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Ministry of Higher Education and Scientific Research

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Title

Generalization of the Gronwall-Bellman lemma for the stability of perturbed systems.

Stream
Applied Mathematics

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By
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Le Titre

Généralisation du lemme de Gronwall-Bellman pour la stabilité des systèmes perturbés.

Filière
Mathématiques Appliquées

Spécialité
Mathématiques Appliquées
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Résumé

Il est bien connu que la théorie des inégalités intégrales joue un rôle essentiel dans l'étude de l'analyse qualitative et quantitative du comportement de divers types de solutions des équations différentielles non linéaires.

L'objectif de cette thèse est d'établir, dans un premier temps, de nouvelles inégalités intégrales fractionnaires pour les fonctions à une variable.

Dans un deuxième temps, nous allons établir de nouvelles inégalités de type Gamidov bidimensionnelles.

Enfin, nous avons utilisé quelques inégalités intégrales pour étudier d'une part la stabilité exponentielle de certains systèmes dynamiques non linéaires perturbés aux échelles de temps arbitraires et d'autre part la h-stabilités d'une classes d'équations dynamique dans un espace de Banach.

Mots-clés: Échelles de temps, Inégalité intégrale de type Gronwall, Inégalité de type Gamidov, Intégrale fractionnaire, Stabilité exponentielle,h-stabilité Systèmes perturbés.

Abstract

It is well known that the theory of integral inequalities plays an essential role in the study of the qualitative and quantitative analysis of the behaviour of various types of solutions of non-linear differential equations.

The aim of this thesis is to establish, in a first step, new fractional integral inequalities for one-variable functions.

Secondly, we will establish new two-dimensional Gamidov-type inequalities.

Finally, we used some integral inequalities to study the exponential stability of some perturbed nonlinear dynamical systems at arbitrary time scales and to study the h-stability of a class of dynamical equations in a Banach space.

Keywords : Time scales, Inequality of Gronwall, Inequality of Gamidov, Fractional integral , Exponential stability,h-stability, Perturbed systems.

ملخـــــص

من الجيد أن نعرف أنّ نظرية عدم المساواة للتكاملات لها دور أساسي في دراسة التحليل النوعي والكمي لسلوك أنواع مختلفة من حلول المعادلات التفاضلية الغير الخطية.

الهدف من هذه الأطروحة هو إنشاء :

أولا عدم المساواة للتكاملات الكسرية جديدة ذات متغير واحد.

ثانيا عدم المساواة للتكاملات جديدة من نوع جاميدوف للدوال ثنائية الأبعاد.

أخيرا، استخدمنا بعضًا من عدم المساواة للتكاملات لدراسة:

أولا الاستقرار الأسي لبعض الأنظمة الديناميكية غير الخطية المضطربة بمستويات زمنية عشوائية، و ثانيا لدراسة استقرار h لفئات من المعادلات الديناميكية في فضاء باناخ.

الكلمات المفتاحية: المقاييس الزمنية، المتباينة التكاملية من نوع جرونوال، الأنظمة h المتباينة من نوع جاميدوف، التكامل الكسري، الثبات الأسي، ثبات المضطربة.

Contents

General Introduction

Inequalities have played an important role in the development of all branches of mathematics and occupy a central place in the attention of many mathematical researchers. Integral inequalities are involved in the theory of differential and integral equations. The first to introduce integral inequalities was Gronwall in 1919([11], Ref of chapter 1)who gave their applications in the study of certain problems concerning ordinary differential equations. After 1919, several researchers showed their attention to this subject and several works have been established. Much information on this subject can be found in a number of monographs published in recent years .

One of the main attempts to unify continuous and discrete analysis is the theory of time scales. This new theory was introduced by Stefan Hilger in his doctoral thesis ([15] ,Ref of chapter 3) in 1988. This constitutes a unique topic for studying both differentials and difference problems of dynamic equations. It can be applicable to any process dynamics that are described by discrete or continuous temporal models or a mixture of both.

Martin Bohner and Allan Peterson developed this theory in two important books ([7, 8] ,Ref of chapter 3).

Stability is the most effective study for designing the behavior of solutions of dynamic equations in relation to their domain of evolution. Aulbach and Hilger began the study of this subject which attracted the attention of several researchers. Various contributions dealing with the stability of linear and nonlinear dynamic systems on time scales, such as ([$2-4, 6, 10-13$], Ref,chapter 3) and ([$11, 17-24, 26, 27$] ,Ref chapter 4).Among the most valid approaches, we cite the integral inequalities approach and Lyapunov analysis to evaluate trajectory behaviors through an energy function.

The objective of this thesis is initially, the study of the estimation, the existence and the uniqueness of certain classes of fractional differential equations associated with a generalized operator and also for certain classes of integral equations and secondly, the study of the exponential stability and the h-stability of certain classes of perturbed nonlinear systems.

The thesis is organized into 4 chapters as follows:

In chapter one, we recall some definitions and notations about fractionnal derivatives and integrals, then we establish some fractional integral inequalities associated with a generalized operator which will be used in the qualitative study of the following initial problem:

$$\begin{aligned} D^{\alpha,\rho}_{t_{0+}} x(t) &= f(t, x(t)), \\ I^{1-\alpha,\rho}_{t_{0+}} x(t) &\ |_{t=t_0} = c, \end{aligned} \tag{IVP}$$

where $0 < \alpha \leq 1$, $0 \leq t < T \leq \infty$ and $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function with respect to all its arguments.

Chapter 2 concerns generalizations of some integral inequalities of the Gamidov type involved in the qualitative study of certain classes of integral equations of the form :

$$x(s,t) = \alpha(s,t) + \beta(s,t) \int_0^s \int_0^t \Phi(\eta, \tau, x) d\eta d\tau + \gamma(s,t) \int_0^A \int_0^B H(\eta, \tau, x) d\eta d\tau, \tag{IE}$$

for $(s,t) \in \Gamma$, where $x(s,t) \in C(\Gamma, \mathbb{R}), \alpha(s,t), \beta(s,t), \gamma(s,t) \in C(\Gamma, \mathbb{R}_+)$ such that $\alpha(s,t)$, $\beta(s,t)$, $\gamma(s,t)$ are nondecreasing in s and t and $\Phi(s,t,x), H(s,t,x) \in C(\Gamma \times \mathbb{R}, \mathbb{R})$ and $\Gamma = [0, A] \times [0, B]$.

In chapter 3, we study the exponential stability of the following system

$$\begin{aligned} z^{\Delta}(t) &= A(t)z + F(t, z(t)), \\ z(t_0) &= z_0, z_0 \neq 0. \end{aligned} \tag{PNS}$$

where $z_0, z \in \mathbb{R}^n$, $F(t, 0) = 0$, $t_0 \in T$, and $F : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is an rd-continuous function.

The chapter 4, deal with problem of h- stability of certain classes of the following abtract dynamic equations :

$$z^{\Delta}(t) = Az(t) + f(t, z), z(t_0) = z_0 \in D(A), t \in \mathbb{T}_0^+, \quad \text{(ADE)}$$

where we derive sufficient conditions on the perturbed term of the dynamic equations to ensure h-stability of system (ADE).

The results of these chapters are published inhe results of theses chapters are published in

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CHAPTER 1

Some new Grőnwall-Bihari type inequalities associated with generalized fractional operators and applications

1.1 Introduction and preliminaries

It is well known that the Grőnwall-Bellman inequality [2, 11] and their generalizations can provide explicit bounds for solutions to differential and integral equations as well as difference equations. Many authors have researched various inequalities and investigated the boundedness, global existence, uniqueness, stability, and continuous dependence on the initial value and parameters of solutions to differential equations, integral equations see $[3-6, 14]$. However, we notice that the existing results in the literature are inadequate for researching the qualitative and quantitative properties of solutions to some fractional integral equations see $[13-15, 17, 22-23]$.

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order see [21]. However, in this branch of Mathematics we are not looking at the usual integer order but at the non-integer order integrals and derivatives. These are called fractional derivatives and fractional integrals. The first appearance of the concept of a fractional derivative is found in a letter written to Guillaume de l'Hôpital by Gottfried Wilhelm Leibniz in 1695. As far as the existence of such a theory is concerned the foundations of the subject were laid by Liouville in a paper from 1832. The autodidact Oliver Heaviside introduce the practical use of fractional differential operators in electrical transmission line analysis circa 1890. Many authors have established a variety of inequalities for those fractional integral and derivative operators, some of which have turned out to be useful in analyzing solutions of certain fractional integral and differential equations, for example, we refer the reader to [14, 15, 22, 23] and the references therein.

In [16], the authors proved the following results :

Theorem 1.1.1 *Let $\pi, \theta \in \mathbb{R}^+$. Also, let Ψ and u be nonnegative and locally integrable functions defined on $[0, \varpi)$ with $\varpi \leq +\infty$. Further, let $\phi(x)$ be a nonnegative, nondecreasing, and continuous function on $[0, \varpi)$ which is bounded on $[0, \varpi)$, that is, $\phi(x) \leq M$ for all $x \in [0, \varpi)$ and some $M \in \mathbb{R}^+$. Suppose that the functions Ψ, u, and ϕ satisfy the following inequality:*

$$u(x) \leq \Psi(x) + \pi\phi(x) \int_0^x (x-\rho)^{\frac{\theta}{\pi}-1} u(\rho)\, d\rho,\ \ x \in [0, \varpi).$$

Then

$$u(x) \leq \Psi(x) + \sum_{n=1}^{\infty} \frac{\{\pi\phi(x)\Gamma_k(\theta)\}^n}{\Gamma_k(n\theta)} \int_0^x (x-\rho)^{n\frac{\theta}{\pi}-1} \Psi(\rho) d\rho\ ,\ x \in [0, \varpi)\ .$$

Corollary 1.1.1 *Let $\pi, \theta \in \mathbb{R}^+$ Also, let Ψ and u be nonnegative and locally integrable functions defined on $[1, \varpi)$ with $\varpi \leq +\infty$. Further, let $\phi(x)$ be a nonnegative, nondecreasing, and continuous function on $[0, \varpi)$ which is bounded on $[1, \varpi)$, that is, $\phi(x) \leq M$*

for all $x \in [1,\varpi)$ and some $M \in \mathbb{R}^+$. Suppose that the functions $\Psi, u,$ and ϕ satisfy the following inequality:

$$u(x) \leq \Psi(x) + \pi\phi(x)\int_0^x \left(\ln\frac{x}{\rho}\right)^{\frac{\theta}{\pi}-1} u(\rho)\,\frac{d\rho}{\rho}\ , \qquad (x \in [1,\varpi)).$$

Then

$$u(x) \leq \Psi(x) + \sum_{n=1}^{\infty}\frac{\{\pi\phi(x)\Gamma_k(\lambda)\}^n}{\Gamma_k(n\theta)}\int_1^x \left(\ln\frac{x}{\rho}\right)^{n\frac{\theta}{\pi}-1}\Psi(\rho)\,\frac{d\rho}{\rho}, \quad (x \in [1,\varpi)).$$

In [1] ,the authors proved the following result :

Theorem 1.1.2 *Let $\alpha > 0$, $x(t), a(t)$ be nonnegative functions and $b(t)$ be nonnegative and nondecreasing function for $t \in [t_0, T)$,$T > 0$, $b(t) \leq M$, where M is a constant. If*

$$x(t) \leq a(t) + b(t)\int_{t_0}^t (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} x(\tau)\frac{d\tau}{\tau^{1-\rho}}, \tag{1.1}$$

then

$$x(t) \leq a(t) + \int_{t_0}^t \sum_{n=1}^{\infty}\frac{(b(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(\frac{t^\rho - \tau^\rho}{\rho})^{n\alpha-1} a(\tau)\frac{d\tau}{\tau^{1-\rho}}, t \in [t_0, T). \tag{1.2}$$

In this chapter, we establish some new Grőnwall-Bihari-type inequalities associated with the generalized fractional integral operator given by (1.12) (see Definition 1.1.3), which generalize some results given in [1] .We also present some nonlinear integral inequalities with singular kernels of Bihari type,we apply the results established to research boundedness, uniqueness for the solution to some certain initial value problems within generalized fractional derivatives given by (1.13) (see Definition 1.1.4).

Now, some important properties for the modified Riemann-Liouville derivative and fractional integral are listed as follows :

Definition 1.1.1 *The Riemann-Liouville fractional integral of order α on the interval $[0,\varpi]$ is defined by*

$$(I^\alpha f) = \frac{1}{\Gamma(\alpha)}\int_0^\varpi (\varpi - \tau)^{\alpha-1} f(\tau)d\tau \qquad (\varpi > 0), \tag{1.3}$$

where

$$\Gamma(\alpha) = \int_0^\infty s^{\alpha-1}\exp(-s)\,ds,$$

which is well defined for $\alpha > 0$.

Definition 1.1.2 *i) The modified Riemann-Liouville derivative of order* α *is defined by*

$$(D_{\varpi}^{\alpha} f) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d}{d\varpi} \int_0^{\varpi} (\varpi - \zeta)^{-\alpha} (f(\zeta) - f(0))\, d\zeta, & 0 < \alpha < 1, \\ \left(f^{(n)}(\varpi)\right)^{(\alpha-n)}, & n \leq \alpha < n+1, \quad n \geq 1. \end{cases} \tag{1.4}$$

ii) The Hadamard fractional integral ${}_H I_{1,\varpi}^{\mu} f$ *of order* $\mu > 0$ *is defined by*

$${}_H I_{1,\varpi}^{\mu} f = \frac{1}{\Gamma(\mu)} \int_1^{\varpi} \left(\ln \frac{\varpi}{\tau}\right)^{\mu-1} f(\tau) \frac{d\tau}{\tau} \qquad (\varpi > 1). \tag{1.5}$$

iii) The Hadamard fractional derivative ${}_H D_{1,\varpi}^{\mu} f$ *of order* $\mu > 0$ *is defined by*

$${}_H D_{1,\varpi}^{\mu} f = \frac{1}{\Gamma(n-\mu)} \left(\varpi \frac{d}{dx}\right)^n \int_1^{\varpi} \left(\ln \frac{\varpi}{\tau}\right)^{n-\mu-1} f(\tau) \frac{d\tau}{\tau} \qquad (\varpi > 1) \tag{1.6}$$

$$[n = [\mu] + 1, \ \varpi > 0),$$

In what follows, we denote by $\mathbb{C}, \mathbb{R}, \mathbb{R}^+, \mathbb{R}_+, \mathbb{N}$, and $\mathbb{Z}_0^-$ the sets of complex numbers, real numbers, positive real numbers, nonnegative real numbers, positive- integers, and non-positive integer, respectively.

Dĭaz and Pariguan [8] introduced k-gamma function Γ_k defined by

$$\Gamma_k(z) = \int_0^{\infty} e^{-\frac{t^k}{k}} t^{z-1} dt \qquad \left[[\mathfrak{R}(z)] > 0, k \in \mathbb{R}^+\right), \tag{1.7}$$

which has the following relationships:

$$\Gamma_k(z+k) = z\Gamma_k(z), \ \Gamma_k(k) = 1, \tag{1.8}$$

and

$$\Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1} \Gamma\left(\frac{\gamma}{k}\right). \tag{1.9}$$

Also, k beta function $B_k(\alpha, \beta)$ is defined by

$$B_k(\alpha, \beta) = \begin{cases} \frac{1}{k} \int_0^1 t^{\frac{\alpha}{k}-1} (1-t)^{\frac{\beta}{k}-1} dt & (\min\{\mathfrak{R}(\alpha), \mathfrak{R}(\beta)\} > 0) \\ \frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} & \left(\alpha, \beta \in \mathbb{C} \diagdown k\mathbb{Z}_0^-\right), \end{cases} \tag{1.10}$$

where $k\mathbb{Z}_0^-$ denotes the set of k-multiples of the elements in $\mathbb{Z}_0^-$.

Among many generalizations of the Mittag-Leffler function, one of them is recalled (see [18, 19]) :

$$E_{\lambda,\beta} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + \beta)} \qquad (\lambda, \beta \in \mathbb{C}; \ \mathfrak{R}(\lambda) > 0), \tag{1.11}$$

Definition 1.1.3 *The generalized fractional integral operator of order* $\alpha \in [n-1, n), \rho > 0, t_0 \geq 0$ *and* $t \in [t_0, \infty)$ *is defined by*

$$(I^{\alpha,\rho}_{t_0+} g)t = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_0}^{t} (t^\rho - \tau^\rho)^{\alpha-1} g(\tau) \frac{d\tau}{\tau^{1-\rho}}. \tag{1.12}$$

Definition 1.1.4 *The generalized fractional derivative operator is defined by*

$$(D^{\alpha,\rho}_{t_0+} g)t = \frac{\gamma^n}{\Gamma(n-\alpha)} \int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{n-\alpha-1} g(\tau) \frac{d\tau}{\tau^{1-\rho}}, \alpha \in [n-1, n), \tag{1.13}$$

where $\gamma = (t^{1-\rho} \frac{d}{dt})$.

The relation between the above latter two fractional operators is as follows:

$$(D^{\alpha,\rho}_{t_0+} g)t = \gamma^n (I^{n-\alpha,\rho}_{t_0+} g)(t), \alpha \in [n-1, n). \tag{1.14}$$

Note that the generalized operators (1.12) – (1.13) are reduced to Riemann–Liouville fractional operators as $\rho \to 1$ and Hadamard fractional operators as $\rho \to 0^+$.

The generalized Caputo fractional derivatives were discussed in [10].

Lemma 1.1.1 ([10]**)** *(i) Let* $\alpha > 0, \beta \geq 0, 0 \leq t_0, \rho > 0$. *Then we have*

$$(I^{\alpha,\rho}_{t_0+})((\frac{\tau^\rho - t_0^\rho}{\rho})^\beta)(t) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (\frac{t^\rho - t_0^\rho}{\rho})^{\alpha+\beta}. \tag{1.15}$$

In particular

$$(I^{\alpha,\rho}_{t_0+} 1)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} \frac{d\tau}{\tau^{1-\rho}} = \frac{1}{\Gamma(\alpha+1)} (\frac{t^\rho - t_0^\rho}{\rho})^\alpha. \tag{1.16}$$

(ii)If $\beta > 0$ *and* $0 < \alpha \leq 1$ *then*

$$D^{\alpha,\rho}_{t_0+}((\frac{\tau^\rho - t_0^\rho}{\rho})^\beta)(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (\frac{t^\rho - t_0^\rho}{\rho})^{\beta-\alpha}. \tag{1.17}$$

In particular

$$(D^{\alpha,\rho}_{t_0+} 1)(t) = \frac{(\frac{t^\rho - t_0^\rho}{\rho})^{-\alpha}}{\Gamma(1-\alpha)}, \tag{1.18}$$

and for $k = 0, 1.., [\alpha] + 1$*,we have*

$$(D^{\alpha,\rho}_{t_0+} (\frac{\tau^\rho - t_0^\rho}{\rho})^{\alpha-k})(t) = 0. \tag{1.19}$$

Lemma 1.1.2 ([9]) *Suppose that $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$, then*

$$a^{\frac{q}{p}} \leq \frac{q}{p}\varepsilon^{\frac{q-p}{p}}a + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}. \tag{1.20}$$

for any $\varepsilon > 0$.

1.2 Main Results

In this section, we establish a new version of Grőnwall type integral inequality, which generalizes some previous ones.

Theorem 1.2.1 *Let $\alpha > 0, x(t), a(t)$ be nonnegative functions and $b(t)$ be nonnegative and nondecreasing function for $t \in [t_0, T)$ $, T > 0, b(t) \leq M$, where M is a constant.If*

$$x^p(t) \leq a(t) + b(t)\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1}x^q(\tau)\frac{d\tau}{\tau^{1-\rho}}, \tag{2.1}$$

where $p \neq 0$,$p \geq q > 0$, are constants. Then

$$x(t) \leq [\ \widetilde{a}(t) + \int_{t_0}^{t}\sum_{n=1}^{\infty}\frac{(\widetilde{b}(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(\frac{t^\rho - \tau^\rho}{\rho})^{n\alpha-1}\widetilde{a}(\tau)\frac{d\tau}{\tau^{1-\rho}}]^{\frac{1}{p}}\ ,\ t \in [t_0, T), \tag{2.2}$$

where

$$\widetilde{a}(t) = a(t) + \frac{p-q}{p\alpha}\varepsilon^{\frac{q}{p}}b(t)(\frac{t^\rho - t_0^\rho}{\rho})^\alpha, \widetilde{b}(t) = \frac{q}{p}\varepsilon^{\frac{q-p}{p}}b(t). \tag{2.3}$$

Preuve. Denote the right-hand side of (2.1) by $z(t)$. Then we have

$$x(t) \leq z^{\frac{1}{p}}(t), \qquad (t \in [t_0, T))\,. \tag{2.4}$$

So it follows that

$$z(t) \leq a(t) + b(t)\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1}z^{\frac{q}{p}}(\tau)\frac{d\tau}{\tau^{1-\rho}},\ \ (t \in [t_0, T))\,. \tag{2.5}$$

Using Lemma 1.1.2, we obtain that

$$z(t) \leq a(t) + b(t)\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1}\left(\frac{q}{p}\varepsilon^{\frac{q-p}{p}}z(\tau) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}\right)\frac{d\tau}{\tau^{1-\rho}}, \qquad (t \in [t_0, T))\ . \tag{2.6}$$

Using Lemma 1.1.1, one gets

$$z(t) \leq \widetilde{a}(t) + \widetilde{b}(t)\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1}z(\tau)\frac{d\tau}{\tau^{1-\rho}}, \tag{2.7}$$

where $\widetilde{a}$ and $\widetilde{b}$ are defined as in (2.3).

