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وزارة التعليم العالي والبحث العلمي

Ministry of Higher Education and Scientific Research

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Stability of Some Coupled Parabolic-Hyperbolic Thermoelastic Systems: Mathematical and Numerical Study

Stream
Applied Mathematics

Speciality
Optimal Control
By
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ملخّص

تقدم هذه الرسالة دراسة شاملة لبعض أنظمة المعادلات التفاضلية الجزئية في سياق الاهتزاز مع تأثير الحرارة المرونية، تحديداً أنظمة تيموشينكو وبريس. نعالج قضايا تتعلق بوجود و وحدانية الحلول باستخدام نظرية أشباه المجموعات، وسلوك استقرار هذه الحلول عبر طريقة الطاقة، وكذلك التقريب العددي وتقدير الخطأ المسبق. تكشف دراسة متعمقة لنظام بريس الحراري المرن عن رؤى أساسية حول خصائص الحلول وخصائص الاستقرار تحت تأثير حرارة. تعزز تكامل آليات التخميد والتبديد الحراري المستندة إلى نظريات جرين ونجدي الاستقرار بشكل كبير. علاوة على ذلك، تعالج الشروط الجديدة للاستقرار الأسي لنظام تيموشينكو الحراري المرن ذو التأخر الثنائي المرحلة أوجه القصور في الافتراض الكلاسيكي لتساوي سرعة انتشار الموجات. من خلال تعميم تحليل المرونة الحرارية ذات التأخر الثنائي المرحلة إلى نظام بريس، يتأكد أن شروط الاستقرار المحددة لنظام تيموشينكو تنطبق أيضًا. بالإضافة إلى ذلك، تُظهر التجارب العددية باستخدام تقريبات العناصر المحدودة والاختلافات المحدودة سلوك واستقرار الحلول، مما يعمق فهمنا لديناميكية هذه الأنظمة.

كلمات مفتاحية: المعادلات التفاضلية الجزئية، نظام تيموشينكو، نظام بريس، الاستقرار الاسي، تقنية المضاعف، المرونة الحرارية، التأخير الثنائي المرحلة، غرين-ناغدي، طريقة العناصر المحدودة، طريقة الفروق المحدودة

Abstract

This thesis presents a comprehensive study of some partial differential equations systems in context of vibrations with effect of thermoelasticity, specifically the Timoshenko and Bresse systems. Through these models, we address issues pertaining to existence and uniqueness of solutions using semigroup theory, the stability behavior of these solutions via the energy method, as well as the numerical approximation and a priori error analysis. An in-depth exploration of the thermoelastic Bresse system reveals fundamental insights into its solution properties and stability characteristics under thermal effects. The integration of damping and thermal dissipation mechanisms based on Green and Naghdi theories significantly enhances stability. Furthermore, novel conditions for exponential stability of the dual phase lag (DPL) thermoelastic Timoshenko system address deficiencies in the classic equal wave propagation velocity assumption. Extending these findings, by generalizing the analysis of the DPL thermoelasticity to the Bresse system confirms that the stability conditions identified for the Timoshenko system also apply. Additionally, numerical experiments using finite element and finite difference approximations demonstrate behavior and stability of the solutions, deepening our understanding of these systems dynamic.

Keywords: Partial differential equations; Timoshenko system; Bresse system; Exponential stability; Multiplier technique; Thermoelasticity; Dual-Phase-Lag; Green–Naghdi; FEM; FDM

Résumé

Dans cette thèse, nous présentons une étude approfondie de certains systèmes d'équations aux dérivées partielles dans le contexte des vibrations avec effet de thermoélasticité, spécifiquement les systèmes de Timoshenko et de Bresse. À travers ces modèles, nous abordons des questions relatives à l'existence et à l'unicité des solutions en utilisant la théorie des semi-groupes, le comportement de stabilité de ces solutions via la méthode de l'énergie, ainsi que l'approximation numérique et l'analyse a priori des erreurs. Une exploration approfondie du système thermoélastique de Bresse révèle des insights fondamentaux sur les propriétés des solutions et les caractéristiques de stabilité sous les effets thermiques. L'intégration des mécanismes d'amortissement et de dissipation thermique basés sur les théories de Green et Naghdi améliore considérablement la stabilité. En outre, de nouvelles conditions pour la stabilité exponentielle du système thermoélastique Timoshenko à double déphasage (DPL) traitent les insuffisances de l'hypothèse classique d'égalité des vitesses de propagation des ondes. En généralisant l'analyse de la thermoélasticité à retard de phase double au système de Bresse, il est confirmé que les conditions de stabilité identifiées pour le système de Timoshenko s'appliquent également. De plus, des expériences numériques utilisant des approximations par éléments finis et par différences finies démontrent le comportement et la stabilité des solutions, approfondissant notre compréhension de la dynamique des systèmes.

Mots-clés : Équations aux dérivées partielles ; Système de Timoshenko ; Système de Bresse ; Stabilité exponentielle ; Technique des multiplicateurs ; Thermoélasticité ; Déphasage double ; Green–Naghdi ; Méthode des éléments finis ; Méthode des différences finies

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- Bouraoui H.A., Djebabla A., El Arwadi T., and Haiour M. Exponential stability for a thermoelastic Bresse system: theoretical and numerical study. Math Meth Appl Sci. 2022;46:1-23. Doi:10.1002/mma.8885
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To Allah. For his satisfaction, I have read and written. All the praises to Allah. To the souls of martyred students from Gaza who were unable to complete their academic journey. To every martyr mother or father who did not live to see his children graduate. To Gaza.

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Notations index

List of Notations

Tensorial Representation

- σ: Cauchy stress tensor
- ε: Infinitesimal strain tensor
- $\mathbf{u}$: Displacement vector
- C: Fourth-order stiffness tensor
- $\mathbf{F}$: Body force per unit volume
- ρ: Mass density
- ∇: Nabla operator
- $(\ddot{\bullet})$: Second derivative with respect to time

Cartesian Coordinate Formulation

- σ_{ij}: Stress components in Cartesian coordinates
- ε_{ij}: Strain components in Cartesian coordinates
- u_i: Displacement components in Cartesian coordinates
- C_{ijkl}: Stiffness tensor components in Cartesian coordinates
- δ_{ij}: Kronecker delta
- ∂_{tt}: Second derivative with respect to time

Isotropic Homogeneous Media

- K: Bulk modulus
- μ: Shear modulus
- λ: Lamé's first parameter

- ν: Poisson's ratio
- E: Young's modulus

Anisotropic Hooke's Law

- S_{ijkl}: Symmetric part of the fourth-order compliance tensor

In terms of sets,

$$\begin{aligned}
C(\Omega) &= \{u : \Omega \to \mathbb{R} \mid u \text{ is continuous } \},\\
C^k(\Omega) &= \{u : \Omega \to \mathbb{R} \mid u \text{ is } k\text{-times continuously differentiable } \},\\
C^\infty(\Omega) &= \{u : \Omega \to \mathbb{R} \mid u \text{ is infinitely differentiable } \},\\
C_0^k(\Omega) &= \left\{ u \in C^k(\Omega) \mid \mathrm{supp}(u) \subset \Omega \text{ is compact } \right\}, \quad k \in \mathbb{N} \text{ or } k = \infty.\\
L^p(\Omega) &= \left\{ u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } \int_\Omega |u(x)|^p dx < \infty \right\};\\
L^\infty(\Omega) &= \{u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } |u(x)| \le K \text{ a.e. in } \Omega\};\\
W^{m,p}(\Omega) &= \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega), 0 \le |\ \alpha\ | \le m\};\\
W_0^{m,p}(\Omega) &= \overline{C_0^\infty(\Omega)}^{W^{m,p}(\Omega)};\\
H^m(\Omega) &= W^{m,2}(\Omega);\\
H_0^m(\Omega) &= W_0^{m,2}(\Omega);\\
L^p(0,T;X) &= \left\{ u : (0,T) \to X \mid u \text{ is measurable and } \int_0^T \|u(t)\|_X^p dt < \infty \right\};\\
L^\infty(0,T;X) &= \{u : (0,T) \to X \mid u \text{ is measurable and } \|u(t)\|_X \le K \text{ a.e. in } (0,T)\};\\
C([0,T],X) &= \{u : [0,T] \to X \mid u \text{ is continuous on } [0,T] \text{ in } X\};\\
C^k([0,T],X) &= \{u : [0,T] \to X \mid u \text{ is } k\text{-times continuously differentiable on } [0,T] \text{ in } X\};\\
\mathscr{L}(X,Y) &= \{T : X \to Y \mid T \text{ is linear and continuous } \}.
\end{aligned}$$

Chapter 1

Introduction

1.1 Aims and motivation

The study of parabolic and hyperbolic partial differential equations (PDEs) holds profound significance in addressing natural phenomena and real-world challenges across various domains, including thermoelasticity. These equations find applications in diverse fields, offering mathematical tools to model and comprehend complex physical phenomena and their stability. In this context, we delve into the motivations behind exploring parabolic and hyperbolic PDEs, showcasing their practical relevance and impact on advancing science and engineering, particularly in understanding the stability of dynamic systems. The interdisciplinary nature of these equations underscores their significance, as they bridge the gap between theory and practical problem-solving. As we explore parabolic and hyperbolic PDEs, we aim to shed light on their pivotal roles in addressing real-world challenges, fostering innovation, and enriching our understanding of the complex phenomena that shape our world, including their role in thermoelasticity.

The concept of thermoelasticity is a field that combines principles from thermodynamics and elasticity to study the coupling between temperature and mechanical deformation in materials. The behaviour of thermoelastic materials is mathematically described through a system of coupled PDEs, which capture the complex interdependencies between temperature, stress, and strain within a material. At its core, thermoelasticity relies on the solution of PDEs to understand how materials respond to temperature variations and mechanical loads. The governing parabolic-hyperbolic PDEs for thermoelasticity typically include equations for heat conduction, conservation of linear momentum (stress equilibrium), and strain compatibility. These PDEs are often coupled with appropriate constitutive relations that describe the material's thermal and mechanical behavior. The mathematical nature of PDEs allows researchers to model and analyze the intricate interactions within thermoelastic materials. Solving these PDEs can predict and study phenomena such as thermal stresses, thermal shock, and heat transfer in various structures and materials. Moreover, numerical techniques, such as finite element methods (FEM) and finite difference methods (FDM), are commonly employed to obtain approximate solutions to these PDEs, enabling engineers and scientists to simulate the thermoelastic properties of different materials.

The primary aims of this thesis revolve around the intricate interplay of parabolic and hyperbolic partial differential equations (PDEs) in the context of thermoelasticity. We delve into the investigation of exponential stability, a crucial aspect in understanding the long-term behavior of dynamic systems, as well as the lack of stability. Employing advanced numerical techniques, including finite element methods and finite difference methods, we seek to provide valuable insights into the behavior of thermoelastic materials subjected to thermal and mechanical influences. Through this research, we aspire to enhance our understanding of the complex interactions within elastic, porous, and viscoelastic materials, paving the way for practical applications and engineering solutions in thermoelasticity.

In this thesis, we examine some linear thermoelastic systems with varying damping characteristics. We establish their well-posedness using semigroup theory and conduct a thorough investigation using the multiplier method to explore general decay behaviors and the lack of stability of solutions. The thesis is structured as follows: Chapter 1 provides an introduction, outlining the motivation, objectives, and the significance of investigating thermoelastic systems. Chapter 2 covers the necessary physical, mechanical and mathematical preliminaries, setting the foundational framework for the analyses conducted in the subsequent chapters. In Chapter 3, an in-depth exploration of the thermoelastic Bresse system in context of Green and Naghdi thermoelasticity, revealing fundamental insights into solution properties and stability characteristics in the presence of thermal effects. This chapter establishes essential properties such as solution uniqueness and exponential stability, irrespective of system coefficients, through theoretical analyses. Numerical experiments using finite element approximation further elucidate discrete energy decay and a priori error analysis. Chapter 4 introduces novel conditions for the exponential stability of a dual-phase-lag thermoelastic Timoshenko system, Chapter 5 extends the findings from Chapter 4 by generalizing the analysis to the Bresse system. This generalization reveals that the same conditions for exponential stability observed in the thermoelastic Timoshenko system also apply to the Bresse system in the context of DPL thermoelasticity. Finally, we conclude with a summary of the main findings, their implications, and potential directions for future research.

Chapter 2

Preliminaries

This initial chapter serves as an introduction, and more experienced readers may choose to skip it and proceed to the following chapters. Here, we will introduce the notations that will be used throughout the thesis and present some of the classical results in the general theory of linear elasticity and thermoelasticity, functional analysis, Sobolev spaces, linear semigroups, and attractors. We will make use of these results in the subsequent chapters to facilitate the understanding of the content covered and also to make this work as self-contained as possible. The results (Theorems, Propositions, Lemmas, etc.) will be presented without formal proof, but we will provide, in each section, classical references within the literature for the mentioned concepts.

