الجمهورية الجزائرية الديمقراطية الشعبية
People's Democratic Republic of Algeria
وزارة التعليم العالي والبحث العلمي
Ministry of Higher Education and Scientific Research

Badji Mokhtar-Annaba University
Faculty of Science
Department of Mathematics

جامعة باجي مختار-عنابة -
كلية العلوم – قسم الرياضيات

THESE

Présentée en vue de l'obtention du diplôme de

Doctorat en Sciences

Option: Mathématiques appliquées

Existence et comportement asymptotique de quelques problèmes de type hyperbolique

Présenté par:

KIBECHE Khoudir

Directeur de thèse: BOUZETTOUTA Lamine MCA Univ. Skikda

Co-directeur de thèse: DJEBABLA Abdelhak Prof U.B.M. Annaba

Devant le jury

PRESIDENT	KILANI Brahim	MCA	U.B.M. Annaba
Examinateur	BOUMAZA Nouri	Prof	Univ. Tébessa
Examinateur	KHOCHEMANE Houssem Eddine	MCA	ENSET Skikda

Année 2024

الجمهورية الجزائرية الديمقراطية الشعبية
People's Democratic Republic of Algeria
وزارة التعليم العالي والبحث العلمي
Ministry of Higher Education and Scientific Research

Badji Mokhtar-Annaba University
Faculty of Science
Department of Mathematics

جامعة باجي مختار-عنابة -
كلية العلوم – قسم الرياضيات

THESE

Présentée en vue de l'obtention du diplôme de

Doctorat en Sciences

Option: Mathématiques appliquées

Existence and asymptotic behavior of some hyperbolic type problems

Présenté par:

KIBECHE Khoudir

Directeur de thèse: BOUZETTOUTA Lamine	MCA	Univ. Skikda
Co-directeur de thèse: DJEBABLA Abdelhak	Prof	U.B.M. Annaba

Devant le jury

PRESIDENT	KILANI Brahim	MCA	U.B.M. Annaba
Examinateur	BOUMAZA Nouri	Prof	Univ. Tébessa
Examinateur	KHOCHEMANE Houssem Eddine	MCA	ENSET Skikda

Année 2024

Acknowledgements

First, I want to thank Allah for all that has been given me strength, courage and above all knowledge. I am very grateful and indebted to my research advisor Dr. Lamine BOUZETTOUTA from 20 August 1955, Skikda University, who supervised this thesis with great patience, guidance, motivation, and kindness. He was able to motivate each step of my work by relevant remarks and let me able to advance in my research, it is my fortune to have a chance to work with him.

My warm thanks go to my co-advisor Pr. Abdelhak DJEBABLA from Annaba University for his guidance, toleration, encouragement, and support.

My thanks go also to Dr. KILANI Brahim from Annaba University for a jury president of this thesis.

I would also like to thank the member of my doctorate degree committee Pr. Nouri BOUMAZA from Tébessa University for his help and encouragement, I am also very grateful to Dr. Houssem Eddine KHOCHEMANE from ENSET Skikda.

I also want to use this occasion to thank my parents for always supporting me spiritually throughout my life, the culmination of their efforts and express my most affectionate gratitude.

I also thank my friends. Finally, a big thank you to all the people who have contributed from near or far to the completion of this thesis.

Dedication

To the memory of my sister

Abstract

This thesis is devoted to the study of the existence, the uniqueness and the asymptotic behavior of some hyperbolic type problems, in the first problem, we study the existence and the exponential stability of the solution of a porous-elastic system with microtemperatures and a distributed internal delay term, the proof which we have established the exponential stability is based on the construction an appropriate Lyapunov function equivalent to the energy of the solution considered. This function checks for a differential inequation leading to the result of the desired decay

The next problem is devoted to the study of the asymptotic behavior of the solution of a damped one-dimensional porous-elastic system with a single weakly nonlinear feedback and a distributed delay term, using a multiplier method and some properties of convex functions, we prove that the energy decreases explicitly and generally for equal propagation speeds. .

Then, in the last problem, we consider a one-dimensional thermoelastic system of full von Kármán beam with a delayed linear frictional damping, where the heat fux is given by Cattaneo's law. Under suitable assumption on the weight of the delay and that of frictional damping, we prove that the system is exponentially stable. The idea here, is to generalize some previous results existing in [19, 20, 21], by considering the delayed problem.

Keywords: Porous system, mecrotemperature, exponential stability, Wave equation, Varying delay term, Exponential stability, Semigroup theory, *Lyapunov* functional, delay terms in the feedbacks, distributed delay, viscoelastic, thermoelastic, decay rate.

2000 Mathematics Subject Classification: 35B40, 35L70, 93D15, 93D20. 35B40, 35L70, 93D15.

Résumé

Cette thèse est consacrée à l'étude de l'existence, de l'unicité et du comportement asymptotique de certains problèmes du type hyperbolique, dans le premier problème, nous étudions l'existence et la stabilité exponentielle de la solution d'un système poreux-élastique à microtempératures et à terme de retard interne distribué, la preuve que nous avons établie pour démontrer la stabilité exponentielle repose sur la construction d'une fonction de *Lyapunov* appropriée équivalente à l'énergie de la solution considérée. Cette fonction vérifie une inéquation différentielle conduisant au résultat.

Le problème suivant est consacré à l'étude du comportement asymptotique de la solution d'un système poreux-élastique unidimensionnel amorti avec une seule rétroaction faiblement non linéaire et un terme de retard distribué, en utilisant une méthode du multiplicateur et certaines propriétés des fonctions convexes, nous prouvons que l'énergie décroît explicitement et généralement pour des vitesses de propagation égales. .

Ensuite, et dans le dernier problème, nous considérons un système thermoélastique unidimensionnel de poutre pleine de von Kármán avec un amortissement de frottement linéaire retardé, où le flux de chaleur est donné par la loi de Cattaneo. Sous des hypothèses appropriées sur le poids du retard et celui de l'amortissement par frottement, nous prouvons que le système est exponentiellement stable. L'idée ici, est de généraliser certains résultats antérieurs existant dans [19, 20, 21], en considérant le problème retardé.

Mots-clés: Système poreux, Equation des ondes, Differentiales à retard, Stabilité exponentielle, Théorie de semi-groupes, Fonction de *Lyapunov*, Viscoelasticité, Thermoélastisité, Retard distribué, .

2000 Mathematics Subject Classification: 35B40, 35L70, 93D15, 93D20. 35B40, 35L70, 93D15.

ملخص

تهدف هذه الأطروحة لدراسة الوجود، الوحدانية والسلوك التقاربي لبعض المسائل من النوع الزائدي، حيث قمنا في المسألة الأولى بدراسة الوجود و الوحدانية و الإستقرار الأسي لحل جملة نظام مرن مسامي مع درجات حرارة دقيقة و بوجود حد التأخير الداخلي الموزع ، أثبتنا الإستقرار الأسي عن طريق بناء دالة *Lyapunov* مناسبة، مكافئة لطاقة النظام و تحقق متراجحة تفاضيلية، هذه الأخيرة

بدورها تؤدي مباشرة لتحقيق الإستقرار الأسي لطاقة النظام.

المسألة التالية ندرس فيها السلوك التقاربي لحل جملة نظام مرن مسامي أحادي البعد مخمد بردود فعل غير خطية ضعيفة و بوجود حد تأخير موزع، باستخدام طريقة المؤثر الضربي و بعض خصائص الدوال المحدبة، أثبتنا أن طاقة الجملة تتناقص بشكل صريح وعام تحت شرط تساوي سرعات الإنتشار.

في المسألة الأخيرة، إعتبرنا نظامًا حراريًا أحادي البعد لشعاع ***von Kármán*** الكامل بحد التخامد الإحتكاكي الخطي المتأخر، حيث التدفق الحراري يتبع قانون Cattaneo

تحت شروط مناسبة على دالتي وزن حد التأخير و حد التخامد الإحتكاكي، نثبت أن النظام مستقر بشكل كبير، النتائج المحصل عليها هي تعميم لبعض النتائج السابقة الموجودة في [19]، [20]، [21] من خلال إعتبار المسألة المتأخرة.

الكلمات المفتاحية:

نظام مسامي، الإستقرار الأسي، المرونة الحرارية، حد التأخير الموزع، معادلة الأمواج.

Publication resulting from this thesis

KIBECHE Khoudir,

BOUZETTOUTA Lamine,

DJEBABLA Abdelhak,

HEBHOUB Fahima,

Stability Result for a Weakly Nonlinearly Damped Porous System with Distributed Delay

- International Journal of Analysis and Applications

Indexation DGRST: SCOPUS**,** WEB OF SCIENCE**.**

Contents

0.1 Introduction

Porous media have become an important subject of study and increasing interest in recent years due to their role in the modeling required by various industrial applications such as geological storage of CO2, storage of waste radioactive, engineering of hydrocarbon reservoirs or geothermal energy. A scientific challenge facing geomechanics in this field is to model the flow in such materials as well as determine their permeability effective. Significant work has recently been devoted by various organizations to these questions, both theoretically and in terms of numerical modeling and experimental. The interest of this question also affects other fields such as hydrogeology (management of water resources), the environment (transport of pollutants in cracked soils) or civil engineering (waterproofing of concrete).

The theory of porous-elastic material has been established by Cowin and Nunziato [1] and Cowin [2] in 1985. In order to obtain a stability results, various types of dissipative mechanisms have been considered by many authors, temperature and microtemperature elements have been introduced in the theory by Lesan [3], and Lesan and Quintanilla [4]. We can refer to some references in this topic [5, 6, 7, 8]. Quintanilla [9] has studied the temporal decay in one-dimensional porous-elastic materials, and has found that porous viscosity was not sufficient to have exponential stability in the solutions. We note that only thermal damping or only porous damping leads to the slow decay of the solutions, but when both of them are considered, Casas and Quintanilla [10] has proved that exponentially stability of the solutions holds. In addition, in another paper [11], they have showed that mixing temperature and microtemperature gives rise to exponential stability. In this way, Santos , A. D. S. Campelo and D. A. J unior.[12] studied the porous elastic system when porous viscosity is coupled with microtemperature.

The main results of this thesis

Our main results in this thesis can be summarized as follows:

Chapter II. In this chapter, we consider a porous-elastic system with microtemperatures and internal distributed delay acting on the first equation

$$\begin{cases} \rho u_{tt}-\mu u_{xx}-b\phi_x+\mu_1 u_t+\int_{\tau_1}^{\tau_2}\mu_2(s)u_t\left(x,t-s\right)ds=0, \\ j\phi_{tt}-\delta\phi_{xx}+bu_x+\xi\phi+dw_x=0, \\ \alpha w_t-kw_{xx}+d\phi_{tx}+kw=0. \end{cases}$$

By using the semigroup approach, we prove the well-posedness of our problem. In addition, we prove that the unique dissipation due to the microtemperature is strong enough to exponentially stabilize the system when the speeds of wave propagation are equal.To achieve the decay estimate, we have introduced an appropriate multiplier method which leads to the desired result.

Chapter III. In this chapter, we consider a one-dimensional porous system damped with a single weakly nonlinear feedback and distributed delay term,

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x + \mu_1 u_t + \int_{\tau_1}^{\tau_2} \mu_2(s) u_t\left(x, t-s\right) ds = 0, \ x \in (0,1), \ t > 0, \\ j\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \alpha\left(t\right) g\left(\phi_t\right) = 0, \qquad x \in (0,1), \ t > 0, \\ u\left(x,0\right) = u_0\left(x\right), \ u_t\left(x,0\right) = u_1\left(x\right), \qquad x \in (0,1), \\ \phi\left(x,0\right) = \phi_0\left(x\right), \ \phi_t\left(x,0\right) = \phi_1\left(x\right), \qquad x \in (0,1), \\ u_x\left(0,t\right) = u_x\left(1,t\right), \ \phi\left(0,t\right) = \phi\left(1,t\right) = 0 \\ u_t\left(x,-t\right) = f_0\left(x,t\right) \qquad \text{in} \qquad (0,1) \times (0,\tau_2) \end{cases}$$

The aim of this chapter is to establish an explicit and general decay rate, using a multiplier method and some properties of convex functions in case of the same speed of propagation in the two equations of the system. The result is new and opens more research areas into porous-elastic system.

Chapter IV. This chapter is devoted to the study of the delayed system with Cattaneo's law and thermoelasticity with second sound

$$\begin{cases} w_{tt} - d_1\left[\left(u_x + \frac{1}{2}\left(w_x\right)^2\right) w_x\right]_x + d_2 w_{xxxx} + \mu_1 w_t + \mu_2 w_t(x, t-\tau) = 0, \\ u_{tt} - d_1\left[\left(u_x + \frac{1}{2}\left(w_x\right)^2\right)\right]_x + \delta\theta_x = 0, \\ \theta_t + q_x + \delta u_{tx} = 0, \\ q_t + \gamma q + \theta_x = 0, \end{cases}$$

In this chapter, we consider a one-dimensional thermoelastic system of full von Kármán beam with a delayed linear frictional damping, where the heat fux is given by Cattaneo's law. Under suitable assumption on the weight of the delay and that of frictional damping, we prove that the system is exponentially stable. The idea here, is to generalize some previous results existing in [19], [20], [21] by considering the delayed problem.

Chapter 1

Preliminary

In this chapter, we recall and state some necessary basic knowledge in fonctional analysis and some basic results which concerning the Layponov functionals and other theorems, most of which will be used in the subsequent chapter. The reader can easily find the detailed in the related literature, see e.g. [13, 14, 15, 16]

1.1 Functional Spaces

We denote by $\mathbb{R}^n$ the Euclid space, $\Omega \subseteq \mathbb{R}^n$ is a bounded smooth domain, $C^k(\Omega)$ is the k^{th} differentiable continuous function space in Ω, $C^\infty(\Omega)$ is the ∞^{th} differentiable continuous functions space in Ω , $C_c^\infty(\Omega)$ is the ∞^{th} differentiable continuous functions space with compact support in Ω

Definition 1.1.1 *Let X be a vector space over the filed K ($K = \mathbb{R}$ or $\mathbb{C}$). Then a semi-norm on X is a function $\|.\| : X \to \mathbb{R}$, such that:*

a) $\|x\| \geq 0$ for all $x \in X$,

b) $\|\alpha x\| = |\alpha| \, \|x\|$ *for all* $x \in X$ *and* $\alpha \in K$,

c) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$.

A norm on X *is a semi-norm wich also satisfies :*

d) $\|x\| = 0 \Longrightarrow x = 0$. *A vector space* X *toghether with a norm* $\|.\|$ *is called a normed vector space, or simply, a normed space.*

Definition 1.1.2 (*Convergent and Cauchy sequences*). *Let* X *be a normed space, and let* $(x_n)_{n \in \mathbb{N}}$ *be a sequence of elements of* X.

a) $(x_n)_{n \in \mathbb{N}}$ *converges to* $x \in X$ *if:* $\lim_{n \longrightarrow \infty} \|x_n - x\| = 0$,

i.e. if: $\forall \epsilon > 0; \exists N \in \mathbb{N}, \forall n \geq N, \|x_n - x\| < \epsilon$

b) $(x_n)_{n \in \mathbb{N}}$ *is a Cauchy sequence if:* $\forall \epsilon > 0; \exists N \in \mathbb{N}, \forall m, n \geq N, \|x_m - x_n\| < \epsilon$

Normed spaces in which every Cauchy sequence is convergent are called complet normed spaces. In general a normed space is not complete.

Definition 1.1.3 *(Banach Spaces). A normed spaces is called a Banach space if it is complet i.e. if any Cauchy sequence inside the space converges to a point of the space.*

Its dual space $X^{'}$ *is the vector space of all continuous linear functional* $f : X \rightarrow \mathbb{R}$

Proposition 1.1.1 $X^{'}$ *equipped with the norm* $\|.\|_{X'}$ *defined by*

$$\forall f \in X^{'}; \|f\|_{X'} = \sup \left\{ |f(u)|, \|u\|_X \leq 1 \right\}$$

is also a Banach space.

Remark 1.1.1 *From* $X^{'}$ *we construct the bidual or second dual* $X^{''} = \left(X^{'}\right)^{'}$*.Furthermore,with each* $u \in X$

we can define $\varphi(u) \in X^{''}$ *by* $(\varphi(u))(f) = f(u), f \in X^{'}$*, this satisfies clearly* $\|\varphi(u)\|_{X^{''}} \leq \|u\|_{X}, u \in X.$

Moreover, for each $u \in X$ *there is an* $f \in X^{'}$ *with* $f(u) = \|u\|_{X}^{2}$ *and* $\|f\|_{X^{'}} = \|u\|_{X}$*, so it follows that* $\|\varphi(u)\|_{X^{''}} = \|u\|_{X}.$

Proposition 1.1.2 *Since* φ *is linear we see that*

$$\varphi : X \to X^{''}$$

Proposition 1.1.3 *is a linear isometry of* X *onto a closed subspace of* $X^{''}$*, we denote this by* $X \hookrightarrow X^{''}$

Definition 1.1.4 *If* φ *(in the above definition) is onto* $X^{''}$ *we say that* X *is reflexive.*

1.1.1 The weak and weak star topologies:

Let X be a Banach space and $f \in X^{'}$. Denot by

$$\varphi_f : X \to \mathbb{R}$$

$$x \longmapsto (\varphi(x))(f)$$

when f cover $X^{'}$, we obtain a family $\left(\varphi_f\right)_{f \in X^{'}}$ of applications to X in $\mathbb{R}$.

Definition 1.1.5 *The weak topology on* X, *denoted by* $\sigma\left(X, X^{'}\right)$ *,is the weakest topology on* X *for which every* $\left(\varphi_f\right)_{f \in X^{'}}$ *is continuous.*

We will define the topology on $X^{'}$,the weak star topology, denoted by $\sigma\left(X^{'},X\right)$.For all $x\in X$.Denote by

$$\varphi_x : X^{'} \to \mathbb{R}$$

$$f \longmapsto \varphi_x\left(f\right) = \langle f,x\rangle_{X',X}$$

Definition 1.1.6 *The weak star topology on $X^{'}$ is the weakest topology on $X^{'}$for wich every $(\varphi_x)_{x\in X'}$is continuous.*

Remark 1.1.2 *Since $X \hookrightarrow X^{''}$, it is clear that, the weak star topology $\sigma\left(X^{'},X\right)$ is weakest*

then the topology $\sigma\left(X^{'},X^{''}\right)$,and this later is weakest then the strong topology.

Definition 1.1.7 *we call that a sequence $(x_n)_{n\in\mathbb{N}}$ in X is weakly convergent to $x\in X$ if and only if:*

$$\lim_{n\longrightarrow\infty} f\left(x_n\right) = f\left(x\right)$$

for evry $f\in X^{'}$, and this is denoted by $x_n \rightharpoonup x$.

Remark 1.1.3 *1) If the weak limit exist, it is unique.*

2) If $x_n \to x\in X$ $\left(strongly\right)$, then $x_n \rightharpoonup x$ $\left(weakly\right)$.

3) If $dimX<\infty$, then the weak convergence implies the strong convergence.

1.1.2 Hilbert spaces

The proper setting for the rigorous theory of partial differential equation turns out to be

the most important function space in modern physics and modern analyse, known as

Hilbert spaces. Then, we most give some impotant result on these spaces here.

Definition 1.1.8 *A Hilbert space H is a vector space supplied with inner product $\langle u, v\rangle$*

such that $\|u\|_H = \sqrt{\langle u, v\rangle}$ is the norm which let H complete.

Theorem 1.1.1 *[13] Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in the Hilbrt space H, then it possess a*

subsequence which converges in the weak topology of H.

Theorem 1.1.2 *[13] In the Hilbrt space, all sequence which converges in the weak topology is bounded.*

Theorem 1.1.3 *[13] Let $(x_n)_{n\in\mathbb{N}}$ be a sequence which converges to x in the weak topology and $(y_n)_{n\in\mathbb{N}}$*

is an other sequence which converges weakly to y, then:

$$\lim_{n\longrightarrow\infty} \langle x_n, y_n\rangle = \langle x, y\rangle .$$

Proposition 1.1.4 *Let H_1 and H_2 be two Hilbert spaces, let $(x_n)_{n\in\mathbb{N}} \subseteq H_1$ be a sequence which converges weakly to $x \in H_1$, let A a bounded linear operator from H_1 to H_2. Then, the sequence $(A(x_n))_{n\in\mathbb{N}} \subseteq H_2$ converges to $A(x)$*

in the weak topology of H_2.

Theorem 1.1.4 *[14] (The Lax-MilgramTheorem)*

Let H *be a Hilbert space and let* $a : H \times H \longrightarrow \mathbb{R}$ *be a bilinear functional. Asume that*

there existe tow constants $C < \infty, \alpha > 0$ *such that:*

(i) $|a(u,v)| \leq C \|u\|_H$

(ii) $a(u,v) \geq \alpha \|u\|_H$ *for all* $u \in H$ *(coerciveness).*

Then, for every $f \in H^{'}$ *(the continuous dual space of* H*), there exists a unique* $u \in H$ *such that*

$$a(u,v) = \langle f, v \rangle \quad \text{for all } v \in H$$

1.1.3 The $L^p(\Omega)$ spaces

Definition 1.1.9 *Let* $1 \leq p \leq \infty$, *and let* Ω *be an open domain in* $\mathbb{R}^n$, $n \in \mathbb{N}$. *Define the*

standard Lebesgue space $L^p(\Omega)$ *by*

$$L^p(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R};\ f \text{ is measurable and } \int\limits_\Omega |f(x)|^p \, dx < \infty \right\}$$

Notation 1.1.1 *for* $p \in [1, \infty[$, *denote by*

$$\|f\|_p = \left(\int\limits_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

.If $p = \infty$, *we have*

$$L^p(\Omega) = \{\ f : \Omega \longrightarrow \mathbb{R};\ f \text{ is measurable and there exists } C \text{ such that}; |f(x)| \leq C \text{ in } \Omega\}$$

Notation 1.1.2 *Let* $p \in [1,\infty[$, *we denote by* q *the conjugate of* p *i.e.*$\frac{1}{p}+\frac{1}{q}=1.$

Theorem 1.1.5 $L^p(\Omega)$ *supplied with the norm* $\|.\|_p$ *is a Banach space, for all* $1 \leq p \leq \infty$

Remark 1.1.4 *In particularly, when* $p=2$, $L^2(\Omega)$ *equipped with the inner product*

$$\langle f,g\rangle_{L^2(\Omega)} = \int\limits_{\Omega} f(x)\,g(x)\,dx,$$

is a Hilbert space.

Theorem 1.1.6 *For* $1<p<\infty$, $L^p(\Omega)$ *is a reflexive space.*

1.1.4 The Sobolev space $W^{m,p}(\Omega)$

Definition 1.1.10 *i) Let* $m \in \mathbb{N}$ N *and* $p \in [1,\infty]$. *The* $W^{m,p}(\Omega)$ *is the space of all* $f \in L^p(\Omega)$, *defined as*

$$W^{m,p}(\Omega) = \{\ f \in L^p(\Omega)\,, \text{such that } \partial^\alpha f \in L^p(\Omega) \text{ for all } \alpha = (\alpha_1,\alpha_2,\alpha_3,...,\alpha_n) \in \mathbb{N}^n,\ \ |\alpha| \leq m\}$$

such that $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}...\partial_n^{\alpha_n}$ *and* $|\alpha| = \sum_{j=1}^n \alpha_j$.

ii) If $f \in W^{m,p}(\Omega)$, *we define its norm to be*

$$\|f\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\leq m} \|\partial^\alpha f\|_p^p\,dx\right)^{\frac{1}{p}}; & (1 \leq p < \infty) \\ \sum_{|\alpha|\leq m} ess\sup|\partial^\alpha f| \qquad ; & (p=\infty) \end{cases}$$

Definition 1.1.11 *We denote by* $W_0^{m,p}(\Omega)$ *the closure of* $C_0^\infty(\Omega)$ *in* $W^{m,p}(\Omega)$

Remark 1.1.5 *i) If* $p=2$ *we usully write*

$$W^{m,2}(\Omega) = H^m(\Omega)\,,\quad W_0^{m,2}(\Omega) = H_0^m(\Omega)$$

Supplied with the norm

$$\|f\|_{H^m(\Omega)} = \left(\sum_{|\alpha|\leq m} \|\partial^\alpha f\|^2_{L^p(\Omega)} \, dx \right)^{\frac{1}{2}}$$

The letter H *is used, since - as we will see -* $H^m(\Omega)$ *is a Hilbert space with usual scalar product*

$$\langle u, v\rangle_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega \partial^\alpha u \partial^\alpha v dx$$

Note that $H^0(\Omega) = L^2(\Omega)$.

