constant is represented by $G_N = l_P^{d-1} = 1/M_P^{d-1}$

- A_H is the area of the region at boundary $z = \epsilon$, where the metric of AdS_{d+1} become

$$ds_d^2 = \frac{L^2}{\epsilon^2} dx_\mu dx^\mu, \sqrt{g} = \left(\frac{L}{\epsilon}\right)^{d-1} \tag{1.2.4}$$

Now, we will calculate this area using the following method:

$$A_H = \int_{z=\epsilon} d^{d-1}x\sqrt{g} = \left(\frac{L}{\epsilon}\right)^{d-1} \int d^{d-1}x \tag{1.2.5}$$

The integral on 3.1.32 is the volume of the box

$$\int_{z=\epsilon} d^{d-1}x = R^{d-1}. \tag{1.2.6}$$

Then, the AdS_{d+1} space has the following number of degrees of freedom :

$$N_{\text{AdS}} = \frac{1}{4}\left(\frac{R}{\epsilon}\right)^{d-1}\left(\frac{L}{l_P}\right)^{d-1} \tag{1.2.7}$$

By using the conjecture AdS/CFT, we compare (3.2) and (1.2.7). These numbers scale the same, and we obtain the central charge.

$$c_{\text{QFT}} = \frac{1}{4}\left(\frac{L}{l_P}\right)^{d-1}. \tag{1.2.8}$$

Hence, semi-classical gravity corresponding to $L >> l_P$ is dual to a CFT with a sizeable central charge, i.e., a large number of degrees of freedom; as a consequence, semi-classical gravity is dual, in this case, to a large N gauge theory.

In summarize:$N_{\text{Ad}S} \sim N_{\text{QFT}}$, which means that the entropy of the ads is equal to the entropy of cft

1.3 The two-point function

Two- and three-point functions act as a demanding obstacle course that a theory must pass to be considered valid. The AdS/CFT conjecture.We begin by recalling the Euclidean AdS_{d+1} presented by :

$$S_{\text{grav}} = -\frac{\eta}{2}\int d^{d+1}x\left[\sqrt{g}g^{MN}\partial_M\phi\partial_N\phi + \sqrt{g}m^2\phi^2\right] \tag{1.3.1}$$

We calculate Euler-Lagrange equations of motion, and we substitute them in the equation of motion (action) above to acquire the ”on-shell action” and variant as follows :

$$\begin{aligned} S_{\text{grav}}^{\text{on-shell}} &= \frac{\eta}{2}\int d^dx\sqrt{g}\left(\phi g^{zz}\partial_z\phi\right)_{z=\epsilon} \\ \delta S_{\text{grav}}^{\text{on-shell}} &= \eta\int d^dx\sqrt{g}\left(\delta\phi g^{zz}\partial_z\phi\right)_{z=\epsilon} \end{aligned} \tag{1.3.2}$$

Defining a canonical momentum regarding z yields the following result:

$$\Pi = -\frac{\delta\mathcal{L}}{\delta\partial_z\phi(z,x)} = \frac{\delta S_{\text{grav}}^{\text{on-shell}}}{\delta\phi(z,x)} = \eta\sqrt{g}g^{zz}\partial_z\phi. \tag{1.3.3}$$

Consequently, the on-shell action takes the form:

$$S_{\text{grav}}^{\text{on-shell}} = \frac{1}{2}\int_{z=\epsilon} d^dx\Pi(z,x)\phi(z,x) \tag{1.3.4}$$

Working in momentum space (in x remain z) ,makes it possible to rewrite $S^{\text{on-shell}}$ as:

$$\begin{aligned} S_{\text{grav}}^{\text{on-shell}} &= \frac{1}{2}\int_{z=\epsilon} d^dx\Pi(z,x)\phi(z,x) \\ &= \frac{1}{2}\int_{z=\epsilon} \frac{d^dk}{(2\pi)^d}\pi_{-k}(z)f_k(z). \end{aligned} \tag{1.3.5}$$

The function $f_k(z)$ obeys the equation of motion represented by (4.8.3). Another function $g_k(z)$ is defined as $f_k(z) = z^{d/2}g_k(z)$ and then substituted into equation (4.8.3) to get the differential equation.

$$z^2\partial_z^2 g_k + z\partial_z g_k - \left(\nu^2 + k^2z^2\right)g_k = 0, \nu^2 = \frac{d^2}{4} + m^2L^2 = \left(\Delta - \frac{d}{2}\right)^2. \tag{1.3.6}$$

It is is the modified Bessel equation [1]. The solutions are therefore the modified Bessel functions $g_k(z) = I_{\pm\nu}(kz)$, viz $f_k(z) = z^{d/2}I_{\pm\nu}(kz)$. Recall the small and large z limits of the modified Bessel functions

$$\begin{aligned} I_{\pm\nu}(kz) &\sim \frac{1}{\Gamma(1\pm\nu)}\left(\frac{kz}{2}\right)^{\pm\nu}, z \longrightarrow 0. \\ I_{\pm\nu}(kz) &\sim \frac{e^{kz}}{\sqrt{2\pi kz}}, z \longrightarrow \infty. \end{aligned} \tag{1.3.7}$$

The two distinct solutions of equation (4.8.3) are considered to be provided by:

[1] Bessel equation It is known for everything; for example, we know how to behave in large and small limits

1. The first solution is proportional to the negative solution

$$z \longrightarrow 0, \phi_1(z,k) = \Gamma(1-\nu)\left(\frac{k}{2}\right)^{\nu} z^{d/2} I_{-\nu}(kz) \tag{1.3.8}$$

we substitute $\nu = \left(\Delta - \frac{d}{2}\right)$

$$\phi_1(z,k) \longrightarrow z^{d-\Delta} \tag{1.3.9}$$

2. The second solution is proportional to the postive solution

$$\phi_2(z,k) = \Gamma(1+\nu)\left(\frac{k}{2}\right)^{-\nu} z^{d/2} I_{\nu}(kz) \longrightarrow z^{\Delta}, z \longrightarrow 0 \tag{1.3.10}$$

In summary, we know what the bulk independent modes are; we knew this was Bessel because they solved the Bessel equation. We then fix the normalization by imposing a behavior limit because we know how they should move in the boundary dimension ($z^{d-\Delta}$,z^{Δ}) Δ is the scaling dimension.

We wrote the particular solution, and now we write the general solution, which is linear composition:

$$\begin{aligned} f_k(z) &= A(k)\phi_1(z,k) + B(k)\phi_2(z,k) \\ &= z^{d/2}\left[\Gamma(1-\nu)\left(\frac{k}{2}\right)^{\nu} A(k) I_{-\nu}(kz) + \Gamma(1+\nu)\left(\frac{k}{2}\right)^{-\nu} B(k) I_{\nu}(kz)\right] \end{aligned} \tag{1.3.11}$$

The small z limit is correct, i.e., it behaves as expected at the boundary (This concurs with the one that was previously discovered in (3.2.23)

Now, we calculate the large z of this solution because $A(k)$ and $B(k)$ are still unknown. We know that the Bessel function behaves as $\frac{e^{kz}}{\sqrt{2\pi kz}}$ at the large limit,We substitute this in (2.1.13) and we get

$$f_k(z) = z^{d/2}\frac{e^{kz}}{\sqrt{2\pi kz}}\left[\Gamma(1-\nu)\left(\frac{k}{2}\right)^{\nu} A(k) + \Gamma(1+\nu)\left(\frac{k}{2}\right)^{-\nu} B(k)\right], z \longrightarrow \infty. \tag{1.3.12}$$

The term within the brackets must vanish in order for this to diverge in the limit $zlongrightarrowinfty$; otherwise, it becomes non-physical.IR limit regular field $zlongrightarrowinfty$ iff:

$$\Gamma(1-\nu)\left(\frac{k}{2}\right)^{\nu} A(k) + \Gamma(1+\nu)\left(\frac{k}{2}\right)^{-\nu} B(k) = 0 \tag{1.3.13}$$

$$\frac{B(k)}{A(k)} = -\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu}. \tag{1.3.14}$$

As a result, the field in large z is zero, provaided 1.3.14 . Now, in the limit when z is small, the field is located on the boundary and acts as we said before.

$$f_k(z) = A(k)z^{d-\Delta} + B(k)z^{\Delta}, z \longrightarrow 0. \tag{1.3.15}$$

Instantly, we find the corresponding field's small z limit by $\pi = \eta\sqrt{g}g^{zz}\partial_z\phi$.

We perform the Fourier transform of ϕ ; the rest are simply numbers we get from metric. After this, we extract the Fourier transform of π, and then we use the behavior of the Fourier transform of the field ϕ at the boundary to extract the behavior of π on the boundary. And, since the action on shelle is evaluated at $z = \epsilon$ (boundary), we can simply compute π at the bondary as follows:

$$\pi_{-k}(z) = \eta L^{d-1}\left[(d-\Delta)A(-k)z^{-\Delta} + \Delta B(-k)z^{\Delta-d}\right], z \longrightarrow 0. \tag{1.3.16}$$

Our goal in this section is to get on-shell action because it is the primary for the compute one-point function, and the last is the basic for the compute two-point function to demonstrate that ϕ is actually a conformal field with a scale dimension Δ, remember that:

$$S^{\text{on-shell}}_{\text{grav}} = \frac{1}{2}\int_{z=\epsilon}\frac{d^dk}{(2\pi)^d}\pi_{-k}(z)f_k(z). \tag{1.3.17}$$

We substitute ϕ and π.We get on-shell action as follows

$$S^{\text{on-shell}}_{\text{grav}} = \frac{\eta}{2}L^{d-1}\int\frac{d^dk}{(2\pi)^d}\left((d-\Delta)A(k)A(-k)\epsilon^{-2\nu} + dA(k)B(-k)\right).) \tag{1.3.18}$$

Because epsilon is small, there are diverging in the initial term, which leads to a necessary renormalization of the action; it must be local because the first term lives on the boundary and quadratic because of the way the first term acts, where it is given by:

$$S_{\text{ct}} = \frac{\eta}{2}\eta_1\int\sqrt{\gamma}d^dx\phi^2. \tag{1.3.19}$$

The symbol γ represents the induced metric on the border, namely gamma.

$$ds^2 = \gamma_{\mu\nu}dx_\mu dx_\mu = L^2dx_\mu dx_\mu/\epsilon^2. \tag{1.3.20}$$

We replace the field, and we do the Fourier transform; we get :

$$S_{\text{ct}} = \frac{\eta}{2}\eta_1 L^d\int\frac{d^dk}{(2\pi)^d}\left(A(k)A(-k)\epsilon^{-2\nu} + 2A(k)B(-k)\right). \tag{1.3.21}$$

Let us change the counterterm action's coefficient to eliminate the leading divergence. As soon as possible, confirm that the following counterterm action is required:

$$(d-\Delta)+\eta_1 L=0 \Longrightarrow \eta_1=-(d-\Delta)/L \tag{1.3.22}$$

$S_{\text{grav}}^{\text{renor}}$ given by :

$$\begin{aligned} S_{\text{grav}}^{\text{renor}} &= S_{\text{grav}}^{\text{on- shell}}+S_{\text{ct}} \\ &= \frac{\eta}{2}L^{d-1}(2\Delta-d)\int \frac{d^dk}{(2\pi)^d}A(k)B(-k) \\ &= -\frac{\eta}{2}L^{d-1}(2\Delta-d)\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\int \frac{d^dk}{(2\pi)^d}A(k)\left(\frac{k}{2}\right)^{2\nu}A(-k) \\ &= -\frac{\eta}{2}L^{d-1}(2\Delta-d)\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\int \frac{d^dk}{(2\pi)^d}\varphi(k)\left(\frac{k}{2}\right)^{2\nu}\varphi(-k), \end{aligned} \tag{1.3.23}$$

In the first step, we replace B with the function of A (1.3.14) because the field must be regular at infinity, and in the second step, we used the limite the ϕ

$$\varphi(x)=\lim_{z\longrightarrow 0} z^{\Delta-d}\phi(z,x)=z^{\Delta-d}(A(k)z^{d-\Delta}+B(k)z^{\Delta})=A(x) \tag{1.3.24}$$

$$B(k)\longrightarrow 0, \text{because } 2\Delta>d \tag{1.3.25}$$

Moreover, through the renormalized action, we get "The one-point correlator" on the boundary as:

$$\langle\mathcal{O}(k)\rangle_{\varphi}=(2\pi)^d\frac{\delta S_{\text{grav}}^{\text{renor}}\,[\phi\longrightarrow\phi_0]}{\delta\varphi(-k)}. \tag{1.3.26}$$

We substitute $S_{\text{grav}}^{\text{renor}}$,we find:

$$\langle\mathcal{O}(k)\rangle_{\varphi}=-\eta L^{d-1}(2\Delta-d)\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu}\varphi(k). \tag{1.3.27}$$

Then, using the formula, the two-point function ($\langle\mathcal{O}(x)\mathcal{O}(0)\rangle$) is expressed in terms of the one-point function.

$$G_E(k)=\frac{\langle\mathcal{O}(k)\rangle_{\varphi}}{\varphi(k)}=-\eta L^{d-1}(2\Delta-d)\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu}. \tag{1.3.28}$$

remember that $\nu=\Delta-\frac{d}{2}$,the $\langle\mathcal{O}(x)\mathcal{O}(0)\rangle$ in position space(Fourier transform) given by :

$$\langle\mathcal{O}(x)\mathcal{O}(0)\rangle=\frac{2\nu L^{d-1}\eta}{\pi^{d/2}}\frac{\Gamma\left(\frac{d}{2}+\nu\right)}{\Gamma(-\nu)}\frac{1}{|x|^{2\Delta}}. \tag{1.3.29}$$

we observe that The behavior $\langle\mathcal{O}(x)\mathcal{O}(0)\rangle\sim|x|^{2\Delta}$, and this is how a "conformal field of scaling dimension" behaves correctly.

$\boxed{\mathcal{O}(x) \textbf{ is a "conformal field of scaling dimension" } \Delta}$

1.4 Non-Commutative AdS_2/CFT_1

The reconciliation of GR and QM requires a radical change in the mathematical concepts of general relativity, which consequently changes the concepts of classical geometry. The combination of general relativity and quantum mechanics suggests that space-time cannot be an ordinary differential variety [131]; we find natural generalizations of differential geometry to be non-commutative differential geometry, in the sense of Connes. There is a generally held belief that the quasiclassical regime of quantum gravity should appear as a quantum field theory on a non-commutative background. [59],

In this thesis, we want to study aspects of the $\mathrm{AdS}^2/\mathrm{CFT}_1$ correspondence in a non-commutative setting, namely when the geometry on the gravity side of the correspondence is replaced by the non-commutative version of (Euclidean) AdS^2. Adding some quantum gravitational corrections should correspond to making the AdS^2 space non-commutative.

In this chapter, we start by briefly reviewing some aspects of the non-commutative differential geometry applied in physics following.

Non-commutative geometry, developed by Alan Connes [49], is a type of algebraic geometry that studies non-commutative elements. Where algebraic functions of phase space are replaced by algebraic matrices and replace the ordinary local point-wise multiplication of fields with the non-local Moyal-Weyl star product. It is a generalization of classical differential geometry.

1.4.1 Symplectic manifolds

Symplectic manifolds are important to geometric quantization, noncommutative geometry, phase space formulation of quantum mechanics, classical dynamics and matrix models. The term ”symplectic” is a Greek term meaning ”complex” introduced by Weyl in 1939 in order to change the name of the ”line complex groups” to ”symplectic groups”. Symplectic structures are intimately connected to complex structures, Poisson structures and topology.

- (1) It is a 2 -form, i.e. $\omega \in \Omega^2(M)$. Hence ω is antisymmetric and linear, viz

$$\omega(X,Y) = -\omega(Y,X), \omega(X, fY + gZ) = f\omega(X,Y) + g\omega(X,Z).$$

- (2) It is non-degenerate, i.e.

$$\omega(X,Y) = 0, \forall X \in \Omega^1(M) \Longleftrightarrow Y = 0.$$

- (3) It is closed, i.e.

$$d\omega = 0.$$

The symplectic form ω and the antisymmetric tensor $(\theta^{-1})_{ij}$ play in symplectic geometry the same role played in Riemannian geometry by the scalar product $\langle ,, rangleandthemetrictensorg_{ij}$.

Indeed, on a Riemannian manifold M the scalar product $\langle,,ranglesimilarlyto\omega$ is a map which takes as input two vectors and produces as output a function, viz

$$\langle,\rangle : \mathcal{V}(M) \times \mathcal{V}(M) \longrightarrow \mathbf{C}^{\infty}.$$

This map is i) symmetric and linear and ii) positive definite, i.e. non-degenerate. This scalar product can be expressed in local coordinates x^i in terms of the metric tensor g_{ij} which is symmetric and invertible as follows

$$\langle X, Y\rangle = \sum_{i,j} g_{ij} X^i Y^j.$$

However,there are several differences between symplectic geometry and Riemannian geometry among them we find the following results.

- In Riemannian geometry the metric tensor allows us to measure lengths of lines and angles between lines. In contrast, in symplectic geometry the symplectic form allows us to measure areas of two-dimensional surfaces and their orientations. In some sense the basic object in Riemannian geometry is a point particle whereas in symplectic geometry it is a string.
- In Riemannian geometry the curves of shortest length are called geodesics whereas in symplectic geometry the surfaces of minimal area are called pseudoholomorphic curves.
- A symplectic space is necessarily i) even dimensional and ii) oriented in contrast to the fact that any smooth manifold is Riemannian.

Indeed, from requirements (1) and (2) the matrix A with components (in the local coordinates x^i) $A_{ij} = \omega\left(\partial_i, \partial_j\right)$ satisfies $A^T = -A$ and $\det A \neq 0$. This implies immediately that the dimension of the space must be even, i.e. $d = 2n$.

1.4.2 The Moyal-Weyl plane

The Moyal-Weyl space $\mathbb{R}^p_{\lambda\theta}$ is a quantization of the Poisson/symplectic manifold $\mathbb{R}^p_\theta$ which is given by the space $\mathbb{R}^p$ together with a symplectic structure given by the usual Poisson brackets, viz

$$\left\{x^a, x^b\right\} = \theta^{ab}.$$

Dirac canonical quantization is given precisley by the replacements

$$x^a \longrightarrow \hat{x}^a, i\{,\} \longrightarrow \frac{1}{\lambda}[,] \Rightarrow \mathbb{R}^p \longrightarrow \mathbb{R}^p_{\lambda\theta}.$$

The non-commutativity parameter λ plays therefore the role of $\hbar$. We define next the Weyl map $\mathcal{F}$ as the map from the algebra $\mathcal{A}$ of functions on $\mathcal{M} = \mathbb{R}^p$ given by $\mathcal{A} = \mathcal{C}(\mathcal{M})$ to the algebra $\mathcal{A}_\theta$ of functions on $\mathcal{M}_\theta = \mathbb{R}^p_{\lambda\theta}$ which takes the coordinate functions x^a to the coordinate operators $\hat{x}^a$, viz [56]

$$F : \mathcal{A} = \mathcal{C}(\mathcal{M}) \longrightarrow \mathcal{A}_\theta$$
$$x^a \longrightarrow \mathcal{F}(x^a) = \hat{x}^a.$$

There is a convenient way to map the algebra $\mathcal{A}_\theta$ to the algebra $\mathcal{A}_0$ generated by the coordinate operator $\hat{x}^a$, via what so-called Weyl map.

1.4.3 The Star-Product

The noncommutative star product of Groenewold is the foundation of deformation (phase-space) quantization,as we know There are three main alternative paths to quantization. The standard formulation that uses operators in Hilbert space, the path integral formulation and the phase-space formulation based on Wigner's quasi probability distribution function in phase space (WF) and Weyl's correspondence between quantum operators and ordinary c-numbers phase-space functions, that relies on the star-product, that was fully understood by Groenewold together with Moyal, which maps products of operators that act in some Hilbert space to product of functions on the phase space, giving an alternative procedure to achieve the quantization [56] [54].when calculated, it means that we controlled the commutative limite . star product is an essential element in understanding the continuum limit. [55], [56].

The star product $*$ is precisely given by the Groenewold-Moyal-Weyl star product 3.4 defined by

$$f_1 * f_2(x) = \exp\left(\frac{i\lambda}{2}\theta^{ab}\frac{\partial}{\partial\xi^a}\frac{\partial}{\partial\eta^b}\right) f_1(x+\xi)f_2(x+\eta)\bigg|_{\xi=\eta=0}.$$

The compatibility condition between the Poisson structure θ and the star product is obviously given (think of $\mathbb{R}^p_{\lambda\theta}$ as a phase space)

$$f_1 * f_2(x) = f_1 f_2 + \frac{i\lambda}{2}\{f_1, f_2\} + O\left(\lambda^2\right).$$

The commutative limit $\lambda \longrightarrow 0$ is precisely the semi-classical limit.

Chapter 2

the Maximally symmetric spaces in two dimensions,their symmetry

2.1 the Maximally symmetric spaces in two dimensions

The only maximally symmetric curved spaces in a given dimension d are the spheres S^d and the hyperbolic spaces H^d (in Euclidean signature) and the de Sitter spaces dS^d and the anti-de Sitter spaces AdS^d (in Lorentzian signature). A sphere is the Euclidean continuation of de Sitter spaces with constant positive scalar curvature. In contrast, hyperbolic spaces H^d are the Euclidean continuation of anti-de Sitter spaces with constant negative scalar curvature.

Maximally symmetric spaces are essential ingredients in quantum gravity theories and cosmological models. We refer to maximally symetric, the maximum number of linearly independent Killing vector fields, i.e., isometries or simply symmetries, which is homogenous and isotropic and contains the most significant number of symmetries. [46], [47], [91]

- Homogeneous: All points are equivalent.
- Isotropic: All directions are equivalent.
- Isometries: Diffeomorphisms which leave the metric invariant.

These spaces are maximally symmetric and therefore they admit $d(d+1)/2$ Killing vector fields representing isometries of the metric with symmetry groups $SO(d-1,2)$ (forAdS^d) $SO(d,1)$ (for H^d and dS^d) and $SO(d+1)$ (forS^d) .See [46] for more detail.

This thesis will use the Euclidean signature to study the two-dimensional case. In this situation, a pseudo-sphere H^2 represents the negative curvature and a sphere S^2 represents the positive curvature. Our main attention will be on the pseudo-sphere H^2, which is also thought of as a Euclidean version of AdS^2.