2.1 Linear Elasticity

Linear deformation, as a mathematical framework, endeavors to elucidate the nuanced interplay between solid objects and their response to applied external forces. It stands as a less intricate alternative within the vast expanse of continuum mechanics, diverging from the intricacies of nonlinear elasticity. At its core, linear deformation hinges on two bedrock assumptions: infinitesimal strains and the linear correlation between stress and strain components. These assumptions find application in scenarios where yielding is not a pivotal concern, rendering linear deformation a versatile tool in engineering analysis and design, often coupled with finite element methodologies.

2.1.1 Foundational Postulates

The underpinning principles of Linear elasticity rest on two crucial premises:

Negligible Deformations

Linear elasticity presumes that the deformations experienced by a material are infinitesimally small. This is encapsulated in the mathematical expression $\varepsilon = \frac{\Delta L}{L} \ll 1$, signifying the insignificance of changes in dimensions and shapes. This assumption proves particularly valid within a material's elastic limits, facilitating a straightforward mathematical formulation.

Linear Stress-Strain Relationships

The linchpin of linear elasticity is the linear relationship between stress (σ) and strain (ε), articulated through Hooke's Law: $\sigma = E \cdot \varepsilon$. This equation signifies a direct correlation between stress and strain, a behavior upheld within a material's elastic threshold. The material's stiffness is encapsulated in E, denoted as Young's modulus.

These foundational assumptions render linear deformation apt for scenarios where yielding is not imminent, providing a reliable foundation for engineering materials and design scenarios.

2.1.2 Mathematical Representation

The mathematical framework governing linear elasticity involves a system of equations orchestrating the equilibrium of linear momentum, strain-displacement relations, and constitutive equations. Presented both in tensor and Cartesian coordinate forms, these equations encapsulate the mechanical behavior of materials.

Tensorial Representation

In a tensorial form independent of coordinates, the governing equations manifest as three differential equations:

1. **Equation of Motion (Newton's Second Law):**

$$\nabla \cdot \sigma + \mathbf{F} = \rho \ddot{\mathbf{u}}$$

2. **Strain-Displacement Relations:**

$$\varepsilon = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}\right]$$

3. **Constitutive Equations:**

$$\sigma = \mathsf{C} : \varepsilon$$

The notations include the Cauchy stress tensor (σ), infinitesimal strain tensor (ε), displacement vector ($\mathbf{u}$), stiffness tensor (C), body force per unit volume ($\mathbf{F}$), mass density (ρ), nabla operator (∇), and second derivative with respect to time ($\ddot{(\bullet)}$).

Cartesian Coordinate Formulation

Expressed in terms of components with respect to a Cartesian coordinate system, the governing equations metamorphose into:

1. **Equation of Motion:**

$$\sigma_{ji,j} + F_i = \rho \partial_{tt} u_i$$

2. **Strain-Displacement Relations:**

$$\varepsilon_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j})$$

3. **Constitutive Equations:**

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

The subscript $(\bullet)_{,j}$ denotes $\frac{\partial(\bullet)}{\partial x_j}$, and $\frac{\partial^2}{\partial t^2}$ is denoted by ∂_{tt}.

2.1.3 Isotropic Uniform Media

Within isotropic media, the stiffness tensor steps into the limelight, orchestrating the connection between stresses and strains. This tensor, for isotropic materials, manifests direction-independent traits, ensuring uniform displacements under forces applied from any direction. The isotropic stiffness tensor is expressed both in matrix form and as constitutive equations.

$$C_{ijkl} = K\,\delta_{ij}\,\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\,\delta_{ij}\,\delta_{kl})$$

$$\sigma_{ij} = K\delta_{ij}\varepsilon_{kk} + 2\mu\left(\varepsilon_{ij} - \tfrac{1}{3}\delta_{ij}\varepsilon_{kk}\right)$$

formulation using Lamé's first parameter (λ):

$$\sigma_{ij} = \lambda\,\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$

The strain-stress relation is distilled to:

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}]$$

Here, ν is Poisson's ratio, and E is Young's modulus.

2.1.4 Elastodynamics and Wave Equations

Elastodynamics delves into the realm of elastic waves, introducing a temporal dimension to linear deformation. Elastic waves, a subset of mechanical waves, traverse elastic or viscoelastic materials. The elastodynamic wave equation derives from the linear momentum equation, assimilating inertial effects

$$\sigma_{ji,j} + F_i = \rho\, \ddot{u}_i = \rho\, \partial_{tt} u_i$$

2.2 Classical Coupled Thermoelasticity

Elasticity concentrates on elucidating the deformative characteristics and responses of solids subjected to external forces, exploring the intricate relationships between stress and strain. In contrast, thermoelasticity introduces an additional dimension by incorporating the nuanced influence of temperature variations. The confluence of thermal effects with mechanical responses in thermoelasticity enhances the depth of our understanding of material behavior, where temperature alterations become inherent to the overarching deformation processes. Consequently, the synergistic interplay between thermoelasticity and elasticity establishes a comprehensive analytical framework for the examination of materials experiencing concurrent mechanical loading and thermal fluctuations. This integrated methodology facilitates a more holistic investigation into the mechanical and thermal attributes of materials. Such an approach proves indispensable across diverse academic disciplines, ranging from structural engineering to materials science, thereby augmenting our analytical capacity to anticipate and decipher the complex interdependencies between mechanical and thermal factors intrinsic to material behavior.

The Duhamel–Neumann relations provide a mathematical expression for the stress, temperature relations, and strain relations in isotropic homogeneous thermoelastic solids. The fundamental relations in the context of linear thermoelasticity can be defined as follows::

$$\sigma_{ij} = (\lambda u_{i,i} - \beta \theta)\, \delta_{ij} + 2\mu e_{ij}; \quad (i, j = 1,2,3) \tag{2.2.1}$$

where $\beta = (3\lambda + 2\mu)\alpha_t, \alpha_t$ is the coefficient of linear thermal expansion of the material, λ and μ are Lamè's constants, the stress tensor σ_{ij}, T_0 is the reference temperature, θ is the temperature increase above T_0, e_{ij} are given by

$$e_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i})\,, \tag{2.2.2}$$

and σ_{ij} is defined by Hooke's law [1] , shown by

$$\sigma_{ij} = c_{ijkl} e_{kl}, \;\; i,j,k,l = 1,2,3 \tag{2.2.3}$$

where c_{ijkl} is the stiffness tensor (see [2] for the proprieties of σ and c).

The thermoelastic behavior is governed by these equations, along with Fourier's law, which provides the constitutive relationship for temperature gradient $\bar{\nabla}\theta$ and the thermal conductivity of the material k with heat flux vector $\bar{q}$ given by the equation $\bar{q} = -k\bar{\nabla}\theta$ or

$$q_i = -k\theta_{,i},\, i = 1,2,3. \tag{2.2.4}$$

If we consider how strain and temperature affect each other, the local energy balance principle leads to

$$\begin{aligned} -q_{i,i} + \rho R &= \rho c_v \dot{\theta} + \beta T_0 \dot{e} \\ \text{i.e.,}\ -\bar{\nabla}\bar{q} + \rho R &= \rho c_v \dot{\theta} + \beta T_0 \dot{e} \end{aligned} \tag{2.2.5}$$

where $e = u_{i,i}$, ρ is the mass density, The text you provided seems to be already clear and grammatically correct. c_v represents the specific heat of a solid when its volume remains constant. Then, by eliminating q_i we obtain the coupled heat conduction:

$$k\nabla^2\theta + \rho R = \rho c_v \dot{\theta} + \theta_0 \beta \dot{u}_{k,k}, \tag{2.2.6}$$

R is a source term, and the term T_0 highlights the coupling between strain and temperature, necessitating the incorporation of this coupling into the balance of linear momentum principle. This leads to the stress equations of motion in a linearized form.

$$\sigma_{ij,j} + \rho f_i = \rho \ddot{u}_i, \quad i,j = 1,2,3 \tag{2.2.7}$$

where $\bar{f}$ is the external body force vector. The equations (2.2.1), (2.2.4), and (2.2.7) (with $\lambda^* = \lambda + \mu$)), result in the following displacement motion equation:

$$\lambda^* u_{k,k} + \mu\nabla^2 u_i + \rho f_i - \beta\theta_{,i} = \rho \ddot{u}_i. \tag{2.2.8}$$

While (2.2.6) governs the coupled interaction of heat and deformation (based on Biot's [3] work), its reliance on classical Fourier's law leads to an unrealistic outcome. Any thermal disturbance instantly affects both temperature and displacement infinitely far away, implying infinitely fast heat propagation. This contradicts reality, especially for short timeframes, highlighting a common drawback of classical theories.

2.3 Thermoelasticity with Finite Wave Speeds

Initially, mathematicians and engineers mainly focused on classical thermoelasticity, in which the heat flux is determined by Fourier's law, and this classical theory predicts an infinite speed of heat propagation. As a result, classical thermoelasticity leads to an unrealistic property that a sudden disturbance will be sensed instantly somewhere else in the materials at some time. However, as our understanding of materials and their behavior advanced, a need arose to account for finite heat propagation speeds, leading to the development of thermoelasticity with finite wave speeds. This more advanced approach recognizes that heat propagation takes time and introduces a temporal dimension to the analysis, making it particularly valuable in scenarios where thermal wave effects play a significant role, such as in microscale systems and advanced materials.

As is known, the Fourier model of heat conduction, first observed in rigid isotropic conductors,

$$q_i = -k\frac{\partial \theta}{\partial x_i} \tag{2.3.1}$$

which leads to a parabolic-type diffusion equation

$$\frac{\partial \theta}{\partial t} = \frac{k}{\rho c_p}\nabla^2\theta, \tag{2.3.2}$$

where θ is the temperature, q_i is the heat flux, k is the thermal conductivity, ρ is the mass density, and c_p is the heat constant pressure. According to this law, (2.3.1), temperature changes propagate instantaneously throughout a material, indicating an infinite speed of thermal signal propagation. However, this contradicts the theory of relativity, as postulated by Albert Einstein. Einstein's theory of relativity, particularly the special theory of relativity, sets a universal speed limit, the speed of light (c), which is approximately 3×10^8 meters per second in a vacuum. This implies that no information or signal, including thermal signals, can travel faster than the speed of light.

The contradictions between Fourier's heat conduction laws and relativity theory can be overcome through various models. Most of them are inspired by a model suggested by Maxwell (1867) and Cattaneo (1948). They propose that Eq. (2.3.1) should be changed to

$$\left(1+t_0\frac{\partial}{\partial t}\right)q_i = -k\frac{\partial \theta}{\partial x_i}, \tag{2.3.3}$$

whereupon, instead of (2.3.3), we have a hyperbolic type equation

$$t_0\frac{\partial^2\theta}{\partial t^2}+\frac{\partial \theta}{\partial t} = \frac{k}{\rho c_p}\nabla^2\theta. \tag{2.3.4}$$

This clearly models heat conduction as a wave, often referred to as a second sound. In Eqs.(2.3.3) and (2.3.4) $c_{\mathrm{T}} = (k/\rho c_p t_0)^{1/2}$ represents the speed of the second sound and t_0 is relaxation time. This model has two notable disadvantages, include:

- The complexity of the hyperbolic differential Equation type poses mathematical challenges, often lacking analytical solutions in the majority of physical scenarios.

- The general unavailability of the relaxation time as a known variable in a given system introduces ambiguity, necessitating caution in interpreting results.

Aside from the paradox of infinite propagation speeds, classical dynamic thermoelasticity theory provides unsatisfactory or poor descriptions of a solid's response to fast transient loading (for example, due to short laser pulses) and low temperatures. The aforementioned limitations have prompted numerous scholars to propose different generalized theories in the field of thermoelasticity. In line with the works of Maxwell [4] and Cattaneo [5], various thermoelastic models have been presentedand. These models features non-fourier models of thermal conductivity.

2.3.1 Lord–Shulman Model (L-S Model (1967))

The L-S theory addresses the limitations of classical Fourier's law, which assumes instantaneous heat diffusion, by incorporating additional terms accounting for finite heat propagation speeds and the interaction between thermal and mechanical fields. By incorporating Eq. (2.3.3) instead of (2.2.4), we obtain

$$k\nabla^2\theta = \left(1+\tau\frac{\partial}{\partial t}\right)[\rho c_v\dot{\theta} + \theta_0\beta u_{\dot{k},k} - \rho R]. \tag{2.3.5}$$

Equation (2.3.5) is a more general formulation of the heat conduction equation with hyperbolic type and is infinite heat propagation speed paradox-free. We note that Lord and Shulman [6] independently proposed the generalized thermoelasticity, which was also presented by Kaliski [7].

