وزارة التعليم العالي والبحث العلمي

BADJI MOKHTAR-ANNABA UNIVERSITY
UNIVERSITE BADJI MOKHTAR-ANNABA

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Department of Mathematics

THESIS

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SEARCH FOR PERIODIC SOLUTIONS OF CERTAIN CLASSES OF PERTURBED DIFFERENTIAL SYSTEMS

Presented by
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THÈSE

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RECHERCHE DE SOLUTIONS PÉRIODIQUES DE CERTAINES CLASSES DE SYSTÈMES DIFFÉRENTIELS PERTURBÉS

Présentée par
Imane ZEMMOURI

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Dedication

بسم الله الرحمن الرحيم

الحمد لله رب العالمين، وصلى الله على صاحب الشفاعة سيدنا محمد النبي الكريم، وعلى آله وصحبه الميامين، ومن تبعهم بإحسان إلى يوم الدين وبعد:

إلى من كان لهم الفضل في كل خطوة، إلى أولئك الذين ساندوني في لحظات التحدي ومنحوني الأمل في أوقات اليأس. إلى من علموني أن الإصرار هو مفتاح النجاح. أهدي هذا العمل:

إلى والديَّ العزيزين، اللذين هما مصدر قوتي وإلهامي

إلى زوجي شريك العمر، الذي كان السند والرفيق في كل خطوة، وأضاء دربي بحبه وصبره وتفهمه.

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Abstract

This thesis is interested in the study of the maximum number of limit cycles for three different classes of perturbed differential systems, using averaging theory. The study is illustrated by application examples.

In first, we study the maximum number of limit cycles which bifurcate from the periodic orbits of the Hamiltonian system $\dot{x} = y^{2p-1}, \dot{y} = -x^{2q-1}$, when it is perturbed in the form

$$\dot{x} = y^{2p-1}, \dot{y} = -x^{2q-1} - \varepsilon(1 + Cs^{2m}\theta)Q(x, y),$$

where $Cs\theta, Sn\theta$ are generalized trigonometric functions, p, q, m are positive integers, the polynomial $Q(x, y)$ has degree $n \geq 2$ with $Q(0, 0) = 0$ and ε is a small parameter.

Afterwards, we study the maximum number of limit cycles for differential systems in form

$$\dot{x} = y, \dot{y} = -x - \sum_{k\geq 1} \varepsilon^k(1 + cos^{2m_k}\theta)Q_k(x, y),$$

where $k = 1, 2$, the polynomial $Q_k(x, y)$ has degree n_k and m_k a non-negative integer.

We study also the bifurcation of limit cycles from the periodic annulus surrounding the origin in a class of cubic polynomial differential systems in form

$$\dot{x} = y - y(x + a)^2, \dot{y} = -x + x(x + a)^2,$$

where a is a positive real number and $a \neq 1$, when perturbed within the class of all polynomial differential systems of degree 7.

Keywords : Averaging theory, Periodic orbit, Hamiltonian system, Limit cycle, Mathieu differential equation, Mathieu–Duffing equation.

Résumé

Cette thèse s'intéresse à l'étude du nombre maximal de cycles limites pour trois classes différentes de systèmes différentiels perturbés, en utilisant la théorie de moyennisation. Cette étude est illustrée par des exemples d'applications.

En effet, nous étudions le nombre maximal de cycles limites qui bifurquent des orbites périodiques du système Hamiltonien $\dot{x} = y^{2p-1}, \dot{y} = -x^{2q-1}$, lorsqu'il est perturbé sous la forme

$$\dot{x} = y^{2p-1}, \dot{y} = -x^{2q-1} - \varepsilon(1 + Cs^{2m}\theta)Q(x, y),$$

où $Cs\theta, Sn\theta$ sont des fonctions trigonométriques généralisées et p, q, m sont des entiers positifs, $Q(x, y)$ est un polynôme de degré $n \geq 2$ avec $Q(0, 0) = 0$ et ε est un petit paramètre.

Ensuite, nous étudions le nombre maximal de cycles limites pour les systèmes différentiels de la forme

$$\dot{x} = y, \dot{y} = -x - \sum_{k\geq 1} \varepsilon^k (1 + cos^{2m_k}\theta)Q_k(x, y),$$

où $k = 1, 2$, $Q_k(x, y)$ est un polynôme de degré n_k et m_k est un entier non négatif.

Nous étudions aussi la bifurcation des cycles limites à partir de l'annulus périodique entourant l'origine dans une classe de systèmes différentiels polynomiaux cubiques de la forme

$$\dot{x} = y - y(x + a)^2, \dot{y} = -x + x(x + a)^2$$

où a est un nombre réel positif et $a \neq 1$, lorsqu'ils sont perturbés au sein de la classe de tous les systèmes différentiels polynomiaux de degré 7.

Mots clés : Théorie de moyennisation, Orbite périodique, Système Hamiltonien, Cycle limite, Equation de Mathieu, Equation Mathieu-Duffing.

ملخص

تهتم هذه الأطروحة بدراسة العدد الأقصى للدورات الحدية لثلاث فئات مختلفة من الأنظمة التفاضلية المضطربة، باستخدام نظرية المتوسط .يتم توضيح هذه الدراسة بأمثلة تطبيقية.

في الفصل الثالث، ندرس العدد الأقصى للدورات الحدية التي تنشأ من ذبذبة المدارات الدورية للنظام الهاميلتوني

$$\dot{x} = \mathrm{y}^{2\mathrm{p}-1}, \dot{\mathrm{y}} = -x^{2\mathrm{q}-1}$$

عند تعرضه للاضطراب بالجملة:

$$\dot{x} = \mathrm{y}^{2\mathrm{p}-1}, \dot{\mathrm{y}} = -x^{2\mathrm{q}-1} - \varepsilon(1 + Cs^{2m}\theta)Q(x, y),$$

حيث $Cs\theta$ و $Sn\theta$ دوال مثلثيه معممة، p,q,m أعداد طبيعية، $Q(x, y)$ كثير حدود من الدرجة $n\geq 2$ بحيث $Q(0,0)=0$، و ε معامل صغير. كما سنرى، فإن النظام التفاضلي أعلاه يمثل تعميماً لمعادلة ماثيو التفاضلية.

في الفصل الرابع، ندرس العدد الأقصى للدورات الحدية للأنظمة التفاضلية من الشكل:

$$\dot{x} = y,\ \dot{y} = -x - \sum_{k\geq 1} \varepsilon^k (1 + Cs^{2m_k}\theta) Q_k(x, y),$$

حيث $k=1,2$، $Q_k(x,y)$ كثير حدود من الدرجة n_k، ε معامل صغير و m_k عدد طبيعي.

في الفصل الخامس، ندرس تفكك الدورات الحدية الناشئة عن الحلقة الدورية المحيطة بالمبدأ في فئة الأنظمة التفاضلية متعددة الحدود التكعيبية من الشكل:

$$\dot{x} = y - y(x + a)^2,\ \dot{y} = -x + x(x + a)^2,$$

حيث $a \neq 1$ عدد حقيقي موجب، عند تعرضها للاضطراب بالأنظمة التفاضلية المتعددة الحدود من الدرجة السادسة.

كلمات مفتاحية : نظرية المتوسط، مدار دوري، جملة هاميلتونيا، دورة الحد، المعادلة التفاضلية لماتيو معادلة ماتيو-دفينغ،

Introduction

In recent years, the study of dynamical systems has gained widespread attention, driven by its extensive applications in both theoretical and practical aspects of mathematics. Since the field of dynamical systems merges tools and concepts from disciplines like analysis, geometry, and topology. A system is considered dynamic when it evolves over time, and studying these systems is crucial for understanding complex behaviors across various fields, including chemistry, physics, biology, and economics. The evolution of these systems is often modeled using differential equations, providing a framework for analyzing their behavior. As such, dynamical systems play a major role in predicting and understanding intricate patterns in both natural and engineered systems.

The analysis of limit cycles in dynamical systems represents one of the most complex and intriguing areas in the qualitative study of differential equations. Almost challenge in this domain is the second part of Hilbert's sixteenth problem, which focuses on identifying the maximum number of limit cycles and their arrangements in planar polynomial differential systems of a specified degree. Despite extensive research spanning decades, this question remains unresolved, even for polynomial systems with relatively simple coefficients (see. [3], [29]). Recently, the investigation of limit cycle behavior has gained substantial interest across a wide range of disciplines, including biology, physics, chemistry, economics, and mechanics (see. [13], [21], [57], [59]). Significant strides have been made in exploring the emergence of limit cycles from periodic orbits in differential systems with centers, whether linear or nonlinear, as evidenced by numerous studies (see. [10], [62], [44], [8], [55]). These limit cycles, defined as isolated periodic trajectories in phase space, were the first introduced by Poincaré in 1881 ([46]). Since then, they have become a cornerstone of research due to their wide-ranging applications across diverse scientific fields.

Mathieu equation, one of the most famous second-order differential equations, is given by

$$\ddot{x} + b\,(1 + \cos t)\,x = 0, \tag{0.1}$$

where b is a real constant (see [27], [35]). This equation, first studied by Mathieu in 1868 during his research on the vibrations of an elliptical drumhead, has become a fundamental model in fields such as quantum mechanics, mechanical engineering, and theoretical physics. Its solutions are expressed in terms of Mathieu functions, which possess a wide range of properties and are extensively used to model physical phenomena (see [31], [49]). Mathieu equation can also exhibit limit cycle behavior, representing stable periodic oscillations under certain conditions, with significant applications in the design of mechanical systems, electronic circuits and other systems exhibiting oscillatory behavior (see. [14]).

The equation (0.1) can be reformulated as the following system of first-order differential equations

$$\begin{cases} \dot{x} = y \\ \dot{y} = -b\,(1 + \cos t)\,x, \end{cases}$$

Afterwards, various generalizations of the Mathieu equation have been proposed. One notable extension is the Mathieu-Duffing equation, which introduces a nonlinear term to account for more complex behaviors. The general form of the Mathieu-Duffing equation is presented as

$$\ddot{x} + b\,(1 + \cos t)\, x - x^3 = 0,$$

where the cubic nonlinearity x^3 models nonlinear stiffness, making this system more suitable for studying complex oscillatory phenomena in mechanical and electrical systems.

In the generalized form (see [16]), the system can be expressed in the following form

$$\begin{cases} \dot{x} = y \\ \dot{y} = -b\,(1 + \cos t)\, x + x^{\beta}, \end{cases} \tag{0.2}$$

where β is a real number, often representing different types of nonlinearity. This form(0.2) of Mathieu-Duffing equation is widely used to study nonlinear oscillations and bifurcation phenomena.

Another noteworthy generalization is presented in [61], where the authors conducted an in-depth study of the limit cycles of a modified system, represented as

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \varepsilon \left(1 + \cos^{l}\theta\right) Q\,(x, y)\,, \end{cases}$$

where $\theta = \operatorname{arc\,tg}(y/x)$, $Q(x, y)$ is a polynomial of degree k and $l \in \mathbb{N}$.

The study of the Mathieu-Duffing equation and its extensions are crucial for understanding more complex dynamical behaviors, such as chaos and bifurcations. Recent works, such as those by Kyziol and Okninski (2023, [24]) on jump phenomena in the asymmetric Duffing oscillator, have demonstrated the applicability of these systems in modeling real-world phenomena. In another study, Mao et al. (2024, [37]) explored the use of the fractional-order Duffing system for weak signal detection, showcasing the system's sensitivity and robustness in noisy environments. These studies underline the importance of both the Mathieu and Mathieu-Duffing equations in modern research, particularly in areas dealing with nonlinear dynamics and complex oscillatory systems.

Among the important results concerning limit cycles are those related to cycles that bifurcate from a single degenerate equilibrium point, such as those arising from Hopf bifurcation. These are known as small amplitude limit cycles, and significant studies on them can be found in the work of Lloyd [34]. Significant progress has also been made in uncovering partial results regarding the number of small-amplitude limit cycles in various classes of generalized Liénard polynomial differential systems (see. [6], [7]). However, obtaining periodic solutions analytically is generally a challenging task, often bordering on the impossible. So the averaging theory can play a crucial role, simplifying this difficult problem for certain ordinary differential equations by transforming it into a problem of finding the zeros of nonlinear functions.

The study of limit cycles in polynomial dynamical systems is a vibrant and diverse area of mathematical research, utilizing various important perturbative methodologies to achieve advancements in this field.Among the diverse methodologies available, averaging theory stands out as one of the most influential and widely utilized approaches. Its roots can be traced back to the pioneering works of Lagrange and Laplace in 1788, who laid the foundation for this method through their insightful and intuitive justifications. Subsequently, the first proof of its asymptotic validity was presented by Fatou in 1928, followed by Krylov and Bogoliubov in 1934 [4] and further developed by Mitropolsky and Bogoliubov in 1961 [5]. The method has since been refined by numerous researchers, including Verhulst [56], Sanders and Verhulst [50], Malkin [36], Roseau [48], and Buic and Llibre [9].

Averaging theory is also a powerful tool in the study of systems with centers, where limit cycles can emerge through the perturbation of a system characterized by a center. This perturbation induces the bifurcation of limit cycles from the periodic orbits encircling the center of the unperturbed system, as detailed in [[38], [53], [43]]. This methodology has undergone various developments over the decades by many scholars, including Verhulst [56], Sanders and Verhulst [51], Malkin [36], Roseau [47], Buic and Llibre [9], Giné, Grau, and Llibre [20], and Llibre, Novaes, and Teixeira [33].
In our work, we concentrate on exploring limit cycles in three distinct classes of differential systems:

(i) Generalized Mathieu Differential Equations,

(ii) Generalized Mathieu- Duffing Equations,

(iii) Septic Polynomial Differential Systems.

The structure of this thesis is organized as follows

Chapter 1: Preliminary Concepts and Definitions

First chapter presents a comprehensive overview of fundamental concepts related to the study of differential systems, while revisiting classical preliminary notions and the essential mathematical tools required for this thesis. It delves into polynomial differential systems, techniques for solving such systems, and the core principles of dynamical systems. Key analytical techniques, such as linearization, are presented to analyze the behavior of systems in the vicinity of equilibrium points. The chapter expands on this foundation by exploring periodic orbits and limit cycles, emphasizing their stability and identifying the conditions that govern their presence or absence. Moreover, Hamiltonian systems are discussed, with a particular emphasis on their significance in energy-conserving dynamics. Lastly, critical equations and definitions are outlined to provide a robust foundation for the detailed analysis conducted throughout this work.

Chapter 2: Averaging Theory

The second chapter focuses on the main tool employed to establish our various results the theory of averaging. This theory is presented through key theorems and illustrated with several examples. In particular, we delve into the first-order and second-order averaging methods, providing practical demonstrations to highlight their application and effectiveness.

Chapter 3: Limit Cycles for a Class of Generalized Mathieu Differential Equations Using Averaging Theory

In this chapter, we delve into the bifurcation analysis of limit cycles within Hamiltonian systems. Utilizing the first-order averaging theory, we determine upper limits on the maximum number of limit cycles that can arise from the periodic orbits of the Hamiltonian system characterized by

$$\dot{x} = y^{2p-1}, \dot{y} = -x^{2q-1},$$

where p and q are positive integers. The study is framed within the context of perturbations belonging to a specific class of generalized Mathieu differential equations, providing a systematic approach to understanding their dynamic behavior. This study was published in the international journal "Nonlinear Studies" titled "Limit cycles of a class of generalized Mathieu differential equations", for more details see [44].

Chapter 4: On the number of limit cycles in a class of generalized Mathieu-Duffing equations

In this chapter, we investigate limit cycles in Mathieu-Duffing equations, which are significant in physics, engineering, and applied mathematics. Using first and second-order averaging theory, we derive upper bounds for the maximum number of limit cycles that can bifurcate from the periodic orbits of the center system

$$\dot{x} = y, \dot{y} = -x,$$

when it is perturbed by generalized Mathieu-Duffing differential equations. Our approach combines averaging theory and Descartes' theorem to determine the maximum number of limit cycles in these systems.

This chapter is submitted for publication in the international journal " Symmetry".

Chapter 5: A number of limit cycle of septic polynomial differential systems via the averaging theory

This chapter explores the upper limits on the number of limit cycles that may arise via bifurcation from the periodic orbits of a specified system, characterized by the equations

$$\dot{x} = y - y(x+a)^2, \dot{y} = -x + x(x+a)^2,$$

where a is a positive parameter not equal to 1. Our analysis focuses on perturbations of this system within the context of septic polynomial differential systems. Using the first-order averaging theory, we demonstrate that the perturbed system can generate up to twenty-two limit cycles. This result highlights the complex bifurcation behavior of the system under specific perturbations, offering deeper insights into its underlying dynamics.

This study was published in the international journal "Partial Differential Equations in Applied Mathematics" titled "Limit cycles of septic polynomial differential systems bifurcating from the periodic annulus of cubic center", for more details (see. [62]).

Chapter 1 Preliminary Concepts and Definitions

In this chapter, we present key foundational concepts essential for the qualitative analysis of polynomial differential systems and dynamical systems.

1.1 Polynomial Differential Systems

Polynomial Differential Systems are differential equations where derivatives depend on polynomial functions, used to study nonlinear dynamics such as stability, oscillations, and chaos. They have applications in natural sciences (e.g., disease modeling), engineering (system stability), economics (market dynamics), and physics (chaotic systems), making them essential in mathematical modeling.

Definition 1.1.1. *A polynomial differential system in* $\mathbb{R}^n$ *is elegantly char-*

acterized as a mathematical construct of the form

$$\begin{cases} \dfrac{dy_1(t)}{dt} = Q_1\left(y_1(t), y_2(t), \ldots, y_n(t)\right), \\ \dfrac{dy_2(t)}{dt} = Q_2\left(y_1(t), y_2(t), \ldots, y_n(t)\right), \\ \vdots \\ \dfrac{dy_n(t)}{dt} = Q_n\left(y_1(t), y_2(t), \ldots, y_n(t)\right), \end{cases} \tag{1.1}$$

where Q_1, $Q_2, ...,$ Q_n are polynomials with real coefficients,where y_i represents the dependent variables and t is the independent variable, typically interpreted as time.

Remark 1.1.1. *The system (1.1) is said to be of degree s, where*

$$s = Max(DegQ_1, DegQ_2, ..., DegQ_n).$$

The degree of a polynomial differential system is an important characteristic. It provides insight into the complexity of the system. For instance, a first-degree polynomial (linear) system is much simpler to analyze than a higher-degree one.

Remark 1.1.2. *An autonomous polynomial differential system is one where the polynomials Q_1, $Q_2, ...,$ Q_n are functions of the state variables only and do not explicitly depend on time t. This implies that the system's behavior is time-invariant, making it easier to analyze the stability and long-term behavior of the system.*

1.1.1 Significance in Qualitative Analysis

Qualitative analysis of polynomial differential systems involves studying the properties of the solutions without necessarily finding the exact solutions. This encompasses the analysis of equilibrium point stability, the behavior of periodic orbits, and the overall dynamics of trajectories within the phase space.

- **Equilibrium Points**: Points where $\frac{dy_1(t)}{dt} = 0, ..., \frac{dy_n(t)}{dt} = 0$. These points are crucial as they represent the steady states of the system.
- **Stability Analysis**: Determines whether small perturbations around equilibrium points grow or decay over time.
- **Phase Portraits**: Graphical representations that provide a visual insight into the trajectories of the system.

By exploring these aspects, researchers can predict the system's long-term behavior under various initial conditions.

1.2 Solution of a Differential System

Differential system solutions are crucial in understanding and predicting phenomena in fields like physics, biology, engineering, and economics, as they provide insights into stability, oscillations, and long-term system behavior.

Definition 1.2.1. *A solution to the system (1.1) is any differentiable function*

$$Y : J \subset \mathbb{R} \longrightarrow \mathbb{R}^n$$

$$t \longmapsto Y(t) = (Y_1(t), Y_2(t), \ldots, Y_n(t)),$$

where J is a non-empty interval such that, for every $t \in J$, the vector $Y(t)$ satisfies the system of differential equations.

1.3 Dynamical Systems

Dynamical Systems refer to mathematical models that describe how a system evolves over time according to a set of fixed rules, typically expressed as differential or difference equations. These systems can represent physical, biological, economic, or engineering processes where the state of the system changes dynamically. Let's delve deeper into the conditions specified in the definition

Definition 1.3.1. *A dynamical system on* $\mathbb{R}^n$ *is a function* $V : \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ *such that*

- **i.** ***Continuity in Time***: $V(.,y) : \mathbb{R} \longrightarrow \mathbb{R}^n$ *is continuous for esch fixed* $y \in \mathbb{R}^n$ *. This ensures that the system's evolution with respect to time is smooth.*

- **ii.** ***Continuity in State***: $V(t,.) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is continuous for each fixed* $t \in \mathbb{R}$ *. This guarantees that the system's state is continuous with respect to its initial conditions.*

- **iii.** ***Identity Condition***: *The condition* $V(0,y) = y$ *for all* $y \in \mathbb{R}^n$ *. This represents the initial state of the system at time* $t = 0$*.*

- **iv.** ***Composition property*** : $V(t_1 + t_2, y) = V(t_1, V(t_2, y))$ *for all* $t_1, t_2 \in \mathbb{R}$ *and all* $y \in \mathbb{R}^n$ *reflects the property of the system's evolution being consistent over time. It means that evolving the system first by time* t_2 *and then by time* t_1 *is equivalent to evolving it by the total time* $t_1 + t_2$ *in one go. This property is known as the semigroup property and is fundamental in the theory of dynamical systems.*

Example 1.1. *Consider the linear system*

$$\begin{cases} \dot{Y} = CY \\ Y(0) = Y_0, \end{cases}$$

where C is a constant matrix, $Y(t)$ is a state vector in $\mathbb{R}^n$, and $t \in R^+$ is time. The solution is: $V(t, Y_0) = \exp(Ct)\, Y_0$.

This function satisfies all the conditions of a dynamical system:

i. Continuity in Time*: The matrix exponential* $\exp(Ct)$ *is continuous in* t.

ii. Continuity in State*: For a fixed* t*,* $\exp(Ct)$ *is a linear transformation, hence continuous.*

iii. Identity Condition*:* $V(0, Y_0) = \exp(C.0)\, Y_0 = IY_0 = Y_0$*, where I is the identity matrix.*

iv. Composition property:

$$\begin{aligned} V(t_1 + t_2, Y_0) &= \exp(C(t_1 + t_2))\, Y_0 \\ &= \exp(Ct_1)\exp(Ct_2)\, Y_0 \\ &= V(t_1, V(t_2, Y_0)). \end{aligned}$$

A linear dynamical system is characterized by a fundamental property called linearity, which guarantees that the system's reaction to a combination of inputs equals the combined responses to each input individually.

Definition 1.3.2. *A dynamical system V defined on $\mathbb{R}^n$ is considered linear if for all scalars α, $\beta \in \mathbb{R}$, for all vectors $y, z \in \mathbb{R}^n$, and for all $t \geq 0$*

$$V(t, \alpha y + \beta z) = \alpha V(t, y) + \beta V(t, z).$$

This implies that the system's evolution preserves the operations of scaling and addition, such that the result of evolving a linear combination of initial states y and z is equivalent to the linear combination of their individually evolved states.

1.4 Linearization of Systems

When dealing with nonlinear dynamic systems, precisely analyzing their behavior can be challenging due to the inherent complexities in the equations. To address this, system linearization is employed, where the system is approximated around an equilibrium point or steady state using a linear expansion (linear approximation) of the equations. This approach enables the application of mathematical tools specific to linear systems, such as eigenvalues and Jacobian matrices, to gain insights into the local dynamics of the system.

Definition 1.4.1. *Let us examine the nonlinear differential system*

$$\dot{Y} = h\left(Y\right), \tag{1.2}$$

where $Y \in \mathbb{R}^n$ and h represents the nonlinear vector field. Let Y_0 be a equilibrium point of this system (i.e., $h\left(Y_0\right) = 0$). The linearized system at the point Y_0 is given by

$$\dot{Y} = DY,$$

where $D = Dh\left(Y_0\right) = \left(\left.\dfrac{\partial h_i}{\partial Y_j}\right|_{Y_0}\right)$, $1 \leq i,\ j \leq n$, is the Jacobian matrix of h evaluated at Y_0. The matrix D is called the Jacobian matrix associated with the system $h\left(Y\right)$ evaluated at Y_0.

Remark 1.4.1. *By linearizing the system of differential equations, we are able to investigate the nature of the equilibrium points.*

Definition 1.4.2. *A critical point Y_0 of the system (1.2) is considered hyperbolic if the real part of none of the eigenvalues of the Jacobian matrix $D = Dh(Y_0)$ is equal to zero..*

Example 1.2. *Consider the nonlinear system*

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -y_1 + y_1^2 y_2 - y_2. \end{cases}$$

To perform the linearization of this system at the equilibrium point $(y_1, y_2) = (0,0)$:

i. Calculate the Jacobian matrix:

$$\begin{aligned} D &= \left.\begin{pmatrix} \dfrac{\partial h_1}{\partial y_1} & \dfrac{\partial h_1}{\partial y_2} \\ \dfrac{\partial h_2}{\partial y_1} & \dfrac{\partial h_2}{\partial y_2} \end{pmatrix}\right|_{(0,0)} \\ &= \left.\begin{pmatrix} 0 & 1 \\ -1 + 2y_1y_2 & y_1^2 - 1 \end{pmatrix}\right|_{(0,0)} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}. \end{aligned}$$

ii. Linearized system:

$$\dot{Y} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} Y.$$

***iii. Eigenvalues of** D: $\lambda_1 = \frac{-1+i\sqrt{3}}{2}\lambda_2 = \frac{-1-i\sqrt{3}}{2}$. Since the eigenvalues have non-zero real parts, the equilibrium point at $(0,0)$ is hyperbolic.*

1.5 Periodic Orbits and Limit Cycles

Understanding periodic orbits and limit cycles is crucial in the study of dynamical systems, as these concepts play a key role in analyzing the long-term behavior of systems. They help determine whether systems settle into steady states, exhibit oscillations, or evolve into more complex patterns.