2.1.1 S^2 sphere

A sphere is a three-dimensional, round, symmetrical, geometric object. It is an equidistant set of a fixed space point representing a center. The distance between the center and any point on the sphere's surface is constant and equal to the radius of the sphere [61]

The sphere $\mathbf{S}^2$ is embedded in Euclidean space $\mathbb{R}^3$ by the quadratic relation:

$$\begin{aligned} &X_1^2 + X_2^2 + X_3^2 = R^2 \\ &X_1 = R\sin\theta\cos\phi, X_2 = R\sin\theta\sin\phi, X_3 = R\cos\theta \end{aligned} \tag{2.1.1}$$

The induced metric on the sphere S^2 is given by:

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu = \left(\eta^{ab}\partial_\mu X_a \partial_\nu X_b\right) dx^\mu dx^\nu \\ &= R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \end{aligned} \tag{2.1.2}$$

The group of symmetries leaving the sphere $\mathbf{S}^2$ invariant is $SO(3)$. Three rotations generate the sphere's SO(3) rotation group, where the generator responsible for the rotation is called "the angular momentum operators" L_a. The su(2) Lie algebra, which these generators satisfy, is given by(2.1.3) [67]: (as calculated in the next section)

$$[L_a, L_b] = i\epsilon_{abc}L_c. \tag{2.1.3}$$

2.1.2 AdS^2 spacetime

The anti-de Sitter spacetime AdS^2 (topology $\mathbb{R}^{d-1}\times\mathbb{S}^1$) is the one-sheeted hyperboloid,as embedded in $\mathbb{R}^{2,1}$ is given by the the quadric form ($\eta_{a,b} = (-1,-1,+1)$

embedded in $\mathbb{R}^{2,1}$,with metric $\eta = (-1,-1,+1)$ by the quadratic relation [62]

$$-X_1^2 - X_2^2 + X_3^2 = -R^2. \tag{2.1.4}$$

Recall that X_3 is spacelike and X_1 and X_2 are timelike. We induce two(cylindrical,conformal) coordinates (τ,σ). on this one-sheeted hyperboloid by the relations [62]

$$\begin{aligned} X_1 &= R\frac{\cos\tau}{\cos\sigma} \\ X_2 &= R\frac{\sin\tau}{\cos\sigma} \\ X_3 &= R\tan\sigma \end{aligned} \tag{2.1.5}$$

The interval for σ is within $]-\pi/2,\pi/2[$, and for τ, it's within $[0,2\pi]$. As a result, time exhibits periodic behavior, signifying the existence of enclosed timelike loops around the constant value of X_3. By extending τ to $[-\infty, \infty]$, we can transition to the universal cover. The metric is expressed in global coordinates as follows:

$$\begin{aligned}
ds^2 &= -dX_1^2 - dX_2^2 + dX_3^2 \\
&= R^2[-\frac{\sin^2\tau}{\cos^2\sigma}d\tau^2 - \frac{\sin^2\sigma\cos^2\tau}{\cos^4\sigma}d\sigma^2 - 2\frac{\sin\tau\sin\sigma\cos\tau}{\cos^3\sigma}d\tau d\sigma - \frac{\cos^2\tau}{\cos^2\sigma}d\tau^2 - \frac{\sin^2\sigma\sin^2\tau}{\cos^4\sigma}d\sigma^2 \\
&+ 2\frac{\sin\tau\sin\sigma\cos\tau}{\cos^3\sigma}d\tau d\sigma] + \frac{1}{\cos^4\sigma}d\sigma^2 \\
&= R^2[-\frac{(\sin^2\tau + \cos^2\tau)}{\cos^2\sigma}d\tau^2 - \frac{\sin^2\sigma(\sin^2\tau + \cos^2\tau)}{\cos^4\sigma}d\sigma^2] + \frac{1}{\cos^4\sigma}d\sigma^2] \\
&= R^2[-\frac{d\tau^2}{\cos^2\sigma} + \frac{\cos^2\sigma}{\cos^4\sigma}d\sigma^2] \\
&= \frac{R^2}{\cos^2\sigma}\left(-d\tau^2 + d\sigma^2\right)
\end{aligned} \tag{2.1.6}$$

Consequently, $\mathbf{AdS}^2$ can be visualized as a rectangular shape, having its starting point at $\tau = -\infty$ and its endpoint at $\tau = +\infty$, Positioned in $\sigma = 0$, the spatial infinities are positioned at $\sigma = \pm\pi/2$. Isometries group is denoted as $SO(2, 1)$.

2.1.3 The pseudo-sphere

the pseudo-sphere H^2 is actually the hyperbolic space which is defined as the upper sheet of the two-sheeted hyperboloid

It will be helpful and crucial in various situations to examine the Euclidean manifestation of AdS along with the Euclidean conformal group, and this is relevant to fields like quantum field theory and noncommutative geometry, where the formal conception of field and geometric quantization mostly adheres to the Euclidean framework. [91].There are theree pseudo-rotations which generate the SO(1, 2) pseudo-rotation group of the pseudo-sphere. These pseudo-rotations are translation, dilatation and special conformal transformation.

- We go from $\mathbf{AdS}^2$ to the pseudo-sphere $\mathbf{H}^2$ by the Wick rotation $X_1 \longrightarrow -iX_1$.
- The pseudo-sphere, which is given by $-X_0^2 + X_1^2 + X_2^2 = -R^2$, is therefore Euclidean anti-de Sitter space, i.e. $\mathbf{H}^2 = \mathbf{AdS}_E^2$.
- The pseudo-sphere $\mathbf{H}^2$ is locally obtained by the Wick rotation $\tau \longrightarrow -i\tau$. The cylindrical coordinates in this case are then defined by

$$X_0 = R\frac{\cosh\tau}{\cos\sigma}, X_1 = R\frac{\sinh\tau}{\cos\sigma}, X_2 = R\tan\sigma \tag{2.1.7}$$

The hyperbolic space known as (Euclidean) AdS^2 is characterized as the upper sheet of the two-sheeted hyperboloid $\mathbf{H}^2$ (a pseudosphere) embedded in $\mathbb{R}^{1,2}$ with metric $\eta = (-1, +1, +1)$.The transformation from the Lorentzian AdS^2 to this Euclidean configuration involves the substitutions $X_2 \longrightarrow -iX_2$ and $\tau \longrightarrow -i\tau$. These substitutions are outlined in detail in the source [91], which is given by :

$$-X_1^2 + X_2^2 + X_3^2 = -R^2. \tag{2.1.8}$$

The cylindrical coordinates, in this case, are then defined by [63] :

$$\begin{aligned} X_1 &= R\frac{\cosh\tau}{\cos\sigma} \\ X_2 &= R\frac{\sinh\tau}{\cos\sigma} \\ X_3 &= R\tan\sigma \end{aligned} \tag{2.1.9}$$

The metric is calculated in the same way as the previous one, and we get :

$$ds^2 = \frac{R^2}{\cos^2\sigma}\left(d\tau^2 + d\sigma^2\right). \tag{2.1.10}$$

the pseudo-rotation group is $SO(1,2)$

2.1.4 The Poincaré patch

Poincaré coordinates prove especially advantageous in examining the holographic dimensions of AdS/CFT. This stems from their ability to streamline the exploration of boundary-based field theories and their associated gravitational counterparts within the voluminous spacetime. These coordinates offer a sophisticated avenue for comprehending how AdS spacetime behaves near its boundary, thereby fostering linkages with the corresponding dual field theory existing on that boundary. The crux of their significance lies in their ability to bring to the forefront the d-dimensional Poincaré subgroup of the conformal group, as expounded in the source [63].

In the Poincaré patch the induced metric of Lorentzian $\mathbf{AdS}^2$ and Euclidean $\mathbf{H}^2$ is given by (with $x^0 = t, x^1 = z$)

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= \left(\eta^{ab}\partial_\mu X_a \partial_\nu X_b\right) dx^\mu dx^\nu \\ &= \frac{R^2}{z^2}\left(dz^2 - dt^2\right) = R^2\left(\frac{du^2}{u^2} - u^2dt^2\right), \mathbf{AdS}^2 \\ &= \frac{R^2}{z^2}\left(dz^2 + dt^2\right) = R^2\left(\frac{du^2}{u^2} + u^2dt^2\right), \mathbf{H}^2. \end{aligned} \tag{2.1.11}$$

For Lorentzian $\mathbf{AdS}^2$ we write the embedding :

$$
\begin{aligned}
X_0 &= \frac{R\cos\tau}{\cos\sigma} = -\frac{z}{2}\left(1+\frac{R^2-t^2}{z^2}\right)\\
X_1 &= \frac{R\sin\tau}{\cos\sigma} = -R\frac{t}{z}\\
X_2 &= R\tan\sigma = -\frac{z}{2}\left(1-\frac{R^2+t^2}{z^2}\right)
\end{aligned}
\tag{2.1.12}
$$

For Eculidient with the embedding [106], [63].

$$
\begin{aligned}
X_1 &= R\frac{\cosh\tau}{\cos\sigma} = -\frac{z}{2}\left(1+\frac{R^2+t^2}{z^2}\right)\\
X_2 &= R\frac{\sinh\tau}{\cos\sigma} = -R\frac{t}{z}\\
X_3 &= R\tan\sigma = -\frac{z}{2}\left(1-\frac{R^2-t^2}{z^2}\right)
\end{aligned}
\tag{2.1.13}
$$

- We can think of the coordinate $u = 1/z$ as the energy scale of the conformal field theory living on the boundary, which is foliated at $z = 1/u$.
- Alternatively, z and u can be thought of as a lattice spacing and an ultraviolet cutoff after Euclidean rotation. In terms of the embedding coordinates, we have

$$
u = \frac{1}{z} = \frac{X_1 - X_3}{R^2}. \tag{2.1.14}
$$

The conformal boundary in the Poincaré patch

- The conformal boundary in the Poincaré patch is located at $z = 0$.
- Indeed, we have
- The point $u = \infty$ or equivalently the point $z = 0$ is the conformal boundary (the spatial infinities $\sigma \longrightarrow \pm\pi/2$).
- The point $u = 0$ or equivalently $z = \infty$ is the horizon (the metric can be extended beyond this point).
- Indeed, in the Minkowski case the Poincaré coordinates are called a patch because they cover only half of the spacetime since the range of the radial Poincare coordinate z is $z \in [0, \infty[$.
- Obviously, Poincaré coordinates can be extended beyond the horizon. In the Euclidean case the horizon at $z = \infty$ shrinks to a point.
- In the case of higher anti-de Sitter spacetimes AdS^{d+1} the conformal boundary $z = 0$ is a Minkowski spacetime $\mathbf{M}^d$ with metric $\eta = (-1, +1, \ldots, +1)$.

2.2 Symmetry group :the su(2), su(1, 1)

Symmetry is essential in the description of physical phenomena [77] [76]. There is an external symmetry and an internal symmetry. External symmetries in spacetime include rotations, Lorentz, discrete, Poincare, and conformal symmetries, and another symmetry is Internal symmetry is symmetry within Hilbert space (for example, the $SU(3)$ group symmetry). Every symmetry involves preserving a physical quantity, resulting in conservation laws, and this is Noether's theorem [78], such as rotational symmetry, which preserves the angular momentum.

We study symmetries through group theory. [80], It is the mathematical basis for symmetry. Group theory is pivotal in elementary particle physics, quantum mechanics, quantum field theory, and superstring theory.

The following four natural axioms must all be satisfied by an operation * for a set G to be considered a group. [82] :

1. Closure:Any two components t_1 and t_2 of G are composed to form the expression $(g_1 * g_2)$ which is another element of G.
2. Associativity:It is necessary to have the formula $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$.
3. Identity:There is a component e in G such that $e * t = t * e = t$
4. Invertibility :There exists for every $g \in G$ an inverse $g^{-1} \in G$ such that $g * g^{-1} = e$

There are three kinds of groups:

- continuous $\longleftrightarrow$ discrete
- abelian when the composition law * is commutative $\longleftrightarrow$ non-abelian
- infinite $\longleftrightarrow$ finite

The amount of independent parameters needed to define or describe a general element g in group G is known as the group's dimension.

There is a type of group called "**lie groups**" (for the name of the person Lie), which is an essential type because it is both a group and a manifold indeed. Continuous groups with finite dimensions are called lie groups. The rotation G is a "lie group"; for instance, symbolized by the capital letters. A Lie algebra is the vector space tangent to the group at the identity, denoted by lowercase letters. In the same sense, it is generated by the lie group. Usually, dealing with linear algebra is easier because it is a vector space.

Lie algebra is represented by **representation theory** [79] [81]. As their name implies, they are matrix mathematical spaces that can be used to solve the fundamental commutation relations defined in Lie algebra. Consequently, it provides the defining representation R of SO(3) of the associated algebra. "The general linear group" $GL(V)$, also known as Aut(V), is made up of all bijective linear operators (automorphisms) acting in V, and V is a map from the group G to this group. [82]

We produce the group elements by taking the exponential map of the algebraic elements, $g = exp(T)$, provided that the unit exists.

The number of representations is unlimited, and when we take the exponential function, we will realistically utilize a particular representation of the group and its algebra as T.

2.3 the Rotations group $SO(3)$

In this section we follow the presentation of [81], [82].

We refer to three-dimensional Euclidean space as R^3. The Euclidean formula provides the distance between two points $\vec{x}$ and $\vec{x} + d\vec{x}$:

$$ds^2 = d\vec{x}^2 = dx_1^2 + dx_2^2 + dx_3^2 = dx^2 + dy^2 + dz^2 \tag{2.3.1}$$

Under the rotation, this distance is invariant (principle of relativity)

$$\begin{aligned} &\vec{x} \xrightarrow{R} \vec{x'} \\ &ds^2 \xrightarrow{R} ds^2 \end{aligned} \tag{2.3.2}$$

From this relationship, we extract that the requirement for R is given:

$$R \cdot R^T = R^T \cdot R = \mathbf{1}_3. \tag{2.3.3}$$

$$\begin{cases} detR = +1 \rightarrow \text{rotations} \\ detR = -1 \rightarrow \text{ reflection} \end{cases} \tag{2.3.4}$$

The group of rotations denoted by $SO(3)$ is composed of all matrices that fulfill the requirement (2.3.3), where "O" stands for 'orthogonal' with a determinant R equal to +1. (where "S" stands for 'special'),the special orthogonal group in 3 dimensions,and hence we have $\mathfrak{so}(3) \cong \mathfrak{su}(2)$,where 'u' stands unitarity $U^{+}.U = U^{+}.U = \mathbf{1}$.it is the group of isometries (rotations) of $\mathbf{S}^2$

We take the rotation around the axis x with the corresponding angle θ_1, and after the projection, we get :

$$\begin{aligned}\hat{y}' &= (\cos\theta_1 \hat{y}^{)} + (\sin\theta_1 \hat{z}) \\ \hat{z}' &= (-\sin\theta_1 \hat{y}^{)} + (\cos\theta_1 \hat{z})\end{aligned} \tag{2.3.5}$$

We write this rotation around the axis x , $\hat{x}' = \hat{x}$ as a matrix; we remember that we took the following notation:

$$\begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= z \end{aligned} \tag{2.3.6}$$

$$\begin{aligned} &\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow R_1 = \mathbf{1}_3 + i\theta_1 L_1, \\ &L_1 = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \end{aligned} \tag{2.3.7}$$

We do the same for the two remaining rotations, and we extract the generators in that direction, i.e., when we take the rotation around the axis y with angle θ_2 and around the axis z with angle θ_3, we get :

$$\begin{aligned} &\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow R_2 = \mathbf{1}_3 + i\theta_2 L_2, \\ &L_2 = -i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \end{aligned} \tag{2.3.8}$$

$$\begin{aligned} &\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow R_3 = \mathbf{1}_3 + i\theta_3 L_3, \\ &L_3 = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.3.9}$$

Now we calculate the commutator between the generators, and we find:

$$\begin{aligned} &[L_1, L_2] = iL_3, \quad [L_3, L_1] = iL_2 \\ &[L_2, L_3] = iL_1 \end{aligned} \tag{2.3.10}$$

We can write this relationship in compact form :

$$[L_i, L_j] = i\varepsilon_{ijk} L_k \tag{2.3.11}$$

ε_{ijk} Levi civita tansor

$$\begin{cases} \varepsilon_{ijk} = 1, i, j, k = 1, 2, 3 \\ \varepsilon_{ijk} = -1, i, j, k = 2, 1, 3 \\ \varepsilon_{ijk} = 0 \end{cases} \tag{2.3.12}$$

It is called $o(3) = su(2)$ lie algebra of angular momentum. The angular moment operator L_i is the rotation generator because the rotation is related to the angular momentum.

$SU(2)$ is a double cover of $SO(3)$,in other words, $SU(2)$ covers $SO(3)$ twice and thus a rotation angle equal 4π in $SU(2)$ corresponds to a rotation angle equal 2π in $SO(3)$, we have $\mathfrak{so}(3) \cong \mathfrak{su}(2)$

2.3.1 SU(2)Representations

Now we want to know the solution to this equation, which is called the representations (which gives all irreducible representations of the rotation group) The rotation group is the lie group; therefore, the representations of this algebra are sufficient to specify the representations of the group.

We know two solutions. The first solution is obvious: $L_i = 0$n is a trivial representation. The second representation is the so-called vector, or adjoint, representation, which we calculated above. Now, for general representation, we changed the symbol to avoid confusion.

$$[J_i, J_j] = i\varepsilon_{ijk} J_k. \tag{2.3.13}$$

We get the representation of the $su(2)$ group from the solutions of the commutation relations 2.3.13, characterized by a single spin quantum number j and which are of dimension $2j + 1$.

A representation U of the group $SU(2)$ is an automorphism acting in some vector space V.In other words, U is a map from the group $G = SU(2)$ to the general linear group GL(V), which preserves the composition law and the group structure. Explicitly, we have $U : G = SU(2) \longrightarrow GL(V)$. $U(T)$ it is representation the element $t \in G = SU(2)$ which obtained simply by the exponential map given by:

$$U(g) = \exp\left(i\theta_i \mathbf{J}_i\right)$$

The J_i is the generator in the representation U which generates a rotation with an angle θ_i around the axis i.

Utilizing Shur's lemma, which ensures that a: U is an irreducible representation (IRR) if and only if the only elements $C^{(1)}$ which commute with $U(t)$ for all t are the elements proportional to the identity; the elements $C^{(1)}$ are called Casimir operators and, their eigenvalues characterize the different IRR's U.In other words, if there is no invariant subspace, the representation is irreducible. Angular momentum generators J_i commute with the squared angular momentum operator $\vec{J}^2 = J_1^2 + J_2^2 + J_3^2$, also known as $C^{(1)} = \vec{J}^2$, according to quantum mechanics. Specifically, this is the rotation group $SO(3)$'s only Casimir operator. Thus, we have

$$\left[J_i, \vec{J}^2\right] = 0, \quad \vec{J}^2 = J_1^2 + J_2^2 + J_3^2 \tag{2.3.14}$$

And we have

$$\hat{J}_+ = \hat{J}_1 + i\hat{J}_2, \hat{J}_- = \hat{J}_1 - i\hat{J}_2 \tag{2.3.15}$$

The Casimir operator is an operator who commutes with any other operator according to quantum mechanics, if two operators commute,they form a complete set of mutual observations, implying a joint subjective group between the Casimir and J_3, which we denote as $|jm\rangle$, we have

$$\begin{aligned}
&\vec{J}^2|jm\rangle = j(j+1)|j,m\rangle \\
&J_3|jm\rangle = m|jm\rangle \\
&\qquad m = j, j-1, \ldots, -j+1, -j, \quad j = 0, 1/2, 1, 3/2, \ldots
\end{aligned} \tag{2.3.16}$$

The ladder operators $\hat{J}_+$ and $\hat{J}_-$ respectively raise and lower the value of m:

$$\begin{aligned}
\hat{J}_+|j,m\rangle &= \sqrt{(j-m)(j+m-1)}|j,m+1\rangle, \\
\hat{J}_-|j,m\rangle &= \sqrt{(j+m)(j-m+1)}|j,m-1\rangle,
\end{aligned} \tag{2.3.17}$$

whil $\hat{J}_+$ annihilates the $|j,j\rangle$ state and $\hat{J}_-$ annihilates $|j,-j\rangle$ state.

Therefore, $N = 2j+1$ is the dimension of these irreducible representations. Since $SU(2)$ is compact, all its irreducible representations are unitary and finite-dimensional.

How many representations do we have? We have an infinite number of representations. Any representation of $SU(2)$ or $SO(3)$ in a "Hilbert space" can be seen as a unitary representation and is completely reducible into a sum of finite-dimensional irreducible representations.

Finally,this group of rotations $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ is the most important for several reasons:

- is the simplest non-abelian group, and the same methodology will be used to study all other Lie groups.

- $SU(2)$ group is a principal group in the non-commutative sphere, also known as the fuzzy sphere, the most important non-commutative space in non-commutative geometry, another approach to quantum gravity. [69]

- This group is associated with the spin group SO (2.1) for the pseudo-sphere, which is the Euclidean rotation of the anti-de Sitter space in two dimensions AdS^2. [106]

- In reality, the fuzzy sphere and the noncommutative AdS^2_θ are closely related. [102] [84]

2.3.2 The Lorentz group $SO(2,1)$ and its Lie algebra $su(1,1)$

The group $SO(2,1) \sim SU(1,1)$ is the group of isometries (translation, dilatation and special conformal transformation) of Lorentzian $\mathbf{AdS}^2$, whereas the isometry group of Euclidean $\mathbf{AdS}^2$ is $SO(1,2)$, which is locally isomorphic to $SU(1,1)$. [106]

In Lorentzian $\mathbf{AdS}^2$, the isometries group is" $SO(2,1) \sim SU(1,1)$". These isometries include translation, dilatation, and special conformal transformations. On the other hand, in the Euclidean, $\mathbf{AdS}^2$ ($\mathbf{H}^2 = \mathbf{AdS}^2_E$) is denoted as $SO(1,2)$, both of which are locally "isomorphic" to $SU(1,1)$.

The following equations can be used to represent the generators of the Lorentzian $\mathbf{AdS}^2$ (Killing vectors) isometry group $SO(2,1)$ [63]:

$$L^A_B = X^A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X_A}. \tag{2.3.18}$$

The raising/lowering of indices is done with the appropriate embedding metric

We will use in this calculation the cylindrical coordinates. Explicitly, we have:

$$\begin{aligned} -iK^3 \equiv L^2_1 &= X^2 \frac{\partial}{\partial X^1} - X_1 \frac{\partial}{\partial X_2} = -\partial_\tau. \\ -iK^2 \equiv L^1_3 &= X^1 \frac{\partial}{\partial X^3} - X_3 \frac{\partial}{\partial X_1} = \sin\tau\sin\sigma\partial_\tau - \cos\tau\cos\sigma\partial_\sigma. \\ iK^1 \equiv L^3_2 &= X^3 \frac{\partial}{\partial X^2} - X_2 \frac{\partial}{\partial X_3} = -\cos\tau\sin\sigma\partial_\tau - \sin\tau\cos\sigma\partial_\sigma. \end{aligned} \tag{2.3.19}$$

$$-iK^3 \equiv L^2_1 = X^2 \frac{\partial}{\partial X^1} - X_1 \frac{\partial}{\partial X_2} = X^2 \left(\frac{\partial\tau}{\partial X^1}\frac{\partial}{\partial\tau} + \frac{\partial\sigma}{\partial X_1}\frac{\partial}{\partial\sigma} \right) - X_1 \left(\frac{\partial\tau}{\partial X_2}\frac{\partial}{\partial\tau} + \frac{\partial\sigma}{\partial X_2}\frac{\partial}{\partial\sigma} \right) \tag{2.3.20}$$

sigma is not related to X^1, X^2 , $\frac{\partial\sigma}{\partial X_1} = \frac{\partial\sigma}{\partial X_2} = 0$ and

$$\begin{aligned}
\cos\tau &= \frac{x_1}{\sqrt{x_1^2+x_2^2}} \quad , \tau = \arccos\frac{x_1}{\sqrt{x_1^2+x_2^2}} \\
\sin\tau &= \frac{x_2}{\sqrt{x_1^2+x_2^2}} \quad , \tau = \arcsin\frac{x_2}{\sqrt{x_1^2+x_2^2}} \\
\frac{\partial\tau}{\partial X^1} &= \frac{-X_2}{X_1^2+X_2^2} \\
\frac{\partial\tau}{\partial X_2} &= \frac{X_1}{X_1^2+X_2^2}
\end{aligned} \tag{2.3.21}$$

We compensate in 2.3.20 we get :

$$L_1^2 = -\frac{X_1^2+X_2^2}{X_1^2+X_2^2}\frac{\partial}{\partial\tau} = -\frac{\partial}{\partial\tau} \tag{2.3.22}$$

These encode the infinitesimal transformations of the conformal group on the boundary of $\mathbf{AdS}^2$. These generators satisfy the $su(1,1)$ algebra given by [68]

$$\begin{aligned}
&\left[\mathcal{K}^2,\mathcal{K}^1\right] = -i\mathcal{K}^0, \left[\mathcal{K}^0,\mathcal{K}^1\right] = -i\mathcal{K}^2, \left[\mathcal{K}^2,\mathcal{K}^0\right] = i\mathcal{K}^1 \\
&\Leftrightarrow \left[\mathcal{K}^a,\mathcal{K}^b\right] = -i\epsilon^{ab}{}_c\mathcal{K}^c, \epsilon^{012} = +1.
\end{aligned}$$

$$\left[K^3,K^2\right] = -iK^1, \left[K^1,K^2\right] = -iK^3, \left[K^3,K^1\right] = iK^2. \tag{2.3.23}$$

We will demonstrate how to compute.

$$\begin{aligned}
&\left[K^3,K^2\right]f = K^3(K^2f) - K^2(K^3f) \\
&= \partial_\tau(\sin\tau\sin\sigma\partial_\tau f - \cos\tau\cos\sigma\partial_\sigma f) - (\sin\tau\sin\sigma\partial_\tau - \cos\tau\cos\sigma\partial_\sigma)\partial_\tau f \\
&= \cos\tau\sin\sigma\partial_\tau f + \sin\tau\sin\sigma\partial_\tau^2 f + \sin\tau\cos\sigma\partial_\sigma f - \cos\tau\cos\sigma\partial_\tau\partial_\sigma f - \sin\tau\sin\sigma\partial_\tau\partial_\tau f + \cos\tau\cos\sigma\partial_\sigma\partial_\tau f
\end{aligned} \tag{2.3.24}$$

After simplification, we get:

$$\left[K^3,K^2\right]f = (\cos\tau\sin\sigma\partial_\tau + \sin\tau\cos\sigma\partial_\sigma)f = -K^2. \tag{2.3.25}$$

We follow the same steps to prove the rest of equation $[K^1,K^2] = -iK^3, [K^3,K^1] = iK^2$.

where in all representations the plus sign corresponds to Lorentzian AdS^2 whereas the minus sign corresponds to Euclidean AdS^2.