While the L-S theory is an improvement over classical Fourier's law, it still has limitations:

- Small Strain and Temperature Gradient: The model is based on small strain and temperature gradient assumptions. This makes it less suitable for situations where large deformations or significant temperature gradients are present.

- Neglects Thermal Relaxation: The model neglects the effects of thermal relaxation time, which can be important in situations where the rate of temperature change is significant.

- Applicability to Specific Materials: The model might not be universally applicable to all materials, and its accuracy can vary depending on the material properties and the specific conditions of the problem.

2.3.2 Green–Lindsay Model (G-L Model)

Green and Lindsay proposed in 1972 a theory that involves two relaxation times (see [8]), where the Fourier's law of heat conduction remains unaltered while adjustments are made to the stress-strain temperature relations and the classical energy equation. Instead of a single relaxation time τ, two constitutive constants, α and α_0, both with dimensions of time, are introduced into the governing equations in this model. The equations of thermoelasticity proposed are:

$$-q_{i,i}+\rho R=\rho c_v\left(\dot{\theta}+\alpha_0\ddot{\theta}\right)+\beta T_0\dot{e} \tag{2.3.6}$$

$$\sigma_{ij}=\lambda u_{i,i}\delta_{ij}+2\mu e_{ij}-\beta(\theta+\alpha\dot{\theta})\delta_{ij}. \tag{2.3.7}$$

These modifications involve the introduction of a temperature rate term into the constitutive equations. Replacing q_i (2.2.1) in (2.3.6) gives the following heat equation:

$$k\nabla^2\theta+\rho R=\rho c_v\left(\dot{\theta}+\alpha_0\ddot{\theta}\right)+\beta\theta_0\dot{u}_{k,k}, \tag{2.3.8}$$

with the motion equation:

$$\mu\nabla^2 u_i+(\lambda+\mu)u_{k,k}-\beta(\theta+\alpha\dot{\theta})_{,i}+\rho f_i=\rho\ddot{u}_i. \tag{2.3.9}$$

The material constants α and α_0 satisfy the inequalities $\alpha\geq\alpha_0\geq 0$. It is evident that Equation (2.3.8) maintains a hyperbolic nature and anticipates a finite speed for transmitting thermoelastic signals. When $\alpha=\alpha_0=0$ is set in (2.3.8) and (2.3.9), the resulting equations correspond to Eqs. (2.2.6) and (2.2.8).

2.3.3 Green–Naghdi Models

Green and Naghdi presented three forms of thermoelastic theories in the 1990s, called thermoelasticity of types I, II, and III. These theories are based on entropy equality rather than the usual entropy inequality. Whereas type I is the same as the classical type, in type II the heat is permitted to move by thermal waves, also known as thermoelasticity without dissipation. Types I and II are thermoelasticity type III 's limiting cases (for more information, see [9]).

Green–Naghdi Type II (1993)

The Type II model is also called thermoelasticity without energy dissipation. Green and Naghdi, in [10], proposed the following:

Equation	**Form**
Energy Equation	$-\bar{\nabla}\bar{q}+\rho R=\rho c_v\dot{\theta}+\beta T_0\dot{e}$
Heat Conduction Law	$-\bar{q}=-k^*\bar{\nabla}\nu$
Governing Equation	$k^*\nabla^2\theta+\rho\dot{R}=\rho c_v\ddot{\theta}+\beta T_0\ddot{e}$

where $\bar{\nabla}\nu$ is the thermal displacement [1] gradient with $k^*>0$ is a material constant. The finite thermal wave speed for model II is given by $\sqrt{\frac{k^*}{\rho c_v}}$.

Green-Naghdi Type III (1993)

This model incorporates energy dissipation effects (thermoelasticity with energy dissipation). The modified energy equation and heat conduction law include terms that account for both thermal and mechanical dissipation (see [10]), i.e., the following equation:

Equation	**Form**
Energy Equation	$-\bar{\nabla}\bar{q}+\rho R=\rho c_v\dot{\theta}+\beta T_0\dot{e}$
Heat Conduction Law	$\bar{q}=-\left(k\bar{\nabla}\theta+k^*\bar{\nabla}\nu\right)$
Governing Equation	$k\nabla^2\dot{\theta}+k^*\bar{\nabla}^2\theta+\rho\dot{R}=\rho c_v\ddot{\theta}+\beta T_0\ddot{e}$

The Green-Naghdi Type III and Type II models surpass the conventional restrictions on heat flow; however, they vary in complexity and predictions. The third kind, which accounts for the dispersion of thermal waves and various effects, is more difficult but provides correct results for materials with complex thermoelastic behavior. The Type II model, which is simpler and more computationally efficient, is designed to forecast non-dissipative waves. This makes it ideal for materials with simpler heat transport mechanisms and limited resources. Ultimately, the decision relies on the material's characteristics, precision, and computational practicality.

2.3.4 Three and Dual Phase Lag Models

In 1997, Tzou proposed the dual-phase-lag (DPL) thermoelasticity theory (see [12–14]), where the heat conduction equation was developed by integrating phase delays into the Fourier law of heat conduction to account for the microstructural changes that occur during high-rate heat transfer, and also used to describe the effect of ultrafast heating on material defects and thermo-mechanical coupling (see [12, 15, 16]), where Tzou's experimental results in [17] confirm the physical significance and application of the dual-phase-lag model. Choudhuri presents the

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[1] H. von Helmholtz introduced the concept of thermal displacement [11], the thermal displacement satisfies $\dot{\nu}=\theta$ or

$$\nu=\int_{t_0}^{t}\theta(s)ds+\nu_0. \tag{2.3.10}$$

three-phase-lag (TPL) thermoelasticity in [18], where he extends the Tzou model (DPL) of heat conduction. In which phase lags in the heat flux vector, temperature gradient, and thermal displacement gradient are considered, where the heat transfer lags behind the temperature change in three different directions. The principal direction where heat is conducted most efficiently experiences a temperature change first, followed by the transverse directions. This lag time is referred to as the ”phase lag.”

Dual Phase-Lag Thermoelasticity

The DPL heat conduction law proposed by Tzou is given by

$$q_i(.,t+\tau_q) = -k\theta_{,i}(.,t+\tau_\theta), \tag{2.3.11}$$

with the phase lag parameters:

τ_θ : The gradient of temperature,

τ_q : Heat flux vector.

These phase lags arise from the inherent characteristics of the material, specifically from the phenomena of ”thermal inertia” and ”microstructural interaction” (see [19]). The phase lag τ_q can be understood as the time delay resulting from the rapid transient impact of thermal inertia. On the other hand, the phase lag τ_θ indicates the impact of interactions between phonons and electrons, as well as phonon scattering during the ultrafast heat transfer process. Both τ_q and τ_θ are natural properties of the material.

The modified energy equation for the DPL model is

$$-\bar{\nabla}\bar{q} + \rho R = \rho c_v \dot{\theta} + \beta T_0 \dot{e}, \tag{2.3.12}$$

when we consider the Taylor series expansion up to the first term on both sides of (2.3.11), we get the DPL type I:

$$\left(1+\tau_q \frac{\partial}{\partial t}\right) q_i = -k\left(1+\tau_\theta \frac{\partial}{\partial t}\right)\theta_{,i}, \tag{2.3.13}$$

distinguish and for Taylor series expansion up to the second order in τ_q and up to the first order in τ_q, (2.3.11) becomes:

$$\left(1+\tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right) q_i = -k\left(1+\tau_\theta \frac{\partial}{\partial t}\right)\theta_{,i}, \tag{2.3.14}$$

Eliminating the heat flux q from (2.3.12) using (2.3.13) and (2.3.14) respectively, we get the corresponding governing equations:

- DPL-I:

$$k\left(\tau_\theta \nabla^2 \dot{\theta} + \bar{\nabla}^2 \theta\right) + \rho \widetilde{R} = \rho c_v \left(\tau_q \ddot{\theta} + \dot{\theta}\right) + \beta T_0 \left(\tau_q \ddot{e} + \dot{e}\right), \tag{2.3.15}$$

- DPL-II:

$$k\left(\tau_\theta \nabla^2 \dot{\theta} + \bar{\nabla}^2 \theta\right) + \rho \tilde{R} = \rho c_v \left(\frac{\tau_q^2}{2} \dddot{\theta} + \tau_q \ddot{\theta} + \dot{\theta}\right) + \beta T_0 \left(\frac{\tau_q^2}{2} \dddot{e} + \tau_q \ddot{e} + \dot{e}\right), \tag{2.3.16}$$

The heat conduction model (2.3.11) describes basic processes in diffusion, thermal waves, interactions between phonons and electrons, and phonon scatterings that happen when reaction times get shorter. Below are some of its key characteristics:

- The heat conduction (2.3.13) reduces to
  - the Fourier law, when $\tau_q = \tau_\theta = 0$.
  - the single phase lag (SPL) models (Maxwell-Cattaneo-Vernott, Lord-Shulman, extra) if $\tau_\theta = 0$.
- The DPL model has four modes of heat propagation: wave mode when $\tau_\theta = 0$; wavelike if $\tau_q > \tau_\theta > 0$; diffusion mode in case $\tau_q = \tau_\theta$; over-diffusion mode when $\tau_q < \tau_\theta$,
- Stability analysis of one-dimensional DPL heat conduction, i.e., (2.3.13) has been conducted in [20–23], where the study examined the conditions on phase-lags τ_q and τ_θ for a stable solution, we mention these points:
  - The system is consistently stable in the approximation (2.3.11) (i.e., the first order in τ_q and τ_θ).
  - If τ_q is approximated up to the second order and τ_θ up to the first order, the system exhibits stability if $\tau_\theta > \frac{\tau_q}{2}$ and instability if $\tau_\theta < \frac{\tau_q}{2}$.
  - When both τ_q and τ_θ are approximated up to the second order, the system remains stable when $\frac{\tau_\theta}{\tau_q} > 2 - \sqrt{3}$. In cases where $\frac{\tau_\theta}{\tau_q} < 2 - \sqrt{3}$, the solution is unstable.

Triple-Phase-Lag Model

By adding the relaxation times for the heat flux τ_q and temperature gradient τ_θ to the DPL, in addition to the relaxation times for the thermal displacement gradient, Choudhuri [18] obtains the triple-phase-lag model. This is sometimes called the “phase lag of the thermal displacement”

$$q_i(.,t+\tau_q) = -k[\theta_{,i}(.,t+\tau_T) - k^* \nu_{,i}(.,t+\tau_\nu)], \tag{2.3.17}$$

k^* represents a positive material constant that defines the rate of thermal conductivity in the TPL theory, and τ_v : is the displacement gradient (thermal).

2.4 Elastic Beams

The resolution of equations governing the mechanics of a three-dimensional continuum, even for a relatively straightforward constitutive model such as isotropic hyperelasticity, poses considerable challenges. Despite advancements in computational methods and finite element analysis in the era of computers, the comprehensive treatment of every solid body as a three-dimensional continuum remains impractical. However, for bodies exhibiting specific geometric characteristics, a reduction from three dimensions to fewer dimensions is feasible concerning the governing differential equations. Typically denoted as beams (1d), plates (2d, flat), and shells (2d, curved), these bodies fall within the realm of reduced theories in structural mechanics. Beam theory, among the various theories in structural mechanics, stands out as the most straightforward.

Beams are structural elements commonly used to support loads and span open spaces. They are crucial in various engineering applications, such as bridges, buildings, and mechanical structures. ”A beam is defined as a structure having one of its dimensions much larger than the other two. The axis of the beam is defined along that longer dimension, and a cross-section normal to this axis is assumed to smoothly vary along the span or length of the beam (Bauchau and Craig, [24]).”. Understanding the behavior of beams under different loading conditions, including vibration, is essential for designing structures that can withstand dynamic forces and maintain stability. Beams are subjected to different applied loads, such as point loads, distributed loads. These applied loads generate internal forces and moments within the beam structure, resulting in deformations and stresses. Applied loads can be categorised as either static or dynamic, and they significantly impact the overall behaviour of the beam.