Applying Theorem 1.1.2 to (2.7) and using (2.4), we can get the desired inequality in (2.2). ■

Remark 1.2.1 *If $p = q = 1$, then Theorem 1.2.1 reduces to Theorem 1.1.2 in [1].*

Theorem 1.2.2 *Let $\alpha > 0, x(t), a(t)$ be nonnegative functions and $b(t)$ be nonnegative and nondecreasing function for $t \in [t_0, T)$,$T > 0$, $b(t) \leq M$, where M is a constant. Furrther, let $S \in C\left(\mathbb{R}_+^2, \mathbb{R}_+\right)$ be a continuous function such that*

$$0 \leq S(t,x) - S(t,y) \leq L(x-y), \quad x \geq y \geq 0, \tag{2.8}$$

for $t \in [t_0, T)$,where $L > 0$. If

$$x^p(t) \leq a(t) + b(t)\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} S(\tau, x^q(\tau))\frac{d\tau}{\tau^{1-\rho}}. \tag{2.9}$$

where $p \neq 0$,$p \geq q > 0$, are constants.Then

$$x(t) \leq [\ \widetilde{a}(t) + \int_{t_0}^{t}\sum_{n=1}^{\infty}\frac{(\widetilde{b}(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(\frac{t^\rho - \tau^\rho}{\rho})^{n\alpha-1}\widetilde{a}(\tau)\frac{d\tau}{\tau^{1-\rho}}]^{\frac{1}{p}}, t \in [t_0, T), \tag{2.10}$$

where

$$\widetilde{a}(t) = a(t) + b(t)\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} S(\tau, \frac{p-q}{p}\varepsilon^{\frac{q}{p}})\frac{d\tau}{\tau^{1-\rho}}, \widetilde{b}(t) = L\frac{q}{p}\varepsilon^{\frac{q-p}{p}}b(t). \tag{2.11}$$

Preuve. Denote the right-hand side of (2.9) by $z(t)$. Then we have

$$x(t) \leq z^{\frac{1}{p}}(t), \qquad (t \in [t_0, T)). \tag{2.12}$$

So it follows that

$$z(t) \leq a(t) + b(t)\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} S(\tau, z^{\frac{q}{p}}(\tau))\frac{d\tau}{\tau^{1-\rho}}, \quad (t \in [t_0, T)). \tag{2.13}$$

By Lemma 1.1.2, we obtain for any $\varepsilon > 0$,

$$z^{\frac{q}{p}}(t) \leq \frac{q}{p}\varepsilon^{\frac{q-p}{p}}z(t) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}. \tag{2.14}$$

Using (2.8) and (2.14), one has for any $\varepsilon > 0$ that

$$\begin{aligned} S(t, z^{\frac{q}{p}}(t)) &\le S(t, \tfrac{q}{p}\varepsilon^{\frac{q-p}{p}} z(t) + \tfrac{p-q}{p}\varepsilon^{\frac{q}{p}}) \\ &\le S(t, \tfrac{p-q}{p}\varepsilon^{\frac{q}{p}}) + L\tfrac{q}{p}\varepsilon^{\frac{q-p}{p}} z(t). \end{aligned} \tag{2.15}$$

From (2.13) and (2.15), we have

$$z(t) \le a(t) + b(t)\int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1}[S(\tau, \frac{p-q}{p}\varepsilon^{\frac{q}{p}}) + L\frac{q}{p}\varepsilon^{\frac{q-p}{p}} z(\tau)]\frac{d\tau}{\tau^{1-\rho}}, \quad (t \in [t_0, T)), \tag{2.16}$$

the inequality (2.16) can be reformulated as

$$z(t) \le \widetilde{a}(t) + \widetilde{b}(t)\int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} z(\tau)\frac{d\tau}{\tau^{1-\rho}}, \tag{2.17}$$

where $\widetilde{a}$ and $\widetilde{b}$ are defined as in (2.11).

Applying Theorem 1.1.2 to (2.17) and using (2.12), we can get the desired inequality in (2.10). ■

Remark 1.2.2 *If $p = q = 1$ and $S(t, x) = x$, then Theorem* 1.2.2 *reduces to Theorem* 1.1.2 *in* [1] .

Theorem 1.2.3 *Let $\alpha > 0, x(t), a(t)$ be nonnegative functions and $b(t)$ be nonnegative and nondecreasing function for $t \in [t_0, T)$,$T > 0$, $b(t) \le M$, where M is a constant. Furrther, let $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable increasing function on $]0, +\infty[$ with continuous nonincreasing first derivative g' on $]0, +\infty[$. If*

$$x^p(t) \le a(t) + b(t)\int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} g(x^q(\tau))\frac{d\tau}{\tau^{1-\rho}}, \tag{2.18}$$

where $p \ne 0$,$p \ge q > 0$, are constants.Then

$$x(t) \le [\ \widetilde{a}(t) + \int_{t_0}^{t} \sum_{n=1}^{\infty} \frac{(\widetilde{b}(\tau)\Gamma(\alpha))^n}{\Gamma(n\alpha)} (\frac{t^\rho - \tau^\rho}{\rho})^{n\alpha-1}\widetilde{a}(\tau)\frac{d\tau}{\tau^{1-\rho}}]^{\frac{1}{p}}, t \in [t_0, T), \tag{2.19}$$

where

$$\widetilde{a}(t) = a(t) + \frac{1}{\alpha}(\frac{t^\rho - t_0^\rho}{\rho})^{\alpha} b(t) g(\frac{p-q}{p}\varepsilon^{\frac{q}{p}}),\ \widetilde{b}(t) = \frac{q}{p}\varepsilon^{\frac{q-p}{p}} g'(\frac{p-q}{p}\varepsilon^{\frac{q}{p}}) b(t). \tag{2.20}$$

Preuve. Denote the right-hand side of (2.18) by $z(t)$. Then we have

$$x(t) \leq z^{\frac{1}{p}}(t), \qquad (t \in [t_0, T))\,. \tag{2.21}$$

So it follows that

$$z(t) \leq a(t) + b(t) \int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} g(z^{\frac{q}{p}}(\tau)) \frac{d\tau}{\tau^{1-\rho}}, \quad (t \in [t_0, T))\,. \tag{2.22}$$

By Lemma 1.1.2, we obtain for any $\varepsilon > 0$ that

$$g(z^{\frac{q}{p}}(t)) \leq g(\frac{q}{p}\varepsilon^{\frac{q-p}{p}} z(t) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}), \tag{2.23}$$

applying the mean value Theorem for the function g, then for every $x \geq y > 0$ there exists $c \in \,]\, y, x[$such that

$$g(x) - g(y) = g^{'}(c)(x-y) \leq g^{'}(y)(x-y),$$

then

$$\begin{aligned} g(z^{\frac{q}{p}}(t)) &\leq g(\tfrac{q}{p}\varepsilon^{\frac{q-p}{p}} z(t) + \tfrac{p-q}{p}\varepsilon^{\frac{q}{p}}) \\ &\leq g(\tfrac{p-q}{p}\varepsilon^{\frac{q}{p}}) + g^{'}(\tfrac{p-q}{p}\varepsilon^{\frac{q}{p}})\tfrac{q}{p}\varepsilon^{\frac{q-p}{p}} z(t). \end{aligned} \tag{2.24}$$

From (2.22) and (2.24), we have

$$z(t) \leq a(t) + b(t) \int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} [g(\frac{p-q}{p}\varepsilon^{\frac{q}{p}}) + \frac{q}{p}\varepsilon^{\frac{q-p}{p}} g^{'}(\frac{p-q}{p}\varepsilon^{\frac{q}{p}}) z(\tau)] \frac{d\tau}{\tau^{1-\rho}}, \quad (t \in [t_0, T))\,, \tag{2.25}$$

the inequality (2.25) can be reformulated as

$$z(t) \leq \widetilde{a}(t) + \widetilde{b}(t) \int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} z(\tau) \frac{d\tau}{\tau^{1-\rho}}, \tag{2.26}$$

where $\widetilde{a}$ and $\widetilde{b}$ are defined as in (2.20).

Applying Theorem 1.1.2 to (2.26) and using (2.21), we can get the desired inequality in (2.19). ■

1.3 Applications

In this section, we will use the Grőnwall inequality mentioned in the previous section in order to investigate the boundedness and uniqueness of a certain fractional differential equation with generalized derivatives, on the order and the initial conditions. Consider the following initial value problem within generalized fractional derivatives:

$$D_{t_0+}^{\alpha,\rho}x(t) = f(t, x(t)), \tag{3.1}$$

and

$$I_{t_0+}^{1-\alpha,\rho}x(t)\mid_{t=t_0}= c, \tag{3.2}$$

where $0 < \alpha \leq 1$, $0 \leq t < T \leq \infty$ and $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function with respect to all its arguments. The Volterra integral equations corresponding to the problem (3.1) − (3.2) is as follows :

$$x(t) = c\frac{(\frac{t^\rho - t_0^\rho}{\rho})^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1}f(\tau, x(\tau))\frac{d\tau}{\tau^{1-\rho}}, 0 \leq t_0 \leq t \leq \infty. \tag{3.3}$$

It is clear that the cauchy problem(3.1) − (3.2) and the problem (3.3) are equivalent.

Exemple 1.3.1 *Assume that $f(t, x(t))$ satisfies*

$$|f(t, x(t))| \leq b(t)g\left(|x(t)|\right), \tag{3.4}$$

where g, b are defined as in Theorem 1.2.3, such that $g(0) = 0$,then we have the following explicit estimate for $x(t)$

$$|x(t)| \leq \widetilde{a}(t) + \int_{t_0}^{t}\sum_{n=1}^{\infty}\frac{(g^{'}(0)b(\tau))^n}{\Gamma(n\alpha)}(\frac{t^\rho - \tau^\rho}{\rho})^{n\alpha-1}\widetilde{a}(\tau)\frac{d\tau}{\tau^{1-\rho}}, t \in [t_0, T), \tag{3.5}$$

where

$$\widetilde{a}(t) = \frac{|c|}{\Gamma(\alpha)}(\frac{t^\rho - t_0^\rho}{\rho})^{\alpha-1} + \frac{1}{\Gamma(\alpha+1)}(\frac{t^\rho - t_0^\rho}{\rho})^{\alpha}b(t)g(0). \tag{3.6}$$

Preuve. The solution of the initial value problem (3.1) − (3.2) is given by

$$x(t) = \frac{c}{\Gamma(\alpha)}(\frac{t^\rho - t_0^\rho}{\rho})^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1}f(\tau, x(\tau))\frac{d\tau}{\tau^{1-\rho}}, 0 \leq t \leq \infty, \tag{3.7}$$

then

$$|x(t)| \leq \frac{|c|}{\Gamma(\alpha)}(\frac{t^\rho - t_0^\rho}{\rho})^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1}b(\tau)g\left(|x(\tau)|\right)\frac{d\tau}{\tau^{1-\rho}}, 0 \leq t_0 \leq t \leq \infty,$$

taking into-account that b is nondecresing fuction, we obtain that

$$|x(t)| \leq \frac{|c|}{\Gamma(\alpha)}(\frac{t^\rho - t_0^\rho}{\rho})^{\alpha-1} + \frac{b(t)}{\Gamma(\alpha)}\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} g\left(|x(\tau)|\right)\frac{d\tau}{\tau^{1-\rho}}, 0 \leq t_0 \leq t \leq \infty,$$

applying Theorem 1.2.3 to the last inequality, we obtain the desired inequality in (3.5).

■

Exemple 1.3.2 *Assume that*

$$|f(t,x) - f(t,\overline{x})| \leq b(t) g\left(|x - \overline{x}|\right),$$

where g, b are defined as in Theorem 1.2.3 *such that $g(0) = 0$ and $b(t)$ is nondecreasing function in $t \geq 0$. Then the Cauchy problem* $(3.1) - (3.2)$ *has has at most one solution.*

Preuve. *Suppose $x(t)$, $\overline{x}(t)$ are two solutions of the Cauchy problem* $(3.1) - (3.2)$*, then we have*

$$\begin{aligned} x(t) &= c\frac{(\frac{t^\rho - t_0^\rho}{\rho})^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} f(\tau, x(\tau))\frac{d\tau}{\tau^{1-\rho}}, 0 \leq t_0 \leq t \leq \infty. \\ \overline{x}(t) &= c\frac{(\frac{t^\rho - t_0^\rho}{\rho})^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} f(\tau, \overline{x}(\tau))\frac{d\tau}{\tau^{1-\rho}}, 0 \leq t_0 \leq t \leq \infty. \end{aligned}$$

Then, we have

$$x(t) - \overline{x}(t) = \frac{1}{\Gamma(\alpha)}\int_{0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1}(f(\tau, x(\tau)) - f(\tau, \overline{x}(\tau)))\frac{d\tau}{\tau^{1-\rho}},$$

which implies that

$$|x(t) - \overline{x}(t)| \leq \frac{1}{\Gamma(\alpha)}\int_{0}^{x}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} b(\tau) g\left(|x(\tau) - \overline{x}(\tau)|\right)\frac{d\tau}{\tau^{1-\rho}}.$$

Taking into account that b is nondecreasing function, one gets

$$|x(t) - \overline{x}(t)| \leq \frac{b(t)}{\Gamma(\alpha)}\int_{0}^{x}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} g\left(|x(\tau) - \overline{x}(\tau)|\right)\frac{d\tau}{\tau^{1-\rho}}. \tag{3.8}$$

Through a suitable application of Theorem 1.2.3 *to* (3.8) (*with* $p = q = 1$), *we obtain that* $|x(t) - \overline{x}(t)| \leq 0$, *which implies* $x(t) = \overline{x}(t)$. ■

Conclusion In this chapter, some new inequalities fractional of Grőnwall-Bihari type,were derived. The obtained inequalities are extensions of many results given in [1]. They can be help in the study of boundedness, uniqueness for the solution to some certain initial value problems.

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CHAPTER 2

Further new generalizations of certain Gamidov integral inequalities in two independent variables and their applications

2.1 Introduction

Integral inequalities play a dominant role in the study quantitative properties of solutions of differential and integral equations. For instance, see $[1, 10, 11, 13-16, 19-22]$ and the references given therein. During the past few years an enormous amount of effort has been devoted to the discovery of new types of inequalities and their applications in various branches of ordinary and partial differential and integral equations see $[2-9, 18]$.

In [11] , Sh.G.Gamidov while studying the boundary value poblem for higher order differential equations, initiated the study of obtaining explicit upper bounds on the integral inequalities of the forms :

$$x(t) \leq \delta + \int_a^t h(s)x(s)ds + \int_a^b m(s)x(s)ds, \tag{1.1}$$

for $t \in [a, b]$, under some suitable conditions on the functions involved in (1.1).

Pachpatte in [17], established the following inequality :

$$x(t) \leq \alpha(t) + \int_a^t \Psi(t, s)x(s)ds + \int_a^b m(s)x(s)ds. \tag{1.2}$$

K. Cheng, C. Guo in [8] discussed the following general version in two independent variables as follows :

$$x(s,t) \leq \alpha(s,t)+\beta(s,t)\int_0^s \int_0^t \Psi(\eta,\tau)x(\eta,\tau)d\eta d\tau+\gamma(s,t)\int_0^A \int_0^B m(\eta,\tau)x(\eta,\tau)d\eta d\tau, \tag{1.3}$$

for $(s,t) \in [0, A] \times [0, B]$.

In this chapter, we are interested in the study of some properties of the following integral equation in two independent variables.

$$x(s,t) = \alpha(s,t) + \beta(s,t)\int_0^s \int_0^t \Phi(\eta,\tau,x)d\eta d\tau + \gamma(s,t)\int_0^A \int_0^B H(\eta,\tau,x)d\eta d\tau, \tag{1.4}$$

where, we derive some new results on Pachpatte-Gamidov-type inequalities, which can be used in the analysis of (I.E)(see General Introduction). To show the feasibility of the obtained inequalities, some illustrative examples are also introduce.

Now, we give some Lemmas which are used in our work.

Lemma 2.1.1 [8] *Assume* $x(s,t)$, $\alpha(s,t)$, $\gamma(x,y), m(s,t) \in C([0,A] \times [0,B], [0,\infty))$ *and*

$$x(s,t) \leq \alpha(s,t) + \gamma(s,t)\int_0^A \int_0^B m(\eta,\tau)x(\eta,\tau)d\eta d\tau,$$

for $(s,t) \in [0, A] \times [0, B]$.

If

$$\int_0^A \int_0^B \gamma(\eta,\tau)m(\eta,\tau)d\eta d\tau < 1,$$

then the following explicit estimate

$$x(s,t) \leq \alpha(s,t) + \frac{\gamma(s,t)\int_0^A \int_0^B \alpha(\eta,\tau)m(\eta,\tau)d\eta d\tau}{1-\int_0^A \int_0^B \gamma(\eta,\tau)m(\eta,\tau)d\eta d\tau},$$

hold for $(s,t) \in [0,A] \times [0,B]$.

Lemma 2.1.2 [12] *Assume that* $\sigma \geq 0$, $p \geq q \geq 0$ *and* $p \neq 0$, *then*

$$\sigma^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} \sigma + \frac{p-q}{p} K^{\frac{q}{p}}, \tag{1.5}$$

for any $K > 0$.

2.2 Main Results

In what follows, $\mathbb{R}$ denotes the set of real numbers $\mathbb{R}_+ = [0,\infty)$, $J_1 = [0,A]$, and $J_2 = [0,B]$ are given subsets of $\mathbb{R}$. Let $\Gamma = J_1 \times J_2$, $C(W,\Sigma)$ denotes the collection of continuous functions from W to Σ. Now let's give the main results of this paper.

Lemma 2.2.1 *Assume* $x(s,t)$, $\alpha(s,t)$, $\gamma(s,t)$, $m(s,t) \in C(\Gamma,\mathbb{R}_+)$ *and* $S : \mathbb{R}^3_{+\to}\mathbb{R}_+$ *be a continuous function which satisfies the condition*

$$0 \leq S(s,t,x) - S(s,t,y) \leq R(s,t,y)(x-y), \tag{2.1}$$

for $x \geq y \geq 0$, where $R(s,t,y)$ is a nonnegative continuous function defined for $s,t,y \in \mathbb{R}_+$.

If $x(s,t)$ satisfies

$$x(s,t) \leq \alpha(s,t) + \gamma(s,t)\int_0^A \int_0^B m(\eta,\tau)S(\eta,\tau,x(\eta,\tau))d\eta d\tau, \tag{2.2}$$

for $(s,t) \in \Gamma$, then

$$x(s,t) \leq \alpha(s,t) + \frac{\gamma(s,t)\int_0^A \int_0^B m(\eta,\tau)S(\eta,\tau,\alpha(s,t))d\eta d\tau}{1-\int_0^A \int_0^B \gamma(\eta,\tau)m(\eta,\tau)R(\eta,\tau,\alpha(s,t))d\eta d\tau}, \tag{2.3}$$

holds for $(s,t) \in \Gamma$, when

$$\int_0^A \int_0^B \gamma(\eta,\tau) m(\eta,\tau) R(\eta,\tau,\alpha(\eta,\tau)) d\eta d\tau < 1.$$

Preuve. It is clear that, $\int_0^A \int_0^B m(\eta,\tau) S(\eta,\tau,x(\eta,\tau)) d\eta d\tau$ is a constant.

Setting

$$\phi = \int_0^A \int_0^B m(\eta,\tau) S(\eta,\tau,\alpha(s,t)) d\eta d\tau, \tag{2.4}$$

from (2.2), we have

$$x(s,t) \leq \alpha(s,t) + \gamma(s,t)\phi. \tag{2.5}$$

and

$$0 \leq S(s,t,x(s,t)) \leq S(s,t,\alpha(s,t) + \gamma(s,t)\phi). \tag{2.6}$$

From (2.1), we have

$$0 \leq S(x,y,x(s,t)) \leq S(x,y,\alpha(s,t)) + R(x,y,\alpha(s,t))\gamma(s,t)\phi. \tag{2.7}$$

As $m(s,t)$ is positive function, then

$$m(s,t)S(s,t,x(s,t)) \leq m(s,t)\left\{S(s,t,\alpha(s,t)) + R(s,t,\alpha(s,t))\gamma(s,t)\phi\right\}. \tag{2.8}$$

Integrating botth sides of (2.8) on Γ, we get

$$\begin{array}{rl} \varphi = & \int_0^A \int_0^B m(\eta,\tau) S(\eta,\tau,x\eta,\tau) d\eta d\tau \\ \leq & \phi \int_0^A \int_0^B \gamma(\eta,\tau) m(\eta,\tau) R(\eta,\tau,\alpha(\eta,\tau)) d\eta d\tau \\ & + \int_0^A \int_0^B m(\eta,\tau) S(\eta,\tau,\alpha(\eta,\tau)) d\eta d\tau \end{array} \tag{2.9}$$

It is easy to see that

$$\phi \leq \frac{\int_0^M \int_0^N m(\eta,\tau) S(s,t,\alpha(s,t)) d\eta d\tau}{1 - \int_0^M \int_0^N \gamma(\eta,\tau) m(\eta,\tau) R(s,t,\alpha(s,t)) d\eta d\tau}. \tag{2.10}$$

Substituting the inequality (2.10) into (2.5), we get the explicit estimate (2.3) this completes the proof. ■

Remark 2.2.1 *Setting $S(x,y,x(s,t)) = x(s,t)$,we see that the obtained inequality is as seen in in Lemma* 1.2.1 *and if we replace $S(x,y,x(s,t))$ by $n(x(s,t))$ where n is is a differentiable increasing function on* $]0,+\infty[$ *with continuous decreasing first derivative on* $]0,+\infty[$*, one can easily derive Lemma* 2.1 *in* [6] .