Indeed, we can define, as usual, the raising and lowering operators $K^\pm = K^1 \pm iK^2$ and rewrite the $su(1,1)$ Lie algebra (2.4.5) in the form [62]

$$\begin{aligned}\left[K^3, K^{\pm}\right] &= \pm K^{\pm}\\ \left[K^+, K^-\right] &= -2K^3.\end{aligned} \tag{2.3.26}$$

The first equation means that $K^{\pm}|km\rangle \sim |km \pm 1\rangle$, i.e. K^+raises the eigenvalue m of K^3 by a single unit whereas K^-lowers it with by a single unit. By defining a lowest weight state by the usual condition $K^-|kk\rangle = 0$ and using the second commutator in the form $K^+K^- = K_3(K_3 - 1) - C$ we arrive at the constraint $C = k(k-1)$. The higher states $|kk+m\rangle$ are obtained by the action of $(K^+)^m$ on $|kk\rangle$. Similarly, by defining the highest weight state by the condition $K^+|kk\rangle = 0$ and using the second commutator in the form $K^-K^+ = K_3(K_3+1) - C$ we arrive at the constraint $C = k(k+1)$. The lower states $|kk-m\rangle$ are obtained by the action of $(K^-)^m$ on $|kk\rangle$.

The k value designates "the Hilbert space" that corresponds to the IRRs, as indicated by

$$\begin{aligned}&K^3|km\rangle = m|km\rangle\\ &K^+|km\rangle = \sqrt{m(m+1) - k(k-1)}|km+1\rangle\\ &K^-|km\rangle = \sqrt{m(m-1) - k(k-1)}|km-1\rangle\\ &C|km\rangle = \pm k(k-1)|km\rangle\end{aligned} \tag{2.3.27}$$

2.4 Representation theory of $su(1,1)$

There are several classes of irreducible representations of $su(1,1)$, Bragmann originally calculated it [85]. A pseudo-spin quantum number k is a defining characteristic of the many classes of (IRR's) of $su(1,1)$. The different IRR's of $su(1,1)$ include the discrete series $D_k^{\pm}$, the principal/complementary continuous series $C_k^{\frac{1}{2}}/C_k^0$ and the finite-dimensional series F_k. As we have said, the possible irreducible representations of $su(1,1)$ are given by the following cases [71]:

- **The discrete series $D_k^{\pm}$:**

  - The pseudo-spin quantum number is given in this case by $k = \{1/2, 1, 3/2, 2, \ldots\}$.

  - Our analysis will solely focus on the integer number possibilities of the $su(1,1)$ pseudo-spin quantum number $j \equiv k - 1$.

  - The highest and lowest weight representations of the discrete series:

$$\begin{aligned}&D_k^+, \quad k \in \mathbb{N}_{>0}: \quad \mathcal{H}_k = \{|k,m\rangle; m = k, k+1, \cdots; m \in \mathbb{N}\}\\ &D_k^-, \quad k \in \mathbb{N}_{>0}: \quad \mathcal{H}_k = \{|k,m\rangle; m = -k, -k-1, \cdots; -m \in \mathbb{N}\}\end{aligned} \tag{2.4.1}$$

  - The Casimir in these representations is positive given by $C = k(k-1)$

- **The principal continuous series** $C_k^{\frac{1}{2}} \equiv P_a^{\frac{1}{2}}$:

  - In this case, the pseudo-spin quantum number k is a complex number given by :

$$k = \frac{1}{2} + ia, \quad a \in \mathbb{R}, , \quad \mathcal{H}_k = \{|k,m\rangle; m = 0, \pm 1, \ldots; m \in \mathbb{Z}\} \tag{2.4.2}$$

  - The Casimir is negative given by $C = -s^2 - \frac{1}{4} < -\frac{1}{4}$.

- **The complementary continuous series** $C_k^0 \equiv P_k^0$:

$$0 < k < 1, \quad k \in \mathbb{R}, \mathcal{H}_k = \{|km\rangle; m = 0, \pm 1, \pm 2, \ldots\} \tag{2.4.3}$$

  - Defined by negative Casimir in the range $-\frac{1}{4} < C < 0$

  All previous representations are unitary and infinite-dimensional. There is a finite representation, but it is not unitary, as follows:

- **The finite-dimensional IRR'S representations** F_k:

  - This corresponds to $k - 1 \in \mathbb{N}/2$, i.e. these IRR's coincide with those of $su(2)$ with $j = k - 1$.

  - These IRR's are not unitary.

  - with Casimir $C = |k|(|k| + 1)$.

Euclidean continuation $\mathbf{H}^2$

The Euclidean anti-de Sitter space is calculable in a similar way.where we have $\cos t \to \cosh t$ and $\sin t \to \sinh t$.The isometry group $SO(1,2)$ generators corresponding to the pseudo-sphere H^2 can be derived within the framework of local cylindrical coordinates in the same way as above, and we get the following result:

$$\begin{aligned} -iK^3 \equiv L_1^2 &= X^2 \frac{\partial}{\partial X^1} - X_1 \frac{\partial}{\partial X_2} = -\partial_\tau \\ iK^2 \equiv L_3^1 &= X^1 \frac{\partial}{\partial X^3} - X_3 \frac{\partial}{\partial X_1} = \sinh\tau \sin\sigma \partial_\tau - \cosh\tau \cos\sigma \partial_\sigma \\ -iK^1 \equiv L_2^3 &= X^3 \frac{\partial}{\partial X^2} - X_2 \frac{\partial}{\partial X_3} = \cosh\tau \sin\sigma \partial_\tau - \sinh\tau \cos\sigma \partial_\sigma \end{aligned} \tag{2.4.4}$$

We write these equations collectively as:

$$\left[K^a, K^b\right] = i f^{ab}{}_c K^c. \tag{2.4.5}$$

In the Euclidean case we have $f^{ab}{}_c = \epsilon^{ab}{}_c$ whereas in the Lorentzian case we had $f^{ab}{}_c = -\epsilon^{ab}{}_c$. The Casimir operator is given by [71]

$$C = -K_1^2 \mp K_2^2 + K_3^2. \tag{2.4.6}$$

2.5 The $su(1,1)$ Lie algebra on $\mathbf{H}^2$

- The generators of the isometry group $SO(1,2)$ of the pseudo-sphere $\mathbf{H}^2$ are given by the "pseudo-angular momentum" differential operators defined by

$$\mathcal{K}_a = -if_{ab}{}^{c}X^b\partial_c, \mathcal{K}_a\left(X_b\right) = if_{ab}{}^{c}X_c.$$

- These generators are the Killing vector fields generating pseudo-rotations on the pseudo-sphere and satisfy the $su(1,1)$ Lie algebra defined by

$$\left[\mathcal{K}^a, \mathcal{K}^b\right] = if^{ab}{}_{c}\mathcal{K}^c.$$

- We will show that the structure coefficients are given by $f^{ab}{}_{c} = \epsilon^{ab}{}_{c}$.

- We have already shown that the isometry group $SO(1,2)$ of de Sitter spacetime $\mathbf{dS}^2$ is generated by the same $su(1,1)$ Lie algebra with $f^{ab}_{c}{}_{c} = \epsilon^{ab}{}_{c}$.

2.6 IRR's of $su(1,1)$: Euclidean $\mathbf{H}^2$

- We consider now the case of Euclidean $\mathbf{H}^2$/ Lorentzian $\mathbf{dS}^2$ (with embedding metric $\eta = (-1,+1,+1)$ and $\epsilon^{012} = +1$).

- The $su(1,1)$ Lie algebra is given by (with $f^{ab}{}_{c} = \epsilon^{ab}{}_{c}$)

$$\left[K^2, K^1\right] = iK^0, \left[K^0, K^1\right] = iK^2, \left[K^2, K^0\right] = iK^1.$$

- Alternatively, the Lie algebra and the quadratic Casimir operator are given by

$$\left[\bar{K}_0, \bar{K}_\pm\right] = \pm\bar{K}_\pm, \left[\bar{K}_+, \bar{K}_-\right] = -2\bar{K}_0$$
$$\bar{K}_0 = K^0, \bar{K}_\pm = K_1 \pm iK_2.$$
$$C = -K_0^2 + K_1^2 + K_2^2 = -\bar{K}_0\left(\bar{K}_0 \pm 1\right) + \bar{K}_\mp\bar{K}_\pm.$$

- The IRR's of this $su(1,1)$ Lie algebra are exactly the same as before, except that the Casimir operator must be multiplied by an overall minus sign.

Chapter 3

Conformal field theory and Anti de Sitter space

3.1 A Primer on Conformal Fields

An essential aspect of the AdS/CFT correspondence is that the quantum field theory involved is a conformal field theory (CFT), which is quantum field theory with conformal symmetry, i.e., there is no preferred length scale(that preserves the angles but changes the lengths), the physics, looks the same at all length scales. We will study this theory through symmetry in two dimensions; there is an infinite-dimensional algebra of local conformal transformations; because it has an unfinished number of symmetries, it makes it an exceptional theory. Conformal symmetry is crucial in: -Statistical mechanics (second-order phase transitions, critical exponents) and quantum field theory (fixed points of the renormalization group equation). -String theory (Polyakov path integral, AdS/CFT correspondence). We will present the fundamentals of CFT in a standard manner, building on the work of [127], [126], and [63], [128], [143], [6].

3.1.1 Symmetry in cft

The conformal group is a subgroup of general coordinate transformations. Conformal transformations' important geometric property is that they preserve angles between any two vectors but not distances. This means that in the Lorentzian case, they always preserve the causal structure of spacetime.

The conformal group contains the following transformations:

- **translations:** $x^\mu \to x'^\mu = x^\mu + a^\mu$, These transformations include variables with the following symbols a^μ, $a^\mu = a^0, a^1, ...a^{d-1}$, there is d variable.in total, there are D generators denoted as P^μ

- **Lorentz transformations:**with anti-symmetry parameters ω^μ_ν, include rotations and boosts. The mathematical representation of Lorentz transformations is given by $x^\mu \to$

$x'^{\mu} = x^{\mu} + \omega^{\mu}_{\nu} x^{\nu}$. The total dimensions of the rotation matrix are $D \times D$, we minus the number of diagonal elements, i.e., $D^2 - D$, $\omega_{\nu\mu}$ it is antisymmetric tensor ($\omega_{\mu\nu} = -\omega_{\nu\mu}$), i.e., the lower elements of the matrix are related to the upper elements. We get the total dimension as $D^2 - D/2$, which represents the number of independent generators of Lorentz transformations and is noted as $M_{\mu\nu}$.

- **The dilatation (scale):**It modifies coordinates as follows: $x^{\mu} \to x'^{\mu} = \lambda x^{\mu}$,λ is a constant while the mass transforms as $m \to m' = \frac{1}{\lambda} m$. The special property of conformal theory allows mass transformation from zero to infinity, resulting in no particles. The conformal theory gains a unique property from this transformation that links physics at different length scales. i.e., the CFT does not contain an S-matrix. [132]. λ is a number that has no index, i.e., there is one number of generators denoted by D.

- **the special conformal transformations:** Combine an inversion and translation (Where its factor is symbolized by b_{μ}) and another inversion. We start with an inversion:

$$\begin{aligned}
&x^{\mu} \longrightarrow x^{u'} = \frac{x^{\mu}}{x^2} \Rightarrow dx^{\mu} \longrightarrow dx^{\mu'} = \frac{dx^{\mu}}{x^2} + x^{\mu} \left(\frac{(dx^{\nu} x_{\nu})^{-1}}{dx^{\mu}} \right) \\
&dx^{u'} = \frac{dx^{\mu}}{x^2} - \frac{2x^{\mu} x_{\nu} dx^{\nu}}{x^4} \\
&= \left(\frac{\eta^{\mu}_{\nu} dx^{\nu}}{x^2} - \frac{2x^{\mu} x_{\nu} dx^{\nu}}{x^{\mu}} \right) \\
&= \frac{1}{x^2} \big(\eta^{\mu}_{\nu} - \frac{2x^{\mu} x_{\nu}}{x^2} \big) dx^{\nu}.
\end{aligned} \tag{3.1.1}$$

Before calculating the special conformal transformations, we will check that the inversion transformation leaves the metric conformally flat.

$$\begin{aligned}
&dx^{\mu'} dx_{\mu'} = \left(\frac{1}{x^2} \big(\eta^{\mu}_{\nu} - \frac{2x^{\mu} x_{\nu}}{x^2} \big) \right) \left(\frac{1}{x^2} \big(\eta_{\nu\lambda} - \frac{2x_{\mu} x_{\lambda}}{x^2} \big) \right) dx^{\nu} dx^{\lambda} \\
&= \frac{1}{x^4} (dx^{u} dx_{u} - \frac{2x_{\mu} x_{\lambda}}{x^2} dx^{\nu} dx^{\lambda} - \frac{2x_{\lambda} x\nu}{x^2} dx^{\nu} dx^{\lambda} \\
&+ \frac{4x^{\mu} x_{\mu} x_{\nu} x_{\lambda}}{x^4} dx^{\nu} dx^{\lambda}) \\
&= \frac{dx^{\mu} dx_{\mu}}{(x^4)}.
\end{aligned} \tag{3.1.2}$$

It is **conformal flats** .

We will now investigate a transformation that consists of an inversion, followed by a translation and an inversion.

$$\begin{aligned} x^\mu \longrightarrow \frac{x^\mu}{x^2} \longrightarrow \frac{x^\mu}{x^2} + b^\mu \longrightarrow & \frac{\frac{x^\mu}{x^2}+b^\mu}{(\frac{x^\mu}{x^2}+b^\mu)^2} \\ = & \frac{\frac{1}{x^2}\left(x^\mu + x^2 b^u\right)}{\frac{1}{x^4}\left(x^u + x^2 b^u\right)\left(x_u + x^2 b_u\right)} \\ = & \frac{x^\mu + x^2 b^4}{\frac{1}{x^2}\left(x^2 + 2x^2 x^\mu b_\mu + x^2 b^2\right)} \\ = & \frac{x^\mu + b^\mu x^2}{1 + 2b^\mu x_\mu + b^2 x^2} \end{aligned} \tag{3.1.3}$$

In summary, the Conformal Group contains an extensive number of transformations that are crucial for understanding spacetime geometry, the total number of conformal transformations in d dimensions is: $\underbrace{d}_{translations} + \underbrace{d^2 - d/2}_{lorentz} + \underbrace{1}_{dilatation} + \underbrace{d}_{specialconformal} = \underbrace{(d+1)(d+2)/2}_{total}$.

All these transformations form the conformal group in d dimensions, which is isomorphic to $SO(d,2)$ in the Lorentzian signature and $SO(d+1,1)$ in the Euclidean signature.

The generators of each transformation are given as follows [126] :

$$\begin{aligned} & P_\mu = -i\partial_\mu && \text{translation} \\ & D = -ix^\mu\partial_\mu && \text{dilation} \\ & L_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) && \text{rotation} \\ & K_\mu = -i\left(2x_\mu x^\nu\partial_\nu - (x\cdot x)\partial_\mu\right) && \text{SCT} \end{aligned} \tag{3.1.4}$$

Moreover, the commutation relations between them are given by : In addition, the commutation relations between them are given by :

3.1.2 Infinitesimal conformal transformations

A manifold is called conformally flat if the metric takes the following form:

$$ds^2 = \exp(\omega(x))dx^\mu dx_\mu. \tag{3.1.5}$$

A Conformal transformation (a generalization of the scale transformation) is given by:

We are interested in researching flat space quantum field theory, which is invariant under the conformal transformation. Traditionally, conformal isometries are described as a change in coordinates or diffeomorphism such that the metric is

$$x^\mu \longrightarrow x^{'\mu}(x)\ ,\ dx^{'\mu}dx^{'}_{\mu} = \Omega^2(x)dx_\mu dx^\mu.,\Omega = 1 + \omega/2. \tag{3.1.6}$$

In the previous section, we study the global transformation; now We consider infinitesimal conformal transformations (it is essential because it is local).

$$
\begin{aligned}
&x'_\mu = x_\mu + v_\mu \Rightarrow dx'_\mu = dx_\mu + dv_\mu, \qquad\qquad \text{sabstitute in (3.1.6)}\\
&(dx_u + dv_u)(dx^u + dv^u) = (1+w)dx_u dx^u\\
&2dx_\mu dv^\mu = \omega dx_\mu dx^\mu\\
&2dx_\mu \frac{\partial v^u}{\partial x^\mu} dx^\nu = \omega dx_\mu dx^\mu\\
&2\partial_\nu v^u dx_\mu dx^\nu = \omega dx_\mu dx^\mu
\end{aligned} \tag{3.1.7}
$$

interchanging $\mu \longleftrightarrow \nu$ and we add it to the last equation of (3.1.7), divide by 2, we get:

$$\partial_\nu v^\mu + \partial^\mu v_\nu = \omega \eta^\mu_\nu. \tag{3.1.8}$$

Taking both sides' traces (η^ν_μ), we obtain:

$$
\begin{aligned}
&\eta^\nu_\mu(\partial_\nu v^\mu + \partial^\mu v_\nu) = \omega \eta^\mu_\nu \eta^\nu_\mu\\
&2\partial_\mu v^\mu = \omega D\\
&\omega = \frac{2\partial_\mu v^\mu}{D}
\end{aligned} \tag{3.1.9}
$$

As a result, the condition on infinitesimal conformal transformations gives rise to the conformal killing equation.

$$\partial_\nu v^\mu + \partial^\mu v_\nu - \frac{2}{D}\partial_\rho v^\rho \eta^\mu_\nu = 0. \tag{3.1.10}$$

It is referred to as the conformal Killing vector equation. [141]; it is essential to note that when we set $d = 2$, we obtain an infinite number of solutions, i.e., an infinite number of conserved quantities. For this reason, the AdS_3/CFT_2 correspondence" is the most studied case [72]. We will take a general overview of this case in the next section.

The conformal killing equation (3.1.10) in d in more than two dimensions has finite solutions, which are provided by:

$$v_\mu = \delta x^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu \mathrm{nn} \tag{3.1.11}$$

All terms in this equation are trivial; the last part is the infinitesimal version of the special conformal transformations; we get it as follows:

We do the infinitesimal transformations of (3.1.3), around the parameters $b^\mu << 1$,and we stop at the first order we obtain:

$$
\begin{aligned}
x'^u &= (x^\mu + b^\mu x^2)(1 + 2b^\mu x_\mu + b^2 x^2)^{-1} \\
&= (x^\mu + b^\mu x^2)(1 - 2b^\mu x_\mu - b^2 x^2) \\
&= x^\mu - 2x^\mu x^\nu b_\nu + b^2 x^2 \\
\delta x^u &= x'^u - x^u \\
&= -2x^\mu bx + x^2 b^\mu .
\end{aligned}
\tag{3.1.12}
$$

Invariance under the transformation (3.1.5 can only hold if the theory has no preferred length scale. However, this means there can be nothing like a mass or a Compton wavelength in the theory. In other words, conformal field theories only support massless excitations. The questions that we ask are not those of particles and S-matrices. Instead, we will be concerned with correlation functions and the behavior of different operators under conformal transformations.

3.1.3 the conserved current

The stress-energy tensor, also referred to as the energy-momentum tensor, is a fundamental component within the framework of QFT.

One of the essential objects in any QFT is the stress-energy tensor (also known as the energy-momentum tensor); it is a tensor physical quantity that describes the density and flux of energy and momentum in spacetime, just as mass density is the source of such a field in Newtonian gravity. It controls the relationship between symmetry and the conserved current.

Let us recall Noether's theorem, which states that for every continuous symmetry in a field theory, a current j_μ is conserved, i.e.$\partial^\mu j_\mu = 0$.

The conserved current J_μ associated with the conformal transformation δx^μ is constructed from the stress-energy tensor $T_{\mu\nu}$ as follows [141] :

$$
J_\mu = T_{\mu\nu} \delta x^\nu . \tag{3.1.13}
$$

The conserved current J_μ associated with:

- Translations yield the conservation of the stress-energy tensor. $\longrightarrow \partial^\mu T_{\mu\nu} = 0$
- Lorentz transformations $\longrightarrow T_{\mu\nu} = T_{\nu\mu}$, i.e., the stress-energy tensor is symmetric.

  Lorentz transformations $\longrightarrow T_{\mu\nu} = T_{\nu\mu}$, i.e. the stress-energy tensor is symmetric.
- dilatation $\longrightarrow T^\mu_\mu = 0$.

Conserving all conformal currents in a Poincaré and scale invariant theory with a symmetric traceless conserved stress-energy tensor is almost trivial.

Proof:

$$\begin{aligned}
\partial u J^u &= \partial_\mu \left(T^{\mu\nu} v_\nu\right) \\
&= \partial_\mu T^{\mu\nu} \cdot v_\nu + T^{\mu\nu} \partial_u v_\nu \\
&= T^{\mu\nu} \partial_u v_v \\
\mu \leftrightarrow \nu &= T^{\mu\nu} \partial_\nu v_u \\
&= \frac{1}{2} T^{\mu\nu} \left(\partial_u v_v + \partial_v v_u\right) \\
&= \frac{1}{2} T^{\mu\nu} \cdot \frac{2}{d} \left(\partial_\lambda, v^\lambda\right) \cdot \eta_{\mu\nu} \\
&= \frac{1}{d} \partial_\lambda V^\lambda T^\mu_\mu \\
&= 0
\end{aligned} \tag{3.1.14}$$

where we used and the fact that $T^{\mu\nu}$ is symmetric.