Shear force and bending moment are internal forces that result from applied loads on beams. They play a crucial role in determining the structural response of beams. Shear forces act parallel to the beam’s cross-section, causing it to deform, while bending moments induce bending deformations. Mathematical models based on partial differential equations (PDEs) often describe the beams’ vibrations, strain, displacement, stress, shear force, and bending moment. Let us consider a 3d beam; if x, y, z are the Cartesian coordinates, with x as the axis of the beam, we will denote the displacements along these axes as $u(x,y,z)$, $v(x,y,z)$, and $w(x,y,z)$ (representing x-, y-, and z-displacements, respectively). The notation employed for the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$, which is also conventional in engineering, is

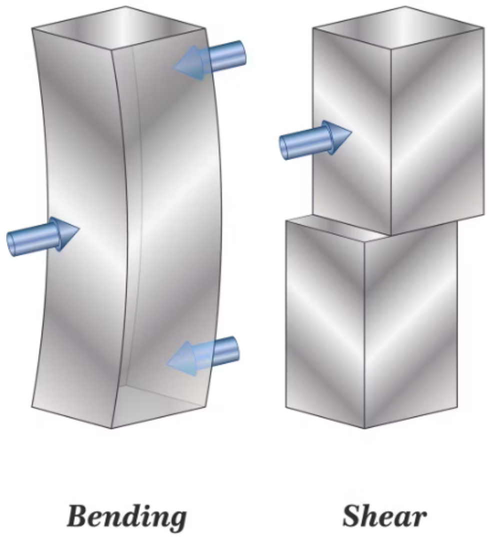

Figure 2.1: Shear and beanding forces

given by:

$$\varepsilon = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zy} & \tau_{zy} & \sigma_z \end{bmatrix}.$$

It is important to note that both $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are symmetric, with $\boldsymbol{\tau}$ is the shear stress. We will consistently assume a linearly elastic material. In the context of a material with Young modulus E, Poisson coefficient ν, and shear modulus G, the strain-stress relationships are described by:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu(\sigma_y + \sigma_z)), \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy} \tag{2.4.1}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu(\sigma_x + \sigma_z)), \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz} \tag{2.4.2}$$

$$\varepsilon_z = \frac{1}{E}(\sigma_z - \nu(\sigma_x + \sigma_y)), \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz} \tag{2.4.3}$$

We will consider the following assumptions:

- **A1:** Consider a straight beam with a length L and a cross-section of area A. For simplicity, we will use A to represent both the geometric surface of the cross-section and its area. The coordinates $y = z = 0$ will be assigned to the center of gravity of the cross-section.

- **A2:** The material is linearly elastic, particularly assuming infinitesimal strains and neglecting Poisson's effect. Consequently, the parameters E and G are Young's modulus and shear modulus, and we employ the uniaxial law $\sigma_x = E\varepsilon_x$.

For discussions not involving beam dynamics:

- **A3:** The loads and boundary conditions remain constant with time. All time derivatives are considered zero, indicating a static problem.

Additionally, in this chapter and subsequent ones, we assume:

- **A4:** All loads act within the xy plane, presumed to be symmetrical. Specifically, forces lie within the xy plane, and moments align exclusively with the z-axis.

2.5 Timoshenko beam

2.5.1 The classical Timoshenko hypotheses for a thin beam/plate

In the study of thin beams and plates, it is often impractical to consider the full three-dimensional complexities of the structures. The classical Timoshenko hypotheses provide a set of simplifications that enable a more manageable analysis while retaining essential mechanical characteristics. These hypotheses are particularly relevant for structures with dimensions significantly larger than their thickness. In order to legitimate the Timoshenko model from a mathematical (and physical) viewpoint, we regard some constitutive laws in mathematical-physics proposed by Timoshenko [25]. let us consider a thin three-dimensional beam

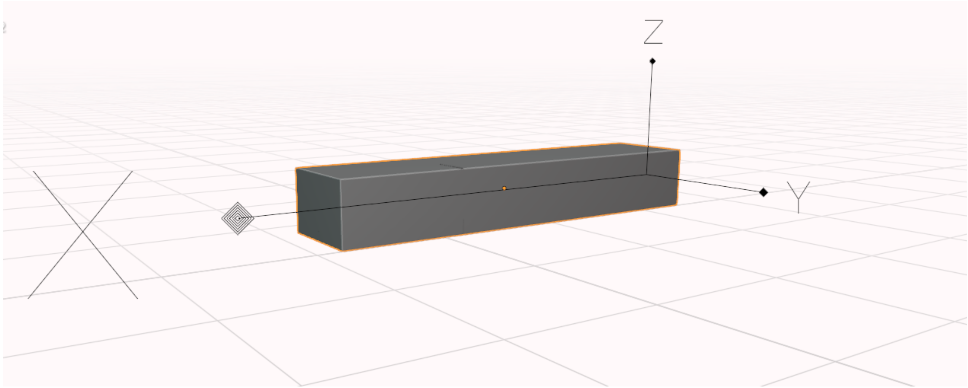

Figure 2.2: 3d beam

In this context, we consider a thin 3D beam defined over the domain $[0,L]\times\Omega$, where Ω represents a uniform cross-section in $\mathbb{R}^2$. The beam, of length $L>0$, is composed of a homogeneous isotropic material, and its behavior is analyzed based on the classical Timoshenko assumptions.

$$[0,L]\times\Gamma := \{(x,y,z) : x\in[0,L] \text{ and } (y,z)\in\Gamma\},$$

The thin beam is subjected to the following Timoshenko assumptions:

- Centering Conditions (S1): The origin $(0,0)$ is considered the center of Γ, ensuring that the integrals over Ω result in zero values for both y and z coordinates.
- Slenderness (S2): The diameter of Γ is significantly smaller than the length L, allowing for the treatment of thin beams.
- Bending Plane (S3): Bending occurs exclusively in the (x,z)-plane, and normal stresses (in the y-axis) are generally negligible.
- Stress Tensor Simplification (S4): The stress tensor matrix $\sigma = (\sigma_{ij})_{1\leq i,j\leq 3}$ is assumed to have only two relevant stresses, namely, σ_{11} and σ_{13}, with all other stresses approximated as zero.

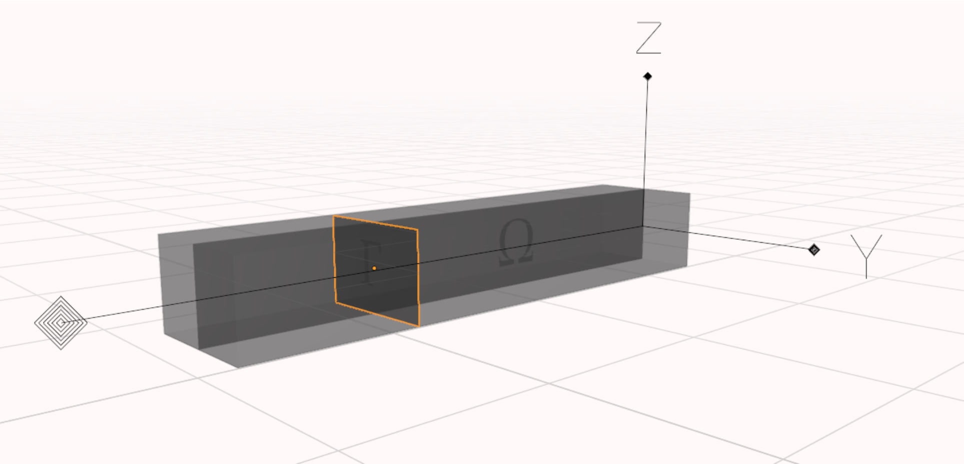

Figure 2.3: Cross section Γ

The analysis focuses on a longitudinal section along the (x,z)-plane, represented by points of the form $(x,0,z)$ and identified as (x,z) for simplicity. Here, bending occurs according to assumption S3.

The displacements and rotation angle in the (x,z)-plane are denoted as follows:

- $u = u(x,t)$: Longitudinal displacement along the x-axis.
- $\psi = \psi(x,t)$: Rotation angle for the normal to the x-axis.
- $w_1(x,z,t) = u(x,t) + z\psi(x,t)$: Longitudinal displacement.
- $w_2(x,z,t) = \varphi(x,t)$: Vertical beam displacement.

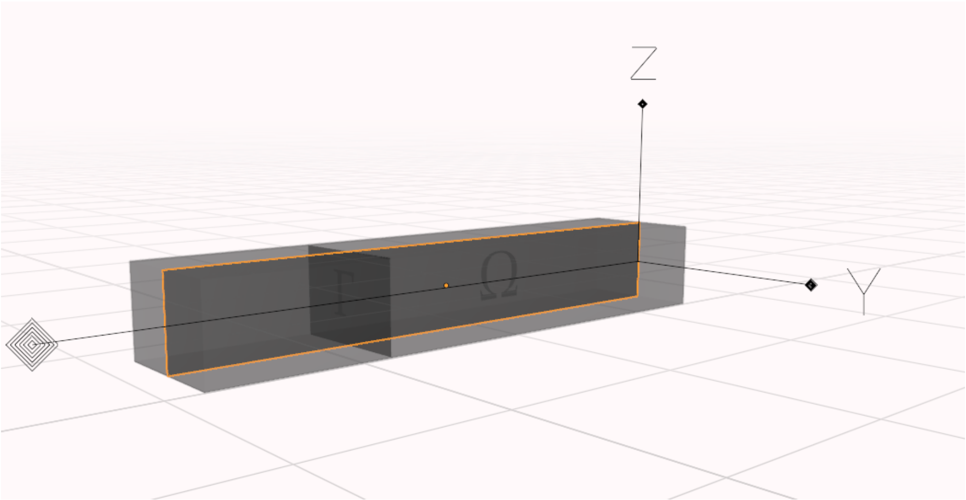

Figure 2.4: Longitudinal section

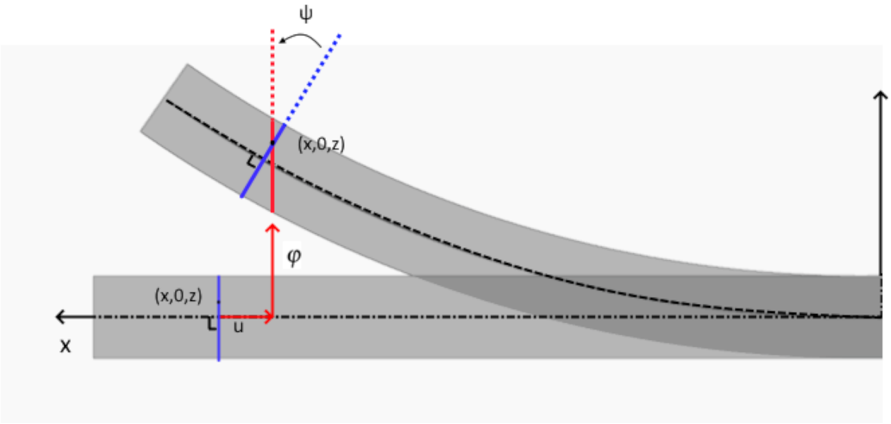

Figure 2.5: Displacements distribution in the (x , z)-plane

Considering a homogeneous, elastically, and thermally isotropic thin beam/plate, the stress-strain relations for the remaining stresses σ_{11} and σ_{13} in **A4** are expressed as follows, in accordance with (6.1) on page 26 in [[26], Chapt.1]:

$$\sigma_{11} = a_{11}\left(\varepsilon_{11} - \varepsilon_{11}^{T}\right) \text{ and } \sigma_{13} = a_{13}\left(\varepsilon_{13} - \varepsilon_{13}^{T}\right), \tag{2.5.1}$$

Notable choices include $a_{11} = E$ and $a_{13} = 2k'G$, where E represents the Young modulus of elasticity and G is the shear modulus, incorporating a shear correction coefficient k'. The elastic strains ε_{11} and ε_{13} are determined following Timoshenko's laws in elasticity. The symbols ε_{11}^{T} and ε_{13}^{T} represent the thermal strains, which will be formulated in accordance with appropriate laws for thermoelastic beams/plates (here we do not take into consideration the thermal effect on beam, i.e.,$\varepsilon_{11}^{T} = \varepsilon_{13}^{T} = 0$). For elastic strains, under the provided notations, the components of the infinitesimal elastic strain tensor can be expressed using standard formulas (refer to e.g., (2.4) on page 339 in [27]):

$$\varepsilon_{11}(x,z,t) = \frac{\partial w_1}{\partial x} = u_x(x,t) + z\psi_x(x,t), \tag{2.5.2}$$

$$\varepsilon_{13}(x,z,t) = \frac{1}{2}\left(\frac{\partial w_1}{\partial z} + \frac{\partial w_2}{\partial x}\right) = \frac{1}{2}\left[\psi(x,t) + \varphi_x(x,t)\right]. \tag{2.5.3}$$

Going back to postulations $S1$-$S4$ and following the identities (9.10)-(9.11) in [28], the conventional formulas to express the bending moment and the shear force are given by

$$\begin{aligned} M(x,t) &= \int_\Omega z\sigma_{11}(x,z,t)\,dy\,dz, \\ S(x,t) &= \int_\Omega \sigma_{13}(x,z,t)\,dy\,dz, \end{aligned} \tag{2.5.4}$$

respectively. Here, we have normalized the equations by the area A and inertial moment I of the cross-section Ω, namely,

$$A = \int_\Omega dy\,dz \quad \text{and} \quad I = \int_\Omega z^2\,dy\,dz. \tag{2.5.5}$$

As a consequence, using identities (2.5.1), (2.5.5), and (2.5.2), one can compute the classical bending moment

$$\begin{aligned} M(x,t) =& E\overbrace{\left(\int_\Omega z\,dy\,dz\right)}^{0=}u_x(x,t) \\ &+ E\overbrace{\left(\int_\Omega z^2\,dy\,dz\right)}^{I=}\psi_x(x,t), \end{aligned}$$

and then

$$M(x,t) = EI\left[\psi_x(x,t)\right], \quad x \in [0,L], t \geq 0 \tag{2.5.6}$$

Moreover, using identities (2.5.1), (2.5.5), and (2.5.2), the following law for the shear force comes into the picture

$$S(x,t) = 2k'G\left(\int_\Omega dy\,dz\right)\left[\frac{1}{2}\left(\psi(x,t)+\varphi_x(x,t)\right)\right]$$

that is,

$$S(x,t) = 2k'GA\left[\frac{1}{2}\left(\psi+\varphi_x\right)(x,t)\right], \quad x\in[0,L], t\geq 0. \tag{2.5.7}$$

The well-known elastic constitutive relations for the bending moment and shear force as follows

$$M(x,t) = EI\psi_x(x,t) \quad \text{and} \quad S(x,t) = k'GA\left(\psi+\varphi_x\right)(x,t) \tag{2.5.8}$$

To derive the desired Timoshenko systems, we incorporate the classical momentum equations (see [25]):

$$\rho A\varphi_{tt} - S_x = 0 \tag{2.5.9}$$

$$\rho I\psi_{tt} - M_x + S = 0 \tag{2.5.10}$$

for $(x,t)\in(0,L)\times(0,\infty)$, where ρ represents the mass density per area unit. Keeping this system in mind, and considering the constitutive laws (2.5.6)-(2.5.7), we can deduce at least three different models for thermoelastic Timoshenko systems.