Theorem 2.2.1 *Assume $\alpha(s,t)$, $\beta(s,t)$, $\gamma(s,t)$, $\psi(x,y)$, $m(s,t) \in C(\Delta, \mathbb{R}_+)$,and $\alpha(s,t)$, $\beta(s,t)$, $\gamma(s,t)$ are nondecreasing in s and t. If $x(s,t) \in C(\Gamma, \mathbb{R}_+)$ satisfies*

$$\begin{aligned} x(s,t) &\leq \alpha(s,t) + \beta(s,t) \textstyle\int_0^s \int_0^t \psi(\eta,\tau) x(\eta,\tau) d\eta d\tau \\ &+\gamma(s,t) \textstyle\int_0^A \int_0^B m(\eta,\tau) S(\eta,\tau,x(\eta,\tau)) d\eta d\tau \end{aligned} \tag{2.11}$$

then,

$$x(s,t) \leq M^*(x,y) + N^*(x,y) \times \frac{\int_0^A \int_0^B m(\eta,\tau) S(\eta,\tau,M^*(\eta,\tau)) d\eta d\tau}{1 - \int_0^A \int_0^B N^*(\eta,\tau) m(\eta,\tau) R(\eta,\tau,M^*(\eta,\tau)) d\eta d\tau}, \tag{2.12}$$

for $(s,t) \in \Gamma$, where

$$\begin{aligned} M^*(x,y) &= \alpha(s,t) \exp\left\{ \beta(s,t) \textstyle\int_0^s \int_0^t \psi(\eta,\tau) d\eta d\tau \right\}, \\ N^*(x,y) &= \gamma(s,t) \exp\left\{ \beta(s,t) \textstyle\int_0^s \int_0^t \psi(\eta,\tau) d\eta d\tau \right\}, \end{aligned} \tag{2.13}$$

and S (also R) is introduced as in Lemma 2.2.1,with

$$\int_0^A \int_0^B N^*(s,t) m(\eta,\tau) R(\eta,\tau,M^*(\eta,\tau)) d\eta d\tau < 1, \tag{2.14}$$

Preuve. Fixing any arbitrary $(S,T) \in \Gamma$, then for $(s,t) \in \Gamma_1 = [0,S] \times [0,T]$, from (2.11) and taking into account that $\alpha(s,t)$, $\beta(s,t)$, and $\gamma(s,t)$ are nondecreasing in s and t, we have

$$\begin{aligned} x(s,t) &\leq \alpha(S,T) + \beta(S,T) \textstyle\int_0^s \int_0^t \psi(\eta,\tau) x(\eta,\tau) d\eta d\tau \\ &+\gamma(S,T) \textstyle\int_0^A \int_0^B m(s,t) S(\eta,\tau,x(\eta,\tau)) d\eta d\tau. \end{aligned} \tag{2.15}$$

Define a function $z(s,t), (s,t) \in \Gamma_1$ by the right side of (2.15). It is easy tp see that $z(s,t)$ is postive and nondecreasing in s and t, then

$$x(s,t) \leq z(s,t) \ , \tag{2.16}$$

Further, we have

$$z(0,y) = \alpha(S,T) + \gamma(S,T)\int_0^A\int_0^B m(\eta,\tau)S(\eta,\tau,x(\eta,\tau)d\eta d\tau, \tag{2.17}$$

$$\begin{aligned} \tfrac{\partial}{\partial s}z(s,t) &= \beta(S,T)\textstyle\int_0^t \psi(s,\tau)x(s,\tau)d\tau \\ &\le \beta(S,T)\textstyle\int_0^t \psi(s,\tau)z(s,\tau)d\tau \\ &\le (\beta(S,T)\textstyle\int_0^t \psi(s,\tau)d\tau)z(s,\tau). \end{aligned} \tag{2.18}$$

Since $z(s,t)$ is nondecreasing in t, from (2.18), one gets

$$\frac{(\partial/\partial s)\, v(s,t)}{z(s,t)} \le B(S,T)\int_0^t \psi(s,\tau)dt, \tag{2.19}$$

keeping t fixed in (2.19), letting $s=\tau$, and integrating it from 0 to s, we obtain

$$z(s,t) \le \ z(0,t)\exp\left\{B(S,T)\int_0^s\int_0^t \psi(\eta,\tau)d\eta d\tau\right\}. \tag{2.20}$$

Combining (2.16) and (2.17) , we have

$$\begin{aligned} x(s,t) \ &\le \left[\alpha(S,T) + \gamma(S,T)\textstyle\int_0^A\int_0^B m(\eta,\tau)S(\eta,\tau,x(\eta,\tau))d\eta d\tau\right] \\ &\times \exp\left\{\beta(S,T)\textstyle\int_0^s\int_0^t \psi(\eta,\tau)d\eta d\tau\right\} \\ &= \alpha(S,T)\exp\left\{\beta(S,T)\textstyle\int_0^s\int_0^t \psi(\eta,\tau)d\eta d\tau\right\} \\ &+\gamma(S,T)\exp\left\{\beta(S,T)\textstyle\int_0^s\int_0^t \psi(\eta,\tau)d\eta d\tau\right\} \\ &\times \textstyle\int_0^A\int_0^B m(\eta,\tau)S(\eta,\tau,x(\eta,\tau))d\eta d\tau \\ &= M_1(s,t,S,T) + N_1(s,t,S,T)\textstyle\int_0^A\int_0^B m(\eta,\tau)S(\eta,\tau,x(\eta,\tau))d\eta d\tau, \end{aligned} \tag{2.21}$$

where

$$\begin{aligned} M_1(s,t,S,T) \ &= \alpha(S,T)\exp\left\{\beta(S,T)\textstyle\int_0^s\int_0^t \psi(\eta,\tau)d\eta d\tau\right\}, \\ N_1(s,t,S,T) \ &= \gamma(S,T)\exp\left\{\beta(S,T)\textstyle\int_0^s\int_0^t \psi(\eta,\tau)d\eta d\tau\right\}. \end{aligned} \tag{2.22}$$

Using Lemma 2.2.1 from (2.21), we obtain

$$\begin{aligned} x(s,t) \ &\le M_1(s,t,S,T) + N_1(s,t,S,T) \\ &\times \tfrac{\int_0^A\int_0^B m(\eta,\tau)S(\eta,\tau,M_1(\eta,\tau,S,T))d\eta d\tau}{1-\int_0^A\int_0^B N_1(s,t,X,Y)m(\eta,\tau)R(s,t,M_1(s,t,S,T))d\eta d\tau}, \end{aligned} \tag{2.23}$$

since the inequality (2.23) holds for all $(s,t)\in\Gamma_1$, taking $s=S$ and $t=T$, we have

$$\begin{aligned} x(s,t) \ &\le M_1(S,T,S,T) + N_1(S,T,S,T) \\ &\times \tfrac{\int_0^A\int_0^B m(\eta,\tau)S(\eta,\tau,M_1(\eta,\tau,S,T))d\eta d\tau}{1-\int_0^A\int_0^B m(\eta,\tau)M_1(\eta,\tau,S,T)R(\eta,\tau,M_1(\eta,\tau,S,T))d\eta d\tau} \\ &= M^*(S,T) + N^*(S,T)\times \tfrac{\int_0^A\int_0^B m(\eta,\tau)S(\eta,\tau,M^*(\eta,\tau))d\eta d\tau}{1-\int_0^A\int_0^B N^*(\eta,\tau)m(\eta,\tau)R(\eta,\tau,M^*(\eta,\tau))d\eta d\tau}, \end{aligned} \tag{2.24}$$

for $(S,T)\in\Gamma$, where $M^*(X,Y), N^*(X,Y)$ are defined as in (2.13).

Taking into account that S and T are arbitrary, we replace S and T by s and t, respectively, one gets the desired inequality.This completes the proof. ■

Remark 2.2.2 *If we take $S(s,t,x(s,t)) = x(s,t)$, Theorem* 2.2.1 *will be reduced to Theorem* 2 *in* [8] *and if we replace $S(s,t,x(s,t))$ by $n(x(s,t))$ where n is defined as in Remark* 2.2.1, *one can easily derive Theorem* 2.1 *in* [6].

Theorem 2.2.2 *Assume $\alpha(s,t)$, $\beta(s,t)$, $\gamma(s,t)$, $\psi(s,t)$ and $m(s,t)$ are defined as in Theorem* 1. *If $x(s,t)\in C(\Gamma,\mathbb{R}_+)$ satisfies*

$$\begin{aligned} x^p(s,t) &\le \alpha(s,t)+\beta(s,t)\int_0^s\int_0^t\psi(\eta,\tau)x^q(\eta,\tau)d\eta d\tau \\ &+\gamma(s,t)\int_0^A\int_0^B m(\eta,\tau)S(\eta,\tau,x(\eta,\tau))d\eta d\tau, \end{aligned} \tag{2.25}$$

where $p\ge q\ge 0$, $p\ge 1$ and p,q are constants, then,

$$x(s,t)\le M^*(s,t)+N^*(s,t)\times\frac{\int_0^A\int_0^B m^*(s,t)S(\eta,\tau,M^*(\eta,\tau))d\eta d\tau}{1-\int_0^A\int_0^B N^*(\eta,\tau)m^*(\eta,\tau)R(\eta,\tau,M^*(\eta,\tau))d\eta d\tau}, \tag{2.26}$$

holds for $(s,t)\in\Gamma$, where

$$\begin{aligned} M^*(s,t) &= M_1(s,t)\exp\left\{B_1(s,t)\int_0^s\int_0^t\psi^*(\eta,\tau)d\eta d\tau\right\}, \\ N^*(s,t) &= N_1(s,t)\exp\left\{B_1(s,t)\int_0^s\int_0^t\psi^*(\eta,\tau)d\eta d\tau\right\}. \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} &M_1(x,y)=\tfrac{1}{p}K^{\frac{1-p}{p}}\beta(s,t)\int_0^s\int_0^t\psi(\eta,\tau)\left[\tfrac{q}{p}K^{(q-p)/p}\alpha(\eta,\tau)+\tfrac{p-q}{p}\ K^{q/p}\right]d\eta d\tau \\ &+\tfrac{1}{p}K^{\frac{1-p}{p}}\alpha(s,t)+\tfrac{p-1}{p}K^{\frac{1}{p}}, \\ &B_1(s,t)=\tfrac{q}{p}K^{(q-p)/p}\beta(s,t),\quad N_1(s,t)=\tfrac{1}{p}K^{\frac{1-p}{p}}\gamma(s,t), \\ &\psi^*(s,t)=\psi(s,t),\quad m^*(s,t)=m(s,t), \end{aligned} \tag{2.28}$$

and S (also R) is introduced as in Lemma 2.2.1,with

$$\int_0^A \int_0^B N^*(\eta,\tau)M^*(\eta,\tau)R(\eta,\tau,M^*(\eta,\tau))d\eta d\tau < 1.$$

Preuve. Define a function $\theta(s,t)$ as follows

$$\begin{aligned}\theta(s,t) &= \beta(s,t)\int_0^s\int_0^t \psi(\eta,\tau)x^q(\eta,\tau)d\eta d\tau \\ &+\gamma(s,t)\int_0^A\int_0^B m(\eta,\tau)S(\eta,\tau,x(\eta,\tau))d\eta d\tau,\end{aligned} \tag{2.29}$$

for $(s,t)\in\Gamma$. Then, from (2.29) ,we obtain

$$x^p(s,t) \le \alpha(s,t)+\theta(s,t). \tag{2.30}$$

Applying Lemma 2.1.2, we have

$$\begin{aligned} x(s,t) &\le (\alpha(s,t)+\theta(s,t))^{1/p} \ \le \tfrac{1}{p}K^{\frac{1-p}{p}}(\alpha(s,t)+\theta(s,t)) + \tfrac{p-1}{p}K^{\frac{1}{p}} \\ &= \lambda(s,t). \\ x^q(s,t) &\le (\alpha(s,t)+\theta(s,t))^{q/p} \ \le \tfrac{q}{p}K^{(q-p)/p}(\alpha(s,t)+\theta(s,t)) \\ &+\tfrac{p-q}{p}\ K^{q/p}, \\ \tfrac{1}{p}K^{\frac{1-p}{p}}\theta(s,t) &\le \lambda(s,t). \end{aligned} \tag{2.31}$$

It follows from (2.31) , we easily obtain

$$\begin{aligned}\theta(s,t) &\le \beta(s,t)\int_0^s\int_0^t \psi(\eta,\tau)\times\left[\tfrac{q}{p}K^{(q-p)/p}(\alpha(\eta,\tau)+\theta(\eta,\tau)) \ + \tfrac{p-q}{p}\ K^{q/p}\right]d\eta d\tau \\ &+\gamma(s,t)\int_0^A\int_0^B m(\eta,\tau)S(\eta,\tau,\lambda(\eta,\tau)d\eta d\tau,\end{aligned} \tag{2.32}$$

from the last inequality of (2.31), we deduce that

$$\begin{aligned}\theta(s,t) &\le \beta(s,t)\int_0^s\int_0^t \psi(\eta,\tau)\left[\tfrac{q}{p}K^{(q-p)/p}\alpha(\eta,\tau)+\tfrac{p-q}{p}\ K^{q/p}\right]d\eta d\tau \\ &+qK^{(q-1)/p}\beta(s,t)\int_0^s\int_0^t \psi(\eta,\tau)\lambda(\eta,\tau)d\eta d\tau \\ &+\gamma(s,t)\ \int_0^A\int_0^B m(\eta,\tau)\ S(\eta,\tau,\lambda(\eta,\tau))d\eta d\tau,\end{aligned} \tag{2.33}$$

By a simple computation, we obtain

$$\begin{aligned}\lambda(s,t) &\le M_1(s,t)+B_1(s,t)\int_0^s\int_0^t \psi^*(\eta,\tau)\lambda(\eta,\tau)d\eta d\tau \\ &+N_1(s,t)\int_0^A\int_0^B m^*(s,t)S(\eta,\tau,\lambda(\eta,\tau))d\eta d\tau,\end{aligned} \tag{2.34}$$

where $M_1(s,t), B_1(s,t), N_1(s,t), \psi^*(s,t)$ and $m^*(s,t)$ are defined as in (2.28).

Remarking that $M_1(s,t), B_1(s,t)$ and $N_1(s,t)$ are nonnegative, continuous, and nondecreasing for $(s,t) \in \Gamma$.A suitable application of Theorem 2.2.1 to (2.34) gives

$$\begin{aligned} x(s,t) &\leq \lambda(s,t) \leq M^*(s,t) + N^*(s,t) \\ &\times \frac{\int_0^A \int_0^B m^*(\eta,\tau) S(\eta,\tau,M^*(\eta,\tau)) d\eta d\tau}{1-\int_0^A \int_0^B N^*(\eta,\tau) m^*(\eta,\tau) R(\eta,\tau,M^*(\eta,\tau)) d\eta d\tau}, \end{aligned} \tag{2.35}$$

where $M^*(x,y)$ and $N^*(x,y)$ are defined as in (2.27). ■

Remark 2.2.3 *If we take* $S(s,t,x(s,t)) = x(s,t)$,$p = 2, q = 1$, *Theorem* 2.2.2 *will be reduced to corollary* 8 *in* [8] *and if we replace* $S(s,t,x(s,t))$ *by* $n(x(s,,t))$ *where* n *is is defined as in Remark* 1.2.1, *one can easily derive Theorem* 2.2 *in* [6].

2.3 Illustrative Examples

In this section, we apply some inequalities obtained in the previous sections to investigate certain properties of the solution of integral equation in two independent variables.

Exemple 2.3.1 *Consider the following intergral equation:*

$$x(s,t) = \alpha(s,t) + \beta(s,t) \int_0^s \int_0^t \Phi(\eta,\tau,x) d\eta d\tau + \gamma(s,t) \int_0^A \int_0^B H(\eta,\tau,x) d\eta d\tau, \tag{3.1}$$

for $(s,t) \in \Gamma$, where $x(s,t) \in C(\Gamma,\mathbb{R}), \alpha(s,t), \beta(s,t), \gamma(s,t) \in C(\Gamma,\mathbb{R}_+)$ such that $\alpha(s,t)$, $\beta(s,t)$, $\gamma(s,t)$ are nondecreasing in s and t and $\Phi(s,t,x), H(s,t,x) \in C(\Gamma \times \mathbb{R},\mathbb{R})$.

Theorem 2.3.1 *Assume that the functions* Φ *and* H *in* (3.1) *satisfy the conditions*

$$|\Phi(s,t,x)| \leq \psi(s,t)\,|x|\,,\ |H(s,t,x)| \leq m(s,t) S(s,t,x), \tag{3.2}$$

where $S, R : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}_+$ are nonnegative continuous function satisfy the following properties

$$\begin{aligned} S(s,t,v) &\leq S(s,t,u), \quad (s,t)\in\mathbb{R}_+^2, v\leq u, u,v\in\mathbb{R} \\ S(s,t,u)-S(s,t,v) &\leq R(s,t,v)(u-v), (s,t)\in\mathbb{R}_+^2, 0\leq v\leq u. \end{aligned} \tag{3.3}$$

where $\psi(s,t)$ and $m(\eta,\tau)$ are defined as in Theorem 2.2.1.

If $x(s,t)$ is the unique solution of (3.1) ,then

$$|x(s,t)| \leq M^*(s,t) + N^*(s,t) \times \frac{\int_0^A \int_0^B m(\eta,\tau)S(\eta,\tau,M^*(\eta,\tau))d\eta d\tau}{1-\int_0^A \int_0^B N^*(\eta,\tau)m(\eta,\tau)R(\eta,\tau,M^*(\eta,\tau))d\eta d\tau}, \tag{3.4}$$

for $(s,t)\in\Gamma$ where $M^*(s,t), N^*(s,t)$ are defined in (2.13), with

$$\int_0^A \int_0^B N^*(\eta,\tau)m(\eta,\tau)R(\eta,\tau,M^*(\eta,\tau))d\eta d\tau < 1. \tag{3.5}$$

Preuve. Assume that $x(s,t)$ is the unique solution of (3.1) , from (3.2) and (3.3) we have

$$\begin{aligned} |x(s,t)| &\leq \alpha(s,t)+\beta(s,t)\int_0^s \int_0^t \psi(\eta,\tau)\,|x(\eta,\tau)|\,d\eta d\tau \\ &+\gamma(s,t)\int_0^A \int_0^B m(\eta,\tau)S(\eta,\tau,|x(\eta,\tau)|)d\eta d\tau. \end{aligned} \tag{3.6}$$

Now by applying Theorem 2.2.1 to (3.6), the required estimation (3.4) is obtained.

■

Proposition 2.3.1 *Assume that the functions* Φ *and* H *in* (3.1) *satisfy the conditionS*

$$|\Phi(s,t,x)-\Phi(s,t,\overline{x})| \leq \psi(s,t)\,|x-\overline{x}|\,, \, |H(s,t,x)-H(s,t,\overline{x})| \leq m(s,t)S(s,t,x-\overline{x}), \tag{3.7}$$

where $\psi(s,t)$, $m(s,t)$ are defined as in Theorem 2.2.1 with $S(s,t,0)=0$. If

$$\int_0^A \int_0^B N^*(\eta,\tau)m(\eta,\tau)R(\eta,\tau,M^*(\eta,\tau))d\eta d\tau < 1,$$

where $M^*(x,y)$and $N^*(x,y)$ are defined as in Theorem 2.2.1, and $x(s,t)$ is a solution of (3.1), then (3.1) has at most one solution.