3.1.4 The conformal group in 2 dimensions

The conformal algebra in two dimensions is thus infinitely dimensional, as we see from (3.1.10). As a result, an infinite number of conserved charges are essentially provided by the Virasoro generators [141], [91], [129],

This is only a brief note because we are only interested in one dimension in this thesis, which we will investigate using free scalar fields in two dimensions with Euclidean signature

The action is:

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma (\partial_1 X^\mu \partial_1 X_\mu + \partial_2 X^\mu \partial_2 X_\mu). \tag{3.1.15}$$

We are adopting complex coordinates, which is quite helpful.

$$z = \sigma^1 + i\sigma^2 \ , \ \bar{z} = \sigma^1 - i\sigma^2, \tag{3.1.16}$$

$\partial = \frac{\partial}{\partial z}$,$\bar{\partial} = \frac{\partial}{\partial \bar{z}}$, we express the action in this notation as follows:

$$S = \frac{1}{2\pi\alpha'} \int d^2 z \partial X^\mu \bar{\partial} X_\mu. \tag{3.1.17}$$

We calculate the equation of motion by the change of action

$$\delta S = \frac{1}{2\pi\alpha'}\int d^2z\delta(\partial X^\mu\bar{\partial}X_\mu) \quad (3.1.18)$$

$$= \frac{1}{2\pi\alpha'}\int d^2zX_\mu)\partial\delta X^\mu\bar{\partial}X_\mu + \partial X^\mu\bar{\partial}\delta X_\mu) \quad (3.1.19)$$

$$= \frac{1}{2\pi\alpha'}\int d^2z\partial(\delta X^\mu\bar{\partial}X_\mu) - \delta X^\mu\partial\bar{\partial}X_\mu + \bar{\partial}(\partial X^\mu\delta X_\mu) - \bar{\partial}\partial X^\mu\delta X_\mu) \quad (3.1.20)$$

$$= -\frac{2}{2\pi\alpha'}\int d^2z((\delta X^\mu\partial\bar{\partial}X_\mu)) = 0 \quad (3.1.21)$$

The equations of motion Whatever the change δX^μ are

$$\partial\bar{\partial}X_\mu = 0. \quad (3.1.22)$$

- ∂X_μ is a all holomorphic functions on the plane,similarly $\bar{\partial}X_\mu$ is an antiholomorphic on z and $\bar{z}$ respectively [11] .
- The symmetries of this action is [129]

$$z \longrightarrow f(z)\ ,\ \bar{z} \longrightarrow \bar{z} = \bar{f}(\bar{z}). \quad (3.1.23)$$

  We notice that there are an infinite number of these coordinate transformations, which confirms that the conformal algebra in two dimensions is infinite-dimensional. [141]
- These are conformal mappings. These modifications keep the angle. For example

$$z \longrightarrow z + a$$

  is a translation

$$z \longrightarrow \zeta z$$

  where $|\zeta| = 1$ is a rotation
- The corresponding infinitesimal transformations are given by

$$z \longrightarrow z + a(z)$$

  and

$$\bar{z} \longrightarrow \bar{z} + \bar{a}(\bar{z})$$

- A general infinitesimal conformal transformation can then be written as [126]

$$z \longrightarrow z' = z - \epsilon_n z^{n+1}\ ,\ \bar{z} \longrightarrow \bar{z}' = \bar{z} - \bar{\epsilon}_n\bar{z}^{n+1}\ ,\ n \in Z. \quad (3.1.24)$$

where the infinitesimal parameters ϵ_n and $\bar{\epsilon}_n$ are constant.

By performing the Laurent expansion, we prove that polynomials are generators following the translation operator, i.e., we obtain z from the derivation just once. The associated infinitesimal generators for these symmetries are now provided by. [141] :

$$l_n = -z^{n+1}\partial_z \ , \ \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \ , \ n \in Z. \tag{3.1.25}$$

Upon encountering a set of conserved charges, our first step should include calculating their algebraic properties. (For example, think of the angular momentum in the hydrogen atom.) As a next step, let us compute the commutators of the generators (3.1.47) to determine the corresponding algebra. We calculate [129] :

$$\begin{aligned}
[l_m, l_n] &= z^{m+1}\partial_z\left(z^{n+1}\partial_z\right) - z^{n+1}\partial_z\left(z^{m+1}\partial_z\right) \\
&= (n+1)z^{m+n+1}\partial_z - (m+1)z^{m+n+1}\partial_z \\
&= -(m-n)z^{m+n+1}\partial_z \\
&= (m-n)l_{m+n} \\
\left[\bar{l}_m, \bar{l}_n\right] &= (m-n)\bar{l}_{m+n} \\
\left[l_m, \bar{l}_n\right] &= 0
\end{aligned} \tag{3.1.26}$$

$\boxed{\text{The algebra of infinitesimal conformal transformations in two-dimensional space is infinite dimensional}}$

This is the classical Virasoro algebra. The set of all conformal transformations defines the two-dimensional (global) conformal group, which is generated by the infinitesimal generators: $\{\ell_{-1}, \ell_0, \ell_1\} \cup \{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1\}$ [141]

$$\begin{aligned}
&l_{-1} \ : \ z \longrightarrow z - \epsilon, \bar{l}_{-1} \ : \ \bar{z} \longrightarrow \bar{z} - \bar{\epsilon} \ \text{ translations,} \\
&l_0 \ : \ z \longrightarrow z - \epsilon z, \bar{l}_0 \ : \ \bar{z} \longrightarrow \bar{z} - \bar{\epsilon}\bar{z}, i(l_0 - \bar{l}_0) \ \text{ rotations}, l_0 + \bar{l}_0 \ \text{ scalings} \\
&l_1 \ : \ z \longrightarrow z - \epsilon z^2., \bar{l}_1 \ : \ \bar{z} \longrightarrow \bar{z} - \bar{\epsilon}\bar{z}^2 \ \text{ special conformal transformations.}
\end{aligned} \tag{3.1.27}$$

The Virasoro algebra with central charge c or so-called quantum Virasoro algebra satisfies the following commutation relation [143]

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{3.1.28}$$

3.1.5 correlation functions

The quantities of interest in conformal field theories are N-point correlation functions of fields. The field, in CFT as we say tong [115] In CFT, the term field refers to any local expression, including ϕ, $\partial\phi$, and composite operators. The correlator function Gives us all the information on the theory. When we find the correlator function, we will have solved the theory. We start with the two-point function. This section examines the consequences of conformal invariance on quasi-primary fields' two- and three-point correlation functions.

The term local operator in quantum field theory (CFT) refers to everything that can be expressed as a field, including derivatives and composite operators. Any local expression, including ϕ, $\partial\phi$, and composite operators, is referred to as a field in CFT. In contrast to quantum field theory, which only discusses finite fundamental objects, the set of all fields in CFT is always infinite. [115]

$$\langle \phi_1(z)\phi_2(w)\rangle = g(z,w) \tag{3.1.29}$$

- The invariance under translations $f(z) = z + a$ requires g to be of the form

$$g(z,w) = g(z-w) \tag{3.1.30}$$

- The invariance under rescalings of the form $f(z) = \lambda z$, implies that

$$g(z-w) = \lambda^{h_1+h_2} g(\lambda(z-w)). \tag{3.1.31}$$

- the two-point function must be invariant under inversions $f(z) = \frac{1}{z}$

$$g(z-w) = \lambda^{-h_1-h_2} g\left(\frac{1}{z} - \frac{1}{w}\right). \tag{3.1.32}$$

  where h_1 is the dimension of ϕ_1 and h_2 is the dimension of ϕ_2,Clearly (3.1.31) and (3.1.32)can only be satisfied simultaneously if $h_1 = h_2$

one can make the ansatz,for an structure constant c_{12} which is determined by the normalization of the fields

$$g(z-w) = \frac{c_{12}}{(z-w)^{h_1+h_2}} \tag{3.1.33}$$

The symmetries of conformal field theory have, therefore, constrained the two-point function to be of the form [132], [126]

$$\langle \phi_i(z)\phi_j(w)\rangle = \frac{d_{ij}\delta_{h_i,h_j}}{(z-w)^{2h_i}} \tag{3.1.34}$$

For the three-point function imposing the exact requirements, one finds that:

$$\langle\phi_1(z_1)\,\phi_2(z_2)\,\phi_3(z_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}\,z_{23}^{h_2+h_3-h_1}\,z_{13}^{h_1+h_3-h_2}} \tag{3.1.35}$$

3.1.6 Radial quantization and state-operator correspondence

In the context of a QFT, considering states involves contemplating their spatial existence and how they undergo changes over time. [115], and By inserting operators or by stating a wavefunction in the past of a particular surface, we can directly generate states; similar to this, we handle future states by directly describing the wavefunction or inserting operators. [63] The evolution of states is controlled by the Hamiltonian, which produces temporal translations. We will try to understand this in conformal field theories; we will discuss foliations of spacetime by foliation.The metric reads (with Ω the solid angle on $\mathbf{S}^{d-1}$ and $\tau = \log r$) [143].

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\Omega^2 \\ &\tau = \log r, r = e^{\tau}, dr = e^{\tau} d\tau \\ &ds^2 = e^{2\tau}\left(d\tau^2 + d\Omega^2\right), \text{metric of a cylinder.} \end{aligned} \tag{3.1.36}$$

The transformation $r \longrightarrow \tau = \log r$ maps $\mathbf{R} \times \mathbf{S}^{d-1}$ to $\mathbf{R}^d$, this map is a different way to quantize a CFT on the plane. The cylinder has two bases: a lower base at the infinite past $\tau \longrightarrow -\infty (r = 0)$and an upper base at the infinite future $\tau \longrightarrow +\infty (r = +\infty)$, where τ is time parameter. From this application, we conclude that the states on the plane can be obtained via their roots' cylindrical shape (constant radius). We call this the convergence of the two theories of radial quantization see(3.1). [143], [115].where the time coordinate and field theory lives in$\mathbf{R} \times \mathbf{S}^{d-1}$ is chosen to be in the radial direction in $\mathbf{R}^d$.

In radial quantization, the operator-state correspondence results from associating states on arbitrarily small circles with operators at the centers of these circles. It does say that the states are in one-to-one correspondence with the local operators . [63] , [115]

We will see this through the conformal transformations on scalar fields. We will follow this reference. [91]

In conformal field theory, states are classified by their scaling dimensions, i.e., by the eigenvalues of the dilatation operator $D \equiv L_0$ which generates scale transformations, i.e.

$$D|\Delta\rangle = -i\Delta|\Delta\rangle. \tag{3.1.37}$$

The scaling dimension Δ of a scalar field Φ is defined by the action of the scale transformations $t \longrightarrow t' = \lambda t$ on the field given by

$$\Phi(t) \longrightarrow \Phi'(t') = \lambda^{-\Delta}\Phi(t). \tag{3.1.38}$$

The infinitesimal transformation is given by :

$$\Phi'(t) = \Phi(t) - \ln\lambda\left(t\partial_t\Phi(t) + \Delta\Phi(t)\right) + \ldots.. \tag{3.1.39}$$

On the one hand and on the other, we have The evolution operator governed by the dilatation operator D, which plays the role of the (conformal) Hamiltonian.

$$U = \exp(i\tau D). \tag{3.1.40}$$

If we act on an eigenstate $|\Delta\rangle$ with this operator, we ge

$$U|\Delta\rangle = \exp(i\tau D)|\Delta\rangle. \tag{3.1.41}$$

and in our case,where $t \longrightarrow t' = \lambda t$ and as we know the scale transformations is $t \longrightarrow t' = \lambda t$, the time parameter transformation as $\tau \longrightarrow \tau' = \tau + \ln\lambda$ and the The evolution operator transformation as $U_\lambda = e^{i\ln\lambda D}$. Then, the quantum mechanical form of the transformation law (3.1.38) emerges as

$$\Phi'(t) = U_\lambda^\dagger \Phi(t) U_\lambda, U_\lambda = e^{i\ln\lambda D}. \tag{3.1.42}$$

In order to obtain the infinitesimal transformations, we make up $U_\lambda = e^{i\ln\lambda D} = 1 + i\ln\lambda D$, and after deployment and simplification we get:

$$\Phi'(t) = \Phi(t) - i\ln\lambda[D, \Phi(t)] + \dots \tag{3.1.43}$$

the action of the dilatation operator D on fields Φ,is given by comparing (3.1.39)and (3.1.43), we get:

$$[D, \Phi(t)] = -i\left(t\partial_t\Phi(t) + \Delta\Phi(t)\right). \tag{3.1.44}$$

We follow the same calculations to get the action of the momentum operator P and special conformal generator K on scalar fields Φ:

$$\begin{aligned} [P, \Phi(t)] &= -i\partial_t\Phi(t). \\ [K, \Phi(t)] &= -i\left(t^2\partial_t + 2\Delta t\right)\Phi(t). \end{aligned} \tag{3.1.45}$$

These three equations demonstrate how to represent a scalar field of the conformal algebra in one dimension.

One-to-one between local operators and states in the radial quantization is one of the unique characteristics of CFTs [132]. Importantly, stating a state on the cylinder in the distant past is identical to specifying a local disturbance at the origin since the distant past on the cylinder maps to a single point $z = 0$ in the complex plane. [115].

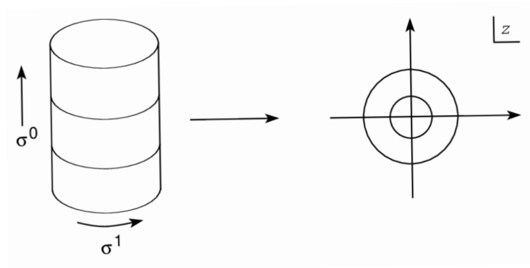

Figure 3.1: The map from the cylinder to the plane.
taken from [115]

3.1.7 de Alfaro-Fubini-Furlan model

The single-dimensional case of AdS/CFT, might seem more straightforward at first glance. However, it holds some surprising complexities around its boundaries and vacuum state, making it the most challenging and least understood case. Luckily, a specific realization of CFT_1, known as the de Alfaro-Fubini-Furlan model (dAFF), acts as conformal quantum mechanics. In this section, we will quickly review some of the well-understood aspects of this model [130], [35]:

The dAFF is a quasi-conformal quantum model in that there is neither an invariant vacuum state nor primary operators. developed in 1976 by de Alfaro, Fubini and Furlan [130].conformally invariant quantum mechanical model possessing $SO(2,1)$ symmetry.At the boundary $z=0$, we have the generators

$$iK^1 = \frac{1}{2R}\left(R^2+t^2\right)\partial_t, iK^2 = -t\partial_t, iK^3 = \frac{1}{2R}\left(R^2-t^2\right)\partial_t. \tag{3.1.46}$$

The $so(1,2)$ Lie algebra is

$$i[D,P] = P, i[D,K] = -K, i[K,P] = 2D. \tag{3.1.47}$$

The Casimir C of the $SL(2,R)$ group takes then the form :

$$-C = \left(K^1\right)^2 - K^1 - K^+K^- = \frac{1}{2}(PK+KP) - D^2 \tag{3.1.48}$$

There are many puzzles of this realization:

- the first one is that there is not an invariant vacuum state $|\Omega\rangle$ which is annihilated by all the generators, i.e. a state satisfying

$$D|\Omega\rangle = P|\Omega\rangle = K|\Omega\rangle = 0. \tag{3.1.49}$$

If such a state existed, then we would have from the definition (36) of the Casimir operator the result $\mathcal{C}|\Omega\rangle = 0$. But in the irreducible representation D_k^+we must have $C = -r_0\left(r_0-1\right)$ which can not be satisfied generically.

- The second difference is the absence in the dAFF quantum mechanics of primary operators [1] satisfying the equations: [91]

$$\begin{aligned} {[P, \mathcal{O}_{\Delta}(0)] } & =-i \dot{\mathcal{O}}_{\Delta}(0) . \\ {[D, \mathcal{O}_{\Delta}(0)] } & =-i \Delta \mathcal{O}_{\Delta}(0) . \\ {[K, \mathcal{O}_{\Delta}(0)] } & =0 . \end{aligned} \tag{3.1.50}$$

Jackiw and his team [35] showed that the absence of a true conformal vacuum and primary operators does not prevent us from calculating the essential properties of the system. They achieved this by utilizing equations (3.1.50) and (3.1.49), which allowed them to compute correlation functions despite these limitations.

Selecting a specific vacuum state corrects the operator's conformal dimension, as it is not optimal for generating the expected behaviour for correlation functions.

we'll start , by using (3.1.50) which define primary operators and (**??**) which defines the ground state $|\Omega\rangle$ we obtain the condition

$$(x K+i D) \mathcal{O}_{\Delta}(0)|\Omega\rangle=\Delta \mathcal{O}_{\Delta}(0)|\Omega\rangle . \tag{3.1.51}$$

Where it show that there is an operator $O_{\Delta}(0)$ and a non-normalizable "state" $|\Omega\rangle$ that together conspire to satisfy (3.1.51)

we introduce a coherent-like state $|t\rangle$ associated with the time variable t

$$|t\rangle=\sum_{n}\langle n \mid t\rangle|n\rangle=\sum_{n} \beta_{n}^{*}(t)|n\rangle .$$

we have obtained [13]

$$\begin{aligned} |t\rangle & =O(t)|0\rangle \\ & =N(t) \exp \left(-\omega K^{+}\right)|0\rangle . \end{aligned} \tag{3.1.52}$$

This is sufficient for the existence of a state-operator correspondence. I Indeed, we have

$$\begin{aligned} & |0\rangle \longrightarrow|\Omega\rangle \\ & O(t)=N(t) e^{-\omega K^{+}} \longrightarrow \mathcal{O}_{\Delta}(t), \Delta=r_{0} \\ & |t\rangle=O(t)|0\rangle \longrightarrow\left|\mathcal{O}_{\Delta}(t)\right\rangle=\mathcal{O}_{\Delta}(t)|\Omega\rangle \end{aligned} \tag{3.1.53}$$

Although these do not satisfy the stronger conditions, they satisfy weaker ones enough to calculate the two- and third-point functions.

$$\left\langle 0\left|O^{\dagger}\left(t_{1}\right) O\left(t_{2}\right)\right| 0\right\rangle=\frac{f_{0}}{\left(t_{1}-t_{2}\right)^{2 r_{0}}} . \tag{3.1.54}$$

[1] The primary operator it is operator annihilated by lowering operator.

$$\left\langle 0\left|O^{\dagger}\left(t_{1}\right)B(t)O\left(t_{2}\right)\right|0\right\rangle=\frac{f_{0}}{\left(t-t_{1}\right)^{\delta}\left(t-t_{2}\right)^{\delta}\left(t_{1}-t_{2}\right)^{2r_{0}-\delta}}. \tag{3.1.55}$$

The surprising result is that the correlation functions calculated on the boundary perfectly match those obtained within the bulk of the two-dimensional anti-de Sitter space (AdS^2) when the space is commutative (standard, non-fuzzy).

.

3.2 AdS Spacetime

This section will outline the primary characteristics of the AdS spacetime. We are adhering to the presentation provided in [64], [159], [63], [140] and [65].

- The maximally symmetric spacetime with constant negative curvature, known as anti-de Sitter spacetime, is non-compact (it is infinite). Maximally symmetric refers to having the most symmetries. It is a solution to Einstein's equations with a negative cosmological constant, and its topology is $\mathbb{S}^{d-1}\times\mathbb{R}$. Many coordinates can be used when describing the anti-de Sitter space. We choose them according to the goal that is to be calculated, and we shall provide two descriptions of the anti-De Sitter space: global coordinates and Poincare patch coordinates.

  In the global coordinate, this space can be conceived as a cylinder with bases at $\tau=-\infty$ and $\tau=+\infty$, a $r=0$, while going around the cylinder is given by the angular variables Ω_{d-1}, the axis of the cylinder is the center of the Anti-de Sitter space. The cylinder walls are the boundary or perimeter of the Anti-de Sitter space. It is a conformal limit. See figure (3.2).

  In terms of global coordinates, this space can be visualized as a cylinder with bases at $\tau=-\infty$ and $\tau=+\infty$, a $r=0$, and rotation around the cylinder determined by the angular variables Ω_{d-1} . The cylinder's axis represents the center of the Anti-de Sitter space, and its walls serve as the perimeter or boundary. The limit is conformal. (3.2) (See Figure)

As embedded in Minkowski spacetime, the anti-de Sitter space is a generalized sphere in that it is the set of all points for which the distance is constant. We embed AdS_{d+1} [2] in the Minkowski spacetime $\mathbf{R}^{2,d}$ by :

$$-X_0^2-X_{d+1}^2+X_1^2+X_2^2+X_3^2+\ldots+X_d^2=-L^2. \tag{3.2.1}$$

[2]Since the counterpart CFT is assumed to have d spacetime dimensions, it is usual to use $d+1$ dimensions while investigating AdS/CFT..

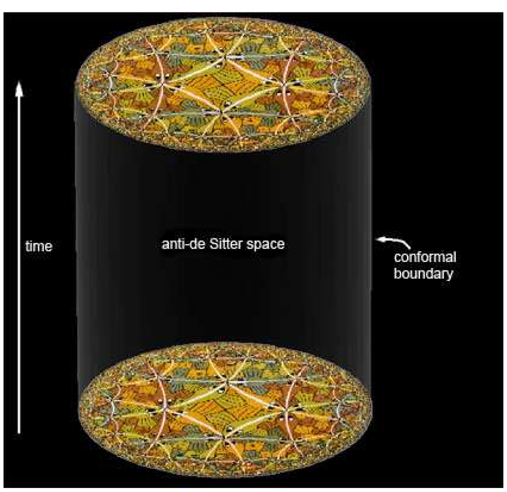

Figure 3.2: AdS in global coordinates.

where the coordinates are $X_1, X_2, ...X_{d+1}$ and L is the AdS radius. Now we can induce global coordinates $(\tau, r, \hat{x}_i)$ by the relations

$$\begin{aligned} X_0 &= L\frac{\cos\tau}{\cos r} \\ X_{d+1} &= L\frac{\sin\tau}{\cos r} \\ X_i &= L\tan r\hat{x}_i \end{aligned} \tag{3.2.2}$$

wher $\hat{x}_i$ are angular coordinates defines a $(d-1)$-dimensional sphere $\mathbf{S}^{d-1}$, i.e. $\sum_{i=1}\hat{x}_i^2 = 1$, and $r \in [0, \pi/2[$, and $\tau \in [0, 2\pi] \longrightarrow \tau \in [-\infty, +\infty]$ as we unwrap to the universal cover The metric is

$$ds^2_{d+1} = \frac{L^2}{\cos^2 r}\left(-d\tau^2 + dr^2 + \sin^2 r d\Omega_{d-1}\right) \tag{3.2.3}$$

The symmetry group of AdS_{d+1} is given by $SO(2, d)$; We will refer to this group as the conformal group because it is the same isometry group for CFT in d dimension with Lorentzian signature, which is an essential space in quantum gravity. For this reason, it plays an important role in the AdS/CFT correspondence.

How many generators are in this algebra? We can count the generators explicitly as we have d translations, one dilatation,d special conformal,$d(d-1)/2$ rotation in total, we have $d(d-1)/2 + 1 + d + d = (d+1)(d+2)/2$ generators.All of these transformations can be represented by :

$$L^A_B = X^A\frac{\partial}{\partial X^B} - X^B\frac{\partial}{\partial X^A} \tag{3.2.4}$$

$$L_{AB} = X_A\frac{\partial}{\partial X_B} - X_B\frac{\partial}{\partial X_A}. \tag{3.2.5}$$

For example, the generator of translations :

$$\begin{aligned} L_{(d+1),0} = X_{d+1}\frac{\partial}{\partial X_0} - X_0\frac{\partial}{\partial X_{d+1}} &= \ell\frac{\sin(\tau)}{\cos(r)}\frac{\partial}{\partial X_0} - \ell\frac{\cos(\tau)}{\cos(r)}\frac{\partial}{\partial X_{d+1}} = \\ &= -\frac{\partial X_0}{\partial t}\frac{\partial}{\partial X_0} - \frac{\partial X_{d+1}}{\partial t}\frac{\partial}{\partial X_{d+1}} = -\frac{\partial}{\partial t}. \end{aligned} \tag{3.2.6}$$

Hence, the Hamiltonian in anti-de Sitter spacetime is given by

$$\text{Hamiltonian } \equiv -i\frac{\partial}{\partial t} \tag{3.2.7}$$

Anti-de Sitter spacetimes have received much interest recently since they are produced by the geometry of the extreme black holes' near horizons. Maldacena's AdS/CFT correspondence conjecture, which puts AdS and even Euclidean quantum gravity in the spotlight, was inspired by this.