By replacing M and S from (2.5.8) in system (2.5.9), it becomes the Timoshenko problem:

$$\begin{cases} \rho A\varphi_{tt} - k'GA(\varphi_x+\psi)_x = 0 \\ \rho I\psi_{tt} - EI\psi_{xx} + k'GA(\varphi_x+\psi) = 0 \end{cases} \tag{2.5.11}$$

2.6 Curved Beams

The study of vibrations in curved beams and subsequent efforts to stabilize them over time have been topics of interest for at least two centuries. Among the various models for curved beams available today, Bresse (1859) stands out as a pivotal figure from the mid-19th century who rigorously derived a reliable set of partial differential equations (PDEs) for extensible curved beams under shear and axial effects in his renowned work ”Cours de Méchanique Appliquée” [29]. Further insights into Bresse’s discoveries can be found in earlier works by Timoshenko [25] and more recent publications by Challamel and Elishakoff [30], which underscore the significance of Bresse’s contributions. Timoshenko (cf. [[25], p. 151]) notably acknowledged Bresse as the first to consider the rotational inertia of bar elements, with additional historical data on Bresse’s advancements provided in [[30], Sect 1].

The mathematical and physical intricacies of the Brass system are already been thoroughly addressed in the aforementioned references. The well-established Bresse system outlined in [29] (refer also to Fig. 2.6 for a visual representation of the curved element and its displacements). This system serves as a renowned archetype in evolutionary partial differential equations (PDEs), capturing vibrational phenomena in elastic curved bodies subjected to pertinent forces inherently linked to Shear force (Q), axial load (N), and Bending Moment (M). These forces, which induce vibrations, manifest in beam motions such as vertical displacement (ϕ), horizontal displacement (w), and angle of rotation (ψ). The mathematical expressions detailing the forces Q, N, M in terms of displacements ϕ, w, ψ are :

$$\begin{aligned} N &= k_0\left(w_x - l\varphi\right), \\ Q &= k\left(\varphi_x + \psi + lw\right), \\ M &= b\psi_x. \end{aligned} \tag{2.6.1}$$

and the evolution equations are given by:

$$\begin{aligned} \rho_1\varphi_{tt} &= Q_x + lN, \\ \rho_2\psi_{tt} &= M_x - Q, \\ \rho_1 w_{tt} &= N_x - lQ, \end{aligned} \tag{2.6.2}$$

where the parameters are as follows:

ρ_1 the mass density of the beam

ρ_2 the moment of mass inertia of the beam

k the shear modulus of elasticity

$k_0 = Eh$, where E is the Young's modulus and h is the cross sectional area

l is the initial curvature

G is the rigidity coefficient of the cross section

For clarity, Figure 2.6 below presents a geometric representation illustrating the displacements ϕ, w, and ψ.

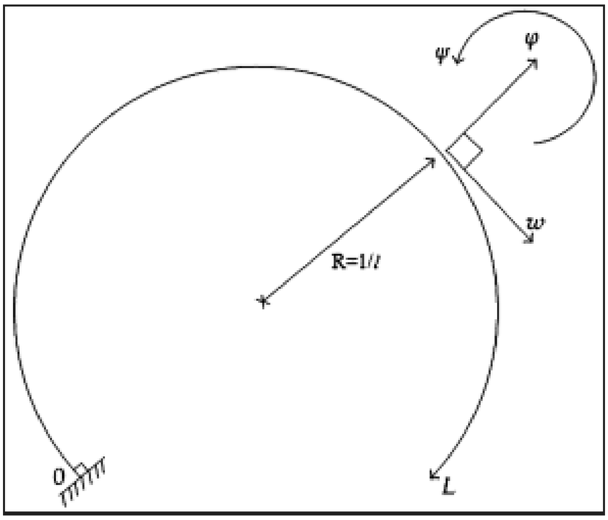

Figure 2.6: Geometric illustration of the displacements φ, w, and ψ

2.7 Banach and Hilbert spaces

In this section, we review fundamental concepts in functional analysis, laying the groundwork for subsequent chapters. Detailed proofs for the presented material can be readily located in relevant literature. We will begin with some basic concepts from the theory of Sobolev spaces and functional analysis (see the references [31–40]).

Let $\Omega \subset \mathbb{R}^N$ be an open set. For $1 \leq p < \infty$, we denote by $L^p(\Omega)$ the set (of classes) of Lebesgue measurable functions u such that $|u|^p$ is integrable over Ω. In terms of sets,

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} \mid u \text{ is measurable, } \int_\Omega |u(x)|^p \, dx < \infty \right\}.$$

In $L^p(\Omega)$, we can define the usual norm as follows:

$$\|u\|_p = \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty.$$

Thus, $(L^p(\Omega); \|\cdot\|_p)$ is a Banach space. In the particular case when $p = 2$, we also have that $L^2(\Omega)$ is a Hilbert space with inner product $(u,v) = \int_\Omega u(x)v(x)\,dx$ and norm $\|u\|_2 = (u,u)^{1/2}$.

For $p = \infty$, we define $L^\infty(\Omega)$ as the set (of classes) of Lebesgue measurable functions u, almost everywhere bounded on Ω. In terms of sets,

$$L^\infty(\Omega) = \{ u : \Omega \to \mathbb{R} \mid u \text{ is measurable, } |u(x)| \leq K \text{ almost everywhere on } \Omega \}.$$

In this case, the real number K is called an essential upper bound of u. Denoting $A = \{K \in$

$\mathbb{R} \mid |u(x)| \le K$ almost everywhere on $\Omega\}$, we can define a norm in $L^\infty(\Omega)$ as

$$\|u\|_\infty = \sup_{x\in\Omega} \mathrm{ess}\, |u(x)| = \inf A.$$

Thus, $(L^\infty(\Omega); \|\cdot\|_\infty)$ is a Banach space.

Given a multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ and a point $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, we define

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N, \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N}, \quad \text{and} \quad \alpha! = \alpha_1!\alpha_2!\cdots\alpha_N!.$$

The operator of differentiation of order α, denoted by D^α, is defined as

$$D^\alpha u = \begin{cases} \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}} & \text{if } \alpha \neq (0, \dots, 0), \\ u & \text{if } \alpha = (0, \dots, 0). \end{cases}$$

When the multi-index is of the form $\alpha = (0, \dots, 0, i, 0, \dots, 0) \in \mathbb{N}^N$, the differentiation operator D^α is also represented by the following notations

$$D^i = D_i = \partial_{x_i} = \frac{\partial}{\partial x_i} = (\cdot)_{x_i}, \quad i = 1, \dots, N.$$

Now, for $k = 0, 1, 2, \dots$, we denote by $C^k(\Omega)$ the set of functions k times continuously differentiable on the open set Ω. When $k = 0$, we simply say that $C^0(\Omega)$ is the set of continuous functions on Ω, and we also denote it as $C(\Omega)$. When $k = \infty$, we say that $C^\infty(\Omega)$ is the set of infinitely differentiable functions on Ω. For a function, we denote the functions u from $C^k(\Omega)$ whose support is compact in $\mathbb{R}^N$.

Definition 2.1. *We will say that a sequence (or succession) of functions $\{\varphi_n\} \subset C_0^\infty(\Omega)$ converges to a function $\varphi \in C_0^\infty(\Omega)$ if, and only if, there exists a compact subset $K \subset \Omega$ such that*

1. $\mathrm{supp}(\varphi) \subset K$ *and* $\mathrm{supp}(\varphi_n) \subset K$ *for all* $n \in \mathbb{N}$.

2. $D^\alpha \varphi_n \to D^\alpha \varphi$ *uniformly over* K *for all* $\alpha \in \mathbb{N}^N$.

With this notion of convergence in the spaces $C_0^\infty(\Omega)$, we also denote it by $\mathscr{D}(\Omega)$ and call it the space of test functions. Furthermore, denoting by $\mathscr{L}(X, \mathbb{R})$ the vector space of linear and continuous functionals from X to $\mathbb{R}$, we define the space of distributions over Ω with values in $\mathbb{R}$ as

$$\mathscr{D}'(\Omega) := \mathscr{L}(\mathscr{D}(\Omega), \mathbb{R}),$$

where continuity is understood in terms of convergence in $\mathscr{D}(\Omega)$.

Regarding the derivative of a distribution with real values, we recall that for $f \in \mathscr{D}'(\Omega)$, its derivative of order $\alpha \in \mathbb{N}^N$ is given by

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle, \quad \forall \varphi \in \mathscr{D}(\Omega).$$

Next, we will recall the definition of Sobolev spaces, which will be of fundamental importance in our future considerations. For $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, we denote by $W^{m,p}(\Omega)$ the set (of classes) of functions u in $L^p(\Omega)$ such that their derivatives $D^\alpha u$, $0 \leq |\alpha| \leq m$, still belong to $L^p(\Omega)$, where the derivative is taken in the sense of distributions. In terms of sets,

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega), \quad 0 \leq |\alpha| \leq m\}.$$

In $W^{m,p}(\Omega)$, a well-defined norm is given by

$$\|u\|_{m,p} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_p^p \right)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$\|u\|_{m,\infty} = \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_\infty.$$

With this, $(W^{m,p}(\Omega), \|\cdot\|_{m,p})$ is a Banach space. Additionally, we define the set $W_0^{m,p}(\Omega)$ as the subspace of $W^{m,p}(\Omega)$ constituted by the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$, i.e.,

$$W_0^{m,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{m,p}(\Omega)}.$$

In the particular case $p = 2$, the space $W^{m,2}(\Omega) \left(W_0^{m,2}(\Omega) \right)$ is also a Hilbert space with the corresponding inner product, usually denoted by $H^m(\Omega)\,(H_0^m(\Omega))$.

In what follows, X will denote a Banach space with norm $\|\cdot\|_X$, and H will denote a Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$.

2.7.1 Some important results

In this section, we collect some of the most classical results from the general theory of functional analysis and Sobolev spaces, which will be extremely useful for studying the problems proposed in this thesis.

Lemma 2.1. *Let Ω be a domain in $\mathbb{R}^N$.*

- *If $1 < p < \infty$, then $L^p(\Omega)$ is reflexive. However, $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive.*
- *If $1 \leq p < \infty$, then $L^p(\Omega)$ is separable. However, $L^\infty(\Omega)$ is not separable*

Theorem 2.1. *Let Ω be a domain in $\mathbb{R}^N$.*

- *If $k \geq 0$, then $C_0^k(\Omega)$ is dense in $L^p(\Omega)$, for $1 \leq p < \infty$.*
- *If Ω is of class $C^m, m \geq 1$, then $C^m(\bar{\Omega})$ is dense in $W^{m,p}(\Omega)$, for $1 \leq p < \infty$.*

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^m boundary.*

- *If $mp < N$, then the following inclusion is continuous*

  $$W^{m,p}(\Omega) \hookrightarrow L^{q^*}(\Omega), \quad \text{where} \quad \frac{1}{q^*} = \frac{1}{p} - \frac{m}{N}$$

  Moreover, the inclusion is compact for any q, with $1 \leq q < q^*$.