Theorem 1.1.7 *[13] 1.* $H^m(\Omega)$ *supplied with inner product* $\langle .,. \rangle_{H^m(\Omega)}$ *is Hilbert space.*

2. If $m \geq m'$, $H^m(\Omega) \hookrightarrow H^{m'}(\Omega)$

Theorem 1.1.8 *[13] Assume that* Ω *is an open domain in* $\mathbb{R}^n, n \geq 1,$*with smooth boundary* Γ*. Then,*

i) if $1 \leq p < n$, *we have* $W^{1,p}(\Omega) \subset L^q(\Omega)$, *for every* $q \in [p, p^*]$, *where* $p = \frac{np}{n-p}$

ii) if $p = n$, *we have* $W^{1,p}(\Omega) \subset L^q(\Omega)$, *for every* $q \in [p, \infty)$.

iii) if $p > n$, *we have* $W^{1,p}(\Omega) \subset L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$ *,where* $\alpha = \frac{p-n}{p}$

1.1.5 The $L^p(0, T, X)$ space

Definition 1.1.12 *Let* X *be a Banach space, denote by* $L^p(0, T, X)$ *the space of measurable*

functions

$$f :]0, T[\to X$$

$$t \longmapsto f(t)$$

such that

$$\|f\|_{L^p(0,T,X)} = \left(\int_0^T \|f(t)\|_X^p \, dt \right)^{\frac{1}{p}} < \infty, \qquad 1 \le p < \infty.$$

If $p = \infty$,

$$\|f\|_{L^\infty(0,T,X)} = \sup_{t \in]0,T[} ess \, \|f(t)\|_X$$

Theorem 1.1.9 *[13] $L^p(0,T,X)$ equipped with the norm $\|.\|_{L^p(0,T,X)}$ is a Banach space.*

Proposition 1.1.5 *Let X be a reflexive Banach space, X' it's dual, and $1 \le p < \infty$, $1 \le q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual of $L^p(0,T,X)$ is identify algebraically and topologically with $L^q(0,T,X')$.*

1.2 Some useful inequalities

In this section, we shall recall some inqualities which will be used in the subsequent chapters.

1.2.1 Young inequalities

Theorem 1.2.1 *[13] Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\forall a, b \in \mathbb{R}_+^*; \quad ab \le \frac{1}{p} a^p + \frac{1}{q} b^q$$

Theorem 1.2.2 *[13] (Young inequality with ε) Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\forall a, b \in \mathbb{R}_+^*, \forall \varepsilon > 0; \quad ab \leq \frac{\varepsilon}{p} a^p + \frac{1}{\varepsilon^{\frac{q}{p}} q} b^q$$

The Young inequality has several variants in the following.

Corollary 1.2.1 *Let $a, b > 0$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned} i)\ a^{\frac{1}{p}} b^{\frac{1}{q}} &\leq \frac{1}{p} a + \frac{1}{q} b. \\ ii)\ a^{\frac{1}{p}} b^{\frac{1}{q}} &\leq \frac{1}{p \varepsilon^{\frac{1}{q}}} a + \frac{\varepsilon^{\frac{1}{p}}}{q} b, \quad \forall \varepsilon > 0. \\ i)\ a^{\alpha} b^{1-\alpha} &\leq \alpha a + (1-\alpha) b, \quad 0 < \alpha < 1. \end{aligned}$$

1.2.2 The Holder inequalities

Theorem 1.2.3 *[13] Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then if $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ we have*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$$

Theorem 1.2.4 *[13] (Generalized Holder inequality)Let $1 \leq p_1, p_2, ..., p_m \leq \infty$, $\frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_m} = 1$,*

then if $f_k \in L^{p_k}(\Omega)$ for $k = 1, 2, ..., m$, we have

$$\int_{\Omega} |f_1 f_2 ... f_m| \, dx \leq \|f_1\|_{L^{p_1}(\Omega)} \|f_2\|_{L^{q_2}(\Omega)} \times ... \times \|f_m\|_{L^{p_m}(\Omega)}$$

Remark 1.2.1 *We have the corresponding weighted Holder inequality of the integral form. Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, $\omega(x) > 0$ on Ω. Then*

$$\int_{\Omega} |fg| \, \omega(x) \, dx \leq \left(\int_{\Omega} |f(x)|^p \, \omega(x) \, dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |g(x)|^q \, \omega(x) \, dx \right)^{\frac{1}{q}}.$$

1.2.3 The Minkowski inequality

Theorem 1.2.5 *[13] Assume* $1 \leq p \leq \infty,\ f, g \in L^p(\Omega)$, *then*

$$\|f+g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^q(\Omega)}$$

if $0 < p < 1,$ *then*

$$\|f\|_{L^p(\Omega)} + \|g\|_{L^q(\Omega)} \leq \|f+g\|_{L^p(\Omega)}$$

In the applications, the integral form from the *Minkowski inequality* is used frequentely.

1.2.4 The Poincaré inequality

In this subsection, we shall recall the Poincaré inequality in different forms.

Theorem 1.2.6 *[13] Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and* $f \in H_0^1(\Omega)$. *Then there is a positive constant* C *such that*

$$\|f\|_{L^2(\Omega)} \leq C \|\nabla f\|_{L^2(\Omega)}$$

Theorem 1.2.7 *[13] Let* Ω *be a bounded domain of* C^1 *in* $\mathbb{R}^n$. *There is a positive constant* C *such that for any* $f \in H^1(\Omega)$.

$$\left\|f - \tilde{f}\right\|_{L^2(\Omega)} \leq C \|\nabla f\|_{L^2(\Omega)}$$

Where $\tilde{f} = \frac{1}{|\Omega|} \int\limits_{\Omega} f(x)\, dx$ *is the integral average of* f *over* Ω *and* $|\Omega|$ *is the volume of* Ω.

Theorem 1.2.8 *[13] Under the assumption of the previous theorem, for any* $f \in H^1(\Omega)$, *we have*

$$\|f\|_{L^2(\Omega)} \leq C \left(\|\nabla f\|_{L^2(\Omega)} + \left| \int_{\Omega} f(x)\, dx \right| \right).$$

1.3 Basic theory of semigroups

In this section, we recall some basic knowledge in semigroups,most of whiche will be used in the subsequent chapters. A general reference to this topic is [15, 16, 17]

The goal of this section is to prove Lumer-Phillips' theorem (see Theorems 1.4.3 and 1.4.6 of [15]) in a Hilbert space setting. For that purpose, we first recall the notion of m-dissipative operators.

Definition 1.3.1 *A linear operator A on a Banach space X is called dissipative if*

$$\|(\lambda I - A)x\|_X \geq \lambda \|x\|_X$$

for all $\lambda > 0$ and $x \in D(A)$.

The dissipative operator A is called m-dissipative if $(\lambda I - A)$ is surjective for some $\lambda > 0$.

Note that it suffices to establish the validity of the inequality above only for unit vectors $x \in X$, $\|x\|_X = 1$.

For $x = 0$ the inequality is trivial, for $x \neq 0$ one can normalise.

Theorem 1.3.1 *[15] Let X be a reflexive Banach space. If $A : D(A) \subseteq X \to X$ is a maximal dissipative operator, then $D(A)$ is dense in X.*

Proposition 1.3.1 *For a dissipative operator A the following properties hold.*

a) $\lambda I - A$ is injective for all $\lambda > 0$ and

$$\left\|(\lambda I - A)^{-1} y\right\| \leq \frac{1}{\lambda} \|y\|$$

for all y in the range $ran\,(\lambda I - A) = (\lambda I - A)\,(D(A))$

b) $\lambda I - A$ is surjective for some $\lambda > 0$ if and only if it is surjective for each $\lambda > 0$. In that case, one has $(0, \infty) \subseteq \rho\,(A)$.

c) A is closed if and only if the range $ran\,(\lambda I - A)$ is closed for some (hence all) $\lambda > 0$.

d) If $ran\,(A) \subseteq \overline{D(A)}$, e.g., if A is densely defined, then A is closable. Its closure $\overline{A}$ is again dissipative and satisfies

$$ran\left(\lambda I - \overline{A}\right) = \overline{(\lambda I - A)}\ for\ all\ \lambda > 0$$

Proposition 1.3.2 *Let H be a Hilbert space. An operator A on H is dissipative if and only if for every*

$x \in D(A)$ we have

$$\operatorname{Re}\langle Ax, x\rangle \leq 0 \tag{1.1}$$

Proof. *Assume (**??**) is satisfied for $x \in D(A), \|x\| = 1$ .Then we have*

$$\|(\lambda I - A)\,x\|_X = \|(\lambda I - A)\,x\|_X\,\|x\|_X \geq |\langle(\lambda I - A)\,x, x\rangle| \geq \operatorname{Re}\langle(\lambda I - A)\,x, x\rangle \geq \lambda$$

for all $\lambda > 0$.This proves one of the implications.

To show the converse, we take $x \in D(A), \|x\| = 1$, and assume that $\|(\lambda I - A)\,x\|_X \geq \lambda$ for all $\lambda > 0$.

Consider the normalised elements

$$y_\lambda = \frac{1}{\|(\lambda I - A)\, x\|_X} (\lambda I - A)\, x$$

Then for all $\lambda > 0$ *we have*

$$\lambda \leq \|(\lambda I - A)\, x\|_X = \langle (\lambda I - A)\, x, y_\lambda \rangle = \lambda \operatorname{Re} \langle x, y_\lambda \rangle - \operatorname{Re} \langle Ax, y_\lambda \rangle$$

By estimating one of the terms on right-hand side trivially we can conclude the following two

inequalities:

$$\lambda \leq \lambda - \operatorname{Re} \langle Ax, y_\lambda \rangle \ \textit{ and } \lambda \leq \lambda \operatorname{Re} \langle x, y_\lambda \rangle + \|Ax\|_X$$

are valid for each $\lambda > 0.$ *These yield for* $\lambda = n$

$$\operatorname{Re} \langle Ax, y_n \rangle \leq 0 \ \textit{ and } 1 - \frac{1}{n} \|Ax\|_X \leq \operatorname{Re} \langle x, y_n \rangle$$

Since the unit ball of a Hilbert space is weakly (sequentially) compact, we can take a weakly

convergent subsequence (y_{n_k}) *with weak limit* $y \in H.$ *Then we obtain*

$$\|y\|_X \leq 1, \operatorname{Re} \langle Ax, y \rangle \leq 0, \textit{and } \operatorname{Re} \langle x, y \rangle \geq 1$$

Combining these facts, it follows that $y = x$ *and that it satisfies* (1.1) ■

Let us now go on with the notion of a semigroup of bounded linear operators on a Banach space.

Definition 1.3.2 *Let $(X, \|.\|)$ be a (real or complex) Banach space. A semigroup of bounded linear operators on X is a map*

$$S : [0, +\infty[\to \mathcal{L}(X)$$

with the following properties:

(a) $S(0) = I,$

(b) $S(t+s) = S(t)\,S(s)$, *for all* $t, s \geq 0.$

We will use the equivalent notation $(S(t))_{t\geq 0}$ and the abbreviated form $S(t)$.

Definition 1.3.3 *The infinitesimal generator of a semigroup of bounded linear operators $S(t)$ is the map $A : D(A) \subseteq X \to X$ defined by*

$$\begin{cases} D(A) = \left\{ u \in X; \lim_{t\longrightarrow 0^+} \frac{S(t)u-u}{t} \text{ exists} \right\} \\ Au = \lim_{t\longrightarrow 0^+} \frac{S(t)u-u}{t}, \qquad \forall u \in D(A) \end{cases}$$

Definition 1.3.4 *A semigroup $S(t)$ of bounded linear operators on X is uniformly continuous if*

$$\lim_{t\longrightarrow 0^+} \|S(t) - I\|_{\mathcal{L}(X)} = 0$$

Proposition 1.3.3 *Let $S(t)$ be a uniformly continuous semigroup of bounded linear operators.*

Then there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t} \qquad \forall t \geq 0$$

Corollary 1.3.1 *A semigroup $S(t)$ is uniformly continuous if and only if*

$$\lim_{s\longrightarrow t} \|S(s) - S(t)\|_{\mathcal{L}(X)} = 0 \qquad \forall t \geq 0$$

Definition 1.3.5 *A semigroup $S(t)$ of bounded linear operators on X is called*

Theorem 1.3.2 *[15] strongly continuous $(or\ of\ class\ C_0,\ or\ even\ a\ C_0 - semigroup)$ if*

$$\lim_{t \longrightarrow 0^+} S(t)\,u = u \qquad \forall u \in X$$

Theorem 1.3.3 *[15] Let $S(t)$ be a $C_0 -$ semigroup of bounded linear operators on X.*

Proposition 1.3.4 *Then there exists $\omega \geq 0$ and $M \geq 1$ such that*

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t} \qquad \forall t \geq 0 \tag{1.2}$$

Corollary 1.3.2 *Let $S(t)$ be a $C_0 -$ semigroup of bounded linear operators on X.*

Then for every $u \in X$, the map $t \longmapsto S(t)\,u$ is continuous from $\mathbb{R}_+$ into X.

Definition 1.3.6 *A $C_0 -$ semigroup of bounded linear operators on X is called uniformly bounded if*

$S(t)$ satisfies (1.2) with $\omega = 0$. If, in addition, $M = 1$, we say that $S(t)$ is a contraction semigroup.

Theorem 1.3.4 *[17] Let $A : D(A) \subseteq X \rightarrow X$ be the infinitesimal generator of a $C_0 -$ semigroup of bounded linear operators on X,*

Theorem 1.3.5 *denoted by $S(t)$ Then the following properties hold true.*

(a) For all $t \geq 0$

$$\lim_{h \longrightarrow 0^+} \frac{1}{h} \int_t^{t+h} S(s)\,u ds = S(t)\,u \qquad \forall u \in X$$

(b) For all $t \geq 0$ *and* $u \in X$

$$\int_0^t S(s)\,uds \in D(A) \quad \text{and} \quad A\left(\int_0^t S(s)\,uds\right) = S(t)\,u - u$$

(c) $D(A)$ *is dense in* X

(d) For all $u \in D(A)$ *and* $t \geq 0$ *we have that* $S(t)\,u \in D(A), t \longmapsto S(t)\,u$ *is continuously differentiable, and*

$$\frac{d}{dt}S(t)\,u = AS(t)\,u = S(t)\,Au$$

(e) For all $u \in D(A)$ *and all* $t > s \geq 0$ *we have that*

$$S(t)\,u - S(s)\,u = \int_s^t S(\tau)\,Aud\tau = \int_s^t AS(\tau)\,ud\tau.$$

Proposition 1.3.5 *The infinitesimal generator of a* C_0-*semigroup* $S(t)$ *is a closed operator.*

Definition 1.3.7 *(of the resolvent and spectrum of a closed operator) Let* $A : D(A) \subseteq X \to X$ *be a closed operator on a complex Banach space* X.

The resolvent set of $A, \rho(A)$, *is the set of all* $\lambda \in \mathbb{C}$,*such that* $\lambda I - A : D(A) \to X$ *is bijective.*

The set $\sigma(A) = \mathbb{C} \diagdown \rho(A)$ *is called the spectrum of* A.

For any $\lambda \in \rho(A)$ *the linear operator* $R(\lambda, A) = (\lambda I - A)^{-1} : X \to X$ *is called the resolvent of* A.

Proposition 1.3.6 *(properties of* $R(\lambda, A)$*) Let* $A : D(A) \subseteq X \to X$ *be a closed operator on a complex Banach space* X.*Then the following holds true.*

(a) $R(\lambda, A) \in \mathcal{L}(X)$ *for any* $\lambda \in \rho(A)$

(b) *For any* $\lambda \in \rho(A)$

$$AR(\lambda, A) = \lambda R(\lambda, A) - I$$

(c) *The resolvent identity holds:*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A) \qquad \forall \lambda, \mu \in \rho(A)$$

(d) *For any* $\lambda, \mu \in \rho(A)$

$$R(\lambda, A) R(\mu, A) = R(\mu, A) R(\lambda, A)$$

Theorem 1.3.6 *[17] (analiticity of* $R(\lambda, A)$*) Let* $A : D(A) \subseteq X \to X$ *be a closed operator on a complex Banach space* X*.Then the resolvent set* $\rho(A)$ *is open in* $\mathbb{C}$ *and for any* $\lambda_0 \in \rho(A)$ *we have that*

$$|\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0, A)\|} \Longrightarrow \lambda \in \rho(A)$$

and the resolvent $R(\lambda, A)$ *is given by the* $(Neumann)$ *series*

$$R(\lambda, A) = \sum_{n \geq 0} (\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1}.$$

Consequently, $\lambda \mapsto R(\lambda, A)$ *is analytic on* $\rho(A)$ *and*

$$\frac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1} \qquad \forall n \in \mathbb{N}.$$

Theorem 1.3.7 *[17] (integral representation of* $R(\lambda, A)$*) Let* $A : D(A) \subseteq X \to X$ *be the infinitesimal generator of a* $C_0 -$ *semigroup of bounded linear operators on* X*,* $S(t)$ *and let* $M \geq 1$ *and* $\omega \in \mathbb{R}$ *be such that*

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t} \qquad \forall t \geq 0$$

Then $\rho(A)$ contains the half-plane

$$\Pi_{\omega} = \{\lambda \in \mathbb{C}, \operatorname{Re}(\lambda) > \omega\}$$

and

$$R(\lambda, A) u = \int_0^{\infty} e^{-\lambda t} S(t) u dt \qquad \forall u \in X, \forall \lambda \in \Pi_{\omega}.$$

Theorem 1.3.8 *[16] (The Hille-Yosida generation theorem) Let $M \geq 1$ and $\omega \in \mathbb{R}$.For a linear operator $A : D(A) \subseteq X \to X$*

the following properties are equivalent:

(a) A is closed, $D(A)$ is dense in X,and

$$\begin{aligned} \Pi_{\omega} &= \{\lambda \in \mathbb{C}, \operatorname{Re}(\lambda) > \omega\} \subseteq \rho(A) \\ \left\| R(\lambda, A)^k \right\| &\leq \frac{M}{(\operatorname{Re}(\lambda) - \omega)^k} \quad \forall k \geq 1, \ \forall \lambda \in \Pi_{\omega} \end{aligned}$$

(b) A is the infinitesimal generator of a $C_0 -$ semigroup, $S(t)$, such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq M e^{\omega t} \qquad \forall t \geq 0.$$

Theorem 1.3.9 *(Lumer–Phillips).Let $A : D(A) \subseteq X \to X$ be a densely defined linear operator. Then the following properties are equivalent:*

(a) A is the infinitesimal generator of a $C_0 -$ semigroup of contractions,

(b) A is maximal dissipative.

Proof. $(a) \Rightarrow (b)$: In view of Theorem 69, we have that $]0, +\infty[\subseteq \rho(A)$.So, $(\lambda I - A) D(A) = X$ for all $\lambda > 0$.Moreover, by the Hille-Yosida theorem for all $\lambda > 0$ and $v \in X$ we have that $\lambda \|R(\lambda, A) v\| \leq \|v\|$ or, setting $u = R(\lambda, A) v$,

$$\lambda \|u\| \leq \|(\lambda I - A) v\| \qquad \forall u \in X$$

So, A is maximal dissipative.

$(b) \Rightarrow (a)$: We have that:

(i) $D(A)$ is dense by hypothesis,

(ii) A is closed by Proposition $46-(c)$

(iii) $]0,+\infty[\subseteq \rho(A)$ and $\|R(\lambda,A)\| \leq \frac{1}{\lambda}$ for all $\lambda > 0$ by **Proposition** 1.3.1

The conclusion follows by the Hille-Yosida theorem. ■

1.4 Lyapunov Stability Theory

The investigation of stability for hereditary systems is often related to the construction of Lyapunov functionals. The general method of Lyapunov functionals construction which was proposed by V. Kolmanovskii and L. Shaikhet [22] and successfully used already for functional differential equations, for difference equations with discrete time, for difference equations with continuous time, is used here to investigate the stability of delay evolution equations, in particular, partial differential equations.

1.4.1 Notations and definitions

Let U and H be two real separable Hilbert spaces such that $U \subset H \equiv H^* \subset U^*$, where the injections are continuous and dense. Let $\|\|$, $||$ and $\|\|_*$ be the norms in U, H and H^*respectively, $((\cdot,\cdot))$ and $(\cdot,\cdot)$ be the scalar products in U and H respectively, and $\langle .,. \rangle$ the duality product between U and $U*$. We assume that

$$|u| \leq \beta \|u\|, u \in U \tag{1.3}$$

Let $C(-h,0,H)$ be the Banach space of all continuous functions from $[-h,0]$ to H, $x_t \in C(-h,0,H)$ for each $t \in [0,\infty)$, be the function defined by $x_t(s) = x(t+s)$ for all $s \in [-h,0]$. The space $C(-h,0,U)$ is similarly defined. Let $A(t,\cdot) : U \to U^*, f_1(t,\cdot) : C(-h,0,H) \to U*$ and $f_2(t,\cdot) : C(-h,0,U) \to U*$ be three families of nonlinear operators defined for $t > 0, A(t,0) = 0, f_1(t,0) = 0, \quad f_2(t,0) = 0$.

Consider the equation

$$\frac{du(t)}{dt} = A\left(t, u(t)\right) + f_1(t,u_t) + f_1(t,u_t), t > 0 \tag{1.4}$$

$$u(s) = \psi(s), s \in [-h,0]$$

Let us denote by $u(\cdot;\psi)$ the solution of Eq. (1.4) corresponding to the initial condition ψ.

Definition 1.4.1 *The trivial solution of Eq. (1.4) is said to be stable if for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left|u\left(t;\psi\right)\right| < \varepsilon \textit{ for all } t \geq 0, \textit{ if } \left|\psi\right|_{C_H} = \sup_{s\in[-h,0]} \left|\psi\left(s\right)\right| < \delta.$$

Definition 1.4.2 *The trivial solution of Eq. (1.4) is said to be exponentially stable if it is stable and there exists a positive constant* λ *such that for any* $\psi \in C(-h,0,U)$ *there exists* C *(which may depend on* ψ*) such that* $\left|u\left(t;\psi\right)\right| \leq Ce^{-\lambda t}$ *for* $t > 0$.

1.4.2 Lyapunov type stability theorem

Let us now prove a theorem which will be crucial in our stability investigation.

Theorem 1.4.1 *Assume that there exists a functional* $V(t, u_t)$ *such that the following conditions hold for some positive numbers* c_1, c_2 *and* λ*:*

$$\left|u\left(t; u_t\right)\right| \leq c_1 e^{\lambda t} \left|u(t)\right|^2, t \geq 0, \tag{1.5}$$

$$\left|u\left(0; u_0\right)\right| \leq c_2 \left|\psi\right|_{C_H}^2, \tag{1.6}$$

$$\frac{d}{dt} V\left(t, u_t\right) \leq 0, t \geq 0. \tag{1.7}$$

Then the trivial solution of Eq. (1.4) is exponentially stable.

Note that Theorem 1.4.1 implies that the stability investigation of Eq. (1.4) can be reduced to the construction of appropriate Lyapunov functionals. A formal procedure to construct Lyapunov functionals is described below.

1.4.3 Procedure of Lyapunov functionals construction

The procedure consists of four steps.

Step 1.

To transform Eq. (1.4) into the form

$$\frac{dz(t, u_t)}{dt} = A_1\left(t, u(t)\right) + A_2\left(t, u_t\right) \tag{1.8}$$

where $z(t,\cdot)$ and $A_2(t,\cdot)$ are families of nonlinear operators, $z(t, 0) = 0, A_2(t, 0) = 0$,operator $A_1(t,\cdot)$ only depends on t and $u(t)$, but does not depend on the previous values $u(t + s), s < 0$.

Step 2.

Assume that the trivial solution of the auxiliary equation without memory

$$\frac{dy(t)}{dt} = A_1\left(t, y(t)\right) \tag{1.9}$$

is exponentially stable and therefore there exists a Lyapunov function $v(t, y(t))$, which satisfies the conditions of Theorem 1.4.1 .

Step 3.

A Lyapunov functional $V(t, ut)$ for Eq. (1.8) is constructed in the form $V = V1+V2$, where $V_1(t, u_t) = v(t, z(t, u_t))$. Here the argument y of the function $v(t, y)$ is replaced on the

functional $z(t, x_t)$ from the left-hand part of Eq. (1.8).

Step 4.

Usually, the functional $V_1(t, u_t)$ almost satisfies the conditions of Theorem 1.4.1. In order to fully satisfy these conditions, it is necessary to calculate $\frac{d}{dt}V_1(t, u_t)$ and estimate it. Then, the additional functional $V_2(t, u_t)$ can be chosen in a standard way.

Note that the representation (1.8) is not unique. This fact allows, using different representations type of (1.8) or different ways of estimating $\frac{d}{dt}V_1(t, u_t)$, to construct different Lyapunov functionals and, as a result, to get different sufficient conditions of exponential stability.

Chapter 2

Stability of a microtemperature porous-elastic system with distributed delay-time

In this chapter, we study the following porous-elastic system with microtemperatures and internal distributed delay acting on the first equation

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x + \mu_1 u_t + \int_{\tau_1}^{\tau_2} \mu_2(s) u_t\left(x, t-s\right) ds = 0, \\ j\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + dw_x = 0, \\ \alpha w_t - kw_{xx} + d\phi_{tx} + kw = 0. \end{cases} \tag{2.1}$$

In order to have a well-posed problem we impose the following Dirichlet-Newmann-Dirichlet boundary conditions

$$u\left(0,t\right) = u\left(1,t\right) = \phi_x\left(0,t\right) = \phi_x\left(1,t\right) = w\left(0,t\right) = w\left(1,t\right) = 0, \;\; t > 0 \tag{2.2}$$

and the initial conditions

$$\begin{cases} u\left(x,0\right)=u_0\left(x\right), u_t\left(x,0\right)=u_1\left(x\right), \\ \phi\left(x,0\right)=\phi_0\left(x\right),\ \phi_t\left(x,0\right)=\phi_1\left(x\right),\ w\left(x,0\right)=w_0\left(x\right), \\ u_t\left(x,-t\right)=f_0\left(x,t\right), \quad x\in(0,1). \end{cases} \tag{2.3}$$

Where the parameter ρ is the mass density and j is the product of the equilibrated inertia by the mass density, the functions u, ϕ and w are the displacement of the solid elastic material, the volume fraction and microtempertaures respectively. The coefficients μ, δ, b, ξ, d, k are positives constants satisfying $\frac{\mu}{\rho}=\frac{\delta}{j}$ and $\mu\xi>b^2$. τ_1 and τ_2 are two real numbers with $0\leq\tau_1<\tau_2$, μ_1 is a positive constant, $\mu_2:[\tau_1,\tau_2]\longrightarrow\mathbb{R}$ is an L^∞ function, $\mu_2\geq 0$ almost everywhere, such that

$$\int_{\tau_1}^{\tau_2}\mu_2\left(s\right)ds<\mu_1 \tag{2.4}$$

and the initial data $(u_0, u_1, \phi_0, \phi_1, w_0, f_0)$ belong to a suitable space.