3.2.1 Euclidean Version and the Poincaré Patch

The best coordinates, easier to understand in Euclidean signature, are Poincaré patches, which cover the entire AdS spacetime, just like global coordinates. The reason for its importance is that it makes the d-dimensional Poincaré subgroup of the conformal group manifest [63]

The Euclidean AdS_{d+1} space (with conformal group $SO(1, d+1)$) is obtained by the Wick rotation $X_{d+1} \longrightarrow -iX_{d+1}$, i.e. it is given by the embedding

$$-X_0^2 + X_{d+1}^2 + \sum_{i=1}^{d} X_i^2 = L^2 \tag{3.2.8}$$

$t \longrightarrow -it$ and $\tau \longrightarrow -i\tau$ (global coordinate) When we consider the global coordinates, the t term of the metric (3.2.3) changes the sign. Additionally, it will simply replace the trigonometric functions in the global mapping (3.2.2)with hyperbolic trigonometric ones [63]

$$\begin{aligned} X_0 &= L\frac{\cosh\tau}{\cos r} \\ X_{d+1} &= L\frac{\sinh\tau}{\cos r} \\ X_i &= L\tan r\hat{x}_i. \end{aligned} \tag{3.2.9}$$

The map between these systems becomes then given by [63], [91]

$$\begin{aligned} X_0 &= L\cosh\tau\cosh\rho = \frac{z}{2}\left(1 + \frac{L^2 + \vec{x}^2 + t^2}{z^2}\right) \\ X_i &= L\sinh\rho\hat{x}_i = L\frac{x_i}{z}, i < d+1 \\ X_{d+1} &= L\sinh\tau\cosh\rho = \frac{z}{2}\left(1 - \frac{L^2 - \vec{x}^2 - t^2}{z^2}\right). \end{aligned} \tag{3.2.10}$$

The metric becomes

$$ds_{d+1}^2 = R^2\left(\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-1}\right) = \Omega^2(z)\left(dz^2 + d\vec{x}^2 + dt^2\right). \tag{3.2.11}$$

The function Ω must then satisfy

$$\Omega(z) = \frac{L}{z} \tag{3.2.12}$$

We get then

$$ds_{d+1}^2 = \frac{L^2}{z^2}\left(dz^2 + d\vec{x}^2 + dt^2\right) \tag{3.2.13}$$

3.2.2 A scalar field in AdS_{d+1}

We examine the most basic scalar field example in AdS in this part and follow [139] presentation.we Take the AdS_{d+1} space in euclidean signature with metric as an example.

$$ds^2_{d+1} = \frac{L^2}{z^2}\left(dz^2 + d\vec{x}^2 + dt^2\right). \tag{3.2.14}$$

In AdS_{d+1} , a scalar field's action is described by

$$S = \int d^{d+1}x\sqrt{g}\left[-\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi - \frac{1}{2}m^2\phi^2\right]. \tag{3.2.15}$$

We derived the Euler-Lagrange equation of motion.

$$\frac{1}{\sqrt{g}}\partial_M\left(\sqrt{g}g^{MN}\partial_N\phi\right) - m^2\phi = 0. \tag{3.2.16}$$

We replace metrics and explicitly receive the Klein-Gordon equation in the AdS_{d+1} reads.

$$z^{d+1}\partial_z\left(z^{1-d}\partial_z\phi\right) + z^2\partial^2\phi - m^2L^2\phi = 0. \tag{3.2.17}$$

In the $x-$ space, we apply the Fourier transform, i.e.

$$\phi(z,x) = \int \frac{d^dk}{(2\pi)^d}\exp(ikx)f_k(z). \tag{3.2.18}$$

$f_k(z)$ associated with k. However, it remains related to z because it remains related to the radial direction; we sabstude (3.2.18) in (3.2.17) the Klein-Gordon equation becomes:

$$z^{d+1}\partial_z\left(z^{1-d}\partial_z f_k\right) - k^2z^2f_k - m^2L^2f_k = 0. \tag{3.2.19}$$

Let us solve (4.8.3) near the boundary $z = 0$; it is proportional to the same degree the z we may expect $f_k \sim z^\beta$. We still know β. replace $f_k \sim z^\beta$ in (4.8.3) and after short derivation we get

$$\beta(\beta - d) - m^2L^2 = 0 \Rightarrow \beta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2}. \tag{3.2.20}$$

1. The first solution is beta postive and denoted by Δ, $\sim z^\Delta$

$$\beta = \Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2L^2} \sim z^\Delta \tag{3.2.21}$$

2. The second solution is beta-negative

$$\beta = \frac{d}{2} - \sqrt{\frac{d^2}{4} + m^2 L^2} = d - \Delta \sim z^{d-\Delta} \tag{3.2.22}$$

The general form The solution near the boundary is

$$f_k(z) \longrightarrow A(k) z^{d-\Delta} + B(k) z^{\Delta}, z \longrightarrow 0. \tag{3.2.23}$$

- for $m^2 > -\frac{d^2}{4L^2}$,Δ is real and we have $d - \Delta \leq \Delta$ and hence $z^{d-\Delta}$ is the dominant term $z^{d-\Delta} \gg z^{\Delta}$.We place the boundary at $z = \epsilon \longrightarrow 0$. Then, the behavior of the scalar field on the boundary is given by

$$\phi(z = \epsilon, x) = A(x) \epsilon^{d-\Delta}. \tag{3.2.24}$$

- this is divergent for $m^2 > 0$ and $d - \Delta$ is negative, and hence the scalar field living on the boundary (source field, live in CFT), is identified with $A(x)$, viz

$$\varphi(x) = \lim_{\epsilon \longrightarrow 0} \epsilon^{\Delta - d} \phi(\epsilon, x). \tag{3.2.25}$$

The field φ is the holographic dual of the AdS field ϕ.

Let $\mathcal{O}(z, x)$ dual operators to the scalar fields $\phi(z, x)$ and $\mathcal{O}(x)$ dual to the $\varphi(x)$,Their coupling is a boundary term of the form

$$S_{\text{bound}} = \int d^d x \sqrt{\gamma} \phi(\epsilon, x) \mathcal{O}(\epsilon, x) \tag{3.2.26}$$

$\sqrt{\gamma}$ is induced metric equal a $\frac{L^d}{\epsilon^d}$ at the $z = \epsilon$ and by plugging $\phi(\epsilon, x)$ we get:

$$\begin{aligned} S_{\text{bound}} &= \int d^d x \sqrt{\gamma} \phi(\epsilon, x) \mathcal{O}(\epsilon, x) \\ &= \int d^d x \frac{L^d}{\epsilon^d} \epsilon^{d-\Delta} \varphi(x) \mathcal{O}(\epsilon, x) \\ &= L^d \int d^d x \varphi(x) \mathcal{O}(x), \end{aligned} \tag{3.2.27}$$

where

$$\mathcal{O}(x) = \epsilon^{-\Delta}\mathcal{O}(\epsilon, x) \tag{3.2.28}$$

$$\mathcal{O}(\epsilon, x) = \epsilon^{\Delta}\mathcal{O}(x). \tag{3.2.29}$$

This relationship demonstrates how operators transform. We understand how the dual operator and field behave in a conformal boundary. This also shows explicitly that Δ is the scaling dimension of the dual operator $\mathcal{O}$

Chapter 4

Phase structure of emergent geometry/gravity on $\mathrm{AdS}^2_\theta/\mathrm{CFT}_1$

4.1 Fuzzy Spaces

Non-commutative geometry (NCG) is a novel method to discretisation [93], where coordinates are no longer commutative when replacing the canonical positions and momenta with Hermitian operators [3]. Geometric spaces (and their extensions) are studied using algebras of fields defined on them in non-commutative geometry [94], which is characterized by a spectral triple $(\mathcal{A}, \mathcal{H}, \Delta)$, define non-commutative algebra, Hilbert space, Dirac operator respectively, The classical geometry is retrieved.

As we know from the Heisenberg uncertainty principle that can not measure the exact position and momentum of the particle, this principle is an initial realizable picture of a noncommutative space, .i.e.the notion of a space-time point stops are replaced by Planck cell [3]. Since it is impossible to localize points there precisely, it is fuzzy [92].In the context of noncommutative geometry, fuzzy spaces [87]

Another option perspective on fuzzy space building is quantizing classical phase spaces of finite volume. The essential example perhaps of a symplectic space is the phase space of classical mechanics; symplectic manifolds of finite volume are given by the co-adjoint orbits of Lie group G, which can, therefore, be quantized when a Dirac-type quantization condition is satisfied, we get a quantum representation of the manifold described by linear operators on irreducible representations of the group [92], [90], [93]

4.1.1 Fuzzy Sphere

This passage introduces the concept of fuzzy spaces, known as matrix geometries. These are a particular type of mathematical object used in non-commutative geometry. The text mentions a specific example of a fuzzy space: the fuzzy sphere, denoted asS_N. This fuzzy sphere is cited as being researched by Madore [87] It's important to understand that the fuzzy sphere differs from the familiar round sphere (S^2). The fuzzy sphere is a quantized phase or non-commutative space, which means it has some mathematical properties that differ from those of the classical sphere. In simpler terms, the fuzzy sphere is an approximation to the regular sphere, but it uses a more complex mathematical framework from non-commutative geometry [95]

Beginning with a sphere S^2 embedded in R^3,by the constraint [98]

$$\sum_{a=1}^{3} n^a n^a = 1 \tag{4.1.1}$$

The algebra of the fuzzy sphere is the space of $n \times n$ matrices, and the geometry was originally specified using an analogue of the Laplace operator. [95]

The sphere S^2 has $SO(3)$ symmetry, generated by the angular momentum $\mathcal{L}_a = -i\epsilon_{abc} n_b \partial_c$, which states the LI algebra, which is calculated in Chapter 1.

$$\mathcal{L}_a = -i\epsilon_{abc} n_b \partial_c, [\mathcal{L}_a, \mathcal{L}_b] = i\epsilon_{abc}\mathcal{L}_c \tag{4.1.2}$$

and the Laplacian is $\mathcal{L}^2 = \mathcal{L}_a\mathcal{L}_a$,The eigenfunctions of the Laplacian Δ are given by the celebrated spherical harmonics Y_{lm}. with eigenvalues equal $l(l+1)$.

The differential geometry of a manifold can be described in terms of an algebra of functions defined on it [87],Using spherical harmonics Y_{lm}., constitute a complete set of functions on the sphere, i.e., they constitute an orthogonal basis of the Hilbert space of square-integrable functions on S^2 [19] the following general function of the sphere can be expanded [3]:

$$f(\vec{n}) = \sum_{l=0}^{\infty} a_{lm} Y_{lm}(\vec{n}) \tag{4.1.3}$$

The fuzzy sphere $\mathbf{S}^2_N$ [87], [88] can be obtained by quantization by quantizing the function algebra over the sphere by using its Poisson structure [19], which is is the co-adjoint orbit $SO(3)/SO(2)$ admits a non-degenerate symplectic two-form ω given by

$$\omega = \frac{R}{\kappa} \sin\theta d\theta \wedge d\phi. \tag{4.1.4}$$

They can be inverted to define Poisson structures because these forms are non-degenerate [62]:

$$\{f, g\} = \frac{\kappa}{R\sin\theta} \left(\partial_\theta f \partial_\phi g - \partial_\phi f \partial_\theta g\right) \tag{4.1.5}$$

The fundamental Poisson brackets are computed immediately :

$$\{X_a, X_b\} = \kappa\epsilon_{abc}X_c \tag{4.1.6}$$

Following the usual procedure for quantization [104].This means that the algebra of functions $\mathcal{C}(\mathcal{M})$ should be mapped to a non-commutative (operator) algebra $\mathcal{A}$

- defining a Hilbert space $\mathcal{H}$, substituting coordinate operators X^a for coordinate functions $\hat{X}^a$
- The basic Poisson brackets $\{.,.\}$ are replaced with the fundamental commutators $[.,.]/i$. The commutation relations are then obtained.

$$\left[\hat{X}_a, \hat{X}_b\right] = i\kappa\epsilon_{abc}\hat{X}_c, \mathbf{S}_N^2. \tag{4.1.7}$$

we obtain the fuzzy sphere $\mathbf{S}_N^2$ [87] , [88],The embedding constraint is given by

$$\hat{X}_1^2 + \hat{X}_2^2 + \hat{X}_3^2 = R^2, \mathbf{S}_N^2 \tag{4.1.8}$$

The coordinate operators $\hat{X}_a$ are given explicitly by

$$\hat{X}_a = \alpha L_a. \tag{4.1.9}$$

Witch is the solution of both the commutator equation and the embedding condition, L_a are the generators of $su(2)$ in the IRR's D_{I} characterized by the spin number $I = \{1/2, 1, 3/2, 2, \ldots\}$. These coordinate operators $\hat{X}_a$ define the algebra of operators $\mathcal{A}$ on the noncommutative $\mathbf{S}_N^2$. In particular, X_a are $N \times N$ Hermitian matrices where $N = 2l + 1$ [95].

This result deforms the algebra of functions to the algebra of matrices cit ahlem,

$$\mathcal{Q} : \mathcal{C}^\infty(S^2) \to Mat_N(\mathbb{C})$$

$\mathcal{A} = \mathrm{Mat}_N$ where Mat_N is the algebra of $N \times N$ Hermitian matrices..spectral tripel $(\mathcal{A}, \mathcal{H}, \Delta)$ encodes all of the sphere's geometric information [98] , [96], [97].

The algebra $\mathcal{A}$ of operators on S_N^2

In summary,The fuzzy sphere S_N^2 is a quantized phase phase, i.e. it is a noncommutative space,The noncommutative geometry of $\mathbf{S}_N^2$ is defined in terms of a spectral triple $(\mathcal{A}, \mathcal{H}, \Delta)$ consisting of an algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$ and a Laplacian Δ.we will follow [98], [3]

the fuzzy sphere is the finite algebra S_N^2 defined by $N \times N$ Hermitian matrix operators X_a where (a=1,2,3) satisfying the commutation relations [99], [100].

$$[X_a, X_b] = i\epsilon_{abc}\theta\frac{X_c}{R}, \tag{4.1.10}$$

$$X_1^1 + X_2^2 + X_3^2 = R^2. \tag{4.1.11}$$

These generators satisfy [71] , [98], [3]

$$[L_a, L_b] = i\epsilon_{abc}L_c, \quad \sum_a L_a^2 = c_2 = \frac{N^2-1}{4}. \tag{4.1.12}$$

Every irreducible representation in the direct sum corresponds to a different polarization tensor T_{LM} on $\mathbf{S}_N^2$; consequently, the following general function can be extended in terms of polarization tensors, Consequently, the following general function on can be extended in terms of polarization tensors [98]:

$$f = \sum_{l=0}^{L}\sum_{m=-l}^{l} f_{lm}\hat{Y}_{lm}. \tag{4.1.13}$$

witch act on this basis as [101]

$$[L_a, [L_a, T_{LM}]] = l(l+1)T_{LM}, \; [L_\pm, T_{LM}] = \sqrt{(l\pm m)(l+-m+1)}T_{LM}, \; [L_3, \hat{Y}_{lm}] = mT_{LM} \tag{4.1.14}$$

- The generators of the isometry group $SO(3)$ acting on the noncommutative $\mathbf{S}_N^2$ are given by the outer derivations $\hat{\mathcal{L}}_a$ defined by

$$\hat{\mathcal{L}}_a(f) = [L_a, f]\,. \tag{4.1.15}$$

- These derivations define natural derivatives (vector fields) on the noncommutative algebra $\mathcal{A}$ of $\mathbf{S}_N^2$.
- Indeed, we have the correct action $\hat{\mathcal{L}}_a\left(\hat{X}_b\right) = i\epsilon_{abc}\hat{X}_c$ on the coordinate operators $\hat{X}_b$.
- The Laplacian operator which fixes the metric structure on S_N^2 is then given in terms of the derivations $\hat{\mathcal{L}}_a$ by

$$\begin{aligned}\hat{\mathcal{L}}^2 &= \hat{\mathcal{L}}_a\hat{\mathcal{L}}_a \\ &= \hat{\mathcal{L}}_1^2 + \hat{\mathcal{L}}_2^2 + \hat{\mathcal{L}}_3^2.\end{aligned} \tag{4.1.16}$$

- The derivations $\hat{\mathcal{L}}_a$ (since they are commutators) act clearly both on the left and on the right of the algebra $\mathcal{A}$.

$$\frac{L}{2} \otimes \frac{L}{2} = 0 \oplus 1 \oplus 2 \oplus .. \oplus L \tag{4.1.17}$$

- This means in particular that the algebra $\mathcal{A}$ will decompose under the action of the isometry group $SO(3)$ as the tensor product of two identical irreducible representations D_l.

The commutative limit of S_N^2

- The deformation parameter α appearing in the fundamental commutation relations is not arbitrary. Clearly, α must approach 0 in the commutative limit.

- For $\mathbf{S}_N^2$ the coordinate operators $\hat{X}_a$ are given in terms of the generators L_a of $su(2)$ in the irreducible representation D_l by the relation [87], [88] $\hat{X}_a = \alpha L_a$.

- Thus, by substituting $\hat{X}_a = \alpha L_a$ in the embedding constraint, it is found that the value of the deformation is quantized in terms of the $su(2)$ spin quantum number $/$ as follows:

$$\frac{R^2}{\alpha^2} = I(I+1) = \frac{N^2-1}{4} \tag{4.1.18}$$

- The commutative limit is then defined by

$$\alpha \longrightarrow 0, N \longrightarrow \infty. \tag{4.1.19}$$

- We have seen that noncommutative functions on $\mathbf{S}_N^2$ are operators which belong to the algebra $\mathcal{A}$ which is viewed as the projective module $\mathcal{A} \equiv D_l \otimes D_l$.

- The algebra $\mathcal{A} \equiv D_I \otimes D_I = \text{Mat}_{\ N}$ approaches the commutative algebra of functions on the ordinary sphere $\mathbf{S}^2$.

- Indeed, in the commutative limit $I \longrightarrow \infty$ the coordinate operators $\hat{X}_a$ approach the coordinate functions X_a and the polarization tensors T_{LM} approach the spherical harmonics Y_{LM}.

- This means in particular that the spectrum of the noncommutative Laplacian $\hat{\Delta} = \hat{\mathcal{L}}_a\hat{\mathcal{L}}_a/R^2$ coincides with the spectrum of the commutative Laplacian $\Delta = \mathcal{L}_a\mathcal{L}_a/R^2$ in the commutative limit.

4.2 The noncommutative $\mathrm{AdS}_\theta^2 \,/\, \mathrm{H}_\theta^2$

Symplectic manifolds are important to geometric quantization, where our goal is to quantize the anti-de sitter space, Lorentzian AdS^2 /Euclidean AdS^2 (the pseudo-sphere H^2),since is the co-adjoint orbit $SO(1,2)/SO(2)$, admits a symplectic two-form given by

$$\omega = \frac{1}{\kappa}\frac{R}{\cos^2\sigma}d\tau \wedge d\sigma \tag{4.2.1}$$

These two forms are non-degenerate (invertible corresponding thus to a Poisson manifold) and closed, [62]:

$$\{f,g\} = \kappa\frac{\cos^2\sigma}{R}\left(\partial_\tau f\partial_\sigma g - \partial_\sigma f\partial_\tau g\right), \text{ Lorentzian/Euclidean } \mathbf{AdS}^2 \tag{4.2.2}$$

The fundamental Poisson brackets are computed immediately :

$$\left\{X^a, X^b\right\} = \kappa f^{ab}{}_c X^c, \text{ Lorentzian/Euclidean } \mathbf{AdS}^2 \tag{4.2.3}$$

Lorentzian AdS^2 are given by $f^{ab}c = -\epsilon^{ab}c$,while Euclidean AdS^2 are given by $f^{ab}c = \epsilon^{ab}c$

Following the usual procedure for quantization

Canonical quantization proceeds by introducing a Hilbert space H, replacing the coordinate functions X^a with coordinate operators $\hat{X}^a$ and replacing the fundamental Poisson brackets $\{.,.\}$ with the fundamental commutators $[.,.]/i$, i.e.obtained:

$$\left[\hat{X}^a, \hat{X}^b\right] = i\kappa f^{ab}{}_c \hat{X}^c, \text{ Lorentzian/Euclidean } \mathbf{AdS}^2_\theta. \tag{4.2.4}$$

In each case, the embedding constraint becomes given by:

$$\begin{aligned} &-\hat{X}_1^2 + \hat{X}_2^2 + \hat{X}_3^2 = -R^2, \mathbb{H}^2_\theta \\ &-\hat{X}_1^2 - \hat{X}_2^2 + \hat{X}_3^2 = -R^2, \mathrm{AdS}^2_\theta \end{aligned} \tag{4.2.5}$$

These coordinate operators are also satisfied :

$$\left[\hat{X}_a, \hat{X}_b\right] = i\kappa f_{abc}\hat{X}^c \tag{4.2.6}$$

The coordinate operators are given explicitly by:

1. Noncommutative Lorentzian AdS$^2_\theta$:n this case the solution is given by [102], [62]

$$\hat{X}^a = \kappa K^a. \tag{4.2.7}$$

   K^a are the generators of the “Lie algebra” $su(1,1)$’ , need to satisfy certain conditions. Firstly, they must be unitary, leading to the exclusion of finite-dimensional representations. Secondly, they must possess a negative Casimir value due to the embedding condition, resulting in the exclusion of the discrete series. Lastly, these generators must have a commutative limit, leading to the exclusion of the complementary series, as their Casimir values are limited. Consequently,K^a serve as the generators of $su(1,1)$ within the “continuous series” $C_k^{\frac{1}{2}} = P_s^{\frac{1}{2}}$ which are specified by a real number s in terms of which K is given by $K = \frac{1}{2} + is$. These are the infinite dimensional representations found in

   Noncommutative Euclidean AdS$^2_\theta$ or noncommutative pseudo-sphere $\mathbb{H}^2_\theta$: In this case the solution is given by [84]

$$\hat{X}^a = \kappa K^a \tag{4.2.8}$$

have specific requirements. These requirements are as follows:

(a) They need to be unitary, leading to the exclusion of finite-dimensional representations.

(b) They must have a negative Casimir value due to the embedding condition, which excludes the continuous and complementary series.

(c) They should possess a commutative limit, further excluding the complementary series.

As a result, Ka serve as "the generators" of su(1, 1) within "the discrete series" denoted as $D_k^\pm$.These are the infinite dimensional representations found in

4.2.1 The commutative limit

- The deformation parameter κ in the fundamental commutation relations is not arbitrary. Clearly, κ must approach 0 in the commutative limit.