- *If $mp = N$, then the following inclusion is continuous and compact*

  $$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{for all} \quad 1 \leq q < \infty$$

  Furthermore, if $p = 1$ and $m = N$, then the same relation holds for $q = \infty$.

- *If $k+1 > m - \frac{N}{p} > k, k \in \mathbb{N}$, then, writing $m - \frac{N}{p} = k + \alpha$, with $0 < \alpha < 1$, the following inclusion is continuous*

  $$W^{m,p}(\Omega) \hookrightarrow C^{k,\alpha}(\bar{\Omega}),$$

where $C^{k,\alpha}(\bar{\Omega})$ represents the space of functions in $C^k(\bar{\Omega})$ whose kth-order derivatives are α-Hölder continuous. Additionally, if $N = m - k - 1, \alpha = 1$, and $p = 1$, then the inclusion holds for $\alpha = 1$, and the inclusion $W^{m,p}(\Omega) \hookrightarrow C^{k,\beta}(\bar{\Omega})$ is compact for all $0 \leq \beta < \alpha$.

Theorem 2.3. *(Hölder's Inequality). Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\Omega \subset \mathbb{R}^N$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$, and*

$$\int_\Omega |u(x)v(x)|dx \leq \|u\|_p \|v\|_q$$

Lemma 2.2. *(Gronwall's Inequality). Let $\alpha \geq 0$ be a constant and $\phi \in L^\infty(a,b), \beta \in L^1(a,b)$ such that $\beta > 0, \phi \geq 0$. If*

$$\phi(t) \leq \alpha + \int_a^b \beta(s)\phi(s)$$

then

$$\phi(t) \leq \alpha e^{\int_a^b \beta(s)ds}, \quad a \leq t \leq b.$$

Lemma 2.3. *(Young's Inequality). Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \forall a, b \geq 0$$

Lemma 2.4. *(Young's Inequality with ε). Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\varepsilon > 0$. Then*

$$ab \leq \varepsilon a^p + C_\varepsilon b^q, \quad \forall a, b \geq 0$$

where $C_\varepsilon = (\varepsilon p)^{-q/p} q^{-1}$.

Remark 2.1. *In the particular case where $p = q = 2$, Young's inequality with $\varepsilon > 0$ reduces to*

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \quad \forall a, b \geq 0$$

which is also known as Cauchy's inequality with ε. This inequality will be extremely useful in all the subsequent chapters of this thesis.

The next result is also of fundamental importance in our future problems.

Lemma 2.5. *Let X be a Banach space. If $f \in L^p(0,T;X)$ and $\frac{df}{dt} \in L^p(0,T;X)$, then $f \in C([0,T],X)$, except for a set of measure zero in $[0,T]$.*

Lemma 2.6. *Let X and Y be Banach spaces such that $X \hookrightarrow Y$. Then,*

$$[L^\infty(0,T;X) \cap C_w([0,T],Y)] \subset C_w([0,T],X).$$

If the inclusion of X in Y is dense and X is reflexive, then the reverse inclusion also holds.

Theorem 2.4. *(Riesz-Fréchet Representation Theorem). Let $(H, \|\cdot\|_H, (\cdot,\cdot)_H)$ be a Hilbert space. For every functional $\varphi \in H'$, there exists a unique $f \in H$ such that*

$$\langle \varphi, v \rangle_{H',H} = (f, v)_H, \quad \forall v \in H$$

Moreover, $\|\varphi\|_{H'} = \|f\|_H$.

Theorem 2.5. *(Lax-Milgram Theorem). Let $(H, \|\cdot\|_H, (\cdot,\cdot)_H)$ be a Hilbert space, and let (u,v) be a continuous and coercive bilinear form. Then, for every functional $\varphi \in H'$, there exists a unique $u \in H$ such that*

$$a(u,v) = \langle \varphi, v \rangle_{H',H}, \quad \forall v \in H$$

Moreover, if $a(u,v)$ is symmetric, then u is characterized by the property

$$u \in H \quad itis \quad \frac{1}{2} a(u,u) - \langle \varphi, u \rangle_{H',H} = \min_{v \in H} \left\{ \frac{1}{2} a(v,v) - \langle \varphi, v \rangle_{H',H} \right\}$$

Theorem 2.6. *(Regularity for the Dirichlet problem) [33] Let Ω be an bounded open set of class*

C^2, $f \in L^2(\Omega)$ and let $u \in H_1^0(\Omega)$ satisfy

$$\int_\Omega \nabla u \cdot \nabla \phi + \int_\Omega u\phi = \int_\Omega f\phi, \quad \forall \phi \in H_0^1(\Omega), \tag{48}$$

Then $u \in H^2(\Omega)$ and $\|u\|_{H^2} \leq C\|f\|_{L^2}$, where C is a constant depending only on Ω. Furthermore, if Ω is of class C^{m+2} and $f \in H^m(\Omega)$, then $u \in H^{m+2}(\Omega)$ and $\|u\|_{H^{m+2}} \leq C\|f\|_{H^m}$.

Remark 2.2. *Similarly, under the same assumptions, the conclusions of Theorem 9.25 also hold for the solution of the Neumann problem, i.e., for $u \in H^1(\Omega)$ such that*

$$\int_\Omega \nabla u \cdot \nabla \phi + \int_\Omega u\phi = \int_\Omega f\phi, \quad \forall \phi \in H^1(\Omega). \tag{2.7.1}$$

2.7.2 Operator Associated with a Bilinear Form

Let $(V, \|\cdot\|_V, (\cdot,\cdot)_V)$ and $(H, \|\cdot\|_H, (\cdot,\cdot)_H)$ be two Hilbert spaces such that V is dense in H, with a continuous and compact inclusion $V \hookrightarrow H$. We denote by V' the dual of V, and $\langle \cdot,\cdot \rangle$ the duality between V' and V. Identifying H with its dual through the Riesz Representation Theorem, we obtain the following chain of inclusions:

$$V \hookrightarrow H \cong H' \hookrightarrow V'$$

Considering a continuous bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$, we can define a linear operator $A : V \to V'$ by

$$\langle Au, v \rangle = a(u,v), \quad \forall u,v \in V.$$

Moreover, the domain of the operator A is defined as

$$D(A) = \{u \in V \mid Au \in H\},$$

and we also say that the linear operator A is defined by the triplet $\{V, H, a(\cdot,\cdot)\}$.

Recalling from functional analysis that if $a(\cdot,\cdot)$ is a continuous, coercive, and symmetric bilinear form, then the linear operator $A : D(A) \subset H \to H$ is closed, unbounded, positive definite, self-adjoint, and a bijection (isomorphism). Furthermore, equipping the domain $D(A)$ with the norm $\|u\|_{D(A)} = \|Au\|_H$ (which is equivalent to the graph norm $\|u\|_G^2 = \|u\|_H^2 + \|Au\|_H^2$), we obtain that $D(A)$ is a dense Hilbert space in H.

1.5.1 C_0-Semigroups of Linear Operators

In this section, $(X, \|\cdot\|_X)$ will always denote a Banach space, $(H, \|\cdot\|_H, (\cdot,\cdot)_H)$ a Hilbert space, and $(\mathscr{L}(X,X), \|\cdot\|)$ the space of linear and continuous operators on X.

Definition 2.2. *A family* $\{T(t)\}_{t\geq 0}$ *of linear and bounded operators defined on a Banach space* X *is called a semigroup of bounded linear operators (or simply a semigroup) when*

(i) $T(0) = I : X \to X$ *(Identity Operator on* X*). (ii)* $T(t+s) = T(t)T(s)$*, for every* $t,s \geq 0$*.*

Moreover, we say that $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup if, in addition to the above items, we also have that (iii) $\lim_{t\to 0} T(t)x = x$, for every $x \in X$.

An operator A is called the infinitesimal generator of a semigroup $\{T(t)\}_{t\geq 0}$ when A is defined as

$$D(A) = \left\{ x \in X \;\middle|\; \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and for each $x \in D(A)$ we have

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t}.$$

Sometimes it is also said that the semigroup $T(t)$ is generated by A. We will frequently use the notation $T(t) = e^{tA}$. Note also that the domain $D(A)$ of the operator A can be rewritten as

$$D(A) = \{x \in X \mid Ax \in X\}.$$

Definition 2.3. *A semigroup* $\{T(t)\}_{t\geq 0}$ *is called uniformly bounded if there exists a constant* $M \geq 1$ *such that*

$$\|T(t)\| \leq M, \quad \forall t \geq 0.$$

When $M = 1$*, we will also say that* $\{T(t)\}_{t\geq 0}$ *is a contraction semigroup.*

Definition 2.4. *A semigroup* $\{T(t)\}_{t\geq 0}$ *is called exponentially stable if there exist constants* $\alpha > 0$ *and* $M \geq 1$ *such that*

$$\|T(t)\| \leq Me^{-\alpha t}, \quad \forall t \geq 0.$$

Definition 2.5. *Let* A *be a linear operator (not necessarily bounded) defined on a Banach space* X*. The resolvent set, denoted by* $\rho(A)$*, is defined as*

$$\rho(A) = \left\{\lambda \in \mathbb{C} \mid \lambda I - A \text{ is invertible and } (\lambda I - A)^{-1} \in \mathscr{L}(X,X)\right\}$$

The set $\sigma(A) = \mathbb{C} \backslash \rho(A)$ *is called the spectrum of* A*.*

Definition 2.6. *Let* A *be a linear operator defined on a Hilbert space* H *with domain* $D(A) \subseteq H$*. We say that* A *is a dissipative operator when*

$$\operatorname{Re}(Ax,x)_H \leq 0, \quad \forall x \in D(A)$$

Now, let's present the results related to an infinitesimal generator of a C_0 semigroup $T(t) = e^{tA}$ defined on Banach or Hilbert spaces.

Theorem 2.7. *(Hille-Yosida). A linear operator (not necessarily bounded) A is an infinitesimal generator of a C_0-contraction semigroup $\{T(t)\}_{t\geq 0}$ if, and only if,*

(a) A is closed, and $\overline{D(A)} = X$.

(b) The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$, and for each $\lambda > 0$,

$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{1}{\lambda}$$

Next, we will see the Lumer-Phillips Theorem, which provides the characterization of infinitesimal generators of C_0-contraction semigroups on Hilbert spaces. The same result holds for Banach spaces, but in this work, we will only need the result for Hilbert spaces.

Theorem 2.8. *(Lumer-Phillips). Let A be a linear operator with domain $D(A)$ dense in a Hilbert space H. We have:*

- *If A is dissipative and there exists a number $\lambda > 0$ such that $\mathrm{Im}(\lambda I - A) = H$, then A is an infinitesimal generator of a C_0-contraction semigroup in H.*
- *If A is the infinitesimal generator of a C_0-contraction semigroup in H, then $\mathrm{Im}(\lambda I - A) = H$ for every $\lambda > 0$, and A is dissipative.*

A fundamental consequence of the Lumer-Phillips Theorem that we will use later is the following:

Corollary 2.1. *Let A be a dissipative linear operator with domain $D(A)$ dense in a Hilbert space H. If $0 \in \rho(A)$, then A is an infinitesimal generator of a C_0-contraction semigroup in H.*

2.7.3 Analyticity and Asymptotic Stability of C_0-Semigroups

In this subsection, we will present some results dealing with analyticity, exponential stability, and polynomial decay of C_0-semigroups defined on Hilbert spaces.

The first result establishes a necessary and sufficient condition for analyticity. Such a result can be found in a general setting for uniformly bounded C_0-semigroups, as proposed by Theorem 5.2 in Pazy [42]. However, we will use a version due to Liu & Young [43], which can also be found in Liu & Zheng [41] (see Theorem 1.3.3).

Theorem 2.9. *Let $T(t) = e^{tA}$ be a C_0-contraction semigroup on a Hilbert space H such that*

$$i\mathbb{R} := \{i\beta; \beta \in \mathbb{R}\} \subseteq \rho(A).$$

Then, $T(t)$ is analytic if, and only if,

$$\overline{\lim_{|\beta|\to\infty}} \left\|\beta \left(i\beta I_d - A\right)^{-1}\right\| < \infty,$$

where the norm $\|\cdot\|$ *is in the space* $\mathscr{L}(H,H)$.

The next result grants exponential stability for a semigroup and is due to Gearhart [44] (see also Prüss [45]). We note that there are two equivalent ways to achieve exponential stability for C_0-contraction semigroups in Hilbert spaces, and a proof of this equivalence can be found in Liu & Zheng [41]. They provide the following version, giving a necessary and sufficient condition for exponential stability.