1.5.1 Periodic Orbits

A periodic orbit in a dynamical system is a path in which the system's state recurs identically after a fixed period T. In other words, starting from any point along the orbit, the system will return to that exact state after a duration of T, and this behavior will repeat endlessly over time.

Definition 1.5.1. *A trajectory $\Psi(t, y)$ of the system (1.1) is defined as a periodic orbit if it repeats itself after a fixed interval, meaning there exists a positive number $T > 0$ such that*

$$\Psi(t + T, y) = \Psi(t, y), \forall y \in \mathbb{R}^n.$$

The period of the orbit, which represents the fundamental time required for one complete cycle, is the smallest positive value of T that satisfies this condition.

1.5.2 Limit Cycles

A limit cycle is an isolated and unique type of periodic orbit, meaning no other periodic orbits exist arbitrarily close to it. In the phase space of a dynamical system, it stands distinct, with nearby trajectories either converging toward it or diverging away, depending on the stability of the limit cycle. This isolation highlights its special role in shaping the system's long-term behavior and stability dynamics.

Definition 1.5.2. *A limit cycle is a closed periodic orbit that is isolated within a set of periodic orbits.*

Definition 1.5.3. *Let* Γ *be a limit cycle of planar dynamical system, parametrized by a periodic solution* $\gamma(t) = (x(t), y(t))$ *with period* T*. The amplitude of the limit cycle is defined as* $A = \max_{t\in[0,T]} |\gamma(t)|$ *where* $|\cdot|$ *denotes the Euclidean norm.*

Example 1.3. *Consider the following differential system*

$$\begin{cases} \dot{x} = -y + x - x\left(x^2 + y^2\right) \\ \dot{y} = y + x - y\left(x^2 + y^2\right). \end{cases} \tag{1.3}$$

In polar coordinates $x = r\cos(\theta)$ *and* $y = r\sin(\theta)$ *with* $r > 0$*, the system becomes:* $\dot{r} = r\left(1 - r^2\right)$ *and* $\dot{\theta} = 1$*. From this transformation, we get that* $\dot{r} = 0$ *when* $r = 0$ *or* $|r| = 1$*. Since* $r > 0$*, we accept only the positive root* $r = 1$*. Thus, for* $r = 1$*, we have the periodic solution* $(x(t), y(t)) = (\cos(t + \theta_1), \sin(t + \theta_1))$*, with* $\theta(0) = \theta_1$*.*

In the phase plane, there is a single limit cycle with the equation $x^2 + y^2 = 1$ *and amplitude* $r = 1$*.*

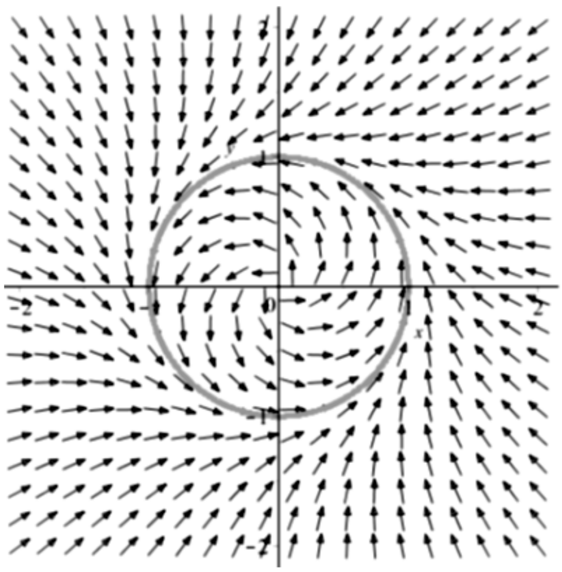

Figure 1.1: Limit cycle of the system (1.3)

1.6 Stability of Limit Cycles

The stability of limit cycles refers to how a system's periodic solutions behave when disturbed by small changes. It determines whether trajectories near the limit cycle converge to, diverge from, or stay at a fixed distance from it. This concept is essential for understanding the long-term behavior of nonlinear systems, with applications in fields such as mechanical engineering, biology and chaos theory (to predict transitions between periodic and chaotic behavior). Studying limit cycle stability helps predict system behavior and design more reliable, controlled systems.

Theorem 1.6.1. *Let C represent the trajectory of a limit cycle. Assume that all neighboring trajectories, both inside and outside C, spiral toward it as $t \longrightarrow \infty$.*

i. *A limit cycle is considered stable if all neighboring trajectories, both inside and outside, converge toward C.*

ii. *The limit cycle is said to be unstable if all neighboring interior and exterior trajectories are repelled from C.*

iii The limit cycle is said to be semi-stable if trajectories approach it on one side and diverge from it on the other.

Example 1.4. *Consider the van der Pol oscillator, a classic example in nonlinear dynamics, which exhibits limit cycles. The van der Pol equation is given by*

$$\ddot{z} - \rho \left(1 - z^2\right) \dot{z} + z = 0,$$

where ρ is a positive parameter. This second-order differential equation can be rewritten as a system of first-order differential equations

$$\begin{cases} \dot{z} = y \\ \dot{y} = \rho\,(1 - z^2)\,y - z. \end{cases}$$

For large values of ρ, the system has a stable limit cycle. To see this, let's analyze the stability of the limit cycle:

- *Equilibrium Points:* *The equilibrium points are obtained by solving: $\dot{z} = \dot{y} = 0$:*

  $$\begin{cases} y = 0 \\ \rho\,(1 - z^2)\,y - z = 0. \end{cases}$$

  This gives the equilibrium points $(0, 0)$.

- *Jacobian Matrix: Compute the Jacobian matrix J at $(0, 0)$:*

  $$J = \begin{pmatrix} 0 & 1 \\ -1 - 2\rho z y & \rho\,(1 - z^2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \rho \end{pmatrix}.$$

  The eigenvalues of J are given by the characteristic equation: $\lambda^2 - \rho\lambda + 1 = 0$, which has roots: $\lambda = \left(\rho \pm \sqrt{\rho^2 - 4}\right) \Big/ 2$.

- *Stability Analysis:*

  - *For $\rho > 2$, the eigenvalues are real and of opposite signs, indicating a saddle point, and hence the origin is unstable.*
  - *For $0 < \rho < 2$, the eigenvalues are complex conjugates with positive real parts, indicating an unstable focus.*
  - *For, $\rho = 0$ the origin $(0, 0)$ is a center point.*

1.7 Existence and Non-existence of Limit Cycles

Examining the presence of limit cycles is essential for understanding the dynamics and trajectory patterns of nonlinear differential systems.. The Poincaré-Bendixson theorem helps in identifying the presence of limit cycles or equilibrium points in planar systems, while Bendixson's criterion aids in ruling out the existence of closed orbits based on the behavior of the divergence of the vector field. These theorems are fundamental in the qualitative analysis of nonlinear differential systems, providing essential tools to understand the possible behaviors of trajectories within these systems.
The Poincaré-Bendixson theorem provides significant insight into the long-term behavior of trajectories in planar systems.

Theorem 1.7.1. ***(Poincaré-Bendixson)***

Consider the planar system

$$\begin{cases} \dot{X} = h_1(X,Y) \\ \dot{Y} = h_2(X,Y), \end{cases} \tag{1.4}$$

where h_1 and h_2 are continuously differentiable functions defined on an open subset G of $\mathbb{R}^2$. Assume that the positive trajectory $\psi^+(q) = \psi(q,t)$, $t \geq 0$ originating from a point q, remains entirely within a compact subset $H \subset G$. Then one of the following three cases holds:

- ***i.*** *$\psi^+(q)$ converges to an equilibrium point.*
- ***ii.*** *$\psi^+(q)$ approaches a periodic orbit.*
- ***iii.*** *$\psi^+(q)$ forms a periodic orbit itself.*

Additionally, if the compact set H does not include any equilibrium points, the system must have a periodic orbit within H.

Bendixson's criterion provides a method to determine the non-existence of limit cycles. It states that if the divergence of the vector field $\mathbf{H}$ (i.e., $\nabla\mathbf{H} = \frac{\partial h_1}{\partial X} + \frac{\partial h_2}{\partial Y}$) is always positive or always negative and not zero within a simply connected region G, then no closed orbits (and hence no limit cycles) can exist within that region.

Theorem 1.7.2. ***(Bendixson's Criterion)*** *Consider the system (1.4), where $\mathbf{H} = (h_1, h_2)^t$ is $\mathcal{C}^1$ on a simply connected region G in $\mathbb{R}^2$. If the divergence of the vector field $\mathbf{H}$, denoted as $\nabla\mathbf{H}$, is not identically zero and does not change sign in G, then the system has no closed orbit entirely contained in G.*

Example 1.5. *Consider the following planar differential system*

$$\begin{cases} \dot{X} = -2X + Y \\ \dot{Y} = -X - 2Y, \end{cases}$$

Let $\mathbf{H} = (-2X + Y, -X - 2Y)^t$. We compute the divergence of the vector field $\mathbf{H}$.

$$\begin{aligned} div\mathbf{H} &= \nabla\mathbf{H} = \frac{\partial}{\partial X}(-2X + Y) + \frac{\partial}{\partial Y}(-X - 2Y) \\ &= (-2) + (-2) \\ &= -4. \end{aligned}$$

Since $div\mathbf{H} < 0$, it follows that the divergence is negative and does not change sign within $\mathbb{R}^2$. Based on Bendixson's Criterion, this system does not possess a limit cycle in R^2.

1.8 Hamiltonian Systems

Hamiltonian systems are a class of dynamical systems that describe the time evolution of physical processes using a function known as the Hamiltonian, which encapsulates the system's total energy.

Definition 1.8.1. *Let G be an open subset of $\mathbb{R}^{2n}$, and let $\mathcal{H} \in \mathcal{C}^2(G)$ such that $\mathcal{H} = \mathcal{H}(X, Y)$ with $X, Y \in \mathbb{R}^n$. A system of the form*

$$\begin{cases} \dot{X} = \frac{\partial \mathcal{H}}{\partial Y} \\ \dot{Y} = -\frac{\partial \mathcal{H}}{\partial X}, \end{cases}$$

where: $\frac{\partial \mathcal{H}}{\partial X} = \left(\frac{\partial \mathcal{H}}{\partial X_1}, \ldots, \frac{\partial \mathcal{H}}{\partial X_n} \right)^t$ *and* $\frac{\partial \mathcal{H}}{\partial Y} = \left(\frac{\partial \mathcal{H}}{\partial Y_1}, \ldots, \frac{\partial \mathcal{H}}{\partial Y_n} \right)^t$, *It is referred to as a Hamiltonian system with n degrees of freedom on G.*

1.9 Some Important Equations

1.9.1 Liénard Equation

The Liénard equation is a second-order nonlinear differential equation widely used to describe dynamical systems, particularly in the analysis of electrical and mechanical oscillators. It serves as a fundamental tool for studying limit cycles, which are closed trajectories representing stable periodic behavior in dynamical systems.

Definition 1.9.1. *The Liénard equation is defined as a nonlinear second-order differential equation represented by*

$$\ddot{y} + f(y)\,\dot{y} + g(y) = 0,$$

where $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are two continuously differentiable functions on $\mathbb{R}$.

Example 1.6. *Consider the Van der Pol oscillator, which is a classic example of a Liénard system. The Van der Pol equation can be written as*

$$\ddot{y} - \mu \left(1 - y^2\right) \dot{y} + y = 0,$$

where μ is a positive constant that controls the nonlinearity and damping of the system.

1.9.2 Duffing Equation

The Duffing equation plays a crucial role in the study of nonlinear dynamical systems. It is vital for modeling and analyzing forced oscillations in systems with nonlinear stiffness and damping, providing insights into phenomena such as chaos and bifurcations in nonlinear oscillators. This equation describes a driven, damped harmonic oscillator with a nonlinear stiffness term and models various physical systems, including certain types of mechanical oscillators and electrical circuits.

Definition 1.9.2. *The Duffing equation is a nonlinear second-order differential equation expressed as*

$$\ddot{y} + c\dot{y} + f(y) = g(t),$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function adhering to the local Lipschitz condition, c is a positive constant, and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function that exhibits periodic behavior with a period T.

Example 1.7. *Consider the classical Duffing oscillator, which is a specific case of the Duffing equation with a cubic nonlinearity*

$$\ddot{y} + c\dot{y} + \alpha y + \beta y^3 = \gamma \cos g(\omega t),$$

where $\alpha, \beta, \gamma,$ and ω are constants.

1.9.3 Mathieu Equation

The Mathieu equation is a second-order linear differential equation with periodic coefficients, playing a pivotal role in analyzing parametric resonance and stability across various physical systems. It emerges in diverse applications, such as evaluating the stability of floating structures, studying the oscillations of elliptical membranes, and exploring certain aspects of quantum mechanical systems. What sets the Mathieu equation apart is its ability to exhibit a wide range of dynamic behaviors depending on the parameters a and q. Understanding the stability zones through the Mathieu chart is essential for predicting the behavior of systems governed by this equation. Its solutions reveal a fascinating spectrum of behaviors, from stable oscillations to parametric resonance, establishing it as a cornerstone in the study of systems with periodic coefficients.

Definition 1.9.3. *The Mathieu equation in its standard form is written as*

$$\ddot{y} + (a - 2q\cos(2t))\, y = 0.$$

Where $\ddot{y}$ is the second derivative of y with respect to time t, a and q are parameters that determine the behavior of the solution.

1.10 Important definitions

In the study of differential equations and dynamical systems, certain mathematical tools and coordinate systems play a crucial role in simplifying and solving complex problems. Among these tools, the use of generalized coordinates, such as $(p, q)-$polar coordinates, and classical theorems like Descartes' Rule of Signs, provide valuable insights into the behavior of systems in both two-dimensional ($\mathbb{R}^2$) and three-dimensional ($\mathbb{R}^3$) spaces. These concepts are particularly useful in analyzing symmetries, singularities, and the distribution of roots in polynomials, thereby enabling a deeper understanding of the underlying structures in mathematical models.

1.10.1 Descartes' Rule of Signs

Descartes' Rule of Signs offers a robust method for approximating the count of positive real roots in a polynomial by analyzing the variations in the signs of its coefficients. This principle establishes an upper limit for the number of positive roots, with the actual count being either equal to this limit or differing by an even number. Furthermore, the rule guarantees the feasibility of constructing a polynomial with an exact number of positive real roots matching the observed sign changes in its coefficients.

Theorem 1.10.1. *[3] Consider the real polynomial*

$$P(x) = p_{j_1} x^{i_1} + p_{j_2} x^{i_2} + \cdots + p_{j_m} x^{i_m}$$

where $0 \leq j_1 < j_2 < \cdots < j_m$ *and* $p_{j_i} \neq 0$ *are real constants for* $1 \leq i \leq m$.

If $p_{j_i} p_{j_{i+1}} < 0$, *it indicates a sign change between* p_{j_i} *and* $p_{j_{i+1}}$. *The total number of such sign changes, denoted as* n, *sets an upper bound for the number of positive real roots of* $P(x)$*such that the polynomial has precisely* n *strictly positive real roots.*

1.10.2 $(p,q)-$Polar Coordinates in $\mathbb{R}^2$

According to Lyapunov ([27]), we introduce the trigonometric functions $v(\theta) = Cs\theta$ and $w(\theta) = Sn\theta$ as solution to the following initial value problem

$$\begin{cases} \dot{v} = -w^{2p-1}, \\ \dot{w} = v^{2q-1}, \end{cases}$$

with initial conditions: $v(0) = \sqrt[2q]{\frac{1}{p}}$ and $w(0) = 0$, p and q are positive integers. These functions satisfy the following properties:

(i) Periodicity: The functions $Cs\theta$ and $Sn\theta$ are periodic with period T:

$$T = 2p^{\frac{-1}{2q}} q^{\frac{-1}{2p}} \frac{\Gamma(\frac{1}{2p})\Gamma(\frac{1}{2q})}{\Gamma(\frac{1}{2p} + \frac{1}{2q})},$$

where Γ is the gamma function.

(ii) Classical Trigonometric Functions: For $p = q = 1$, we have $Cs\theta = \cos\theta$ and $Sn\theta = \sin\theta$.

(iii) Unit Circle: The identity $pCs^{2p}\theta + qSn^{2q}\theta = 1$ holds.

(iv) Polar Coordinates $(\mathbf{p}, \mathbf{q})$: Introducing the polar coordinates (p, q) defined by (r, θ) where $v = r^p Cs\theta$ and $w = r^q Sn\theta$, the general expression of a differential system in these coordinates is:

$$\begin{cases} \dot{r} = \frac{1}{r^{2pq-1}}(v^{2q-1}\dot{v} + w^{2p-1}\dot{w}), \\ \dot{\theta} = \frac{1}{r^{p+q}}(pv\dot{w} - q\dot{v}w). \end{cases}$$

(v) Integral of Trigonometric Functions:

Let $Cs\theta$ and $Sn\theta$ be the $(1.q)$-trigonometrical functions such that the indices i and j are even (see. [18]),

$$\alpha_{i,j} = \int_0^T Cs^i\theta Sn^j\theta d\theta = 2q^{-\frac{j+1}{2}} \frac{\Gamma(\frac{i+1}{2q})\Gamma(\frac{j+1}{2})}{\Gamma(\frac{i+1}{2q} + \frac{j+1}{2})}. \tag{1.5}$$

This formulation in polar coordinates allows for a more general analysis of the differential system, incorporating specific properties of the trigonometric functions $Cs\theta$ and $Sn\theta$.

1.10.3 $(p, q)-$Polar Coordinates in $\mathbb{R}^3$

Extending $(p, q)-$polar coordinates to $\mathbb{R}^3$ involves introducing an additional angular coordinate. In this three-dimensional space, $(p, q)-$polar coordinates can describe systems with radial symmetry around a central point. The coordinates are defined as: $x = r^p Cs\theta$, $y = r^q Sn\theta \cos\varphi$, $z = r^q Sn\theta \sin\varphi$, where r is the radial distance, θ is the polar angle, and φ is the azimuthal angle. The differential equations governing the dynamics in these coordinates can be derived similarly to the two-dimensional case, accounting for the additional angular dependence.

Example 1.8. ***Special Case*** $p = q = 1$

Consider the special case when $p = q = 1$*, where the functions* $Cs\theta = \cos\theta$ *and* $Sn\theta = \sin\theta$*. The original Liouville system is*

$$\begin{cases} \dot{v} = -w \\ \dot{w} = v, \end{cases}$$

with initial conditions: $v(0) = 1$ *and* $w(0) = 0$*. We can solve this equation using basic integrals to obtain the solutions:* $v(\theta) = \cos(\theta)$ *and* $w(\theta) = \sin(\theta)$*.*

The functions $Cs\theta$ *and* $Sn\theta$ *are periodic with period* T :

$$T = 2.1^{-1/2}.1^{-1/2}.\Gamma(1/2)\Gamma(1/2)/\Gamma(1) = 2\pi.$$

The identity $pCs^{2p}\theta + qSn^{2q}\theta = 1$ *holds because* $\cos^2\theta + \sin^2\theta = 1$*. Introducing the polar coordinates* $(1, 1)$ *defined by* (r, θ) *where* $v = rCs\theta$ *and* $w = rSn\theta$*,* *the general expression of a differential system in these coordinates is*

$$\begin{cases} \dot{r} = \frac{1}{r^1}(v^1\dot{v} + w^1\dot{w}) = \frac{1}{r}(v\dot{v} + w\dot{w}), \\ \dot{\theta} = \frac{1}{r^2}(v\dot{w} - \dot{v}w). \end{cases}$$

This simplifies to

$$\begin{cases} \dot{r} = 0, \\ \dot{\theta} = \frac{1}{r^2}. \end{cases}$$

We evaluate the integral $\alpha_{i,j}$ when $i = 2$ and $j = 2$

$$\alpha_{2,2} = \int_0^T Cs^2\theta Sn^2\theta d\theta = 2.1^{-3/2}\frac{\Gamma(3/4)\Gamma(3/2)}{\Gamma(9/4)} = \pi^{1/2}.\Gamma(3/4)/\Gamma(9/4).$$

Example 1.9. ***General Case*** $p = 2$, $q = 1$

Consider the case when $p = 2$, $q = 1$. The Liouville system is

$$\begin{cases} \dot{v} = -w \\ \dot{w} = v^3, \end{cases}$$

subject to the initial condition: $v(0) = \sqrt{2}$ and $w(0) = 0$. To calculate the period T when $p = 2$ and $q = 1$: $T = 2.2^{-1/2}.1^{-1/2}.\Gamma(1/4)\Gamma(1/2)/\Gamma(3/4)$, then

$$T = 2.\,(2\pi)^{1/2}.\Gamma(1/4)/\Gamma(3/4).$$

Introducing the polar coordinates $(2, 1)$ defined by (r, θ) where $v = r^2 Cs\theta$ and $w = rSn\theta$, the general expression of a differential system in these coordinates is

$$\begin{cases} \dot{r} = \frac{1}{r^5}(v^1\dot{v} + w^3\dot{w}) = \frac{1}{r^5}(v\dot{v} + w^3\dot{w}), \\ \dot{\theta} = \frac{1}{r^3}(2v\dot{w} - \dot{v}w). \end{cases}$$

We calculate the integral $\alpha_{i,j}$ when $i = j = 2$:

$$\alpha_{2,2} = \int_0^T Cs^2\theta Sn^2\theta d\theta = 2.1^{-3/2}\frac{\Gamma(3/4)\Gamma(3/2)}{\Gamma(9/4)} = \pi^{1/2}.\Gamma(3/4)/\Gamma(9/4).$$

Chapter

2 Averaging Theory

Averaging theory is a traditional approach employed to analyze the presence of periodic solutions in differential equations. Beyond equilibrium points, periodic orbits and their stability types provide valuable insights into the dynamics of differential systems. Averaging methods offer conditions under which singular points of the averaged system yield limit cycles in systems with a center. The historical roots of this method trace back to the works of Lagrange and Fatou in the late 18th century. The first rigorous proofs of its asymptotic validity were presented by Fatou in (1928, [17]), Bogoliubov and Krylov in (1934, [4]), and later by Bogoliubov and Mitropolsky in (1961, [5]). Subsequent developments were made by Sanders and Verhulst [50], Roseau [47], Verhulst [56] and Malkin [36]. This chapter delves into the averaging theory with its various theorems.

2.1 First-Order Averaging Theorem

The first-order averaging theorem is a crucial tool for studying dynamical systems, as it establishes conditions linking the equilibrium points of the averaged system to periodic solutions in the original system. This technique is essential for understanding and predicting periodic behavior in nonlinear systems.

Let us consider the following differential system:

$$\begin{cases} \dot{x}(t) = \varepsilon f(t,x) + \varepsilon^2 g(t,x,\varepsilon), \\ y(0) = x_0, \end{cases} \tag{2.1}$$

where $x \in B \subseteq \mathbb{R}^n$ is within a bounded domain and $t \geq 0$. Assume that $f(t,x)$ and $g(t,x,\varepsilon)$ are $T-$periodic functions in t.

The averaged system associated with the system (2.1) is defined by

$$\begin{cases} \dot{y} = \varepsilon F(y), \\ y(0) = x_0, \end{cases} \tag{2.2}$$

where

$$F(y) = \frac{1}{T}\int_0^T f(t,y)dt. \tag{2.3}$$

The theorem below establishes the conditions for which the equilibrium points of the averaged system (2.2) coincide with periodic solutions in the original system (2.1).

Theorem 2.1.1. *Consider the system (2.1) and suppose that*

1. *f, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ and , $\frac{\partial g}{\partial x}$ are continuous functions and bounded by a constant K independent of ε in $[0,+\infty[\times B$ and $\varepsilon \in]0,\varepsilon_0[$.*

2. *f and g are $T-$periodic in t (with T independent of ε).*

Then,

i. *If the point q is an equilibrium point for the averaged system (2.2) such that*

$$det\left(\frac{\partial F}{\partial y}\right)\Big|_{y=q} \neq 0, \tag{2.4}$$

then for $|\varepsilon| > 0$ sufficiently small, there exists a $T-$periodic solution $x_\varepsilon(t)$ of the system (2.1) such that $x_\varepsilon(t) \longrightarrow q$.

ii. *Provided that the equilibrium point q of the averaged system (2.2) is hyperbolic, there exists a sufficiently small $|\varepsilon| > 0$ such that the corresponding periodic solution in the original system (2.1) is both unique and exhibits stability properties identical to those of q.*

Proof. For more details on the proof of this theorem, see reference [51]. □

Example 2.1. *Consider the Rayleigh equation :*

$$\ddot{x} - \epsilon\left(\dot{x} - \frac{\dot{x}^3}{3}\right) + x = 0. \tag{2.5}$$

We recast equation (2.5) in the form of a differential system as follows:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \epsilon\left(y - \frac{y^3}{3}\right) - x. \end{cases} \tag{2.6}$$

In polar coordinates (r, θ) where $x = r\cos\theta$, $y = r\sin\theta$, the system (2.6) becomes

$$\begin{cases} \dot{r} = \epsilon\left(r\sin^2\theta - \frac{r^3\sin^4\theta}{3}\right), \\ \dot{\theta} = -1 + \epsilon\left(\sin\theta\cos\theta - \frac{r^2\sin^3\theta\cos\theta}{3}\right). \end{cases} \tag{2.7}$$

The system (2.7) is equivalent to

$$\frac{dr}{d\theta} = -\epsilon\left(r\sin^2\theta - \frac{r^3\sin^4\theta}{3}\right) + O(\epsilon^2). \tag{2.8}$$

Equation (2.8) is in the standard form (2.1) so we can apply the averaging theory with $x = r$, $t = \theta$, $T = 2\pi$, $f(t, x) = f(\theta, r) = -r\sin^2\theta + \frac{r^3\sin^4\theta}{3}$ From (2.3), we obtain the averaged function

$$F_0(r) = \frac{1}{2\pi}\int_0^{2\pi}\left(-r\sin^2\theta + \frac{r^3\sin^4\theta}{3}\right)d\theta = -\frac{r}{2} + \frac{r^3}{8}.$$

$F_0(r) = 0$ has a unique positive root $r = 2$. According to Theorem 2.1.1, the Rayleigh equation (2.5) has a limit cycle bifurcating from the periodic solution of radius $r = 2$ for $|\varepsilon| > 0$ sufficiently small because $\frac{dF_0}{dr}(2) = 1 \neq 0$. Since $\frac{dF_0}{dr}(2) > 0$, this cycle is unstable (see Figure 2.1).