Preuve. Let $x(s,t)$ and $\overline{x}(s,t)$ be two solutions of (3.1), then

$$\begin{aligned} x(s,t) &= \alpha(s,t) + \beta(s,t) \int_0^s \int_0^t \Phi(\eta,\tau,x) d\eta d\tau + \gamma(s,t) \int_0^A \int_0^B H(\eta,\tau,x) d\eta d\tau, \\ \overline{x}(s,t) &= \alpha(\eta,\tau) + \beta(s,t) \int_0^s \int_0^t \Phi(\eta,\tau,\overline{x}) d\eta d\tau + \gamma(s,t) \int_0^A \int_0^B H(\eta,\tau,\overline{x}) d\eta d\tau, \end{aligned} \tag{3.8}$$

From (3.8) , we have

$$\begin{aligned} |x(s,t) - \overline{x}(s,t)| &\leq \beta(s,t) \int_0^s \int_0^t |\Phi(\eta,\tau,x) - \Phi(\eta,\tau,\overline{x})| \, d\eta d\tau \\ &+ \gamma(s,t) \int_0^A \int_0^B |H(\eta,\tau,x) - H(\eta,\tau,\overline{x})| \, d\eta d\tau \\ &\leq \beta(s,t) \int_0^s \int_0^t \psi(\eta,\tau) \, |x - \overline{x}| \, d\eta d\tau \\ &+ \gamma(s,t) \int_0^A \int_0^B m(\eta,\tau) S(\eta,\tau,|x - \overline{x}|) d\eta d\tau. \end{aligned} \tag{3.9}$$

According to Theorem 2.2.1, we obtain that $|x(s,t) - \overline{x}(s,t)| \leq 0$, which implies $x(s,t) = \overline{x}(s,t)$ for $(s,t) \in \Gamma$. ■

Conclusion In this chapter, some new inequalities of Pachpatte-Gamidov type in two independent variables were derived .The obtained inequalities are extensions of many results given in [6] . They can be help in the study of some classes of integral equations.

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CHAPTER 3

Exponential stability for nonlinear perturbed time scales systems with Grőnwall-Bihari-inequalities

3.1 Introduction and preliminaries

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [15] in his Ph. D. thesis in 1988 in order to unify continuous and discrete analysis. A great deal of work has been done since 1988, unifying the theory of differential equations and the theory of difference equations by establishing the corresponding results in time scale setting. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. During the last decades, time scale methods have rapidly been developed, and have received a lot of attention by several authors, not only to unify continuous and discrete processes, but also help reveal diversities in the corresponding results.The analysis of nonlinear perturbations of linear systems is not only important for its own sake but also has a broad range of applications. One of the analytic methods of the perturbation theory was referred to integral inequalities to quest some type of stability. Latterly, there have been several papers[2, 3, 4, 6, 9, 10, 11, 12, 13, 17, 18, 19], studying various types of stability of dynamical time scale systems.

In this chapter , we investigate uniform exponential stability for nonlinear perturbed systems on time scales by using the Grönwall-Bihari type inequality and Pachpatte type inequality.

The chapter is organized as follows: in Section 2, provides a brief review of the time scale theory and integral inequalities which play an important rôle in our analysis. In Section 3, contains the statements and proofs of our main results. Section 4 shows the applicability of the theoretical results by numerical examples.

First, we will briey mention some basic definitions and results of time scale calculus for reader's convenience, as they are detailed in the books of M. Bohner and A. Peterson [7, 8].

3.1.1 Time scale calculus

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, \infty)$ is the given subset of $\mathbb{R}$ and $\mathbb{T}$ is an arbitrary time scale. The forward and backward jump operators $\sigma, \rho : T \to T$ are defined by $\sigma(t) := \inf \{s \in \mathbb{T} : s > t\}$, $\rho(t) := \sup \{s \in \mathbb{T} : s < t\}$. C_{rd} denotes the set of rd-continuous functions and the set T^k which is derived from the time scale $\mathbb{T}$as follows: If T has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}$-$\{m\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$, The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu p \neq 0$ on $\mathbb{T}^k$. $\mathfrak{R}$ denotes the set of all regressive and rd-continuous functions. We define the set of all positively regressive functions by

$$\mathfrak{R}^+ = \{p \in \mathfrak{R} : 1 + \mu(t)p(t) > 0 \text{ for } \text{ all } t \in \mathbb{T}\}.$$

Also, we define the interval $[a, b]$ means the set $\{t \in \mathbb{T} : a \leq t \leq b\}$ for the points $a < b$ in $\mathbb{T}$, If $b = +\infty$, we denote $\mathbb{T}_a^+ = [a, +\infty[_{\mathbb{T}}$.

Definition 3.1.1 [7] *If $p \in \Re(\mathbb{T},\mathbb{R})$, then we define the generalized exponential function $e_p(t,t_0)$ by*

$$e_p(t,t_0) = \exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \text{ for } t,t_0 \in \mathbb{T},$$

where $\xi_h(z)$ is the cylinder transformation given by

$$\begin{aligned} \xi_h(z) &= \frac{1}{h}\log(1+zh), \quad \text{if } h \neq 0, \\ \xi_0(z) &= z, \quad \text{if } h = 0, \end{aligned}$$

Lemma 3.1.1 [7]*For a nonnegative p whith $-p \in \Re^+$, we have the inequalities*

$$1 - \int_{t_0}^{t} p(u)\,\Delta u \le e_{-p}(t,t_0) \le \exp\left\{-\int_{t_0}^{t} p(u)\,\Delta u\right\} \quad \text{for all } t \in \mathbb{T}_{t_0}^{+}.$$

Lemma 3.1.2 [7] *Let $t_0 \in \mathbb{T}^k$ and assume $L : \mathbb{T}\times\ \mathbb{T}^k \to \mathbb{R}$ is continuous at (t,t), where $t \in \mathbb{T}^k$ with $t > t_0$. Also assume that for each $t \in \mathbb{T}^k$, $L^{\Delta_t}(t,.)$ is rd-continuous on $[t_0,\sigma(t)]$ and for each $\varepsilon > 0$, there exists a neighborhood U_t of t, independent of $\tau \in [t_0,\sigma(t)]$, such that*

$$\left|L(\sigma(t),\tau) - L(s,\tau) - L^{\Delta_t}(t,\tau)(\sigma(t)-s)\right| \le \varepsilon\,|\sigma(t)-s| \quad \text{for all } s \in U_t$$

where L^{Δ_t} denotes the derivative of L with respect to the first variable. Then

$$g(t) := \int_{t_0}^{t} L(t,\eta)\Delta\eta \quad \text{implies} \quad g^{\Delta}(t) := \int_{t_0}^{t} L^{\Delta_t}(t,\eta)\Delta\eta + L(\sigma(t),t).$$

Lemma 3.1.3 [6]*Assume that $z,m \in C_{rd}, n \in R^+$. If*

$$z^{\Delta}(t) \le n(t)z(t) + m(t)\ ,\ t \ge t_0, t \in \mathbb{T}^k.$$

Then

$$z(t) \le z(t_0)e_n(t,t_0) + \int_{t_0}^{t} m(s)e_n(t,\sigma(s))\Delta s, t \ge t_0 \qquad t \in \mathbb{T}^k.$$

Corollary 3.1.1 [7] *Let* $p \in \Re$ *and* $t, t_0, s \in T$, *then*

(i)$e_0(t, t_0) \equiv 1$ *and* $e_p(t, t) \equiv 1$,

(ii)$e_p(\sigma(t), t_0) = (1 + \mu(t) p(t)) e_p(t, t_0)$,

(iii)$e_p(t, t_0) e_p(t_0, s) = e_p(t, s)$,

(iv)$e_p(t, t_0) = \frac{1}{e_p(t_0, t)}$,

(v)If $p \in \Re^+$, *then* $e_p(t, t_0) > 0$ *for all* $t \in \mathbb{T}$

Exemple 3.1.1 *Let* $p \in \Re$, $t, s \in \mathbb{T}$ *and* $t \geq s$.

If $\mathbb{T} = \mathbb{R}$, *then* $e_p(t, s) = \exp\left(\int_s^t p(\tau)\, d\tau\right)$,

If $\mathbb{T} = \mathbb{R}$ *and* $p(t) \equiv \alpha$, *then* $e_p(t, s) = e^{\alpha(t-s)}$,

If $\mathbb{T} = \mathbb{Z}$, *then* $e_p(t, s) = \prod_{\tau=s}^{\tau=t-1} (1 + p(\tau))$,

If $\mathbb{T} = h\mathbb{Z}$, *with* $h > 0$ *and* $p(t) \equiv \alpha$, *then* $e_p(t, s) = (1 + \alpha h)^{\frac{t-s}{h}}$.

Note that if $p, q \in \mathfrak{R}$, then we define $p \oplus q$ and $\oplus p$ by

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t) p(t) q(t), \qquad \text{for all} \quad t \in \mathbb{T}^k,$$

and

$$(\ominus p)(t) := \frac{-p(t)}{1 + \mu(t) p(t)}, \qquad \text{for all} \quad t \in \mathbb{T}^k.$$

For more details about the time scale exponential function properties, the readers may read [8, *chapter* 2].

Definition 3.1.2 [14] *A function* $w : \mathbb{R}_+ \to \mathbb{R}_+$ *is said to belong to class* F, *if it satisfies the following conditions :*

$$\begin{aligned} w(x) &> 0 \text{ is nondecreasing and continuous for } z \geq 0, \\ \frac{1}{a} w(z) &\leq w\left(\frac{z}{a}\right) \text{ for } a > 0 . \end{aligned}$$

For example, if $w(z) = z^p, p \geq 1$, *then* $w(\frac{z}{a}) = (\frac{z}{a})^p \geq \frac{z^p}{a} = \frac{w(z)}{a}$ *for* $a \in (0, 1]$.

3.1.2 Stability definitions

For our purpose, we will assume that the time scale $\mathbb{T}$ is unbounded above, i.e., $\sup \mathbb{T} = +\infty$. Let $t_0 \in \mathbb{T}$and $t \in \mathbb{T}^+_{t_0}$. Let us consider time scale dynamic equations of the form

$$\begin{aligned} z^{\Delta}(t) &= f(t, z(t)), \\ z(t_0) &= z_0, \end{aligned} \tag{1.1}$$

where $z : \mathbb{T}^+_{t_0} \to R^n$ is the state vector and $f : \mathbb{T}^+_{t_0} \times \mathbb{R}^n \to \mathbb{R}^n$ is a rd-continuous vector-valued function. It is assumed that the conditions for the existence of a unique solution of system (1.1) are satisfied. For the existence, uniqueness and extensibility of its solutions, one can refer to [7]. Designate any solution of (1.1) with the initial state (t_0, z_0) by $z(t) = z(t, t_{0,} z_0)$.

Definition 3.1.3 [8] *A mapping $A : \mathbb{T} \to M_n(\mathbb{R})$ is called regressive if for each $t \in \mathbb{T}^k$ the $n \times n$ matrix $I_n + \mu(t)A$ is invertible, where I_n is the identity matrix.*

The class of all regressive and rd-continuous functions A from $\mathbb{T}$to $M_n(R)$ is denoted by $C_{rd}R(\mathbb{T}, M_n(\mathbb{R}))$.

Definition 3.1.4 [17] *Let $t_0 \in \mathbb{T}$. The unique matrix-valued solution of the IVP*

$$Z^{\Delta} = A(t)Z \qquad Z(t_0) = I_n, \tag{1.2}$$

where $A \in C_{rd}R(T, M_n(R))$, is called the matrix eXponential function and it denoted by $\phi_A(t, t_0)$.

Accordingly, the matrix function $\phi_A(t, t_0)$ possesses the following two properties:

$$\begin{aligned} \phi_A^{\Delta}(t, t_0) &= A(t)\phi_A(t, t_0), \\ \phi_A(t_0, t_0) &= I_n. \end{aligned}$$

This matrix function is referred to as the state transition matrix, and our assumption in the nature of $A(t)$ turns out that the state transition matrix exists and is unique.

Theorem 3.1.1 [7] *Suppose* $A, B \in C_{rd}R(\mathbb{T}; M_n(\mathbb{R}))$ *are matrix-valued functions on* $\mathbb{T}$, *then*

(*i*) $\phi_A(t,r)\phi_A(r,s) = \phi_A(t,s)$ *for all* $r,\ s,\ t \in \mathbb{T}$.

(*ii*) $\phi_A(\sigma(t),s) = (I + \mu(t)A(t))\phi_A(t,s)$.

(*iii*) *If* $\mathbb{T} = R$ *and* A *is constant, then* $\phi_A(t,s) = e_A(t,s) = e^{A(t-s)}$.

(*iv*) *If* $\mathbb{T} = hZ$, *with* $h > 0$, *and* A *is constant, then* $\phi_A(t,s) = (I + hA)^{\frac{(t-s)}{h}}$.

Definition 3.1.5 [13] *The system of dynamic equations* (1.1) *is said to be uniformly exponentially stable if there exist constants* $\gamma \geq 1$ *(independent of* t_0*)* , $\lambda > 0$ $(-\lambda \in \mathfrak{R}^+)$ *such that*

$$\|z(t)\| \leq \gamma \|z_0\| e_{-\lambda}(t,t_0).$$

The positif reals γ *and* λ *are the so-called growth constants.*

Now, we give the following characterization in terms of the transition matrix for system (1.2).

Theorem 3.1.2 [13] *The system of dynamic equations* (1.2) *is uniformly exponentially stable with respect to* $t \in \mathbb{T}^+_{t_0}$ *if and only if there exist constants* $\lambda > 0$ $(-\lambda \in \Re^+)$ *and* $\gamma \geq 1$ *such that for any* t_0 *and* $z(t_0)$, *the corresponding solution satisfies*

$$||\phi_A(t,t_0)|| \leq \gamma e_{-\lambda}(t,t_0), \text{ for all } t \in \mathbb{T}^+_{t_0}.$$

3.1.3 integrals dynamic Inequalities

The following lemmas are useful in our main results.

Lemma 3.1.4 [16] *Let* $z \in C_{rd}(\mathbb{T},\ \mathbb{R}_+)$. *Suppose that* $h, g \in \mathfrak{R}^+$ *with* $h \geq 0, g \geq 0$ *and* $z_0 \in \mathbb{R}$. *If*

$$z(t) \leq z_0 + \int_{t_0}^{t} h(s)[z(s) + \int_{t_0}^{s} g(\tau)z(\tau)\Delta\tau]\Delta s \quad \text{for } t \in \mathbb{T}^k,$$

then

$$z(t) \leq z_0[1 + \int_{t_0}^{t} h(s)e_{h+g}(s,t_0)\Delta s] \text{ for } t \in \mathbb{T}^k,$$

In particular, if $z(t_0) = z_0 = 0$*, then* $z(t) \equiv 0$ *on* $\mathbb{T}^k$.

Lemma 3.1.5 [16] *Let* $H \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing function such that* $H(z) > 0$ *for* $z > 0$ *and* $g \in \mathfrak{R}^+$ *with* $g \geq 0$*. If* $z(t) \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$ *satisfies*

$$z(t) \leq M + \int_{t_0}^{t} g(s)H(z(s))\Delta s \text{ for } t \in \mathbb{T}^k,$$

where $M > 0$ *is a constant, then*

$$z(t) \leq G^{-1}(G(M) + \int_{t_0}^{t} g(s)\Delta s) \text{ for } t \in \mathbb{T}^k,$$

where G *satisfies condition:*

a) G *is a solution of* $G^{\Delta}(u(t)) = \frac{u^{\Delta}(t)}{H(u(t))}$ *and* G *is strictly increasing with* $G(M) + \int_{t_0}^{t} g(s)\Delta s$ *which is in the domain of* G^{-1} *for* $t \in T^k$.

Lemma 3.1.6 *Suppose that* $z, f, g, d, m \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$*. If*

$$z(t) \leq f(t) + g(t)\int_{t_0}^{t} [d(s)z^p(s) + m(s)]\,\Delta s \text{ for all } t \in \mathbb{T}_{t_0}^{+},\ p \in]0,1[,$$

then

$$z(t) \leq f(t) + g(t)\int_{t_0}^{t} L(\tau)\exp\left[\int_{\sigma(\tau)}^{t} M(s)\Delta s\right]\Delta\tau \text{ for all } t \in \mathbb{T}_{t_0}^{+},$$

where

$$\begin{aligned} L(t) &= d(t)(pf(t) + 1 - p) + m(t), \\ M(t) &= pd(t)g(t). \end{aligned} \tag{1.3}$$

Preuve. *Define a function* $z(t)$ *on* $\mathbb{T}_{t_0}^{+}$ *by:*

$$z(t) = \int_{t_0}^{t} \{d(s)z^p(s) + m(s)\}\,\Delta s,$$

then,

$$z(t) \leq f(t) + g(t)z(t), \tag{1.4}$$

for $t \in \mathbb{T}_{t_0}^+ \cap \mathbb{T}^k$, *we have*

$$z^{\Delta}(t) = d(t)z^p(t) + m(t) \leq d(t)(f(t) + g(t)z(t))^p + m(t),$$

then,

$$z^{\Delta}(t) \leq d(t)(p(f(t) + g(t)z(t)) + 1 - p) + m(t),$$

further,

$$z^{\Delta}(t) \leq pd(t)g(t)z(t) + d(t)(pf(t) + 1 - p) + m(t), \tag{1.5}$$

the inequality (1.5) *can be reformulated as*

$$z^{\Delta}(t) \leq M(t)z(t) + L(t), z(t_0) = 0,$$

where $L(t), M(t)$ *are defined as in* (1.3).

Applying Lemma 3.1.3 *to the last inequality, we obtain that*

$$z(t) \leq \int_{t_0}^{t} L(\tau)e_M(t, \sigma(\tau))\Delta\tau,$$

It follows from Lemma 3.1.1 *that*

$$z(t) \leq \int_{t_0}^{t} L(\tau) \exp\left[\int_{\sigma(\tau)}^{t} M(s)\Delta s\right] \Delta\tau, \tag{1.6}$$

substituting (1.6) *in* (1.4), *we obtain the desired inequality.* ■

Lemma 3.1.7 [6] *Let us consider* $z, f \in C_{rd}(T, \mathbb{R}_+)$ *and* c *is a positive constant. Let* $w : \mathbb{R}_+ \to \mathbb{R}_+$ *be a continuous function which is nondecreasing positive on* $]0, +\infty[$, $L : \mathbb{T} \times \mathbb{T} \to \mathbb{R}_+$ *be a rd-continuous function and* G *be given by*

$$G(z) = \int_{z_0}^{z} \frac{ds}{w(s)}, z > 0, z_0 > 0.$$

If

$$z(t) \leq c + \int_{t_0}^{t} f(\eta)\left[w(z(\eta)) + \int_{t_0}^{\eta} L(\eta, \tau)w(z(\tau))\Delta\tau\right]\Delta\eta,$$

for $t \in \mathbb{T}$, *then for all* $t \in \mathbb{T}$*satisfying*

$$G(c) + \int_{t_0}^{t} f(\eta)[1 + \int_{t_0}^{\eta} L(\eta, \tau)\Delta\tau]\Delta\eta \in Dom(G^{-1}),$$

we have

$$z(t) \leq G^{-1}(G(c) + \int_{t_0}^{t} f(\eta)[1 + \int_{t_0}^{\eta} L(\eta,\tau)\Delta\tau]\Delta\eta),$$

where $G^{-1}$is the inverse function of G.

3.2 Main Rresult

In this section, we consider a particular class of systems (1.1), i.e the system

$$\begin{aligned} z^{\Delta}(t) &= A(t)z + F(t, z(t)), \\ z(t_0) &= z_0, z_0 \neq 0. \end{aligned} \tag{3.1}$$

where $z_0, z \in \mathbb{R}^n$, $F(t,0) = 0$, $t_0 \in \mathbb{T}$ and $F : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is an rd-continuous function. F represents the disturbance of the time-varying linear system :

$$\begin{aligned} z^{\Delta}(t) &= A(t)z, \\ z(t_0) &= z_0, z_0 \neq 0. \end{aligned} \tag{2.2}$$

Lemma 3.2.1 [3]*Consider the regressive time-varying perturbed system of the form* (2.1). *Then, every solution can be written in this form*

$$z(t) = \Phi_A(t,t_0)z_0 + \int_{t_0}^{t} \Phi_A(t,\sigma(s))F(s,z(s))\Delta s, \ t \in \mathbb{T}^+_{t_0}. \tag{2.3}$$

Now, we investigate the uniform exponential stability of such time-varying perturbed systems under different conditions on the perturbed term using Grönwall-Bihari and Pachpatte- type inequality.