Theorem 2.10. *Let* $T(t)=e^{tA}$ *be a* C_0*-semigroup of contractions on a Hilbert space* H. *Then,* $T(t)$ *is exponentially stable if and only if the following conditions hold:*

$$i\mathbb{R} := \{i\beta;\beta\in\mathbb{R}\}\subseteq\rho(A) \tag{2.7.2}$$

and

$$\overline{\lim_{|\beta|\to\infty}}\left\|(i\beta I_d - A)^{-1}\right\| < \infty. \tag{2.7.3}$$

We are interested in applying these general results to dissipative evolution problems, viewing the solution of the system through a semigroup.

2.7.4 Abstract Cauchy Problem

As is well known, the general theory of semigroups allows us to study initial value problems for abstract evolution equations of the form

$$\begin{cases} \frac{d}{dt}U(t) = AU(t), & t>0 \\ U(0)=U_0 \end{cases} \tag{2.7.4}$$

where A is a linear operator with domain $D(A)\subset X$, and X is a Banach (or Hilbert) space.

Definition 2.7. *A solution to the Cauchy problem* (2.7.4) *is a function* $U:[0,+\infty)\to X$ *such that* $U(t)$ *is continuous for* $t\geq 0$, *continuously differentiable with* $U(t)\in D(A)$ *for* $t>0$, *and satisfies* (2.7.4) *almost everywhere on* $[0,+\infty)$.

The following result is a classic in the literature and can be found in Brézis [15, Chapter 7], Pazy [85, Chapter 4], or Zheng [94, Chapter 2]. Note that in [15, 94], the authors use the language of m-accretive operators with respect to the operator A.

Theorem 2.11. *Let* A *be an infinitesimal generator of a* C_0*-contraction semigroup* $T(t):=e^{tA}$ *on* X. *If* $U_0\in D(A)$, *then the abstract Cauchy problem* (2.7.4) *has a unique solution* U *in* $D(A)$ *given by*

$$U(t)=T(t)U_0 := e^{tA}U_0, \quad \forall t\geq 0$$

such that

$$U \in C([0,+\infty),D(A)) \cap C^1([0,+\infty),X)$$

To obtain more regular solutions, we introduce the following Banach (or Hilbert) space

$$D\left(A^k\right) = \left\{U \in D\left(A^{k-1}\right) \mid AU \in D(A)^{k-1}\right\}, \quad k \in \mathbb{N}$$

equipped with the norm $\|U\|^2_{D\left(A^k\right)} = \sum_{j=0}^{k} \|U\|^2_{D(A^j)}$. With this, we have the

Theorem 2.12. *Under the assumptions of Theorem 2.11 if* $U_0 \in D\left(A^k\right)$ *for* $k \in \mathbb{N}$, *then the solution* U *of problem* (2.7.4) *satisfies the following condition*

$$U \in \bigcap_{j=0}^{k} C^{k-j}\left([0,+\infty),D\left(A^j\right)\right)$$

2.8 Finite elements methods

The finite element (FE) method stands as a well-established numerical technique for approximating differential equations, finding widespread use, especially in solid mechanics. These notes do not aim to provide a comprehensive introduction to FE approximation but rather to explain the significance of approximating beam problems. The necessity for this approximation becomes apparent in the numerical part of the thesis, where problems lacking analytical solutions will be examined.

2.8.1 Fundamentals

The Finite Element Method (FE) involves several conceptual steps applied to our specific problems of interest:

1. Create a partition of the domain where the problem is defined, breaking it into subdomains. In our scenario, the domain is the interval $[0,L]$. The partition comprises points $0 = x_0 < x_1 < \cdots < x_{n_{\text{el}}} = L$, with $h_i = x_i - x_{i-1}$ representing the length of each subdomain $[x_{i-1},x_i]$ and h denoting the diameter of the finite element (FE) partition.

2. Select a set of points (nodes) within the subdomains. These nodes will serve for approximating the unknown functions, and the values at these nodes are termed as nodal values.

3. Employ the approximated unknown function as the interpolation from the nodal values within each subdomain. The finite element method (FE) utilizes polynomials of local

support for interpolation, vanishing outside a small region formed by the union of a few subdomains.

4. Ensure that the integral form of the problem is satisfied in the space of interpolated functions, constituting the FE space. Nodal values in this space are specifically denoted with a subscript h (representing the diameter of the FE partition).

A finite element comprises the subdomain, the set of nodes, and a collection of basis functions for interpolation. Two finite elements sharing the same subdomain and nodes but featuring different interpolation functions are considered distinct. Nevertheless, it is common to refer to the subdomains of the partition as ’finite elements.’

2.8.2 Discretization and Interpolation in Finite Element Method

As previously mentioned, the discretization of the computational domain involves creating a partition, denoted as $0 = x_0 < x_1 < \cdots < x_{n_{\mathrm{el}}} = L$. The next step is defining Finite Element (FE) interpolation, which emerges after selecting nodes and basis functions for interpolating from the nodal values. For the axial deformation problem, continuity is essential due to the continuity requirement of the axial displacement $\varphi(x)$ in the continuous problem. This adherence to continuity is termed conformity.

Linear interpolation

- **Nodes:** Within each subdomain $[x_{i-1}, x_i]$, nodes are chosen as the end-points x_{i-1}, x_i (two nodes per subdomain).

- **Nodal values:** Nodal values are taken as approximations to $\varphi(x)$ at the nodes, denoted as φ_{i-1} and φ_i for x_{i-1} and x_i, respectively. These values are shared by adjacent subdomains, ensuring continuity in the approximate solution $\varphi_h(x)$.

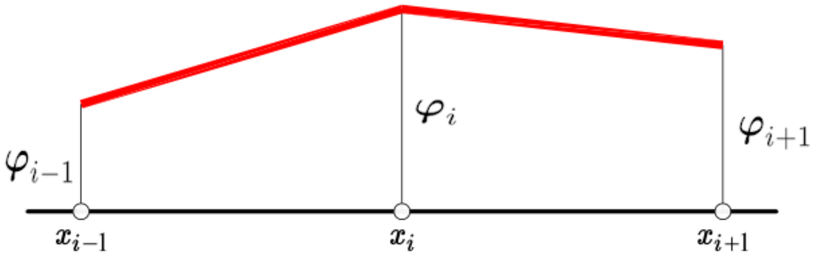

Figure 2.7: Linear interpolation

Interpolation basis: the basis of the interpolation constructed this way is: - For $i = 1,2,\ldots,n_{\mathrm{el}}-1$:

$$N_i(x) = \begin{cases} \frac{x-x_{i-1}}{h_i} & \text{if } x_{i-1} \leq x \leq x_i \\ \frac{x_{i+1}-x}{h_{i+1}} & \text{if } x_i \leq x \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

- For $i = 0, n_{\mathrm{el}}$:

$$N_0(x) = \begin{cases} \frac{1}{h_1}\left(x_1 - x\right) & \text{if } 0 \leq x \leq x_1 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{n_{\mathrm{el}}}(x) = \begin{cases} \frac{1}{h_{n_{\mathrm{el}}}}\left(x - x_{n_{\mathrm{el}}-1}\right) & \text{if } x_{n_{\mathrm{el}}-1} \leq x \leq L \\ 0 & \text{otherwise} \end{cases}$$

These basis functions are continuous, linear inside each subdomain and verify that

$$N_i\left(x_j\right) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

where δ_{ij} is the Krönecker delta. The basis functions allow one to write the FE approximation $\varphi_h(x)$ as

$$\varphi_h(x) = \sum_{i=0}^{n_{\mathrm{el}}} N_i(x)\varphi_i = \mathbf{N}^T(x)\varphi \approx \varphi(x) \tag{2.8.1}$$

where

$$\mathbf{N}(x) := \left[N_0(x), N_1(x), \ldots, N_{n_{\mathrm{el}}}(x)\right]^T$$
$$\varphi := \left[\varphi_0, \varphi_1, \ldots, \varphi_{n_{\mathrm{el}}}\right]^T$$

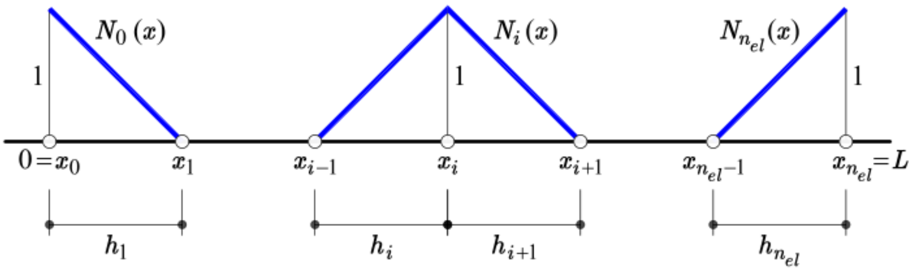

Figure 2.8: basis functions

- Local interpolation is a key aspect of the Finite Element (FE) approximation, enabling localized calculations within each subdomain of the partition. This characteristic significantly enhances the practical implementation and computational efficiency of the method. When dealing with element number i and its nodes $i-1$ and i, a notation distinction is introduced—elements are enclosed in parentheses and marked with a superscript. Nodes within the element are then renumbered as 1 and 2, facilitating the consideration of local interpolation within each subdomain for improved computational efficiency.

$$\varphi_h^{(i)}(x) = \varphi_h(x)|_{[x_{i-1},x_i]} = N_1^{(i)}(x)\varphi_1^{(i)} + N_2^{(i)}(x)\varphi_2^{(i)}, \quad x_{i-1} \le x \le x_i$$

where

$$\begin{aligned} N_1^{(i)}(x) &= N_{i-1}(x)|_{[x_{i-1},x_i]}, & N_2^{(i)}(x) &= N_i(x)|_{[x_{i-1},x_i]} \\ \varphi_1^{(i)} &= \varphi_{i-1}, & \varphi_2^{(i)} &= \varphi_i \end{aligned}$$

Formulation of the discrete problem: Let's consider this scenario where the displacement is anchored at $x=0$, and a load is applied at $x=L$. The extension to different boundary conditions is straightforward. Hence, the problem at hand is to determine a function $\varphi(x)$ in the interval $[0,L]$ that satisfies:

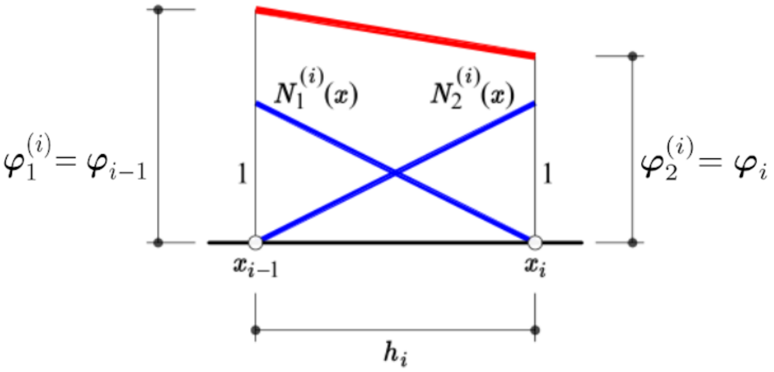

Figure 2.9: basis functions over subdomain (x_{i-1},x_i)

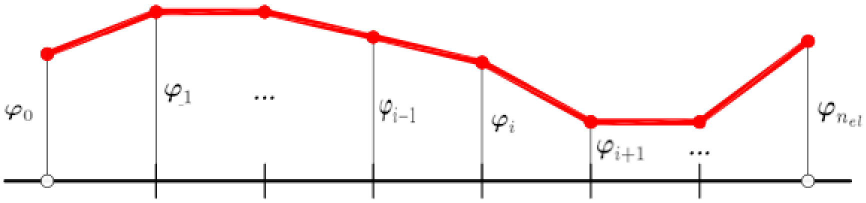

Figure 2.10: The interpolations of φ on all domain

$$-\varphi_{xx} + \varphi_x + \varphi = f \quad \text{for } 0 < x < L \tag{2.8.2}$$

Under the following boundary conditions:

$$\varphi(0) = \varphi_1 \quad \text{at } x = 0$$

$$\varphi_x \Big|_{x=L} = g \quad \text{at } x = L$$

The FE approximation to variational problem

$$\varphi(0) = \bar{\varphi}_1 \tag{2.8.3}$$

$$\int_0^L v_x \varphi_x dx + \int_0^L v\varphi dx = \int_0^L v f\, dx + v(L)g \tag{2.8.4}$$

subject to the constraint: $v(0) = 0$, consists of seeking $\varphi_h(x)$ of the form (2.8.1) that satisfies (2.8.3) and an equation analogous to (2.8.4). Obviously, this equation cannot hold for all virtual displacements of the continuous problem, since $\varphi_h(x)$ would be overdetermined. However, we can impose it for virtual displacements in the FE space, i.e., of the form:

$$v = \delta\varphi_h(x) = \mathbf{N}^T(x)\delta\boldsymbol{\varphi} = \mathbf{N}^T(x)\mathbf{v}. \tag{2.8.5}$$

where

$$\mathbf{v}^T = \delta\boldsymbol{\varphi}^T = [\delta\varphi_0, \delta\varphi_1, \ldots, \delta\varphi_{n_{\text{el}}}] = [v_0, v_1, \ldots, v_{n_{\text{el}}}].$$