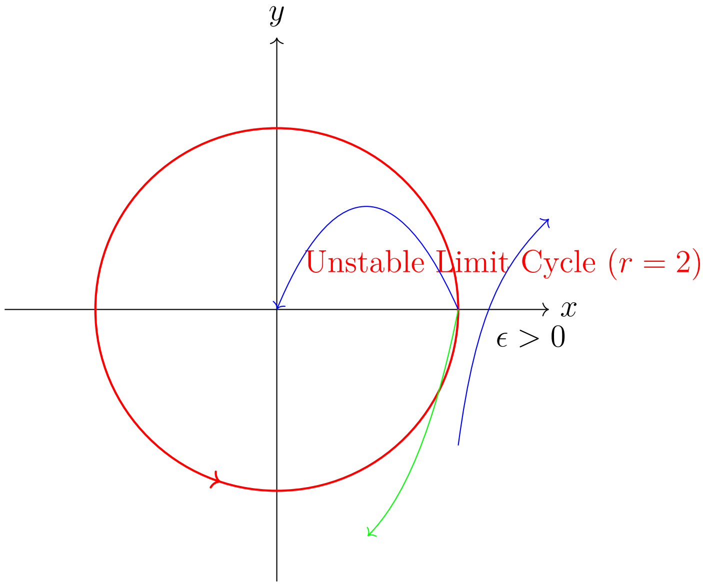

Figure 2.1: Unstable limit cycle, for $\epsilon = 0.1$ of the system (2.7)

2.2 Second-Order Averaging Theory

Second-order averaging theory extends the first-order method to provide a second-order approximation of periodic solutions for certain differential systems. This higher-order method allows for more accurate approximations and is particularly useful for systems with more complex dynamics.

Theorem 2.2.1. *Consider the initial value problems:*

$$\begin{cases} \dot{x}(t) = \varepsilon f_1(t,x) + \varepsilon^2 f_2(t,x) + \varepsilon^3 f_3(t,x,\varepsilon), \\ x(0) = x_0, \end{cases} \tag{2.9}$$

and

$$\begin{cases} \dot{y} = \varepsilon g_0(y) + \varepsilon^2 \left(g_{10}(y) + G_0(y)\right), \\ y(0) = x_0, \end{cases} \tag{2.10}$$

where $f_1, f_2 : [0,+\infty[\, \times B \longrightarrow \mathbb{R}^n$ *and* $f_3 : [0,+\infty[\, \times B \times [0,\varepsilon_0[\longrightarrow \mathbb{R}^n$ *are continuous functions,* $T-$*periodic in* t*, and* B *is an open subset of* $\mathbb{R}^n$*. Let*

$$F_1\left(t,x\right) = \frac{\partial f_1}{\partial x} y_1(t,x) - \frac{\partial y_1}{\partial x} g_0(x), \tag{2.11}$$

where

$$y_1(t,x) = \int_0^t \left(f_1\left(u,x\right) - g_0(x)\right) du + h\left(x\right), \tag{2.12}$$

with $h(x)$ *being a* $\mathcal{C}^1$ *function such that the mean is zero.* g_0*,* g_{10} *and* G_0 *are the averaged functions of* f_1*,* F_1 *and* f_2*, respectively. Suppose that:*

- $\frac{\partial f_1}{\partial x}$*,* f_2 *and* f_3 *are continuous functions and Lipschitz continuous in* x*.*
- $|f_3(t,x,\varepsilon)|$ *is uniformly bounded by a constant* K *in* $[0, K/\varepsilon[\times B\times]0,\varepsilon_0]$*.*
- T *is independent of* ε
- $y(t) \in B$ *for a time scale of* $1/\varepsilon$*.*

Then $x(t) = y(t) + \varepsilon y_1(t,y(t)) + O\left(\varepsilon^2\right)$*, for a time scale of* $1/\varepsilon$*.*

Proof. For more details on the proof of this theorem, see Verhulst [56]. □

Corollary 2.1. *If the assumptions of Theorem 2.2.1 are satisfied and additionally* $F_0(y) = 0$*, then:*

i. *If q is an equilibrium point of the averaged system (2.10) such that*

$$\frac{\partial}{\partial y}\left(g_{10}(y)+G_0(y)\right)\Big|_{y=q}\neq 0, \tag{2.13}$$

then there exists a $T-$periodic solution $x_\varepsilon(t)$ of (2.9) such that $x_\varepsilon(t)\longrightarrow q$ as $\varepsilon\longrightarrow 0$.

ii. *If q is hyperbolic, then for sufficiently small $|\varepsilon|$, the periodic solution $x_\varepsilon(t)$ of (2.9) is unique, hyperbolic, and has the same stability as q.*

Remark 2.2.1. ***i.*** *If $F_0(y)=0$, then in polar coordinates $x=r\cos\theta$ and $y=r\sin\theta$, we obtain $F_0(r)=0$, leading to $\int_0^{2\pi}\frac{\partial}{\partial r}f_1(\theta,r)\,d\theta=0$. Calculating F_{10}, we find*

$$F_{10}=(2\pi)^{-1}\int_0^{2\pi}\left(\frac{\partial}{\partial r}f_1(\theta,r)\int_0^{\theta}f_1(\varphi,r)\,d\varphi\right)d\theta+h(r)(2\pi)^{-1}\int_0^{2\pi}\frac{\partial}{\partial r}f_1(\theta,r)\,d\theta.$$

Since $\int_0^{2\pi}\frac{\partial}{\partial r}f_1(\theta,r)\,d\theta=0$, we get $F_{10}=(2\pi)^{-1}\int_0^{2\pi}\left(\frac{\partial}{\partial r}f_1(\theta,r)\int_0^{\theta}f_1(\varphi,r)\,d\varphi\right)d\theta$.

ii. *For $F_0(y)=0$, we denote the second-order averaged function as*

$$F_2(r)=F_{10}(r)+G_0(r). \tag{2.14}$$

Example 2.2. *Consider the system:*

$$\begin{cases}\dot{x}=y+\varepsilon x^2+\varepsilon^2x^2,\\ \dot{y}=-x+\varepsilon^2\left(x^4+x^3+y^3-y/3\right).\end{cases} \tag{2.15}$$

In polar coordinates $x=r\cos\theta$ and $y=r\sin\theta$ with $r>0$, system (2.15) becomes:

$$\begin{cases}\dot{r}=\varepsilon r^2\cos^3\theta+\varepsilon^2\left(3r^3\cos^3\theta\sin\theta+3r^2\cos^4\theta\right.\\ \qquad\qquad +3r^2\cos^3\theta\sin\theta+3r\cos^3\theta-6r^2\cos^2\theta\\ \qquad\qquad \left.+\cos^2\theta+3r^2-1\right)/r,\\ \dot{\theta}=-1+\varepsilon r\cos^2\theta\sin\theta+\varepsilon^2\left(r^3\cos^5\theta+r^2\cos^4\theta\right.\\ \qquad\qquad -r^2\cos^3\theta\sin\theta-r\cos^2\theta\sin\theta+r^2\cos^2\theta\sin\theta\\ \qquad\qquad \left.-\cos\theta\sin\theta/3\right).\end{cases}$$

Considering θ as an independent variable, we obtain:

$$\frac{dr}{d\theta}=\varepsilon r^2\cos^3\theta-\varepsilon^2\left(r^4\cos^3\theta\sin\theta+r^3\cos^4\theta+r^2\cos^3\theta\sin\theta-r^3\cos^5\theta\sin\theta\right).$$

This is equivalent to: $\frac{dr}{d\theta}=\varepsilon f_1(\theta,r)+\varepsilon^2 f_2(\theta,r)+o\left(\varepsilon^2\right)$, *with* $f_1(\theta,r)=r^2\cos^3\theta$ *and* $-f_2(\theta,r)=r^4\cos^3\theta\sin\theta+r^3\cos^4\theta+r^2\cos^3\theta\sin\theta-r^3\cos^5\theta\sin\theta$. *Applying Theorem 2.2.1, we calculate the averaged function*

$$F_0\left(r\right)=\left(2\pi\right)^{-1}\int_0^{2\pi}f_1\left(\theta,r\right)d\theta=r^2\left(2\pi\right)^{-1}\int_0^{2\pi}\cos^3\theta d\theta=0.$$

Since $F_0\left(r\right)=0$, *we proceed to the second-order averaging theory. We calculate* F_1 *such that* $F_1\left(\theta,r\right)=\frac{\partial f_1}{\partial r}\left(\theta,r\right)y_1\left(\theta,r\right)$, *with* $y_1\left(\theta,r\right)=\int_0^{\theta}f_1\left(u,r\right)du=r^2\sin\theta\left(\cos^2\theta+2\right)/3$, *hence*

$$F_1\left(\theta,r\right)=r^4\cos^3\theta\sin\theta\left(\cos^2\theta+2\right)/3.$$

We now calculate the function F_2 *such that*

$$F_2(r)=F_{10}(r)+G_0(r)=\left(2\pi\right)^{-1}\int_0^{2\pi}\left(F_1\left(\theta,r\right)+f_2\left(\theta,r\right)\right)d\theta,$$

on obtient $F_2(r)=-r\left(9r^2-4\right)/24$. *The equation* $F_2(r)=0$ *has a single positive root* $r=2/3$. *Since* $F_2'(2/3)=-1/3$, *the system (2.15) has a single stable limit cycle with amplitude* $2/3$ *for sufficiently small* $|\varepsilon|\neq 0$ *(see., Figure 2.2).*

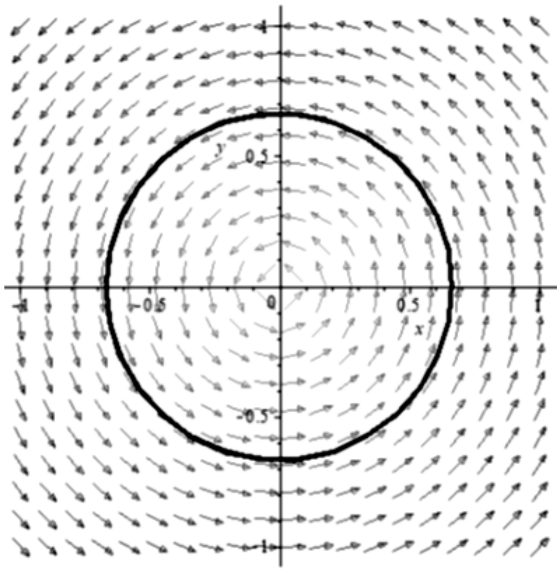

Figure 2.2: Stable limit cycle, for $\varepsilon = 0.01$ of the system (2.15).

Chapter 3

Analysis of Limit Cycles in Generalized Mathieu Differential Equations via Averaging Theory

3.1 Introduction and Main Results

Studying limit cycles in the Mathieu equation is crucial for solving problems in various scientific and engineering fields, especially those involving periodic systems like electronic circuits. This chapter uses first-order averaging theory to establish upper bounds on the number of limit cycles that can emerge from periodic orbits in a specific Hamiltonian system $\dot{x} = y^{2p-1}$, $\dot{y} = -x^{2q-1}$ with p, q being positive integers (see chapter 2), when subjected to perturbations characterized by a generalized form of the Mathieu differential equation

$$\ddot{x} + b\,(1 + \cos t)\,x = 0,$$

where b is a real constant (see. [35]).

The system under investigation includes two differential equations:

$$\begin{cases} \dot{x} = y^{2p-1}, \\ \dot{y} = -x^{2q-1} - \varepsilon(1 + Cs^{2m}\theta)Q(x, y), \end{cases} \tag{3.1}$$

where $x(\theta) = Cs\theta$ and $y(\theta) = Sn\theta$ is the solution of the initial problem

$$\begin{cases} \dot{x} = y^{2p-1}, \\ \dot{y} = -x^{2q-1}, \end{cases}$$

with $x(0) = \sqrt[2q]{\frac{1}{p}}$ and $y(0) = 0$, and p, q, m are positive integers, defining a dynamic system that can exhibit periodic behavior under certain perturbations. The polynomial $Q(x, y)$ in the system has a degree n, $n \geq 2$, and the condition $Q(0, 0) = 0$ ensures that the polynomial passes through the origin. The parameter ε is a small positive value that introduces a perturbation into the system. The system (3.1) can be written as

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y}, \\ \dot{y} = -\frac{\partial H}{\partial x} - \varepsilon(1 + Cs^{2m}\theta)Q(x, y). \end{cases} \tag{3.2}$$

This form clearly indicates that when $\varepsilon = 0$, the system (3.2) is a Hamiltonian system, simplifying to:

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y}, \\ \dot{y} = -\frac{\partial H}{\partial x}, \end{cases}$$

with the Hamiltonian function: $H(x, y) = \frac{1}{2q}x^{2q} + \frac{1}{2p}y^{2p}$. In particular, our main findings were as follows

Theorem 3.1.1. *Suppose that $n = 2s$ or $n = 2s - 1$ when n is even or odd respectively, $\varepsilon \neq 0$ sufficiently small, the system (3.1) has at most*

$$\delta_{p,q} = \begin{cases} qs \text{ if } p \leq q \\ p(s-1) + q \text{ if } p \geq q, \end{cases}$$

limit cycles bifurcating from the periodic orbits of the center $\dot{x} = y^{2p-1}$, $\dot{y} = -x^{2q-1}$, using first-order averaging theory.

Remark 3.1.1. *System (3.1) with $q = p = 1$ has been studied in the paper [11].*

Theorem 3.1.2. *Consider system (3.1) with $p = 1$, $q \geq 1$ and $|\varepsilon| \neq 0$ sufficiently small, let $s = \frac{n}{2}$ or $s = \frac{n+1}{2}$ when n is even or odd respectively and l denote the maximum number of limit cycles of the polynomial differential system (3.1) bifurcating from the periodic orbits of the center $\dot{x} = y, \dot{y} = -x^{2q-1}$ using first-order averaging theory, then*

a) For $s \leq q$, $l = \frac{(s-1)(s+2)}{2}$.

b) For $s \geq q+1$, $l = qs - \frac{q(q-1)}{2} - 1$.

3.2 Proofs of the Results

3.2.1 Proof of Theorem 3.1.1

To apply the averaging method, we must transform system (3.1) into canonical form (??). By performing the change of coordinates to $(p, q)-$polar coordinates $x = r^p Cs\theta$ and $y = r^q Sn\theta$ and taking θ as the independent variable, if we write

$$\begin{aligned} Q(x,y) &= \sum_{d=1}^{n}\left(\sum_{j=0}^{d} a_{j,d}x^{d-j}y^{j}\right) \\ &= g_1(x,y) + g_2(x,y) + g_3(x,y) + g_4(x,y), \end{aligned}$$

where

$$g_1(x,y) = \sum_{l=1}^{\left[\frac{n}{2}\right]}\left(\sum_{i=1}^{l} a_{2i-1,2l}x^{2l-2i+1}y^{2i-1}\right),$$

$$g_2(x,y) = \sum_{l=1}^{s}\left(\sum_{i=0}^{l} a_{2i,2l-1}x^{2l-2i-1}y^{2i}\right),$$

$$g_3(x,y) = \sum_{l=1}^{\left[\frac{n}{2}\right]}\left(\sum_{i=0}^{l} a_{2i,2l}x^{2l-2i}y^{2i}\right),$$

and

$$g_4(x,y) = \sum_{l=1}^{s}\left(\sum_{i=1}^{l} a_{2i-1,2l-1}x^{2l-2i}y^{2i-1}\right),$$

for every $s = \frac{n}{2}$ or $s = \frac{n+1}{2}$ when n is even or odd respectively, then system (3.1) can be written as

$$\begin{cases} \dot{r} = -\varepsilon r^{-q+1} Sn^{2p-1}\theta(1 + Cs^{2m}\theta)Q\left(r^p Cs\theta, r^q Sn\theta\right) \\ \dot{\theta} = -r^{2pq-p-q} - \varepsilon p r^{-q} Cs\theta(1 + Cs^{2m}\theta)Q\left(r^p Cs\theta, r^q Sn\theta\right), \end{cases} \tag{3.3}$$

where

$$\begin{aligned} Q\left(r^p Cs\theta, r^q Sn\theta\right) = & \sum_{l=1}^{\left[\frac{n}{2}\right]}\left(\sum_{i=1}^{l} a_{2i-1,2l} r^{2lp+2i(q-p)+p-q} Cs^{2l-2i+1}\theta Sn^{2i-1}\theta\right) \\ & + \sum_{l=1}^{s}\left(\sum_{i=0}^{l} a_{2i,2l-1} r^{2lp+2i(q-p)-p} Cs^{2l-2i-1}\theta Sn^{2i}\theta\right) \\ & + \sum_{l=1}^{\left[\frac{n}{2}\right]}\left(\sum_{i=0}^{l} a_{2i,2l} r^{2lp+2i(q-p)} Cs^{2l-2i}\theta Sn^{2i}\theta\right) \\ & + \sum_{l=1}^{s}\left(\sum_{i=1}^{l} a_{2i-1,2l-1} r^{2lp+2i(q-p)-q} Cs^{2l-2i}\theta Sn^{2i-1}\theta\right). \end{aligned} \tag{3.4}$$

Now taking θ as independent variable, system (3.3) becomes

$$\frac{dr}{d\theta} = \varepsilon R(r,\theta) + O(\varepsilon^2),$$

where

$$R(r,\theta) = r^{p-2pq+1} Sn^{2p-1}\theta(1 + Cs^{2m}\theta)Q\left(r^p Cs\theta, r^q Sn\theta\right).$$

Using the notation introduced in section two, we have

$$h^0(r) = \frac{r^{p-2pq+1}}{T}[\int_0^T [Sn^{2p-1}\theta(1 + Cs^{2m}\theta)Q\left(r^p Cs\theta, r^q Sn\theta\right)]d\theta,$$

we have the formula

$$\int_0^T Cs^i\theta Sn^j\theta d\theta = 0 \text{ if } i \text{ or } j \text{ is odd.} \tag{3.5}$$

By using (3.4), the formulaes in (3.5) show that

$$h^0(r) = \frac{r^{-2pq+p-q+1}}{T}\left[\sum_{l=1}^{s}\left(\sum_{i=1}^{l} a_{2i-1,2l-1} r^{2lp+2i(q-p)} \int_0^T Cs^{2(l-i)}\theta Sn^{2(i+p-1)}\theta d\theta\right)\right.$$
$$\left. + \sum_{l=1}^{s}\left(\sum_{i=1}^{l} a_{2i-1,2l-1} r^{2lp+2i(q-p)} \int_0^T Cs^{2(l-i+m)}\theta Sn^{2(i+p-1)}\theta d\theta\right)\right],$$

we have

$$h^0(r) = \frac{r^{-2pq+p-q+1}}{T} \times \sum_{l=1}^{s}\left(\sum_{i=1}^{l}(a_{2i-1,2l-1}(\alpha_{2(l-i),2(i+p-1)} + \alpha_{2(l-i+m),2(i+p-1)})r^{2(p(l-i)+qi)}\right), \tag{3.6}$$

with

$$\alpha_{a,b} = \int_0^T Cs^a\theta Sn^b\theta d\theta.$$

For the simplicity of calculation, let

$$w_{i,l} = a_{2i-1,2l-1}(\alpha_{2(l-i),2(i+p-1)} + \alpha_{2(l-i+m),2(i+p-1)}),$$

therefore, (3.6) can be reduced to

$$g^0(r) = \frac{r^{-2pq+p-q+1}}{T}\sum_{l=1}^{s}\left(\sum_{i=1}^{l} w_{i,l} r^{2(p(l-i)+qi)}\right). \tag{3.7}$$

As is common knowledge, the number of positive roots of $g^0(r)$ is equal to that of

$$T(r) = \sum_{l=1}^{s}\left(\sum_{i=1}^{l} w_{i,l} r^{2(p(l-i)+qi)}\right),$$

then, to find the real positive roots of $T(r)$, we must find the zeros of a polynomial in the variable $\rho = r^2$,

$$S(\rho) = \sum_{l=1}^{s}\left(\sum_{i=1}^{l} w_{i,l} \rho^{p(l-i)+qi}\right). \tag{3.8}$$

Now, we stretch the polynomial (3.8) as the following

$$\begin{aligned}
S(\rho) = \sum_{l=1}^{s} \big(& w_{1,l}\rho^{p(l-1)+q} + w_{2,l}\rho^{p(l-2)+2q} + w_{3,l}\rho^{p(l-3)+3q} + \ldots + w_{l-1,l}\rho^{p+(l-1)q} \\
& + w_{l,l}\rho^{ql}\big) \\
& = w_{1,1}\rho^{q} + [w_{1,2}\rho^{p+q} + w_{2,2}\rho^{2q}] + [w_{1,3}\rho^{2p+q} + w_{2,3}\rho^{p+2q} + w_{3,3}\rho^{3q}] \\
& + \ldots\ldots \\
& + [w_{1,s-2}\rho^{p((s-2)-1)+q} + w_{2,s-2}\rho^{p((s-2)-2)+2q} + w_{3,s-2}\rho^{p((s-2)-3)+3q} \\
& + \ldots + w_{s-3,s-2}\rho^{p+((s-2)-1)q} + w_{s-2,s-2}\rho^{q(s-2)}] \\
& + [w_{1,s-1}\rho^{p((s-1)-1)+q} + w_{2,s-1}\rho^{p((s-1)-2)+2q} + w_{3,s-1}\rho^{p((s-1)-3)+3q} \\
& + \ldots + w_{s-2,s-1}\rho^{p+((s-1)-1)q} + w_{s-1,s-1}\rho^{q(s-1)}] \\
& + [w_{1,s}\rho^{p(s-1)+q} + w_{2,s}\rho^{p(s-2)+2q} + w_{3,s}\rho^{p(s-3)+3q} \\
& + \ldots + w_{s-1,s}\rho^{p+(s-1)q} + w_{s,s}\rho^{sq}].
\end{aligned}$$

So, the degree of $S(\rho)$ is bounded by

$$\delta_{p,q} = \begin{cases} qs \text{ if } p \leq q \\ p(s-1)+q \text{ if } p \geq q, \end{cases}$$

we conclude tha $g^0(r)$ has at most $\delta_{p,q}$ simple positive real roots. This completes the proof of Theorem 3.1.1.