Theorem 3.2.1 *Suppose that* (2.2) *is uniformly exponentially stable with positive constants λ and γ and*

$$\|F(t,z(t))\| \leq g(t)H(\|z(t)\|), \tag{2.4}$$

where $H \in F$ and $g \in \Re^+$ with $g \geq 0$ such that $\int_{t_0}^{\infty} \frac{\gamma}{1-\lambda\mu(s)} g(s)\Delta s < +\infty$.
Then the perturbed system (1.1) *is uniformly exponentially stable.*

Preuve. Let $t_0 \in \mathbb{T}$, $0 \neq z_0 \in \mathbb{R}^n$ and $t \in \mathbb{T}^+_{t_0}$. For any t_0 and $z_0 = z(t_0)$ and from (2.3), the solution of the perturbed system (2.1) is given by :

$$z(t) = \Phi_A(t, t_0)z(t_0) + \int_{t_0}^{t} \Phi_A(t, \sigma(s))F(s, z(s))\Delta s.$$

Taking into account the fact that system (2.2) is uniformly exponentially stable and the growth rate perturbation (2.4), we can estimate the solution $z(t)$ as

$$\begin{aligned}
\|z(t)\| &\leq \|\Phi_A(t, t_0)\| \|z_0\| + \int_{t_0}^{t} \|\Phi_A(t, \sigma(s))\| \|F(s, z(s))\| \Delta s \\
\|z(t)\| &\leq \|\Phi_A(t, t_0)\| \|z_0\| + \int_{t_0}^{t} \|\Phi_A(t, \sigma(s))\| g(s)H(\|z(s)\|)\Delta s \\
\|z(t)\| &\leq \gamma e_{-\lambda}(t, t_0) \|z_0\| \\
&\quad + \gamma e_{-\lambda}(t, t_0) \|z_0\| \int_{t_0}^{t} \gamma e_{-\lambda}(t_0, \sigma(s))e_{-\lambda}(s, t_0)g(s)\frac{1}{\gamma e_{-\lambda}(s, t_0) \|z_0\|}H(\|z(s)\|)\Delta s.
\end{aligned}$$

Setting $u(t) = \frac{\|z(t)\|}{\gamma \|z_0\| e_{-\lambda}(t,t_0)}$ and taking into account the fact that $H \in \mathfrak{F}$, one can get

$$u(t) \leq 1 + \int_{t_0}^{t} \frac{\gamma}{1 - \lambda\mu(s)} g(s)H(u(s))\Delta s.$$

Applying Lemma 3.1.5 to the above inequality, one obtain that

$$u(t) \leq G^{-1}(G(1) + \int_{t_0}^{t} \frac{\gamma}{1 - \lambda\mu(s)} g(s)\Delta s),$$

where G is defined as in Lemma 3.1.5.

Then, we have

$$\|z(t)\| \leq \gamma \|z_0\| e_{-\lambda}(t, t_0)G^{-1}(G(1) + \int_{0}^{+\infty} \frac{\gamma}{1 - \lambda\mu(s)} g(s)\Delta s),$$

let

$$d = \gamma G^{-1}(G(1) + \int_{0}^{+\infty} \frac{\gamma}{1 - \lambda\mu(s)} g(s)\Delta s).$$

It is clear that $G^{-1}(G(1) + \int_{0}^{+\infty} \frac{\gamma}{1-\lambda\mu(s)} g(s)\Delta s) \geq 1$, then $d \geq 1$, which proof that the perturbed system (3.1) is uniformly exponentially stable. ■

Theorem 3.2.2 *Assume that the following assumptions are satisfied:*

i) the system (2.2) *is uniformly exponentially stable with positive constants* λ *and* γ *and*

$$\begin{aligned} \|F(t,z(t))\| &\leq h(t)(\|z(t)\|+y(t)), \\ y^{\Delta}(t) &\leq g(t)\|z(t)\|, \ y(t_0)=0, \end{aligned} \tag{2.5}$$

where h, y, g *are nonnegative rd-continuous functions,*

ii) there exists a positive constant m *such that*

$$\int_{t_0}^{+\infty} \widetilde{h} e_{\widetilde{h}+g}(s,t_0)\Delta s \leq m < +\infty, \tag{2.6}$$

where

$$\widetilde{h}(t) = \gamma e_{-\lambda}(t_0,\sigma(t))h(t). \tag{2.7}$$

Then the perturbed system (2.1) *is uniformly exponentially stable.*

Preuve. Let $t_0 \in \mathbb{T}$, $0 \neq z_0 \in \mathbb{R}^n$ and $t \in \mathbb{T}^+_{t_0}$. For any t_0 and $z_0 = z(t_0)$ and from (2.3), the solution of the perturbed system (2.1) is given by :

$$z(t) = \Phi_A(t,t_0)z(t_0) + \int_{t_0}^{t} \Phi_A(t,\sigma(s))F(s,z(s))\Delta s.$$

Taking into account that system (2.2) is uniformly exponentially stable and the growth rate perturbation (2.5), we can estimate the solution $z(t)$ as

$$\begin{aligned} \|z(t)\| &\leq \|\Phi_A(t,t_0)\|\|z_0\| + \int_{t_0}^{t} \|\Phi_A(t,\sigma(s))\|\|F(s,z(s))\|\Delta s \\ &\leq \gamma e_{-\lambda}(t,t_0)\|z_0\| \\ &\quad + e_{-\lambda}(t,t_0)\int_{t_0}^{t} \gamma e_{-\lambda}(t_0,\sigma(s))h(s)\left(\|z(s)\| + \int_{t_0}^{s} g(\tau)\|z(\tau)\|\Delta\tau\right)\Delta s \\ &\leq \gamma e_{-\lambda}(t,t_0)\|z_0\| \\ &\quad + e_{-\lambda}(t,t_0)\int_{t_0}^{t} \gamma e_{-\lambda}(t_0,\sigma(s))h(s)\left(\frac{\|z(s)\|}{e_{-\lambda}(s,t_0)} + \int_{t_0}^{s} g(\tau)\frac{\|z(\tau)\|}{e_{-\lambda}(\tau,t_0)}\Delta\tau\right)\Delta s \end{aligned}$$

Since from Lemma 3.1.1, we have $e_{-\lambda}(t,t_0) \leq 1$ and setting $u(t) = \frac{\|z(t)\|}{e_{-\lambda}(t,t_0)}$,one can obtain

$$u(t) \leq \gamma\|z_0\| + \int_{t_0}^{t} \gamma e_{-\lambda}(t_0,\sigma(s))h(s)\left(u(s) + \int_{t_0}^{s} g(\tau)u(\tau)\Delta\tau\right)\Delta s,$$

Applying Lemma 3.1.7 to the last inequality one can get

$$u(t) \leq \gamma \|z_0\| [1 + \int_{t_0}^{t} \widetilde{h}(s) e_{\widetilde{h}+g}(s, t_0) \Delta s],$$

where $\widetilde{h}$ is defined as in (2.7).

From (2.6), one can obtain

$$\|z(t)\| \leq \gamma(m+1) \|z_0\| e_{-\lambda}(t, t_0).$$

Therefore, the perturbed system (2.1) is uniformly exponentially stable. ■

Theorem 3.2.3 *Assume that there the following conditions are satisfies :*

i) $\|F(t,x)\| \leq \eta(l(t)\|z\|^p + k(t))$, *where* l, $k \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$ *and* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ *is a differentiable increasing function on* $]0, +\infty[$ *with continuous nonincreasing first derivative* η' *on* $]0, \infty[$, $p \in]0, 1[$.

ii) Suppose that the linear system (2.2) *is uniformly exponentially stable with growth constants* λ *and* γ,

iii) $\int_{t_0}^{+\infty} \frac{\eta'(k(s))l(s)}{1-\lambda\mu(s)} \Delta s \leq \widetilde{d} < +\infty$, $\int_{t_0}^{+\infty} ((1-p)\eta'(k(s))l(s) + \eta(k(s))) e_{-\lambda}(t_0, \sigma(s)) \Delta s \leq \widetilde{k} < +\infty$.

Then the perturbed system (2.1) *is uniformly exponentially stable.*

Preuve. Let $t_0 \in \mathbb{T}, z_0 \in \mathbb{R}^n$ and $t \in \mathbb{T}^+_{t_0}$. For any t_0 and $z_0 = z(t_0)$ and from (2.3), the solution of the perturbed system (2.1) is given by :

$$z(t) = \Phi_A(t, t_0) z_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s)) F(s, z(s)) \Delta s.$$

Taking into account hypotheses i) and ii), we obtain that

$$\|z(t)\| \leq \gamma e_{-\lambda}(t, t_0) \|z_0\| + \gamma e_{-\lambda}(t, t_0) \int_{t_0}^{t} e_{-\lambda}(t_0, \sigma(s)) \eta(l(s)\|z(s)\|^p + k(s)) \Delta s. \quad (2.8)$$

Applying the mean value Theorem for the function η, then for every $z_1 \geq y_1 > 0$, there exists $c \in]y_1, z_1[$ such taht

$$\eta(z_1) - \eta(y_1) = \eta'(c)(z_1 - y_1) \leq \eta'(y_1)(z_1 - y_1),$$

which gives:

$$\eta \left(l(s) \left\|z(s)\right\|^p + k(s)\right) \leq \eta'(k(s)) \times l(s) \left\|z(s)\right\|^p + \eta(k(s)). \tag{2.9}$$

From (2.8) and (2.9),one gets

$$\|z(t)\| \leq \gamma e_{-\lambda}(t,t_0) \|z_0\| + \gamma e_{-\lambda}(t,t_0) \int_{t_0}^{t} e_{-\lambda}(t_0,\sigma(s)) \left[\eta'(k(s)) \times l(s) \left\|z(s)\right\|^p + \eta(k(s))\right] \Delta s.$$

Applying Lemma 3.1.6 to the above inequality, we obtain

$$\begin{aligned} \|z(t)\| \;& \leq \gamma e_{-\lambda}(t,t_0) \Big[\|z_0\| + \textstyle\int_{t_0}^{t} \{e_{-\lambda}(t_0,\sigma(s))\eta'(k(s))l(s)(p\gamma e_{-\lambda}(s,t_0)\|z_0\| + 1 - p) \\ & + e_{-\lambda}(t_0,\sigma(s))\eta(k(s))\} \\ & \times \exp\left(p\gamma \textstyle\int_{\sigma(s)}^{t} e_{-\lambda}(t_0,\sigma(\tau))\eta'(k(\tau)) \times l(\tau) e_{-\lambda}(\tau,t_0)\Delta\tau\right) \Delta s\Big], \end{aligned} \tag{2.10}$$

It is clear that inequality (2.10) can be reformulated as :

$$\begin{aligned} \|z(t)\| \;& \leq \gamma e_{-\lambda}(t,t_0) \Big[\|z_0\| + \textstyle\int_{t_0}^{t} \{e_{-\lambda}(t_0,\sigma(s))\eta'(k(s))l(s)(p\gamma e_{-\lambda}(s,t_0)\|z_0\| + 1 - p) \\ & + e_{-\lambda}(t_0,\sigma(s))\eta(k(s))\} \\ & \times \exp\left(p\gamma \textstyle\int_{\sigma(s)}^{t} e_{-\lambda}(t_0,\sigma(\tau))\eta'(k(\tau)) \times l(\tau) e_{-\lambda}(\tau,t_0)\Delta\tau\right) \Delta s\Big], \end{aligned} \tag{2.11}$$

According to the hypothesis iii) and from (1.11), we obtain the following estimate

$$\begin{aligned} \|z(t)\| \;& \leq \; \gamma\left(1 + p\gamma\widetilde{d}\exp\left(p\gamma\widetilde{d}\right)\right) e_{-\lambda}(t,t_0)\|z_0\| + \gamma\widetilde{k}\exp\left(p\gamma\widetilde{d}\right) e_{-\lambda}(t,t_0), \\ & = \; \gamma e_{-\lambda}(t,t_0)\|z_0\| \left[1 + p\gamma\widetilde{d}\exp\left(p\gamma\widetilde{d}\right) + \frac{\widetilde{k}\exp\left(p\gamma\widetilde{d}\right)}{\|z_0\|}\right]. \end{aligned}$$

Then the perturbed system (2.1) is uniformly exponentially stable. ■

Theorem 3.2.4 *Assume that there exist $l,\ y, g \in C_{rd}(T, R_+)$ and $w \in F$ that satisfy the following conditions :*

i)

$$\begin{aligned} \|F(t,z(t))\| \;& \leq l(t)(w(\|z(t)\|) + y(t)), \\ y^{\Delta}(t) \;& \leq g(t)w(\|z(t)\|),\ y(t_0) = 0. \end{aligned} \tag{2.12}$$

ii) Suppose that the linear system (2.2) *is uniformly exponentially stable with growth constants* λ *and* γ,

iii) There exists a positive constant m *such that*

$$\int_{t_0}^{+\infty} \frac{\gamma}{1-\lambda\mu(s)} l(s)[1+\int_{t_0}^{s} e_{-\lambda}(\eta,s)g(\eta)\Delta\eta]\Delta s \leq m < +\infty.$$

Then the perturbed system (2.1) is uniformly exponentially stable.

Preuve. Let $t_0 \in \mathbb{T}$, $0 \neq z_0 \in \mathbb{R}^n$ and $t \in \mathbb{T}^+_{t_0}$. For any t_0 and $z_0 = z(t_0)$ and from (2.3), the solution of the perturbed system (2.1) is given by :

$$z(t) = \Phi_A(t,t_0)z(t_0) + \int_{t_0}^{t} \Phi_A(t,\sigma(s))F(s,z(s))\Delta s.$$

Taking into account that $w \in \mathfrak{F}$ and the growth rate perturbation (2.12), then we have

$$\begin{aligned}
\|z(t)\| &\leq \|\Phi_A(t,t_0)\| \|z_0\| + \int_{t_0}^{t} \|\Phi_A(t,\sigma(s))\| \|F(s,z(s))\| \Delta s \\
&\leq \gamma e_{-\lambda}(t,t_0) \|z_0\| \\
&\quad + e_{-\lambda}(t,t_0) \int_{t_0}^{t} \gamma e_{-\lambda}(t_0,\sigma(s)) l(s) \left(w(\|z(s)\|) + \int_{t_0}^{s} g(\eta) w(\|z(\eta)\|)\Delta\eta \right) \Delta s \\
&\leq \gamma e_{-\lambda}(t,t_0) \|z_0\| \\
&\quad + e_{-\lambda}(t,t_0) \int_{t_0}^{t} \gamma e_{-\lambda}(t_0,\sigma(s)) l(s) \left(\begin{array}{c} \|z_0\| e_{-\lambda}(s,t_0) w(\frac{\|z(s)\|}{\|z_0\| e_{-\lambda}(s,t_0)}) \\ + \int_{t_0}^{s} g(\eta) \|z_0\| e_{-\lambda}(\eta,t_0) w(\frac{\|z(\eta)\|}{\|z_0\| e_{-\lambda}(\eta,t_0)})\Delta\eta \end{array} \right) \Delta s \\
&\leq \gamma e_{-\lambda}(t,t_0) \|z_0\| \\
&\quad + e_{-\lambda}(t,t_0) \int_{t_0}^{t} \gamma l(s) \left(\begin{array}{c} \frac{\|z_0\|}{1-\lambda\mu(s)} w(\frac{\|z(s)\|}{\|z_0\| e_{-\lambda}(s,t_0)}) \\ + e_{-\lambda}(t_0,\sigma(s)) \int_{t_0}^{s} g(\eta) \|z_0\| e_{-\lambda}(\eta,t_0) w(\frac{\|z(\eta)\|}{\|z_0\| e_{-\lambda}(\eta,t_0)})\Delta\eta \end{array} \right) \Delta s \\
&\leq e_{-\lambda}(t,t_0) \|z_0\| \left[\gamma + \gamma \int_{t_0}^{t} l(s) \left(\begin{array}{c} \frac{1}{1-\lambda\mu(s)} w(\frac{\|z(s)\|}{\|z_0\| e_{-\lambda}(s,t_0)}) \\ + \int_{t_0}^{s} g(\eta) \frac{e_{-\lambda}(\eta,s)}{1-\lambda\mu(s)} w(\frac{\|z(\eta)\|}{\|z_0\| e_{-\lambda}(\eta,t_0)})\Delta\eta \end{array} \right) \Delta s \right].
\end{aligned}$$

Setting $u(t) = \frac{\|z(t)\|}{\|z_0\| e_{-\lambda}(t,t_0)}$, then we obtain that

$$u(t) \leq \gamma + \int_{t_0}^{t} \frac{\gamma}{1-\lambda\mu(s)} l(s) \left(w(u(s)) + \int_{t_0}^{s} e_{-\lambda}(\eta,s) g(\eta) w(u(\eta))\Delta\eta \right) \Delta s,$$

an application of Lemma 3.1.7, gives

$$u(t) \leq W^{-1}(W(\gamma) + \int\limits_{t_0}^{t} \frac{\gamma}{1-\lambda\mu(s)} l(s)[1 + \int_{t_0}^{s} e_{-\lambda}(\eta, s) g(\eta)\Delta\eta]\Delta s),$$

where $W(\gamma) + \int\limits_{t_0}^{t} \frac{\gamma}{1-\lambda\mu(s)} l(s)[1 + \int_{t_0}^{s} e_{-\lambda}(\eta, s) g(\eta)\Delta\eta]\Delta s) \in Dom(W^{-1})$,then, we obtain that

$$\|z(t)\| \leq \|z_0\| \, e_{-\lambda}(t, t_0) W^{-1}(W(\gamma) + \int\limits_{t_0}^{+\infty} \frac{\gamma}{1-\lambda\mu(s)} l(s)[1 + \int_{t_0}^{s} e_{-\lambda}(\eta, s) g(\eta)\Delta\eta]\Delta s),$$

which ensure that the perturbed system (2.1) is uniformly exponentially stable. ■

3.3 Numerical examples

In order to illustrate the performance of the proposed stability criteria in Theorem 3.2.1 and Theorem 3.2.3, we will provide a numerical examples.

Exemple 3.3.1 *Let $\mathbb{T}$ be a mixed continuous-discrete time scale and $t_0 = 0$. The discrete part has non-uniform step size. The graininess function is bounded as follows: $\forall t \in \mathbb{T}_0^+$*

$$0 \leq \mu(t) < \mu_{\max} = \frac{1}{2}.$$

Consider the following time-varying system:

$$\begin{aligned} z_1^{\Delta} &= -z_1 + \tfrac{\sqrt{3}}{2} \tfrac{1}{(t+1)(\sigma(t)+1)} \left|z_2\right|^2, \\ z_2^{\Delta} &= -z_2 + \tfrac{1}{2} \tfrac{1}{(t+1)(\sigma(t)+1)} \left|z_1\right|^2 \\ z(0) &= (z_{1,0}, z_{2,0}), \end{aligned} \tag{3.1}$$

where $z = (z_1, z_2)^T \in \mathbb{R}^2$.

System (3.1) *can be written as system* (2.1) :

$$z^{\Delta} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \frac{\sqrt{3}}{2} \frac{1}{(t+1)(\sigma(t)+1)} \left|z_2\right|^2 \\ \frac{1}{2} \frac{1}{(t+1)(\sigma(t)+1)} \left|z_1\right|^2 \end{pmatrix}, \tag{3.2}$$

where $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in C_{rd}\Re(\mathbb{T}, M_n(\mathbb{R}))$, $\mu \neq 1, F(t,0) = 0$ *and*

$$\Phi_A(t,0) = \begin{pmatrix} e_{-1}(t,0) & 0 \\ 0 & e_{-1}(t,0) \end{pmatrix}, \ t \in \mathbb{T}_0^+, \tag{3.3}$$

we deduce that

$$\|\Phi_A(t,t_0)\| = \sqrt{2}e_{-1}(t,0), \tag{3.4}$$

then the varying linear system of (3.1) *is uniformly exponentially stable with* $(\lambda, \gamma) = \left(1, \sqrt{2}\right)$.

The perturbation satisfies conditions of Theorem 3.2.1 *with*

$$\|F(t,z(t))\| \leq \frac{1}{(t+1)(\sigma(t)+1)} \|z(t)\|^2, \tag{3.5}$$

Here $g(t) = \frac{1}{(t+1)(\sigma(t)+1)}$ *and* $H(z) = z^2$.*It is clear that* $H \in F$. *Moreover, one can verify that* $\gamma \|z_0\| e_{-\lambda}(t,t_0)$

$$\begin{aligned} \int_0^{+\infty} \frac{\gamma g(s)}{1-\lambda\mu(s)} \Delta s &= \int_0^{+\infty} \frac{\gamma}{(1-\mu(s))} \frac{1}{(s+1)(\sigma(s)+1)} \Delta s, \\ &\leq 2\gamma \int_0^{+\infty} \frac{1}{(s+1)(\sigma(s)+1)} \Delta s = 2\sqrt{2} < +\infty. \end{aligned} \tag{3.6}$$

From Theorem 3.2.1, *one can conclude that system* (3.1) *is uniformly exponentially stable.*

Exemple 3.3.2 *Let* $\mathbb{T}$*be a mixed continuous-discrete time scale and* $t_0 = 0$.*The discrete part has non-uniform step size. The graininess function is bounded as follows:* $\forall t \in T_0^+$

$$0 \leq \mu(t) < \mu_{\max} = \frac{1}{2}.$$

Consider the following time-varying system:

$$\begin{aligned} z_1^\Delta &= -z_1 + \tfrac{1}{3}\arctan(l(t)\left|z_2\right|^{\frac{1}{3}} + \tfrac{k(t)|z_1|}{\sqrt{z_1^2+z_2^2+1}}), \\ z_2^\Delta &= -z_2 + \tfrac{\sqrt{8}}{3}\arctan(l(t)\left|z_1\right|^{\frac{1}{3}} + \tfrac{k(t)|z_2|}{\sqrt{z_1^2+z_2^2+1}}), \\ z(0) &= (z_{1,0}, z_{2,0}), \end{aligned} \tag{3.7}$$

where $z = (z_1, z_2)^T \in \mathbb{R}^2$, $l(t) = \frac{(e_{-1}(\sigma(t),0)(t+\sigma(t)+2)}{(t+1)^2(\sigma(t)+1)^2}$ *and* $k(t) = \frac{e_{-1}(\sigma(t),0)}{(t+1)(\sigma(t)+1)}$.