The basic evolution equations for one-dimensional theories of porous materials with microtemperatures is given by

$$\begin{aligned} \rho u_{tt} &= T_x + R, \\ j\phi_{tt} &= H_x + G, \\ \rho E_t &= P_x + q - Q, \end{aligned} \tag{2.5}$$

where T is the stress tensor, R is the distributed delay, H is the equilibrated stress vector, G is the equilibrated body force, q is the heat flux vector, P is the first heat flux moment, Q is the mean heat flux, and E is the first moment of energy. The constitutive equations needed to construct (2.1) are:

$$
\begin{aligned}
&T=\mu u_x+b\phi, \quad R=-\mu_1 u_t-\int_{\tau_1}^{\tau_2}\mu_2(s)u_t\left(x,t-s\right)ds \qquad H=\delta\phi_x-dw,\\
&G=-bu_x-\xi\phi, \qquad \rho E=-\alpha w-d\phi_x, \quad P=-kw_x, \qquad q=k_1w, \quad Q=k_2w.
\end{aligned}
\tag{2.6}
$$

By substituting (2.6) into (2.5) we get system (2.1), with $k=k_1-k_2>0$.

The introduction of the distributed delay term in porous-elastic system, was discussed recently by KHOCHMENE et AL in [18], where they considered a one-dimensional porous-elastic system with distributed delay term acting on the porous equation.

$$
\begin{cases}
\rho u_{tt}-\mu u_{xx}-b\phi_x=0,\\
j\phi_{tt}-\delta\phi_{xx}+bu_x+\xi\phi+\mu_1\phi_t+\int_{\tau_1}^{\tau_2}\mu_2(s)\phi_t\left(x,t-s\right)ds=0,
\end{cases}
$$

they showed that the dissipation given by this complementary control stabilizes exponentially the system for the case of equal speeds of wave propagation. Most phenomena naturally depend on the present state and also on some past occurrences, that is why time delay arise in many applications. Introducing distributed delay, constant delay or varying delay have been a major research subject in EDPs, and has attracted a great deal of attention in the last decades (see, e.g, [10, 12, 18, 21]). We recall that delay term became a source of instability, as it was showed that a small delay in a boundary control could turn such well-behave hyperbolic system into a wild one and therefore. The aim of this work is to show that microtemperatures effect is powerful enough to uniformly stabilize the system (2.1) even in the presence of time delay.

The needed assumptions and the study of existence and uniqueness of solutions for system (2.1)-(2.3) are described in section 2, the exponential stability result under some conditions, using the energy method is given in section 3.

2.1 Preliminaries and Well-posedness

In this section we first prove the existence and uniqueness of regular solutions to problem (2.1)-(2.3) by using a semigroup theory as in [15].

As in [20], introducing the following new variable

$$z(x,\sigma,t,s)=u_t\left(x,t-\sigma s\right), \quad (x,\sigma,t,s)\in(0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty), \tag{2.7}$$

which satisfies

$$sz_t(x,\sigma,s,t)+z_\sigma(x,\sigma,s,t)=0.$$

So problem (2.1) is equivalent to

$$\begin{cases} \rho u_{tt}-\mu u_{xx}-b\phi_x+\mu_1 u_t+\int_{\tau_1}^{\tau_2}\mu_2(s)z\left(x,1,s,t\right)ds=0, \\ j\phi_{tt}-\delta\phi_{xx}+bu_x+\xi\phi+dw_x=0, \\ \alpha w_t-kw_{xx}+d\phi_{tx}+kw=0, \\ sz_t+z_\sigma=0. \end{cases} \tag{2.8}$$

With (2.2) and the initial conditions:

$$\begin{cases} u\left(x,0\right)=u_0\left(x\right), u_t\left(x,0\right)=u_1\left(x\right), \\ \phi\left(x,0\right)=\phi_0\left(x\right),\ \phi_t\left(x,0\right)=\phi_1\left(x\right),\ w\left(x,0\right)=w_0\left(x\right), \\ z\left(x,0,t,s\right)=u_t\left(x,t\right) \text{ on } (0,1)\times(0,\infty)\times\left(\tau_1,\tau_2\right), \\ z\left(x,\sigma,0,s\right)=f_0\left(x,\sigma,s\right) \text{ on } (0,1)\times(0,1)\times\left(\tau_1,\tau_2\right) \end{cases} \tag{2.9}$$

Remark 2.1.1 *from de second equation of (2.8) and the boundary conditions, it follows*

$$\frac{d^2}{dt^2}\int_0^1 \phi(x,t)\,dx + \frac{\xi}{j}\int_0^1 \phi(x,t)\,dx = 0.$$

So, by solving the above equation and using the initial data of (2.8), we obtain

$$\int_0^1 \phi(x,t)\,dx = \left(\int_0^1 \phi_0(x,t)\,dx\right) cos\left(\sqrt{\frac{\xi}{j}}t\right) + \left(\sqrt{\frac{j}{\xi}}t\right)\left(\int_0^1 \phi_1(x,t)\,dx\right) sin\left(\sqrt{\frac{\xi}{j}}t\right).$$

Consequently, if we set

$$\bar{\phi}(x,t) = \phi(x,t) - \left(\int_0^1 \phi_0(x,t)\,dx\right) cos\left(\sqrt{\frac{\xi}{j}}t\right) + \left(\sqrt{\frac{j}{\xi}}t\right)\left(\int_0^1 \phi_1(x,t)\,dx\right) sin\left(\sqrt{\frac{\xi}{j}}t\right),$$

we get

$$\int_0^1 \bar{\phi}(x,t)\,dx = 0 \quad \forall t \geq 0. \tag{2.10}$$

Therefore, the use of Poincaré's inequality for $\bar{\phi}$ is justified. In addition, simple substitution shows that $\left(u, \bar{\phi}, w\right)$ satisfies system (2.8) with initial data for $\bar{\phi}$ given as

$$\begin{aligned} \bar{\phi}_0(x) &= \phi_0(x) - \int_0^1 \phi_0(x)\,dx, \\ \bar{\phi}_1(x) &= \phi_1(x) - \int_0^1 \phi_1(x)\,dx. \end{aligned} \tag{2.11}$$

Henceforth, we work with $\bar{\phi}$ but we write ϕ for simplicity of notation.

In this section, we give an existence and uniqueness results for the system (2.8)-(2.9) using the semigroup theory.

Introducing the vector function $U = (u, v, \phi, \varphi, w, z)^T$, where $v = u_t$ and $\varphi = \phi_t$, system (2.8)-(2.9) can be written as

$$\begin{cases} U'(t) - \mathcal{A}U(t) = 0 & t > 0 \\ U_0 = U(0) = (u_0, v_0, \phi_0, \varphi_0, w_0, z_0)^T, \end{cases} \tag{2.12}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \phi \\ \varphi \\ w \\ z \end{pmatrix} = \begin{pmatrix} v \\ \frac{1}{\rho}(\mu u_{xx} + b\phi_x - \mu_1 v - \int_{\tau_1}^{\tau_2} \mu_2(s) z\left(x, 1, s, t\right) ds) \\ \varphi \\ \frac{1}{j}(\delta\phi_{xx} - bu_x - \xi\phi - dw_x) \\ \frac{1}{\alpha}(kw_{xx} - d\varphi_x - kw) \\ -\frac{1}{s} z_\sigma \end{pmatrix}.$$

We consider the following spaces

$$L_a^2\left(0,1\right) = \left\{\psi \in L^2\left(0,1\right) : \int_0^1 \psi\left(x\right) = 0\right\},$$

$$H_a^1\left(0,1\right) = H^1\left(0,1\right) \cap L_a^2\left(0,1\right),$$

$$H_a^2\left(0,1\right) = \left\{\psi \in H^2\left(0,1\right) : \psi_x\left(0\right) = \psi_x\left(1\right) = 0\right\},$$

and

$$\mathcal{H} = H_0^1\left(0,1\right) \times L^2\left(0,1\right) \times H_a^1\left(0,1\right) \times L_a^2\left(0,1\right) \times L^2\left(0,1\right) \times L_\omega^2\left(\left(0,L\right) \times \left(0,1\right) \times \left(\tau_1,\tau_2\right)\right),$$

with

$$L_\omega^2\left(\left(0,1\right) \times \left(0,1\right) \times \left(\tau_1,\tau_2\right)\right) = \left\{z \text{ measurable } / \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu_2\left(s\right) z^2\left(x,\sigma,s\right) ds d\rho dx < \infty\right\}.$$

We will show that A generates a C_0 semigroup on $\mathcal{H}$.

Let $U = \left(u, v, \phi, \varphi, w, z\right)^T$, $\overline{U} = \left(\overline{u}, \bar{v}, \overline{\phi}, \bar{\varphi}, \overline{w}, \overline{z}\right)^T$ and under the assumption $b^2 <$

$\mu\xi$ we equipped the Hilbert space $\mathcal{H}$ with the following inner product

$$\langle U,\overline{U}\rangle_{\mathcal{H}}=\int_0^1\left\{\rho v\overline{v}+j\varphi\overline{\varphi}+\alpha w\overline{w}+\delta\phi_x\overline{\phi}_x+\mu u_x\overline{u}_x+\xi\phi\overline{\phi}+b(\phi\overline{u}_x+\overline{\phi}u_x)\right\}dx \tag{2.13}$$

$$+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|z(x,\sigma,s,t)\overline{z}(x,\sigma,s,t)dsd\sigma dx.$$

$\mathcal{H}$ is a Hilbert space, in this case, the above inner product is equivalent to the natural inner product defined on $\mathcal{H}$.

The domain of A is given by

$$D\left(\mathcal{A}\right)=\left\{\begin{array}{c} U\in\mathcal{H}/u,w\in H^2\left(0,1\right)\cap H_0^1\left(0,1\right);\ \phi\in H_a^2\left(0,1\right)\cap H_a^1\left(0,1\right), \\ v\in H_0^1\left(0,1\right),\ \varphi\in H_a^1\left(0,1\right), \\ z\in L_\omega^2\left(\left(0,1\right)\times\left(0,1\right)\times\left(\tau_1,\tau_2\right)\right),v\left(x\right)=z\left(x,0,s\right)\ \text{in}\ \left(0,1\right). \end{array}\right\}.$$

Theorem 2.1.1 *Let $(u_0,u_1,\phi_0,\phi_1,w_0,w_1,f_0)\in\mathcal{H}$. Assume that the hypothesis (2.4) holds. Then, for any initial datum $U_0\in\mathcal{H}$ there exists a unique solution $U\in C\left([0,\infty),\mathcal{H}\right)$ for problem (2.12). Moreover, if $U_0\in D(\mathcal{A})$, then $U\in C([0,\infty),D(A))\cap C^1([0,\infty),\mathcal{H})$.*

Proof. To obtain the above result, we need to prove that $A:D(A)\to\mathcal{H}$ is a maximal monotone operator. For this purpose, we need the following two steps: A is dissipative and $Id-A$ is surjective.

Step 1: In this step, we prove that the operator A is dissipative. Let $U=$

$(u, v, \phi, \varphi, w, z)^T$,

$$\begin{aligned}
\langle AU, U\rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} v \\ \frac{1}{\rho}(\mu u_{xx} + b\phi_x - \mu_1 v - \int_{\tau_1}^{\tau_2} \mu_2(s) z\left(x, 1, s, t\right) ds) \\ \varphi \\ \frac{1}{j}(\delta\phi_{xx} - b u_x - \xi\phi - d w_x) \\ \frac{1}{\alpha}(k w_{xx} - d\varphi_x - kw) \\ -\frac{1}{s} z_\sigma \end{pmatrix}, \begin{pmatrix} u \\ v \\ \phi \\ \varphi \\ w \\ z \end{pmatrix} \right\rangle \\
&= \mu \int_0^1 u_{xx} v dx + b \int_0^1 \phi_x v dx - \mu_1 \int_0^1 v^2 dx \\
&\quad - \int_0^1 \int_{\tau_1}^{\tau_2} v \mu_2(s) z\left(x, 1, t, s\right) ds dx + \delta \int_0^1 \phi_{xx} \varphi dx - b \int_0^1 u_x \varphi dx \\
&\quad -k \int_0^1 w_x^2 dx - k \int_0^1 w^2 dx + \mu \int_0^1 u_x v_x dx + \delta \int_0^1 \phi_x \varphi_x dx + b \int_0^1 v_x \phi dx \\
&\quad +b \int_0^1 \varphi u_x dx - \int_0^1 \int_{\tau_1}^{\tau_2} \left|\mu_2\left(s\right)\right| \int_0^1 z\left(x, \sigma, s\right) z_\sigma\left(x, \sigma, s\right) d\sigma ds dx,
\end{aligned}$$

with integration by parts we obtain,

$$\begin{aligned}
\langle AU, U\rangle_{\mathcal{H}} = -\mu_1 \int_0^1 u_t^2 dx - k \int_0^1 w_x^2 dx - k \int_0^1 w^2 dx \\
- \int_0^1 \int_{\tau_1}^{\tau_2} v \mu_2(s) z\left(x, 1, t, s\right) ds dx \\
- \int_0^1 \int_{\tau_1}^{\tau_2} \left|\mu_2\left(s\right)\right| \int_0^1 z\left(x, \rho, s\right) z_\sigma\left(x, \sigma, s\right) d\sigma ds dx, \qquad (2.14)
\end{aligned}$$

and Integrating by parts in σ, we have

$$\begin{aligned}
\int_0^1 z_\sigma\left(x, \sigma, s\right) z\left(x, \sigma, s\right) d\sigma &= \frac{1}{2} \int_0^1 \frac{\partial}{\partial\sigma} z^2\left(x, \sigma, s\right) d\sigma, \\
&= \frac{1}{2}\left[z^2\left(x, 1, s\right) - z^2\left(x, 0, s\right)\right],
\end{aligned}$$

then

$$\begin{aligned}&\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\int_0^1 z_\sigma(x,\sigma,s)\,z(x,\sigma,s)\,d\sigma ds dx\\&=\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\left[z^2(x,1,s)-z^2(x,0,s)\right].\end{aligned}\tag{2.15}$$

Therefore, from (2.14) and (2.15),

$$\begin{aligned}\langle AU,U\rangle=&-\mu_1\int_0^1 u_t^2(x)\,dx-k\int_0^1 w_x^2dx-k\int_0^1 w^2dx\\&-\int_0^1 v(x)\left(\int_{\tau_1}^{\tau_2}\mu_2(s)\,z(x,1,s)\,ds\right)dx\\&-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s)\,dsdx\\&+\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\int_0^1 u^2(x)\,dx\end{aligned}$$

Now, by using Cauchy-Schwarz's inequality, we can estimate,

$$\begin{aligned}\left|\int_0^1 v(x)\left(\int_{\tau_1}^{\tau_2}\mu_2(s)\,z(x,1,s)\,ds\right)dx\right|\le&\frac{1}{2}\int_0^1 v^2(x)\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\right)dx\\&+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s)\,dsdx\end{aligned}\tag{2.16}$$

Therefore, from the assumption (2.4) we have,

$$\langle AU,U\rangle\le-\left(\mu_1-\int_{\tau_1}^{\tau_2}\mu_2(s)\right)\int_0^1 v^2(x)\,dx-k\int_0^1 w_x^2dx-k\int_0^1 w^2dx\le 0\tag{2.17}$$

that is, the operator A is dissipative.

Step 2: To prove that the operator $Id-A$ is surjective, that is, for any $G=(g_1,g_2,g_3,g_4,g_5,g_6)\in\mathcal{H}$. We seek $U=(u,v,\phi,\varphi,w,z)^T\in D(A)$ satisfying

$$(Id-A)\,U=G,$$

which is equivalent to

$$\begin{cases} u-v=g_1 \\ v-\dfrac{\mu}{\rho}u_{xx}-\dfrac{b}{\rho}\phi_x+\dfrac{\mu_1}{\rho}v+\dfrac{1}{\rho}\int_{\tau_1}^{\tau_2}\mu_2(s)z\left(x,1,s,t\right)ds=g_2 \\ \phi-\varphi=g_3 \\ \varphi-\dfrac{\delta}{j}\phi_{xx}+\dfrac{b}{j}u_x+\dfrac{\xi}{j}\phi+\dfrac{d}{j}w_x=g_4 \\ w-\dfrac{k}{\delta}w_{xx}+\dfrac{d}{\delta}\phi_{tx}+\dfrac{k}{\delta}w=g_5 \\ z+\frac{1}{s}z_\sigma=g_6 \end{cases} \tag{2.18}$$

Suppose that we have found u and ϕ. Therefore, the first, and the third equation in (2.18) give

$$\begin{cases} v=u-g_1 \\ \varphi=\phi-g_3. \end{cases} \tag{2.19}$$

It is clear that $v\in H_0^1\left(0,1\right),\ \varphi\in H_a^1\left(0,1\right).$

Following the same approach as in [20], and using the fact that,

$$z(x,0,s)=v(x),\ \text{ for } x\in\left(0,1\right),s\in\left(\tau_1,\tau_2\right), \tag{2.20}$$

we obtain the solution of $(2.18)_6$

$$z\left(x,\sigma,s\right)=u(x)e^{-\sigma s}-g_1(x)e^{-\sigma s}+se^{-\sigma s}\int_0^\sigma g_6\left(x,\nu,s\right)e^{\nu s}d\nu. \tag{2.21}$$

By using (2.18), (2.19) and (2.21) the functions u,ϕ and w satisfying the following system,

$$\begin{cases} (1+\dfrac{\mu_1}{\rho})u-\dfrac{\mu}{\rho}u_{xx}-\dfrac{b}{\rho}\phi_x+\dfrac{1}{\rho}\int_{\tau_1}^{\tau_2}\mu_2(s)u(x)e^{-s}ds=g_2+g_1(1+\dfrac{\mu_1}{\rho}) \\ +\dfrac{1}{\rho}\int_{\tau_1}^{\tau_2}\mu_2(s)g_1(x)e^{-s}ds-\dfrac{1}{\rho}\int_{\tau_1}^{\tau_2}s\mu_2(s)e^{-s}\int_0^1 g_6\left(x,\nu,s\right)e^{\nu s}d\nu ds \\ (1+\dfrac{\xi}{j})\phi-\dfrac{\delta}{j}\phi_{xx}+\dfrac{b}{j}u_x+\dfrac{d}{j}w_x=g_4+g_3 \\ w\left(1+\dfrac{k}{\delta}\right)-\dfrac{k}{\delta}w_{xx}+\dfrac{d}{\delta}\phi_x=g_5+\dfrac{d}{\delta}g_{3x} \end{cases} \tag{2.22}$$

Solving system (2.22) is equivalent to finding,

$$(u, \phi, w) \in \left[H_0^1(0,1) \times H_a^1(0,1) \times H_0^1(0,1)\right],$$

such that

$$\begin{cases} \int_0^1 \left((\rho+\mu_1)u - \mu u_{xx} - b\phi_x + \int_{\tau_1}^{\tau_2} \mu_2(s)u(x)e^{-s}ds\right)\epsilon dx = \int_0^1 (\rho g_2 + g_1(\rho+\mu_1) \\ + \int_{\tau_1}^{\tau_2} \mu_2(s)g_1(x)e^{-s}ds)\epsilon dx - \int_0^1 \left(\int_{\tau_1}^{\tau_2} s\mu_2(s)e^{-s}\int_0^1 g_6(x,\nu,s)e^{\nu s}d\nu ds\right)\epsilon dx \\ \int_0^1 ((j+\xi)\phi - \delta\phi_{xx} + bu_x + dw_x)\eta dx = \int_0^1 j(g_4+g_3)\eta dx \\ \int_0^1 (w(\delta+k) - kw_{xx} + d\phi_x)\zeta dx = \int_0^1 (\delta g_5 + dg_{3x})\zeta dx \end{cases} \tag{2.23}$$

for all $(\epsilon, \eta, \zeta) \in H_0^1(0,1) \times H_a^1(0,1) \times H_0^1(0,1)$ .Consequently, problem(2.23) is equivalent to the problem

$$a((u,\phi,w),(\epsilon,\eta,\zeta)) = L(\epsilon,\eta,\zeta), \tag{2.24}$$

where the bilinear form

$$a : \left[H_0^1(0,1) \times H_a^1(0,1) \times H_0^1(0,1)\right]^2 \longrightarrow \mathbb{R},$$

and the linear form

$$L : \left[H_0^1(0,1) \times H_a^1(0,1) \times H_0^1(0,1)\right] \longrightarrow \mathbb{R},$$

are defined by

$$\begin{aligned} a((u,\phi,w),(\epsilon,\eta,\zeta)) = \int_0^1 & ((\rho+\mu_1)u\epsilon + \mu u_x\epsilon_x + b\phi\epsilon_x + (j+\xi)\phi\eta + \delta\phi_x\eta_x + bu_x\eta - dw\eta_x \\ & + (\delta+k)w\zeta + kw_x\zeta_x + d\phi_x\zeta + \epsilon u\int_{\tau_1}^{\tau_2} \mu_2(s)e^{-s}ds\right) dx \end{aligned}$$

and

$$\begin{aligned} L(\epsilon,\eta,\zeta) = \int_0^1 & \left[\left(\rho + \mu_1 + \int_{\tau_1}^{\tau_2} \mu_2(s)e^{-s}ds\right) g_1\epsilon + \rho g_2\epsilon + j\eta(g_3+g_4) + \delta\zeta g_5 + d\zeta g_{3x} \right. \\ & \left. -\epsilon\int_{\tau_1}^{\tau_2} s\mu_2(s)e^{-s}\int_0^1 g_6(x,\nu,s)e^{\nu s}d\nu ds\right] dx. \end{aligned}$$

It is easy to verify that a and L are continuous. In addition, we have

$$a\left((u,\phi,w),(u,\phi,w)\right)=\int_0^1\left((\rho+\mu_1)u^2+\mu u_x^2+2b\phi u_x+(j+\xi)\phi^2+\delta\phi_x^2+kw_x^2\right.$$
$$\left.+(\delta+k)\,w^2\right)dx+\int_0^1 u^2(\int_{\tau_1}^{\tau_2}\mu_2(s)e^{-s}ds)dx,$$

by considering

$$\mu u_x^2+2b\phi u_x+\xi\phi^2=\frac{1}{2}\left[\mu(u_x+\frac{b}{\mu}\phi)^2+\xi(\phi+\frac{b}{\xi}u_x)^2+(\mu-\frac{b^2}{\xi})u_x^2+(\xi-\frac{b^2}{\mu})\phi^2\right]$$

and using the fact that $\mu\xi-b^2>0$, we get

$$\mu u_x^2+2b\phi u_x+(j+\xi)\phi^2>\frac{1}{2}\left[(\mu-\frac{b^2}{\xi})u_x^2+(2j+\xi-\frac{b^2}{\mu})\phi^2\right].$$

Then, for some $l>0$

$$a\left((u,\phi,w),(u,\phi,w)\right)\geq l\left\|(u,\phi,w)\right\|_{H_0^1(0,1)\times H_a^1(0,1)\times H_0^1(0,1)}.$$

Thus a is coercive So applying the Lax-Milgram theorem, we deduce that for all

$(\epsilon,\eta,\zeta)\in H_0^1(0,1)\times H_a^1(0,1)\times H_0^1(0,1)$, problem

(2.24) admits a unique solution.

Now, if we take $(\epsilon,0,0)$ in (2.24), we get

$$\begin{aligned}\int_0^1\left((\rho+\mu_1+\int_{\tau_1}^{\tau_2}\mu_2(s)e^{-s}ds)u\epsilon+\mu u_x\epsilon_x\right)dx &= \int_0^1\left((\rho+\mu_1+\int_{\tau_1}^{\tau_2}\mu_2(s)e^{-s}ds)g_1\epsilon-b\phi_x\epsilon+\rho g_2\epsilon\right.\\ &\left.-\epsilon\int_{\tau_1}^{\tau_2}s\mu_2(s)e^{-s}\int_0^1 g_6\left(x,\nu,s\right)e^{\nu s}d\nu ds\right)dx.\end{aligned}$$

which gives

$$\mu \int_0^1 u_x \epsilon_x dx = \int_0^1 X \epsilon dx. \ \forall \epsilon \in H_0^1(0,1)$$

with

$$X = (\rho + \mu_1 + \int_{\tau_1}^{\tau_2} \mu_2(s) e^{-s} ds)(g_1 - u) + \rho g_2 - b\phi_x - \int_{\tau_1}^{\tau_2} s\mu_2(s) e^{-s} \int_0^1 g_6(x, \nu, s) e^{\nu s} d\nu ds \in L^2(0,1).$$

Thus, by the definition of weak derivatives, $u_x \in H^1(0,1)$, hence $u \in H^2(0,1)$.