3.2.2 Proof of Theorem 3.1.2

Consider the polynomial differential system (3.1) with $p = 1$, from equation (3.7) we obtain

$$g^0(r) = \frac{r^{-3q+2}}{T} \sum_{l=1}^{s} \left(\sum_{i=1}^{l} w_{i,l} r^{2(l+(q-1)i)} \right), \tag{3.9}$$

where

$$w_{i,l} = a_{2i-1,2l-1}(\alpha_{2(l-i),2i} + \alpha_{2(l-i+m),2i}),$$

with

$$\alpha_{a;b} = \int_0^T Cs^a\theta Sn^b\theta d\theta.$$

The number of positive roots of $g^0(r)$ is equal to that of

$$\digamma(r) = \sum_{l=1}^{s}\left(\sum_{i=1}^{l} w_{i,l} r^{2(l+(q-1)i)}\right), \tag{3.10}$$

we write (3.10) as follows

$$\begin{aligned}
\digamma(r) = &\sum_{l=1}^{s}[w_{1,l}r^{2(l+(q-1))} + w_{2,l}r^{2(l+2(q-1))} + w_{3,l}r^{2(l+3(q-1))} + \\
&\quad ... + w_{l-1,l}r^{2(l+(l-1)(q-1))} + w_{l,l}r^{2(l+l(q-1))}] \\
&= w_{1,1}r^{2(1+(q-1))} + w_{1,2}r^{2(2+(q-1))} + w_{2,2}r^{2(2+2(q-1))} \\
&+ w_{1,3}r^{2(3+(q-1))} + w_{2,3}r^{2(3+2(q-1))} + w_{3,3}r^{2(3+3(q-1))} \\
&+ ...... \\
&+ [w_{1,s-2}r^{2((s-2)+(q-1))} + w_{2,s-2}r^{2((s-2)+2(q-1))} + w_{3,s-2}r^{2((s-2)+3(q-1))} + ... \\
&+ w_{s-3,s-2}r^{2((s-2)+(s-3)(q-1))} + w_{s-2,s-2}r^{2((s-2)+(s-2)(q-1))}] \\
&+ [w_{1,s-1}r^{2((s-1)+(q-1))} + w_{2,s-1}r^{2((s-1)+2(q-1))} + w_{3,s-1}r^{2((s-1)+3(q-1))} + ... \\
&+ w_{s-2,s-1}r^{2((s-1)+(s-2)(q-1))} + w_{s-1,s-1}r^{2((s-1)+(s-1)(q-1))}] \\
&+ [w_{1,s}r^{2(s+(q-1))} + w_{2,s}r^{2(s+2(q-1))} + w_{3,s}r^{2(s+3(q-1))} + ... \\
&+ w_{s-1,s}r^{2(s+(s-1)(q-1))} + w_{s,s}r^{2(s+s(q-1))}].
\end{aligned}$$

Let us write $F(r)$ as

$$\begin{aligned}F(r) = & [w_{1,1}r^{2(1+(q-1))} + w_{1,2}r^{2(2+(q-1))} + w_{1,3}r^{2(3+(q-1))} + \dots \\ & + w_{1,s-2}r^{2((s-2)+(q-1))} + w_{1,s-1}r^{2((s-1)+(q-1))} + w_{1,s}r^{2(s+(q-1))}] \\ & + [w_{2,2}r^{2(2+2(q-1))} + w_{2,3}r^{2(3+2(q-1))} + \dots \\ & + w_{2,s-2}r^{2((s-2)+2(q-1))} + w_{2,s-1}r^{2((s-1)+2(q-1))} + w_{2,s}r^{2(s+2(q-1))}] \\ & + [w_{3,3}r^{2(3+3(q-1))} + \dots \\ & + w_{3,s-2}r^{2((s-2)+3(q-1))} + w_{3,s-1}r^{2((s-1)+3(q-1))} + w_{3,s}r^{2(s+3(q-1))}] \\ & + \dots \\ & + [w_{s-2,s-2}r^{2((s-2)+(s-2)(q-1))} + w_{s-2,s-1}r^{2((s-1)+(s-2)(q-1))} + w_{s-2,s}r^{2(s+(s-2)\lambda)}] \\ & + [w_{s-1,s-1}r^{2((s-1)+(s-1)(q-1))} + w_{s-1,s}r^{2(s+(s-1)(q-1))}] \\ & + w_{s,s}r^{2(s+s(q-1))}.\end{aligned} \tag{3.11}$$

To find the number of positive roots of polynomials $F(r)$, we distinguish 2 cases

Case 1: For $s \leq q$, the number terms in polynomial (*3.11*) is

$$s + (s-1) + (s-2) + \dots + 1 = \frac{s(s+1)}{2}.$$

Now, we shall apply the Descartes Theorem, we can choose the appropriate coefficients $w_{i,l}$ in order that the simple positive roots number of $F(r)$ is at most

$$\frac{s(s+1)}{2} - 1 = \frac{(s-1)(s+2)}{2}.$$

Hence (a) of Theorem *3.1.2* is proved.

Case 2: For $s \geq q+1$, let $\mu = s - q$ the number terms in polynomial

(3.11) is

$$
\begin{aligned}
&s+(s-1)+(s-2)+...+2+1-\mu-(\mu-1)-(\mu-2)-...-2-1\\
&=\frac{s(s+1)-\mu(\mu+1)}{2}\\
&=qs-\frac{q(q-1)}{2},
\end{aligned}
$$

by Descartes Theorem, we can choose the appropriate coefficients $w_{i,l}$ in order that the simple positive roots number of $\mathcal{F}(r)$ is at most

$$qs-\frac{q(q-1)}{2}-1.$$

Hence (b) of Theorem 3.1.2 is proved.

3.3 Example

We consider the system

$$
\begin{cases}
\dot{x}=y,\\
\dot{y}=-x^5-\varepsilon(1+Cs^2\theta)(x+2.4551y-28.761x^2y+2y^3).
\end{cases}
\tag{3.12}
$$

In this cas $p=1$, $q=3$, $m=1$, $n=3$, $Cs\theta$ and $Sn\theta$ are $T-$periodic function with period $T=8.4131$. From equation (3.9) we obtain

$$g^0(r)=\frac{1}{rT}(w_{1,1}+w_{1,2}r^2+w_{2,2}r^6),$$

where

$$w_{i,l}=a_{2i-1,2l-1}(\alpha_{2(l-i),2i}+\alpha_{2(l-i+1),2i}).$$

Using (**??**), we get

$$\alpha_{0,2}=2.1033, \alpha_{2,2}=0.60460, \alpha_{4,2}=0.32339,$$

$$\alpha_{0,4}=0.63098, \alpha_{2,4}=0.15115.$$

So

$$g^0(r) = \frac{1}{8.4131r}(2.7079a_{1,1} + 0.92799a_{1,3}r^2 + 0.78213a_{3,3}r^6)$$
$$= \frac{1}{8.4131r}(6.6482 - 26.690r^2 + 1.5643r^6).$$

This polynomial has two positive real roots, $r_1 = 0.5$ and $r_2 = 2$. Using the first-order averaging theory, system (3.12) has exactly two limit cycles bifurcating from the periodic orbits of the center $\dot{x} = y$, $\dot{y} = -x^5$.

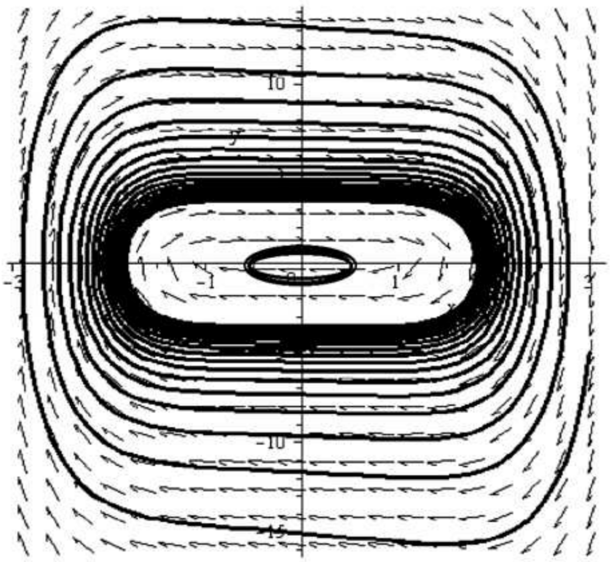

Figure 3.1: The system (3.12) has two limit cycles, for $\varepsilon = 0.01$

3.4 Summary and Future Directions

The generation of limit cycles through perturbations of Hamiltonian systems with centers is a significant and intricate method in the study of dynamical systems.Through precise perturbations of such a system, it becomes possible

to trigger the bifurcation of limit cycles from the periodic orbits characteristic of the original, unperturbed system. In this study, we have systematically determined upper bounds for the maximum number of limit cycles that can arise from the periodic orbits of a particular Hamiltonian system. This system is defined by the Hamiltonian function $H(x,y) = \frac{x^{2q}}{2q} + \frac{y^{2p}}{2p}$, where p and q are positive integers. The perturbations considered fall within the scope of generalized Mathieu differential systems of the form (3.1), extending and building upon earlier studies on differential systems, as referenced in [11], [51]. This study not only contributes to the theoretical understanding of limit cycle bifurcations in perturbed Hamiltonian systems but also opens avenues for further exploration using advanced mathematical techniques. In future research, we plan to apply higher-order averaging theory and the Melnikov method to delve deeper into the bifurcation phenomena associated with such systems. Additionally, numerical simulations will be conducted to validate and exemplify the theoretical predictions, providing a more comprehensive understanding of the dynamics at play. Through these efforts, we aim to contribute to the broader field of dynamical systems and enhance the toolbox available for analyzing complex nonlinear phenomena.

Chapter 4
On the number of limit cycles in a class of generalized Mathieu-Duffing equations

4.1 Introduction and Main Results

The study and examination of limit cycles in Mathieu-Duffing equations are crucial for numerous disciplines, including physics, engineering, and applied mathematics, where periodic phenomena are of paramount importance. These equations often model systems that exhibit complex dynamic behaviors such as oscillations, bifurcations, and chaos, making their analysis essential for both theoretical understanding and practical applications. In this chaper, we investigate the maximum number of limit cycles in generalized Mathieu-Duffing equations using first- and second-order averaging theory. Our methodology involves a detailed examination of two second-order differential equations, employing averaging theory and Descartes' theorem to uncover the intricate dynamics of these systems. We provide a theorem that determines the maximum number of possible limit cycles in the generalized differential systems through polynomial averaging associated with the problem. Additionally, we present lemmas detailing polynomial integrals related to the functions $F_{20}^1(r)$ and $F_{20}^2(r)$, which are crucial to our analysis. The results indicate that the maximum number of limit cycles in the studied system can be determined based on polynomial analysis and second-order averaging theory, considering the equation parameters and system conditions.

In recent years, significant progress has been made in understanding the dynamics of the Duffing equation and chaos in dynamical systems. For instance, Kyziol and Okninski (2023, [24]) studied the jump phenomenon in the forced asymmetric Duffing oscillator, introducing the concept of a jump manifold, which encodes global information about all possible jumps. Their work demonstrated the advantages of this method by calculating several examples of jumps. In another study, Kyziol and Okninski (2024, [25]) investigated escape phenomena, bistability, jumps, and other bifurcations in the twin-well Duffing equation, providing predictions for bifurcations using the steady-state asymptotic solution. Furthermore, Mao et al. (2024, [37]) examined the application of the incommensurate fractional-order Duffing system for weak signal detection, highlighting the system's sensitivity to specific signal functions and its robustness against noise. In [11] Chen and Llibre, using the averaging theory of the first, studied the limit cycles of the differential system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \varepsilon(1 + cos^m\theta)Q(x, y). \end{cases}$$

Building on these foundational studies, the primary aim of this paper is to examine the upper limit on the number of limit cycles that can arise in a specialized generalization of the differential system (3.1) under the specific case where $p = q = 1$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \sum\limits_{k\geq 1} \varepsilon^k (1 + \cos^{2m_k}\theta) Q_k(x, y), \end{cases} \tag{4.1}$$

where for each k, the polynomial $Q_k(x, y)$ is characterized by having a degree of n_k, ε is a small parameter and m_k a non-negative integer. By leveraging polynomial analysis and advanced averaging techniques, we aim to elucidate the maximum number of limit cycles that can arise in this generalized differential system, providing new insights and extending the existing body of knowledge on the dynamic behavior of Mathieu-Duffing equations.

The core finding of this paper is presented in the following theorem, which succinctly outlines the upper bound on the number of limit cycles in the differential system (4.1).

Theorem 4.1.1. *Let $k = 1, 2$, and assume that the polynomials $Q_k(x, y)$ are of degree n_k, with $n_k \geq 1$. For $|\varepsilon|$ sufficiently small the maximum number of limit cycles of the polynomial differential systems (4.1) bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, using averaging theory*

*(**i**) of first order is*

*(**a**) if n_1 is even, we find $l_1 - 1$ limit cycles,*

*(**b**) if n_1 is odd, we find l_1 limit cycles,*

*(**ii**) of second order is*

$$\lambda = \max\{2l_1 - 1, l_2 - 1\},$$

where $n_k = 2l_k$ or $n_k = 2l_k + 1$ when n_k is even or odd respectively.

Remark 4.1.1. *The function F_{20} represents the averaged behavior of the system over one period, encapsulating key dynamics that help determine the conditions for periodic solutions, known as limit cycles. Our approach employs classical methods such as the averaging method and Descartes' rule of signs.*

However, it innovates in its application to generalized Mathieu-Duffing equations. This specific application, along with the detailed polynomial analysis we provide, offers new insights and significantly contributes to the understanding of such systems. To contextualize our findings within the broader field, we compare our results with existing literature. For example, previous studies, like those by Melki and Makhlouf (2021, [12]), used first-order averaging methods to estimate the number of limit cycles. Our study extends this by incorporating second-order averaging and detailed polynomial analysis, providing a more comprehensive and precise understanding of the limit cycles in generalized Mathieu-Duffing equations. Unlike some previous approaches that relied on numerical methods, our analytical framework offers exact bounds and conditions for the existence of limit cycles.

4.2 Proofs of the Results

4.2.1 Proof of Theorem 4.1.1

Proof of Statement (i) of Theorem 4.1.1

To use the averaging method we express system (*4.1*) with $k = 2$, in polar coordinates (r, θ) where $x = r\cos\theta$, $y = r\sin\theta$, $r > 0$. If we take $Q_k(x, y) = \sum_{i+j=0}^{n_k} q_{k,i,j} x^i y^j$ the expression for system (*4.1*) can be formulated as follows:

$$\begin{cases} \dot{r} = -\varepsilon \sin\theta(1+\cos^{2m_1}\theta)Q_1(r\cos\theta, r\sin\theta) + \varepsilon^2 \sin\theta(1+\cos^{2m_2}\theta) \\ \qquad \times Q_2(r\cos\theta, r\sin\theta) \\ \dot{\theta} = -1 - \frac{\varepsilon}{r}\cos\theta(1+\cos^{2m_1}\theta)Q_1(r\cos\theta, r\sin\theta) + \frac{\varepsilon^2}{r}\cos\theta(1+\cos^{2m_2}\theta) \\ \qquad \times Q_2(r\cos\theta, r\sin\theta). \end{cases} \tag{4.2}$$

Now taking θ as the new independent variable, system (4.2) becomes

By considering θ as the new independent variable, the system (4.2) transforms into

$$\frac{dr}{d\theta} = \varepsilon F_1\left(r,\theta\right) + \varepsilon^2 F_2\left(r,\theta\right) + o\left(\varepsilon^3\right),$$

where

$$F_1\left(r,\theta\right) = \sin\theta(1+\cos^{2m_1}\theta)Q_1\left(r\cos\theta, r\sin\theta\right), \tag{4.3}$$

$$\begin{aligned} F_2\left(r,\theta\right) = &-\sin\theta\left(1+\cos^{2m_2}\theta\right)Q_2\left(r\cos\theta, r\sin\theta\right) \\ &-r^{-1}\cos\theta\sin\theta\left(1+\cos^{2m_1}\theta\right)^2 Q_1^2\left(r\cos\theta, r\sin\theta\right). \end{aligned} \tag{4.4}$$

Applying the precedent notation, it follows that

$$\begin{aligned} F_{10}\left(r\right) &= \frac{1}{2\pi}\int_0^{2\pi} F_1\left(r,\theta\right)d\theta \\ &= \frac{1}{2\pi}\sum_{i+j=0}^{n_1} q_{1,i,j}r^{i+j}\left(\int_0^{2\pi}\cos^i\theta\sin^{j+1}\theta d\theta + \int_0^{2\pi}\cos^{i+2m_1}\theta\sin^{j+1}\theta d\theta\right). \end{aligned}$$

We know that

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^i\theta\sin^j\theta d\theta = \begin{cases} I_{ij}\neq 0 \text{ if } i \text{ and } j \text{ are even} \\ 0 \text{ otherwise.} \end{cases}. \tag{4.5}$$

To investigate the number of positive roots for F_{10}, we consider two cases:

Case (**1**): Let n_1 be even,

$$F_{10}\left(r\right) = \sum_{i+j=1}^{n_1/2} q_{1,2i,2j-1}r^{2(i+j)-1}\left(I_{2i,2j} + I_{2i+2m_1,2j}\right), \tag{4.6}$$

then $F_{10}\left(r\right) = \sum\limits_{h=1}^{n_1/2} A_h r^{2h-1}$. Since F_{10} is a polynomial generated by a linear combination of $\{r, r^3, r^5, ..., r^{n_1-1}\}$, Using the Descartes Theorem, the polynomial F_{10} has at most $\left(n_1-2\right)/2$ positive roots.

Case (**2**): Let n_1 be odd,

$$F_{10}(r) = \sum_{i+j=1}^{(n_1+1)/2} q_{1,2i,2j-1} r^{2(i+j)-1} \left(I_{2i,2j} + I_{2i+2m_1,2j} \right), \tag{4.7}$$

then $F_{10}(r) = \sum\limits_{h=1}^{(n_1+1)/2} B_h r^{2h-1}$. Since F_{10} is a polynomial generated by a linear combination of $\{r, r^3, r^5, ..., r^{n_1}\}$, Applying Descartes' Rule of Signs, the polynomial F_{10} can have at most $(n_1-1)/2$ positive roots. Consequently, statement (i) in Theorem 4.1.1 is validated.

Proof of Statement (ii) of Theorem 4.1.1

For proving ii. of Theorem 4.1.1 we shall use the second-order averaging theory. To calculate $F_{20}(r)$, $F_{10}(r)$ must be identically zero. Then from (*4.5*) and (*4.6*), we have:

$$q_{1,2i,2j-1} = 0, \;\; i+j \in \left\{1, ..., \frac{n_1}{2}\right\} \text{ if } n_1 \text{ is even}, \tag{4.8}$$

$$q_{1,2i,2j-1} = 0, \;\; i+j \in \left\{1, ..., \frac{n_1+1}{2}\right\} \text{ if } n_1 \text{ is odd}. \tag{4.9}$$

First, using (*4.8*) − (*4.9*) and by substituting into the expression $F_1(r,\theta)$, we obtain

$$\begin{aligned} F_1(r,\theta) = & \sum_{i+j=0}^{n_1/2} q_{1,2i,2j} r^{2(i+j)} (1+\cos^{2m_1}\theta) \cos^{2i}\theta \sin^{2j+1}\theta \\ & + \sum_{i+j=0}^{n_1/2} q_{1,2i+1,2j-1} r^{2(i+j)} (1+\cos^{2m_1}\theta) \cos^{2i+1}\theta \sin^{2j}\theta \\ & + \sum_{i+j=0}^{(n_1-2)/2} q_{1,2i+1,2j} r^{2(i+j)+1} (1+\cos^{2m_1}\theta) \cos^{2i+1}\theta \sin^{2j+1}\theta, \text{ if } n_1 \text{ is even}, \end{aligned} \tag{4.10}$$

and

$$F_1(r,\theta) = \sum_{i+j=0}^{(n_1-1)/2} q_{1,2i,2j} r^{2(i+j)} (1+\cos^{2m_1}\theta)\cos^{2i}\theta\sin^{2j+1}\theta \tag{4.11}$$
$$+ \sum_{i+j=0}^{(n_1-1)/2} q_{1,2i+1,2j-1} r^{2(i+j)} (1+\cos^{2m_1}\theta)\cos^{2i+1}\theta\sin^{2j}\theta$$
$$+ \sum_{i+j=0}^{(n_1-1)/2} q_{1,2i+1,2j} r^{2(i+j)+1} (1+\cos^{2m_1}\theta)\cos^{2i+1}\theta\sin^{2j+1}\theta, \text{ if } n_1 \text{ is odd.}$$

Now that we have the expression $\frac{d}{dr}F_1(r,\theta)$,

$$\frac{d}{dr}F_1(r,\theta) = \sum_{i+j=1}^{n_1/2} 2(i+j)\, q_{1,2i,2j} r^{2(i+j)-1} (1+\cos^{2m_1}\theta)\cos^{2i}\theta\sin^{2j+1}\theta$$
$$+ \sum_{i+j=1}^{n_1/2} 2(i+j)\, q_{1,2i+1,2j-1} r^{2(i+j)-1} (1+\cos^{2m_1}\theta)\cos^{2i+1}\theta\sin^{2j}\theta$$
$$+ \sum_{i+j=0}^{(n_1-2)/2} (2(i+j)+1)\, q_{1,2i+1,2j} r^{2(i+j)} (1+\cos^{2m_1}\theta)\cos^{2i+1}\theta$$
$$\times \sin^{2j+1}\theta, \text{ if } n_1 \text{ is even,} \tag{4.12}$$

and

$$\frac{d}{dr}F_1(r,\theta) = \sum_{i+j=1}^{(n_1-1)/2} 2(i+j)\, q_{1,2i,2j} r^{2(i+j)-1} (1+\cos^{2m_1}\theta)\cos^{2i}\theta\sin^{2j+1}\theta$$
$$+ \sum_{i+j=1}^{(n_1-1)/2} 2(i+j)\, q_{1,2i+1,2j-1} r^{2(i+j)-1} (1+\cos^{2m_1}\theta)\cos^{2i+1}\theta\sin^{2j}\theta$$
$$+ \sum_{i+j=0}^{(n_1-1)/2} (2(i+j)+1)\, q_{1,2i+1,2j} r^{2(i+j)} (1+\cos^{2m_1}\theta)\cos^{2i+1}\theta$$
$$\times \sin^{2j+1}\theta, \text{ if } n_1 \text{ is odd.} \tag{4.13}$$

Furthermore, by employing the integrals provided in Appendix A, we obtain:

Case (**1**): Let n_1 be even,

$$
\begin{aligned}
y(r,\theta) &= \textstyle\int_0^\theta F_1(r,\varphi)\,d\varphi \\
&= \sum_{i+j=0}^{n_1/2} q_{1,2i,2j} r^{2(i+j)} \sum_{l=0}^{j}\sum_{k=1}^{l+j} \widetilde{\gamma}_{l,k}(i,m_1,j)\cos((2k+1)\theta) \\
&\quad + \sum_{i+j=0}^{(n_1-2)/2} q_{1,2i+1,2j} r^{2(i+j)+1}(\widetilde{\rho}_{i,m_1,j} + \sum_{l=0}^{j}\bar{\rho}_{l,i,j}\cos^{2(i+l+1)}\theta \\
&\quad + \sum_{l=0}^{j}\bar{\rho}_{l,i+m_1,j}\cos^{2(i+m_1+l+1)}\theta) + \sum_{i+j=0}^{n_1/2} q_{1,2i+1,2j-1} r^{2(i+j)} \\
&\quad \times(\sum_{l=0}^{i}\sum_{k=1}^{l+i}\gamma_{l,k}^{(1)}(i,j)\sin((2k+1)\theta) \\
&\quad + \sum_{l=0}^{i+m_1}\sum_{k=1}^{l+i+m_1}\gamma_{l,k}^{(1)}(i+m_1,j)\sin((2k+1)\theta)),
\end{aligned}
\tag{4.14}
$$

Case (**2**): Let n_1 be odd,

$$
\begin{aligned}
y(r,\theta) &= \textstyle\int_0^\theta F_1(r,\varphi)\,d\varphi, \\
&= \sum_{i+j=0}^{(n_1-1)/2} q_{1,2i,2j} r^{2(i+j)} \sum_{l=0}^{j}\sum_{k=1}^{l+j} \widetilde{\gamma}_{l,k}(i,m_1,j)\cos((2k+1)\theta) \\
&\quad + \sum_{i+j=0}^{(n_1-1)/2} q_{1,2i+1,2j} r^{2(i+j)+1}(\widetilde{\rho}_{i,m_1,j} + \sum_{l=0}^{j}\bar{\rho}_{l,i,j}\cos^{2(i+l+1)}\theta \\
&\quad + \sum_{l=0}^{j}\bar{\rho}_{l,i+m_1,j}\cos^{2(i+m_1+l+1)}\theta) + \sum_{i+j=0}^{(n_1-1)/2} q_{1,2i+1,2j-1} r^{2(i+j)} \\
&\quad \times(\sum_{l=0}^{i}\sum_{k=1}^{l+i}\gamma_{l,k}^{(1)}(i,j)\sin((2k+1)\theta) \\
&\quad + \sum_{l=0}^{i+m_1}\sum_{k=1}^{l+i+m_1}\gamma_{l,k}^{(1)}(i+m_1,j)\sin((2k+1)\theta)),
\end{aligned}
\tag{4.15}
$$

where

$$\widetilde{\gamma}_{l,k}(i,m_1,j) = \gamma_{l,k}^{(0)}(i,j) + \gamma_{l,k}^{(0)}(i+m_1,j),$$

$$\widetilde{\rho}_{i,m_1,j} = \rho_{i,j} + \rho_{i+m_1,j},$$

with $\rho_{i,j},\ \rho_{i+m_1,j},\ \bar{\rho}_{l,i,j},\ \bar{\rho}_{l,i+m_1,j},\ \gamma_{l,k}^{(0)}(i,j),\ \gamma_{l,k}^{(0)}(i+m_1,j),\ \gamma_{l,k}^{(1)}(i,j)$ and $\gamma_{l,k}^{(1)}(i+m_1,j)$ are non-zero constants. To implement the second-order averaging method, it is necessary to calculate the associated function $F_{20}(r)$, which can be expressed as $F_{20}(r) = F_{20}^1(r) + F_{20}^2(r)$ with

$$F_{20}^1(r) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{d}{dr}F_1(r,\theta)\right)y(r,\theta)\,d\theta \text{ and } F_{20}^2(r) = \frac{1}{2\pi}\int_0^{2\pi}F_2(r,\theta)\,d\theta.$$

Lemma 4.1. *The integrale $F_{20}^{1}(r)$ is expressed as a polynomial in the variable r, defined by:*

$$F_{20}^{1}(r)=\sum_{i+j=0}^{l_1}\sum_{u+v=0}^{l_1}Q_{(i,j)}^{(m_1)}(u,v)\,r^{2(i+j+u+v)-1},$$

where $n_1=2l_1$ or $n_1=2l_1+1$ when n_1 is even or odd respectively, and

$$\begin{aligned}Q_{(i,j)}^{(m_1)}(u,v)=\frac{i+j}{\pi}[&q_{2i,2j}q_{2u+1,2v-1}\sum_{l=0}^{u}\sum_{k=1}^{l+u}\gamma_{l,k}^{(1)}(u,v)\left(\Lambda_{i,j,k}+\Lambda_{m_1+i,j,k}\right)\\&+q_{2i,2j}q_{2u+1,2v-1}\sum_{l=0}^{u+m_1}\sum_{k=1}^{l+u+m_1}\gamma_{l,k}^{(1)}(u+m_1,v)\left(\Lambda_{i,j,k}+\Lambda_{m_1+i,j,k}\right)\\&+q_{2i+1,2j-1}q_{2u,2v}\sum_{l=0}^{v}\sum_{k=1}^{l+v}\widetilde{\gamma}_{l,k}(u,m_1,v)\,\Gamma_{i,j,k}\\&+q_{2i+1,2j-1}q_{2u,2v}\sum_{l=0}^{v}\sum_{k=1}^{l+v}\widetilde{\gamma}_{l,k}(u,m_1,v)\,\Gamma_{m_1+i,j,k}],\end{aligned}$$

where $\Lambda_{i,j,k}$, $\Lambda_{m_1+i,j,k}$, $\Gamma_{i,j,k}$, and $\Gamma_{m_1+i,j,k}$ are real constants.