System (3.7) *can be rewritten as :*

$$z^{\Delta} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{3}\arctan(l(t)\,|z_2|^{\frac{1}{3}} + \frac{k(t)|z_1|}{\sqrt{z_1^2+z_2^2+1}}) \\ \frac{\sqrt{8}}{3}\arctan(l(t)\,|z_1|^{\frac{1}{3}} + \frac{k(t)|z_2|}{\sqrt{z_1^2+z_2^2+1}}) \end{pmatrix}. \tag{3.8}$$

The varying linear system of (3.7) *is uniformly exponentially stable with* $(\lambda, \gamma) = \left(1, \sqrt{2}\right)$. *Hence assumption (ii) of Theorem* 3.2.3 *is verified.*

The perturbation satisfies

$$\begin{aligned} \|F(t, z(t))\| &\leq \arctan(l(t)\,\|z\|^{\frac{1}{3}} + k(t)) = n(l(t)\,\|z\|^{\frac{1}{3}} + k(t)), \\ l(t) &= \tfrac{e_{-1}(\sigma(t),0)(t+\sigma(t)+2)}{(t+1)^2(\sigma(t)+1)^2},\ k(t) = \tfrac{e_{-1}(\sigma(t),0)}{(t+1)(\sigma(t)+1)}, \end{aligned} \tag{3.9}$$

hence assumption (i) *of Theorem* 3.2.3 *is verified, with* $n(z) = \arctan(z)$ *is a differentiable increasing function on*$]0, \infty[$ *with continuous nonincreasing first derivative.*

Now, we verify iii) as follow :

$$\begin{aligned} \int_{t_0}^{+\infty} \frac{\eta'(k(s))l(s)}{1-\lambda\mu(s)}\Delta s &= \int_0^{+\infty} \frac{\frac{1}{1+k^2(s)}\frac{e_{-1}(\sigma(s),0)(s+\sigma(s)+2)}{(s+1)^2(\sigma(s)+1)^2}}{1-\mu(s)}\Delta s \\ &\leq \int_0^{+\infty} \frac{s+\sigma(s)+2}{(s+1)^2(\sigma(s)+1)^2}\Delta s = \int_0^{+\infty} (\frac{-1}{(s+1)^2})'\Delta s = 1 < +\infty, \end{aligned}$$

$$\begin{aligned} & \int_{t_0}^{+\infty} ((1-p)\eta'(k(s))l(s) + \eta(k(s)))e_{-\lambda}(t_0, \sigma(s))\Delta s \\ \leq & \int_0^{+\infty} \frac{2}{3}\frac{s+\sigma(s)+2}{(s+1)^2(\sigma(s)+1)^2} e_{-1}(\sigma(s),0)e_{-1}(0,\sigma(s))\Delta s \\ & + \int_0^{+\infty} \arctan(\frac{1}{(s+1)(\sigma(s)+1)} e_{-1}(\sigma(s),0))e_{-1}(0,\sigma(s))\Delta s \\ \leq & \int_0^{+\infty} \frac{2}{3}\frac{s+\sigma(s)+2}{(s+1)^2(\sigma(s)+1)^2}\Delta s + \int_0^{+\infty} \frac{1}{(s+1)(\sigma(s)+1)}\Delta s = \frac{2}{3} + 1 = \frac{5}{3} < +\infty. \end{aligned}$$

From Theorem 3.2.3 , *one can conclude that system* (3.7) *is uniformly exponentially stable.*

Conclusion In this chapter, the exponential stability problem for some classes of non linear perturbed system has been studied by using appropriate integral inequalities on time scales.

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CHAPTER 4

Further new refinements in h-Stability conditions for nonlinear Abstract Dynamic Equations on Time Scales and applications

4.1 Introduction and preliminaries

The theory of dynamic equations on time scale was introduced in [2] whose main objective is to provide a unified approach to continuous and discrete analysis.The calculus on time scales and dynamic equations on time scales have applications in any field that requires simultaneous modeling of continuous and discrete processes, because they bridge the divide between continuous and discrete aspects of processes. The applications include insect population models, epidemic models, neural networks, and heat transfer. Foundational definitions and results from the time scale calculus appear in an excellent introductory text by Bohner and Peterson [8, 9]. During the last few years, several studies were achieved on stability, $h-$ stability of certain classes of dynamical equations and oscillation of dynamic equations on time scales, see[3, 4, 5, 6, 7, 11, 12, 13, 14, 15, 17, 18, 20, 22, 23, 24, 26, 27, 28, 29, 30, 31] and references cited therein. The notion of h-stability was introduced by Pinto [26] which is an extension of the notions of exponential stability and uniform stability.

In this chapter we discuss the h$-$stability for the following abstract dynamic equation on time scale :

$$z^{\Delta}(t) = Az(t) + f(t, z), z(t_0) = z_0 \in D(A),\ t \in \mathbb{T}_0^+, t \geq t_0 \tag{1.1}$$

in terms of the h-stability of the homogeneous equation :

$$z^{\Delta}(t) = Az(t), z(t_0) = z_0 \in D(A), t \in \mathbb{T}_0^+, t \geq t_0, \tag{1.2}$$

where A is the generator of a C_0-semi-group T , $f : \mathbb{T}_0^+ \times X \to X$ is an rd-continuous with $f(t, 0) = 0$ and x^{Δ} is the delta derivative of $x : \mathbb{T}_0^+ \to X$, X is a Banach space.

We derive sufficient conditions for the h-stability notion of certain classes of abstract dynamic perturbed equations on time scales using time scale versions of some Pachpatte type and Grönwall Bihari inequalities.

Thanks to the definition of C_0-semigroup based on the concept of Laplace transforms on time scales given in [21], we will avoid the following condition used in $[18 - 20, 24]$: If $a, b \in \mathbb{T}$, then $a - b \in \mathbb{T}$.

This condition is very restrictive because many time scales do not satisfy this condition. For exemple, the quantum time scale $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}, q > 1$.

This chapter is organized as follows. In Section 2, we present definitions, properties and some necessary concepts on the theory of C_0-semigroup on time scale which are useful for our study.In Section 3, we prove some integral dynamic inequality which can be used in stydy of the h$-$stability of system (1.1). In section 4, some examples are given to illustrate the obtained results.

4.2 C_0-SEMIGROUPS AND THE ABSTRACT CAUCHY PROBLEM

In [21], the authors introduce a definition of C_0-semigroup based on the concept of Laplace transform on time scales, definition which is more general and encompass all the time scales $\mathbb{T}_0$ satisfying $0 \in \mathbb{T}_0$ and $\sup \mathbb{T}_0 = +\infty$. Let us denote by $\mathbb{T}_0^+ = \mathbb{T}_0 \cap [0, +\infty)$.

We begin by recalling the definition of the Laplace transform on time scales.

Definition 4.2.1 [8]*Assume that $z : \mathbb{T}_0 \to \mathbb{R}$ is a regulated function. Then the Laplace transform of z is defined by*

$$\widehat{z}(\lambda) = \mathcal{L}\{z\}(\lambda) := \int_0^\infty z(t) e_{\ominus\lambda}^{\sigma}(t, 0)\, \Delta t,$$

for $\lambda \in D\{z\}$, where $D\{z\}$ consists of all complex numbers $\lambda \in \mathcal{R}$ for which the improper integral exists.

In what follows, we present some properties of the Laplace transform.

Theorem 4.2.1 (*Linearity*, [8]) *Assume f nd g are regulated functions on $\mathbb{T}_0$ and α and β are constants. Then,*

$$\mathcal{L}\{\alpha z + \beta y\}(\lambda) = \alpha \mathcal{L}\{z\}(\lambda) + \beta \mathcal{L}\{y\}(\lambda),$$

for $\lambda \in D\{z\} \cap D\{y\}$.

Theorem 4.2.2 [16]*The function $f\colon \mathbb{T} \to \mathbb{R}$ is said to be of exponential type **II** if there exist constants M, $c > 0$ such that*

$$|f(t)| \leq M e_c(t, 0). \tag{2.1}$$

Theorem 4.2.3 [16] *(Domain of the transform). Assume that $f\colon \mathbb{T}_0 \to \mathbb{R}$ is a regulated function of exponential type **II** with exponential constant $c > 0$.*

Then the integral $\int_0^\infty e_{\ominus\lambda'}^{\sigma}(t, 0) f(t) \Delta t$ converges absolutely for

$$\lambda' \in D = \{\lambda \in \mathbb{C}\colon \quad Re_\mu(\lambda)(t) > c, \quad \forall \; \in \mathbb{T}_0\}.$$

where

$$Re_h(\boldsymbol{z}) := \frac{|\boldsymbol{z}\boldsymbol{h} + \mathbf{1}| - 1}{h},$$

Definition 4.2.2 *We say that $T : \mathbb{T}_0^+ \to \mathcal{L}(X)$ is strongly continuous if T satisfies $\|T(t)z - z\|_z \to 0$ as $t \to 0^+$ for each $z \in X$.*

In [21], the authors use this property to establish concept of C_0-semigroup on time scales.

Definition 4.2.3 *We say that $T : \mathbb{T}_0^+ \to \mathcal{L}(X)$ is a C_0-semigroup with infinitesimal generator A if the following conditions are satisfied :*

i)$T(0) = I$ and for every $z \in X$, the function $t \to T(t)z$ is strongly continuous.

ii)There exists a λ_0 such that $(\lambda_0, \infty) \subset \rho(A)$, $\lambda \in D\{T\} \cap \mathcal{R}$ and

$$\widehat{T}(\lambda)z = (\lambda - A)^{-1} z, \qquad z \in X,$$

for all $Re_\mu(\lambda)(t) > \lambda_0$ for $t \in \mathbb{T}_0^+$.

Corollary 4.2.1 [21] *Let $A : D(A) \subset X \to X$ be a closed linear operator. Then there exists at most a unique C_0-semigroup with infinitesimal generator A.*

Also the authors in [21] *characterize the infinitesimal generator of the semigroup $T : \mathbb{T}_0^+ \to \mathcal{L}(X)$. To this object, there begin by introducing the operator B as follows :*

(i) If 0 is right-scattered, then

$$Bz = \frac{T(\sigma(t))\, z - z}{\mu(t)} \,|_{t=0}= \frac{T(0)\, z - z}{\mu(0)}.$$

(ii)If 0 is right-dense , then

$$Bz = \lim_{t \to 0} \frac{T(t)\, z - z}{t},$$

on the domain $D(B)$ consisting of all $z \in X$ for which the limit exists.

In [21]*, the authors establish that definition of C_0-semigroup is equivalent to the existence of mild solutions of an abstract Cauchy problem. In what follows, let us consider that $A : D(A) \sqsubseteq X \to X$ is a closed linear operator and λ_0 is the constant involved in Definition 4.2.3.*

Theorem 4.2.4 [21] *Let $T : \mathbb{T}_0^+ \to \mathcal{L}(X)$ is a C_0-semigroup on X and let A be its generator. Then the following properties hold :*

(i) *For all $z \in X$, the function $u_z(t) = T(t)z$, $t \in \mathbb{T}_0^+$, is a mild solution of (1.2), where u_z satifies (2.1).*

(ii) $(\lambda - A)^{-1} T(t) = T(t) (\lambda - A)^{-1}$ *for all* $\lambda \in (\lambda_{0,}\infty) \subset \rho(A)$, *where* $\lambda \in \mathcal{R} \cap \mathbb{C} \setminus \{0\}$, *for all $t \in \mathbb{T}_0^+$.*

(iii) *If $z \in D(A)$, then $T(t)z \in D(A)$ and $AT(t)z = T(t)Az$, for all $t \in \mathbb{T}_0^+$.*

(iv) $\int_0^t T(s)z\Delta s \in D(A)$ *and* $A \int_0^t T(s)z\Delta s = T(t)z - z$ *for all $z \in X$ and $t \in \mathbb{T}_0^+$.*

(v) *Let $z, y \in X$. Then $z \in D(A)$ and $Az = y$, if and only if $\int_0^t T(s)y\Delta s = T(t)z - z$ for all $t \in \mathbb{T}_0^+$.*

(vi) $A = B$.

(vii) *$T(\cdot)z$ is a classical solution of (1.2), if and only if, $z \in D(A)$.*

Remark 4.2.1 *For more details about the properties of a C_0 -semi-group T and its generator A in arbitrary time scale, we refer the reader to [21] and in the case $\mathbb{T} = \mathbb{R}_+$, we refer to [25].*

Now, we present the concepts of $h-$stability. We consider the dynamic equation defined on a time scale $\mathbb{T}$ as below

$$z^{\Delta}(t) = f(t, z), \ z(t_0) = z_0, \ t, t_0 \in \mathbb{T}, \tag{2.2}$$

where X is a Banach space and $f : \mathbb{T} \times X \to X$ is rd-continuous in the first argument.

Definition 4.2.4 *[12] Equation (2.2) is called globally uniformly h-stable if there exist a positive bounded rd-continuous function $h : \mathbb{T} \to \mathbb{R}$, and a constant $\gamma \geq 1$ such that, any solution $z(t) = z(t, t_0, x_0)$ of equation (2.2) satisfies*

$$\|z(t, t_0, z_0)\| \leq \gamma \|z_0\| h(t) (h(t_0))^{-1}, \ t, t_0 \in \mathbb{T}, \tag{2.3}$$

(here $(h(t))^{-1} = \frac{1}{h(t)}$).

For the various definitions of stability, we refer to [15].

4.3 Statement of results

4.3.1 Integral dynamic inequalities

Now, we state some nonlinear integral Grönwall type inequalities on time scales, which are useful in our study.

Lemma 4.3.1 *([3]) Suppose that* $z, f, g, d, m \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$ *.If*

$$z(t) \le f(t) + g(t) \int_{t_0}^{t} \{d(s)z(s) + m(s)\} \Delta s, \ \text{for all } t \in \mathbb{T}_{t_0}^{+}, \tag{3.1}$$

then

$$z(t) \le f(t)+g(t) \int_{t_0}^{t} \{d(s)f(s) + m(s)\} \exp\left[\int_{\sigma(s)}^{t} d(\tau)g(\tau)\Delta\tau\right] \Delta s, \ \text{for all } t \in \mathbb{T}_{t_0}^{+}. \tag{3.2}$$

Lemma 4.3.2 *Suppose that* $z, f, g, d, m \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$ *.If*

$$z(t) \le f(t) + g(t) \int_{t_0}^{t} \{d(s)z^p(s) + m(s)\} \Delta s, \ \text{for all } t \in \mathbb{T}_{t_0}^{+}, p \in]0, 1[\,,$$

then

$$z(t) \le f(t) + g(t) \int_{t_0}^{t} L(\tau) \exp\left[\int_{\sigma(\tau)}^{t} M(s)\Delta s\right] \Delta\tau, \ \text{for all } t \in \mathbb{T}_{t_0}^{+},$$

where

$$\begin{aligned} L(t) &= d(t)(pf(t) + 1 - p) + m(t), \\ M(t) &= pd(t)g(t). \end{aligned} \tag{3.3}$$

Preuve. Define a function $z(t)$ on $\mathbb{T}_{t_0}^{+}$ by :

$$z(t) = \int_{t_0}^{t} \{d(s)z^p(s) + m(s)\} \Delta s,$$

then,

$$z(t) \le f(t) + g(t)z(t), \tag{3.4}$$

for $t \in \mathbb{T}_{t_0}^{+} \cap \mathbb{T}^k$, we have

$$z^{\Delta}(t) = d(t)z^p(t) + m(t) \le d(t)(f(t) + g(t)z(t))^p + m(t),$$

then

$$z^{\Delta}(t) \leq d(t)(p(f(t)+g(t)z(t))+1-p)+m(t),$$

further

$$z^{\Delta}(t) \leq pd(t)g(t)z(t)+d(t)(pf(t)+1-p)+m(t), \tag{3.5}$$

the inequality (3.5) can be reformulated as

$$z^{\Delta}(t) \leq M(t)z(t)+L(t), z(t_0)=0,$$

where $L(t), M(t)$ are defined as in (3.3).

Applying Lemma 3.1.3 (*chapter* 3) to the last inequality, we get

$$z(t) \leq \int_{t_0}^{t} L(\tau)e_M(t,\sigma(\tau))\Delta\tau,$$

from Lemma 3.1.1 (*chapter* 3), it follows that :

$$z(t) \leq \int_{t_0}^{t} L(\tau)\exp\left[\int_{\sigma(\tau)}^{t} M(s)\Delta s\right]\Delta\tau, \tag{3.6}$$

substituting (3.6) in (3.4), we obtain the desired inequality. ■

Lemma 4.3.3 *[1] Suppose that g is continuous and nondecreasing, p is rd-continuous and nonnegative, and y is rd-continuous. Let w be the solution of*

$$w^{\Delta}=p(t)g(w(t)),\ w(t_0)=\beta$$

and suppose there is a bijective function G with $(G\circ w)^{\Delta}=p$. Then

$$y(t) \leq \beta+\int_{t_0}^{t} p(\tau)g(y(\tau))\Delta\tau,\ \textit{for all } t\in\mathbb{T}$$

implies

$$y(t) \leq G^{-1}\left[G(\beta)+\int_{t_0}^{t} p(\tau)\Delta\tau\right],\ \textit{for all } t\in\mathbb{T}.$$

Lemma 4.3.4 *([6]) Let $t_0 \in \mathbb{T}$ and assume that $u, f \in C_{rd}(\mathbb{T}^+_{t_0}, \mathbb{R}_+)$ and $S : \mathbb{T}^+_{t_0} \times \mathbb{R}_+ \to \mathbb{R}_+$ be a rd-continuously Δ-differentiable with respect to the first variable, which verifies*

$$0 \leq S(t,z) - S(t,y) \leq R(t,y)(z-y), \tag{3.7}$$

for $t \in \mathbb{T}^+_{t_0}, z \geq y \geq 0$, and

$$S^{\Delta_t}(t,0) \geq 0, \quad R^{\Delta_t}(t,0) \geq 0 \quad \text{for all } t \in mathbbT^{k,+}_{t_0}, \tag{3.8}$$

where $R : \mathbb{T}^+_{t_0} \times \mathbb{R}_+ \to \mathbb{R}^*_+$ a rd-continuously Δ-differentiable function with respect to t. Let u, f be nonnegative rd-continuous functions on $mathbbT^+_{t_0}$. If $L(t,s)$ is defined as in Lemma 1 such that $L(t,s) \geq 0$ and $L^{\Delta_t}(t,s) \geq 0$ for $t, s \in \mathbb{T}^{k,+}_{t_0}$ with $s \leq t$,then,

$$u(t) \leq c + \int_{t_0}^{t} f(\eta) \left(S(\eta, u(\eta)) + \int_{t_0}^{\eta} L(\eta,\tau) S(\tau, u(\tau)) \Delta\tau \right) \Delta\eta,$$

with $c \geq 0$,for all $t \in \mathbb{T}^{k,+}_{t_0}$, implies

$$u(t) \leq c + \int_{t_0}^{t} f(\eta)((S(t_0,0) + R(t_0,0)c)\, e_{A^*}(\eta, t_0) + \int_{t_0}^{\eta} e_{A^*}(\eta, \sigma(\tau)) B^*(\tau)\Delta\tau)\Delta\eta,$$

for all $t \in \mathbb{T}^{k,+}_{t_0}$, with

$$A^*(t) := R(\sigma(t),0) f(t) + \frac{R^{\Delta_t}(t,0)}{R(t,0)} + L(\sigma(t),t) + \int_{t_0}^{t} L^{\Delta_t}(t,\tau)\Delta\tau$$

and

$$B^*(t) := S^{\Delta_t}(t,0) + L(\sigma(t),t) S(t,0) + \int_{t_0}^{t} L^{\Delta_t}(t,\tau) S(\tau,0)\Delta\tau.$$

4.3.2 h-stability via integral inequalities

In this Section, we study h- stability of the abstract dynamic equation (1.1) in term of the h-stability of the homogeneous equation (1.2) i.e.

$$z^{\Delta}(t) = Az(t) + f(t,z), 0 \neq z(t_0) = z_0 \in D(A),\ t, t_0 \in \mathbb{T}^+_0,\ t \geq t_0.$$

in terms of the h-stability of the homogeneous equation :

$$z^{\Delta}(t) = Az(t), 0 \neq z(t_0) = z_0 \in D(A), t, t_0 \in \mathbb{T}^+_0, t \geq t_0.$$

Here, we assume that the values $z(t) \in X$ and $f : \mathbb{T}_0^+ \to X$ is an rd-continuous function. Assume initially that A is the infinitesimal generator of a C_0-semigroup $T : \mathbb{T}_0^+ \to X$.

Proceeding as indicated in [[21], Theorem 4.13], we can show that problem (1.2) has a unique solution $u(t,s)$ for all $z \in X$. We define $z(t,t_0) = T(t,t_0)z$. It is not difficult to see that

$$T : \left\{(t,t_0) : \ t_0, t \in \mathbb{T}_0^+, \ t \geq t_0\right\} \to \mathcal{L}(X),$$

is a strongly continuous map, for every initial value $z_0 \in D(A)$.