Similarly, if we take $\zeta \in H_0^1([0,1])$ and substitute with $(0,0,\zeta)$ in (2.24), we get

$$k \int_0^1 w_x \zeta_x dx = \int_0^1 (\delta g_5 + d g_{3x} - (\delta + k) w - d\phi_x) \zeta dx, \ \forall \zeta \in H_0^1([0,1]) \tag{2.25}$$

so $w \in H^2(0,1)$ and we have

$$k w_{xx} = -(\delta g_5 + d g_{3x} - (\delta + k) w - d\phi_x)$$

This gives $(2.22)_3$

Next, to show that $\phi \in H_a^2(0,1)$, we define $\tilde{\eta}(x) = \eta(x) - \int_0^1 \eta(x) dx$ with $\eta \in H_0^1(0,1)$. it is clear that $\tilde{\eta} \in H_a^1(0,1)$

Replacing $(\varepsilon, \tilde{\eta}, \zeta) = (0, \tilde{\eta}, 0)$ in (2.24), we obtain

$$\int_0^1 (\delta\phi_x - dw) \tilde{\eta}_x dx = \int_0^1 (j(g_3 + g_4) - b u_x - j\phi - \zeta\phi) \tilde{\eta} dx, \ \forall \tilde{\eta} \in H_a^1(0,1)$$

which gives

$$\int_0^1 (\delta\phi_x)\eta_x dx = \int_0^1 \left(j\left(g_3+g_4\right) - bu_x - j\phi - \zeta\phi - dw_x\right)\eta dx,\ \forall\eta \in H_0^1\left(0,1\right)$$

thus $\phi \in H^2(0,1)$.with

$$\delta\phi_{xx} = -\left(j\left(g_3+g_4\right) - bu_x - j\phi - \zeta\phi - dw_x\right)$$

This gives $(2.22)_2$. In addition when $\eta \in C^1\left(0,1\right)$, we get

$$\int_0^1 (\delta\phi_x)\eta_x dx = \int_0^1 \left(j\left(g_3+g_4\right) - bu_x - j\phi - \zeta\phi - dw\right)\eta dx,\ \forall\eta \in C^1\left(0,1\right)$$

using integration by parts, we obtain

$$\phi_x(1)\eta_x(1) - \phi_x(0)\eta(0) = 0,$$

since $\eta \in C^1\left(0,1\right)$ is arbitrary, then $\phi_x(1) = \phi_x(0) = 0$. Hence $\phi \in H_a^2(0,1)$.

Finally from (2.18).1,(2.18).3, we conclude $v \in H_0^1\left(0,1\right)$, $\varphi \in H_0^1\left(0,1\right)$, and it is also clear from (2.21) that $z \in L_\omega^2\left((0,1)\times(0,1)\times(\tau_1,\tau_2)\right)$. Hence, there exists a unique solution $U \in D(A)$ such that (2.18) is satisfied. Therefore, the operator $Id - A$ is surjective. Consequently, the existence result follows from the Hille-Yosida theorem. ■

2.2 Stability results

In this section, we show that, the solution of the system (2.8)-(2.9) decays exponentially if and only if $\chi = \frac{\mu}{\rho} - \frac{\delta}{j} = 0$. To achieve our goal we use the energy method to produce a suitable Lyapunov functional which leads to an exponential decay result.

We can prove that the energy is decreasing. More precisely, we have the following result.

Theorem 2.2.1 *Let (u, ϕ, w, z) be the solution of (2.8)-(2.9).Assume that* $\frac{\mu}{\rho} = \frac{\delta}{j}$. *Then there exist two positive constants α and γ such that*

$$E(t) \leq \alpha E(0)\, e^{-\gamma t}, t \geq 0 \tag{2.26}$$

In order to prove such result, we need several lemmas.

Lemma 2.2.1 *Let (u, ϕ, w, z) be the solution of (2.8)-(2.9). and under the assumption (2.4). The energy functional, defined by*

$$\begin{aligned} E(t) = &\frac{1}{2}\int_0^1 \left\{\rho u_t^2 + j\phi_t^2 + \alpha w^2 + \delta\phi_x^2 + \mu u_x^2 + \xi\phi^2 + 2b\phi u_x\right\} dx \\ &+ \frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\sigma,s,t)\,ds\,d\sigma\,dx. \end{aligned} \tag{2.27}$$

satisfies,

$$\frac{d}{dt}(E(t)) \leq -k\int_0^1 w^2 dx - k\int_0^1 w_x^2 dx - (\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds)\int_0^1 u_t^2 dx. \tag{2.28}$$

Proof. Multiplying $(2.8)_1$, $(2.8)_2$ and $(2.8)_3$ by u_t, ϕ_t and w_t, respectively, and integrating over $(0, 1)$, using integration by parts and the boundary conditions and summing them we obtain

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left\{\rho u_t^2 + j\phi_t^2 + \alpha w^2 + \delta\phi_x^2 + \mu u_x^2 + \xi\phi^2 + 2b\phi u_x\right\} dx + k\int_0^1 w^2 dx + k\int_0^1 w_x^2 dx \\ &+\mu_1\int_0^1 u_t^2 dx + \int_0^1 u_t\int_{\tau_1}^{\tau_2} \mu_2(s)z(x,1,s,t)\,ds\,dx = 0. \end{aligned} \tag{2.29}$$

Multiplying $(2.8)_4$ by $|\mu_2(s)|\, z$, integrating over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, and using (2.15) we get

$$\begin{aligned}\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\sigma,s,t)dsd\sigma dx &= -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|u^2(x,t)dsdx\\ &\quad+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx.\end{aligned}$$

Which, together with (2.29), (2.4) and using young's inequality, gives us (2.28).

■

Lemma 2.2.2 *Let (u,ϕ,w,z) be the solution of (2.8)-(2.9). Then the functional*

$$I_1(t) = -\rho\int_0^1 u_t u dx - \frac{\mu_1}{2}\int_0^1 u^2 dx,\ \ t\geq 0 \tag{2.30}$$

satisfies, the estimate

$$I_1'(t) \leq -\rho\int_0^1 u_t^2 dx + \frac{3}{2}\mu_1\int_0^1 u_x^2 dx + \frac{b}{\mu_1}\int_0^1 \phi^2 dx + c\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\, z^2(x,1,s,t)dsdx \tag{2.31}$$

Proof. Taking the derivative of (2.30), using the first equation in (2.8) and integrating by parts, we get

$$I_1'(t) = \mu_1\int_0^1 u_x^2 dx - \rho\int_0^1 u_t^2 dx + b\int_0^1 u_x\phi dx + \int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s,t)u(x,t)dsdx. \tag{2.32}$$

Hence, by using the Young inequality, Poincaré inequality, we conclude the following estimates

$$b\int_0^1 u_x\phi dx \leq \frac{b}{\mu_1}\int_0^1\phi^2 dx + \frac{\mu_1}{4}\int_0^1 u_x^2 dx, \tag{2.33}$$

and,

$$\int_0^1 \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) u(x,t) ds dx \le c \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, z^2(x,1,s,t) ds dx + \frac{\mu_1}{4} \int_0^1 u_x^2 dx, \tag{2.34}$$

substituting (2.33), (2.34) into (2.32), we conclude (2.31). ■

Lemma 2.2.3 *Let (u, ϕ, w, z) be the solution of (2.8)-(2.9). Then the functional*

$$I_2(t) = -\alpha \int_0^1 w \int_0^x \phi_t(y,t) dy dx \,, \tag{2.35}$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$\begin{aligned} I_2'(t) \quad &\le \quad -\frac{d}{2} \int_0^1 \phi_t^2 dx + \varepsilon_1 \int_0^1 \phi_x^2 dx + \varepsilon_1 \int_0^1 u_x^2 dx + \varepsilon_1 \int_0^1 \phi^2 dx \\ &\qquad + c(1 + \frac{1}{\varepsilon_1}) \int_0^1 w^2 dx + \frac{k^2}{d} \int_0^1 w_x^2 dx \end{aligned} \tag{2.36}$$

Proof. Direct computation, using the first and the second equations in system (2.8)-(2.9), yields

$$\begin{aligned} I_2'(t) = k \int_0^1 w_x \phi_t dx - d \int_0^1 \phi_t^2 dx + \frac{d\alpha}{j} \int_0^1 w^2 dx + \frac{b\alpha}{j} \int_0^1 w u dx \\ + k \int_0^1 w \left(\int_0^x \phi_t(y,t) dy \right) dx - \frac{\alpha\delta}{j} \int_0^1 w \phi_x dx + \frac{\alpha\xi}{j} \int_0^1 w \left(\int_0^x \phi(y,t) dy \right) dx. \end{aligned} \tag{2.37}$$

Using Young's and Poincare's inequalities, we get

$$k \int_0^1 w_x \phi_t dx \le \frac{k^2}{d} \int_0^1 w_x^2 dx + \frac{d}{4} \int_0^1 \phi_t^2 dx, \tag{2.38}$$

$$-\frac{\alpha\delta}{j} \int_0^1 w \phi_x dx \le \frac{(\alpha\delta)^2}{j^2 \varepsilon_1} \int_0^1 w^2 dx + \varepsilon_1 \int_0^1 \phi_x^2 dx, \tag{2.39}$$

$$\begin{aligned}\frac{b\alpha}{j}\int_0^1 wu dx &\leq \frac{(b\alpha)^2}{j^2\varepsilon_1}\int_0^1 w^2dx+\varepsilon_1\int_0^1 u^2dx,\\ &\leq \frac{(b\alpha)^2}{j^2\varepsilon_1}\int_0^1 w^2dx+\varepsilon_1\int_0^1 u_x^2dx, \qquad (2.40)\end{aligned}$$

and,

$$k\int_0^1 w\left(\int_0^x \phi_t(y,t)dy\right)dx \leq \frac{k^2}{d}\int_0^1 w^2dx+\frac{d}{4}\int_0^1\left(\int_0^x \phi_t(y,t)dy\right)^2dx \qquad (2.41)$$

$$\frac{\alpha\xi}{j}\int_0^1 w\left(\int_0^x \phi(y,t)dy\right)dx \leq \frac{(\alpha\xi)^2}{j^2\varepsilon_1}\int_0^1 w^2dx+\varepsilon_1\int_0^1\left(\int_0^x \phi(y,t)dy\right)^2dx \qquad (2.42)$$

By Cauchy-Schwartz inequality, it is clear that

$$\left(\int_0^x \phi_t(y,t)dy\right)^2 \leq \left(\int_0^1 \phi_t(x,t)dx\right)^2 \leq \int_0^1 \phi_t^2(x,t)dx \qquad (2.43)$$

$$\left(\int_0^x \phi(y,t)dy\right)^2 \leq \left(\int_0^1 \phi(x,t)dx\right)^2 \leq \int_0^1 \phi^2(x,t)dx \qquad (2.44)$$

Inserting (2.38) -(2.44) into (2.37) , we conclude (2.36). ■

Lemma 2.2.4 *Let (u,ϕ,w,z) be the solution of (2.8)-(2.9). Then the functional*

$$I_3(t)=j\int_0^1 \phi_t\phi dx+\frac{b\rho}{\mu}\int_0^1 \phi\int_0^x u_t(y,t)dydx, \qquad (2.45)$$

satisfies, for any positive constant ε_2 and $\delta_1>0$, the estimate

$$\begin{aligned}I_3'(t) &\leq c\left(1+\frac{1}{\varepsilon_2}\right)\int_0^1 \phi_t^2dx-\frac{1}{2}\delta_1\int_0^1 \phi^2dx+c(\varepsilon_2+1)\int_0^1 u_t^2dx \qquad (2.46)\\ &-\frac{\delta}{2}\int_0^1 \phi_x^2dx+\frac{d^2}{2\delta}\int_0^1 w^2dx+\frac{b^2}{\mu\delta_1}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s,t)dsdx\end{aligned}$$

Proof. By differentiating the functional $I_3(t)$ and using the first and second equations in system (2.8)-(2.9), we get,

$$I_3'(t) = j\int_0^1 \phi_t^2 dx - \delta\int_0^1 \phi_x^2 dx - \xi\int_0^1 \phi^2 dx + d\int_0^1 w\phi_x dx + \frac{b^2}{\mu}\int_0^1 \phi^2 dx \tag{2.47}$$

$$+ \frac{b\rho}{\mu}\int_0^1 \phi_t\left(\int_0^x u_t(y,t)dy\right)dx - \frac{b\mu_1}{\mu}\int_0^1 \phi\left(\int_0^x u_t(y,t)dy\right)dx$$

$$- \frac{b}{\mu}\int_0^1 \phi\int_0^x\int_{\tau_1}^{\tau_2} \mu_2(s)z(y,1,s,t)dsdydx$$

Using Young's and Cauchy Schwartz inequalities, we conclude, for all $\delta_1 > 0$,

$$d\int_0^1 w\phi_x dx \le \frac{d^2}{2\delta}\int_0^1 w^2 dx + \frac{\delta}{2}\int_0^1 \phi_x^2 dx,$$

$$-\frac{b\mu_1}{\mu}\int_0^1 \phi\left(\int_0^x u_t(y,t)dy\right)dx \le \frac{\delta_1}{4}\int_0^1 \phi^2 dx + \frac{(b\mu_1)^2}{\mu^2\delta_1}\int_0^1\left(\int_0^x u_t(y,t)dy\right)^2 dx,$$

$$\le \frac{\delta_1}{4}\int_0^1 \phi^2 dx + \frac{(b\mu_1)^2}{\mu^2\delta_1}\int_0^1 u_t^2 dx. \tag{2.48}$$

And

$$-\frac{b}{\mu}\int_0^1 \phi\int_0^x\int_{\tau_1}^{\tau_2} \mu_2(s)z(y,1,s,t)dsdydx \le \frac{\delta_1}{4}\int_0^1 \phi^2 dx + \frac{b^2}{\mu^2\delta_1}\int_0^1\left(\int_0^x\int_{\tau_1}^{\tau_2} \mu_2(s)z(y,1,s,t)dsdy\right)^2 dx$$

$$\le \frac{\delta_1}{4}\int_0^1 \phi^2 dx + \frac{b^2}{\mu^2\delta_1}\int_0^1\left(\int_{\tau_1}^{\tau_2} \mu_2(s)z(x,1,s,t)ds\right)^2 dx,$$

$$\le \frac{\delta_1}{4}\int_0^1 \phi^2 dx + \frac{b^2}{\mu^2\delta_1}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)dsdx. \tag{2.49}$$

Similarly, we have for $\varepsilon_2 > 0$,

$$\begin{aligned}\frac{b\rho}{\mu}\int_0^1 \phi_t\left(\int_0^x u_t(y,t)dy\right)dx &\le \frac{(b\rho)^2}{4\mu^2\varepsilon_2}\int_0^1 \phi_t^2dx+\varepsilon_2\int_0^1\left(\int_0^x u_t(y,t)dy\right)^2dx,\\ &\le \frac{(b\rho)^2}{4\mu^2\varepsilon_2}\int_0^1 \phi_t^2dx+\,\varepsilon_2\int_0^1 u_t^2dx.\end{aligned} \tag{2.50}$$

Substituting (2.48)-(2.50)into (2.47), and choosing $\delta_1=\left(\xi-\dfrac{b^2}{\mu}\right)$ then we can obtain (2.46). ■

Lemma 2.2.5 *Let (u,ϕ,w,z) be the solution of system (2.8)-(2.9). Then the functional*

$$I_4\left(t\right)=\frac{\rho\delta}{b}\int_0^1 u_t\phi_x dx+\frac{j\mu}{b}\int_0^1 \phi_t u_x dx, \tag{2.51}$$

satisfies,

$$\begin{aligned}I_4'\left(t\right) \;&\le\; c\int_0^1 \phi_x^2dx-\frac{\mu}{2}\int_0^1 u_x^2dx+\frac{d^2\mu}{b^2}\int_0^1 w_x^2dx+\frac{\mu_1}{b}\int_0^1 u_t^2dx\\ &\quad+\frac{\delta\mu}{2b}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|z^2(x,1,s,t)dsdx\;-\frac{\rho j}{b}\chi\int_0^1 \phi_t u_{tx}dx.\end{aligned} \tag{2.52}$$

Proof. By differentiating $I_4\left(t\right)$, using the first and the second equation in system (2.8)-(2.9) and integrating by parts, we obtain

$$\begin{aligned}I_4'\left(t\right) \;&=\; \delta\int_0^1 \phi_x^2dx-\mu\int_0^1 u_x^2dx-\frac{d\mu}{b}\int_0^1 w_x u_x dx-\frac{\xi\mu}{b}\int_0^1 \phi u_x dx-\frac{\delta\mu_1}{b}\int_0^1 u_t\phi_x dx\\ &\quad-\frac{\rho j}{b}\chi\int_0^1 \phi_t u_{tx}dx\;-\frac{\delta}{b}\int_0^1 \phi_x\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s,t)dsdx\,.\end{aligned} \tag{2.53}$$

Using Young's and Poincare's inequalities, we get,

$$-\frac{d\mu}{b}\int_0^1 w_x u_x dx \leq \frac{d^2\mu}{b^2}\int_0^1 w_x^2 dx + \frac{\mu}{4}\int_0^1 u_x^2 dx, \tag{2.54}$$

$$-\frac{\xi\mu}{b}\int_0^1 \phi u_x dx \leq c\int_0^1 \phi_x^2 dx + \frac{\mu}{4}\int_0^1 u_x^2 dx,$$

$$-\frac{\delta\mu_1}{b}\int_0^1 u_t\phi_x dx \leq \frac{\mu_1}{b}\int_0^1 u_t^2 dx + \frac{\delta^2\mu_1}{4b}\int_0^1 \phi_x^2 dx.$$

and

$$-\frac{\delta}{b}\int_0^1 \phi_x \int_{\tau_1}^{\tau_2} \mu_2(s) z(y,1,s,t) ds dy dx \leq \frac{\delta}{2b}\int_0^1 \phi_x^2 dx + \frac{\delta}{2b}\int_0^1 \left(\int_{\tau_1}^{\tau_2} \mu_2(s) z(y,1,s,t) ds dy\right)^2 dx$$

$$\leq \frac{\delta}{2b}\int_0^1 \phi_x^2 dx + \frac{\delta\mu}{2b}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t) ds dx. \tag{2.55}$$

Substituting (2.54)-(2.55) in (2.53) we conclude (2.52). ■

Lemma 2.2.6 *Let (u,ϕ,w,z) be the solution of (2.8)-(2.9). Then the functional*

$$I_5(t) := \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\sigma} \mu_2(s)\, z^2(x,\sigma,s,t)\, ds d\sigma dx \tag{2.56}$$

satisfies , for $m > 0$, the estimate

$$I_5'(t) \leq -m \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)\, ds dx \tag{2.57}$$
$$-\, m \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu_2(s)\, z^2(x,\sigma,s,t)\, ds d\sigma dx + \mu_1 \int_0^1 \int u_t^2 dx.$$

Proof. Differentiating $I_5(t)$, and using (2.7), we obtain,

$$\begin{aligned} I_5'(t) &= -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\sigma}\mu_2(s)\, z(x,\sigma,s,t)\, z_\sigma(x,\sigma,s,t)\, ds d\sigma dx \\ &= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\sigma}\mu_2(s)\frac{d}{d\sigma} z^2(x,\sigma,s,t)\, ds d\sigma dx. \end{aligned}$$

Integration by parts gives,

$$\begin{aligned} I_5'(t) &= -\int_0^1\int_0^1 \frac{d}{d\sigma}\int_{\tau_1}^{\tau_2}\mu_2(s)\left(e^{-s\sigma} z^2(x,\sigma,s,t)\right) ds d\sigma dx \\ &\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu_2(s)\, e^{-s\sigma} z^2(x,v,s,t)\, ds d\sigma dx \\ &= \int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\left(z^2(x,0,s,t) - e^{-s\rho} z^2(x,1,s,t)\right) ds d\sigma dx \\ &\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu_2(s)\, e^{-s\sigma} z^2(x,\sigma,s,t)\, ds d\sigma. \end{aligned}$$

Therefore,

$$\begin{aligned} I_5'(t) &\le -e^{-\tau_2}\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\, z^2(x,1,s,t)\, ds dx + \left(\int_{\tau_1}^{\tau_2}\mu_2(s)\, ds\right)\int_0^1 u_t^2 dx \\ &\quad - e^{-\tau_2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu_2(s)\, z^2(x,\sigma,s,t)\, ds d\sigma dx. \end{aligned}$$

Where $m = e^{-\tau_2}$, recalling (2.4) we obtain (2.57). ■

Now, for some positive constants; N, N_1, N_2, N_3 and N_4 to be chosen appropriately later. We define the Lyapunov functional by

$$\pounds(t) = NE(t) + I_1(t) + N_1 I_2(t) + N_2 I_3(t) + N_3 I_4(t) + I_5(t),\ \forall t > 0. \qquad (2.58)$$

Next, by taking into account (2.28) , (2.31), (2.36), (2.46) , (2.52) , (2.57), we obtain,

$$\begin{aligned}
\pounds'(t) \leq & -\left[+\frac{\mu}{2}N_3 - \frac{3}{2}\mu_1 - N_1\varepsilon_1\right]\int_0^1 u_x^2 dx \\
& -\left[N(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds) + \rho - N_2 c(1+\frac{1}{\varepsilon_2}) - \frac{N_3\mu_1}{b} - N_4\mu_1\right]\int_0^1 u_t^2 dx \\
& -\left[\frac{N_1 d}{2} - N_2 j\right]\int_0^1 \phi_t^2 dx \\
& -\left[N_2\frac{\delta}{2} - N_1\varepsilon_1 - N_3 c\right]\int_0^1 \phi_x^2 dx \\
& -\left[\frac{N_2}{2}\left(\xi - \frac{b^2}{\mu}\right) - \frac{b}{\mu_1} - N_1\varepsilon_1\right]\int_0^1 \phi^2 dx \\
& -\left[kN - N_1(1+\frac{1}{\varepsilon_1}) - \frac{N_2 d^2}{2\delta}\right]\int_0^1 w^2 dx \\
& -\left[kN - N_1\frac{k^2}{d}\right]\int_0^1 w_x^2 dx \\
& -\left[mN_4 - N_3\frac{\delta\mu}{2b} - c\right]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)\,ds dx \\
& -[mN_4]\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, z^2(x,\sigma,s,t)\, ds d\sigma dx.
\end{aligned} \tag{2.59}$$

We must choose the coefficients very carefully in order to make them negative. First, choosing N_2 large enough, such that

$$N_2 \geq \frac{2b\mu}{\mu_1(\xi\mu - b^2)},$$

second, we choose N_1 large enough, such that

$$N_1 \geq \frac{2jN_2}{d},$$

then, we choose N_3 large enough, such that

$$\frac{3\mu_1}{\mu} \leq N_3 \leq \frac{\delta N_2}{2c}.$$

Next, we select ε_1 and ε_2 smalls enough, so that

$$\begin{aligned} \varepsilon_1 &\leq \min\left\{\frac{\mu N_3 - 3\mu_1}{2N_1}, \frac{N_2\delta - 2cN_3}{2N_1}, \frac{N_2(\xi\mu - b^2)\mu_1 - 2b\mu}{2\mu\mu_1 N_1}\right\} \\ \varepsilon_2 &\leq \frac{N_1 d - 2jN_2}{2}. \end{aligned}$$

Moreover, we pick N_4 large enough, such that

$$N_4 \geq \frac{\delta\mu N_3 + 2bc}{m}.$$

Finally, once all the above constants are fixed, we choose N large enough, such that

$$\begin{aligned} N(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds) + \rho - N_2 c(1 + \frac{1}{\varepsilon_2}) - \frac{N_3\mu_1}{b} - N_4\mu_1 &\geq 0, \\ kN - N_1(1 + \frac{1}{\varepsilon_1}) - \frac{N_2 d^2}{2\delta} &\geq 0, \\ kN - N_1\frac{k^2}{d} &\geq 0. \end{aligned}$$

Consequently, there exists a positive constant ϱ, such that (2.59) becomes

$$\begin{aligned} \frac{d}{dt}\pounds\,(t) \;\leq\; & -\varrho\left\{\int_0^1 u_x^2 dx + \int_0^1 u_t^2 dx + \int_0^1 \phi_x^2 dx + \int_0^1 \phi_t^2 dx + \int_0^1 \phi_t^2 dx + \int_0^1 w^2 dx\right\} \\ & -\varrho\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, z^2\,(x,\sigma,s,t)\, ds d\sigma dx. \qquad (2.60) \end{aligned}$$

On the other hand, recalling (2.26), using Young's inequality, we get

$$
\begin{aligned}
E(t) &\leq \frac{1}{2}\int_0^1 \left\{\rho u_t^2 + j\phi_t^2 + \alpha w^2 + \delta\phi_x^2 + (\mu+b)u_x^2 + (\xi+b)\phi^2\right\} dx \\
&+ \frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\sigma,s,t)dsd\sigma dx. \\
&\leq \varsigma \int_0^1 \left\{u_t^2 + \phi_t^2 + w^2 + \phi_x^2 + u_x^2 + \phi^2\right\} dx + \varsigma \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\sigma,s,t)dsd\sigma dx,
\end{aligned}
$$

then

$$
-\int_0^1 \left\{u_t^2 + \phi_t^2 + w^2 + \phi_x^2 + u_x^2 + \phi^2\right\} dx - \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\sigma,s,t)dsd\sigma dx \leq \frac{-1}{\varsigma}E(t). \tag{2.61}
$$

From (2.60) and (2.61), we find

$$
\frac{d}{dt}\pounds\,(t) \leq \frac{-\varrho}{\varsigma}E(t). \tag{2.62}
$$

Moreover, we have the following Lemma

Lemma 2.2.7 *For N large enough, we have*

$$
c_1E(t) \leq \mathcal{L}(t) \leq c_2E(t), \forall t \geq 0 \tag{2.63}
$$

for two positive constants c_1 *and* c_2.