4.2.2 Proof of Lemma 4.1

From (4.12) − (4.15), we have

$$F_{20}^{1}(r)=\begin{cases}\sum\limits_{i=1}^{6}\widehat{h}_i(r)\,, & \text{if } n_1 \text{ si even},\\ \sum\limits_{i=1}^{6}\widetilde{h}_i(r)\,, & \text{if } n_1 \text{ is odd},\end{cases}\tag{4.16}$$

where $\widehat{h}_i(r)$, $\widetilde{h}_i(r)$, $i=1,...,6$, are given by

$$\widehat{h}_1(r)=\frac{1}{2\pi}\sum_{i+j=1}^{n_1/2}2(i+j)\,q_{1,2i,2j}r^{2(i+j)-1}\int_0^{2\pi}\cos^{2i}\theta\sin^{2j+1}\theta y(r,\theta)\,d\theta,$$

$$\widehat{h}_2(r)=\frac{1}{2\pi}\sum_{i+j=1}^{n_1/2}2(i+j)\,q_{1,2i,2j}r^{2(i+j)-1}\int_0^{2\pi}\cos^{2m_1+2i}\theta\sin^{2j+1}\theta y(r,\theta)\,d\theta,$$

$$\widehat{h}_3(r) = \frac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2(i+j) q_{1,2i+1,2j-1} r^{2(i+j)-1} \int_0^{2\pi} \cos^{2i+1}\theta \sin^{2j}\theta y(r,\theta)\, d\theta,$$

$$\widehat{h}_4(r) = \frac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2(i+j) q_{1,2i+1,2j-1} r^{2(i+j)-1} \int_0^{2\pi} \cos^{2m_1+2i+1}\theta \sin^{2j}\theta y(r,\theta)\, d\theta,$$

$$\widehat{h}_5(r) = \frac{1}{2\pi} \sum_{i+j=0}^{(n_1-2)/2} (2(i+j)+1) q_{1,2i+1,2j} r^{2(i+j)} \int_0^{2\pi} \cos^{2i+1}\theta \sin^{2j+1}\theta y(r,\theta)\, d\theta,$$

$$\widehat{h}_6(r) = \frac{1}{2\pi} \sum_{i+j=0}^{(n_1-2)/2} (2(i+j)+1) q_{1,2i+1,2j} r^{2(i+j)} \int_0^{2\pi} \cos^{2m_1+2i+1}\theta \sin^{2j+1}\theta y(r,\theta)\, d\theta,$$

$$\widetilde{h}_1(r) = \frac{1}{2\pi} \sum_{i+j=1}^{(n_1-1)/2} 2(i+j) q_{1,2i,2j} r^{2(i+j)-1} \int_0^{2\pi} \cos^{2i}\theta \sin^{2j+1}\theta y(r,\theta)\, d\theta,$$

$$\widetilde{h}_2(r) = \frac{1}{2\pi} \sum_{i+j=1}^{(n_1-1)/2} 2(i+j) q_{1,2i,2j} r^{2(i+j)-1} \int_0^{2\pi} \cos^{2m_1+2i}\theta \sin^{2j+1}\theta y(r,\theta)\, d\theta,$$

$$\widetilde{h}_3(r) = \frac{1}{2\pi} \sum_{i+j=1}^{(n_1-1)/2} 2(i+j) q_{1,2i+1,2j-1} r^{2(i+j)-1} \int_0^{2\pi} \cos^{2i+1}\theta \sin^{2j}\theta y(r,\theta)\, d\theta,$$

$$\widetilde{h}_4(r) = \frac{1}{2\pi} \sum_{i+j=1}^{(n_1-1)/2} 2(i+j) q_{1,2i+1,2j-1} r^{2(i+j)-1} \int_0^{2\pi} \cos^{2m_1+2i+1}\theta \sin^{2j}\theta y(r,\theta)\, d\theta,$$

$$\widetilde{h}_5(r) = \frac{1}{2\pi} \sum_{i+j=0}^{(n_1-1)/2} (2(i+j)+1) q_{1,2i+1,2j} r^{2(i+j)} \int_0^{2\pi} \cos^{2i+1}\theta \sin^{2j+1}\theta y(r,\theta)\, d\theta,$$

and

$$\widetilde{h}_6(r) = \frac{1}{2\pi} \sum_{i+j=0}^{(n_1-1)/2} (2(i+j)+1) q_{1,2i+1,2j} r^{2(i+j)} \int_0^{2\pi} \cos^{2m_1+2i+1}\theta \sin^{2j+1}\theta y(r,\theta)\, d\theta.$$

To simplify the representation of the polynomial $\widehat{h}_1(r)$, utilizing the integrals provided in Appendix B, we obtain:

$$\widehat{\Delta}_1 = \frac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2\left(i+j\right) q_{2i,2j} r^{2(i+j)-1} \sum_{u+v=0}^{n_1/2} q_{2u,2v} r^{2(u+v)}$$

$$\times \sum_{l=0}^{v} \sum_{k=1}^{l+v} \widetilde{\gamma}_{l,k}\left(u, m_1, v\right) \int_0^{2\pi} \cos^{2i}\theta \sin^{2j+1}\theta \cos\left(\left(2k+1\right)\theta\right) d\theta$$

$$= 0,$$

$$\widehat{\Delta}_2 = \frac{1}{2\pi} \sum_{i+j=1}^{n1/2} 2\left(i+j\right) q_{2i,2j} r^{2(i+j)-1} \sum_{u+v=0}^{n_1/2} q_{2u+1,2v-1} r^{2(u+v)}$$

$$\times \sum_{l=0}^{u} \sum_{k=1}^{l+u} \gamma_{l,k}^{(1)}\left(u, v\right) \int_0^{2\pi} \cos^{2i}\theta \sin^{2j+1}\theta \sin\left(\left(2k+1\right)\theta\right) d\theta$$

$$= \sum_{i+j=1}^{n_1/2} \sum_{u+v=0}^{n_1/2} R_{(i,j)}\left(u, v\right) r^{2(i+j+u+v)-1},$$

where $R_{(i,j)}\left(u, v\right) = \frac{i+j}{\pi} q_{2i,2j} q_{2u+1,2v-1} \sum\limits_{l=0}^{u} \sum\limits_{k=1}^{l+u} \gamma_{l,k}^{(1)}\left(u, v\right) \Lambda_{i,j,k},$

$$\widehat{\Delta}_3 = \frac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2\left(i+j\right) q_{2i,2j} r^{2(i+j)-1} \sum_{u+v=0}^{n_1/2} q_{2u+1,2v-1} r^{2(u+v)}$$

$$\times \sum_{l=0}^{u+m_1} \sum_{k=1}^{l+u+m_1} \gamma_{l,k}^{(1)}\left(u+m_1, v\right) \int_0^{2\pi} \cos^{2i}\theta \sin^{2j+1}\theta \sin\left(\left(2k+1\right)\theta\right) d\theta$$

$$= \sum_{i+j=1}^{n_1/2} \sum_{u+v=0}^{n_1/2} S_{(i,j)}^{(m_1)}\left(u, v\right) r^{2(i+j+u+v)-1},$$

where

$$S_{(i,j)}^{(m_1)}\left(u, v\right) = \frac{i+j}{\pi} q_{2i,2j} q_{2u+1,2v-1} \sum_{l=0}^{u+m_1} \sum_{k=1}^{l+u+m_1} \gamma_{l,k}^{(1)}\left(u+m_1, v\right) \Lambda_{i,j,k},$$

$$\widehat{\Delta}_4 = \tfrac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2\left(i+j\right) q_{2i,2j} r^{2(i+j)-1} \sum_{u+v=0}^{(n_1-2)/2} q_{2u+1,2v} r^{2(u+v)+1} \widetilde{\rho}_{u,m_1,v}$$

$$\times \int_0^{2\pi} \cos^{2i}\theta \sin^{2j+1}\theta d\theta = 0,$$

$$\widehat{\Delta}_5 = \tfrac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2\,(i+j)\, q_{2i,2j} r^{2(i+j)-1} \sum_{u+v=0}^{(n_1-2)/2} q_{2u+1,2v} r^{2(u+v)+1} \sum_{l=0}^{v} \bar{\rho}_{l,u,v}$$
$$\times \int_0^{2\pi} \cos^{2(i+u+l+1)} \theta \sin^{2j+1} \theta d\theta = 0,$$

$$\widehat{\Delta}_6 = \tfrac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2\,(i+j)\, q_{2i,2j} r^{2(i+j)-1} \sum_{u+v=0}^{(n_1-2)/2} q_{2u+1,2v} r^{2(u+v)+1} \sum_{l=0}^{v} \bar{\rho}_{l,u+m_1,v}$$
$$\times \int_0^{2\pi} \cos^{2(i+m_1+u+l+1)} \theta \sin^{2j+1} \theta d\theta = 0.$$

The polynomial $\widehat{h}_1(r)$ is expressed as the sum of integrals $\left(\widehat{\Delta}_1\right)$ to $\left(\widehat{\Delta}_6\right)$. Utilizing the integrals provided in Appendix B, we have:

$$\widehat{\Delta}_7 = \frac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2\,(i+j)\, q_{2i,2j} r^{2(i+j)-1} \sum_{u+v=0}^{n_1/2} q_{2u,2v} r^{2(u+v)}$$
$$\times \sum_{l=0}^{v} \sum_{k=1}^{l+v} \widetilde{\gamma}_{l,k}\,(u, m_1, v) \int_0^{2\pi} \cos^{2(m_1+i)} \theta \sin^{2j+1} \theta \cos\left((2k+1)\,\theta\right) d\theta = 0,$$

$$\widehat{\Delta}_8 = \frac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2\,(i+j)\, q_{2i,2j} r^{2(i+j)-1} \sum_{u+v=0}^{n_1/2} q_{2u+1,2v-1} r^{2(u+v)}$$
$$\times \sum_{l=0}^{u} \sum_{k=1}^{l+u} \gamma_{l,k}^{(1)}\,(u,v) \int_0^{2\pi} \cos^{2(m_1+i)} \theta \sin^{2j+1} \theta \sin\left((2k+1)\,\theta\right) d\theta$$
$$= \sum_{i+j=1}^{n_1/2} \sum_{u+v=0}^{n_1/2} R_{(i,j)}^{(m_1)}\,(u,v)\, r^{2(i+j+u+v)-1},$$

where $R_{(i,j)}^{(m_1)}\,(u,v) = \frac{i+j}{\pi} q_{2i,2j} q_{2u+1,2v-1} \sum_{l=0}^{u} \sum_{k=1}^{l+u} \gamma_{l,k}^{(1)}\,(u,v)\, \Lambda_{m_1+i,j,k}$,

$$\widehat{\Delta}_9 = \frac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2\,(i+j)\, q_{2i,2j} r^{2(i+j)-1} \sum_{u+v=0}^{n_1/2} q_{2u+1,2v-1} r^{2(u+v)}$$
$$\times \sum_{l=0}^{u+m_1} \sum_{k=1}^{l+u+m_1} \gamma_{l,k}^{(1)}\,(u+m_1,v) \int_0^{2\pi} \cos^{2(m_1+i)} \theta \sin^{2j+1} \theta \sin\left((2k+1)\,\theta\right) d\theta$$
$$= \sum_{i+j=1}^{n_1/2} \sum_{u+v=0}^{n_1/2} \widetilde{R}_{(i,j)}^{(m_1)}\,(u,v)\, r^{2(i+j+u+v)-1},$$

where

$$\widetilde{R}_{(i,j)}^{(m_1)}(u,v)=\frac{i+j}{\pi}q_{2i,2j}q_{2u+1,2v-1}\sum_{l=0}^{u+m_1}\sum_{k=1}^{l+u+m_1}\gamma_{l,k}^{(1)}(u+m_1,v)\Lambda_{m_1+i,j,k},$$

$$\widehat{\Delta}_{10}=\tfrac{1}{2\pi}\sum_{i+j=1}^{n_1/2}2(i+j)q_{2i,2j}r^{2(i+j)-1}\sum_{u+v=0}^{(n_1-2)/2}q_{2u+1,2v}r^{2(u+v)+1}\widetilde{\rho}_{u,m_1,v}$$
$$\times\int_0^{2\pi}\cos^{2(m_1+i)}\theta\sin^{2j+1}\theta d\theta=0,$$

$$\widehat{\Delta}_{11}=\tfrac{1}{2\pi}\sum_{i+j=1}^{n_1/2}2(i+j)q_{2i,2j}r^{2(i+j)-1}\sum_{u+v=0}^{(n_1-2)/2}q_{2u+1,2v}r^{2(u+v)+1}\sum_{l=0}^{v}\bar{\rho}_{l,u,v}$$
$$\times\int_0^{2\pi}\cos^{2(i+m_1+u+l+1)}\theta\sin^{2j+1}\theta d\theta=0,$$

$$\widehat{\Delta}_{12}=\tfrac{1}{2\pi}\sum_{i+j=1}^{n_1/2}2(i+j)q_{2i,2j}r^{2(i+j)-1}\sum_{u+v=0}^{(n_1-2)/2}q_{2u+1,2v}r^{2(u+v)+1}\sum_{l=0}^{v}\bar{\rho}_{l,u+m_1,v}$$
$$\times\int_0^{2\pi}\cos^{2(2m_1+i+u+l+1)}\theta\sin^{2j+1}\theta d\theta=0.$$

The polynomial $\widehat{h}_2(r)$ is obtained as the sum of integrals $\left(\widehat{\Delta}_7\right)$ through $\left(\widehat{\Delta}_{12}\right)$. Similarly, to construct the expression for the polynomial $\widehat{h}_3(r)$, we make use of the integrals specified in Appendix B, resulting in:

$$\widehat{\Delta}_{13}=\frac{1}{2\pi}\sum_{i+j=1}^{n_1/2}2(i+j)q_{2i+1,2j-1}r^{2(i+j)-1}\sum_{u+v=0}^{n/_12}q_{2u,2v}r^{2(u+v)}$$
$$\times\sum_{l=0}^{v}\sum_{k=1}^{l+v}\widetilde{\gamma}_{l,k}(u,m_1,v)\int_0^{2\pi}\cos^{2i+1}\theta\sin^{2j}\theta\cos((2k+1)\theta)d\theta$$
$$=\sum_{i+j=1}^{n_1/2}\sum_{u+v=0}^{n_1/2}\overline{R}_{(i,j)}^{(m_1)}(u,v)r^{2(i+j+u+v)-1},$$

where $\overline{R}_{(i,j)}^{(m_1)}(u,v)=\frac{i+j}{\pi}q_{2i+1,2j-1}q_{2u,2v}\sum_{l=0}^{v}\sum_{k=1}^{l+v}\widetilde{\gamma}_{l,k}(u,m_1,v)\Gamma_{i,j,k}$,

$$\widehat{\Delta}_{14}=\frac{1}{2\pi}\sum_{i+j=1}^{n_1/2}2(i+j)q_{2i+1,2j-1}r^{2(i+j)-1}\sum_{u+v=0}^{n_1/2}q_{2u+1,2v-1}r^{2(u+v)}$$
$$\times\sum_{l=0}^{u}\sum_{k=1}^{l+u}\gamma_{l,k}^{(1)}(u,v)\int_0^{2\pi}\cos^{2i+1}\theta\sin^{2j}\theta\sin((2k+1)\theta)d\theta=0,$$

$$\widehat{\Delta}_{15} = \frac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2(i+j) q_{2i+1,2j-1} r^{2(i+j)-1} \sum_{u+v=0}^{n_1/2} q_{2u+1,2v-1} r^{2(u+v)}$$
$$\times \sum_{l=0}^{u+m_1} \sum_{k=1}^{l+u+m_1} \gamma_{l,k}^{(1)}(u,v) \int_0^{2\pi} \cos^{2i+1}\theta \sin^{2j}\theta \sin((2k+1)\theta) d\theta = 0,$$

$$\widehat{\Delta}_{16} = \tfrac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2(i+j) q_{2i+1,2j-1} r^{2(i+j)-1} \sum_{u+v=0}^{(n_1-2)/2} q_{2u+1,2v} r^{2(u+v)+1} \widetilde{\rho}_{u,m_1,v}$$
$$\times \int_0^{2\pi} \cos^{2i+1}\theta \sin^{2j}\theta d\theta = 0,$$

$$\widehat{\Delta}_{17} = \tfrac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2(i+j) q_{2i+1,2j-1} r^{2(i+j)-1} \sum_{u+v=0}^{(n_1-2)/2} q_{2u+1,2v} r^{2(u+v)+1} \sum_{l=0}^{v} \bar{\rho}_{l,u,v}$$
$$\times \int_0^{2\pi} \cos^{2(i+u+l+1)+1}\theta \sin^{2j}\theta d\theta = 0,$$

$$\widehat{\Delta}_{18} = \tfrac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2(i+j) q_{2i+1,2j-1} r^{2(i+j)-1} \sum_{u+v=0}^{(n_1-2)/2} q_{2u+1,2v} r^{2(u+v)+1} \sum_{l=0}^{v} \bar{\rho}_{l,u+m_1,v}$$
$$\times \int_0^{2\pi} \cos^{2(i+u+m_1+l+1)+1}\theta \sin^{2j}\theta d\theta = 0.$$

The polynomial $\widehat{h}_3(r)$ is expressed as the sum of integrals $\left(\widehat{\Delta}_{13}\right)$ to $\left(\widehat{\Delta}_{18}\right)$. Using the integrals of Appendix B, we have:

$$\widehat{\Delta}_{19} = \frac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2(i+j) q_{2i+1,2j-1} r^{2(i+j)-1} \sum_{u+v=0}^{n_1/2} q_{2u,2v} r^{2(u+v)}$$
$$\times \sum_{l=0}^{v} \sum_{k=1}^{l+v} \widetilde{\gamma}_{l,k}(u,m_1,v) \int_0^{2\pi} \cos^{2(m_1+i)+1}\theta \sin^{2j}\theta \cos((2k+1)\theta) d\theta$$
$$= \sum_{i+j=1}^{n_1/2} \sum_{u+v=0}^{n_1/2} \widehat{R}_{(i,j)}^{(m_1)}(u,v) r^{2(i+j+u+v)-1},$$

where

$$\widehat{R}_{(i,j)}^{(m_1)}(u,v) = \frac{i+j}{\pi} q_{2i+1,2j-1} q_{2u,2v} \sum_{l=0}^{v} \sum_{k=1}^{l+v} \widetilde{\gamma}_{l,k}(u,m_1,v) \Gamma_{m_1+i,j,k},$$

$$\widehat{\Delta}_{20} = \frac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2(i+j) q_{2i+1,2j-1} r^{2(i+j)-1} \sum_{u+v=0}^{n_1/2} q_{2u+1,2v-1} r^{2(u+v)}$$
$$\times \sum_{l=0}^{u} \sum_{k=1}^{l+u} \gamma_{l,k}^{(1)}(u,v) \int_0^{2\pi} \cos^{2(m_1+i)+1}\theta \sin^{2j}\theta \sin((2k+1)\theta) d\theta = 0,$$

$$\widehat{\Delta}_{21} = \frac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2(i+j) q_{2i+1,2j-1} r^{2(i+j)-1} \sum_{u+v=0}^{n_1/2} q_{2u+1,2v-1} r^{2(u+v)}$$
$$\times \sum_{l=0}^{u+m_1} \sum_{k=1}^{l+u+m_1} \gamma_{l,k}^{(1)}(u+m_1,v) \int_0^{2\pi} \cos^{2(m_1+i)+1}\theta \sin^{2j}\theta \sin((2k+1)\theta)\, d\theta = 0,$$

$$\widehat{\Delta}_{22} = \frac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2(i+j) q_{2i+1,2j-1} r^{2(i+j)-1} \sum_{u+v=0}^{(n_1-2)/2} q_{2u+1,2v} r^{2(u+v)+1} \widetilde{\rho}_{u,m_1,v}$$
$$\times \int_0^{2\pi} \cos^{2(m_1+i)+1}\theta \sin^{2j}\theta d\theta = 0,$$

$$\widehat{\Delta}_{23} = \frac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2(i+j) q_{2i+1,2j-1} r^{2(i+j)-1} \sum_{u+v=0}^{(n_1-2)/2} q_{2u+1,2v} r^{2(u+v)+1} \sum_{l=0}^{v} \bar{\rho}_{l,u,v}$$
$$\times \int_0^{2\pi} \cos^{2(i+u+m_1+l+1)+1}\theta \sin^{2j}\theta d\theta = 0,$$

$$\widehat{\Delta}_{24} = \frac{1}{2\pi} \sum_{i+j=1}^{n_1/2} 2(i+j) q_{2i+1,2j-1} r^{2(i+j)-1} \sum_{u+v=0}^{(n_1-2)/2} q_{2u+1,2v} r^{2(u+v)+1} \sum_{l=0}^{v} \bar{\rho}_{l,u+m_1,v}$$
$$\times \int_0^{2\pi} \cos^{2(2m_1+i+u+l+1)+1}\theta \sin^{2j}\theta d\theta = 0.$$

The polynomial $\widehat{h}_4(r)$ is expressed as the sum of integrals $\left(\widehat{\Delta}_{19}\right)$ through $\left(\widehat{\Delta}_{24}\right)$. In formulating the expression for the polynomial $\widehat{h}_5(r)$, employing the integrals outlined in Appendix B, we have:

$$\widehat{\Delta}_{25} = \frac{1}{2\pi} \sum_{i+j=0}^{(n_1-2)/2} (2(i+j)+1) q_{2i+1,2j} r^{2(i+j)} \sum_{u+v=0}^{n_1/2} q_{2u,2v} r^{2(u+v)}$$
$$\times \sum_{l=0}^{v} \sum_{k=1}^{l+v} \widetilde{\gamma}_{l,k}(u,m_1,v) \int_0^{2\pi} \cos^{2i+1}\theta \sin^{2j+1}\theta \cos((2k+1)\theta)\, d\theta = 0,$$

$$\widehat{\Delta}_{26} = \frac{1}{2\pi} \sum_{i+j=0}^{(n_1-2)/2} (2(i+j)+1) q_{2i+1,2j} r^{2(i+j)} \sum_{u+v=0}^{n_1/2} q_{2u+1,2v-1} r^{2(u+v)}$$
$$\times \sum_{l=0}^{u} \sum_{k=1}^{l+u} \gamma_{l,k}^{(1)}(u,v) \int_0^{2\pi} \cos^{2i+1}\theta \sin^{2j+1}\theta \sin((2k+1)\theta)\, d\theta = 0,$$

$$\widehat{\Delta}_{27} = \frac{1}{2\pi} \sum_{i+j=0}^{(n_1-2)/2} (2(i+j)+1) q_{2i+1,2j} r^{2(i+j)} \sum_{u+v=0}^{n_1/2} q_{2u+1,2v-1} r^{2(u+v)}$$
$$\times \sum_{l=0}^{u+m_1} \sum_{k=1}^{l+u+m_1} \gamma_{l,k}^{(1)}(u+m_1,v) \int_0^{2\pi} \cos^{2i+1}\theta \sin^{2j+1}\theta \sin((2k+1)\theta)\, d\theta = 0,$$

$$\widehat{\Delta}_{28} = \frac{1}{2\pi} \sum_{i+j=0}^{(n_1-2)/2} (2(i+j)+1)\, q_{2i+1,2j} r^{2(i+j)} \sum_{u+v=0}^{(n_1-2)/2} q_{2u+1,2v} r^{2(u+v)+1} \widetilde{\rho}_{u,m_1,v} \times \int_0^{2\pi} \cos^{2i+1}\theta \sin^{2j+1}\theta d\theta = 0,$$

$$\widehat{\Delta}_{29} = \frac{1}{2\pi} \sum_{i+j=0}^{(n_1-2)/2} (2(i+j)+1)\, q_{2i+1,2j} r^{2(i+j)} \sum_{u+v=0}^{(n_1-2)/2} q_{2u+1,2v} r^{2(u+v)+1} \sum_{l=0}^{v} \bar{\rho}_{l,u,v} \times \int_0^{2\pi} \cos^{2(i+u+l+1)+1}\theta \sin^{2j+1}\theta d\theta = 0,$$

$$\widehat{\Delta}_{30} = \frac{1}{2\pi} \sum_{i+j=0}^{(n_1-2)/2} (2(i+j)+1)\, q_{2i+1,2j} r^{2(i+j)} \sum_{u+v=0}^{(n_1-2)/2} q_{2u+1,2v} r^{2(u+v)+1} \sum_{l=0}^{v} \bar{\rho}_{l,u+m_1,v} \times \int_0^{2\pi} \cos^{2(i+u+m_1+l+1)+1}\theta \sin^{2j+1}\theta d\theta = 0.$$