- the solution (4.2.8) is substituted in the embedding constraint (1.2.7) [71]: by substituting (4.2.8)in the embedding constraint (1.2.7) it is found that the value of the deformation is quantized in terms of the su(1, 1) pseudo-spin quantum number as follows:

$$\frac{R^2}{\kappa^2} = \pm k(k-1) \tag{4.2.9}$$

The minus sign corresponds to Lorentzian AdS^2 whereas the plus sign corresponds to Euclidean AdS^2 or $\mathbb{H}^2$. These coordinate operators also satisfy

$$\left[\hat{X}_a, \hat{X}_b\right] = i\kappa f_{abc}\hat{X}^c \tag{4.2.10}$$

- The commutative limit is then defined by:

$$\kappa \longrightarrow 0, k \longrightarrow \infty$$

- $\mathbf{S}^2_N \longrightarrow \hat{X}_a = \alpha L_a$: The finite-dimensional IRR's of su(2) specified by the spin number $I = (N-1)/2$.

- $\mathbf{H}^2_\theta \longrightarrow \hat{X}_a = \kappa K_a$: Infinite-dimensional IRR's of $su(1,1)$ specified by the pseudo-spin number k in the discrete series $D_k^\pm$.

- $\mathbf{AdS}^2_\theta \longrightarrow \hat{X}_a = \kappa K_a$: Infinite-dimensional IRR's of $su(1,1)$ specified by the number $k = \frac{1}{2}+$ is in the continous series $C_k^{\frac{1}{2}} = P_s^{\frac{1}{2}}$.

4.3 The noncommutative AdS^2_θ and the fuzzy sphere $\mathbb{S}^2_N$

The matrices L_a appearing in the solution (4.6.12) are the angular momentum operators which generates the group of rotations $SO(3)$ of the sphere $\mathbb{S}^2$. They satisfy the $su(2)$ Lie algebra

$$[L_a, L_b] = i\epsilon_{abc}L_c. \tag{4.3.1}$$

The irreducible representations of this algebra are characterized by the eigenvalues $l(l+1)$ of the Casimir operator

$$C = L_1^2 + L_2^2 + L_3^2. \tag{4.3.2}$$

It is not difficult to show that the allowed values are $l = 0, 1/2, 1, 3/2, 2....$ with degeneracy equals $N = 2l + 1$ for each spin l representation which can be labeled by the eigenvalues m of L_3 given by $l, l-1, ..., -l+1, -l$. In other words, the corresponding Hilbert spaces is given by $\mathcal{H}_l = \{|lm\rangle\}$.

Similarly, the operators K_a appearing in the solution (4.6.13) are the generators of the group of pseudo-rotations $SO(1,2)$ of the pseudo-sphere $\mathbb{H}^2$. They satisfy the $su(1,1)$ Lie algebra

$$[K_a, K_b] = if_{ab}\;^cK_c. \tag{4.3.3}$$

The structure constants are given by $f^{ab}\;c = -\epsilon^{ab}\;c$ for Lorentzian AdS^2 and $f^{ab}\;c = \epsilon^{ab}\;c$ for Euclidean AdS^2.

The irreducible representations of the above $su(1,1)$ algebra are characterized by the eigenvalues $\pm k(k-1)$ of the Casimir operator

$$C = -K_1^2 \mp K_2^2 + K_3^2. \tag{4.3.4}$$

The plus sign corresponds to Lorentzian AdS^2 whereas the minus sign corresponds to Euclidean AdS^2.

The traces Tr_S and Tr_S appearing in the actions (4.6.5), (4.6.6) and (4.6.7) should now be understood to be defined in the Hilbert spaces $\mathcal{H}_l$ and $\mathcal{H}_k$ respectively. In particular, the trace Tr_S is finite dimensional, i.e. $\mathrm{Tr}_\mathrm{S}\mathbf{1} = N$ and thus we must have the identification

$$N_\mathrm{S} \equiv N = 2l + 1. \tag{4.3.5}$$

The AdS trace Tr_H on the other hand will be regularized in such a way that only $2k-1$ states are included, i.e. $\mathrm{Tr}_\mathrm{H}\mathbf{1} = 2k - 1$ and correspondingly we will choose the overall normalization N_H such that $N_\mathrm{H} \equiv 2k - 1$ for a complete parallel with the sphere sector [106]. Thus, the pseudo-spin quantum number $k - 1$ is the analogue of the spin quantum number l and the operators D_a become therefore $(2k-1)\times(2k-1)$ matrices. This regularization is discussed further in [106]. We have then

$$N_{\mathrm{H}} \equiv 2k-1. \tag{4.3.6}$$

From the metric (5.2.3) we see that the AdS spacetime has the same radius as the sphere and hence we will also impose the natural identification

$$N_{\mathrm{H}} \equiv N. \tag{4.3.7}$$

The coordinate operators $\hat{x}_a$ on the fuzzy sphere $\mathbb{S}^2_N$ are defined by

$$C_a = \varphi_S \hat{x}_a. \tag{4.3.8}$$

We want them to satisfy the embedding relation

$$\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = r^2 \ , \ \mathbb{S}^2_N. \tag{4.3.9}$$

This holds if and only if

$$\frac{r^2}{\alpha^2} = l(l+1). \tag{4.3.10}$$

They also satisfy

$$[\hat{x}_a, \hat{x}_b] = i\alpha\epsilon_{abc}\hat{x}_c. \tag{4.3.11}$$

Similarly, the coordinate operators $\hat{X}_a$ on the noncommutative pseudo-sphere $\mathbb{H}^2_\theta$ and on the noncommutative AdS^2_θ are defined by

$$D_a = \varphi_H \hat{X}_a. \tag{4.3.12}$$

We want them to satisfy the embedding relations

$$\begin{aligned} -\hat{X}_1^2 + \hat{X}_2^2 + \hat{X}_3^2 &= -R^2 \ , \ \mathbb{H}^2_\theta \\ -\hat{X}_1^2 - \hat{X}_2^2 + \hat{X}_3^2 &= -R^2 \ , \ \mathrm{AdS}^2_\theta. \end{aligned} \tag{4.3.13}$$

This is indeed true if and only if

$$\frac{R^2}{\kappa^2} = \pm k(k-1). \tag{4.3.14}$$

The minus sign corresponds to Lorentzian AdS^2 whereas the plus sign corresponds to Euclidean AdS^2 or $\mathbb{H}^2$. These coordinate operators also satisfy

$$[\hat{X}_a, \hat{X}_b] = i\kappa f_{abc}\hat{X}^c. \tag{4.3.15}$$

4.4 The noncommutative $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$

We are now in a position to write down more explicitly the coordinate operators on the near-horizon noncommutative geometry $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$ as the tensor product of the operator algebras and Hilbert spaces associated with the noncommutative anti-de Sitter space AdS^2_θ and the fuzzy sphere $\mathbb{S}^2_N$.

First, we must remember that the sphere regulator (4.3.5) is a fully $SO(3)-$symmetric regulator as opposed to the pseudo-sphere regulator (4.3.7) which is an ordinary hard cutoff which breaks $SO(1,2)$ invariance. These two cutoffs are given explicitly by $N_{\mathrm{S}} = 2l+1$ and $N_{\mathrm{H}} = 2k-1$. Thus, the size N of the matrix model (4.6.7) is given by

$$N = N_{\mathrm{S}} N_{\mathrm{H}} = (2l+1)(2k-1). \tag{4.4.1}$$

We can even employ the choice (4.3.7) in which case we will have $N_{\mathrm{H}} = N_{\mathrm{S}}$ or equivalently $k = l-1$ and $N = N_{\mathrm{S}}^2 = (2l+1)^2$.

The solution of the equations of motion which follow from the action $S_{\mathrm{HS}}[D,C]$, which is given by equation (4.6.7), is given immediately by

$$C_a = \phi_S L_a \otimes \mathbf{1}_H \ , \ D_a = \mathbf{1}_S \times \phi_H K_a. \tag{4.4.2}$$

These are $N_{\mathrm{S}} N_{\mathrm{H}} \times N_{\mathrm{S}} N_{\mathrm{H}}$ Hermitian matrices. Since the action (4.6.7) is well behaved only in the Euclidean signature we are only going to consider here the case of the noncommutative Euclidean AdS^2_θ, i.e. the case of the noncommutative pseudo-sphere $\mathbb{H}^2_\theta$. In other words, the K_a in the above equation are the generators of $su(1,1)$ in the discrete series $D^\pm_k$.

The coordinate operators ($\hat{x}^a$,$\hat{X}^a$) on the noncommutative $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$ are then given explicitly by

$$C_a = \varphi_S \hat{x}_a \ , \ D_a = \varphi_H \hat{X}_a. \tag{4.4.3}$$

Equivalently

$$\hat{x}_a = \alpha L_a \otimes \mathbf{1}_H \ , \ \hat{X}^a = \mathbf{1}_S \otimes \kappa K^a. \tag{4.4.4}$$

These coordinate operators satisfy the embedding relations and commutator equations given by

$$-\hat{X}_1^2 + \hat{X}_2^2 + \hat{X}_3^2 = -R^2 \ , \ \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = r^2. \tag{4.4.5}$$

$$\begin{aligned} [\hat{X}_a, \hat{X}_b] &= i\kappa f_{abc} \hat{X}^c \\ [\hat{x}_a, \hat{x}_b] &= i\alpha \epsilon_{abc} \hat{x}_c \\ [\hat{x}_a, \hat{X}_b] &= 0. \end{aligned} \tag{4.4.6}$$

Again, we must also have the quantization of the two deformation parameters α and κ in terms of the radii R and r and in terms of the spin and pseudo-spin quantum numbers l and k, viz

$$\frac{r^2}{\alpha^2} = l(l+1)\ ,\ \frac{R^2}{\kappa^2} = k(k-1). \tag{4.4.7}$$

By expanding the sphere action (4.6.5) around the background (4.6.12) we obtain a $U(1)$ gauge field on the fuzzy sphere $\mathbb{S}^2_N$ where the gauge coupling constant $1/g_S^2$ is given by

$$\frac{1}{g_S^2} = \tilde{\alpha}^4 = 4\alpha^4 l(l+1). \tag{4.4.8}$$

The full gauge dynamics obtained is actually a noncommutative $U(1)$ gauge field on the fuzzy sphere coupled to a scalar field normal to the sphere, i.e. a noncommutative Higgs theory on $\mathbb{S}^2_N$. See for example [147].

Similarly, by expanding the pseudo-sphere action (4.6.6) around the background (4.6.13) we obtain a $U(1)$ gauge field on the noncommutative pseudo-sphere $\mathbb{H}^2_N$ where the gauge coupling constant $1/g_H^2$ is given by

$$\frac{1}{g_H^2} = \tilde{\kappa}^4 = 4\kappa^4 k(k-1). \tag{4.4.9}$$

Again, the full gauge dynamics is a noncommutative $U(1)$ gauge field on the noncommutative pseudo-sphere coupled to a scalar field normal to the pseudo-sphere, i.e. a noncommutative Higgs theory on Euclidean AdS^2_θ.

Next we expand the action (4.6.7) around the background (4.4.2) to obtain a four-dimensional noncommutative gauge theory on the noncommutative near-horizon geometry $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$ with a gauge coupling constant given in the sphere and pseudo-sphere sectors by [149]

$$\frac{1}{g_{\mathrm{HS}}^2} = \bar{\kappa}^4 = \kappa^4 k(k-1)\ ,\ \frac{1}{g_{\mathrm{HS}}^2} = \bar{\alpha}^4 = \alpha^4 l(l+1). \tag{4.4.10}$$

Hence, in order to get a uniform gauge coupling constant across the sphere and the pseudo-sphere sectors we must have the constraint (we have also used here the choice (4.3.7))

$$\frac{1}{g_{\mathrm{HS}}^2} = \bar{\alpha}^4 = \bar{\kappa}^4 = \frac{\tilde{\kappa}^4}{4N} = \frac{\tilde{\alpha}^4}{4N}. \tag{4.4.11}$$

In this equation $\tilde{\alpha}^4 = \alpha^4 N^2$ and $\tilde{\kappa}^4 = \kappa^4 N^2$ in agreement with (4.4.8) and (4.4.9) respectively but with N defined by (4.4.1). This result summarizes the main difference between 2 and 4 dimensions, namely the fact that the gauge coupling constant in 4 dimensions scales differently with the size of the matrices than in 2 dimensions.

The backgrounds (4.6.12), (4.6.13) and (4.4.2) are global minima of their corresponding Yang-Mills matrix actions yielding in the gauge sector a $U(1)$ gauge group. We can also obtain

$U(n)$ gauge groups on the fuzzy sphere $\mathbb{S}^2_N$, on the noncommutative anti-de Sitter AdS^2_θ and on the noncommutative near-horizon geometry $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$ by considering other backgrounds with smaller group representations. See for example [163].

The commutative limit of the sphere $\mathbb{S}^2$ is obtained by taking $l \longrightarrow \infty$ whereas the commutative limit of anti-de Sitter spacetime AdS^2 is obtained by taking $k \longrightarrow \infty$. The commutative limit of the near-horizon geometry $\mathrm{AdS}^2 \times \mathbb{S}^2$ is given by taking together $l \longrightarrow \infty$ and $k \longrightarrow \infty$ in an almost obvious way.

4.5 Yang-Mills matrix models

Imagine a theory extending beyond Maxwell's electromagnetism, delving into strong coupling gauge theories; this is the world of Yang-Mills, which has been thorough in the setting of "gauge/gravity duality" [134]. Contains The emergence of space-time gravity, and geometry [107], another exciting thing about matrix models is that they lie in their connection to string theory, the leading candidate for a unified theory of all fundamental forces and particles. Specific matrix models can describe the dynamics of D-branes, string-like objects that are crucial in understanding string theory's non-perturbative aspects [108]. The discussion of quantized spaces is currently being explored about M(atrix) theory, a proposed non-perturbative formulation of string theory. In contrast [19], the Yang-Mills matrix models involve "second quantization" and describe the quantum gravitational fluctuations occurring in these noncommutative backgrounds [71], and the primary motivation for investigating these matrix models is the emergence of geometric transitions [109].

We start immediately by presenting the fundamental Yang-Mills matrix models. We start with the Yang-Mills matrix models.

$$S = -\frac{\Lambda^4}{4g^2}\,\mathrm{Tr}\,[X_\mu, X_\nu]\,[X^\mu, X^\nu] \tag{4.5.1}$$

We have the following properties:

- The X_μ are $N \times N$ hermitian matrices with N large and $X^\mu = \eta^{\mu\nu} X_\nu$ where the metric is $\eta_{\mu\nu} = \eta^{\mu\nu} = (-1, +1, \ldots, +1)$ (Lorentzian) and $\eta^{\mu\nu} = \eta_{\mu\nu} = (+1, +1, \ldots, +1)$ (Euclidean).

- There are d hermitian matrices X_μ, i.e. the index μ takes the values $\mu = 0, \ldots, d-1$ (Lorentzian) and $\mu = 1, \ldots, d$ (Euclidean). The Euclidean case $d = 10$ is the celebrated type IIB or IKKT model 2 .

- The above action is invariant under global rotations $SO(1, d-1)$ (Lorentzian) or $SO(d)$ (Euclidean) given by $X^\mu \longrightarrow X'_\mu = \Lambda^\mu{}_\nu X^\nu$.

- The above action is invariant under global translations given by $X_\mu \longrightarrow X'_\mu = X_\mu + c_\mu 1$.

- The above action is invariant under the local $U(N)$ gauge symmetry given by

$$X_\mu \longrightarrow X'_\mu = UX_\mu U^\dagger. \tag{4.5.2}$$

- We have the following dimensions:

$$[\Lambda] = L^{-1}, [X] = L, \left[g^2\right] = [S] = 1. \tag{4.5.3}$$

- The probability distribution is given by

$$P[X] = \frac{\exp(iS[X])}{Z}, Z = \int [dX] \exp(iS[X]), \text{ Lorentzian.} \tag{4.5.4}$$

$$P[X] = \frac{\exp(-S[X])}{Z}, Z = \int [dX] \exp(-S[X]), \text{ Euclidean} \tag{4.5.5}$$

The Euclidean measure is completely well defined.

- This model is obtained from the dimensional reduction to zero dimension of Yang-Mills gauge theory in d dimensions.

The equations of motion can be derived from the principle of least action

$$X \longrightarrow X + \delta X \Rightarrow S \longrightarrow S + \delta S, \delta S = 0. \tag{4.5.6}$$

We compute

$$\delta S = -\frac{\Lambda^4}{g^4} \operatorname{Tr} \delta X_\mu \left[X_\nu, [X^\mu, X^\nu]\right] = 0, \forall \delta X_\mu. \tag{4.5.7}$$

The equations of motion are then given by

$$[X_\nu, [X^\mu, X^\nu]] = 0. \tag{4.5.8}$$

The most obvious solutions are commuting matrices satisfying :

$$[X^\mu, X^\nu] = 0. \tag{4.5.9}$$

soulustion of equation of motion is:

$$\begin{aligned} S_{YM} &= -tr[X^a, X^b][X_a, X_b] \\ &= -([X^a, X^b][X_a, X_b])^i_{\ i} \\ &= -[X^a, X^b]^i_{\ j}[X_a, X_b]^j_{\ i} \end{aligned}$$

$$\begin{aligned} \frac{\partial S_{YM}}{\partial X^c} = 0 \Leftrightarrow = & \frac{\partial}{\partial (X^c)^l_{\ m}}([X^a, X^b]^i_{\ j}[X_a, X_b]^j_{\ i}) = 0 \\ & \frac{\partial}{\partial (X^c)^l_{\ m}}([X^a, X^b]^i_{\ j}[X_a, X_b]^j_{\ i}) = 2(\frac{\partial}{\partial (X^c)^l_{\ m}}[X^a, X^b]^i_{\ j})[X_a, X_b]^j_{\ i}, \quad \text{(due to symmetry)} \\ & = 2(\frac{\partial}{\partial (X^c)^l_{\ m}}[(X^a)^i_{\ k}(X^b)^k_{\ j} - (X^b)^i_{\ k}(X^a)^k_{\ j}])[X_a, X_b]^j_{\ i} \\ & = 2[\delta^a_c\delta^i_l\delta^m_k(X^b)^k_{\ j} + \delta^b_c\delta^k_l\delta^m_j(X^a)^i_{\ k} - \delta^b_c\delta^i_l\delta^m_k(X^a)^k_{\ j} - \delta^a_c\delta^k_l\delta^m_j(X^b)^i_{\ k}][X_a, X_b]^j_{\ i} \\ & = 2\big((X^b)^m_{\ j}[X_c, X_b]^j_{\ l} + (X^a)^i_{\ l}[X_a, X_c]^m_{\ i} - (X^a)^m_{\ j}[X_a, X_c]^j_{\ l} - (X^b)^i_{\ l}[X_c, X_b]^m_{\ i}\big) \\ & = 2\big(X^b[X_c, X_b] + [X_a, X_c]X^a - X^a[X_a, X_c] - [X_c, X_b]X^b\big)^m_{\ l} \\ & = 4\big(X^a[X_c, X_a] - [X_c, X_a]X^a\big)^m_{\ l} \\ & = 4[X^a, [X_c, X_a]]^m_{\ l} = 0. \end{aligned}$$

equation of motion of the field X given by:

$$[X^a, [X_b, X_a]] = 0. \tag{4.5.10}$$

4.6 The IKKT-type Yang-Mills matrix models $\mathbb{S}^2_N$, AdS^2_θ, $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$

The embedding coordinates of $\mathbb{S}^2$ and AdS^2 are denoted respectively by x^a and X^a. We will consider mostly Euclidean AdS^2 which is the pseudo-sphere $\mathbb{H}^2$. These coordinates satisfy the constraints

$$x_1^2 + x_2^2 + x_3^2 = r^2\ ,\ \mathbb{S}^2 \in \mathbb{R}^3. \tag{4.6.1}$$

$$-X_1^2 + X_2^2 + X_3^2 = -R^2\ ,\ \mathbb{H}^2 \in \mathbb{M}^{1,2}. \tag{4.6.2}$$

$$-X_1^2 - X_2^2 + X_3^2 = -R^2\ ,\ \mathrm{AdS}^2 \in \mathbb{M}^{2,1}. \tag{4.6.3}$$

From the metric (5.2.3) we can see that $\mathbb{S}^2$ and AdS^2 are characterized by the same radius and hence we must also have

$$R = r. \tag{4.6.4}$$

We can now immediately write down the Yang-Mills matrix models which enjoy the fuzzy $\mathbb{S}^2_N$, the noncommutative AdS^2_θ and the noncommutative $\mathrm{AdS}^2_\theta\times\mathbb{S}^2_N$ respectively as their global minimum. These Yang-Mills matrix models are essentially truncation of the IKKT matrix model [160] to lower dimensions. But they involve in an essential way a cubic Myers term [161] which is responsible for the condensation or emergence of matrix/noncommutative geometry. The fuzzy $\mathbb{S}^2_N$, the noncommutative AdS^2_θ and the noncommutative $\mathrm{AdS}^2_\theta\times\mathbb{S}^2_N$ present "first quantization" of the classical (commutative) geometry of the commutative sphere $\mathbb{S}^2$, the commutative anti-de Sitter AdS^2 and the commutative near-horizon geometry $\mathrm{AdS}^2\times\mathbb{S}^2$ respectively whereas the corresponding Yang-Mills matrix models present "second quantization" which captures the quantum gravitational fluctuations around these noncommutative backgrounds.

These Yang-Mills IKKT-type matrix models are given respectively by the following three actions (the first two are $D=3$ matrix models while the third is a $D=6$ matrix model)

$$S_{\rm S}[C] = N_{\rm S}{\rm Tr}_{\rm S}L_{\rm S}[C]\ ,\ \mathbb{S}^2. \tag{4.6.5}$$

$$S_{\rm H}[D] = N_{\rm H}{\rm Tr}_{\rm H}L_{\rm H}[D]\ ,\ \mathbb{H}^2. \tag{4.6.6}$$

$$\begin{aligned} S_{\rm HS}[D,C] &= N_{\rm H}N_{\rm S}{\rm Tr}_{\rm H}{\rm Tr}_{\rm S}\bigg(L_{\rm S}[C]+L_{\rm H}[D] \\ &- \frac{1}{4}[D_a,C_b][D^a,C_b]\bigg)\ ,\ \mathbb{H}^2\times\mathbb{S}^2. \end{aligned} \tag{4.6.7}$$

The Lagrangian terms are given in terms of three $(2l+1)\times(2l+1)$ Hermitian matrices C_a (and thus the trace ${\rm Tr}_{\rm S}$ is finite dimensional, i.e. ${\rm Tr}_{\rm S}=2l+1\equiv N_{\rm S}$) and three Hermitian operators D_a (and thus the trace ${\rm Tr}_{\rm H}$ is infinite dimensional) by the equations

$$\begin{aligned} L_{\rm S}[C] &= -\frac{1}{4}[C_a,C_b]^2+\frac{2i}{3}\alpha\epsilon_{abc}C_aC_bC_c \\ &+ \beta_S C_a^2. \end{aligned} \tag{4.6.8}$$

$$\begin{aligned} L_{\rm H}[D] &= -\frac{1}{4}[D_a,D_b][D^a,D^b]+\frac{2i}{3}\kappa f_{abc}D^aD^bD^c \\ &+ \beta_H D_aD^a. \end{aligned} \tag{4.6.9}$$

The action $S_{\rm S}[C]$ in the sphere sector is considered for example in [163]. The ambient metric in the sphere sector is naturally Euclidean and the Levi-Civita tensor ϵ_{abc} provides the structure constants of the rotation group $SO(3)=SU(2)/\mathbb{Z}_2$.

The ambient metric in the AdS sector is Lorentzian given by $\eta=(-1,+1,+1)$ (for Euclidean AdS^2, i.e. for the pseudo-sphere $\mathbb{H}^2$) or by $\eta=(-1,-1,+1)$ (for Lorentzian AdS^2) and as a consequence f_{abc} are the structure constants of $SO(1,2)=SU(1,1)/\mathbb{Z}_2$ or $SO(2,1)=SU(1,1)/\mathbb{Z}_2$ respectively with a Lie algebra given by $su(1,1)$ in both cases.