Proof. Let

$$
H(t) = I_1(t) + N_1I_2(t) + N_2I_3(t) + N_3I_4(t) + N_4I_5(t), \tag{2.64}
$$

$$\begin{aligned}|H(t)| = {} & \rho \int_0^1 |u_t u|\, dx + \frac{\mu_1}{2} \int_0^1 u^2 dx + \alpha N_1 \int_0^1 \left| w \int_0^x \phi_t(y,t) dy \right| dx \\ & + N_2 j \int_0^1 |\phi_t \phi|\, dx + \frac{b\rho}{\mu} N_2 \int_0^1 \left| \phi \int_0^x u_t(y,t) dy \right| dx \\ & + N_3 \frac{\rho\delta}{b} \int_0^1 |u_t \phi_x|\, dx + \frac{j\mu}{b} N_3 \int_0^1 |\phi_t u_x|\, dx \\ & + N_4 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)|\, z^2(x,\sigma,s,t)\, ds d\sigma dx,\end{aligned}$$

using Young's, Poincare's and Cauchy-schwartz inequalities, we obtain

$$\begin{aligned}|H(t)| \leq {} & \frac{1}{2} \int_0^1 \left\{ \lambda_1 \rho u_t^2 dx + \lambda_2 \mu u_x^2 dx + \lambda_3 \alpha w^2 dx + \lambda_4 j \phi_t^2 dx + \lambda_5 \xi \phi^2 dx + \lambda_6 \delta \phi_x^2 dx \right. \\ & \left. + \lambda_7 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x,\sigma,s,t)\, ds d\sigma \right\} dx,\end{aligned}$$

where

$$\begin{aligned}\lambda_1 &= 1 + \frac{b}{\mu} N_2 + N_3 \frac{\delta}{b},\ \lambda_2 = \frac{\rho c}{\mu} + \frac{j}{b} N_3 + \frac{\mu_1 c}{\mu}, \\ \lambda_3 &= N_1,\ \lambda_4 = \frac{\alpha}{j} N_1 + N_2 + \frac{\mu}{b} N_3, \\ \lambda_5 &= \frac{N_2 j}{\xi} + \frac{b\rho}{\xi\mu} N_2,\ \lambda_6 = N_3 \frac{\rho}{b},\ \lambda_7 = 2N_4.\end{aligned}$$

Therefore, we get

$$|H(t)| \leq C E(t),$$

such that $C = \max\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, 1\}$.

So, we can choose N large enough so that $c_1 = N - C$, and $c_2 = N + C$. which complete the proof.

Finally, recalling (2.62), we get

$$\frac{d}{dt}\pounds\left(t\right)\leq-\vartheta\pounds\left(t\right) \tag{2.65}$$

a simple integration of (2.65) over $(0,t)$ yields

$$\pounds\left(t\right)\leq\pounds\left(0\right)e^{-\vartheta t}$$

Using again (2.63), we find that.

$$E\left(t\right)\leq\frac{c_2}{c_1}E\left(0\right)e^{-\vartheta t}.$$

Thus we complete the proof of Theorem 2.2.1 ■

Chapter 3

Stability Result for a Weakly Nonlinearly Damped Porous System with Distributed Delay

In this chapter, we consider a one-dimensional porous system damped with a single weakly nonlinear feedback and distributed delay term. Without imposing any restrictive growth assumption near the origin on the damping term, we establish an explicit and general decay rate, using a multiplier method and some properties of convex functions in case of the same speed of propagation in the two equations of the system. The result is new and opens more research areas into porous-elastic system.

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x + \mu_1 u_t + \int_{\tau_1}^{\tau_2} \mu_2(s) u_t\left(x, t-s\right) ds = 0, \ x \in (0,1), \ t > 0, \\ j\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \alpha\left(t\right) g\left(\phi_t\right) = 0, \qquad x \in (0,1), \ t > 0, \\ u\left(x,0\right) = u_0\left(x\right), \ u_t\left(x,0\right) = u_1\left(x\right), \qquad x \in (0,1), \\ \phi\left(x,0\right) = \phi_0\left(x\right), \ \phi_t\left(x,0\right) = \phi_1\left(x\right), \qquad x \in (0,1), \\ u_x\left(0,t\right) = u_x\left(1,t\right), \ \phi\left(0,t\right) = \phi\left(1,t\right) = 0 \\ u_t\left(x,-t\right) = f_0\left(x,t\right) \qquad \text{in} \qquad (0,1) \times (0,\tau_2) \end{cases} \tag{3.1}$$

Firstly, to deal with the delay term, we introduce the new variable [20]

$$z\left(x,\rho,s,t\right) = u_t\left(x, t-\rho s\right), \ x \in \left(0,1\right), \ \rho \in \left(0,1\right), \ \rho \in \left(\tau_1,\tau_2\right), \ t > 0$$

Then we obtain

$$sz_t\left(x,\rho,s,t\right) + z_\rho\left(x,\rho,s,t\right) = 0, x \in \left(0,1\right), \ \rho \in \left(0,1\right), \ \rho \in \left(\tau_1,\tau_2\right), \ t > 0$$

Then problem (3.1) is equivalent to

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x + \mu_1 u_t + \int_{\tau_1}^{\tau_2} \mu_2(s) z\left(x,1,t,s\right) ds = 0, \ x \in (0,1), \ t > 0, \\ j\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \alpha\left(t\right) g\left(\phi_t\right) = 0, \qquad x \in (0,1), \ t > 0, \\ sz_t\left(x,\rho,s,t\right) + z_\rho\left(x,\rho,s,t\right) = 0, x \in \left(0,1\right), \ \rho \in \left(0,1\right), \ \rho \in \left(\tau_1,\tau_2\right), \ t > 0 \\ u\left(x,0\right) = u_0\left(x\right), \ u_t\left(x,0\right) = u_1\left(x\right), \qquad x \in (0,1), \\ \phi\left(x,0\right) = \phi_0\left(x\right), \ \phi_t\left(x,0\right) = \phi_1\left(x\right), \qquad x \in (0,1), \\ u_x\left(0,t\right) = u_x\left(1,t\right), \ \phi\left(0,t\right) = \phi\left(1,t\right) = 0 \\ z\left(x,\rho,s,0\right) = f_0\left(x,\rho s\right), \qquad (x,\rho,s) \in (0,1) \times (0,1) \times (\tau_1,\tau_2) \end{cases} \tag{3.2}$$

In recent paper, Apalara in [23] considered the following on-dimensional porous system damped with a single weakly nonlinear feedback

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \ x \in (0,1), \ t > 0, \\ j\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \alpha(t)\, g(\phi_t) = 0, & x \in (0,1), \ t > 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in (0,1), \\ \phi(x,0) = \phi_0(x), \ \phi_t(x,0) = \phi_1(x), & x \in (0,1), \\ u_x(0,t) = u_x(1,t), \ \phi(0,t) = \phi(1,t) = 0 \end{cases}$$

Without in pasing an explicit and general decay rate, he used a multiplier method and some proprieties of convex function in case of same speed of propagation in the both equation of the system. The same author, in [24] considered a porous-elastic system with memory term acting only on the porous equation, with the mixed boundary Neumann-Direchlet conditions, he prove a general decay result, for which exponential and polynomial decay results are special cases.

Back to system (3.1), it is to be noted that when $\mu_1 = \mu_2 = 0$ and replacing the term $\alpha(t)\, g(\phi_t)$ by the term $\int_0^t g(t-s) u_{xx}(x,s)\, ds$ then (3.1) is equivalent to the well-known Timoshenko system of memory type which is exponentially stable depending of the relaxation function g and provided that the wave speeds of the system are equal (See [25, 26]).

Messaoudi and Fareh [27] investigated the following system:

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x - \beta\theta_x, \ \text{in } (0,1) \times (0,\infty), \\ j\phi_{tt} = \alpha\phi_{xx} - bu_x + \xi\phi + m\theta + \tau\phi_t, \quad \text{in } (0,1) \times (0,\infty), \\ c\phi_t = -q_x - \beta u_{tx} - m\phi_t, \quad \text{in } (0,1) \times (0,\infty), \\ \tau_0 q_t - q + k\theta_x = 0, \quad \text{in } (0,1) \times (0,\infty), \end{cases}$$

and established, using the energy method, an exponential decay result. For more results on the subject, we refer the reader to [28, 29, 30, 31].

Concerning the weight of the delay, we assume that

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds < \mu_1$$

and establish the well-posedness as well as the exponential stability results of the energy $E(t)$, defined by

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \left[\rho u_t^2 + \mu u_x^2 + \xi\phi^2 + \delta\phi_x^2 + j\phi_t^2 + 2b\phi u_x\right] dx \\ &+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, z^2(x,\rho,s,t)\, ds d\rho dx \end{aligned} \tag{3.3}$$

3.1 Preliminaries

In this section, we present some materials needed in the proof of our result. We assume α and g satisfy the following hypotheses:

(H1) $\alpha : \mathbb{R}^+ \to \mathbb{R}_*^+$ is a non-increasing differentiable function;

(H2) $g : \mathbb{R} \to \mathbb{R}$ is a non-decreasing C^0-function such that there exist positive constants c_1, c_2, η and $G \in C^1([0,\infty))$, with $G(0) = 0$, and G is linear or strictly convex C^2-function on $(0,\eta]$ such that

$$\begin{cases} s^2 + g^2(s) \le G^{-1}(sg(s)) \ \text{ for all } \ |s| \le \eta \\ c_1|s| \le |g(s)| \le c_2|s| \ \text{ for all } \ |s| \ge \eta \end{cases}$$

Remark 3.1.1 *Hypothesis (H2) implies that* $sg(s) > 0$ *for all* $s \neq 0$.

* *According to our knowledge, hypothesis (H2) with* $\eta = 1$ *was first introduced by Lasiecka and Tataru [32]. They established a decay result, which depends on the solution of an explicit nonlinear ordinary differential equation. Furthermore, they proved*

that the monotonicity and continuity of g guarantee the existence of the function G defined in (H2).

For completeness purpose we state, without proof, the existence and regularity result of system (3.1). First, we introduce the following spaces:

$$\mathcal{H} = H_*^1(0,1) \times L_*^2(0,1) \times H^1(0,1) \times L^2(0,1) \times L^2((0,1) \times (0,1) \times (\tau_1,\tau_2)), \quad (3.4)$$

and

$$\begin{aligned} \widetilde{\mathcal{H}} &= \phi_0 \in \left[H_*^2(0,1) \cap H_*^1(0,1)\right] \times H_*^1(0,1) \times \left[H^2(0,1) \cap H^1(0,1)\right] \\ &\quad \times H^1(0,1) \times L^2((0,1) \times (0,1) \times (\tau_1,\tau_2)), \end{aligned}$$

where

$$L_*^2(0,1) = \left\{\psi \in L^2(0,1) : \int_0^1 \psi(x)\,dx = 0\right\},$$

$$H_*^1(0,1) = H^1(0,1) \times L_*^2(0,1),$$

$$H_*^2(0,1) = \left\{\psi \in H^2(0,1) : \psi_x(0) = \psi_x(1) = 0\right\}.$$

For $U = (u, u_t, \phi, \phi_t, z)$, *we have the following existence and regularity result:*

Proposition 3.1.1 *Assume that (H1) and (H2) are satisfied. Then for all* $U_0 \in \mathcal{H}$, *the system (3.1) has a unique global (weak) solution*

$$u \in C\left(\mathbb{R}_+; H_*^1(0,1)\right) \cap C^1\left(\mathbb{R}_+; L_*^2(0,1)\right), \ \phi \in C\left(\mathbb{R}_+; H^1(0,1)\right) \cap C^1\left(\mathbb{R}_+; L^2(0,1)\right).$$

Moreover, if $U_0 \in \widetilde{\mathcal{H}}$, *then the solution satisfies*

$$u \in L^\infty\left(\mathbb{R}_+; H_*^2(0,1) \cap H_*^1(0,1)\right) \cap W^{1,\infty}\left(\mathbb{R}_+; H_*^1(0,1)\right) \cap W^{2,\infty}\left(\mathbb{R}_+; L_*^2(0,1)\right),$$

$$\phi \in L^\infty\left(\mathbb{R}_+; H^2(0,1) \cap H_0^1(0,1)\right) \cap W^{1,\infty}\left(\mathbb{R}_+; H_0^1(0,1)\right) \cap W^{2,\infty}\left(\mathbb{R}_+; L^2(0,1)\right)$$

Remark 3.1.2 *This result can be proved using the theory of maximal nonlinear monotone operators (see [33]).*

3.2 Technical Lemmas

In this section, we state and prove our stability results for the energy of system (3.1) by using the multiplier technique. To achieve our goal, we need the following lemmas.

Lemma 3.2.1 *Let (u, ϕ, z) be the solution of (3.2), then we have*

$$E'(t) \leq -m_e \int_0^1 u_t^2 dx - \int_0^1 \alpha(t) \phi_t g(\phi_t) dx \leq 0 \tag{3.5}$$

Proof. Multiplying $(3.2)_1$, and $(3.2)_2$ by u_t, ϕ_t respectively, and integrating over $(0,1)$, using integration by parts and the boundary conditions, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\rho u_t^2 + \mu u_x^2 + \xi \phi^2 + \delta \phi_x^2 + j\phi_t^2 + 2b\phi u_x\right) dx = \\ &-\int_0^1 \alpha(t) \phi_t g(\phi_t) dx - \mu_1 \int_0^1 u_t^2 dx - \int_0^1 u_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s) ds dx \end{aligned} \tag{3.6}$$

Multiplying $(3.2)_3$ by $|\mu_2(s)| z$, integrating the product over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, and recall that $z(x,0,s,t) = u_t$, yield

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,t,s) ds d\rho dx &= -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s) ds dx \\ &\quad + \frac{1}{2}\int_0^1 u_t \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds dx. \end{aligned} \tag{3.7}$$

A combination of (3.6) and (3.7) gives

$$E'(t) = -\int_0^1 \alpha(t) \phi_t g(\phi_t) dx - \mu_1 \int_0^1 u_t^2 dx - \int_0^1 u_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s) ds dx$$

with

$$-\int_0^1 u_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\,ds dx \leq \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 u_t^2 dx + \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,t,s)\,ds dx$$

then

$$E'(t) \leq -\int_0^1 \alpha(t)\,\phi_t g(\phi_t)\,dx - \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\right)\int_0^1 u_t^2 dx$$

by $\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\right) = m_e$ we obtain (3.5). ■

Lemma 3.2.2 *Assume that (H1) and (H2) hold. Then, for all* $U_0 \in \mathcal{H}$*, the functional*

$$F_1(t) = j\int_0^1 \phi_t \phi dx + \frac{b\rho}{\mu}\int_0^1 \phi \int_0^x u_t(y)\,dy dx \tag{3.8}$$

estimate

$$\begin{aligned} F_1'(t) &\leq \left(j + \frac{\varepsilon_1 b\rho}{\mu}\right)\int_0^1 \phi_t^2 dx - j\delta\int_0^1 \phi_x^2 dx + bj\varepsilon_1 \int_0^1 u_x^2 dx + \frac{b\rho}{4\varepsilon_1\mu}\int_0^1 u_t^2 dx \\ &\quad + \left(j\alpha(0)\varepsilon_1 + \frac{bj}{4\varepsilon_1} - \xi j\right)\int_0^1 \phi^2 dx + \frac{j\alpha(0)}{4\varepsilon_1}\int_0^1 g^2(\phi_t)\,dx \end{aligned} \tag{3.9}$$

Proof.

$$\begin{aligned} F_1'(t) &\leq j\int_0^1 \phi_t^2 dx - j\delta\int_0^1 \phi_x^2 dx + bj\varepsilon_1\int_0^1 u_x^2 dx + \frac{c_p bj}{4\varepsilon_1}\int_0^1 \phi_x^2 dx - \xi j c_p \int_0^1 \phi_x^2 dx \\ &\quad -j\int_0^1 \alpha(t)\,\phi g(\phi_t)\,dx + \frac{b\rho}{\mu}\int_0^1 \phi_t \int_0^x u_t(y)\,dy dx \\ &\quad +\frac{b\rho}{\mu}\int_0^1 \phi \frac{d}{dt}\left(\int_0^x u_t(y)\,dy\right) dx \end{aligned}$$

By Caucy-Schwartz inequality, it is clear that

$$\int_0^1 \left(\int_0^x u_t(y)\,dy\right)^2 dx \leq \int_0^1 \left(\int_0^1 u_t dx\right)^2 dx \leq \int_0^1 u_t^2 dx$$

then

$$\begin{aligned}
F_1'(t) \;\leq\;& j\int_0^1 \phi_t^2dx - j\delta\int_0^1 \phi_x^2dx + bj\varepsilon_1\int_0^1 u_x^2dx + \frac{c_pbj}{4\varepsilon_1}\int_0^1 \phi_x^2dx - \xi jc_p\int_0^1 \phi_x^2dx \\
& -j\int_0^1 \alpha(t)\,\phi g(\phi_t)\,dx \\
& +\frac{\varepsilon_1 b\rho}{\mu}\int_0^1 \phi_t^2dx + \frac{b\rho}{4\varepsilon_1\mu}\int_0^1\left(\int_0^x u_t(y)\,dy\right)^2dx \\
& +\frac{b\rho}{\mu}\int_0^1 \phi\frac{d}{dt}\left(\int_0^x u_t(y)\,dy\right)dx
\end{aligned}$$

we get

$$\begin{aligned}
F_1'(t) \;\leq\;& \left(j+\frac{\varepsilon_1 b\rho}{\mu}\right)\int_0^1 \phi_t^2dx - j\delta\int_0^1 \phi_x^2dx + bj\varepsilon_1\int_0^1 u_x^2dx + \frac{b\rho}{4\varepsilon_1\mu}\int_0^1 u_t^2dx \\
& +\left(j\alpha(t)\,\varepsilon_1 + \frac{bj}{4\varepsilon_1} - \xi j\right)\int_0^1 \phi^2dx + \frac{j\alpha(t)}{4\varepsilon_1}\int_0^1 g^2(\phi_t)\,dx
\end{aligned}$$

■

Lemma 3.2.3 *Assume that (H1), (H2) and (3.12) hold. Then, for all $U_0 \in \mathcal{H}$, the functional*

$$F_2(t) = b\int_0^1 \phi_x u_t dx + b\int_0^1 \phi_t u_x dx \tag{3.10}$$

satisfies, for any $\varepsilon_2 > 0$,

$$\begin{aligned}
F_2'(t) \;\leq\;& \left(\frac{b^2}{\rho} + \varepsilon_2\frac{b\mu_1}{\rho} + \frac{bn_0}{2\rho}\right)\int_0^1 \phi_x^2dx - \left(\frac{b^2}{j} - \frac{b\xi}{4\varepsilon_2 j} - \frac{b}{j}\alpha(t)\right)\int_0^1 u_x^2dx \\
& +\frac{b\mu_1}{4\varepsilon_2\rho}\int_0^1 u_t^2dx + \varepsilon_2\frac{b\xi}{j}\int_0^1 \phi^2dx + \frac{b}{j}\alpha(t)\int_0^1 g^2(\phi_t)\,dx \\
& +\frac{1}{2}\frac{b}{\rho}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)\,dsdx
\end{aligned} \tag{3.11}$$

Proof. Simple computaions give

$$\begin{aligned} F_2^{'}(t) &= \frac{b^2}{\rho}\int_0^1 \phi_x^2 dx - \frac{b^2}{j}\int_0^1 u_x^2 dx \\ &+\frac{b\mu}{\rho}\int_0^1 u_{xx}\phi_x dx - \frac{b\mu_1}{\rho}\int_0^1 \phi_x u_t dx \\ &+\frac{b\delta}{j}\int_0^1 \phi_{xx}u_x dx - \frac{b\xi}{j}\int_0^1 \phi u_x dx \\ &-\frac{b}{\rho}\int_0^1 \phi_x \int_{\tau_1}^{\tau_2} \mu_2(s)u_t(x,1,t,s)\,dsdx - \frac{b}{j}\int_0^1 \alpha(t)\,u_x g(\phi_t)\,dx \end{aligned}$$

taking into account the fact that

$$\frac{\mu}{\rho} = \frac{\delta}{j} \tag{3.12}$$

and using Young's inequality

$$\begin{aligned} F_2'(t) &\le \left(\frac{b^2}{\rho} + \varepsilon_2\frac{b\mu_1}{\rho}\right)\int_0^1 \phi_x^2 dx + \left(\frac{b\xi}{4\varepsilon_2 j} - \frac{b^2}{j} + \frac{b}{j}\alpha(t)\right)\int_0^1 u_x^2 dx \\ &+\frac{b\mu_1}{4\varepsilon_2\rho}\int_0^1 u_t^2 dx + \varepsilon_2\frac{b\xi}{j}\int_0^1 \phi^2 dx + \frac{b}{j}\alpha(t)\int_0^1 g^2(\phi_t)\,dx \\ &-\frac{b}{\rho}\int_0^1 \phi_x \int_{\tau_1}^{\tau_2} \mu_2(s)z(x,1,t,s)\,dsdx \end{aligned}$$

$$\begin{aligned} -\frac{b}{\rho}\int_0^1 \phi_x \int_{\tau_1}^{\tau_2} \mu_2(s)z(x,1,t,s)\,dsdx &\le \frac{1}{2}\frac{b}{\rho}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds \int_0^1 \phi_x^2 dx + \\ &\frac{1}{2}\frac{b}{\rho}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,z^2(x,1,s,t)\,dsdx \end{aligned}$$

$$\begin{aligned} F_2'(t) &\le \left(\frac{b^2}{\rho} + \varepsilon_2\frac{b\mu_1}{\rho} + \frac{bn_0}{2\rho}\right)\int_0^1 \phi_x^2 dx - \left(\frac{b^2}{j} - \frac{b\xi}{4\varepsilon_2 j} - \frac{b}{j}\alpha(t)\right)\int_0^1 u_x^2 dx \\ &+\frac{b\mu_1}{4\varepsilon_2\rho}\int_0^1 u_t^2 dx + \varepsilon_2\frac{b\xi}{j}\int_0^1 \phi^2 dx + \frac{b}{j}\alpha(t)\int_0^1 g^2(\phi_t)\,dx \\ &+\frac{1}{2}\frac{b}{\rho}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,z^2(x,1,s,t)\,dsdx \end{aligned}$$

with $\int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds = n_0$ ■

Lemma 3.2.4 *The functional*

$$F_3(t) = -\rho\int_0^1 u_t u dx \tag{3.13}$$

satisfies, for any $\varepsilon_3 > 0$,

$$\begin{aligned} F_3'(t) &\leq \left(\mu + \frac{n_0 c_p}{2} + c_p b \varepsilon_3\right) \int_0^1 u_x^2 dx \\ &+ \frac{b}{4\varepsilon_3} \int_0^1 \phi_x^2 dx - \left(\rho - \frac{\mu_1}{4\varepsilon_3}\right) \int_0^1 u_t^2 dx \\ &+ \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,t,s)\, ds dx \end{aligned} \tag{3.14}$$

Proof. A simple differentiation of $F_3(t)$, using the first equation in (3.2) , gives

$$\begin{aligned} F_3'(t) &= -\rho \int_0^1 u_t^2 dx + \mu \int_0^1 u_x^2 dx \\ &+ c_p b \varepsilon_3 \int_0^1 u_x^2 dx + \frac{b}{4\varepsilon_3} \int_0^1 \phi_x^2 dx \\ &+ \mu_1 \varepsilon_3 c_p \int_0^1 u_x^2 dx + \frac{\mu_1}{4\varepsilon_3} \int_0^1 u_t^2 dx \\ &+ \int_0^1 \int_{\tau_1}^{\tau_2} \mu_2(s) u u_t(x,1,t,s)\, ds dx \end{aligned}$$

$$\int_0^1 u \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(x,1,t,s)\, ds dx \leq \frac{c_p}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 u_x^2 dx + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,t,s)\, ds dx$$

then

$$\begin{aligned} F_3'(t) &\leq \left(\mu + \mu_1 \varepsilon_3 c_p + \frac{n_0 c_p}{2} + c_p b \varepsilon_3\right) \int_0^1 u_x^2 dx \\ &+ \frac{b}{4\varepsilon_3} \int_0^1 \phi_x^2 dx - \left(\rho - \frac{\mu_1}{4\varepsilon_3}\right) \int_0^1 u_t^2 dx \\ &+ \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,t,s)\, ds dx \end{aligned}$$

■

Lemma 3.2.5 *The functional*

$$F_4(t) = \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)|\, z^2(x,\rho,t,s)\, ds d\rho dx \tag{3.15}$$

satisfies, for some positive constant m_1, the following estimate

$$\begin{aligned} F_4'(t) &\leq -m_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,t,s)\, dsdx + \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_0^1 u_t^2 dx \\ &\quad -m_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\, |\mu_2(s)|\, z^2(x,\rho,t,s)\, dsd\rho dx \end{aligned} \tag{3.16}$$

Proof. With

$$\begin{aligned} sz_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) &= 0 \text{ in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty) \\ z_t(x,\rho,t,s) &= -\frac{1}{s} z_\rho(x,\rho,t,s) \end{aligned} \tag{3.17}$$

Differentiating $F_4(t)$, and using the equation (3.17), we obtain

$$\begin{aligned} F_4'(t) &= 2\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu_2(s)|\, zz_t(x,\rho,t,s)\, dsd\rho dx \\ &= -\frac{\partial}{\partial\rho} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_2(s)|\, z^2(x,\rho,t,s)\, dsd\rho dx \\ &\quad - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu_2(s)|\, z^2(x,\rho,t,s)\, dsd\rho dx \end{aligned}$$

$$\begin{aligned} F_4'(t) &= -\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left[e^{-s\rho} z^2(x,1,t,s) - z^2(x,0,t,s) \right] dsdx \\ &\quad - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu_2(s)|\, z^2(x,\rho,t,s)\, dsd\rho dx \end{aligned}$$