The polynomial $\widehat{h}_5(r)$ is expressed as the sum of integrals $\left(\widehat{\Delta}_{25}\right)$ to $\left(\widehat{\Delta}_{30}\right)$. In formulating the expression for the polynomial $\widehat{h}_6(r)$, employing the integrals outlined in Appendix B, we have:

$$\widehat{\Delta}_{31} = \frac{1}{2\pi} \sum_{i+j=0}^{(n_1-2)/2} (2(i+j)+1)\, q_{2i+1,2j} r^{2(i+j)} \sum_{u+v=0}^{n_1/2} q_{2u,2v} r^{2(u+v)} \times \sum_{l=0}^{v} \sum_{k=1}^{l+v} \widetilde{\gamma}_{l,k}(u, m_1, v) \int_0^{2\pi} \cos^{2(m_1+i)+1}\theta \sin^{2j+1}\theta \cos((2k+1)\theta)\, d\theta = 0,$$

$$\widehat{\Delta}_{32} = \frac{1}{2\pi} \sum_{i+j=0}^{(n_1-2)/2} (2(i+j)+1)\, q_{2i+1,2j} r^{2(i+j)} \sum_{u+v=0}^{n_1/2} q_{2u+1,2v-1} r^{2(u+v)} \times \sum_{l=0}^{u} \sum_{k=1}^{l+u} \gamma_{l,k}^{(1)}(u, v) \int_0^{2\pi} \cos^{2(m_1+i)+1}\theta \sin^{2j+1}\theta \sin((2k+1)\theta)\, d\theta = 0,$$

$$\widehat{\Delta}_{33} = \frac{1}{2\pi} \sum_{i+j=0}^{(n_1-2)/2} (2(i+j)+1)\, q_{2i+1,2j} r^{2(i+j)} \sum_{u+v=0}^{n_1/2} q_{2u+1,2v-1} r^{2(u+v)} \times \sum_{l=0}^{u+m_1} \sum_{k=1}^{l+u+m_1} \gamma_{l,k}^{(1)}(u+m_1, v) \int_0^{2\pi} \cos^{2(m_1+i)+1}\theta \sin^{2j+1}\theta \sin((2k+1)\theta)\, d\theta = 0,$$

$$\widehat{\Delta}_{34} = \frac{1}{2\pi} \sum_{i+j=0}^{(n_1-2)/2} (2(i+j)+1)\, q_{2i+1,2j} r^{2(i+j)} \sum_{u+v=0}^{(n_1-2)/2} q_{2u+1,2v} r^{2(u+v)+1} \widetilde{\rho}_{u,m_1,v} \times \int_0^{2\pi} \cos^{2(m_1+i)+1}\theta \sin^{2j+1}\theta d\theta = 0,$$

$$\widehat{\Delta}_{35} = \frac{1}{2\pi} \sum_{i+j=0}^{(n_1-2)/2} \left(2\left(i+j\right)+1\right) q_{2i+1,2j} r^{2(i+j)} \sum_{u+v=0}^{(n_1-2)/2} q_{2u+1,2v} r^{2(u+v)+1} \sum_{l=0}^{v} \bar{\rho}_{l,u,v}$$
$$\times \int_0^{2\pi} \cos^{2(i+u+2m_1+l+1)+1}\theta \sin^{2j+1}\theta d\theta = 0,$$

$$\widehat{\Delta}_{36} = \frac{1}{2\pi} \sum_{i+j=0}^{(n_1-2)/2} \left(2\left(i+j\right)+1\right) q_{2i+1,2j} r^{2(i+j)} \sum_{u+v=0}^{(n_1-2)/2} q_{2u+1,2v} r^{2(u+v)+1} \sum_{l=0}^{v} \bar{\rho}_{l,u+m_1,v}$$
$$\times \int_0^{2\pi} \cos^{2(i+2m_1+u+l+1)+1}\theta \sin^{2j+1}\theta d\theta = 0.$$

Similarly to the preceding method, we obtain:

$$\widetilde{h}_1(r) = \sum_{i+j=1}^{(n_1-1)/2} \sum_{u+v=0}^{(n_1-1)/2} R_{(i,j)}\left(u,v\right) r^{2(i+j+u+v)-1} + \sum_{i+j=1}^{(n_1-1)/2} \sum_{u+v=0}^{(n_1-1)/2} S_{(i,j)}^{(m_1)}\left(u,v\right) r^{2(i+j+u+v)-1},$$

$$\widetilde{h}_2(r) = \sum_{i+j=1}^{(n_1-1)/2} \sum_{u+v=0}^{(n_1-1)/2} R_{(i,j)}^{(m_1)}\left(u,v\right) r^{2(i+j+u+v)-1} + \sum_{i+j=1}^{(n_1-1)/2} \sum_{u+v=0}^{(n_1-1)/2} \widetilde{R}_{(i,j)}^{(m_1)}\left(u,v\right) r^{2(i+j+u+v)-1},$$

$$\widetilde{h}_3(r) = \sum_{i+j=1}^{(n_1-1)/2} \sum_{u+v=0}^{(n_1-1)/2} \overline{R}_{(i,j)}^{(m_1)}\left(u,v\right) r^{2(i+j+u+v)-1},$$

$$\widetilde{h}_4(r) = \sum_{i+j=1}^{(n_1-1)/2} \sum_{u+v=0}^{(n_1-1)/2} \widehat{R}_{(i,j)}^{(m_1)}\left(u,v\right) r^{2(i+j+u+v)-1},$$

and $\widetilde{h}_5(r) = \widetilde{h}_6(r) = 0$. This ends the proof of the lemma 4.1.

Lemma 4.2. *The polynomial in the variable r represented by the integral $F_{20}^2\left(r\right)$ is given by*

$$F_{20}^2\left(r\right) = \sum_{i+j=1}^{l_2} L_{i,j}\left(m_2\right) r^{2(i+j)-1} + \sum_{i+j=1}^{l_1} \sum_{u+v=0}^{l_1} U_{i,j}^{(m_1)}\left(u,v\right) r^{2(i+j+u+v)-1},$$

where $n_k = 2l_k$ or $n_k = 2l_k + 1$ with n_k is even or odd respectively.

4.2.3 Proof of Lemma 4.2

Using (*4.8*), (*4.9*) and, substituting in (*4.4*) we have

$$F_2\left(r,\theta\right) = \begin{cases} \sum_{j=0}^{6} \widehat{g}_j\left(r,\theta\right), & \text{if } n_1 \text{ si even,} \\ \sum_{j=0}^{6} \widetilde{g}_j\left(r,\theta\right), & \text{if } n_1 \text{ is odd,} \end{cases}$$

where

$$\widehat{g}_0(r,\theta)=\widetilde{g}_0(r,\theta)=-\left(1+\cos^{2m_2}\theta\right)\sum_{i+j=0}^{n_2} q_{2,i,j}r^{i+j}\cos^i\theta\sin^{j+1}\theta,$$

$$\widehat{g}_1(r,\theta)=-\left(1+\cos^{2m_1}\theta\right)^2\sum_{i+j=1}^{n_1/2}\sum_{u+v=0}^{n_1/2} q_{2i,2j}q_{2u,2v}r^{2(i+j+u+v)-1}\cos^{2(i+u)+1}\theta$$
$$\times\sin^{2(j+v)+1}\theta,$$
$$\widehat{g}_2(r,\theta)=-2\left(1+\cos^{2m_1}\theta\right)^2\sum_{i+j=0}^{n_1/2}\sum_{u+v=0}^{(n_1-2)/2} q_{2i,2j}q_{2u+1,2v}r^{2(i+j+u+v)}\cos^{2(i+u+1)}\theta$$
$$\times\sin^{2(j+v)+1}\theta,$$
$$\widehat{g}_3(r,\theta)=-2\left(1+\cos^{2m_1}\theta\right)^2\sum_{i+j=1}^{n_1/2}\sum_{u+v=0}^{n_1/2} q_{2i,2j}q_{2u+1,2v-1}r^{2(i+j+u+v)-1}\cos^{2(i+u+1)}\theta$$
$$\times\sin^{2(j+v)}\theta,$$
$$\widehat{g}_4(r,\theta)=-\left(1+\cos^{2m_1}\theta\right)^2\sum_{i+j=0}^{(n_1-2)/2}\sum_{u+v=0}^{(n_1-2)/2} q_{2i+1,2j}q_{2u+1,2v}r^{2(i+j+u+v)+1}\cos^{2(i+u+1)+1}\theta$$
$$\times\sin^{2(j+v)+1}\theta,$$
$$\widehat{g}_5(r,\theta)=-2\left(1+\cos^{2m_1}\theta\right)^2\sum_{i+j=0}^{(n_1-2)/2}\sum_{u+v=0}^{n_1/2} q_{2i+1,2j}q_{2u+1,2v-1}r^{2(i+j+u+v)}\cos^{2(i+u+1)+1}\theta$$
$$\times\sin^{2(j+v)}\theta,$$
$$\widehat{g}_6(r,\theta)=-\left(1+\cos^{2m_1}\theta\right)^2\sum_{i+j=1}^{(n_1-2)/2}\sum_{u+v=0}^{(n_1-2)/2} q_{2i+1,2j-1}q_{2u+1,2v-1}r^{2(i+j+u+v)-1}\cos^{2(i+u+1)+1}\theta$$
$$\times\sin^{2(j+v)-1}\theta,$$

$$\widetilde{g}_1(r,\theta) = -\left(1+\cos^{2m_1}\theta\right)^2 \sum_{i+j=1}^{(n_1-1)/2} \sum_{u+v=0}^{(n_1-1)/2} q_{2i,2j} q_{2u,2v} r^{2(i+j+u+v)-1} \cos^{2(i+u)+1}\theta$$

$$\sin^{2(j+v)+1}\theta,$$

$$\widetilde{g}_2(r,\theta) = -2\left(1+\cos^{2m_1}\theta\right)^2 \sum_{i+j=0}^{(n_1-1)/2} \sum_{u+v=0}^{(n_1-1)/2} q_{2i,2j} q_{2u+1,2v} r^{2(i+j+u+v)} \cos^{2(i+u+1)}\theta$$

$$\sin^{2(j+v)+1}\theta,$$

$$\widetilde{g}_3(r,\theta) = -2\left(1+\cos^{2m_1}\theta\right)^2 \sum_{i+j=1}^{(n_1-1)/2} \sum_{u+v=0}^{(n_1-1)/2} q_{2i,2j} q_{2u+1,2v-1} r^{2(i+j+u+v)-1} \cos^{2(i+u+1)}\theta$$

$$\times \sin^{2(j+v)}\theta,$$

$$\widetilde{g}_4(r,\theta) = -\left(1+\cos^{2m_1}\theta\right)^2 \sum_{i+j=0}^{(n_1-1)/2} \sum_{u+v=0}^{(n_1-1)/2} q_{2i+1,2j} q_{2u+1,2v} r^{2(i+j+u+v)+1} \cos^{2(i+u+1)+1}\theta$$

$$\times \sin^{2(j+v)+1}\theta,$$

$$\widetilde{g}_5(r,\theta) = -2\left(1+\cos^{2m_1}\theta\right)^2 \sum_{i+j=0}^{(n_1-1)/2} \sum_{u+v=0}^{(n_1-1)/2} q_{2i+1,2j} q_{2u+1,2v-1} r^{2(i+j+u+v)} \cos^{2(i+u+1)+1}\theta$$

$$\times \sin^{2(j+v)}\theta,$$

and

$$\widetilde{g}_6(r,\theta) = -\left(1+\cos^{2m_1}\theta\right)^2 \sum_{i+j=1}^{(n_1-1)/2} \sum_{u+v=0}^{(n_1-1)/2} q_{2i+1,2j-1} q_{2u+1,2v-1} r^{2(i+j+u+v)-1} \cos^{2(i+u+1)+1}\theta$$

$$\times \sin^{2(j+v)-1}\theta.$$

Using (4.5), the polynomial $F_{20}^2(r)$ can be explicitly expressed, where

$$\widehat{\delta}_l = \frac{1}{2\pi}\int_0^{2\pi} \widehat{g}_l(r,\theta)\,d\theta,\ \ \widetilde{\delta}_l = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{g}_l(r,\theta)\,d\theta,\ \ l = 1,2,...,6,$$

we have

$$\widetilde{\delta}_0 = \widehat{\delta}_0 = \sum_{i+j=1}^{l_2} L_{i,j}(m_2)\, r^{2(i+j)-1},$$

$$\widehat{\delta}_3 = \widetilde{\delta}_3 = \sum_{i+j=1}^{l_1} \sum_{u+v=0}^{l_1} U_{i,j}^{(m_1)}(u,v)\, r^{2(i+j+u+v)-1},$$

where $L_{i,j}(m_2)$ and $U_{i,j}^{(m_1)}(u,v)$ are non-zero constants and

$$\widehat{\delta}_i = 0, \widetilde{\delta}_i = 0 \text{ for } i = 1,2,4,5,6.$$

With this, the proof of Lemma 4.2 is concluded.

Relying on Lemma 4.1 and Lemma 4.2, we derive

$$\begin{aligned} F_{20}(r) &= \sum_{i+j=1}^{l_1} \sum_{u+v=0}^{l_1} Q_{(i,j)}^{(m_1)}(u,v)\, r^{2(i+j+u+v)-1} + \sum_{i+j=1}^{l_2} L_{i,j}(m_2)\, r^{2(i+j)-1} \\ &+ \sum_{i+j=1}^{l_1} \sum_{u+v=0}^{l_1} U_{i,j}^{(m_1)}(u,v)\, r^{2(i+j+u+v)-1}, \end{aligned}$$

we have

$$\begin{aligned} F_{20}(r) = & \sum_{i+j=0}^{l_1-1} \sum_{u+v=0}^{l_1} Q_{(i,j)}^{(m_1)}(u,v)\, r^{2(i+j+u+v)+1} + \sum_{i+j=0}^{l_2-1} L_{i,j}(m_2)\, r^{2(i+j)+1} \\ & + \sum_{i+j=0}^{l_1-1} \sum_{u+v=0}^{l_1} U_{i,j}^{(m_1)}(u,v)\, r^{2(i+j+u+v)+1} \\ = r(& \sum_{i+j=0}^{l_1-1} \sum_{u+v=0}^{l_1} Q_{(i,j)}^{(m_1)}(u,v)\, r^{2(i+j+u+v)} + \sum_{i+j=0}^{l_2-1} L_{i,j}(m_2)\, r^{2(i+j)} \\ & + \sum_{i+j=0}^{l_1-1} \sum_{u+v=0}^{l_1} U_{i,j}^{(m_1)}(u,v)\, r^{2(i+j+u+v)}). \end{aligned}$$

We can conclude that F_{20} has at most $\max\{2l_1 - 1, l_2 - 1\}$ positive roots. Therefore, statement (**ii**) in Theorem 4.1.1 holds.

4.2.4 Examples

Example 4.1. *Consider the following system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon(1+\cos^4\theta)(-y + 5x + \frac{20}{21}x^2 y + \frac{16}{51}y^3). \end{cases} \tag{4.17}$$

The averaged function as follows

$$F_{10}(r) = \frac{9r}{32}(r^2 - 2),$$

The unique positive root of $F_{10}(r)$ *is* $r = \sqrt{2}$ *Since* $\left(\frac{dF_{10}}{dr}\right)(\sqrt{2}) = 1.125 \neq 0$, *by Theorem* **??**, *it follows that system* (4.17) *has for* $|\varepsilon| \neq 0$ *sufficiently small a limit cycle bifurcating from the periodic orbit of radius* $\sqrt{2}$ *of the unperturbed system* (**??**) *with* $\varepsilon = 0$. *Moreover, since* $\left(\frac{dF_{10}}{dr}\right)(\sqrt{2}) = 1.125 > 0$, *by Theorem* **??**, *this limit cycle is unstable, (see., Figure 4.1.).*

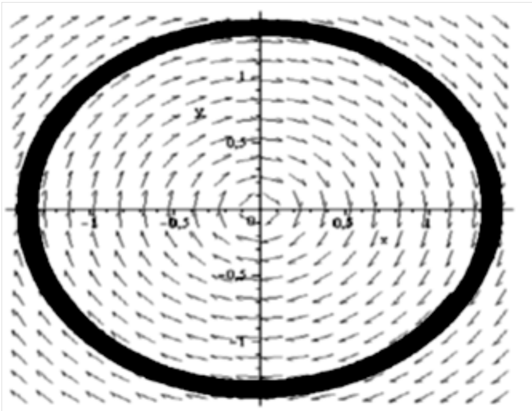

Figure 4.1: The unstable limit cycle of system (4.17)

Example 4.2. *Consider the following system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon(1+\cos^4\theta)(\frac{1}{2}x^3) - \varepsilon^2(1+\cos^2\theta)\left(\frac{1}{10}y - \frac{13}{27}x^2y + \frac{3}{2}xy^3 + \frac{32}{117}x^4y\right), \end{cases} \tag{4.18}$$

here $n_1 = 3, n_2 = 5$ *and* $m_1 = 2, m_2 = 1$. *From (4.3) and (4.4), we get that*

$$\begin{aligned} F_1(r,\theta) &= \sin\theta(1+\cos^4\theta)\left(\frac{1}{2}r^3\cos^3\theta\right), \\ F_2(r,\theta) &= -\sin\theta\left(1+\cos^2\theta\right)\left(\frac{1}{10}r\sin\theta - \frac{13}{27}r^3\cos^2\theta\sin\theta + \frac{3}{2}r^4\cos\theta\sin^3\theta\right. \\ &\quad \left. + \frac{32}{117}r^5\cos^4\theta\sin\theta\right) - \frac{1}{r}\cos\theta\sin\theta\left(1+\cos^4\theta\right)^2\left(\frac{1}{2}r^3\cos^3\theta\right)^2, \end{aligned}$$

We now compute the first averaged function and arrive at

$$F_{10}(r) = \frac{1}{2\pi}\int_0^{2\pi} F_1\left(r,\theta\right) = 0,$$

Since $F_{10}(r) = 0$, we proceed to the second-order averaging theory and obtain

$$\begin{aligned} F_{20}(r) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{d}{dr}F_1\left(r,\theta\right)\right)\left(\int_0^{\theta} F_1\left(r,s\right)ds\right)d\theta \; + \frac{1}{2\pi}\int_0^{2\pi} F_2\left(r,\theta\right)d\theta \\ &= -\frac{1}{144}r\left(4r^4 - 13r^2 + 9\right). \end{aligned}$$

The equation $F_{20}(r) = 0$ has the two simple positive roots $r_1 = \frac{3}{2}, r_2 = 1$. Since $\frac{dF_{20}}{dr}(\frac{3}{2}) = -0.093\,75 < 0$ and $\frac{dF_{20}}{dr}(1) = 0.131\,94 > 0$, then, the differential system (4.18) has one stable limit cycles for r_1 and one unstable limit cycle for r_2 bifurcating from the origin (see., Figure 4.2.).

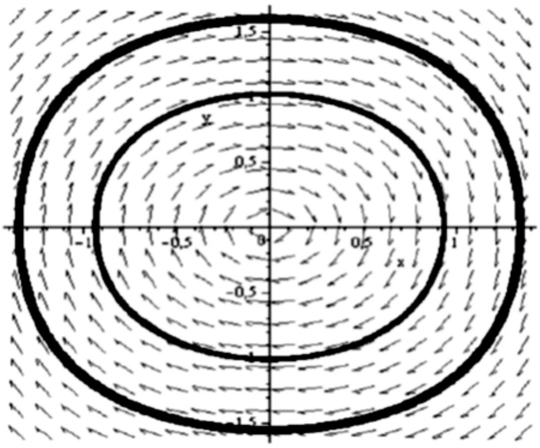

Figure 4.2: The two limit cycles of system (4.18) for $\varepsilon = 0.1$,

4.3 Final Thoughts and Outlook

In this study, we analyzed the maximum number of limit cycles in generalized Mathieu-Duffing equations using both first- and second-order averaging methods. Our primary contributions include establishing explicit bounds on the number of limit cycles and utilizing Descartes' Rule of Signs to confirm the results. We showed that the polynomial differential systems under consideration can generate up to $\max\{2l_1 - 1, l_2 - 1\}$ limit cycles bifurcating from the periodic orbits of the linear center characterized by $x = y$ and $y = -x$, where $n_k = 2l_k$ if n_k is even or $n_k = 2l_k + 1$ if n_k odd .

Our approach is distinguished by its application of classical methods in a new context, providing a rigorous mathematical framework that not only deepens the theoretical understanding but also offers practical insights for applications in fields such as physics, engineering, and applied mathematics. We particularly emphasized the importance of the second averaged function F_{20}, demonstrating its crucial role in determining the number and stability of limit cycles in nonlinear systems. This aspect is vital for predicting and controlling the behavior of such systems. Compared to previous studies, our work extends the application of averaging theory to a broader class of equations and offers a detailed polynomial analysis that yields new insights, highlighting the potential of our investigation to advance the understanding of limit cycles in Mathieu-Duffing equations.

Despite the significant advancements made in understanding the limit cycles of generalized Mathieu-Duffing equations, several open questions remain. One major challenge lies in assessing the robustness of averaging theory when applied to more complex and highly nonlinear systems. Investigating the boundaries and potential extensions of averaging theory could provide deeper insights into the dynamics of such systems. Furthermore, while our study primarily focuses on local bifurcations, extending the analysis to global bifurcation scenarios could reveal more intricate system behaviors and potentially uncover new types of limit cycles.

Another promising area for future research is examining the impact of higher-order terms in the differential equations on the stability and number of limit cycles. A thorough analysis of these effects could lead to a more comprehensive understanding of the dynamics at play. Addressing these open problems will further advance the field, uncover new phenomena, and broaden the scope of applications for averaging theory in the study of nonlinear dynamical systems.

Chapter 5 A number of limit cycle of septic polynomial differential systems via the averaging theory

5.1 Introduction and Main Results

The analysis of limit cycles plays a vital role in understanding the intricate dynamics and behaviors exhibited by differential systems.

This chapter focuses on analyzing the maximum number of limit cycles that can emerge from the periodic orbits of a particular cubic system when perturbed within the framework of septic polynomial differential systems. The cubic system under consideration is defined as:

$$\begin{cases} \dot{x} = y - y\left(x+a\right)^2 \\ \dot{y} = -x + x\left(x+a\right)^2, \end{cases} \tag{5.1}$$

where a is a positive number not equal to 1. The main goal is to estimate the upper bound on the number of limit cycles using the averaging method when the system is subjected to perturbations. Upon perturbation, the cubic system assumes the form:

$$\begin{cases} \dot{x} = y - y\,(x+a)^2 + \varepsilon P\,(x,y)\,, \\ \dot{y} = -x + x\,(x+a)^2 + \varepsilon Q\,(x,y)\,, \end{cases} \tag{5.2}$$

where $P\,(x,y)$ and $Q\,(x,y)$ are polynomials of degree n, $|\varepsilon|$ represents a small perturbation parameter, and a is different from 1. The main result of this study is derived using the first-order averaging theory. The primary finding of this chapter is encapsulated in the following theorem

Theorem 5.1.1. *Let us examine the perturbed system (5.2) with $a \in\,]0,1[\, \cup\,]1,+\infty[$, and $P(x,y)$ and $Q(x,y)$ being polynomials of degree 7. For sufficiently small values of $\varepsilon \neq 0$, the first-order averaging theory predicts that the system can exhibit a maximum of twenty-two limit cycles bifurcating from the periodic orbits of the cubic center.*

This result demonstrates a significant increase in the number of limit cycles compared to previous studies with lower-degree polynomials.

Remark 5.1.1. *In the work by [19], he authors explored a comparable scenario where $P(u,v)$ and $Q(u,v)$ are real polynomials of degree three. Their results showed that the maximum number of limit cycles in this case is restricted to eight. This comparison underscores the impact of the polynomial degree on the number of limit cycles that can bifurcate.*

5.2 Proofs of the Results

5.2.1 Proof of Theorem 5.1.1

For $a \in\,]0,1[\, \cup\,]1,+\infty[$, the cubic system (5.1) possesses a single period annulus denoted by $A = \{(x,y) : 0 < x^2 + y^2 < r^2\}$, where $r = \min\{|a -$

$1|, |a+1|\}$. This result is derived using the first-order averaging theory method in polar coordinates (r,θ), where $x = r\cos\theta$ and $y = r\sin\theta$, with $r > 0$. Let us now consider:

$$P(x,y) = \sum_{k=1}^{n} \sum_{i+j=k} p_{ij} x^i y^j, Q(x,y) = \sum_{k=1}^{n} \sum_{i+j=k} q_{ij} x^i y^j,$$

The system (5.2) can be rewritten in the following manner

$$\begin{cases} \dot{r} = \varepsilon \sum\limits_{k=1}^{n} \left(\cos\theta \sum_{i+j=k} p_{ij} \cos^i\theta \sin^j\theta + \sin\theta \sum_{i+j=k} q_{ij} \cos^i\theta \sin^j\theta\right) r^k, \\ \dot{\theta} = (r\cos\theta + a)^2 - 1 + \varepsilon \sum\limits_{k=1}^{n} \left(\cos\theta \sum_{i+j=k} q_{ij} \cos^i\theta \sin^j\theta \right. \\ \qquad \left. - \sin\theta \sum_{i+j=k} p_{ij} \cos^i\theta \sin^j\theta\right) r^{k-1}. \end{cases}$$

Therefore, we have

$$\begin{aligned} \frac{dr}{d\theta} &= \varepsilon \sum_{k=1}^{n} \left(\cos\theta \sum_{i+j=k} p_{ij} \cos^i\theta \sin^j\theta + \sin\theta \sum_{i+j=k} q_{ij} \cos^i\theta \sin^j\theta\right) \\ &\quad \times \tfrac{r^k}{(r\cos\theta + a)^2 - 1} + \varepsilon^2 R(r,\theta,\varepsilon). \end{aligned} \tag{5.3}$$

The average function of (5.3) is

$$g^0(r) = \frac{1}{2\pi} \sum_{k=1}^{n} r^k \int_0^{2\pi} \frac{\cos\theta \sum_{i+j=k} p_{ij} \cos^i\theta \sin^j\theta + \sin\theta \sum_{i+j=k} q_{ij} \cos^i\theta \sin^j\theta}{(r\cos\theta + a)^2 - 1} d\theta.$$

For $n = 7$, we get

$$g^0(r) = \sum_{k=1}^{7} \sum_{i+j=k} \left(p_{ij} B_{i+1,j}(r) + q_{ij} B_{i,j+1}(r)\right) r^k, \tag{5.4}$$

where

$$B_{i,j} = \frac{1}{2\pi} \int_0^{2\pi} \tfrac{\cos^i\theta \sin^j\theta}{H(r,\theta)} d\theta, \tag{5.5}$$

with

$$H(r,\theta) = (r\cos\theta + a)^2 - 1.$$

Based on Theorem **??**, each simple zero of the average function $g^0(r)$ is associated with a limit cycle of system (5.2). We now proceed to demonstrate

Theorem 5.1.1. The proof consists of two steps: The proof is carried out in two stages: initially, we compute the integral $g^0(r)$, followed by an analysis of the number of its simple zeros.