Lemma 4.3.5 [21]*Let $A \in \mathcal{L}(X)$ and assume that $f : \mathbb{T}_0^+ \to X$ is rd-continuous. Then the mild solution $z(t)$ of the problem* (1.2) *is given by*

$$z(t) = T(t,t_0)z + \int_{t_0}^{t} T(t,\sigma(\tau))f(\tau, z(\tau))\Delta\tau. \tag{3.9}$$

Remark 4.3.1 *If $\mathbb{T} = \mathbb{R}_+$, then (3.9) coincides with the classical solution of non-homogeneous IVP*

$$\begin{aligned} z'(t) &= Az(t) + f(t, z(t)), \ t > 0, \\ z(t_0) &= z_0. \end{aligned} \tag{3.10}$$

i.e.

$$z(t) = e^{A(t-t_0)}z_0 + \int_{t_0}^{t} e^{A(t-s)}f(s, z(s))\Delta s. \tag{3.11}$$

If $\mathbb{T} = h\mathbb{Z}_+$ $(h > 0)$,then (3.9) take the form of

$$z(t) = (I + hA)^{\frac{t-t_0}{h}}z_0 + \int_{t_0}^{t} (I + A)^{\frac{t-s-h}{h}} f(s, z(s))\Delta s, \tag{3.12}$$

for more examples, we can see [21].

The following definition is important in establishing the following theorem.

Definition 4.3.1 *([1]) A function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to calss $\widehat{H}$ if*

(H_1) $g(u)$ is non decreasing and continuous for $u \geq 0$ and positive for $u > 0$,

(H_2) there exists a continuous function ϕ on $\mathbb{R}_+$ with $g(\alpha u) \leq \phi(\alpha)g(u)$ for $\alpha > 0, u \geq 0$,

Now, we investigate the $h-$stability of such time-varying perturbed systems under different conditions on the perturbed term using Grönwall-Bihari and Pachpatte- type integral inequality.

Theorem 4.3.1 *If the following conditions are satisfied*

i) Equation (1.2) is globally uniformly h-stable.

ii) $f(t,z) \leq g(t)w(\|z\|), t \in \mathbb{T}_0^+$, *where* g *is a positive and rd-continuous,and* $w \in \widehat{H}$ *with corresponding multiplier function* Φ *and* r *be the solution of*

$$r^{\Delta}(t) = p(t)w(r(t)),\ r(t_0) = \gamma,$$

we assume that there is a bijective function W *satisfying*

$$(W \circ r)^{\Delta} = p \qquad \text{with} \int_{t_0}^{\infty} p(s)\Delta s < \infty, \tag{3.13}$$

where

$$p(t) = \frac{\gamma g(t)}{\|z_0\| \left(h(t_0)\right)^{-1} h(\sigma(t))} \phi\left(\left(h(t_0)\right)^{-1} h(t) \|z_0\|\right), \tag{3.14}$$

then equation (1.1) is globally uniformly h-stable.

Preuve. Eq (1.2) is h-stable. Then, there exists a positive bounded function h defined on $\mathbb{T}_0^+$ and there is a constant $\gamma \geq 1$ such that for any solution $z(t,t_0) = T(t,t_0)z_0$ of (1.2) with intial value $z_0 \in D(A)$, we have

$$\|T(t,t_0)z_0\| \leq \gamma \|z_0\| h(t) \left(h(t_0)\right)^{-1},\ t \in \mathbb{T}_0^+. \tag{3.15}$$

The solution of equation (1.1) is given by

$$z(t) = T(t,t_0)z_0 + \int_{t_0}^{t} T(t,\sigma(s))f(s,z(s))\Delta s,\ t \in \mathbb{T}_0^+.$$

From condition $ii)$, and for any $t_0 \in \mathbb{T}_0^+$ and $z_0 = z(t_0)$ and from (3.15), we obtain

$$\begin{aligned}
\|z(t)\| &= \|T(t,t_0)z_0\| + \int_{t_0}^{t} \|T(t,\sigma(s))\| \|f(s,(z(s))\| \Delta s \\
&\leq \gamma \|z_0\| h(t) \left(h(t_0)\right)^{-1} + \gamma h(t) \int_{t_0}^{t} \frac{g(s)}{h(\sigma(s))} w(\|z(s)\|)\Delta s \\
&\leq h(t) \left(h(t_0)\right)^{-1} \|z_0\| \left[\gamma + \gamma \int_{t_0}^{t} \frac{g(s)}{\left(h(t_0)\right)^{-1} h(\sigma(s)) \|z_0\|} w(\|z(s)\|)\Delta s\right].
\end{aligned}$$

The last inequality can be reformulated as

$$\frac{h(t_0)\,\|z(t)\|}{h(t)\,\|z_0\|} \leq \gamma + \gamma \int_{t_0}^{t} \frac{g(s)}{\|z_0\|\,(h(t_0))^{-1}\,h(\sigma(s))} w\left((h(t_0))^{-1}\,\|z_0\|\,h(s) \frac{h(t_0)\,\|z(s)\|}{h(s)\,\|z_0\|} \right) \Delta s.$$

Letting $u(t) = \frac{h(t_0)\|z(t)\|}{(h(t))\|z_0\|}$, then the above inequality becomes

$$u(t) \leq \gamma + \int_{t_0}^{t} \frac{\gamma g(s)}{\|z_0\|\,(h(t_0))^{-1}\,h(\sigma(s))} w\left((h(t_0))^{-1}\,\|z_0\|\,h(s)u(s)\right) \Delta s.$$

Since $w \in \widehat{H}$, then we have

$$\begin{aligned} u(t) &\leq \gamma + \int_{t_0}^{t} \frac{\gamma g(s)}{\|z_0\|\,(h(t_0))^{-1}\,h(\sigma(s))} \phi\left((h(t_0))^{-1}\,\|z_0\|\,h(s)\right) w(u(s))\Delta s, \\ u(t) &\leq \gamma + \int_{t_0}^{t} p(s)w(u(s))\Delta s, \end{aligned}$$

where p is defined as in in (3.14). Applying Lemma 4.3.3, we get

$$\begin{aligned} u(t) &\leq W^{-1}\left[W(\gamma) + \int_{t_0}^{t} p(s)\Delta s\right] \\ &\leq W^{-1}\left[W(\gamma) + \int_{t_0}^{\infty} p(s)\Delta s\right], \end{aligned}$$

for all $t \in \mathbb{T}_0^+$.

Then, we have

$$\|z(t)\| \leq d\frac{h(t)\,\|z_0\|}{h(t_0)},$$

where

$$d = W^{-1}\left[W(\gamma) + \gamma \int_{t_0}^{\infty} p(s)\Delta s\right].$$

It is easy to prove that $d \geq 1$.Which characterizes the h-stability of equation (1.1). ■

Theorem 4.3.2 *If the following conditions are satisfied:*

i) Equation (1.2) is globally uniformly h-stable.

ii) The perturbed term satisfies $\|f(t, z(t))\| \leq l(t)\,\|z\| + e(t)$,*where* l, e *are nonnegative rd-continuous functions.*

iii) There exists $a, b \geq 0$ *such that*
$\int_{t_0}^{\infty} \frac{l(s).h(s)}{h(\sigma(s))}\Delta s \leq a < +\infty$ *and* $\frac{h(t_0)}{\|z_0\|} \int_{t_0}^{\infty} \frac{e(s)}{h(\sigma(s))}\Delta s \leq b < +\infty$,
then equation (1.1)is globally uniformly h-stable.

Preuve. Equation (1.2) is globally uniformly h-stable. Then, there exists a positive bounded function h defined on $\mathbb{T}_0^+$ and there is a constant $\gamma \geq 1$ such that for any solution $z(t) = T(t - t_0)z_0$ of (1.2) with intial value $z_0 \in D(A)$, we have

$$\|T(t, t_0)z_0\| \leq \gamma \|z_0\| h(t) (h(t_0))^{-1}, \ t \in \mathbb{T}_0^+.$$

The solution of equation (1.1) satisfies

$$\begin{aligned} \|z(t)\| &\leq \|T(t, t_0)z_0\| + \int_{t_0}^{t} \|T(t, \sigma(s))\| \|f(s, z(s)\| \Delta s \\ &\leq \gamma \|z_0\| h(t) (h(t_0))^{-1} + \gamma h(t) \int_{t_0}^{t} \left\{ \frac{l(s)}{h(\sigma(s))} \|z(s)\| + \frac{e(s)}{h(\sigma(s))} \right\} \Delta s. \end{aligned}$$

Using Lemma 4.3.1 to the above inequality, we obtain that

$$\begin{aligned} \|z(t)\| &\leq \gamma \|z_0\| h(t)(h(t_0))^{-1} + \gamma h(t) \int_{t_0}^{t} \frac{l(s)}{h(\sigma(s))} (\gamma \|z_0\| h(s) (h(t_0))^{-1}) + \frac{e(s)}{h(\sigma(s))}) \\ &\times \exp \left[\gamma \int_{\sigma(s)}^{t} \frac{l(\tau)h(\tau)}{h(\sigma(\tau))} \Delta\tau \right] \Delta s. \end{aligned}$$

It follows that

$$\begin{aligned} \|z(t)\| &\leq \gamma \|z_0\| h(t) (h(t_0))^{-1} + \gamma^2 \|z_0\| h(t) (h(t_0))^{-1} \\ &\int_{t_0}^{t} (\frac{l(s)h(s)}{h(\sigma(s))} + \frac{e(s)}{\gamma h(\sigma(s))(h(t_0))^{-1} \|z_0\|}) \times \exp \left(\gamma \int_{\sigma(s)}^{t} \frac{l(\tau)h(\tau)}{h(\sigma(\tau))} \Delta\tau \right) \Delta s. \end{aligned}$$

From conditions $ii)$ and $iii)$, and taking into-account the fact that $\gamma \geq 1$,we obtain

$$\|z(t)\| \leq \gamma \|z_0\| h(t) (h(t_0))^{-1} [1 + \gamma(a + b)e^{\gamma a}],$$

$$\|z(t)\| \leq c \|z_0\| h(t) (h(t_0))^{-1}, \text{ where } c = \gamma(1 + \gamma(a + b)e^{\gamma a}) \geq 1.$$

Then equation (1.1) is globally uniformly h-stable. ■

Theorem 4.3.3 *If the following conditions are satisfied:*

i) Equation (1.2) iss globally uniformly h-stable,

ii) The perturbed term satisfies $\|f(t, z(t))\| \leq l(t) \|z\|^p + e(t), p \in]0, 1[$,

iii) There exists $a', b' \geq 0$ *such that* $\int_{t_0}^{\infty} \frac{l(s).h(s)}{h(\sigma(s))} \Delta s \leq a < +\infty'$ *and* $\frac{h(t_0)}{p\|z_0\|} \int_{t_0}^{\infty} \frac{(1-p)l(s)+e(s)}{h(\sigma(s))} \Delta s < b' < +\infty$,

then equation (1.1) is s globally uniformly h-stable.

Preuve. Equation (1.2) is s globally uniformly h-stable. Then, there exists a positive bounded function h defined on $\mathbb{T}_0^+$ and there is a constant $\gamma \geq 1$ such that for any solution $z(t) = T(t-t_0)z_0$ of (1.2) with intial value $z_0 \in D(A)$, we have

$$\|T(t,t_0)z_0\| \leq \gamma \|z_0\| h(t) (h(t_0))^{-1}, \ t \in \mathbb{T}_0^+.$$

The solution of equation (1.1) satisfies

$$\begin{aligned} \|z(t)\| &\leq \|T(t,t_0)z_0\| + \int_{t_0}^t \|T(t,\sigma(s))\| \|f(s,z(s)\| \Delta s \\ &\leq \gamma \|z_0\| h(t) (h(t_0))^{-1} + \gamma h(t) \int_{t_0}^t \left\{ \frac{l(s)}{h(\sigma(s))} \|z(s)\|^p + \frac{e(s)}{h(\sigma(s))} \right\} \Delta s. \end{aligned}$$

Using Lemma 4.3.2 to the above inequality, we obtain

$$\begin{aligned} \|z(t)\| \leq\ & \gamma \|z_0\| h(t) (h(t_0))^{-1} + \gamma h(t) \int_{t_0}^t (\frac{l(s)}{h(\sigma(s))} (p\gamma \|z_0\| h(s) (h(t_0))^{-1} + 1 - p) \\ & + \frac{e(s)}{h(\sigma(s))}) \times \exp \left[p\gamma \int_{\sigma(s)}^t \frac{l(\tau)h(\tau)}{h(\sigma(\tau))} \Delta\tau \right] \Delta s. \end{aligned}$$

It follows that

$$\begin{aligned} \|z(t)\| \leq\ & \gamma \|z_0\| h(t) (h(t_0))^{-1} + p\gamma^2 \|z_0\| h(t) (h(t_0))^{-1} \\ & \times \int_{t_0}^t (\frac{l(s)h(s)}{h(\sigma(s))} + \frac{(1-p)l(s) + e(s)}{p\gamma h(\sigma(s))(h(t_0))^{-1} \|z_0\|}) \times \exp \left[p\gamma \int_{\sigma(s)}^t \frac{l(\tau)h(\tau)}{h(\sigma(\tau)} \Delta\tau \right] \Delta s. \end{aligned}$$

From conditions *ii)* and *iii)* and taking into account the fact that $\gamma \geq 1$,we get

$$\|z(t)\| \leq \gamma \|z_0\| h(t) (h(t_0))^{-1} \left[1 + p\gamma(a' + b') e^{p\gamma a'} \right],$$

the last inequality can be reformulated as

$$\|z(t)\| \leq c \|z_0\| h(t) (h(t_0))^{-1}, \text{ where } c = \gamma(1 + p\gamma(a' + b')e^{p\gamma a'}) \geq 1.$$

Then equation (1.1) is s globally uniformly h-stable. ■

Theorem 4.3.4 *If the following conditions are satisfied:*

i) Equation (1.2) is s globally uniformly h-stable.

ii) $\|f(t,z)\| \leq \eta (l(t) \|z\|^p + e(t))$, *where* $l,\ e \in C_{rd}(\mathbb{T}_0^+, \mathbb{R}_+)$ *and* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ *is a differentiable increasing function on* $]0, +\infty[$ *with continuous nonincreasing first derivative* η' *on* $]0, \infty[$, $p \in]0, 1[$.

iii) $\int_{t_0}^{+\infty} \frac{\eta'(e(s))l(s)h(s)\Delta s}{h(\sigma(s))} \leq \mu < +\infty,\ \frac{h(t_0)}{p\|z_0\|}\int_{t_0}^{+\infty} \frac{(1-p)\eta'(e(s))l(s)+\eta(e(s))}{h(\sigma(s))}\Delta s \leq \nu < +\infty,$

then equation (1.1) is s globally uniformly h-stable.

Preuve. Equation (1.2) is s globally uniformly h-stable. Then, there exists a positive bounded function h defined on $\mathbb{T}_0^+$ and there is a constant $\gamma \geq 1$ such that for any solution $z(t) = T(t-t_0)z_0$ of (1.2) with intial value $z_0 \in D(A)$, we have

$$\|T(t,t_0)z_0\| \leq \gamma \|z_0\| h(t) (h(t_0))^{-1},\ t \in \mathbb{T}_0^+.$$

The solution of equation(1.1) satisfies

$$\|z(t)\| \leq \gamma \|z_0\| h(t) (h(t_0))^{-1} + \gamma h(t) \int_{t_0}^{t} \tfrac{1}{h(\sigma(s))}\eta\left(l(s)\|z\|^p + e(s)\right)\Delta s,\ \text{with } p \in]0,1[\,. \tag{3.16}$$

Applying the mean value Theorem for the function η, then for every $z_1 \geq y_1 > 0$, there exists $c \in]y_1, z_1[$ such taht

$$\eta(z_1) - \eta(y_1) = \eta'(c)(z_1 - y_1) \leq \eta'(y_1)(z_1 - y_1).$$

which gives:

$$\eta\left(l(s)\|z\|^p + e(s)\right) \leq \eta'(e(s)) \times l(s)\|z\|^p + \eta(e(s)). \tag{3.17}$$

Then, inequality (3.16) implies

$$\|z(t)\| \leq \gamma \|z_0\| h(t) (h(t_0))^{-1} + \gamma h(t) \int_{t_0}^{t} \left\{ \tfrac{\eta'(e(s))l(s)}{h(\sigma(s))}\|z\|^p + \tfrac{\eta(e(s))}{h(\sigma(s))} \right\}\Delta s\ . \tag{3.18}$$

Applying Lemma 4.3.2, we obtain

$$\begin{aligned}\|z(t)\| \leq\ & \gamma \|z_0\| h(t) (h(t_0))^{-1} + \gamma h(t) \int_{t_0}^{t} \left(\tfrac{\eta'(e(s))l(s)}{h(\sigma(s))}(p\gamma h(s)\|z_0\|(h(t_0))^{-1} + 1 - p)\right. \\ & \left. + \frac{\eta(e(s))}{h(\sigma(s))}\right) \exp\left[p\gamma \int_{\sigma(s)}^{t} \frac{\eta'(e(\tau))l(\tau)h(\tau)}{h(\sigma(\tau))}\Delta\tau\right]\Delta s.\end{aligned}$$

From conditions $ii)$ and $iii)$ and taking into-account the fact that $\gamma \geq 1$, we get

$$\|z(t)\| \leq \gamma \|z_0\| h(t) (h(t_0))^{-1}\left[(1 + p\gamma(\mu+\nu)e^{p\gamma\mu}\right],$$

$$\|z(t)\| \leq c\|z_0\| h(t) (h(t_0))^{-1},\ \text{where } c = \gamma(1 + p\gamma(\mu+\nu)e^{p\gamma\mu}) \geq 1.$$

Then equation (1.1) is s globally uniformly h-stable. ■

Theorem 4.3.5 *If the following conditions are satisfied*

i) Equation (1.2) is s globally uniformly h-stable,

ii)$\|f(t,z)\| \leq g(t)w(\|z\|), t \in \mathbb{T}_0^+$,*where g is a positive and rd-continuous function,* $w \in \mathfrak{F}$ *and r be the solution of*

$$r^{\Delta}(t) = p_1(t)w(r(t)),\ r(t_0) = \gamma,$$

we assume that there is a bijective function W satisfying

$$(W \circ r)^{\Delta} = p_1 \qquad \text{with} \int_{t_0}^{\infty} p_1(s)\Delta s < \infty,$$

where

$$p_1(s) = \frac{\gamma g(s)h(s)}{h(\sigma(s))}.$$

then equation (1.1) iss globally uniformly h-stable.

Preuve. Equation (1.2) is s globally uniformly h-stable, Then, there exists a positive bounded function h defined on $\mathbb{T}_0^+$ and there is a constant $\gamma \geq 1$ such that for any solution $z(t) = T(t-t_0)z_0$ of (1.2) with intial value $z_0 \in D(A)$, we have

$$\|T(t,t_0)z_0\| \leq \gamma \|z_0\| h(t) (h(t_0))^{-1},\ t \in \mathbb{T}_0^+.$$

The solution of equation (1.1) satisfies

$$\begin{aligned} \|z(t)\| &\leq \|T(t,t_0)z_0\| + \int_{t_0}^{t} \|T(t,\sigma(s))\| \, \|f(s,z(s)\| \, \Delta s \\ &\leq \|T(t,t_0)z_0\| + \int_{t_0}^{t} \|T(t,\sigma(s))\| \, g(s) \, (h(t_0))^{-1} \|z_0\| \, h(s) \frac{w(\|z(s)\|)}{(h(t_0))^{-1}h(s) \, \|z_0\|} \Delta s, \end{aligned}$$

taking unto- account the fact that $w \in \mathfrak{F}$,the last inequality can be rewritten as

$$\|z(t)\| \leq \gamma \|z_0\| h(t) (h(t_0))^{-1} + \gamma h(t) \int_{t_0}^{t} \frac{g(s) (h(t_0))^{-1} \|z_0\| h(s)}{h(\sigma(s))} w\left(\frac{1}{(h(t_0))^{-1} \|z_0\| h(s)} \|z(s)\|\right)\Delta s,$$

the above inequality can be reformulated as

$$u(t) \leq \gamma + \int_{t_0}^{t} p_1(s)w(u(s))\Delta s, t \in \mathbb{T}_0^+,$$

where

$$\begin{aligned} u(t) &= \frac{\|z(t)\|}{h(t_0)^{-1}\,(h(t))\,\|z_0\|}, \\ p_1(s) &= \frac{\gamma g(s)h(s)}{h(\sigma(s))}. \end{aligned}$$

Applying Lemma 4.3.3 to the above inequality, we obtain

$$\begin{aligned} u(t) &\leq W^{-1}\left[W(\gamma) + \int_{t_0}^{t} p_1(s)\Delta s\right] \\ &\leq W^{-1}\left[W(\gamma) + \int_{t_0}^{\infty} p_1(s)\Delta s\right], \end{aligned}$$

for all $t \in \mathbb{T}_0^+$.Then, we have

$$\|z(t)\| \leq dh(t)\,(h(t_0))^{-1}\,\|z_0\|,$$

where

$$d = W^{-1}\left[W(\gamma) + \int_{t_0}^{\infty} p_1(s)\Delta s\right].$$

Then equation (1.1) is globally uniformly h-stable. ■

Theorem 4.3.6 *If the following conditions are satisfied*

i) Equation (1.2) is globally uniformly h-stable,

ii) Assume that there exist $l,\ y, \in Crd.(\mathbb{T}_0^+, \mathbb{R}_+)$ *that satisfy the following conditions*

$$\begin{aligned} \|f(t,z(t))\| &\leq l(t)(S(t,\|z(t)\|) + y(t)), \\ y^{\Delta}(t) &\leq g(t)S(t,\|z(t)\|),\ y(t_0) = 0, \end{aligned} \tag{3.19}$$

iii) There exists positive constants m *and* m' *such that*

$$\begin{aligned} \int_{t_0}^{t} \frac{l(\eta)}{h(\sigma(\eta))} & \left((S(t_0,0) + R(t_0,0)c)\, e_{A^*}(\eta,t_0) + \int_{t_0}^{\eta} e_{A^*}(\eta,\sigma(\tau))B^*(\tau)\Delta\tau \right)\Delta\eta \\ & \leq m\delta, \forall\delta \in \mathbb{R}_+, \forall t_0 \in \mathbb{T}_0^+, \end{aligned} \tag{3.20}$$

$$\begin{aligned} \int_{t_0}^{t} \frac{l(\eta)h(\eta)}{h(\sigma(\eta))} & \left((M(t_0,0) + R(t_0,0)c)\, e_{A_1^*}(\eta,t_0) + \int_{t_0}^{\eta} e_{A_1^*}(\eta,\sigma(\tau))B_1^*(\tau)\Delta\tau \right)\Delta\eta \\ & \leq m'\delta', \forall\delta' \in \mathbb{R}_+, \forall t_0 \in \mathbb{T}_0^+, \end{aligned} \tag{3.21}$$

where $M(t,u(t)) = \frac{1}{h(t)} S(t, h(t)u(t))$ *where* S *satisfies (3.7) and the following condition*

$$S^{\Delta_t}(t,0)h(t) \geq S(t,0)h^{\Delta}(t), \tag{3.22}$$

then equation (1.1) is globally uniformly h-stable.