Using the fact that $z(x,0,t,s) = u_t$ and $e^{-s} \leq e^{-s\rho} \leq 1$, for all $\rho \in [0,1]$, we obtain

$$\begin{aligned} F_4'(t) &\leq -\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, e^{-s\rho} z^2(x,1,t,s)\, dsdx + \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_0^1 u_t^2 dx \\ &\quad - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu_2(s)|\, z^2(x,\rho,t,s)\, dsd\rho dx \end{aligned}$$

Because $-se^{-s}$ is an increasing function, we have $-se^{-s} \leq -se^{-\tau_2}$, for all $s \in [\tau_1,\tau_2]$

Finally, setting $m_1 = e^{-\tau_2}$, with $\int_{\tau_1}^{\tau_2} |\mu_2(s)| < \mu_1$, we obtain

$$\begin{aligned} F_4'(t) &\leq -m_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,t,s)\, dsdx + \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_0^1 u_t^2 dx \\ &\quad -m_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\, |\mu_2(s)|\, z^2(x,\rho,t,s)\, dsd\rho dx \end{aligned}$$

■

Lemma 3.2.6 *Suppose (H1), (H2), and Eq. (3.12) hold. Let* $U_0 \in \mathcal{H}$*. Then, for* $N, N_1, N_2, N_3 > 0$ *sufficiently large, the Lyapunov functional defined by*

$$\mathcal{L}(t) := NE(t) + N_1 F_1(t) + N_2 F_2(t) + F_3(t) + N_3 F_4(t)$$

satisfies, for some positive constants d_1, d_2 *and* k_1

$$d_1 \mathcal{L}(t) \leq E(t) \leq d_2 \mathcal{L}(t), \ \forall t \geq 0 \tag{3.18}$$

and

$$\mathcal{L}'(t) \leq -k_1 E(t) + c \int_0^1 \left(\phi_t^2 + g^2(\phi_t)\right) dx, \ \forall t \geq 0 \tag{3.19}$$

with

$$\begin{aligned}
\mathcal{L}'(t) \ \leq \ & \left[\frac{b\rho}{4\varepsilon_1 \mu} N_1 - N m_e + N_3 \mu_1 + \frac{b\mu_1}{4\varepsilon_2 \rho} N_2 - \left(\rho - \frac{\mu_1}{4\varepsilon_3}\right)\right] \int_0^1 u_t^2 dx \\
& + \left(N_1 \left(j + \frac{\varepsilon_1 b\rho}{\mu}\right)\right) \int_0^1 \phi_t^2 dx \\
& + \left(bj\varepsilon_1 N_1 + \left(\mu + \frac{n_0}{2} + b\varepsilon_3\right) - N_2 \left(\frac{b^2}{j} - \frac{b\xi}{4\varepsilon_2 j} - \frac{b}{j}\alpha(t)\right)\right) \int_0^1 u_x^2 dx \\
& + \left(N_2 \frac{1}{2\rho}\left(2b^2 + 2\varepsilon_2 b\mu_1 + bn_0\right) - j\delta N_1 + \frac{b}{4\varepsilon_3}\right) \int_0^1 \phi_x^2 dx \\
& + \left(\varepsilon_2 \frac{b\xi}{j} N_2 + N_1 \left(j\alpha(t)\varepsilon_1 + \frac{bj}{4\varepsilon_1} - \xi j\right)\right) \int_0^1 \phi^2 dx \\
& + \left(N_1 \frac{j\alpha(t)}{4\varepsilon_1} + \frac{b}{j}\alpha(t) N_2\right) \int_0^1 g^2(\phi_t) dx \\
& + \left(\frac{1}{2}\left(\frac{bN_2}{\rho} + 1\right) - m_1 N_3\right) \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx \\
& - m_1 N_3 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, t, s) ds d\rho dx - N \int_0^1 \alpha(t) \phi_t g(\phi_t) dx
\end{aligned}$$

At this point, we have to choose our constants very carefully. First, choosing $\varepsilon_3 << 1$, and $\varepsilon_1, \varepsilon_2$ small enough such that

$$\varepsilon_1 \leq \frac{b\rho N_1}{4\mu\left(N m_e - N_3 \mu_1\right)}, \ \varepsilon_2 \leq \frac{b\mu_1 N_2}{4\rho}$$

Moreover, we pick N_i $i = 1, 2, 3$ large enough so that

$$N_2 \geq \frac{bj\varepsilon_1 N_1 + \left(\mu + \frac{n_0}{2} + b\varepsilon_3\right)}{\dfrac{b^2}{j} - \dfrac{b\xi}{4\varepsilon_2 j} - \dfrac{b}{j}\alpha(t)}$$

and

$$N_3 \geq \frac{\left(\dfrac{bN_2}{\rho} + 1\right)}{2m_1}.$$

After that, we can choose N large enough such that

$$N > \frac{1}{m_e}\left[\frac{b\rho N_1}{4\varepsilon_1 \mu} + N_3\mu_1 + \frac{N_2 b\mu_1}{4\varepsilon_2 \rho} - \left(\rho - \frac{\mu_1}{4\varepsilon_3}\right)\right].$$

Consequently, there exists a positive constant η_1 such that (3.19) becomes

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \quad &\leq \quad -c_1\int_0^1 \left(u_t^2 + u_x + \varphi_x^2 + \phi^2\right)dx + c_2\int_0^1 \left(\phi_t^2 + g^2(\phi_t)\right)dx \\ &\quad -c_3\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)\, dsdx. \qquad (3.20)\end{aligned}$$

In this section, we state and prove our stability result.

3.3 Stability Result

Theorem 3.3.1 *[29] Suppose (H1), (H2), and (3.12) hold. Let $U_0 \in \mathcal{H}$. there exist positive constants a_1, a_2, a_3 and η_0 such that the solution of (3.2) satisfies*

$$E(t) \leq a_1 G_1^{-1}\left(a_2\int_0^t \alpha(s)\,ds + a_3\right),\ t \geq 0, \qquad (3.21)$$

where

$$G_1^{-1} = \int_t^1 \frac{1}{G_0(s)}ds \text{ and } G_0(s) = tG'(\eta_0 t).$$

Remark 3.3.1 *G_1 strictly decreases and is convex on $(0,1]$ and $\lim\limits_{t\to 0} G_1(t) = +\infty$.*

Proof. We multiply (3.19) by $\alpha(t)$ to get

$$\alpha(t)\mathcal{L}'(t) \leq -k_1\alpha(t)E(t) + c\alpha(t)\int_0^1 \left(\phi_t^2 + g^2(\phi_t)\right)dx. \tag{3.22}$$

Now, we discuss two cases:

Case I: G is linear on $[0,\eta]$. In this case, using (H2) and Eq.(3.5), we deduce that

$$\alpha(t)\mathcal{L}'(t) \leq -k_1\alpha(t)E(t) + c\alpha(t)\int_0^1 \left(\phi_t^2 + g^2(\phi_t)\right)dx = -k_1\alpha(t)E(t) - cE'(t),$$

which can be rewritten as

$$\left(\alpha(t)\mathcal{L}(t) + cE(t)\right)' - \alpha'(t)\mathcal{L}(t) \leq -k_1\alpha(t)E(t).$$

Using (H1), we obtain

$$\left(\alpha(t)\mathcal{L}(t) + cE(t)\right)' \leq -k_1\alpha(t)E(t).$$

By exploiting (3.18), it can easily be shown that

$$\mathcal{S}_0(t) := \alpha(t)\mathcal{L}(t) + cE(t) \sim E(t). \tag{3.23}$$

So, for some positive constant λ_1, we obtain

$$\mathcal{S}_0'(t) + \lambda_1\alpha(t)\mathcal{S}_0(t) \leq 0, \ \forall t \geq 0 \tag{3.24}$$

The combination of Eq. (3.23) and (3.24), gives

$$E(t) \leq E(0)e^{-\lambda_1\int_0^t \alpha(s)ds} = E(0)G_1^{-1}\left(\lambda_1\int_0^t \alpha(s)ds\right). \tag{3.25}$$

Case II: G is nonlinear on $[0,\eta]$. In this case, we first choose $0 < \eta_1 < \eta$ such that

$$sg(s) \leq \min\{\eta, G(\eta)\}, \ \forall |s| \leq \eta_1. \tag{3.26}$$

Using (H2) along with fact that g is continuous and $|g(s)| > 0$, for $s \neq 0$, it follows that

$$\begin{cases} s^2 + g^2(s) \leq G^{-1}(sg(s)), \ \forall |s| \leq \eta_1 \\ c_1 |s| \leq |sg(s)| \leq c_2 |s|, \ \forall |s| \geq \eta_1 \end{cases} \tag{3.27}$$

To estimate the last integral in Eq. (3.22), we consider the following partition of $(0,1)$:

$$I_1 = \{x \in (0,1) : |\phi_t| \leq \eta_1\}, \ I_2 = \{x \in (0,1) : |\phi_t| > \eta_1\}.$$

Now, with $I(t)$ defined by

$$I(t) = \int_{I_1} \phi_t g(\phi_t)\, dx,$$

we have, using Jensen inequality (note that G^{-1} is concave and recall (3.26))

$$G^{-1}(I(t)) \geq c \int_{I_1} G^{-1}(\phi_t g(\phi_t))\, dx. \tag{3.28}$$

The combination of Eq. (3.27) and (3.28) yields

$$\begin{aligned} \alpha(t) \int_0^1 \left(\phi_t^2 + g^2(\phi_t)\right) dx &= \alpha(t) \int_{I_1} \left(\phi_t^2 + g^2(\phi_t)\right) dx + \alpha(t) \int_{I_2} \left(\phi_t^2 + g^2(\phi_t)\right) dx \\ &\leq \alpha(t) \int_{I_1} G^{-1}(\phi_t g(\phi_t))\, dx + c\alpha(t) \int_{I_2} \phi_t g(\phi_t)\, dx \\ &\leq c\alpha(t) G^{-1}(I(t)) - cE'(t). \end{aligned} \tag{3.29}$$

So, by substituting (3.29) into (3.22) and using (3.23) and (H1), we have

$$\mathcal{S}_0'(t) \leq -k_1 \alpha(t) E(t) + c\alpha(t) G^{-1}(I(t)) \tag{3.30}$$

Now, for $\eta_1 < \eta$ and $\delta_0 > 0$, using (3.30) and the fact that $E' \leq 0$, $G' > 0$, $G'' > 0$ on $(0, \eta)$, we find that the functional $\mathcal{S}_1$, defined by

$$\mathcal{S}_1(t) := G'\left(\eta_0 \frac{E(t)}{E(0)}\right) \mathcal{S}_0(t) + \delta_0 E(t),$$

satisfies, for some b_1, $b_2 > 0$,

$$b_1 \mathcal{S}_1(t) \leq E(t) \leq b_2 \mathcal{S}_1(t) \tag{3.31}$$

and

$$\begin{aligned}
\mathcal{S}_0'(t) \quad &: \quad = \eta_0 \frac{E'(t)}{E(0)} G'' \left(\eta_0 \frac{E(t)}{E(0)} \right) \mathcal{S}_0(t) + G' \left(\eta_0 \frac{E(t)}{E(0)} \right) \mathcal{S}_0'(t) + \delta_0 E'(t) \\
&\leq \quad -k_1 \alpha(t) E(t) G' \left(\eta_0 \frac{E(t)}{E(0)} \right) + c\alpha(t) G' \left(\eta_0 \frac{E(t)}{E(0)} \right) G^{-1}(I(t)) + \delta_0 E'(t)
\end{aligned} \tag{3.32}$$

Let G^* be the convex conjugate of G defined by

$$G^*(s) = s \left(G' \right)^{-1}(s) - G \left[\left(G' \right)^{-1}(s) \right], \text{ if } s \in (0, G'(\eta)],$$

satisfying the following general Young's inequality

$$AB \leq G^*(A) + G(B), \text{ if } A \in (0, G'(\eta)], \ B \in (0, \eta].$$

With

$$A = G' \left(\eta_0 \frac{E(t)}{E(0)} \right) \text{ and } B = G^{-1}(I(t)),$$

using (3.26), we obtain

$$c\alpha(t) G' \left(\eta_0 \frac{E(t)}{E(0)} \right) G^{-1}(I(t)) \leq c\alpha(t) G^* \left(G' \left(\eta_0 \frac{E(t)}{E(0)} \right) \right) + c\alpha(t) I(t).$$

By exploiting (3.5) and the fact that

$$G^*(s) \leq s (G')^{-1}(s), \text{ we get}$$

$$c\alpha(t) G' \left(\eta_0 \frac{E(t)}{E(0)} \right) G^{-1}(I(t)) \leq c\alpha(t) \eta_0 \frac{E(t)}{E(0)} G' \left(\eta_0 \frac{E(t)}{E(0)} \right) - cE'(t) \tag{3.33}$$

By substituting (3.32) into Eq. (3.33), we obtain

$$\mathcal{S}_1'(t) \leq -k\alpha(t) \frac{E(t)}{E(0)} G' \left(\eta_0 \frac{E(t)}{E(0)} \right) = -k_1 \alpha(t) G_0 \left(\frac{E(t)}{E(0)} \right) \tag{3.34}$$

where $k > 0$ and $G_0(t) = tG'(\eta_0 t)$.

Note that

$$G_0'(t) = G'(\eta_0 t) + \eta_0 t G''(\eta_0 t).$$

So, using the strict convexity of G on $(0, \eta]$, we find that $G_0(t), G_0'(t) > 0$ on $(0, 1]$.With $\mathcal{S}(t) := \frac{b_1 \mathcal{S}_1(t)}{E(0)}$ it is obvious that $\mathcal{S}(t) \leq \frac{E(t)}{E(0)} \leq 1$. Now, using (3.31) and (3.34), we have

$$\mathcal{S}(t) \sim E(t) \tag{3.35}$$

and, for some $a_2 > 0$

$$\mathcal{S}'(t) \leq -a_2 \alpha(t) G_0(\mathcal{S}(t)). \tag{3.36}$$

Inequality (3.36) implies that

$$\frac{d}{dt} G_1(\mathcal{S}(t)) \geq a_2 \alpha(t),$$

where

$$G_1(t) = \int_1^t \frac{1}{G_0(s)} ds.$$

Thus, by integrating over $[0, t]$, we obtain, for some $a_3 > 0$,

$$\mathcal{S}(t) \leq G_1^{-1}\left(a_2 \int_0^t \alpha(s) ds + a_3\right). \tag{3.37}$$

Here, we used, based on the properties of G_0, the fact that G_1 is strictly decreasing on $(0, 1]$.Finally, using (3.37) and (3.35), we obtain (3.21). ■

Chapter 4

Well-posedness and exponential decay of the thermoelastic Ful von Kármán beam with discret delay term

In the present chapter, we will consider a delayed system with Cattaneo's law and thermoelasticity with second sound

$$\begin{cases} w_{tt} - d_1\left[\left(u_x + \frac{1}{2}\left(w_x\right)^2\right)w_x\right]_x + d_2 w_{xxxx} + \mu_1 w_t + \mu_2 w_t(x, t-\tau) = 0, \\ u_{tt} - d_1\left[\left(u_x + \frac{1}{2}\left(w_x\right)^2\right)\right]_x + \delta\theta_x = 0, \\ \theta_t + q_x + \delta u_{tx} = 0, \\ q_t + \gamma q + \theta_x = 0, \end{cases} \tag{4.1}$$

in $\Omega \times (0,\infty)$, where $\Omega = [0,L]$ and d_1, d_2, δ, l, and γ are a positive constants and, μ_1, μ_2 are positive real numbers. We complement system (4.1) with boundary conditions

$$\begin{cases} u = 0, w = 0,\ \theta_x = 0 \text{ at } x = 0, L \text{ for any } t > 0, \\ w_x = 0 \text{ at } x = 0, L \text{ for any } t > 0, \end{cases} \tag{4.2}$$

and the initial data

$$\begin{cases} u\left(0,.\right) = u_0, u_t\left(0,.\right) = u_1, w\left(0,.\right) = w_0, w_t\left(0,.\right) = w_1, \\ \theta\left(0,.\right) = \theta_0, \theta_t\left(0,.\right) = \theta_1, \\ w_t(x,t-\tau) = f_0\left(x,t-\tau\right) \text{ in } \left(0,L\right) \times \left(0,\tau\right). \end{cases} \tag{4.3}$$

The chapter is organized as follows. In section 2 we state several useful results and give a well-posedness theorem (Theorem 4.1.1) by using a semi-group approach for linear and nonlinear cases. In sections 3 we prove exponential stability result (Theorem 4.2.1), under appropriate conditions (4.4), (4.9). The stability results are established by using an appropriate Lyapunov functions.

4.1 Preliminaries and Well-posedness

First assume the following hypotheses:

$$\left|\mu_2\right| < \left|\mu_1\right|, \tag{4.4}$$

and we will prove that system (4.1)-(4.3) is well posed using semigroup theory by introducing the following new variable [20]

$$z(x,\rho,t) = w_t\left(x, t-\tau\rho\right), x \in \left(0,L\right), \rho \in \left(0,1\right), t > 0, \tag{4.5}$$

then we have,

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \text{ in } (0,L)\times(0,1)\times(0,\infty)\,. \tag{4.6}$$

Therefore, problem (4.1) takes the form

$$\begin{cases} w_{tt} - d_1\left[\left(u_x + \frac{1}{2}\left(w_x\right)^2\right)w_x\right]_x + d_2 w_{xxxx} + \mu_1 w_t\left(x,t\right) + \mu_2 z\left(x,1,t\right) = 0, \\ u_{tt} - d_1\left[\left(u_x + \frac{1}{2}\left(w_x\right)^2\right)\right]_x + \delta\theta_x = 0, \\ \theta_t + q_x + \delta u_{tx} = 0, \\ q_t + \gamma q + \theta_x = 0, \end{cases} \tag{4.7}$$

with the initial conditions:

$$\begin{cases} u\left(0,.\right) = u_0, u_t\left(0,.\right) = u_1, w\left(0,.\right) = w_0, w_t\left(0,.\right) = w_1, \\ \theta\left(0,.\right) = \theta_0, \theta_t\left(0,.\right) = \theta_1, \\ z\left(x,1,t\right) = f\left(x,t-\tau\right) \text{ in } (0,L)\times(0,\tau)\,. \end{cases} \tag{4.8}$$

Now, let ξ be positive constant such that:

$$\tau\left|\mu_2\right| < \xi < \tau\left(2\mu_1 - \left|\mu_2\right|\right), \tag{4.9}$$

and let $U = \left(w, w_t, u, u_t, \theta, q, z\right)^T$, then $U_t = \left(w_t, w_{tt}, u_t, u_{tt}, \theta_t, q_t, z_t\right)^T$. Introducing the vector function $\varphi = w_t$, and $\psi = u_t$, system (4.1)-(4.3) can be written as

$$\begin{cases} U_t = \mathcal{A}U + \mathcal{F}\left(U\right) \\ U\left(0\right) = \left(w_0, w_1, u_0, u_1, \theta_0, q_0, f_0\right), \end{cases} \tag{4.10}$$

and the linear operator $\mathcal{A}$ is defined by:

$$\mathcal{A}\begin{pmatrix} w \\ \varphi \\ u \\ \psi \\ \theta \\ q \\ z \end{pmatrix} = \begin{pmatrix} \varphi \\ -d_2 w_{xxxx} - \mu_1 \varphi - \mu_2 z\left(.,1\right) \\ \psi \\ d_1 u_{xx} - \delta \theta_x \\ -q_x - \delta \psi_x \\ -\gamma q - \theta_x \\ -\frac{1}{\tau} z_\rho \end{pmatrix}, \tag{4.11}$$

and

$$\mathcal{F}\left(U\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ d_1\left[\left(u_x + \frac{1}{2}\left(w_x\right)^2\right) w_x\right]_x \\ 0 \\ \frac{d_1}{2}\left(w_x\right)_x^2 \\ 0 \end{pmatrix}, U_0 = \begin{pmatrix} w_0 \\ w_1 \\ u_0 \\ u_1 \\ \theta_0 \\ q_0 \\ f_0 \end{pmatrix}, \tag{4.12}$$

with domain

$$\begin{aligned} D\left(\mathcal{A}\right) \quad = \quad & \left\{ \begin{array}{c} \left(w, \varphi, u, \psi, \theta, q, z\right)^T \in \left[H^4\left(0,L\right) \cap H_0^2\left(0,L\right)\right] \times H_0^1\left(0,L\right) \\ \times \left[H^2\left(0,L\right) \cap H_0^2\left(0,L\right)\right] \end{array} \right. \qquad (4.13) \\ & \left. \begin{array}{c} \times H_0^1\left(0,L\right) \times L^2\left(0,L\right) \times L^2\left(0,L\right) \times L^2\left(\left(0,L\right), H_0^1\left(0,1\right)\right), \\ \varphi = z\left(.,0\right) \text{ in } \left(0,L\right) \end{array} \right\}. \end{aligned}$$

Denote by H the Hilbert space

$$\begin{aligned} H \ : & = \left\{ \left[H^4(0,L) \cap H_0^2(0,L) \right] \times H_0^1(0,L) \times \left[H^2(0,L) \cap H_0^2(0,L) \right] \right. \\ & \left. \times H_0^1(0,L) \times L^2(0,L) \times L^2(0,L) \times L^2\left((0,L), H_0^1(0,1) \right) \right\}. \end{aligned}$$

We will show that $\mathcal{A}$ generates a C_0 semigroup on H. Let us define on the Hilbert space H the inner product, for $U = (w, \varphi, u, \psi, \theta, q, z)^T$, $\widetilde{U} = \left(\widetilde{w}, \widetilde{\varphi}, \widetilde{u}, \widetilde{\psi}, \widetilde{\theta}, \widetilde{q}, \widetilde{z} \right)^T$

$$\begin{aligned} \left\langle U, \widetilde{U} \right\rangle &= \int_0^L \varphi \widetilde{\varphi} dx + \int_0^L \psi \widetilde{\psi} dx + \int_0^L \theta \widetilde{\theta} dx + \int_0^L q \widetilde{q} dx + d_2 \int_0^L w_{xx} \widetilde{w}_{xx} dx \\ & + d_1 \int_0^L u_x \widetilde{u}_x dx + \int_0^L \zeta \int_0^1 z \widetilde{z} d\rho dx \end{aligned}$$

The next result is our first main goal in this pape

Theorem 4.1.1 *Let* $(w, \varphi, u, \psi, \theta, q, z)^T \in H$. *for any initial datum* $U_0 \in H$ *there exists a unique solution* $U \in C([0,\infty), H)$ *for problem (4.10). Moreover, if* $U_0 \in D(\mathcal{A})$, *then* $U \in C([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty), H)$.