The classical equations of motion which follow from the sphere action $S_{\rm S}[C]$ and the pseudo-sphere action $S_{\rm H}[D]$ are given respectively by

$$[C^b, B_{ab}] - 2\beta_S C_a = 0 \ , \ B_{ab} = [C_a, C_b] - i\alpha\epsilon_{abc}C^c \ , \ \mathbb{S}^2. \tag{4.6.10}$$

$$[D^b, F_{ab}] - 2\beta_H D_a = 0 \ , \ F_{ab} = [D_a, D_b] - i\kappa f_{abc}D^c \ , \ \mathbb{H}^2. \tag{4.6.11}$$

A solution of these equations of motion is given by

$$C_a = \phi_S L_a \ , \ \phi_S = \alpha\varphi_S. \tag{4.6.12}$$

$$D_a = \phi_H K_a \ , \ \phi_H = \kappa\varphi_H. \tag{4.6.13}$$

The order parameters φ_S and φ_H are functions of the parameters $\tau_S = \beta_S/\alpha^2$ and $\tau_H = -\beta_H/\kappa^2$ respectively. Explicitly, we have

$$\varphi^3 - \varphi^2 + \tau\varphi = 0 \Rightarrow \varphi_0 = 0 \ , \ \varphi_\pm = \frac{1}{2}\big(1 \pm \sqrt{1-4\tau}\big). \tag{4.6.14}$$

The matrices L_a and the operators K_a appearing in (4.6.12) and (4.6.13) are essentially what defines the fuzzy sphere $\mathbb{S}^2_N$ and the noncommutative pseudo-sphere $\mathbb{H}^2_\theta$ (and the noncommutative anti-de Sitter spacetime ${\rm AdS}^2_\theta$) respectively.

4.7 Effective potentials and phase structure of $\mathbb{S}^2_N$ and ${\rm AdS}^2_\theta$

Starting now from the actions (4.6.5) and (4.6.6) we compute the one-loop effective actions on the fuzzy sphere $\mathbb{S}^2_N$ and on the noncommutative fuzzy pseudo-sphere $\mathbb{H}^2_\theta$ using the background field method. The one-loop effective potential around the sphere background (4.6.12) is computed in [147] whereas the one-loop effective potential around the pseudo-sphere background (4.6.13) is computed (using the regularization (4.3.6)) in [106]. We obtained the effective potentials

$$\begin{aligned}\frac{2V_S}{N_{\rm S}^2} &= 4\alpha^4 l(l+1)\left[\frac{1}{4}\varphi_S^4 - \frac{1}{3}\varphi_S^3 + \frac{1}{2}\tau_S\varphi_S^2\right] + \log\varphi_S^2 \\ \tau_S &= \frac{\beta_S}{\alpha^2}.\end{aligned} \tag{4.7.1}$$

$$\begin{aligned} \frac{2V_H}{N_{\rm H}^2} &= 4\kappa^4 k(k-1)\left[\frac{1}{4}\varphi_H^4 - \frac{1}{3}\varphi_H^3 + \frac{1}{2}\tau_H\varphi_H^2\right] + \log\varphi_H^2 \\ \tau_H &= -\frac{\beta_H}{\kappa^2}. \end{aligned} \tag{4.7.2}$$

These potentials are of the same mathematical form and thus the discussion of the corresponding phase diagrams is effectively the same. Here, the scalar fields φ_S and φ_H play the role of the order parameters characterizing the phase diagrams while the role of the temperatures T_S and T_H is played by the gauge coupling constants squared, i.e. $T_S \equiv 1/\tilde{\alpha}^4 = g_S^2$ and $T_H \equiv 1/\tilde{\kappa}^4 = g_H^2$ with $\tilde{\alpha}^4 = 4\alpha^4 l(l+1)$ and $\tilde{\kappa}^4 = 4\kappa^4 k(k-1)$.

Again we stress the fact that the cutoff $N_{\rm S} = N$ on the fuzzy sphere is a natural $SO(3)-$invariant cutoff whereas the cutoff $N_{\rm H} = N$ on the noncommutative pseudo-sphere is only a regulator which breaks explicitly $SO(1,2)$ invariance as the noncommutative pseudo-sphere and the noncommutative anti-de Sitter spacetime are really infinite-dimensional operator algebras. See equations (4.3.6) and (4.3.7) and also the discussion in [106]. This might be related to the fact that the matrix model (4.6.5) is truly an Euclidean action where all its phases are accessible by the Monte Carlo method whereas the matrix model (4.6.6) is only Euclidean in the sense that it gives an anti-de Sitter spacetime in the noncommutative pseudo-sphere phase.

The model on the fuzzy sphere is extensively studied by analytical and Monte Carlo methods for both $\tau_S =$ and $\tau_S \neq 0$ in [18], [150]. The phase structure in this case can be summarized as follows:

- We start by setting the logarithmic quantum correction to zero. The classical equation of motion admits three solutions:

$$\varphi_0 = 0\ ,\ \varphi_\pm = \frac{1 \pm \sqrt{1-4\tau_S}}{2}. \tag{4.7.3}$$

  The solution $\varphi_0 = 0$ (the Yang-Mills or matrix phase) is the global minimum (ground state) of the system in the regime $\tau_S > 1/4$. The solution φ_- (the geometric or fuzzy sphere phase) is the global minimum in the regime $0 < \tau_S < 1/4$. The model has no ground state for $\tau_S < 0$, i.e. $\beta_S < 0$. The two global minima $C_a = 0$ and $C_a = \phi_- L_a$ are separated by a potential barrier whose maximum height is reached at the local maximum φ_+.

- We should also mention here that the configuration $C_a = \phi_S J_a$ is also a local minimum of the system. The J_a are the generators of $SO(3)$ in a reducible representation characterized by the spin quantum numbers $j_i < l = (N-1)/2$ satisfying $\sum_i (2j_i+1) = N$. More precisely, we find that the configuration $C_a = \phi_- L_a$ has a negative energy and thus lower than the zero energy of the configuration $C_a = 0$ only in the regime $0 < \tau_S < 2/9$. This negative energy is minimized when $J_a = L_a$. In this regime the fuzzy sphere is indeed stable and the expansion of the matrix model around the fuzzy background $C_a = \phi_- L_a$ gives a noncommutative gauge theory which also includes coupling to a normal scalar field, i.e. a noncommutative Higgs system.

- At $\tau_S = 2/9$ the two configurations $C_a = \phi_- L_a$ and $C_a = 0$ become degenerate. Thus, in the regime $2/9 < \tau_S < 1/4$ the fuzzy sphere becomes unstable. The coexistence curve between the geometric fuzzy sphere phase and the Yang-Mills matrix phase asymptotes therefore to the line $\tau_S = 2/9$ (and not to the line $\tau_S = 1/4$) where the energy functional becomes a complete square.

- If we include the logarithmic quantum correction the potential becomes unbounded from below near $\varphi = \varphi_0 = 0$, i.e. the effective potential (4.7.1) is really valid only in the fuzzy sphere phase $\varphi = \varphi_- \neq 0$. But the Yang-mills phase can still be accessed by Monte Carlo simulation of the matrix model (4.6.5).

- In the quantum case the minimum φ_- (corresponding to the geometric fuzzy sphere phase) becomes a function of both τ_S and $\tilde{\alpha}^4 = 4\alpha^4 l(l+1)$. The critical coexistence curve exists therefore in the $(\tau_S, \tilde{\alpha})$ plane where the local minimum φ_- disappears. The conditions determining this curve are obviously given by $V_S' = 0$ and $V_S'' = 0$. Explicitly, we obtain the curve $\tilde{\alpha}_* = \tilde{\alpha}_*(\tau_S)$ defined by the equations

$$\frac{1}{\tilde{\alpha}_*^4} = \frac{\varphi_*^2(\varphi_* - 2\tau_S)}{8}$$
$$\varphi_* = \frac{3}{8}\left(1 + \sqrt{1 - \frac{32\tau_S}{9}}\right). \tag{4.7.4}$$

  Thus, as we increase τ_S from 0 to 1/4 the critical value $\tilde{\alpha}_*$ increases from around 2 to infinity. Thus, the critical temperature $T_* \equiv 1/\tilde{\alpha}_*^4 = g_{S*}^2$ decreases towards zero as we increase τ_S to 1/4. In other words, the geometric fuzzy sphere phase exists in the region of low temperatures T (or large $\tilde{\alpha}$) and $\tau_S < 1/4$.

- Hence, as the temperature is increased the fuzzy sphere phase evaporates to a pure matrix phase with no background geometrical structure. In this model the geometry condenses or emerges only as the system cools.

- These predictions, which are based on the one-loop effective potential (4.7.1), are confirmed by Monte Carlo simulation only for $\tau_S < 2/9$. It is observed (in Monte Carlo simulation) that the coexistence curve between the geometric fuzzy sphere phase (low temperatures) and the Yang-Mills matrix phase (high temperatures) for $2/9 < \tau_S < 1/4$ asymptotes very rapidly to the line $\tau_S = 2/9$ for $\tilde{\alpha} > \tilde{\alpha}_* = 4.02$ [150]. In other words, the region $2/9 < \tau_S < 1/4$ corresponds to the Yang-Mills matrix phase for all values of $\tilde{\alpha}$.

- In fact for $\tau_S > 2/9$ the geometric fuzzy sphere background is a metastable state with an observable decay to the Yang-Mills matrix background. This decay is not observable for $\tau_S = 2/9$ although the fuzzy sphere is not the true ground state even here.

- In the Yang-Mills matrix phase the ground state is given by $\varphi = \varphi_0 = 0$ and fluctuations are insensitive to the value of $\tilde{\alpha}$ and are dominated by commuting matrices. In fact, in this phase the matrix model (4.6.5) is dominated by the Yang-Mills term [153].

- More precisely, the Yang-Mills matrix phase is characterized by a joint eigenvalue distribution, for the three matrices C_1, C_2 and C_3, which is uniform inside a solid ball of some radius $R = 2.0$ in $\mathbb{R}^3$. The eigenvalue distribution of a single matrix is then given by the so-called parabolic law, viz [151–154]

$$\rho(x) = \frac{3}{4R^3}(R^2 - x^2). \tag{4.7.5}$$

- The transition from the geometric fuzzy sphere phase to the Yang-Mills matrix phase is of an exotic character in the sense that by crossing the coexistence curve at fixed τ_S from the fuzzy sphere side we encounter divergent specific heat with critical exponent equal 1/2. However, by crossing the coexistence curve at fixed $\tilde{\alpha} > \tilde{\alpha}_* = 4.02$ we find no critical fluctuations and the transition is associated with a continuous internal energy and discontinuous specific heat.

The description of the phase structure of the noncommutative pseudo-sphere $\mathbb{H}^2_\theta$ using the effective potential (4.7.2) is formally identical to the fuzzy sphere case. However, here we have at our disposal only the effective potential (4.7.2) since Monte Carlo simulations of the infinite-dimensional Lorentzian matrix model (4.6.6) are very difficult if not impossible. Nevertheless, the phase structure of the noncommutative pseudo-sphere $\mathbb{H}^2_\theta$ can be summarized as follows:

- The solution $\varphi_H = \varphi_0 = 0$ (the Yang-Mills or matrix phase) is the global minimum (ground state) of the system in the regime $\tau_H > 1/4$. The solution $\varphi = \varphi_- \neq 0$ (the geometric or noncommutative pseudo-sphere phase) is the global minimum in the regime $0 < \tau_H < 1/4$. The model has no ground state for $\tau_H < 0$, i.e. $\beta_H > 0$. The two global minima $D_a = 0$ and $D_a = \phi_- K_a$ are separated by a potential barrier whose maximum height is reached at the local maximum φ_+.

- In the regime of the noncommutative pseudo-sphere the expansion of the matrix model around the noncommutative background $D_a = \phi_- K_a$ gives a noncommutative gauge theory which also includes coupling to a normal scalar field.

- In the regime $2/9 < \tau_H < 1/4$ the noncommutative pseudo-sphere becomes unstable. The coexistence curve between the geometric noncommutative pseudo-sphere phase and the Yang-Mills matrix phase asymptotes therefore to the line $\tau_H = 2/9$.

- If we include the logarithmic quantum correction the potential becomes unbounded from below near $\varphi_H = \varphi_0 = 0$, i.e. the effective potential (4.7.2) is really valid only in the noncommutative pseudo-sphere phase $\varphi_H = \varphi_- \neq 0$.

- In the quantum case the minimum φ_- (corresponding to the geometric noncommutative pseudo-sphere phase) becomes a function of both τ_H and $\tilde{\kappa}^4 = 4\kappa^4 k(k-1)$. The critical coexistence curve exists therefore in the $(\tau_H, \tilde{\kappa})$ plane where the local minimum φ_- disappears. This curve is given by equation (4.7.4) with the substitution $\tilde{\alpha} \longrightarrow \tilde{\kappa}$, $\tau_S \longrightarrow \tau_H$.

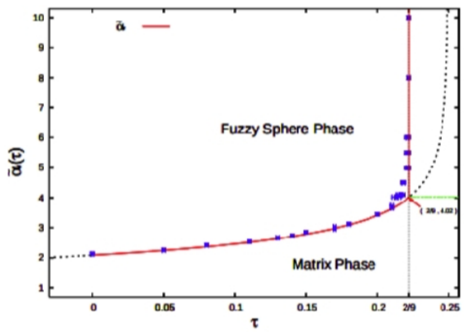

Figure 4.1: The actual phase diagram of the Yang-Mills matrix model on the fuzzy sphere [150]. The phase diagram of the Yang-Mills matrix model on the noncommutative pseudo-sphere is conjectured to share the same characteristics.

- Hence, as the temperature T_H is increased the noncommutative pseudo-sphere phase evaporates to a pure matrix phase with no background geometrical structure, i.e. the geometry condenses or emerges only as the system cools.
- It is also expected that the coexistence curve between the geometric noncommutative pseudo-sphere phase (low temperatures) and the Yang-Mills matrix phase (high temperatures) for $2/9 < \tau_S < 1/4$ will asymptote very rapidly to the line $\tau_H = 2/9$ for $\tilde{\kappa} > \tilde{\kappa}_* = 4.02$. In other words, the region $2/9 < \tau_H < 1/4$ corresponds to the Yang-Mills matrix phase for all values of $\tilde{\kappa}$.
- It is also conjectured that fluctuations in the Yang-Mills matrix phase are insensitive to the value of $\tilde{\kappa}$ and are dominated by commuting matrices, i.e. the matrix model (4.6.6) is dominated by the Yang-Mills term. As a consequence the Yang-Mills matrix phase is characterized by a joint eigenvalue distribution which is uniform inside a solid ball of some radius.

4.8 Phase diagram of $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$

The determination of the precise content of the phase diagram of the noncommutative near-horizon geometry $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$, where the noncommutative anti-de Sitter spacetime AdS^2_θ is Wick-rotated into the pseudo-sphere $\mathbb{H}^2_\theta$, requires the computation of the effective potential in the background (4.4.2). Monte Carlo simulations are useless in this case since the basic action here given by (4.6.7) is an infinite-dimensional (from the perspective of the symmetry group $SO(1,2)$) and Lorentzian (from the perspective of the embedding spacetime) Yang-Mills matrix model. In some sense the Yang-Mills matrix model (4.6.7) is genuinely Euclidean only in the four-dimensional geometric phase.

Let us simply start by writing the classical potential computed using the action (4.6.7) in the background configuration (4.4.2). The sphere and pseudo-sphere sectors are not geometrically entangled at the classical level and thus we obtain

$$\begin{aligned}\frac{V_{HS}}{2N_{\rm S}^2N_{\rm H}^2} &= \bar{\alpha}^4\left[\frac{1}{4}\varphi_S^4-\frac{1}{3}\varphi_S^3+\frac{1}{2}\tau_S\varphi_S^2\right]\\ &+ \bar{\kappa}^4\left[\frac{1}{4}\varphi_H^4-\frac{1}{3}\varphi_H^3+\frac{1}{2}\tau_H\varphi_H^2\right].\end{aligned} \tag{4.8.1}$$

The order parameters are still given by the two scalar fields φ_S and φ_H which are measuring the sizes of the sphere and pseudo-sphere respectively. Also recall that the scaling of the deformation parameters (gauge coupling constants) in four dimension are given by $\bar{\alpha}^4=\tilde{\alpha}^4/4N=\alpha^4N/4$ and $\bar{\kappa}^4=\tilde{\kappa}^4/4N=\kappa^4N/4$.

Before we sketch the calculation of the quantum effective potential we can immediately state the possible phases of the noncommutative near-horizon geometry $\mathrm{AdS}^2_\theta\times\mathbb{S}^2_N$ in the space $(\tau_S,\tau_H,\bar{\alpha},\bar{\kappa})$. The expected phases in this case are as follows:

- A Yang-Mills matrix phase with no background geometrical structure which is expected at high temperature (both $\bar{\alpha}$ and $\bar{\kappa}$ approach zero) or $2/9<\tau_{S,H}<1/4$.

- A $2-$dimensional geometric fuzzy sphere phase ($\bar{\alpha}$ approaches infinity, $\bar{\kappa}$ approaches zero and $0<\tau_{S,H}<2/9$).

- A $2-$dimensional geometric noncommutative pseudo-sphere phase ($\bar{\alpha}$ approaches zero, $\bar{\kappa}$ approaches infinity and $0<\tau_{S,H}<2/9$).

- A $4-$dimensional geometric noncommutative near-horizon geometry $\mathrm{AdS}^2_\theta\times\mathbb{S}^2_N$ phase at low temperature (both $\bar{\alpha}$ and $\bar{\kappa}$ approach infinity and $0<\tau_{S,H}<2/9$).

However, in order to have a single unified temperature $T_{\rm HS}$ we must have a single unified gauge coupling constant $g_{\rm HS}$ as in equation (4.4.11). This together with the regularization (4.3.6) and (4.3.7) allows us to set

$$\alpha=\kappa. \tag{4.8.2}$$

The temperature is then given by $T_{\rm HS}\equiv 1/\bar{\alpha}^4=1/\bar{\kappa}^4=g^2_{\rm HS}$ and the phase diagram becomes three-dimensional in the space $(\tau_S,\tau_H,\bar{\alpha}=\bar{\kappa})$.

The calculation of the effective potential on the noncommutative near-horizon geometry $\mathrm{AdS}^2_\theta\times\mathbb{S}^2_N$ which is based on the matrix model (4.6.7) is much more involved than the analogous calculation on the fuzzy $\mathbb{S}^2_N\times\mathbb{S}^2_N$ which is done in [148,149]. The difficulty can be traced to the fact that the sphere and the pseudo-sphere sectors become entangled quantum-mechanically through the third term in the action (4.6.7). This geometric quantum entanglement is of course essential for the emergence of a four-dimensional space. However, there exists a special case where this geometric quantum entanglement can be removed while keeping the background emergent geometry four-dimensional. A very important situation is the case when the coupling constants $\tau_S = \beta_S/\alpha^2$ and $\tau_H = -\beta_H/\kappa^2$ are identical, viz

$$\tau_S = \tau_H \iff \beta_S = -\beta_H. \tag{4.8.3}$$

From the actions (4.6.5), (4.6.6) and (4.6.7) it is obvious that these coupling constants couple to the radii of the fuzzy sphere $\mathbb{S}^2_N$ and the noncommutative pseudo-sphere $\mathbb{H}^2_\theta$ given respectively by $\mathrm{Tr}_S C_a^2$ and $\mathrm{Tr}_H D_a^2$.

In the special case (4.8.3) the 2−dimensional geometric phases are now expected to disappear and we end up with a single phase transition between a geometric noncommutative near-horizon geometry $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$ phase and a Yang-Mills matrix phase.

Let now A_α stands for the sphere and the pseudo-sphere configurations, i.e. $A_\alpha = (A^S_a, A^H_a)$ where $A^S_a \equiv C_a$ and $A^H_a \equiv D_a$. In the Feynman-'t Hooft background field method we decompose the field as $A_\alpha = B_\alpha + Q_\alpha$ where Q_α stands for the sphere and the pseudo-sphere quantum fluctuations and B_α stands for the sphere and the pseudo-sphere background fields which solve the classical equations of motion, i.e. $B_\alpha = (B^S_a, B^H_a)$ where $B^S_a \equiv C_a = \varphi_S \hat{x}_a$ and $B^H_a \equiv D_a = \varphi_H \hat{X}_a$. We expand the classical action (4.6.7) around the background fields B_α and keep only terms up to quadratic in the fluctuation fields B_α. In other words, we write the action (4.6.7) in the Gaussian form

$$S_{\mathrm{HS}}[A] \;=\; S_{\mathrm{HS}}[B] + N_{\mathrm{H}} N_{\mathrm{S}} \mathrm{Tr_H Tr_S} Q_\alpha \tilde{\Omega}_{\alpha\beta} Q_\beta + O(Q^3). \tag{4.8.4}$$

The linear term vanishes by the classical equations of motion. The local symmetry of this action (which is $U(N)$ by the regularization of anti-de Sitter spacetime employed here in this article) is gauge-fixed using the Lorentz gauge

$$[B_\alpha, Q_\alpha] \sim [B^S_a, Q^S_a] + \eta^{ab}[B^H_a, Q^H_b] = 0. \tag{4.8.5}$$

We thus add to the above quadratic action (4.8.4) the usual gauge fixing and Faddeev-Popov terms given by

$$\begin{aligned} S \;&\sim\; -N_{\mathrm{H}} N_{\mathrm{S}} \mathrm{Tr_H Tr_S} \frac{[B_\alpha, Q_\alpha]^2}{2\xi} \\ &+\; N_{\mathrm{H}} N_{\mathrm{S}} \mathrm{Tr_H Tr_S} c[B_\alpha, [B_\alpha, b]]. \end{aligned} \tag{4.8.6}$$

We will employ the Feynman gauge $\xi = 1$. We also note that the gauge-covariant Laplacian operator $\mathcal{B}^2$ is the sum of the sphere and the pseudo-sphere Laplacian operators, viz

$$\mathcal{B}^2(f) = [B_\alpha, [B_\alpha, f]] \sim [B^S_a, [B^S_a, f]] + \eta^{ab}[B^H_a, [B^H_b, f]]. \tag{4.8.7}$$

By performing the Gaussian path integral we obtain the one-loop effective action

$$\Gamma_{\mathrm{HS}}[B] \;=\; S_{\mathrm{HS}}[B] + \frac{1}{2} Tr \log \Omega - Tr \log \mathcal{B}^2. \tag{4.8.8}$$

The first term gives the classical potential (4.8.1). The most important term in the full gauge-covariant Laplacian operator Ω is of the frm

$$\Omega_{\alpha\beta} = \mathcal{B}^2\delta_{\alpha\beta} + ... \tag{4.8.9}$$

As it turns out, all the other terms in the gauge-covariant Laplacian operator Ω, when evaluated in the configuration (4.4.2), i.e. in $B_a^S = \varphi_S\hat{x}_a$ and $B_a^H = \varphi_H\hat{X}_a$, and using the Feynman gauge $\xi = 1$, are diagonal in the total Hilbert space associated with the tensor product of the fuzzy sphere and the noncommutative pseudo-sphere Hilbert spaces. These terms are also subleading compared to the first term written in (4.8.9). The geometric quantum entanglement between the sphere and the pseudo-sphere sectors is then only found in the gauge-covariant Laplacian operator $\mathcal{B}^2$. Indeed, we compute in the configuration (4.4.2) the trace

$$Tr\log\mathcal{B}^2 = Tr\log\left(\varphi_S^2\Delta_S + \varphi_H^2\Delta_H^2\right). \tag{4.8.10}$$

Δ_S and Δ_H are essentially the Laplacian operators on the sphere and the pseudo-sphere respectively, namely $\Delta_S = [\hat{x}_a, \hat{x}_a, .]]$ and $\Delta_H = [\hat{X}_a, [\hat{X}^a,]]$.