To derive the precise expression for $g^0(r)$ we utilize the following formulas:

(i)-

$$\begin{aligned}\frac{\cos^{k+1}\theta}{H(r,\theta)} &= \frac{\cos^{k+1}\theta}{(r\cos\theta+a)^2-1}\\ &= \frac{H(r,\theta)\cos^{k-1}\theta-2ar\cos^k\theta+(1-a^2)\cos^{k-1}\theta}{r^2H(r,\theta)},\end{aligned}$$

we have

$$\frac{\cos^{k+1}\theta}{H(r,\theta)} = \frac{1}{r^2}\cos^{k-1}\theta-\frac{2a}{r}\frac{\cos^k\theta}{H(r,\theta)}+\frac{1-a^2}{r^2}\frac{\cos^{k-1}\theta}{H(r,\theta)}. \tag{5.6}$$

(ii)- For $s=1,...,\left[\frac{k+1}{2}\right]$

$$\begin{aligned}\frac{\cos^{k-2s+1}\theta\sin^{2s}\theta}{H(r,\theta)} &= \frac{\cos^{k-2s+1}\theta\left(1-\cos^2\theta\right)^s}{H(r,\theta)}\\ &= \sum_{i=0}^{s}(-1)^i C_s^i\frac{\cos^{k-2s+2i+1}\theta}{H(r,\theta)},\end{aligned}$$

we have

$$\frac{\cos^{k-2s+1}\theta\sin^{2s}\theta}{H(r,\theta)} = \sum_{i=0}^{s}(-1)^i C_s^i\frac{\cos^{k-2s+2i+1}\theta}{H(r,\theta)}, \tag{5.7}$$

(iii)-

$$\int_0^{2\pi}\frac{\sin^{k+1}\theta}{H(r,\theta)} = \begin{cases}0 & \text{if } \; k\in 2\mathbb{N},\\ \frac{\left(1-\cos^2\theta\right)^{\frac{k+1}{2}}}{H(r,\theta)} = \sum_{j=0}^{\frac{k+1}{2}}(-1)^j C^j_{\frac{k+1}{2}}\frac{\cos^{2j}\theta}{H(r,\theta)} & \text{if } \quad k\in 2\mathbb{N}+1,\end{cases} \tag{5.8}$$

(iv)-

$$\int_0^{2\pi}\frac{\cos^k\theta\sin^{2s+1}\theta}{H(r,\theta)}d\theta = 0, \;\; \forall k,s \text{ with } k+2s\leq 7. \tag{5.9}$$

Lemma 5.1. *For* $0 < a < 1$, *we have*

$$\int_0^{2\pi} \frac{1}{H(r,\theta)} d\theta = \frac{-u-v}{uv} \pi r^2,$$
$$\int_0^{2\pi} \frac{\cos\theta}{H(r,\theta)} d\theta = \frac{-u(1-a)+v(1+a)}{uv} \pi r,$$

where

$$u = \sqrt{(a+1)^2 - r^2}, \qquad v = \sqrt{(a-1)^2 - r^2}.$$

5.2.2 Proof of Lemma 5.1

By putting $z = e^{i\theta}$ and $C = \{z : |z| = 1\}$, we obtain

$$\int_0^{2\pi} \frac{1}{H(r,\theta)} d\theta = \frac{4}{i} \int_C \frac{z}{(rz^2 + 2az + r)^2 - 4z^2} dz,$$
$$\int_0^{2\pi} \frac{\cos\theta}{H(r,\theta)} d\theta = \frac{2}{i} \int_C \frac{z^2+1}{(rz^2 + 2az + r)^2 - 4z^2} dz,$$

the poles are

$$z_{1,2} = \frac{1-a \pm v}{r}, \; z_{3,4} = \frac{-1-a \pm u}{r}.$$

Using the residue theorem with $|z_2| < 1$, $|z_3| < 1$, $|z_1| > 1$ and $|z_4| > 1$ for $a \in]0,1[$, we obtain

$$\int_0^{2\pi} \frac{1}{H(r,\theta)} d\theta = \frac{-u-v}{uv} \pi r^2,$$
$$\int_0^{2\pi} \frac{\cos\theta}{H(r,\theta)} d\theta = \frac{-u(1-a)+v(1+a)}{uv} \pi r.$$

Lemma 5.2. *For* $a > 1$, *we have*

$$\int_0^{2\pi} \frac{1}{H(r,\theta)} d\theta = \frac{u-v}{uv} \pi r^2,$$
$$\int_0^{2\pi} \frac{\cos\theta}{H(r,\theta)} d\theta = \frac{u(1-a)+v(1+a)}{uv} \pi r,$$

where

$$u = \sqrt{(a+1)^2 - r^2}, \qquad v = \sqrt{(a-1)^2 - r^2}.$$

5.2.3 Proof of Lemma 5.2

To verify this hypothesis, we rely on the same previous proof with $|z_1| < 1$, $|z_3| < 1$, $|z_2| > 1$ and $|z_4| > 1$ for $a > 1$.

For $0 < a < 1$, from the equations (5.6) to (5.9) and Lemma 5.1 in (5.5), it yields the following

$$\begin{aligned} B_{0,0} &= \frac{-u-v}{2uv} r^2, \\ B_{1,0} &= \frac{-u(1-a)+v(1+a)}{2uv} r, \end{aligned}$$

$$\begin{aligned} B_{2,0} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos^2\theta}{f} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{r^2} d\theta - \frac{2a}{r} B_{1,0} + \frac{1-a^2}{r^2} B_{0,0} \\ &= -\frac{(a^2-2a+1)\,ur^2 + (a^2+2a+1)\,vr^2 - 2uv}{2uvr^2}, \end{aligned}$$

then

$$\begin{aligned} B_{0,2} &= B_{0,0} - B_{2,0} \\ &= \frac{\left((a+1)^2 - r^2\right) vr^2 + \left((a-1)^2 - r^2\right) ur^2 - 2uv}{2uvr^2}, \end{aligned}$$

and we aslo have

$$B_{3,0} = \frac{(a^3-3a^2+3a-1)\,ur^2 + (a^3+3a^2+3a+1)\,vr^2 - 4auv}{2r^3uv},$$

$$B_{1,2} = -\frac{(1-a)\left((a+1)^2 - r^2\right) vr^2 + (a+1)\left((a-1)^2 - r^2\right) ur^2 - 4auv}{2uvr^3},$$

$$B_{4,0} = -\frac{(a^4-4a^3+6a^2-4a+)\,ur^2 + (a^4+4a^3+6a^2+4a+)\,vr^2\,(-r^2-6a^2-2)\,uv}{2uvr^4},$$

$$B_{2,2} = \frac{1}{2uvr^4}[\left(a^2+2a+\right)\left((a+1)^2-r^2\right)vr^2$$
$$+\left(a^2-2a+1\right)\left((a-1)^2-r^2\right)ur^2+\left(r^2-6a^2-2\right)uv],$$
$$B_{0,4} = -\frac{1}{2uvr^4}[\left(-r^2+a^2+2a+1\right)\left((a+1)^2-r^2\right)vr^2$$
$$+\left(-r^2+a^2-2a+1\right)\left((a-1)^2-r^2\right)ur^2$$
$$+\left(3r^2-6a^2-2\right)uv],$$
$$B_{5,0} = \frac{1}{2uvr^5}[\left(a^5-5a^4+10a^3-10a^2+5a-1\right)ur^2$$
$$+\left(a^5+5a^4+10a^3+10a^2+5a+1\right)vr^2$$
$$+\left(-2ar^2-8a^3-8a\right)uv],$$

$$B_{3,2} = -\frac{1}{2uvr^5}[\left(a^3+3a^2+3a+1\right)\left((a+1)^2-r^2\right)vr^2$$
$$+\left(a^3-3a^2+3a-1\right)\left((a-1)^2-r^2\right)ur^2$$
$$+\left(2ar^2-8a^3-8a\right)uv],$$

$$B_{1,4} = \frac{1}{2uvr^5}[\left((-a-1)r^2+\left(a^3+3a^2+3a+1\right)\right)\left((a+1)^2-r^2\right)vr^2$$
$$+\left((1-a)r^2+\left(a^3-3a^2+3a-1\right)\right)\left((a-1)^2-r^2\right)ur^2$$
$$+\left(6ar^2-8a^3-8a\right)uv],$$

$$B_{6,0} = -\frac{1}{8uvr^6}[\left(4a^6-24a^5+60a^4-80a^3+60a^2-24a+4\right)ur^2$$
$$+\left(4a^6+24a^5+60a^4+80a^3+60a^2+24a+4\right)vr^2$$
$$+\left(\left(-12a^2-4\right)r^2-3r^4-\left(40a^4+80a^2+8\right)\right)uv],$$

$$B_{4,2} = \frac{1}{8uvr^6}[\left(4a^4+16a^3+24a^2+16a+4\right)\left((a+1)^2-r^2\right)vr^2$$
$$+\left(4a^4-16a^3+24a^2-16a+4\right)\left((a-1)^2-r^2\right)ur^2$$
$$+\left(r^4+\left(12a^2+4\right)r^2-\left(40a^4+80a^2+8\right)\right)uv],$$

$$B_{2,4} = \frac{1}{8uvr^6}[\big(\big(4a^2+8a+4\big)\,r^2 - \big(4a^4+16a^3+24a^2+16a+4\big)\big)\,\big((a+1)^2-r^2\big)\,vr^2$$
$$+\big(\big(4a^2-8a+4\big)\,r^2+\big(16a^3-4a^4-24a^2+16a-4\big)\big)\,\big((a-1)^2-r^2\big)\,ur^2$$
$$+\big(3r^4+\big(-36a^2-12\big)\,r^2+\big(40a^4+80a^2+8\big)\big)\,uv],$$

$$B_{0,6} = \frac{1}{8uvr^6}[\big(4r^4+\big(-8a^2-16a-8\big)\,r^2+\big(4a^4+16a^3+24a^2+16a+4\big)\big)$$
$$\times\big((a+1)^2-r^2\big)\,vr^2+\big(4r^4+\big(16a-8a^2-8\big)\,r^2+\big(4a^4-16a^3+24a^2-16a+4\big)\big)$$
$$\times\big((a-1)^2-r^2\big)\,ur^2+\big(\big(60a^2+20\big)\,r^2-15r^4-\big(40a^4+80a^2+8\big)\big)\,uv],$$

$$B_{7,0} = \frac{1}{4uvr^7}[\big(2a^7-14a^6+42a^5-70a^4+70a^3-42a^2+14a-2\big)\,ur^2$$
$$+\big(2a^7+14a^6+42a^5+70a^4+70a^3+42a^2+14a+2\big)\,vr^2$$
$$+\big(-3ar^4+\big(-8a^3-8a\big)\,r^2-\big(24a^5+80a^3+24a\big)\big)\,uv],$$

$$B_{5,2} = -\frac{1}{4uvr^7}[\big(2a^5+10a^4+20a^3+20a^2+10a+2\big)\,\big((a+1)^2-r^2\big)\,vr^2$$
$$+\big(2a^5-10a^4+20a^3-20a^2+10a-2\big)\,\big((a-1)^2-r^2\big)\,ur^2$$
$$+\big(ar^4+\big(8a^3+8a\big)\,r^2-\big(24a^5+80a^3+24a\big)\big)\,uv],$$

$$B_{3,4} = -\frac{1}{4uvr^7}[\big(\big(2a^3+6a^2+6a+2\big)\,r^2-\big(2a^5+10a^4+20a^3+20a^2+10a+2\big)\big)$$
$$\times\big((a+1)^2-r^2\big)\,vr^2+\big(\big(2a^3-6a^2+6a-2\big)\,r^2+\big(10a^4-2a^5-20a^3+20a^2$$
$$-10a+2)) \times\big((a-1)^2-r^2\big)\,ur^2+\big(3ar^4+\big(-24a^3-24a\big)\,r^2$$
$$+\big(24a^5+80a^3+24a\big)\big)\,uv],$$

$$B_{1,6} = -\frac{1}{4uvr^7}[\big((2a+2)\,r^4+\big(-4a^3-12a^2-12a-4\big)\,r^2$$
$$+\big(2a^5+10a^4+20a^3+20a^2+10a+2\big)\big)\times\big((a+1)^2-r^2\big)\,vr^2$$
$$+\big((2a-2)\,r^4+\big(12a^2-4a^3-12a+4\big)\,r^2+\big(2a^5-10a^4+20a^3-20a^2+10a-2\big)\big)$$
$$\times\big((a-1)^2-r^2\big)\,ur^2+\big(-15ar^4+\big(40a^3+40a\big)\,r^2-\big(24a^5+80a^3+24a\big)\big)\,uv],$$

$$B_{8,0} = -\frac{1}{16r^8uv}[\left(8a^8 + 64a^7 + 224a^6 + 448a^5 + 560a^4 + 448a^3 + 224a^2 + 64a + 8\right) vr^2$$
$$+ \left(8a^8 - 64a^7 + 224a^6 - 448a^5 + 560a^4 - 448a^3 + 224a^2 - 64a + 8\right) ur^2$$
$$+ \left(-5r^6 + \left(-18a^2 - 6\right) r^4 + \left(-40a^4 - 80a^2 - 8\right) r^2\right.$$
$$\left. - \left(112a^6 + 560a^4 + 336a^2 + 16\right)\right) uv],$$

$$B_{6,2} = \frac{1}{16uvr^8}[\left(8a^6 + 48a^5 + 120a^4 + 160a^3 + 120a^2 + 48a + 8\right) \left((a+1)^2 - r^2\right) vr^2$$
$$+ \left(8a^6 - 48a^5 + 120a^4 - 160a^3 + 120a^2 - 48a + 8\right) \left((a-1)^2 - r^2\right) ur^2$$
$$+ \left(r^6 + \left(6a^2 + 2\right) r^4 + \left(40a^4 + 80a^2 + 8\right) r^2 - \left(112a^6 + 560a^4 + 336a^2 + 16\right)\right) uv],$$

$$B_{4,4} = \frac{1}{16uvr^8}[\left(- \left(8a^6 + 48a^5 + 120a^4 + 160a^3 + 120a^2 + 48a + 8\right)\right.$$
$$\left. + \left(8a^4 + 32a^3 + 48a^2 + 32a + 8\right) r^2\right) \times \left((a+1)^2 - r^2\right) vr^2$$
$$+ \left(\left(48a^5 - 8a^6 - 120a^4 + 160a^3 - 120a^2 + 48a - 8\right)\right.$$
$$\left. + \left(\left(8a^4 - 32a^3 + 48a^2 - 32a + 8\right) r^2\right)\right) \times \left((a-1)^2 - r^2\right) ur^2$$
$$+ \left(r^6 + \left(18a^2 + 6\right) r^4 + \left(-120a^4 - 240a^2 - 24\right) r^2\right.$$
$$\left. + \left(112a^6 + 560a^4 + 336a^2 + 16\right)\right) uv],$$

$$B_{2,6} = \frac{1}{16uvr^8}[[\left(8a^2 + 16a + 8\right) r^4 + \left(-16a^4 - 64a^3 - 96a^2 - 64a - 16\right) r^2$$
$$+ \left(8a^6 + 48a^5 + 120a^4 + 160a^3 + 120a^2 + 48a + 8\right)] \left((a+1)^2 - r^2\right) vr^2$$
$$+ [\left(8a^2 - 16a + 8\right) r^4 + \left(64a^3 - 16a^4 - 96a^2 + 64a - 16\right) r^2$$
$$+ \left(8a^6 - 48a^5 + 120a^4 - 160a^3 + 120a^2 - 48a + 8\right)] \left((a-1)^2 - r^2\right) ur^2$$
$$+ [5r^6 + \left(-90a^2 - 30\right) r^4 + \left(200a^4 + 400a^2 + 40\right) r^2 - 112a^6 + 560a^4$$
$$+ 336a^2 + 16]uv],$$

$$
\begin{aligned}
B_{0,8} = & -\frac{1}{16uvr^8}[[-8r^6 + \left(24a^2 + 48a + 24\right) r^4 + \left(-24a^4 - 96a^3 - 144a^2 - 96a - 24\right) r^2 \\
& + \left(8a^6 + 48a^5 + 120a^4 + 160a^3 + 120a^2 + 48a + 8\right)] \left((a+1)^2 - r^2\right) vr^2 \\
& + [-8r^6 + \left(24a^2 - 48a + 24\right) r^4 + \left(-24a^4 + 96a^3 - 144a^2 + 96a - 24\right) r^2 \\
& + \left(8a^6 - 48a^5 + 120a^4 - 160a^3 + 120a^2 - 48a + 8\right)] \left((a-1)^2 - r^2\right) ur^2 \\
& + [35r^6 + \left(-210a^2 - 70\right) r^4 + \left(280a^4 + 560a^2 + 56\right) r^2 \\
& - \left(112a^6 + 560a^4 + 336a^2 + 16\right)]uv],
\end{aligned}
$$

and

$$B_{k,2s+1} = 0, \forall k, s \text{ with } k + 2s \leq 7.$$

Substituting all integrals $B_{i,j}$, for $2 \leq i + j \leq 8$ into (5.4), we have

$$g^0(r) = \frac{Auv + Bur^2 + Cvr^2}{16uvr},$$

where

$$
\begin{aligned}
A &= \left(a_6 r^6 + a_4 r^4 + a_2 r^2 + a_0\right), \\
B &= \left(b_8 r^8 + b_6 r^6 + b_4 r^4 + b_2 r^2 + b_0\right), \\
C &= \left(c_8 r^8 + c_6 r^6 + c_4 r^4 + c_2 r^2 + c_0\right),
\end{aligned}
$$

with the coeffcients a_i, b_i and c_i are polynomials in the coeffcients of $a, p_{i,j}$ and $q_{i,j}$.

Indeed, there are precisely ten independent parameters between a_i, b_i and c_i in relation to p_{ij}, q_{ij} and a. To establish bounds on the zeros of the numerator of $g^0(r)$, it suffices to limit the number of zeros of the function $l(r)$, which is given by

$$l(r) = A^2u^2v^2 - B^2u^2r^4 - C^2v^2r^4 - 2BCuvr^4.$$

Ultimately, to limit the number of zeros of the expression above, it is essential to limit the zeros of the corresponding polynomial:

$$\begin{aligned} w\left(r\right) = {} & A^4u^4v^4 - 2A^2B^2v^2u^4r^4 - 2A^2C^2v^4u^2r^4 + B^4u^4r^8 \\ & -2B^2C^2u^2v^2r^8 + C^4v^4r^8, \end{aligned} \tag{5.10}$$

therefore, we get

$$w\left(r\right) = d_{44}r^{44} + ... + d_2r^2 + d_0,$$

where d_i are polynomials in a, a_i, b_i and c_i. We conclude that $g^0(r)$ has at most twenty-two (22) simple zeros.

For $a > 1$, we can apply an analogous method to the one used for $0 < a < 1$, which results in the same determination of the maximum number of limit cycles.

Therefore, Theorem 5.1.1 is proved.

5.3 Example (At most 20 limit cycles)

Consider the system

$$\begin{cases} \dot{x} = y - y\left(x + \frac{1}{2}\right)^2 + \varepsilon(b_2x^4y + b_4x^7), \\ \dot{y} = -x + x\left(x + \frac{1}{2}\right)^2 + \varepsilon(b_1y + b_3xy^6 + b_5y^7), \end{cases}$$

Given that $\varepsilon > 0$ is a small parameter and b_1, b_2, b_3, b_4, b_5 are real constants. Then

$$g^0(r) = \frac{1}{512ruv}\left(Auv + Bur^2 + Cvr^2\right),$$

where

$$\begin{aligned} A &= \left(160b_4 - 1120b_5\right)r^6 + \left(336b_4 + 3920b_5\right)r^4 + \left(-6832b_5 + 976b_4\right)r^2 - 512b_1 \\ &\quad +4376b_4 + 4376b_5, \\ B &= -256b_5r^8 + 256b_5r^6 - 96b_5r^4 + \left(16b_5 - 256b_1\right)r^2 + \left(-b_4 + 64b_1 - b_5\right), \\ C &= -256b_5r^8 + 2304b_5r^6 - 7776b_5r^4 - \left(256b_1 - 11\,664b_5\right)r^2 \\ &\quad - \left(6561b_4 - 576b_1 + 6561b_5\right), \end{aligned}$$

$$u=\left(\sqrt{\left(\frac{1}{2}+1\right)^{2}-r^{2}}\right) \text{ and } v=\left(\sqrt{\left(\frac{1}{2}-1\right)^{2}-r^{2}}\right).$$

Bounding the zeros of the numerator of $g^0(r)$ can be achieved by limiting the number of roots of the function $w(r)$, as defined in (5.10), Thus, we arrive at

$$w\left(r\right)=T_{40}r^{40}+T_{38}r^{38}...+T_{2}r^{2}+T_{0},$$

where

$$T_{40}=\ 841\,813\,590\,016b_5^4\ ,$$

$$T_{38}=-\,1342\,177\,280b_5^2\left(-5b_4^2+70b_4b_5+12\,299b_5^2\right),$$

$$T_{36}=67\,108\,864b_5^2\left(-955b_4^2+21\,210b_4b_5+2350\,677b_5^2\right),$$

$$T_{34}=16\,777\,216b_5^2\left(25\,519b_4^2-1089\,970b_4b_5-57\,482\,145b_5^2+114\,688b_1b_5\right),$$

$$\begin{aligned}T_{32}=&655\,360\,000b_4^4-18\,350\,080\,000b_4^3b_5-1440\,389\,005\,312b_4^2b_5^2\\&+13\,421\,772\,800b_1b_4^2b_5+186\,777\,620\,971\,520b_4b_5^3-230\,854\,492\,160b_1b_4b_5^2\\&+4164\,749\,504\,806\,912b_5^4-30\,309\,047\,336\,960b_1b_5^3,\end{aligned}$$

$$\begin{aligned}T_{30}=&2228\,224\,000b_4^4+212\,389\,068\,800b_4^3b_5-1174\,166\,044\,672b_4^2b_5^2\\&-77\,846\,282\,240b_1b_4^2b_5-1313\,425\,701\,470\,208b_4b_5^3+2642\,478\,628\,864b_1b_4b_5^2\\&-13\,249\,140\,080\,246\,784b_5^4+225\,835\,286\,003\,712b_1b_5^3,\end{aligned}$$

$$\begin{aligned}T_{28}=&1340\,029\,796\,352b_1^2b_5^2+300\,110\,839\,808b_1b_4^2b_5-29\,250\,337\,898\,496b_1b_4b_5^2\\&-1039\,271\,490\,224\,128b_1b_5^3+10\,639\,769\,600b_4^4-1462\,910\,320\,640b_4^3b_5\\&+56\,967\,428\,833\,280b_4^2b_5^2+6346\,496\,344\,588\,288b_4b_5^3+31\,340\,774\,086\,279\,168b_5^4,\end{aligned}$$

$$
\begin{aligned}
T_{26} =&6710\,886\,400 b_1^2 b_4^2 - 179\,851\,755\,520 b_1^2 b_4 b_5 - 15\,751\,524\,122\,624 b_1^2 b_5^2 \\
&- 8388\,608\,000 b_1 b_4^3 - 154\,081\,951\,744 b_1 b_4^2 b_5 + 246\,040\,758\,321\,152 b_1 b_4 b_5^2 \\
&+ 3213\,854\,234\,378\,240 b_1 b_5^3 + 68\,813\,783\,040 b_4^4 + 3462\,122\,569\,728 b_4^3 b_5 \\
&- 563\,227\,093\,368\,832 b_4^2 b_5^2 - 21\,500\,868\,355\,293\,184 b_4 b_5^3 - 53\,716\,510\,985\,879\,552 b_5^4,
\end{aligned}
$$

$$
\begin{aligned}
T_{24} =& - 13\,757\,317\,120 b_1^2 b_4^2 + 1397\,340\,766\,208 b_1^2 b_4 b_5 + 81\,522\,975\,571\,968 b_1^2 b_5^2 \\
&+ 161\,113\,702\,400 b_1 b_4^3 - 6799\,621\,095\,424 b_1 b_4^2 b_5 - 1301\,385\,985\,392\,640 b_1 b_4 b_5^2 \\
&- 6731\,328\,116\,490\,240 b_1 b_5^3 + 69\,274\,177\,536 b_4^4 + 1233\,156\,489\,216 b_4^3 b_5 \\
&+ 3392\,506\,686\,791\,680 b_4^2 b_5^2 + 51\,011\,087\,661\,645\,824 b_4 b_5^3 + 59\,809\,722\,882\,166\,784 b_5^4,
\end{aligned}
$$