Proof. We show that the operator $\mathcal{A}$ generates a C_0-semigroup in H. In this step, we prove that the operator $\mathcal{A}$ is dissipative. Let $U = (w, \varphi, u, \psi, \theta, q, z)^T$

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle &= \left\langle \begin{pmatrix} \varphi \\ -d_2 w_{xxxx} - \mu_1 \varphi - \mu_2 z\,(.,1) \\ \psi \\ d_1 u_{xx} - \delta \theta_x \\ -q_x - \delta \psi_x \\ -\gamma q - \theta_x \\ -\frac{1}{\tau} z_\rho \end{pmatrix}, \begin{pmatrix} w \\ \varphi \\ u \\ \psi \\ \theta \\ q \\ z \end{pmatrix} \right\rangle \\
&= -\mu_1 \int_0^L \varphi^2 dx - d_2 \int_0^L \varphi w_{xxxx} dx + d_1 \int_0^L \psi u_{xx} dx \qquad (4.14) \\
&\quad -\delta \int_0^L \psi \theta_x dx - \int_0^L q_x \theta dx - \delta \int_0^L \theta \psi_x dx - \gamma \int_0^L q^2 dx \\
&\quad - \int_0^L q \theta_x dx + d_2 \int_0^L w_{xx} \varphi_{xx} dx + d_1 \int_0^L \psi_x u_x dx \\
&\quad -\mu_2 \int_0^L \varphi z\,(x,1)\, dx - \frac{\zeta}{\tau} \int_0^L \int_0^1 z\,(x,\rho)\, z_\rho\,(x,\rho)\, d\rho dx,
\end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle &= -\mu_1 \int_0^L \varphi^2 dx - \delta \int_0^L \psi \theta_x dx - \int_0^L q_x \theta dx \\
&\quad -\delta \int_0^L \theta \psi_x dx - \gamma \int_0^L q^2 dx - \int_0^L q \theta_x dx \\
&\quad -\mu_2 \int_0^L \varphi z\,(x,1)\, dx - \frac{\zeta}{\tau} \int_0^L \int_0^1 z\,(x,\rho)\, z_\rho\,(x,\rho)\, d\rho dx,
\end{aligned}$$

thus,

$$\begin{aligned}\langle \mathcal{A}U,U\rangle &= -\mu_1\int_0^L \varphi^2dx-\mu_2\int_0^L \varphi z\left(x,1\right)dx-\gamma\int_0^L q^2dx\\ &\quad -\frac{\zeta}{\tau}\int_0^L\int_0^1 z\left(x,\rho\right)z_\rho\left(x,\rho\right)d\rho dx. \qquad (4.15)\end{aligned}$$

Now, thanks to Young's inequality, (4.6) and (4.9),we get

$$\begin{aligned}\langle \mathcal{A}U,U\rangle &= -\left(\mu_1-\frac{\zeta}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^L \varphi^2dx-\gamma\int_0^L q^2dx\\ &\quad -\left(\frac{\zeta}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^L z^2\left(x,1\right)dx \le 0. \qquad (4.16)\end{aligned}$$

Consequently, the operator $\mathcal{A}$ is dissipative. Now, we will prove that the operator $\lambda I-\mathcal{A}$ is surjective for $\lambda>0$. For this purpose, let

$$\left(f_1,f_2,f_3,f_4,f_5,f_6,f_7\right)^T\in H,$$

we seek

$$U=\left(w,\varphi,u,\psi,\theta,q,z\right)^T\in D\left(\mathcal{A}\right)$$

solution of the following system of equations

$$\begin{cases} \lambda w - \varphi = f_1 \\ \\ \lambda \varphi + d_2 w_{xxxx} + \mu_1 \varphi + \mu_2 z\left(.,1\right) = f_2 \\ \\ \lambda z + \frac{1}{\tau} z_\rho = f_3 \\ \\ \lambda u - \psi = f_4 \\ \\ \lambda \psi - d_1 u_{xx} + \delta \theta_x = f_5 \\ \\ \lambda \theta + q_x + \delta \psi_x = f_6 \\ \\ \lambda q + \gamma q + \theta_x = f_7 \end{cases} \tag{4.17}$$

Suppose that we have found w, u. Therefore, the first and the third equation in (4.17) give

$$\begin{cases} \varphi = \lambda w - f_1 \\ \\ \psi = \lambda u - f_3 \end{cases}, \tag{4.18}$$

It is clear that $\varphi \in H_0^1\left(0,L\right)$ and $\psi \in H_0^1\left(0,L\right)$. Furthermore, by (4.17) we can find $z\left(x,0\right) = \varphi\left(x\right)$ for $x \in \left(0,L\right)$. Following the same approach as in [46], we obtain, by using equations for z in (4.17),

$$z\left(x,\rho\right) = \varphi\left(x\right) e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho} \int_0^\rho f_3\left(x,s\right) e^{\lambda\tau\rho} ds. \tag{4.19}$$

From (4.18), we obtain

$$z\left(x,\rho\right)=\lambda w\left(x\right)e^{-\lambda\tau\rho}-f_{1}e^{-\lambda\tau\rho}+\tau e^{-\lambda\tau\rho}\int_{0}^{\rho}f_{3}\left(x,s\right)e^{\lambda\tau\rho}ds.$$

By using (4.17) and (4.18) the functions w and u satisfying the following system,

$$\begin{cases}\lambda^{2}w+d_{2}w_{xxxx}+\mu_{1}\varphi+\mu_{2}z\left(.,1\right)=\lambda f_{1}+f_{2}\\ \\ \lambda^{2}u-d_{1}u_{xx}+\delta\theta_{x}=\lambda f_{3}+f_{4}\end{cases}$$

Solving (4.17) system is equivalent to finding

$$\left(w,u\right)\in\left[H^{4}\left(0,L\right)\cap H_{0}^{2}\left(0,L\right)\right]\times\left[H^{2}\left(0,L\right)\cap H_{0}^{2}\left(0,L\right)\right]$$

such that,

$$\begin{cases}\int_{0}^{L}\left(\lambda^{2}w\eta+\mu_{1}\varphi\eta+d_{2}w_{xx}\eta_{xx}+\mu_{2}z\left(.,1\right)\eta\right)dx=\int_{0}^{L}\left(\lambda f_{1}+f_{2}\right)\eta dx\\ \\ \int_{0}^{L}\left(\lambda^{2}u\zeta-d_{1}u_{x}\zeta_{x}+\delta\theta\zeta_{x}\right)dx=\int_{0}^{L}\left(\lambda f_{3}+f_{4}\right)\zeta dx\end{cases}\tag{4.21}$$

for all $\left(\eta,\zeta\right)\in H_{0}^{1}\left(0,L\right)\times H_{0}^{1}\left(0,L\right)$. From (4.5), we have

$$z\left(x,1\right)=\lambda w\left(x\right)e^{-\lambda\tau}-f_{1}e^{-\lambda\tau}+\tau e^{-\lambda\tau}\int_{0}^{1}f_{3}\left(x,s\right)e^{\lambda\tau}ds.$$

Consequently, problem (4.21) is equivalent to the problem

$$a\left(\left(w,u\right),\left(\eta,\zeta\right)\right)=L\left(\eta,\zeta\right)\tag{4.22}$$

where the bilinear form

$$a:\left[H_{0}^{2}\left(0,L\right)\times H_{0}^{1}\left(0,L\right)\right]^{2}\rightarrow\mathbb{R}$$

and the linear form

$$L:H_{0}^{2}\left(0,L\right)\times H_{0}^{1}\left(0,L\right)\rightarrow\mathbb{R}$$

are defined by

$$\begin{aligned} a\left((w,u),(\eta,\zeta)\right) &= \int_0^L \left(\lambda^2\left(w+\mu_1\varphi+\mu_2 z\left(.,1\right)\right)\eta + d_2 w_{xx}\eta_{xx}\right)dx \\ &\quad + \int_0^L \left(\lambda^2 u\zeta + \left(\delta\theta - d_1 u_x\right)\zeta_x\right)dx, \end{aligned} \tag{4.23a}$$

and

$$L\left(\eta,\zeta\right) = \int_0^L \left(\lambda f_1 + f_2\right)\eta dx + \int_0^L \left(\lambda f_3 + f_4\right)\zeta dx.$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(\eta,\zeta) \in H_0^1(0,L) \times H_0^1(0,L)$ the problem (4.22) admits a unique solution $(w,u) \in H^2(0,L) \times H_0^1(0,L)$. Applying the classical elliptic regularity, it follows from (4.21) that $(w,u) \in H^4(0,L) \times H_0^2(0,L)$. Therefore, the operator $(\lambda I - \mathcal{A})$ is surjective for any $\lambda > 0$. Consequently, the existence result of theorem follows from the Hille-Yosida theorem. To prove existence and uniqueness of local solutions for the nonlinear problem, it remains to show that $\mathcal{F}(U)$ is locally Lipschitz continuous in H. In fact, if $U = (w,\varphi,u,\psi,\theta,q,z)^T$, $\widetilde{U} = \left(\widetilde{w},\widetilde{\varphi},\widetilde{u},\widetilde{\psi},\widetilde{\theta},\widetilde{q},\widetilde{z}\right)^T$ belong to H, we have

$$\left\|\mathcal{F}\left(U\right) - \mathcal{F}\left(\widetilde{U}\right)\right\|_H^2 = d_1\left(|h|^2 + |g|^2\right) \tag{4.24}$$

where $h = \left[\left(u_x + \frac{1}{2}w_x^2\right)w_x - \left(\widetilde{u}_x + \frac{1}{2}\widetilde{w}_x^2\right)\widetilde{w}_x\right]_x$and $g = \frac{1}{2}\left(w_x^2 - \widetilde{w}_x^2\right)_x$. Adding and subtracting the term $\left(u_x + \frac{1}{2}w_x^2\right)\widetilde{w}_x$ inside the norm $|h|$, we gets

$$\begin{aligned} |h| &\leq \left\|w_x - \widetilde{w}_x\right\|_{L^\infty(0,L)}\left|u_x + \frac{1}{2}w_x^2\right| + \left\|\widetilde{w}_x\right\|_{L^\infty(0,L)}\left|u_x - \widetilde{u}_x\right| \\ &\quad + \frac{1}{2}\left\|\widetilde{w}_x\right\|_{L^\infty(0,L)}\left|w_x + \widetilde{w}_x\right|\left\|w_x - \widetilde{w}_x\right\|_{L^\infty(0,L)}. \end{aligned} \tag{4.25}$$

Using the embedding of $H^1(0,L)$ into $L^\infty(0,L)$ one has from (4.25) that

$$|h| \leq k\left(\|U\|_H, \left\|\widetilde{U}\right\|_H\right)\left\|U - \widetilde{U}\right\|_H \tag{4.26}$$

Using once again the embedding of $H^{1}(0,L)$ into $L^{\infty}(0,L)$, one also sees that

$$|g| \leq k\left(\|U\|_{H}, \left\|\widetilde{U}\right\|_{H}\right)\left\|U-\widetilde{U}\right\|_{H}. \tag{4.27}$$

Combining (4.24), (4.26) and (4.27), it follows that $\mathcal{F}(U)$ is locally Lipschitz continuous in H. ■

4.2 Exponential decay

The associated energy of this system is defined by

$$\begin{aligned} E(t) &= \frac{1}{2}\int_{0}^{L}\left\{w_{t}^{2}+u_{t}^{2}+\theta^{2}+q^{2}+d_{2}w_{xx}^{2}+d_{1}\left(u_{x}+\frac{1}{2}(w_{x})^{2}\right)^{2}\right\}dx \\ &+ \frac{\xi}{2}\int_{0}^{L}\int_{0}^{1}z^{2}(x,\rho,t)\,d\rho dx. \end{aligned} \tag{4.28}$$

Theorem 4.2.1 *Let us suppose that the initial data are given in H. Then, the energy $E(t)$ decays exponentially, i.e., there exist two positive constants α and β independent of the initial data such that*

$$E(t) \leq \alpha E(0)\, e^{-\beta t}, \textit{for all } t \geq 0.$$

In this section we state and prove our result on exponential decay for the nonlinear system (4.7)-(4.8). We start by introducing some functionals and preparing some lemmas.

Lemma 4.2.1 *Let (w,u,θ,q,z) be the solution of (4.7)-(4.8). Then the energy func-*

tional $E(t)$, *defined by (4.28) satisfies*

$$\frac{d}{dt}E(t) \le -\gamma\int_0^L q^2dx-\left(\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^L w_t^2dx \quad (4.29)$$

$$-\left(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^L z^2(x,1,t)\,dx.$$

Proof. We multiplying the first at four equations in (4.7) by u_t, w_t, θ, q respectively, integrating over $(0,L)$, and the equation (4.6) by $\frac{\xi}{\tau}z$ over $(0,L)\times[0,1]$, using integration by parts with the boundary conditions (4.2), we obtain.

$$\frac{d}{2dt}\int_0^L w_t^2dx-d_1\int_0^L\left[\left(u_x+\frac{1}{2}(w_x)^2\right)w_x\right]_x w_tdx+d_2\frac{d}{2dt}\int_0^L(w_{xx})^2dx$$

$$+\mu_1\int_0^L w_t^2dx+\mu_2\int_0^L z(x,1,t)\,w_tdx=0,$$

$$\frac{d}{2dt}\int_0^L u_t^2dx-d_1\int_0^L\left[\left(u_x+\frac{1}{2}(w_x)^2\right)\right]_x u_tdx+\delta\int_0^L\theta_xu_tdx=0,$$

$$\frac{d}{2dt}\int_0^L\theta^2dx+\int_0^L q_x\theta dx-\delta\int_0^L u_t\theta_xdx=0,$$

$$\frac{d}{2dt}\int_0^L q^2dx+\gamma\int_0^L q^2dx-\int_0^L\theta q_xdx=0,$$

$$\xi\frac{d}{2dt}\int_0^L\int_0^1 z^2(x,\rho,t)d\rho dx+\frac{\xi}{2\tau}\int_0^L\left(z^2(x,1,t)-z^2(x,0,t)\right)dx=0, \quad (4.30)$$

with

$$\frac{\xi}{\tau}\int_0^L\int_0^1\frac{d}{2d\rho}z^2(x,\rho,t)d\rho dx=\frac{\xi}{2\tau}\int_0^L\left(z^2(x,1,t)-z^2(x,0,t)\right)dx.$$

Summing up, we get

$$\frac{d}{2dt}\int_0^L w_t^2dx+\frac{d}{2dt}\int_0^L u_t^2dx+d_2\frac{d}{2dt}\int_0^L (w_{xx})^2\,dx+\frac{d}{2dt}\int_0^L \theta^2dx$$

$$+\frac{d}{2dt}\int_0^L q^2dx+\xi\frac{d}{2dt}\int_0^L\int_0^1 z^2(x,\rho,t)d\rho dx$$

$$+d_1\int_0^L\left(u_x+\frac{1}{2}(w_x)^2\right)u_{xt}dx+d_1\int_0^L\left(u_x+\frac{1}{2}(w_x)^2\right)w_xw_{xt}dx$$

$$+\gamma\int_0^L q^2dx+\mu_1\int_0^L w_t^2dx+\mu_2\int_0^L z\,(x,1,t)\,w_tdx$$

$$+\frac{\xi}{2\tau}\int_0^L\left(z^2(x,1,t)-z^2(x,0,t)\right)dx=0.$$

Now, by using the fact that

$$\frac{d}{2dt}d_1\int_0^L\left(u_x+\frac{1}{2}(w_x)^2\right)^2dx=d_1\int_0^L\left(u_x+\frac{1}{2}(w_x)^2\right)u_{xt}dx$$

$$+d_1\int_0^L\left(u_x+\frac{1}{2}(w_x)^2\right)w_xw_{xt}dx,$$

we arrive at

$$\frac{d}{2dt}\int_0^L\Big[w_t^2+u_t^2+d_2\,(w_{xx})^2+\theta^2+q^2+d_1\left(u_x+\frac{1}{2}(w_x)^2\right)^2$$

$$+\xi\int_0^1 z^2(x,\rho,t)d\rho\Big]+\gamma\int_0^L q^2dx+\mu_1\int_0^L w_t^2dx$$

$$+\mu_2\int_0^L z\,(x,1,t)\,w_tdx+\frac{\xi}{2\tau}\int_0^L\left(z^2(x,1,t)-z^2(x,0,t)\right)dx=0.$$

Thus,

$$\begin{aligned} E'(t) &= -\gamma \int_0^L q^2 dx - \mu_1 \int_0^L w_t^2 dx - \mu_2 \int_0^L z\left(x,1,t\right) w_t dx \\ &- \frac{\xi}{2\tau} \int_0^L \left(z^2(x,1,t) - z^2(x,0,t)\right) dx. \end{aligned}$$

Finally, by using Young's inequality, the relation (4.9) and (4.7), we get

$$E'\left(t\right) \leq -\left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right) \int_0^L z^2(x,1,t) - \left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right) \int_0^L w_t^2 dx - \gamma \int_0^L q^2 dx.$$

■

Lemma 4.2.2 *Let*

$$I_1(t) = \int_0^L \left(u_t u + \frac{1}{2} w_t w + \frac{\mu_1}{4} w^2\right) dx, t \geq 0. \tag{4.31}$$

and let (u, w, θ, q, z) *be a solution of (4.7)-(4.8). Then we have, for any* $\varepsilon_1 > 0$,

$$\begin{aligned} I_1'(t) &\leq -d_1 \int_0^L \left(u_x + \frac{1}{2}\left(w_x\right)^2\right)^2 dx - \frac{d_2}{4} \int_0^L \left(w_{xx}\right)^2 dx \\ &+\varepsilon_1 \int_0^L u_x^2 dx + \frac{\delta^2}{4\varepsilon_1} \int_0^L \theta^2 dx + \int_0^L u_t^2 dx \\ &+\frac{\mu_2^2 c_p^2}{2d_2} \int_0^L z^2\left(x,1,t\right) dx + \frac{1}{2} \int_0^L w_t^2 dx. \end{aligned} \tag{4.32}$$

for all $t \geq 0$.

Proof. Differentiating the functional $I_1(t)$ and using the first and the second equation of (4.7), we get

$$\begin{aligned} I_1'(t) &= -d_1 \int_0^L \left(u_x + \frac{1}{2}(w_x)^2\right) u_x dx - \delta \int_0^L \theta u_x dx + \int_0^L u_t^2 dx \\ &\quad -\frac{d_1}{2} \int_0^L \left(u_x + \frac{1}{2}(w_x)^2\right) w_x^2 dx - \frac{d_2}{2} \int_0^L w_{xx}^2 dx - \mu_2 \frac{1}{2} \int_0^L z(x,1,t)\, w dx \\ &\quad +\frac{1}{2} \int_0^L w_t^2 dx. \end{aligned} \tag{4.33}$$

Using Young's and Poincaré inequalities, we get

$$-\mu_2 \frac{1}{2} \int_0^L z(x,1,t)\, w dx \le \frac{d_2}{4} \int_0^L w_{xx}^2 dx + \frac{\mu_2^2 c_p^2}{2d_2} \int_0^L z^2(x,1,t)\, dx, \tag{4.34}$$

and

$$-\delta \int_0^L \theta u_x dx \le \varepsilon_1 \int_0^L u_x^2 dx + \frac{\delta^2}{4\varepsilon_1} \int_0^L \theta^2 dx. \tag{4.35}$$

By inserting (4.34) and (4.35) into (4.33), then, we obtain (4.32).

$$\begin{aligned} I_1'(t) = & \int_0^L u_t^2 dx - d_1 \int_0^L \left(u_x + \frac{1}{2}(w_x)^2\right) u_x dx + \delta \int_0^L \theta u_x dx \\ & +\frac{1}{2} \int_0^L w_t^2 dx - \frac{d_1}{2} \int_0^L \left(u_x + \frac{1}{2}(w_x)^2\right) w_x^2 dx \\ & -\frac{d_2}{2} \int_0^L (w_{xx})^2 dx + \frac{\mu_2}{2} \int_0^L z^2(x,1,t)\, dx + \frac{\mu_2}{8} \int_0^L w^2 dx. \end{aligned} \tag{4.36}$$

We conclude by using Young and Poincaré's inequalities. ■

Lemma 4.2.3 *Let*

$$I_2(t) := \int_0^L \left(\int_0^x \theta(t,y)\, dy\right) u_t dx, \qquad t \ge 0, \tag{4.37}$$

and let (u, w, θ, q, z) be a solution of (4.7)-(4.8). Then we have, for any $\varepsilon_2 > 0$,

$$I_2'(t) \leq -\frac{\delta}{2}\int_0^L u_t^2 dx + \varepsilon_2 \int_0^L \left(u_x + \frac{1}{2}\left(w_x\right)^2\right)^2 dx$$

$$+\frac{1}{2\delta}\int_0^L q^2 dx + C(\varepsilon_2)\int_0^L \theta^2 dx + \varepsilon_2 u_x^2\left(L\right), \qquad t \geq 0, \tag{4.38}$$

where, $C(\varepsilon_2) = \left[\frac{d_1^2}{4\varepsilon_2}\left(1+L\right)+\delta\right].$

Proof. A differentiation of (4.37) and with the boundary conditions gives

$$\begin{aligned} I_2'(t) &= -\delta\int_0^L u_t^2 dx - \int_0^L q u_t dx + \delta\int_0^L \theta^2 dx \\ &- d_1\int_0^L \left(u_x + \frac{1}{2}\left(w_x\right)^2\right)\theta dx + d_1\left(\int_0^L \theta dx\right) u_x\left(L\right). \end{aligned} \tag{4.39}$$

By recalling Young's inequality for the last term of (4.39), we get for any $\varepsilon_2 > 0$,

$$d_1\left(\int_0^L \theta dx\right) u_x\left(L\right) \leq \varepsilon_2 u_x^2\left(L\right) + \frac{d_1^2 L}{4\varepsilon_2}\int_0^L \theta^2 dx,$$

and our conclusion follows. ■

Lemma 4.2.4 *Let*

$$I_3\left(t\right) := \int_0^L \left(\int_0^x q\left(t,y\right)dy\right)\theta dx, \qquad t \geq 0, \tag{4.40}$$

and let (u, w, θ, q, z) be a solution of (4.7)-(4.8). Then we have, for any $\varepsilon_3 > 0$

$$I_3'(t) \leq -\frac{1}{2}\int_0^L \theta^2 dx + \varepsilon_3\int_0^L {u_t}^2 dx + C_1\left(\varepsilon_3\right)\int_0^L q^2 dx, t \geq 0, \tag{4.41}$$

where, $C_1(\varepsilon_3) = \left(1 + \frac{\delta^2}{4\varepsilon_3} + \frac{\gamma^2}{2}\right).$

Proof. By exploiting (4.40) and integrating by parts, we have

$$I_3'(t) = \int_0^L q^2 dx - \delta \int_0^L \left(\int_0^x q\left(t,y\right)dy \right) u_{tx} dx$$

$$-\gamma \int_0^L \left(\int_0^x q\left(t,y\right)dy \right) \theta dx - \int_0^L \theta^2 dx. \tag{4.42}$$

Using Young's and Poincaré inequalities, we obtain

$$-\gamma \int_0^L \left(\int_0^x q\left(t,y\right)dy \right) \theta dx \leq \frac{1}{2} \int_0^L \theta^2 dx + \frac{\gamma^2}{2} \int_0^L q^2 dx, \tag{4.43}$$

$$-\delta \int_0^L \left(\int_0^x q\left(t,y\right)dy \right) u_{tx} dx \leq \varepsilon_3 \int_0^L {u_t}^2 dx + \frac{\delta^2}{4\varepsilon_3} \int_0^L q^2 dx. \tag{4.44}$$

By substituting (4.43) and (4.44) in (4.42), we obtain immediately (4.41). ■

In order to eliminate the boundary term in (4.38), we introduce the following function

$$m(x) = 2 - \tfrac{4}{L}x, \quad x \in \left[0, L\right],$$

Lemma 4.2.5 *Let*

$$I_4(t) := \int_0^L u_t m u_x dx + \int_0^L w_t m w_x dx - \int_0^L \left(\theta + \delta u_x\right) m q dx, \ t \geq 0. \tag{4.45}$$

and let (u, w, θ, q, z) *be a solution of (4.7)-(4.8). Then we have, the following estimate*

$$
\begin{aligned}
& I_4'(t) \leq \gamma \int_0^L \theta^2 dx + \left(\gamma + \frac{\gamma^2 \delta^2}{4}\right) \int_0^L q^2 dx + \frac{2}{L} \int_0^L u_t^2 dx \\
& -d_1 \left[u_x^2 (L) + u_x^2 (0)\right] + \left(1 + \frac{2d_1}{L}\right) \int_0^L u_x^2 dx \\
& + \left(\frac{2}{L} + \mu_1\right) \int_0^L w_t^2 dx + \mu_1 \int_0^L w_x^2 dx \\
& + \frac{8d_1}{L} \int_0^L \left(u_x + \frac{1}{2} (w_x)^2\right)^2 dx + \frac{6d_2}{L} \int_0^L w_{xx}^2 dx \\
& + \mu_2 \int_0^L z^2 (x, 1, t)\, dx + \mu_2 \int_0^L w_x^2 dx.
\end{aligned}
\tag{4.46}
$$

Proof. Direct differentiation, using the second equation of (4.7) and the integration by parts, lead to

$$
\begin{aligned}
\frac{d}{dt} \int_0^L m u_t u_x dx &= d_1 \int_0^L \left[\left(u_x + \frac{1}{2} (w_x)^2\right)\right]_x m u_x dx - \delta \int_0^L \theta_x m u_x dx + \int_0^L u_t m u_{tx} dx \\
&= d_1 \int_0^L u_{xx} m u_x dx + d_1 \int_0^L w_x w_{xx} m u_x dx - \delta \int_0^L \theta_x m u_x dx + \int_0^L u_t m u_{tx} dx \\
&= \frac{d_1}{2} \left[m u_x^2\right]_{x=0}^{x=L} - \frac{d_1}{2} \int_0^L m_x u_x^2 dx + d_1 \int_0^L w_x w_{xx} m u_x dx \\
&- \delta \int_0^L \theta_x m u_x dx - \frac{1}{2} \int_0^L m_x u_t^2 dx \\
&= -d_1 \left[u_x^2 (L) + u_x^2 (0)\right] + \frac{2d_1}{L} \int_0^L u_x^2 dx + d_1 \int_0^L w_x w_{xx} m u_x dx \\
&- \delta \int_0^L \theta_x m u_x dx + \frac{2}{L} \int_0^L u_t^2 dx
\end{aligned}
$$

then

$$\frac{d}{dt}\int_0^L u_t m u_x dx = d_1\int_0^L u_{xx} m u_x dx + d_1\int_0^L w_x w_{xx} m u_x dx$$
$$-\delta\int_0^L \theta_x m u_x dx + \int_0^L u_t m u_{tx} dx$$
$$\frac{d}{dt}\int_0^L u_t m u_x dx = \frac{d_1}{2}\left[m u_x^2\right]_{x=0}^{x=L} - \frac{d_1}{2}\int_0^L m_x u_x^2 dx - \delta\int_0^L \theta_x m u_x dx$$
$$+d_1\int_0^L w_x w_{xx} m u_x dx - \frac{1}{2}\int_0^L m_x u_t^2 dx$$
$$\frac{d}{dt}\int_0^L u_t m u_x dx \leq -d_1\left[u_x^2(L) + u_x^2(0)\right] + \frac{2d_1}{L}\int_0^L u_x^2 dx + \frac{2}{L}\int_0^L u_t^2 dx$$
$$+d_1\int_0^L w_x w_{xx} m u_x dx - \delta\int_0^L \theta_x m u_x dx \tag{4.47}$$