However, if we assume that the coupling constants τ_S and τ_H are equal as in (4.8.3) then one can check that the order parameters φ_S and φ_H must solve identical equations of motion. The choice (4.8.3) is also motivated by the form of the metric (5.2.3). A simplified model which captures the dynamics of the theory is then obtained by simply setting the two order parameters φ_S and φ_H equal from the outset, i.e.

$$\varphi_S = \varphi_H \equiv \varphi. \tag{4.8.11}$$

Thus, we get the logarithmic potential

$$\frac{1}{2}Tr_dTr\log\varphi^2 - Tr\varphi^2 = \frac{d}{2}N^2\log\varphi^2 - N^2\log\varphi^2. \tag{4.8.12}$$

Clearly, $d = 6$ here. The $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$ effective potential is then given by

$$\frac{V_{HS}}{2N_{\mathrm{S}}^2N_{\mathrm{H}}^2} = 2\bar{\alpha}^4\left[\frac{1}{4}\varphi_S^4 - \frac{1}{3}\varphi_S^3 + \frac{1}{2}\tau_S\varphi_S^2\right] + \log\varphi^2. \tag{4.8.13}$$

This effective potential is of the same form as the sphere effective potential (4.7.1) with the substitution

$$\tilde{\alpha}^4 \longrightarrow 2\bar{\alpha}^4. \tag{4.8.14}$$

Hence the discussion of the resulting phase structure proceeds along the same lines. The main result is the fact that as the system cools down the noncommutative geometry of $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$ emerges. Equivalently, as the temperature $T_{\mathrm{HS}} = 1/\bar{\alpha}^4 = g^2_{\mathrm{HS}}$ is increased the geometric noncommutative $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$ phase evaporates to a pure Yang-Mills matrix phase with no background geometrical structure. The coexistence curve is still given by (4.7.4) with the substitution (4.8.14).

Chapter 5

AdS ^{2} black holes and dilaton gravity

5.1 Black Holes

The study of black holes involves the convergence of QM and GR in physics [8], [20], the black hole is a massive star that has collapsed under its weight into a compact region of space [141], or, in other words, a region where space is falling faster than light. The first photograph of a black hole taken with the Event Horizon Telescope of the supermassive black hole in M87 in April 2019.

Thus, black holes are holes in space (since matter falls into them) that are black (there is no possibility of escape). It is the picture of classical black holes provided by Einstein's general theory of relativity. However, in 1974, Hawking showed that black holes are quantum objects that can emit radiation (the so-called Hawking radiation).

We shall outline the fundamental source of this radiation in a concise manner utilizing three key points. [142]

1. We initiate our investigation by examining the movement of a scalar particle, which resembles a quantum particle governed by the renowned "Schrodinger equation". In the tortoise coordinate system, the potential is

2. It is evident and trivial that this potential vanishes at ∞ and in event horizon r. means that the potential is free at these points. It exhibits a barrier in $r \sim 3r_s/2$. The point at which the potential has its highest value is. The barrier's height $\sim l^2$

3. However, these particles reach a state of equilibrium at the Hawking temperature, meaning that the energy v is directly proportional to a T_H. Hence, it is evident from this concise reasoning that particles without angular momentum, namely with l = 0, could traverse the potential barrier and liberate themselves from the gravitational pull of the black hole, ultimately reaching infinity. These are particularly Hawking particles. However, these particles reach a state of equilibrium at the Hawking temperature, meaning that the

energy v is directly proportional to a T_H. Hence, it is evident from this concise reasoning that particles without angular momentum, namely with l = 0, could traverse the potential barrier and liberate themselves from the gravitational pull of the black hole, ultimately reaching infinity. These are particularly Hawking particles.

The motion of a scalar particle of energy ν and angular momentum l is exactly equivalent to the motion of a quantum particle, i.e., a particle obeying the Schrodinger equation, with energy $E = \nu^2$ in a scattering potential given in the tortoise coordinate r_* with the expression

$$V(r_*) = \frac{r - r_s}{r}\left(\frac{r_s}{r^3} + \frac{l(l+1)}{r^2}\right) \ , \ r_* = r + r_s \ln\left(\frac{r}{r_s} - 1\right). \tag{5.1.1}$$

This potential vanishes at infinity and in the event horizon r_s; thus, the particle is free at infinity and in the event horizon. This potential is characterized by a barrier at $r \sim 3r_s/2$ where the potential reaches its maximum, and the height of this barrier is proportional to the square of the angular momentum, viz

$$V_{\max}(r_*) \sim \frac{l^2+1}{G^2M^2} \sim (l^2+1)T_H, \tag{5.1.2}$$

where $T_H = 1/(8\pi GM)$ is the Hawking temperature.

On the other hand, these particles are in thermal equilibrium at the Hawking temperature, and thus, the energy ν is proportional to T_H. Thus, we can immediately see from this simple argument that only particles with no angular momentum, i.e., $l = 0$, can go through the potential barrier and escape from the black hole to infinity. These particles are precisely Hawking particles.

The primary concern in quantum gravity is the phenomenon known as Hawking radiation [137]. Only thermal Hawking radiation remains once a black hole completely evaporates, leading to a loss of predictability or information. Consequently, the overall outcome of this process involves transforming an initially pure quantum state into a mixed state. This transition from a pure state to a mixed state is non-unitary and thus violates a fundamental principle of quantum mechanics. The concept of unity is compromised, resulting in the loss of information. This particular scenario is commonly referred to as the "information loss paradox" [124], [43]. In the context of black hole singularities, both matter and radiation suffer from this loss of information. Numerous solutions have been proposed to address this issue [137].

Various proposed resolutions have been put forth in order to address the complex nature of quantum gravity, one of which is the ADS/CFT correspondence. Known as the AdS/CFT correspondence [134] or the "gauge/gravity duality," this concept stands as a remarkable and pivotal discovery within the realm of string theory, providing profound insights into the field of quantum gravity. Not only does it offer explanations for the formation and disappearance of black holes, but the AdS/CFT conjecture also suggests that a gravitational system, which elucidates these processes, is inherently dual to a unitary QFT. This duality arises from the fact that if the evolution on the field theory side is defined as unitary, then the evolution on the gravity side must necessarily exhibit the same property [157]. Based on the tenets of superstring theory, it is postulated that a surface or boundary theory encompassing a general quantum gauge theory equipped with the principles of supersymmetry and conformality can encode all the essential information about black holes, space-time, or general quantum gravity. Nevertheless, this indirect resolution fails to provide a precise elucidation regarding how a black hole relinquishes the information that has fallen into it [125], [1].

5.2 1.1 $\mathrm{AdS}^2 \times \mathrm{S}^2$ as a near-horizon geometry

Einstein's gravity coupled to Maxwell's electromagnetism; This is a significant gravity. In Einstein's theory of gravity in two or three dimensions, we get topological gravity, and there is no no metric; there are just fields without energy. Things are different in this gravity, which contains gauge and dilaton fields in metrics. cite

We write the action of this theory and the equation of motion. We find one of the famous solutions, "the Reissner-Nordstrom black hole" Withe metric [118], [122]:

$$ds^2 = -t(r)d\tau^2 + \frac{dr^2}{t(r)} + r^2 d\Omega_2^2, \ t(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{5.2.1}$$

where: $\underbrace{r}_{Newton\ potential}$, $\underbrace{M}_{Newton\ potential}$, $\underbrace{Q}_{charge}$

It is a Schwarzschild black hole. With electric charge Q = 0.

As previously mentioned, It contains an electric field, which is important in describing geometry. This solution addresses three cases:

1. $M < Q$ This solution was rejected by cosmic censorship [119]).
2. $M = Q$ it is " extremal black holes" $(T = 0)$ which is called Bps states we have :
   $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$, $M = Q$ we get:
   $$f(r) = 1 - \frac{2M}{r} + \frac{M^2}{r^2} = (1 - M/r)^2$$
   ,
3. $M > Q$ in fact ther is
   - inner horizon r_-
   - outer horizon r_+ in case $M = Q$, $r_+ = r_- = M$

Thus, a quantum black hole with mass $M > Q$ will evaporate until it reaches the extremal mass $M = Q$ where the temperature vanishes, and the evaporation stops, i.e., the extremal quantum black hole acts as a stable ground state in the case of a charged black hole [120].

The near-horizon geometry is $\mathrm{AdS}^2 \times \mathbb{S}^2$,how can we see this?.We use new coordinates (canonical coordinates (t, z)), and we know them:

$$\begin{cases} r = Q\left(1 + \frac{\lambda}{z}\right) \longrightarrow Q \\ \tau = \frac{Qt}{\lambda}. \end{cases} \tag{5.2.2}$$

By replacing these definitions in the metric equation and calculating the limit $\lambda \longrightarrow 0$, we obtain:

$$ds^2 = \underbrace{\frac{Q^2}{z^2}\left(-dt^2 + dz^2\right)}_{AdS^2} + \underbrace{Q^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)}_{s^2}. \tag{5.2.3}$$

As it appears the near-horizon geometry is $\mathrm{AdS}^2 \times \mathbb{S}^2$, where the charge Q appears as the radius of both factors AdS^2 and $\mathbb{S}^2$ [121] .

Further, the $AdS2$ appears in high-dimensional black holes, so it is of great importance, and we devoted it to a study.

5.3 AdS^2 Black Holes in Dilaton Gravity

In the previous paragraph, we obtained the event horizon from Einstein's gravity coupled to Maxwell's electromagnetism", The second idea is to obtain it from the dilaton field. The latter plays an essential role in understanding the paradox of information loss.

First, let us look at dilaton gravity theory in four dimensions, which is shown by the following [24], [25]:

$$S = \int d^4x\sqrt{-\det g^{(4)}}e^{-2\phi}\left(R^{(4)} - F_{\mu\nu}F^{\mu\nu}\right) \tag{5.3.1}$$

After writing the equation of motion, which is trivial, we write the spherically symmetric non-singular black hole solution:

$$ds^2 = -\left(1 - \frac{r_+}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)} + r^2 d\Omega_2^2 \tag{5.3.2}$$

Which dilaton takes the form :

$$e^{2(\phi-\phi_0)} = \frac{1}{\sqrt{1-\frac{r_-}{r}}}. \tag{5.3.3}$$

When we write the equations of motion, we will get a real black hole in two dimensions, not just a limit of AdS. When there is a dilaton field, a black hole can exist, as we will see in the solutions to the equation of motion.

The temperature and the entropy of the black hole are given, on the other hand, by the relations [24]

$$T = \frac{1}{4\pi r_+}\sqrt{1-\frac{r_-}{r_+}}, S = \pi r_+^2 \tag{5.3.4}$$

The radii $r_\pm$are given explicitly by

$$2M = r_+, Q_M^2 = \frac{3}{4} r_+ r_-. \tag{5.3.5}$$

The extremal limit $T \longrightarrow 0$ of this black hole configuration is then given by $r_+ = r_- = Q = 2Q_M/\sqrt{3}$ or equivalently $M = Q_M/\sqrt{3}$.

For the extremal solution $r_+ = r_- = Q$ we introduce the coordinates

$$r = Q\left(1+\frac{4\lambda^2}{z^2}\right), t = \frac{QT}{\lambda}. \tag{5.3.6}$$

The metric and the dilaton in the near-horizon limit $\lambda \longrightarrow 0$ take then the form

$$\begin{gathered} ds^2 = \frac{4Q^2}{z^2}\left(-dT^2+dz^2\right) + Q^2 d\Omega_2^2 \\ e^{2(\phi-\phi_0)} = \frac{z}{2\lambda} \end{gathered} \tag{5.3.7}$$

It shows explicitly that the near-horizon geometry of the extremal black hole is indeed $\mathrm{AdS}^2 \times \mathbb{S}^2$.

The dilaton field does not exist in Reissner-Nordstrom and plays a key role here. We now have four dimensions of this field, allowing us to solve in two dimensions.

First, we take the metric in four dimensions and make a spherical reduction (Dimanche réduction) [1]. By decomposing the metric as follows :

[1]Dimensional reduction is the limit of a compactified theory where the size of the compact dimension goes to zero

$$\begin{aligned} ds^2 &= g^{(4)}_{\mu\nu} dx^\mu dx^\nu \\ &= g^{(2)}_{ab} dx^a dx^b + \Phi^2 \left(x^a\right) \gamma_{ij} dn^i dn^j \end{aligned} \tag{5.3.8}$$

The scalar field Φ is a dilaton field due to the spherical reduction. We compute then (see [25] and references therein). Now, we calculate that metric; it is diagonal.

$$\begin{aligned} &\sqrt{-\det g^{(4)}} = \Phi^2 \sqrt{-\det g^{(2)}} \sqrt{\det \gamma} \\ &R^{(4)} = R^{(2)} - \frac{2}{\Phi^2} \left(-1 + \partial_a \Phi \partial^a \Phi\right) - \frac{4}{\Phi} \Delta\Phi \end{aligned} \tag{5.3.9}$$

We have to divide it into dimensions; Ritchie Scalar is in two dimensions, plus stuff related to the sphere.

$$\int d^4x \sqrt{-\mathrm{detg}^{(4)}} R^{(4)} \quad = 4\pi \int d^2x \sqrt{-\det g^{(2)}} \left(\Phi^2 R^{(2)} \ +2\partial_a \Phi \partial^a \Phi + 2\right) \tag{5.3.10}$$

Hence, the action reduces to

$$S = 4\pi \int d^2x \sqrt{-\det g^{(2)}} e^{-2\phi} \left(\Phi^2 R^{(2)} + 2\partial_a \Phi \partial^a \Phi \quad +2 - \Phi^2 F^2\right) \tag{5.3.11}$$

We note that this field, from the point of view of two dimensions, appears as a scalar field.

The dilaton field $\Phi = r$ gives Φ for Schwarzschild-like coordinates. However, in the current case, the spherical reduction is performed on a sphere of constant radius $r = Q = 2Q_M/\sqrt{3}$, i.e. $\Phi = Q$ because we are talking on the event horizon.

We then get the action

$$S = 4\pi Q^2 \int d^2x \sqrt{-\det g^{(2)}} e^{-2\phi} \left(R^{(2)} + 2\Lambda^2\right) \tag{5.3.12}$$

It is called the Jackiw-Teitelboim action [22], which is one of the essential dilatonic gravity models because, in Einstein's theory of gravity in two or three dimensions, we get topological gravity, and there is no metric; there are just fields without energy. Things are different in this gravity, which contains gauge fields and dilaton fields in metries. The distribution of matter does not determine the local geometry of the space, as usually happens for gravity theories, but its global properties, i.e., its topology. In some ways, the dynamics of the gravitational and dilaton fields are trivial. [24]

The most general solution of the equations of motion stemming from the Jackiw-Teitelboim action [26] is given by the metric field (in the so-called Schwarzschild coordinates) and the dilaton field (with $\Phi = \exp(-2\phi)$)

In what are known as Schwarzschild coordinates, the metric field provides the most general solution of the equations of motion resulting from the Jackiw-Teitelboim action [26].

$$\begin{aligned} ds^2 &= -\left(\Lambda^2 r^2 - a^2\right) d\tau^2 + \frac{dr^2}{\Lambda^2 r^2 - a^2} \\ \Phi &= e^{-2\phi} = \Phi_0 \Lambda r \end{aligned} \tag{5.3.13}$$

When we calculate the scalar curvature we get $R = -2\Lambda^2$ i.e. it is AdS^2

This solution varies depending on the values of a^2 (which is integration constant associated with the mass M) , and we see that:

- when we put $a^2 = 0$,we obtain AdS^2 spacetime ,we will denote with AdS$_0$
- This solution looks like an AdS^2 black hole configuration when the $a^2 > 0$ with a horizon at $r_H = a/\Lambda$ denote with AdS$_+$

There can be no distinction between AdS$_0$ and AdS$_+$ locally,but globally there exists a coordinate transformation that brings AdS$_+$ into AdS$_0$ because of the way the dilaton field behaves. Moreover,this transformation is as follows :

$$\begin{cases} r \longrightarrow r' = a\Lambda\tau r \\ \tau \longrightarrow \tau' = \frac{1}{2a\Lambda} \ln\left(\Lambda^2\tau^2 - \frac{1}{\Lambda^2 r^2}\right) \end{cases} \tag{5.3.14}$$

Then, we can confirm right away that:

$$ds^2 = \underbrace{-\left(\Lambda^2 r'^2 - a^2\right) d\tau'^2 + \frac{dr'^2}{\Lambda^2 r'^2 - a^2}}_{a^2>0} = \underbrace{-\Lambda^2 r^2 d\tau^2 + \frac{dr^2}{\Lambda^2 r^2}}_{a^2=0} \tag{5.3.15}$$

This means that AdS$_+$ and AdS$_0$ are locally equivalent, but due to the previous coordinate translation, the dilaton field changes, thus.

$$\Phi_0 \sqrt{\frac{\Lambda^2 r'^2}{a^2} - 1} e^{-a\Lambda\tau'} = \Phi_0 \Lambda r \tag{5.3.16}$$

means that the $a^2 > 0$ solution is physically differnt from the $a = 0$ solution.

The above configuration is a solution of the equations of motion of the Jackiw-Teitelboim action representing an AdS^2 black hole configuration but with an additional scalar field. Since a negative dilaton field will produce a negative transverse sphere area in four dimensions, the black hole solution must stop at $r = 0$ with the horizon at $r_H = a/\Lambda$.

- Since $a^2 < 0$ corresponds to a negative mass, it is evident that this value is unphysical.

AdS_0 and AdS_+ have a relationship that is identical to the one between Minkowski spacetime and the Rindler wedge, where the parameter a^2 represents the acceleration. Consequently, the process of quantizing fields and making semi-classical considerations regarding Hawking radiation in the background of AdS_+, with AdS_0 as the ground state, follows the same procedure as the calculation of Hawking radiation in the Rindler wedge with Minkowski spacetime as the ground state. In simpler terms, the computation of the Hawking effect in this scenario is mathematically equivalent to the Unruh effect.

5.4 proposal to resolve the black hole information loss problem

After obtaining the black hole from Delaton gravity, we now want to know the effect of quantization on this hole in this gravity. To provide a solution to the information loss in the hole, this is a summary of the work presented inn [106]

We will start by quantizing the dilaton field because it distinguishes between the black hole and AdS space, which was previously defined as (5.3.13) . We use tortoise coordinate [2],which is written in terms of the radial coordinate as follows:

$$\sigma^{\pm} = \tau \pm \sigma, \sigma = -\frac{1}{a\Lambda} \operatorname{arctanh}\left(\frac{a}{\Lambda r}\right) \tag{5.4.1}$$

It is clear that in quantization, we use the Euclidean sign resulting from the Wick rotation $\tau \longrightarrow -i\tau$.

The relationship between coordinates global coordinates X_2 and Poincare coordinates (2.1.13), We recall that the temperature of a black hole is given by:

$$T = \frac{a\Lambda}{2\pi}$$

[2]They are just compounds that decrease the speed at which the sentence grows

This connection provides the dilaton field of the AdS^2 black hole in respect to the global coordinate X_2 of the AdS^2, our starting point will be to develop AdS$^2_\theta$ and their quantum gravity equivalent using a matrix model.

We quantize the coordinates in the usual canonical way, where:

$$\hat{\Phi} = 2\pi T \hat{X}_2. \tag{5.4.2}$$

which achieve (4.2.5,4.2.6 ,4.2.7)
We may express the equation as the following three commutators:

$$\begin{aligned}\left[\hat{\Phi}, \hat{X}^1\right] &= -i\kappa_1 \hat{X}^3 \\ \left[\hat{\Phi}, \hat{X}^2\right] &= 0 \\ \left[\hat{\Phi}, \hat{X}^3\right] &= -i\kappa_1 \hat{X}^1\end{aligned} \tag{5.4.3}$$

The matrix model describing quantum gravitational fluctuations about the noncommutative black hole is then given by the following D = 4 Yang-Mills matrix model: [31], [31]

$$\begin{aligned}S[D] = N\,\mathrm{Tr}\Big(&-\frac{1}{4}\left[D_\mu, D_\nu\right]\left[D^\mu, D^\nu\right] + 2i\kappa D^2\left[D^1, D^3\right. \\ &+ 2i\kappa_1\hat{\Phi}\left[D^1, D^3\right] + \beta D_a D^a \\ &-\frac{1}{2}\kappa_1^2 D_1 D^1 - \frac{1}{2}\kappa_1^2 D_3 D^3\Big)\end{aligned} \tag{5.4.4}$$

We are interested in the phase structure of the model with the parameters.

$$\beta = 0, \kappa \neq 0, \kappa_1 = 2\pi T\kappa. \tag{5.4.5}$$

In the context of Yang-Mills matrix models of the IKKT type, the loss of geometrical and gravitational information is essentially a unitary process within the path integral quantization. This process manifests as phase transitions, including a gravitational phase with dilaton and gauge field expectation values, a geometrical phase where only the gauge field condenses, and a Yang-Mills phase without condensation. Essentially, there is no information loss issue in these noncommutative models. However, in the commutative limit, the matrix description becomes invalid, and we revert to conventional descriptions of black hole evaporation where information loss seems apparent.

Chapter 6

Conclusion

This thesis concentrated on investigating gauge/gravity duality besides the Yang-Mills matrix model, which is considered the second quantization of geometry, and its most important characteristic is emergent geometry and Gravity. We specifically focused on AdS^2/CFT_1 because it is associated with a black hole.

We discussed the noncommutative $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$ and presented the phase structure of the corresponding Yang-Mills matrix models through The effective Potential using the background field method (in the special case where this geometric quantum entanglement can be removed). This effective potential is of the same form as the sphere effective potential. The critical coexistence curve exists, therefore, in the $(\tau_S, \tilde{\alpha})$ plane where the local minimum φ_-disappears. $V_S' = 0$ and $V_S'' = 0$ are obviously the conditions that determine this curve. Explicitly, we obtain the curve $\tilde{\alpha}_* = \tilde{\alpha}_*(\tau_S)$ defined by the equations:

$$\frac{1}{\alpha_*^4} = \frac{\varphi_*^2(\varphi_* - 2\tau_S)}{8}$$
$$\varphi_* = \frac{3}{8}\left(1 + \sqrt{1 - \frac{32\tau_S}{9}}\right)$$

$$\tilde{\alpha}^4 \longrightarrow 2\alpha^4.$$

This fundamental result is confirmed in the case of the sphere by Monte Carlo simulations of the Euclidean Yang-Mills matrix model, but in our case, we have only at our disposal the effective potential because of the Lorentzian signature of the embedding spacetime and the Hilbert spaces $\mathcal{H}_k$ infinite-dimensional.

Our analysis reveals an emergent geometry. The main result is the fact that as the system cools down, the noncommutative geometry of $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$ emerges. Equivalently, as the temperature $T_{\mathrm{HS}} = 1/\bar{\alpha}^4 = g^2_{\mathrm{HS}}$ is increased, the geometric noncommutative $\mathrm{AdS}^2_\theta \times \mathbb{S}^2_N$ phase evaporates to a pure Yang-Mills matrix phase with no background geometrical structure. Gauge/Gravity correspondence is currently the most successful proposal for quantum Gravity. And we see the

ADS Black hole. When this result is extended to Gravity, The resolution of the information loss problem in quantum black holes in the context of the noncommutative AdS^2/CFT_1 where unitarity is firmly preserved.

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