$$
\begin{aligned}
T_{22} =&274\,877\,906\,944 b_1^3 b_5 + 26\,742\,882\,304 b_1^2 b_4^2 - 13\,303\,862\,525\,952 b_1^2 b_4 b_5 \\
&- 233\,741\,884\,588\,032 b_1^2 b_5^2 - 199\,732\,756\,480 b_1 b_4^3 + 65\,797\,732\,958\,208 b_1 b_4^2 b_5 \\
&+ 4319\,514\,325\,942\,272 b_1 b_4 b_5^2 + 8733\,490\,006\,720\,512 b_1 b_5^3 + 2474\,436\,603\,904 b_4^4 \\
&- 87\,293\,617\,504\,256 b_4^3 b_5 - 13\,138\,631\,886\,315\,520 b_4^2 b_5^2 - 80\,429\,373\,341\,401\,088 b_4 b_5^3 \\
&- 21\,260\,007\,980\,175\,360 b_5^4,
\end{aligned}
$$

$$
\begin{aligned}
T_{20} =& - 42\,949\,672\,960 b_1^3 b_4 - 1967\,095\,021\,568 b_1^3 b_5 + 185\,899\,941\,888 b_1^2 b_4^2 \\
&+ 82\,034\,831\,654\,912 b_1^2 b_4 b_5 + 354\,586\,334\,330\,880 b_1^2 b_5^2 + 59\,057\,111\,040 b_1 b_4^3 \\
&- 486\,334\,773\,854\,208 b_1 b_4^2 b_5 - 8781\,865\,378\,643\,968 b_1 b_4 b_5^2 - 3642\,592\,899\,629\,056 b_1 b_5^3 \\
&+ 3957\,743\,111\,168 b_4^4 + 675\,831\,991\,468\,032 b_4^3 b_5 + 32\,282\,291\,101\,710\,336 b_4^2 b_5^2 \\
&+ 66\,022\,936\,966\,668\,288 b_4 b_5^3 - 64\,924\,921\,473\,655\,808 b_5^4
\end{aligned}
$$

$$
\begin{aligned}
T_{18} =&178\,241\,142\,784 b_1^3 b_4 + 4438\,848\,700\,416 b_1^3 b_5 - 2276\,050\,599\,936 b_1^2 b_4^2 \\
&- 251\,981\,761\,347\,584 b_1^2 b_4 b_5 - 68\,804\,305\,485\,824 b_1^2 b_5^2 + 5119\,616\,090\,112 b_1 b_4^3 \\
&+ 1998\,262\,281\,961\,472 b_1 b_4^2 b_5 + 9025\,828\,111\,646\,720 b_1 b_4 b_5^2 - 10\,630\,093\,125\,713\,920 b_1 b_5^3 \\
&+ 9426\,945\,266\,432 b_4^4 - 3527\,718\,582\,565\,888 b_4^3 b_5 - 45\,061\,159\,695\,459\,840 b_4^2 b_5^2 \\
&+ 29\,062\,400\,903\,853\,056 b_4 b_5^3 + 148\,777\,765\,163\,617\,024 b_5^4,
\end{aligned}
$$

$$
\begin{aligned}
T_{16} =& 17\,179\,869\,184 b_1^4 - 1163\,936\,137\,216 b_1^3 b_4 + 1997\,159\,792\,640 b_1^3 b_5 \\
&+ 16\,562\,769\,362\,944 b_1^2 b_4^2 + 336\,712\,311\,504\,896 b_1^2 b_4 b_5 - 870\,744\,880\,054\,272 b_1^2 b_5^2 \\
&- 38\,487\,929\,380\,864 b_1 b_4^3 - 4032\,477\,910\,310\,912 b_1 b_4^2 b_5 + 2438\,928\,920\,109\,056 b_1 b_4 b_5^2 \\
&+ 25\,331\,516\,791\,267\,328 b_1 b_5^3 + 84\,775\,104\,919\,488 b_4^4 + 10\,040\,354\,918\,608\,640 b_4^3 b_5 \\
&+ 15\,613\,119\,214\,681\,728 b_4^2 b_5^2 - 161\,712\,555\,497\,959\,680 b_4 b_5^3 - 157\,580\,391\,127\,300\,160 b_5^4,
\end{aligned}
$$

$$
\begin{aligned}
T_{14} =& - 17\,179\,869\,184 b_1^4 + 4067\,065\,593\,856 b_1^3 b_4 - 30\,227\,174\,522\,880 b_1^3 b_5 \\
&- 71\,079\,375\,667\,200 b_1^2 b_4^2 + 124\,419\,879\,993\,344 b_1^2 b_4 b_5 + 1878\,169\,731\,727\,360 b_1^2 b_5^2 \\
&+ 318\,313\,920\,471\,040 b_1 b_4^3 + 2417\,575\,165\,747\,200 b_1 b_4^2 b_5 - 21\,247\,546\,419\,486\,720 b_1 b_4 b_5^2 \\
&- 25\,859\,115\,590\,836\,224 b_1 b_5^3 - 123\,203\,181\,814\,144 b_4^4 - 11\,259\,541\,088\,000\,512 b_4^3 b_5 \\
&+ 60\,450\,029\,867\,268\,864 b_4^2 b_5^2 + 210\,561\,490\,727\,042\,560 b_4 b_5^3 + 73\,428\,678\,536\,841\,856 b_5^4,
\end{aligned}
$$

$$
\begin{aligned}
T_{12} =& - 302\,795\,194\,368 b_1^4 + 209\,748\,754\,432 b_1^3 b_4 + 64\,812\,499\,337\,216 b_1^3 b_5 \\
&+ 87\,043\,377\,397\,760 b_1^2 b_4^2 - 1079\,466\,372\,759\,552 b_1^2 b_4 b_5 - 1792\,299\,311\,366\,144 b_1^2 b_5^2 \\
&- 779\,794\,349\,078\,528 b_1 b_4^3 + 5320\,692\,329\,601\,024 b_1 b_4^2 b_5 + 27\,700\,832\,200\,415\,232 b_1 b_4 b_5^2 \\
&+ 9124\,677\,684\,905\,984 b_1 b_5^3 + 1784\,696\,850\,928\,752 b_4^4 - 6737\,096\,032\,344\,128 b_4^3 b_5 \\
&- 110\,341\,502\,034\,266\,976 b_4^2 b_5^2 - 111\,356\,747\,069\,078\,592 b_4 b_5^3 + 36\,991\,622\,335\,619\,696 b_5^4,
\end{aligned}
$$

$$
\begin{aligned}
T_{10} =& 887\,984\,488\,448 b_1^4 - 19\,768\,065\,130\,496 b_1^3 b_4 - 58\,715\,768\,094\,720 b_1^3 b_5 \\
&+ 134\,612\,837\,728\,256 b_1^2 b_4^2 + 1371\,741\,790\,142\,464 b_1^2 b_4 b_5 + 427\,009\,721\,237\,504 b_1^2 b_5^2 \\
&- 164\,025\,292\,040\,192 b_1 b_4^3 - 10\,778\,854\,894\,467\,072 b_1 b_4^2 b_5 - 11\,279\,736\,504\,023\,040 b_1 b_4 b_5^2 \\
&+ 10\,110\,472\,662\,702\,080 b_1 b_5^3 - 817\,575\,021\,177\,600 b_4^4 + 28\,297\,544\,413\,774\,336 b_4^3 b_5 \\
&+ 65\,112\,947\,947\,259\,904 b_4^2 b_5^2 - 43\,546\,467\,092\,505\,088 b_4 b_5^3 - 92\,156\,753\,490\,148\,608 b_5^4,
\end{aligned}
$$

$$
\begin{aligned}
T_8 =& -752\,625\,909\,760b_1^4 + 25\,058\,588\,229\,632b_1^3b_4 + 5874\,781\,257\,728b_1^3b_5 \\
& - 313\,813\,539\,913\,728b_1^2b_4^2 - 353\,041\,251\,631\,104b_1^2b_4b_5 + 885\,271\,623\,991\,296b_1^2b_5^2 \\
& + 1743\,816\,061\,256\,704b_1b_4^3 + 4718\,751\,055\,825\,920b_1b_4^2b_5 - 10\,395\,406\,226\,224\,128b_1b_4b_5^2 \\
& - 16\,536\,940\,144\,606\,208b_1b_5^3 - 3610\,514\,252\,269\,244b_4^4 - 18\,127\,281\,173\,559\,536b_4^3b_5 \\
& + 24\,562\,518\,270\,133\,656b_4^2b_5^2 + 115\,541\,352\,357\,649\,168b_4b_5^3 + 77\,638\,058\,908\,590\,916b_5^4,
\end{aligned}
$$

$$
\begin{aligned}
T_6 =& -80\,262\,201\,344b_1^4 - 989\,406\,953\,472b_1^3b_4 + 33\,733\,860\,130\,816b_1^3b_5 \\
& + 60\,000\,884\,686\,848b_1^2b_4^2 - 696\,397\,347\,815\,424b_1^2b_4b_5 - 1053\,940\,413\,431\,808b_1^2b_5^2 \\
& - 607\,464\,429\,177\,856b_1b_4^3 + 4525\,813\,953\,616\,896b_1b_4^2b_5 + 15\,844\,461\,531\,110\,400b_1b_4b_5^2 \\
& + 11\,019\,639\,998\,772\,224b_1b_5^3 + 1854\,933\,701\,687\,424b_4^4 - 8879\,570\,196\,078\,592b_4^3b_5 \\
& - 58\,514\,864\,771\,248\,896b_4^2b_5^2 - 85\,607\,626\,291\,258\,368b_4b_5^3 - 37\,827\,265\,417\,775\,488b_5^4,
\end{aligned}
$$

$$
\begin{aligned}
T_4 =& 4\,(547b_4 - 64b_1 + 547b_5)^2\,[29\,274\,112b_1^2 - 408\,278\,912b_1b_4 \\
& - 1165\,023\,104b_1b_5 + 1355\,849\,203b_4^2 + 8716\,544\,486b_4b_5 + 9212\,503\,027b_5^2],
\end{aligned}
$$

$$
T_2 = -1152\,(547b_4 - 64b_1 + 547b_5)^3\,(4921b_4 - 640b_1 + 9313b_5)
$$

and

$$
T_0 = 1296\,(547b_4 - 64b_1 + 547b_5)^4\,.
$$

We deduce that $g^0(r)$ has at most 20 positive roots. (The computations for this example were carried out using Matlab).

Conclusion

This thesis has deepened our understanding of limit cycles in polynomial differential systems using the averaging theory. Through a detailed analysis of various systems, such as the generalized Mathieu differential equations and septic polynomial differential systems, we have demonstrated the importance of advanced analytical techniques in exploring the complexity of dynamic behaviors. The results obtained confirm the richness and diversity of phenomena observed in these systems while highlighting the open challenges that persist in the field.

This research paves the way for new explorations, particularly in the application of averaging methods to higher-dimensional systems or those involving more complex nonlinearities. It also strengthens the interest in theoretical studies in the context of practical problems in physics, mechanics, and other applied disciplines.

Appendix

In this Appendix, we present essential integral formulas and mathematical results that underpin the analyses conducted in the main body of the work. These appendices contain detailed derivations and expressions for integrals involving trigonometric functions, which play a pivotal role in the polynomial averaging method employed to determine the maximum number of limit cycles in the studied differential systems. The provided integral expressions and constants are instrumental in streamlining and solving polynomial differential equations, thereby enhancing the overall understanding of the system's dynamics.

5.4 Appendix A

This appendix includes a mathematical formulation of a set of integrals involving sin and cos functions raised to various powers. These integrals have played a crucial role in utilizing the averaging theory to determine the upper bound of limit cycles in dynamical systems. The provided formulations aim to simplify computations and enhance accuracy in the study of limit cycles.

$$\int_0^\theta \cos^{2i+n}\varphi \sin^{2j+1-n}\varphi d\varphi = \sum_{l=0}^{in+j(1-n)} \sum_{k=1}^{l+in+j(1-n)} \gamma_{l,k}^{(n)}(i,j)$$

$$\times \cos\left(n\tfrac{\pi}{2} - (2k+1)\,\theta\right),\ n = 0, 1,$$

$$\int_0^{\theta} \cos^{2(i+h)+n}\varphi \sin^{2j+1-n}\varphi d\varphi = \sum_{l=0}^{(i+h)n+j(1-n)} \sum_{k=1}^{l+(i+h)n+j(1-n)} \gamma_{l,k}^{(n)}\,(i+h, j)$$

$$\times \cos\left(n\frac{\pi}{2} - (2k+1)\,\theta\right),\ n = 0, 1,$$

$$\int_0^{\theta} \cos^{2i+1}\varphi \sin^{2j+1}\varphi d\varphi = \rho_{i,j} + \sum_{l=0}^{j} \bar{\rho}_{l,i,j} \cos^{2(i+l+1)}\theta,$$

$$\int_0^{\theta} \cos^{2(i+h)+1}\varphi \sin^{2j+1}\varphi d\varphi = \rho_{i+h,j} + \sum_{l=0}^{j} \bar{\rho}_{l,i+h,j} \cos^{2(i+l+1)}\theta,$$

where $\gamma_{l,k}^{(n)}\,(i, j)$, $\gamma_{l,k}^{(n)}\,(i+h, j)$, $\bar{\rho}_{l,i,j}$, $\bar{\rho}_{l,i+h,j}$, $\rho_{i,j}$ and $\rho_{l,i,j}$ are non-zero constants.

5.5 Appendix B

This appendix provides a mathematical formulation of integrals involving sin and cos functions raised to various powers. The results are categorized based on the parity of the indices i and j, with specific coefficients identified to describe nonzero contributions under certain conditions. These integrals have greatly facilitated the computation process when applying the averaging theory to assess the maximum number of limit cycles in dynamical systems.

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^i\theta \sin^j\theta \cos(2k+1)\,\theta d\theta = \begin{cases} 0 \text{ if } j = 2l+1 \\ \Lambda_{l,l',k} \text{ if } j = 2l,\ i = 2l'+1 \end{cases},$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^i\theta \sin^j\theta \sin(2k+1)\,\theta d\theta = \begin{cases} 0 \text{ if } j = 2l \\ \Gamma_{l,l',k} \text{ if } j = 2l+1,\ i = 2l' \end{cases},$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^i\theta \sin^j\theta \sin(2k+1)\,\theta d\theta = 0 \text{ if } j = 2l+1,\ i = 2l'+1,$$

where $\Lambda_{l,l',k}$ and $\Gamma_{l,l',k}$ are real constants.

Bibliography

[1] Abdalla, M., Ahmed Idris S., Mekawy I. On Behavior of the Periodic Orbits of a Hamiltonian System of Bifurcation of Limit Cycles, Mathematical Problems in Engineering. 2021;ID:1952706.

[2] Leach, P. G. L., Andriopoulos, K. The Ermakov equation: a commentary. App. Anal. Discrete Math, 2, 146–157 (2008).

[3] Berezin, I. S., Zhidkov, N. P. Computing Methods, Volume II. Pergamon Press, Oxford, 1964.

[4] Bogoliubov, N. N., Krylov, N. The application of methods of nonlinear mechanics in the theory of stationary oscillations. Publ. 8 of the Ukrainian Acad. Sci. Kiev, 1934.

[5] Bogoliubov, N. N., Mitropolsky, Yu. A. Asymtotic methods in the theory of nonlinear oscillations. Gordon and Breach, New York, 1961.

[6] Bouattia, Y., Boudjehem, D., Makhlouf, A., Zubair, S. A., Idris, S. A. A note on small amplitude limit cycles of Liénard equations theory. Mathematical Problems in Engineering, 2021 (2021).

[7] Boulfoul, A., Mellahi, N. On the maximum number of limit cycles of generalized polynomial Liénard differential systems via averaging theory. Applied Mathematics E-Notes, 20, 167–187 (2020).

[8] Boulfoul, A., Saifia, O. On the number of limit cycles in a class of planar differential systems. Nonlinear Stud, 30(3), 855–870 (2023).

[9] Buică, A., Llibre, J. Averaging methods finding periodic orbits via Brouwer degree. Bull. Sci. Math., 128, 7–22 (2004).

[10] Buică, A., Giné, J., Llibre, J. Bifurcation of limit cycles from a polynomial degenerate center. Adv. Nonlinear Stud, 10, 597–609 (2010).

[11] Chen, T., Llibre, J. Limit cycles of a second-order differential equation. Appl. Math. Lett., 88, 111–117 (2019).

[12] Melki, H., Makhlouf, A. Limit cycles of a planar differential system via averaging theory. Open Journal of Mathematical Analysis, 5(2), 73–83 (2021).

[13] Chow SN, Li C, Wang D. Normal Forms and Bifurcation of Planar Vector Fields. Cambridge University Press: Cambridge; 1994.

[14] Tzemos, A. C., Contopoulos, G. Integrals of motion in time-periodic Hamiltonian systems, The case of the Mathieu equation. Regul. Chaotic Dyn, 26(1), 89–104 (2021).

[15] Diab, Z., Guirao, J. L. G., Llibre, J., Makhlouf, A. Limit cycles of a generalised Mathieu differential system. Applied Mathematics and Nonlinear Sciences, 88, 1–8 (2021).

[16] Esmailzadeh, E., Nakhaie-Jazar, G. Periodic solutions of a Mathieu-Duffing type equation. J. Non. Linear Mech, 32, 905–912 (1997).

[17] Fatou, P. Sur le mouvement d'un système soumis à des forces à courte période. Bulletin de la Société Mathématique de France, 56, 98–139 (1928).

[18] Gasull, A., Torregrosa, G. A new algorithm for the computation of the Lyapunov constants for some degenerate critical points. Nonlin. Anal, 47, 4479–4490 (2001).

[19] Giné, J., Llibre, J. Limit cycles of cubic polynomial vector fields via the averaging theory. Nonlinear Anal, 66, 1707–1721 (2007).

[20] Giné, J., Grau, M., Llibre, J. Averaging theory at any order for computing periodic orbits. Phys. D, 250, 58–65 (2013).

[21] Hong Z, Chen W, Tonghua Z. Perturbation from a cubic Hamiltonian with three figure eight-loops. Chaos Solitons Fractals, 22(1), 61–74 (2004).

[22] Kong X, Xiong Y. Limit cycle bifurcations in a class of quintic Z_2-equivariant polynomial systems. Nonlinear Dyn, 73, 1271–1281 (2013).

[23] Kovacic, I., Rand, R., Sah, S. M. Mathieu's equation and its generalizations: Overview of Stability Charts and Their Features, Appl. Mech. Rev, 70(2), 1–22 (2018).

[24] Kyziol, J., Okninski, A. Asymmetric Duffing Oscillator: Jump Manifold and Border Set. Nonlinear Dyn. Syst. Theory, 23, 46–57 (2023).

[25] Kyziol, J., Okninski, A. The Twin-Well Duffing Equation: Escape Phenomena, Bistability, Jumps, and Other Bifurcations. Nonlinear Dynamics and Systems Theory, 24(2), 181–192 (2024).

[26] Leach, P. G. L., Andriopoulos, K. The Ermakov equation: a commentary. App. Anal. Discrete Math, 2, 146–157 (2008).

[27] Liapunov, A. M. Stability of motion. Mathematics in Science and Engineering, 30, Academic Press, New York–London, (1966).

[28] Li F, Wang M. Bifurcation of limit cycles in a quintic system with ten parameters. Nonlinear Dyn, 71, 213–222 (2013).

[29] Li J. Hilbert's 16th problem and bifurcations of planar polynomial vector fields. Int. J. Bifurc. Chaos, 13(1), 47–106 (2003).

[30] Li H. Limit cycles in a sextic Lyapunov system. Nonlinear Dyn, 72, 555–559 (2013).

[31] Llibre, J., Rodrguez, G. Configurations of limit cycles and planar polynomial vector fields. J. Differential Equations, 198(2), 374–380 (2004).

[32] Llibre, J., Mereu, A. Limit cycles for generalized Kukles polynomial differential systems. Nonlinear Anal, 74(4), 1261–1271 (2011).

[33] Llibre, J., Novaes, D. D., Teixeira, M. A. Higher order averaging theory for finding periodic solutions via Brouwer degree. Lon. Math. Soc, 27, 563–583 (2014).

[34] Lloyd, N. G. Limit cycles of polynomial systems-some recent developments. London Math. Soc. Lecture Note Ser, 127, 192–234 (1988).

[35] Mathieu, E. Mémoire sur le mouvement vibratoire d'une membrane de forme elliptique. Math. Pures Appl, 13, 137–203 (1868).

[36] Malkin, I. G. Some problems of the theory of nonlinear oscillations. Gosudarstv.Izdat. Tehn.-Teor. Lit, Moscow, 1956 (in Russian).

[37] Mao, H-C., Feng, Y-L., Wang, X-Q., Yao, Z-H. Weak signal detection application based on incommensurate fractional-order Duffing system. Journal of Nonlinear Mathematical Physics, 31, 1–16 (2024).

[38] Menaceur, A., Boulaaras, S., Makhlouf, A., Rajagobal, K., Abdalla, M. Limit cycles of a class of perturbed differential systems via the first-order averaging method. Complexity, Vol 2021, Article ID 5581423, 6 pages (2021).

[39] Menaceur, A., Boulaaras, S. A number of limit cycle of sextic polynomial differential systems via the averaging theory. Bol. Soc. Paran. Mat, 39(4), 181–197 (2021).

[40] Menaceur, A., Boulaaras, S., Alkhalaf S, Jain S. Limit cycles of a class of polynomial differential systems bifurcating from the periodic orbits of a linear center. Symmetry, 12(8), 15 pages (2020).

[41] Menaceur A, Abdalla M, Ahmed Idris S, Mekawy I. On Behavior of the Periodic Orbits of a Hamiltonian System of Bifurcation of Limit Cycles, Mathematical Problems in Engineering, 2021, ID:1952706 (2021).

[42] Menaceur A, Alrawashdeh M, Ahmed Idris S, Abd-Elmageed H. Fifteen Limit Cycles Bifurcating from a Perturbed Cubic Center. Discrete Dynamics in Nature and Society, 2021, ID:8178729 (2021).

[43] Menaceur, A., Zemmouri, I. Limit cycles of a class of generalized Mathieu differential equations. Nonlinear Stud, 31(1), 279–290 (2024).

[44] Poincaré, H. Mémoire sur les courbes définies par une équation différentielle I, II. J. Math. Pures Appl, 7, 375–422 (1881), 8, 251–296 (1882).

[45] Zounes, R. S., Rand, R. H. Global behavior of a nonlinear quasiperiodic Mathieu equation. Nonlinear Dyn, 27(1), 87–105 (2002).

[46] Roseau, M. Vibrations non linéaires et théorie de la stabilité. Springer Tracts in Natural Philosophy, Vol. 8, Springer, New York, 1985.

[47] Roseau, M. Vibrations non linéaires et théorie de la stabilité, volume 8. Springer, 1966.

[48] Ruby, L. Applications of the Mathieu equation. Amer. J. Phys, 64(39), 39–44 (1996).

[49] Sanders, J. A., Verhulst, F. Averaging methods in nonlinear dynamical systems. Applied Mathematical Sciences, New York, Springer-Verlag, 1985.

[50] Sanders, J. A., Verhulst, F., Murdock, J. Averaging method in nonlinear dynamical systems. Applied Mathematical Sciences, Vol. 59, Springer, New York, 2007.

[51] Shi, H., Bai, Y., Han, M. On the maximum number of limit cycles for a piecewise smooth differential system. Bull. Sci. Math, 163 (102887), 1–17 (2020).

[52] Li S, Zhao Y, Li J. On the number of limit cycles of a perturbed cubic polynomial differential center. J. Math. Anal, 404(2), 212–220 (2013).

[53] Blows TR, Perko LM. Bifurcation of limit cycles from centers and separatrix cycles of planar analytic systems. SIAM Rev, 36(3), 341–376 (1994).

[54] Tabet, A. E., Makhlouf, A. Periodic solutions of some polynomial differential systems in dimension 5 via averaging theory. Nonautonomous Dynamical Systems, 10, 20230104, 14 pages (2023).

[55] Verhulst, F. Nonlinear differential equations and dynamical systems. Universitext. New York: Springer; 1996.

[56] Viano M, Llibre J, Giacomini H. Arbitrary order bifurcations for perturbed Hamiltonian planar systems via the reciprocal of an integrating factor. Nonlinear Anal. Theory Methods Appl, 48, 117–136 (2002).

[57] Yakovenko, S. Quantitative theory of ordinary differential equations and tangential Hilbert 16th problem. in: On Finiteness in Differential Equations and Diophantine Geometry, CRM Monogr. Ser. 24, American Mathematical Society, Providence, 41–109 (2005).

[58] Yanqian Y. Theory of Limit Cycles. Translations of Math: Monographs, 66 (Providence, RI Amer. Math. Soc.); 1986.

[59] Yang, Y., Zhao, L. Bifurcation of limit cycles of a piecewise smooth Hamiltonian system. Qual. Theory Dyn. Syst, 21, 142 (2022).

[60] Zamora, M. A note on the periodic solutions of a Mathieu-Duffing type equations. Math. Nachr, 290, 1113–1118 (2017).

[61] Zemmouri, I., Menaceur, A., Laouar, A., Boulaaras, S. Limit cycles of septic polynomial differential systems bifurcating from the periodic annulus of cubic center. Partial Differential Equations in Applied Mathematics, 9(100622), 5 pages (2024).

[62] Zhao Y, Zhu H. Bifurcation of limit cycles from a non-Hamiltonian quadratic integrable system with homoclinic loop. Infinite Dimensional

Dynamical Systems, Fields Inst. Commun. Springer, New York, 64, 445–479 (2013).

[63] Zounes, R. S., Rand, R. H. Global behavior of a nonlinear quasiperiodic Mathieu equation. Nonlinear Dyn, 27(1), 87–105 (2002).