Similarly, using the first equation of (4.7), we have

$$
\begin{aligned}
& \frac{d}{dt}\int_0^L w_t m w_x dx = +d_1 \int_0^L \left(\left(u_x + \frac{1}{2}\left(w_x\right)^2\right) w_x\right)_x m w_x dx \\
& -\mu_1 \int_0^L w_t m w_x dx - d_2 \int_0^L w_{xxxx} m w_x dx \\
& +\int_0^L w_t m w_{tx} dx - \mu_2 \int_0^L z\left(x,1,t\right) m w_x dx \\
= \ & -d_1 \int_0^L \left(\left(u_x + \frac{1}{2}\left(w_x\right)^2\right) w_x\right)\left(m_x w_x + m w_{xx}\right) dx \\
& -\mu_1 \int_0^L w_t m w_x dx + d_2 \int_0^L w_{xxx}\left(m_x w_x + m w_{xx}\right) dx \\
& -\frac{1}{2}\int_0^L m_x w_t^2 dx - \mu_2 \int_0^L m w_x z\left(x,1,t\right) dx \\
= \ & +\frac{4d_1}{L}\int_0^L \left(u_x + \frac{1}{2}\left(w_x\right)^2\right) w_x^2 dx - d_1 \int_0^L w_x w_{xx} m u_x dx \\
& -\mu_1 \int_0^L w_t m w_x dx - \frac{d_1}{2}\int_0^L w_x^2 w_x m w_{xx} dx - \frac{4d_2}{L}\int_0^L w_{xxx} w_x dx \\
& +d_2 \int_0^L w_{xxx} m w_{xx} dx + \frac{2}{L}\int_0^L w_t^2 dx - \mu_2 \int_0^L m w_x z\left(x,1,t\right) dx \\
= \ & +\frac{4d_1}{L}\int_0^L \left(u_x + \frac{1}{2}\left(w_x\right)^2\right) w_x^2 dx - d_1 \int_0^L w_x w_{xx} m u_x dx \\
& -\mu_1 \int_0^L w_t m w_x dx - \frac{d_1}{4}\int_0^L w_x^2 q\left(\left(w_x\right)^2\right)_x dx \\
& +\frac{4d_2}{L}\int_0^L w_{xx}^2 dx - d_2\left[w_{xx}^2(L) + w_{xx}^2(0)\right] + \frac{2d_2}{L}\int_0^L w_{xx}^2 dx \\
& +\frac{2}{L}\int_0^L w_t^2 dx - \mu_2 \int_0^L m w_x z\left(x,1,t\right) dx \\
\leq \ & -\mu_1 \int_0^L w_t m w_x dx + \frac{4d_1}{L}\int_0^L \left(u_x + \frac{1}{2}\left(w_x\right)^2\right) w_x^2 dx - d_1 \int_0^L w_x w_{xx} m u_x dx \\
& -\frac{d_1}{2L}\int_0^L w_x^4 dx + \frac{6d_2}{L}\int_0^L w_{xx}^2 dx + \frac{2}{L}\int_0^L w_t^2 dx \\
& +\mu_2 \int_0^L z^2\left(x,1,t\right) dx + \frac{\mu_2}{4}\int_0^L \left(m w_x\right)^2 dx,
\end{aligned}
$$

using Young's inequality we find

$$\frac{d}{dt}\int_0^L w_t m w_x dx \leq \left(\frac{2}{L}+\mu_1\right)\int_0^L w_t^2 dx + \mu_1 \int_0^L w_x^2 dx$$

$$+\frac{8d_1}{L}\int_0^L \left(u_x + \frac{1}{2}\left(w_x\right)^2\right)^2 dx - d_1 \int_0^L w_x w_{xx} m u_x dx$$

$$+\frac{6d_2}{L}\int_0^L w_{xx}^2 dx + \mu_2 \int_0^L z^2\left(x,1,t\right) dx + \frac{\mu_2}{4}\int_0^L \left(m w_x\right)^2 dx. \tag{4.48}$$

Finally,

$$\begin{aligned}
-\frac{d}{dt}\int_0^L \left(\theta+\delta u_x\right) m q dx &= -\int_0^L \left(\theta_t + \delta u_{tx}\right) m q dx - \int_0^L \left(\theta+\delta u_x\right) m q_t dx \\
&= \int_0^L q_x m q dx - \int_0^L \left(\theta+\delta u_x\right) m\left(-\gamma q - \theta_x\right) dx \\
&= \int_0^L q_x m q dx + \gamma \int_0^L \theta m q dx + \int_0^L \theta m \theta_x dx \\
&\quad +\gamma\delta \int_0^L q m u_x dx + \delta \int_0^L \theta_x m u_x dx \\
&= \frac{1}{2}\int_0^L m_x q^2 dx + \gamma \int_0^L \theta m q dx + \frac{1}{2}\int_0^L m_x \theta^2 dx \\
&\quad +\gamma\delta \int_0^L q m u_x dx + \delta \int_0^L \theta_x m u_x dx \\
&= -\frac{2}{L}\int_0^L q^2 dx + \gamma \int_0^L \theta m q dx - \frac{2}{L}\int_0^L \theta^2 dx \\
&\quad +\gamma\delta \int_0^L q m u_x dx + \delta \int_0^L \theta_x m u_x dx \\
&\leq \gamma \int_0^L \theta m q dx + \gamma\delta \int_0^L q m u_x dx + \delta \int_0^L \theta_x m u_x dx,
\end{aligned}$$

using, Young's inequality we find

$$-\frac{d}{dt}\int_0^L \left(\theta+\delta u_x\right) m q dx \leq \int_0^L u_x^2 dx + \left(\gamma + \frac{\gamma^2\delta^2}{4}\right)\int_0^L q^2 dx$$

$$+\gamma \int_0^L \theta^2 dx + \delta \int_0^L \theta_x m u_x dx. \tag{4.49}$$

By adding (4.47), (4.48) and (4.49), we conclude

$$I_4'(t) \leq \gamma \int_0^L \theta^2 dx + \left(\gamma + \frac{\gamma^2\delta^2}{4}\right) \int_0^L q^2 dx + \frac{2}{L} \int_0^L u_t^2 dx$$

$$-d_1 \left[u_x^2(L) + u_x^2(0)\right] + \left(1 + \frac{2d_1}{L}\right) \int_0^L u_x^2 dx$$

$$+\left(\frac{2}{L} + \mu_1\right) \int_0^L w_t^2 dx + \mu_1 \int_0^L w_x^2 dx$$

$$+\frac{8d_1}{L} \int_0^L \left(u_x + \frac{1}{2}(w_x)^2\right)^2 dx + \frac{6d_2}{L} \int_0^L w_{xx}^2 dx$$

$$+\mu_2 \int_0^L z^2(x,1,t)\, dx + \mu_2 \int_0^L w_x^2 dx.$$

■

Lemma 4.2.6 *The functional $I_5(t)$ satisfies the estimate*

$$I_5(t) = \int_0^L \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\, d\rho dx, \tag{4.50}$$

for some positive constant m_1

$$I_5'(t) \leq -I_5(t) - \frac{m_1}{2\tau} \int_0^L z^2(x,1,t)\, dx + \frac{1}{2\tau} \int_0^L w_t^2(x,t)\, dx. \tag{4.51}$$

Proof. Differentiating $I_5(t)$, and using the equation (4.6), we obtain

$$\frac{d}{dt}\left(\int_0^L\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho dx\right) = -\frac{1}{\tau}\int_0^L\int_0^1 e^{-2\tau\rho}zz_\rho(x,\rho,t)\,d\rho dx$$

$$= \quad -\int_0^L\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho dx$$

$$-\frac{1}{2\tau}\int_0^L\int_0^1 \frac{d}{d\rho}\left(e^{-2\tau\rho}z^2(x,\rho,t)\right)d\rho dx.$$

The above estimate implies that there exists a positive constant m_1 such that (4.51) holds. ■

Now we are in the position to prove the second main result in this chapter (Theorem 4.2.1). First, we introduce the functional

$$\mathcal{L}(t) = NE(t) + \delta\tfrac{d_1}{4}I_1 + d_1I_2 + N_1I_3 + \varepsilon_2I_4 + I_5. \tag{4.52}$$

Proof. (of Theorem 4.2.1)

taking into account (4.29), (4.32), (4.38), (4.41), (4.46), (4.51) and the relations

$$\begin{aligned}\int_0^L u_x^2dx &\leq 2\int_0^L\left(u_x+\frac{1}{2}\left(w_x^2\right)\right)dx+\frac{1}{2}\int_0^L w_x^2dx\\ &\leq 2\int_0^L\left(u_x+\frac{1}{2}\left(w_x^2\right)\right)dx+\frac{L}{4}\int_0^L w_x^2dx,\end{aligned}$$

we get

$$\begin{aligned}\mathcal{L}'(t) \leq & -\left[\gamma N-\frac{d_1}{2\delta}-C_1(\varepsilon_3)N_1-\varepsilon_2\left(\gamma+\frac{\gamma^2\delta^2}{4}\right)\right]\int_0^L q^2dx \\ & -\left[N\left(\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)-\frac{\delta d_1}{8}-\varepsilon_2\left(\frac{2}{L}+\mu_1\right)-\frac{1}{2\tau}\right]\int_0^L w_t^2dx \\ & -\left[N\left(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)-\frac{\delta d_1\mu_2^2c_p^2}{8d_2}-\mu_2\varepsilon_2+\frac{m_1}{2\tau}\right]\int_0^L z^2(x,1,t)\,dx \\ & -\left[\frac{\delta d_1^2}{4}-\varepsilon_2 d_1-\frac{8\varepsilon_2 d_1}{L}-\frac{\delta\varepsilon_1 d_1}{2}-2\varepsilon_2\left(1+\frac{2d_1}{L}\right)\right]\int_0^L\left(u_x+\frac{1}{2}(w_x)^2\right)^2dx \\ & -\left[\frac{\delta d_1 d_2}{16}-\frac{6\varepsilon_2 d_2}{L}-\frac{C_p\delta\varepsilon_1 d_1}{16}-\frac{L\varepsilon_2 C_p}{4}\left(1+\frac{2d_1}{L}\right)-\varepsilon_2 C_p(\mu_1+\mu_2)\right]\int_0^L(w_{xx})^2dx \\ & -\left[\frac{1}{2}N_1-\frac{\delta^3 d_1}{16\varepsilon_1}-C(\varepsilon_2)d_1-\gamma\varepsilon_2\right]\int_0^L\theta^2dx \\ & -\left[\frac{\delta d_1}{4}-\varepsilon_3 N_1-\frac{2\varepsilon_2}{L}\right]\int_0^L u_t^2dx \\ & -I_5(t)\end{aligned}$$

$$\begin{aligned}&\mathcal{L}'(t)\leq-\eta_1\textstyle\int_0^L q^2dx-\eta_2\int_0^L w_t^2dx-\eta_3\int_0^L z^2(x,1,t)\,dx \\ &-\eta_4\textstyle\int_0^L\left(u_x+\frac{1}{2}(w_x)^2\right)^2dx-\eta_5\int_0^L w_{xx}^2dx-\eta_6\int_0^L\theta^2dx \\ &-\eta_7\textstyle\int_0^L u_t^2dx-I_5(t)\end{aligned} \tag{4.53}$$

where

$$\eta_1 = \left[\gamma N - \tfrac{d_1}{2\delta} - C_1(\varepsilon_3) N_1 - \varepsilon_2\left(\gamma + \tfrac{\gamma^2\delta^2}{4}\right)\right]$$

$$\eta_2 = \left[N\left(\mu_1 - \tfrac{\xi}{2\tau} - \tfrac{|\mu_2|}{2}\right) - \tfrac{\delta d_1}{8} - \varepsilon_2\left(\tfrac{2}{L} + \mu_1\right) - \tfrac{1}{2\tau}\right]$$

$$\eta_3 = \left[N\left(\tfrac{\xi}{2\tau} - \tfrac{|\mu_2|}{2}\right) - \tfrac{\delta d_1 \mu_2^2 c_p^2}{8d_2} - \mu_2\varepsilon_2 + \tfrac{m_1}{2\tau}\right]$$

$$\eta_4 = \left[\tfrac{\delta d_1^2}{4} - \varepsilon_2 d_1 - \tfrac{8\varepsilon_2 d_1}{L} - \tfrac{\delta\varepsilon_1 d_1}{2} - 2\varepsilon_2\left(1 + \tfrac{2d_1}{L}\right)\right]$$

$$\eta_5 = \left[\tfrac{\delta d_1 d_2}{16} - \tfrac{6\varepsilon_2 d_2}{L} - \tfrac{C_p\delta\varepsilon_1 d_1}{16} - \tfrac{L\varepsilon_2 C_p}{4}\left(1 + \tfrac{2d_1}{L}\right) - \varepsilon_2 C_p(\mu_1 + \mu_2)\right]$$

$$\eta_6 = \left[\tfrac{N_1}{2} - \tfrac{\delta^3 d_1}{16\varepsilon_1} - C(\varepsilon_2)d_1 - \gamma\varepsilon_2\right]$$

$$\eta_7 = \left[\tfrac{\delta d_1}{4} - \varepsilon_3 N_1 - \tfrac{2\varepsilon_2}{L}\right]$$

First, let us choose $\varepsilon_1$and ε_2 small enough

$$\varepsilon_1 < \min\left(\tfrac{d_1}{8}, \tfrac{d_2}{4C_p}\right),$$

$$\varepsilon_2 < \min\left[\frac{\delta d_1 d_2}{16C_p(L + 2d_1)}, \frac{\delta d_1 L}{384}, \frac{\delta d_1}{16}, \frac{\delta d_1 d_2}{64C_p(\mu_1 + \mu_2)}, \frac{\delta d_1}{128L}, \frac{\delta d_1^2}{32(L + 2d_1)}\right]$$

and, we choose N_1

$$\left[\tfrac{N_1}{2} - \tfrac{\delta^3 d_1}{16\varepsilon_1} - C(\varepsilon_2)d_1 - \gamma\varepsilon_2\right] > 0.$$

Next, we pick ε_3 so small that,

$$\varepsilon_3 < \tfrac{\delta d_1}{8N_1}$$

Finally, we choose N large enough so that,

$$\gamma N - \tfrac{d_1}{2\delta} - C_1(\varepsilon_3) N_1 - \varepsilon_2\left(\gamma + \tfrac{\gamma^2\delta^2}{4}\right) > 0,$$

$$N\left(\mu_1 - \tfrac{\xi}{2\tau} - \tfrac{|\mu_2|}{2}\right) - \tfrac{\delta d_1}{8} - \varepsilon_2\left(\tfrac{2}{L} + \mu_1\right) - \tfrac{1}{2\tau} > 0,$$

and

$$N\left(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)-\frac{\delta d_1\mu_2^2c_p^2}{8d_2}-\mu_2\varepsilon_2+\frac{m_1}{2\tau}>0.$$

Therefore, which implies by (4.53), that there exists also $\eta_2 > 0$, such that

$$\frac{d}{dt}\mathcal{L}\left(t\right)\leq-\eta_2E\left(t\right),\qquad\forall t>0. \tag{4.54}$$

for some positive η_2. On the other hand it is easy to verify that

$$\beta_2E(t)\leq\mathcal{L}(t)\leq\beta_1E(t),\ \forall t\geq0, \tag{4.55}$$

for some positive constants β_1 and β_2. Combining (4.54) and the right hand side of (4.55), we conclude that

$$\tfrac{d}{dt}\mathcal{L}\left(t\right)\leq-\Lambda\mathcal{L}\left(t\right),\ \forall t\geq0, \tag{4.56}$$

for $\Lambda > 0$. A simple integration of (4.56) leads to

$$\mathcal{L}\left(t\right)\leq\mathcal{L}\left(0\right)e^{-\Lambda t},\qquad\forall t>0. \tag{4.57}$$

Again, use of (4.54) and (4.56) yields the desired result. This completes the proof of Theorem 4.2.1. ■

Conclusion and perspectives

At the end of this research work, we believe that the results presented will contribute to the development of the study of evolutionary problems including partial differential equations of the hyperbolic type, by opening new horizons for scientific research on this emerging theme.

First, we have defined two new porous-elastic systems, one with micro-temperatures and internal distributed delay acting on the first equation and the other is a one-dimensional damped system with a single nonlinear weak feedback and a term of distributed delay

Subsequently, and after having demonstrated the existence and the uniqueness of the solution of the two systems, we obtained, under the assumption that the speeds of propagation are equal, that the energy of the first system is exponentially stable, while than that of the second system, decreases explicitly and generally.

Considering the applications in physics, hydraulics, mechanics, etc. . . , the field of partial differential equations of the hyperbolic type is booming and the improvement of new methods to study the asymptotic behavior of such systems is just as important.

Finally, the avenues of research regarding these PDEs are numerous, and they can always be diversified. Therefore, different perspectives can be cast as a result of this work.

REFERENCES

[**1**] SC. Cowin and JW. Nunziato "Linear elastic materials with voids". J Elast;13(2):125–147. (1983). https://doi.org/10.1007/BF00041230.

[**2**] SC. Cowin, " The viscoelastic behavior of linear elastic materials with voids". J Elast;15(2):185–191.(1985).

[**3**] D. I. Lesan, " A theory of thermoelastic materials with voids", Acta Mech., vol. 60, no. 1–2, pp. 67–89, (1986). DOI: 10.1007/BF01302942.

[**4**] D. I. Lesan, and R. Quintanilla, "A theory of porous thermo-viscoelastic mixtures", J. Thermal Stresses, vol. 30, no. 7, pp. 693–714,(2007).DOI:10.1080/01495730701212880.

[**5**] T. Apalara, "Exponential decay in one-dimensional porous dissipation elasticity". Q J Mech Appl Math. ;70(4):553–555. (2017).

[**6**] S. Chirita, M. Ciarletta, C. D'Apice, "On the theory of thermoelasticity with microtemperatures". J. Math. Anal. Appl. 397(1), 349–361 (2013).

[**7**] A.M. Magana, R. Quintanilla," On the time decay of solutions in one-dimensional theories of porous materials", Int. J. Solids Struct. 43 3414–3427 (2006).

[**8**] P.X. Pamplona, J.E. Munoz Rivera, R. Quintanilla , "Stabilization in elastic solids with voids", J. Math. Anal. Appl. 350 37–49. (2009).

[**9**] R. Quintanilla, “Slow decay for one-dimensional porous dissipation elasticity,” Appl. Math. Lett., vol. 16, no. 4, pp. 487–491, (2003). DOI: 10.1016/S0893-9659(03)00025-9.

[**10**] P.S. Casas, R. Quintanilla, "Exponential decay in one-dimensional porous-thermo-elasticity", Mech. Res. Commun. 32 652–658. (2005).

[**11**] P.S. Casas, R. Quintanilla," Exponential stability in thermoelasticity with microtemperatures", Int. J. Eng. Sci. 43 33–47. (2005).

[**12**] M. L. Santos, A. D. S. Campelo and D. A. J unior, "On the decay rates of porous elastic systems", J. Elast., vol. 127, no. 1, pp. 79–101, (2017). DOI: 10.1007/s10659-016-9597-y.

[**13**] R.A. Adams, Sobolev Spaces, Acadimic Press, New York., (1975).

[**14**] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Diferential Equations, Springer, New York .,(2010).

[**15**] A. Pazy. Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences, 44. New York etc.:Springer-Verlag. VIII, 279 p. DM 88.00; $ 34.20 (1983)., 1983.

[**16**] K. Yosida, Functional Analysis, Springer, Berlin.,6th edn. (1980).

[**17**] Klaus-Jochen Engel and Rainer Nagel. One-parameter semigroups for linear evolution equations, volume 194 of Graduate Texts in Mathematics.Springer-Verlag, New York, 2000. With contributions by S. Brendle, M.Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A.Rhandi, S. Romanelli and R. Schnaubelt.

[**18**] H. E. Khochemane, L. Bouzettouta, A. Guerouah, " Exponential decay and well-posedness for a one-dimensional porous-elastic system with distributed delay", Applicable Analysis, DOI: 10.1080/00036811.2019.1703958.

[**19**] T. A. Apalara, Well-posedness and exponential stability for a linear damped Timoshenko system with second sound and internal distributed delay, Electronic Journal of Differential Equations, Vol. 2014(2014) , No. 254, pp. 1-15.

[**20**] S. Nicaise, C. Pignotti, "Stabilization of the wave equation with boundary or internal distributed delay", Diff Int Equ, 21(9–10):935–958. (2008).

[**21**] E. Borges Filho M. L. Santos, " On porous-elastic system with a time-varying delay term in the internal feedbacks", Z Angew Math Mech. e201800247. (2020).

[**22**] T. Caraballo, J. Real, L. Shaikhet b, J. Math. Anal. Appl. 334 (2007) 1130–1145

[**23**] T. A. Apalara, A General Decay for aWeakly Nonlinearly Damped Porous System (2018).

[**24**] T. A. Apalara,General decay of solutions in one-dimensional porous-elastic system with memory (2017).

[**25**] F. Ammar-Khodja, A. Benabdallah, J. M. Rivera and R. Racke, Energy decay for Timoshenko systems of memory type, J. Diff. Equa., 194(1), 82–115 (2003).

[**26**] S. A. Messaoudi and M. I. Mustafa, A stability result in a memory-type Timoshenko system, Dyn. Sys. Appl., 18(3), 457 (2009).

[**27**] S. A. Messaoudi and A. Fareh, Exponential decay for linear damped porous thermoelastic systems with second sound, Discrete Contin. Dyn. Syst. Ser. B 20 (2015), no. 2, 599–612.

[**28**] L. Bouzettouta. A. Djebabla, Exponential stabilization of the full von Kármán beam by a thermal effect and a frictional damping and distributed delay, Journal of Mathematical Physics, 60, 041506 (2019).

[29] H. E. Khochemane, L. Bouzettouta, A. Guerouah, Exponential decay and well-posedness for a one-dimensional porous-elastic system with distributed delay, Applicable Analysis, DOI: 10.1080/00036811.2019.1703958.

[30] H. E. Khochemane, A. Djebabla , S. Zitouni and L. Bouzettouta, Well-posedness and general decay of a nonlinear damping porous-elastic system with infinite memory, J. Math. Phys., 61, 021505 (2020).

[31] A. Soufyane, Energy decay for porous-thermo-elasticity systems of memory type, Appl. Anal. 87 (2008), no. 4, 451–464.

[32] I. Lasiecka , D. Tataru, Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. Differ Integral Equ. 1993;6(3):507–533.

[33] A. Haraux, Nonlinear evolution equations–q-global behavior of solutions, Lecture Notes in Mathematics 841. Berlin: Springer; 1981.

[34] Ammari. K, Nicaise. and C. Pignotti. C, Feedback boundary stabilization of wave equations with interior delay, Systems Cont. Letters, 59 (2010), 623-628.

[35] Apalara. T. a, Well-posedness and exponential stability for a linear damped Timoshenko system with second sound and internal distributed delay,Elect. J. Diff. Equ., Vol. 2014 (2014), No. 254, pp. 1-15.

[36] Araruna. F. D, Braz E silva. P, Zuazua. E, Asymptotic limits and stabilization for the 1d nonlinear mindlin-timoshenko system, J. Syst. Sci. Complex (2010) 23: 1-17.

[37] Araruna. F. D, Braz, E Silva. P, and Zuazua. E, Asymptotics and stabilization for dynamic models of nonlinear beams, Proc. Estonian Acad. Sci., 59, (2) (2010), 150–155.

[38] Benabdallah. A and Lasiecka. L, Exponential decay rates for a full von Kármán system of dynamic thermoelasticity, J. Diff. Eqns, 160 (2000), 51-93.

[39] Benabdallah. A and Teniou. D, Exponential stability of a Von Kármán model with thermal effects, Electron. J. Diff. Eqns, 1998 (07) (1998), 1-13.

[40] Datko. R, Two questions concerning the boundary control of certain elastic systems, J. Di. Equa., 1 (1991), 27-44.

[41] Djebabla. A and Tatar. N. E, Exponential stabilization of the full von Kármán beam by a thermal effect and a frictional damping, Georgian Math. J. 20 (2013), 427-438.

[42] Guesmia. A, Some well-posedness and stability results for abstract hyperbolic equations with infinite memory and distributed time delay, Communications on pure and Aapplied analysis, 14 (2) (2015),1-35.

[**43**] Guesmia. A, Well-posedness and exponential stability of an abstract evolution equation with infinite memory and time delay, IMA J. Mathematical Control and Information, 30 (2013), 507-526.

[**44**] Haraux. A, Martinez. P and Vancostenoble. J, Asymptotic stability for intermittently controlled second order evolution equations, SIAM J. Control Optim., 43:2089-2108, 2005.

[**45**] Lasiecka.L, Uniform decay rates for a full von Kármán system of dynamics thermoelasticity with free boundary conditions and partial boundary dissipation, Comm. in Partial Diff. Eqns, 24 (9-10)(1999), 1801-1847.

[**46**] Nicaise. S and Pignotti. C, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, SIAM J. Control Optim., 45(5)(2006): 1561–1585.

[**47**] Perla Menzala. G, Pazoto. A. F and Zuazua. E, Stabilization of Berger-Timoshenko's equation as limit of the uniform stabilization of the von Kármán system of beams and plates, ESAIM, Mathematical Modelling and Numerical Analysis, Vol. 36, No 4, 2002, 657-691.

[**48**] Perla Menzala. G and Zuazua. E, Timoshenko beam equation as limit of a nonlinear one-dimensional von Kármán system. Proc. Roy. Soc. Edinburg Sect. A 130 (2000), 855-875.

[**49**] Liu. W, Chen. K and Yu. J, Existence and general decay for the full von Kármán beam with a thermo-viscoelastic damping, frictional dampings and a delay term, IMA Journal of Mathematical Control and Information (2015) Page 1 of 22.

[**50**] D. Hanni, A. Djebabla, N.Tatar, "Well-posedness and exponential stability for the von Kármán systems with second sound", Eurasian Journal of Mathematical and Computer Applications, 7:4 (2019), 52–65