Preuve. Equation (1.2) is globally uniformly h-stable. Then, there exists a positive bounded function h defined on $\mathbb{T}_0^+$ and there is a constant $\gamma \geq 1$ such that for any solution $z(t) = T(t-t_0)z_0$ of (1.2) with intial value $z_0 \in D(A)$, we have

$$\|T(t,t_0)z_0\| \leq \gamma \|z_0\| h(t) (h(t_0))^{-1}, \ t \in \mathbb{T}_0^+.$$

The solution of equation (1.1) satisfies

$$\begin{aligned}
\|z(t)\| &\leq \|T(t,t_0)z_0\| + \int_{t_0}^{t} \|T(t,\sigma(s))\| \|f(s,z(s)\| \Delta s \\
&\leq \|T(t,t_0)z_0\| + \int_{t_0}^{t} \|T(t,\sigma(s))\| l(s) \left(S(s, \|z(s)\|) + \int_{t_0}^{s} g(\tau) S(\tau, \|z(\tau)\|) \Delta\tau \right) \Delta s \\
&\leq \gamma \|z_0\| h(t) (h(t_0))^{-1} + \gamma h(t) \int_{t_0}^{t} \frac{l(s)}{h(\sigma(s))} \left(S(s, \|z(s)\|) + \int_{t_0}^{s} g(\tau) S(\tau, \|z(\tau)\|) \Delta\tau \right) \Delta s,
\end{aligned}$$

the above inequality can be reformulated as

$$u(t) \leq c + \int_{t_0}^{t} \frac{\gamma l(s)}{h(\sigma(s))} \left(S(s, h(s) \|u(s)\|) + \int_{t_0}^{s} g(\tau) S(\tau, h(\tau) \|u(\tau)\|) \Delta\tau \right) \Delta s, \tag{3.23}$$

where $u(t) = \frac{\|z(t)\|}{(h(t))}$, and $c = \gamma h(t_0)^{-1} \|z_0\|$,

-if $h(t) \leq 1$, then $S(t, h(t)\|u(t)\|) \leq S(t, \|u(t)\|)$,

then, we obtain

$$u(t) \leq c + \int_{t_0}^{t} \frac{\gamma l(s)}{h(\sigma(s))} \left(S(s, \|u(s)\|) + \int_{t_0}^{s} g(\tau) S(\tau, \|u(\tau)\|) \Delta\tau \right) \Delta s, \tag{3.24}$$

applying Lemma 4.3.4, with $L(t,s) = g(s)$,we obtain

$$\begin{aligned}
\|z(t)\| &\leq \gamma h(t) h(t_0)^{-1} \|z_0\| + \gamma h(t) \textstyle\int_{t_0}^{t} \frac{l(s)}{h(\sigma(s))} ((S(t_0,0) + R(t_0,0)c)\, e_{A^*}(s,t_0) \\
&\quad + \textstyle\int_{t_0}^{s} e_{A^*}(s,\sigma(\tau)) B^*(\tau) \Delta\tau) \Delta s,
\end{aligned} \tag{3.25}$$

for all $t \in \mathbb{T}_0^{k,+}$, with

$$A^*(t) := R(\sigma(t),0)\frac{l(t)}{h(\sigma(t))} + \frac{R^{\Delta_t}(t,0)}{R(t,0)} + g(t),$$

and

$$B^*(t) := S^{\Delta_t}(t,0) + g(t)S(t,0),$$

Using inequality (3.20) of Assumption (iii), from (3.25) one can obtain for $\delta = \|z_0\| h(t_0)^{-1}$, $\|z(t)\| \leq \gamma(1+m)h(t)h(t_0)^{-1}\|z_0\|$ .Therefore, the equation (1.1) is globally uniformly h-stable.

-If $h(t) \geq 1$,then

$$\begin{aligned}\|z(t)\| &\leq \gamma \|z_0\| h(t) (h(t_0))^{-1} \\ &+\gamma h(t) \int_{t_0}^t \tfrac{l(s)h(s)}{h(\sigma(s))} \left(\tfrac{1}{h(s)} S\left(s, h(s)\tfrac{\|z(s)\|}{h(s)}\right) + \tfrac{1}{h(s)} \int_{t_0}^s g(\tau)\tfrac{1}{h(\tau)} S\left(\tau, h(\tau)\tfrac{\|z(\tau)\|}{h(\tau)}\right)\Delta\tau\right)\Delta s,\end{aligned} \tag{3.26}$$

the inequality (3.26) can be rewritten as :

$$\begin{aligned}\tfrac{\|z(t)\|}{h(t)} &\leq \gamma \|z_0\| (h(t_0))^{-1} \\ &+ \int_{t_0}^t \tfrac{\gamma l(s)h(s)}{h(\sigma(s))} \left(\tfrac{1}{h(s)} S\left(s, h(s)\tfrac{\|z(s)\|}{h(s)}\right) + \tfrac{1}{h(s)} \int_{t_0}^s g(\tau)\tfrac{1}{h(\tau)} S\left(\tau, h(\tau)\tfrac{\|z(\tau)\|}{h(\tau)}\right)\Delta\tau\right)\Delta s,\end{aligned} \tag{3.27}$$

one can reformulate (3.27) as :

$$u(t) \leq c + \int_{t_0}^t \frac{\gamma l(s)h(s)}{h(\sigma(s))}\left(M\left(s,u(s)\right) + \int_{t_0}^s g(\tau)M\left(\tau,u(\tau)\right)\Delta\tau\right)\Delta s,$$

where $M(t,u(t)) = \frac{1}{h(t)}S(t,h(t)u(t))$ and $u(t) = \frac{\|z(t)\|}{h(t)}$.Clearly, M verifes relation (3.7), i.e.

$M(t,z) - M(t,y) \leq R_1(t,y)(z-y), z \geq y \geq 0, t \in \mathbb{T}_0^+$, with $R_1(t,y) = R(t,h(t)y)$. From our hypothesis we see that M and R_1 verify relation (3.8). Using Lemma 4.3.5 , it yields

$$\begin{aligned}\|z(t)\| &\leq \gamma h(t)h(t_0)^{-1}\|z_0\| + \gamma h(t)\int_{t_0}^t \tfrac{l(s)h(s)}{h(\sigma(s))}\left(\left(M(t_0,0) + R(t_0,0)c\right)e_{A_1^*}(s,t_0)\right. \\ &+ \left.\int_{t_0}^s e_{A_1^*}(s,\sigma(\tau))B_1^*(\tau)\Delta\tau\right)\Delta s,\end{aligned} \tag{3.28}$$

$$A_1^*(t) := R(\sigma(t),0)\frac{\gamma l(t)h(t)}{h(\sigma(t))} + \frac{R^{\Delta_t}(t,0)}{R(t,0)} + g(t),$$

and

$$B_1^*(t) := M^{\Delta_t}(t,0) + g(t)M(t,0),$$

Using inequality (3.21) of Assumption (iii), from (3.28), we obtain with $\delta' = h(t_0)^{-1}\|z_0\|$, that $\|z(t)\| \leq \gamma(1+m')h\hat{(t)}h(t_0)^{-1}\|z_0\|$ .Therefore, the equation (1.1) is h-stable. ■

4.4 Applications

In this section, we give two illustrative examples to highlight the utility of our results.

Exemple 4.4.1 *Let us consider the following perturbed problem :*

$$\begin{aligned} z^{\Delta}(t) &= A\ z(t) + e_{-\frac{4}{5}}(t,0)z^{\frac{2}{3}}(t) + \tfrac{e_{-\frac{2}{5}}(t,0)}{(t+1)^2}, t, t_0 \in \mathbb{T} = \tfrac{1}{2}\mathbb{Z}_+, t \geq t_0, \\ z(t_0) &= z_0 \neq 0, \end{aligned} \tag{4.1}$$

where $\mathbb{Z}_+$ *is the set of positive or zero integers,* $z \in C_{rd}\left(\mathbb{T}_0^+, \mathbb{R}^2\right)$, $A = \begin{pmatrix} \frac{-2}{5} & -2 \\ 0 & -2 \end{pmatrix} \in M_2(\mathbb{R})$ *and* $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $f(t,z(t)) = e_{-\frac{4}{5}}(t,0)z^{\frac{2}{3}}(t) + \frac{e_{-\frac{2}{5}}(t,0)}{(t+1)^2}$

It is easy to see that the matrix A *is the generator of the following* C_0 *semi-group,*

$$T(t,0) = (I+\mu A)^{2t} = (I + \frac{1}{2}A)^{2t},$$

then,

$$T(t,0) = \begin{pmatrix} \frac{4}{5} & -1 \\ 0 & 0 \end{pmatrix}^{2t},$$

and

$$T(t,t_0) = \begin{pmatrix} \frac{4}{5} & -1 \\ 0 & 0 \end{pmatrix}^{2(t-t_0)} = (\frac{4}{5})^{2(t-t_0)}\begin{pmatrix} 1 & -\frac{5}{4} \\ 0 & 0 \end{pmatrix}, t \in \mathbb{T}_0^+,$$

therefore,

$$\|T(t,t_0)\| = \frac{\sqrt{41}}{4}(\frac{4}{5})^{2(t-t_0)} = \frac{\sqrt{41}}{4}e_{-\frac{2}{5}}(t,t_0),$$

by taking $\delta \geq \frac{\sqrt{41}}{2}$ *and* $h(t) = e_{-\frac{2}{5}}(t,0)$,*we get*

$$\|T(t,t_0)\| \leq \delta e_{-\frac{2}{5}}(t,0)e_{-\frac{2}{5}}(0,t_0) = \delta h(t)\left(h(t_0)\right)^{-1},$$

which ensures that the homogenous equation of the form $z^{\Delta}(t) = A\ z(t)$ *is globally uniformly h-stable.Furthermore, we have*

$$h(\sigma(t)) = e_{-\frac{2}{5}}(\sigma(t),0) = (1+\frac{1}{2}(\frac{-2}{5}))e_{-\frac{2}{5}}(t,0) = \frac{4}{5}e_{-\frac{2}{5}}(t,0).$$

Let know verify the others two conditions of Theorem 4.3.3.
It's clear that $\|f(t,z(t))\| \ \leq e_{-\frac{4}{5}}(t,0).\ \|z(t)\|^{\frac{2}{3}} + \frac{e_{-\frac{2}{5}}(t,0)}{(t+1)^2}$ *, also*

$$\begin{aligned}
\int_{t_0}^{\infty}\left(\frac{l(s)h(s)}{h(\sigma(s))}\right)\Delta s &= \int_{t_0}^{\infty}\left(\frac{e_{-\frac{4}{5}}(s,0)e_{\frac{-2}{5}}(s,0)}{\frac{4}{5}e_{-\frac{2}{5}}(s,0)}\right)\Delta s \\
&= \frac{5}{4}\int_{t_0}^{\infty} e_{-\frac{4}{5}}(s,0)\Delta s \\
&= \frac{5}{4}\sum_{s\in\frac{1}{2}\mathbb{Z}}\left(\frac{3}{5}\right)^{2s} \leq \frac{5}{4}\sum_{s=0}^{\infty}\left(\frac{9}{25}\right)^{s} < \infty.
\end{aligned}$$

$$\begin{aligned}
\frac{h(t_0)}{p\,\|z_0\|}\int_{t_0}^{\infty}\frac{\frac{1}{3}l(s)+e(s)}{h(\sigma(s))}\Delta s &= \frac{\frac{3}{2}e_{-\frac{2}{5}}(t_0,0)}{\|z_0\|}\int_{t_0}^{\infty}\left(\frac{\frac{1}{3}e_{-\frac{4}{5}}(s,0)+\frac{e_{-\frac{2}{5}}(s,0)}{(s+1)^2}}{\frac{4}{5}e_{-\frac{2}{5}}(s,0)}\right)\Delta s = \\
&\quad \frac{\frac{3}{2}e_{-\frac{2}{5}}(t_0,0)}{\|z_0\|}\int_{t_0}^{\infty}\left(\frac{\frac{1}{3}e_{-\frac{1}{2}}(s,0)e_{-\frac{2}{5}}(s,0)+\frac{e_{-\frac{2}{5}}(s,0)}{(s+1)^2}}{\frac{4}{5}e_{-\frac{2}{5}}(s,0)}\right)\Delta s \\
&= \frac{5e_{-\frac{2}{5}}(t_0,0)}{8\,\|z_0\|}\Big(\sum_{s\in\frac{1}{2}\mathbb{Z}_+}(\frac{9}{16})^{s} + 3\sum_{s\in\frac{1}{2}\mathbb{Z}_+}\frac{1}{(s+1)^2}\Big) < +\infty
\end{aligned}$$

All statements of Theorem 4.3.3 are approved. So, we deduce that the equation (4.1) is globally uniformly h-stable.

Exemple 4.4.2 *Consider the following time-varying system:*

$$\begin{aligned}
z_1^{\Delta} &= -z_1 + 2z_2 + \tfrac{1}{2}\arctan(e_{-\frac{4}{3}}(t,0)\,|z_1|^{\frac{1}{3}} + \tfrac{e_{-\frac{4}{3}}(t,0)|z_2|}{\sqrt{z_1^2+z_2^2+1}}), \\
z_2^{\Delta} &= -2z_2 + \tfrac{\sqrt{3}}{2}\arctan(e_{-\frac{4}{3}}(t,0)\,|z_1|^{\frac{1}{3}} + \tfrac{e_{-\frac{4}{3}}(t,0)|z_2|}{\sqrt{z_1^2+z_2^2+1}}), \\
z(0) &= (z_{1,0}, z_{2,0}),
\end{aligned} \tag{4.2}$$

where $z = (z_1, z_2)^T \in \mathbb{R}^2$.

System (4.2) can be written as :

$$z^{\Delta} = \begin{pmatrix} -1 & 2 \\ 0 & -2 \end{pmatrix} \times \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\arctan(e_{-\frac{4}{3}}(t,0)\,|z_1|^{\frac{1}{3}} + \frac{e_{-\frac{4}{3}}(t,0)|z_2|}{\sqrt{z_1^2+z_2^2+1}}) \\ \frac{\sqrt{3}}{2}\arctan(e_{-\frac{4}{3}}(t,0)\,|z_1|^{\frac{1}{3}} + \frac{e_{-\frac{4}{3}}(t,0)|z_2|}{\sqrt{z_1^2+z_2^2+1}}) \end{pmatrix} \tag{4.3}$$

where $A = \begin{pmatrix} -1 & 2 \\ 0 & -2 \end{pmatrix} \in M_2(\mathbb{R})$, $T(t,0) = (I+\mu A)^{2t} = (I + \frac{1}{2}A)^{2t} = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & 0 \end{pmatrix}^{2t}$,

$T(t,t_0) = (\frac{1}{2})^{2(t-t_0)} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}^{2t}$, therefore,

$$\|T(t,t_0)\| = \sqrt{5}(\frac{1}{2})^{2(t-t_0)} = \sqrt{5}e_{-1}(t,t_0),$$

by taking $\delta \geq \sqrt{5}$ and $h(t) = e_{-1}(t,0)$,we get

$$\|T(t,t_0)\| \leq \delta e_{-1}(t,0)e_{-1}(0,t_0) = \delta h(t)\,(h(t_0))^{-1},$$

which ensures that the homogenous equation of the form $z^{\Delta}(t) = A\ z(t)$ is globally uniformly h-stable.Furthermore, we have

$$h(\sigma(t)) = e_{-1}(\sigma(t),0) = (1 + \frac{-1}{2})e_{-1}(t,0) = \frac{1}{2}e_{-1}(t,0).$$

The perturbation satisfies condition $(\mathfrak{i})$ of Theorem 4.3.4 with $n(z) = \arctan(z)$ is a differentiable increasing function on$]0,\infty[$ with continuous nonincreasing first derivative.

$$\begin{aligned} \|f(t,z)\| &\leq \arctan(e_{-\frac{4}{3}}(t,0)\,\|z\|^{\frac{1}{3}} + e_{-\frac{4}{3}}(t,0)) = n(l(t)\,\|z\|^{\frac{1}{3}} + e(t)), \\ l(t) &= e_{-\frac{4}{3}}(t,0), e(t) = e_{-\frac{4}{3}}(t,0), \end{aligned} \tag{4.4}$$

we have

$$\begin{aligned} \int_{t_0}^{+\infty} \frac{\eta'(e(s))l(s)h(s)}{h(\sigma(s))}\Delta s &= 2\int_{t_0}^{+\infty} \frac{e_{-\frac{4}{3}}(s,0)e_{-1}(s,0)}{e_{-1}(s,0)(1+(e_{-\frac{4}{3}}(s,0))^2)}\Delta s \\ &\leq 2\int_{t_0}^{+\infty} e_{-\frac{4}{3}}(s,0)e_{-1}(s,0)\Delta s = \sum_{s\in\mathbb{T}_0^+}(\frac{1}{3})^{2s} \leq \sum_{s=0}^{\infty}(\frac{1}{9})^s < \infty, \end{aligned}$$

and

$$\begin{aligned}
\frac{h(t_0)}{p\|z_0\|}\int_{t_0}^{+\infty}\frac{\frac{2}{3}\eta'(e(s))l(s)+n(e(s))}{h(\sigma(s))}\Delta s &= \frac{3e_{-1}(t_0,0)}{\|z_0\|}\times \\
&\int_{t_0}^{+\infty}\Big(\frac{\frac{2}{3}e_{-\frac{4}{3}}(s,0)}{\frac{1}{2}e_{-1}(s,0)(1+(e_{-\frac{4}{3}}(s,0))^2}+\frac{\arctan(e_{-\frac{4}{3}}(s,0))}{\frac{1}{2}e_{-1}(s,0)}\Big)\Delta s \\
&\leq \frac{10e_{-1}(t_0,0)}{\|z_0\|}\int_{t_0}^{+\infty}\frac{e_{-1}(s,0)e_{-\frac{2}{3}}(s,0)}{e_{-1}(s,0)}\Delta s \qquad (4.5)\\
&= \frac{10e_{-1}(t_0,0)}{\|z_0\|}\int_{t_0}^{+\infty}e_{-\frac{2}{3}}(s,0)\Delta s \\
&= \frac{10e_{-1}(t_0,0)}{\|z_0\|}\sum_{s\in\frac{1}{2}\mathbb{Z}_+}\Big(\frac{2}{3}\Big)^{2s} \leq \frac{10e_{-1}(t_0,0)}{\|z_0\|}\sum_{s=0}^{\infty}\Big(\frac{4}{9}\Big)^{s} < +\infty.
\end{aligned}$$

All statements of Theorem 4.3.4 are approved. So, we deduce that the equation (4.2) is globally uniformly h-stable.

Conclusion This paper has been concerned with the problem of h-stability for perturbed system in bannach space.Sufficient conditions for h-stability of a class of abstract dynamic equations on arbitrary time scales are obtained using integral inequalities approach. On the mentioned topics, new theorems are proven. The obtained results include and improve some results in the literature. Moreover, two examples are given to illustrate the applicability of the main result.

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