Here and below, we have considered a linear interpolation to simplify the notation, but everything carries over to arbitrary orders of interpolation. The boundary condition (2.8.3) forces us to take:

$$\varphi_0 = \bar{\varphi}_1, \quad v_0 = 0.$$

Thus, taking the virtual displacements as indicated by (2.8.5) will lead to n_{el} conditions (not $n_{\text{el}} + 1$, which is the number of nodes) for the n_{el} unknowns contained in $\boldsymbol{\varphi}$. The discrete form of (2.8.4) is then

$$\int_0^L \mathbf{v}^T \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}\frac{\mathrm{d}\mathbf{N}^T}{\mathrm{d}x}\boldsymbol{\varphi}\mathrm{d}x + \int_0^L \mathbf{v}^T\mathbf{N}\mathbf{N}^T\boldsymbol{\varphi}\mathrm{d}x = \int_0^L \mathbf{v}^T\mathbf{N}p\ \mathrm{d}x + \mathbf{v}^T\mathbf{N}(L)g \tag{2.8.6}$$

If we call

$$\mathbf{P} := \int_0^L \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}\frac{\mathrm{d}\mathbf{N}^T}{\mathrm{d}x}\ \mathrm{d}x, \mathbf{M} := \int_0^L \mathbf{N}\mathbf{N}^T\ \mathrm{d}x,$$

$$\mathbf{l} := \int_0^L \mathbf{N}f\ \mathrm{d}x + \mathbf{N}(L)g$$

Equation (2.8.6) can be reformulated as

$$\mathbf{v}^T\left(\mathbf{P}+\mathbf{M}\right)\varphi = \mathbf{v}^T\mathbf{l}. \tag{2.8.7}$$

This scalar equation, applicable to all arrays v, leads to the system $(\mathbf{P}+\mathbf{M})\varphi = \mathbf{l}$, where $\mathbf{P}$ represents the stiffness matrix [2], $\mathbf{M}$ is the mass matrix and $\mathbf{l}$ is the forcing term. Initially indeterminate, the system has dimensions $(n_{el}+1)\times(n_{el}+1)$, with the degree of freedom associated with the rigid body motion of the bar. Setting $v_0 = 0$ eliminates the first row. Additionally, applying the condition $\varphi_0 = \bar{\varphi}_1$ transforms the system into a determinate $n_{el}\times n_{el}$ form. This transformation involves multiplying the first column of $\mathbf{P}+\mathbf{M}$ by $\bar{\varphi}_1$ and adjusting the forcing term $\mathbf{l}$. The notation $(\mathbf{P}+\mathbf{M})\varphi = \mathbf{l}$ is retained, indicating the imposition of any necessary boundary conditions.

Assembly and element calculations: The stiffness and the mass matrices can be expressed as:

$$\mathbf{P} := \int_0^L \mathbf{B}^T\mathbf{B}\,dx \tag{2.8.8}$$

$$\mathbf{M} := \int_0^L \mathbf{N}\mathbf{N}^T\ \mathrm{d}x, \tag{2.8.9}$$

where $\mathbf{B} := \frac{dN^T}{dx}$.

This formulation involves element-wise calculations and assembly, providing a concise representation of the stiffness matrix. Let P_{ij} and M_{ij} represent the components of $\mathbf{P},\mathbf{M}$ before imposing the boundary conditions $(i,j = 0,1,\ldots,n_{el})$, given by:

$$P_{ij} = \int_{x_{i-1}}^{x_i} \frac{dN_i}{dx}\frac{dN_j}{dx}\,dx = \begin{cases} \int_{x_{i-1}}^{x_i} \frac{dN_i}{dx}\frac{dN_{i-1}}{dx}\,dx & \text{if } j = i-1 \\ \int_{x_{i-1}}^{x_i} \frac{dN_i}{dx}\frac{dN_i}{dx}\,dx + D\int_{x_i}^{x_{i+1}} \frac{dN_i}{dx}\frac{dN_i}{dx}\,dx & \text{if } j = i \\ \int_{x_i}^{x_{i+1}} \frac{dN_i}{dx}\frac{dN_{i+1}}{dx}\,dx & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

---

[2] the popular notation in the structural mechanics community for the stiffness matrix is K

$$M_{ij} = \int_{x_{i-1}}^{x_i} N_i N_j \, dx = \begin{cases} \int_{x_{i-1}}^{x_i} N_i N_{i-1} \, dx & \text{if } j = i-1 \\ \int_{x_{i-1}}^{x_i} N_i N_i \, dx + D \int_{x_i}^{x_{i+1}} N_i N_i \, dx & \text{if } j = i \\ \int_{x_i}^{x_{i+1}} N_i N_{i+1} \, dx & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

These expressions define the individual components of the stiffness matrix P before applying any boundary conditions. For each element i (noting that, informally, this corresponds to the subdomain), we define the arrays:

$$\mathbf{N}^{(i)} = [N_1^{(i)}, N_2^{(i)}], \quad \mathbf{B}^{(i)} = \frac{d\mathbf{N}^{(i)T}}{dx},$$

$$\mathbf{P}^{(i)} = \int_{x_{i-1}}^{x_i} \mathbf{B}^{(i)T} \mathbf{B}^{(i)} \, dx \quad \mathbf{M}^{(i)} = \int_{x_{i-1}}^{x_i} \mathbf{N}^{(i)T} \mathbf{N}^{(i)} \, dx$$

These arrays include the shape function array $\mathbf{N}^{(i)}$, the gradient of the shape function array $\mathbf{B}^{(i)}$, and the element stiffness matrix $\mathbf{P}^{(i)}$.

The components of $\mathbf{P}$ can be computed from the components of the 2×2 matrix $\mathbf{P}^{(i)}$ as follows:

$$P_{ij} = \begin{cases} P_{21}^{(i)} & \text{if } j = i-1 \\ P_{22}^{(i)} + P_{11}^{(i+1)} & \text{if } j = i \\ P_{12}^{(i+1)} & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

and for M

$$M_{ij} = \begin{cases} M_{21}^{(i)} & \text{if } j = i-1 \\ M_{22}^{(i)} + P_{11}^{(i+1)} & \text{if } j = i \\ M_{12}^{(i+1)} & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

The operator facilitating the computation of the global stiffness matrix $\mathbf{P}$ from the element stiffness matrices $\mathbf{P}^{(i)}$ is known as the assembly operator, denoted as:

$$\mathbf{P} = \widehat{P}_{i=1}^{n_{el}} \left(\mathbf{P}^{(i)} \right)$$

The assembly operator $\widehat{P}$ involves two main steps: first, the inclusion of the 2×2 element matrices $\mathbf{P}^{(i)}$ into a $(n_{el}+1) \times (n_{el}+1)$ matrix, and second, the summation of the components of the element matrices contributing to the same global degree of freedom(the same process for $\mathbf{M} = \widehat{M}_{i=1}^{n_{el}}\left(\mathbf{M}^{(i)}\right)$. Let's now define:

$$l^{(i)} = \int_{x_{i-1}}^{x_i} N^{(i)} f\, dx$$

The components of the forcing vector f can be computed as:

$$l_i = \int_{x_{i-1}}^{x_i} N_i f\, dx + \int_{x_i}^{x_{i+1}} N_i f\, dx = f_2^{(i)} + f_1^{(i+1)}$$

for $i = 1, 2, \ldots, n_{el} - 1$, with $l_0 = f_1^{(1)}$ and $l_{n_{el}} = f_2^{(n_{el})} + g$. Thus, the assembly operator can also be defined for arrays, not only for matrices. Note that f_0 will not be needed once the boundary conditions are prescribed. The element stiffness ans mass matrices that arise in each case are given by:

$$P^{(i)} = \frac{1}{h_i}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and

$$M^{(i)} = \frac{h_i}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Approximation of linear problems in dimension $\Omega = (0,1)$

Let us consider the space V_h corresponding to the finite element methods given in (2.8.2) ,

$$V_h = \left\{ v \in C^0[0,1] : v(0) = v(1) = 0 \text{ and } v|_{(x_i, x_{i+1})} \in P_k, i \leqslant M \right\}.,$$

where k is a positive integer and P_k is the space of all polynomial functions of degree $\leqslant k$. It is well known that $\mathrm{V_h} \subset \mathrm{W}^{1,\mathrm{P}}(\Omega)$ for $1 \leqslant \mathrm{p} \leqslant +\infty$, and that for all φ in $\mathrm{W}_0^{1,1}(\Omega)$, there exists a unique element $Q_h\varphi \in V_h$ such that :

$$\int_0^1 (Q_h\varphi)'(x) v_h'(x) dx = \int_0^1 \varphi'(x) v_h'(x) dx, \forall v_h \in v_h. \tag{2.8.10}$$

Remark that $Q_h\varphi$ is the orthogonal projection of φ onto V_h, with respect to the inner product $(\varphi, v)_1 = \int_0^1 \varphi'(x) v'(x) dx$ of $H_0^1(\Omega)$, when $\varphi \in H_0^1(\Omega)$, but $Q_h\varphi$ is defined as soon as $\varphi \in W_0^{1,1}(\Omega)$. We shall say that Q_h is the "elliptic projector" on to $\mathrm{V_h}$.

Lemma 2.7. *[46] For all* p *with* $1 \leqslant \mathrm{p} \leqslant \infty$ *and all* φ *in* $\mathrm{w}_\mathrm{o}^{1,\mathrm{p}}(\Omega)$ *the following proprieties hold :*

(a) $Q_h\varphi(x_i) = \varphi(x_i), \quad \forall i = 0, 1, \ldots M+1;$

(b) $\int_{x_i}^{x_{i+1}} (Q_h\varphi)'(x) q'(x) dx = \int_{x_i}^{x_{i+1}} \varphi'(x) q'(x) dx, \forall q \in P_k,\ \forall i = 0, 1 \ldots, M$

(c)

$$|\mathrm{Q_h}\varphi|_{1,\mathrm{p},\Omega} \leqslant \mathrm{c}|\varphi|_{1,\mathrm{p},\Omega}, \tag{2.8.11}$$

where c *is a constant independent of* h *and* φ*. Moreover, if* φ *belongs to* $\mathrm{w}^{\mathrm{r}+1,\mathrm{P}}(\Omega)$*, with* $0 \leqslant \mathrm{r} \leqslant \mathrm{k}$*, we have:*

(d)

$$|\varphi - Q_h\varphi|_{0,p,\Omega} \leqslant Ch^{r+1}|\varphi|_{r+1,p,\Omega}, \tag{2.8.12}$$

and

$$|\varphi - \mathrm{q_h}\varphi|_{1,\mathrm{p},\Omega} \leqslant \mathrm{Ch^r}|\varphi|_{\mathrm{r}+1,\mathrm{p},\Omega}, \tag{2.8.13}$$

where C *is a constant independent of* h *and* φ*.*

Proof. We first remark that $W^{1,P}(\Omega) \subseteq c^0[0,1]$. With each $\varphi \in W_0^{1,P}(\Omega)$, we associate $\mathrm{v_h} \in \mathrm{V_h}$ given by :

$$v_h(x) = \begin{cases} x/x_i & \text{if } 0 \leqslant x \leqslant x_i \\ (1-x)/(1-x_i) & \text{if } x_i \leqslant x \leqslant 1 \end{cases}, i = 1,\ldots,M-1;$$

we derive from (2.8.10)that

$$\left(\frac{1}{x_i} + \frac{1}{1-x_i}\right)(Q_h\varphi - \varphi)(x_i) = 0, \quad i = 12,\ldots,M-1,$$

which proves the statement (a). Thanks to statement (a), we have for all q in P_k

$$\int_{x_i}^{x_{i+1}} (Q_h\varphi - \varphi)'(x)q'(x)dx = -\int_{x_i}^{x_{i+1}} (Q_h\varphi - \varphi)(x)q''(x)dx.$$

Therefore, if q is a polynomial of degree $\leqslant 1$, the assertion (b) is true. For proving the general case, it is sufficient to show that

$$\int_{x_i}^{x_i+1} (Q_h\varphi - \varphi)'(x)q'(x)dx = 0 \tag{2.8.14}$$

for all $q \in P_k$ such that $q(x_i) = q(x_{i+1}) = 0$.

For such a function q, there exists $v_h \in v_h$ with $v_h(x) = q(x)$ for $x_i \leqslant x \leqslant x_{i+1}$ and $v_h(x) = 0$ for $x \notin [x_i, x_{i+1}]$. Taking this function v_h in the definition (2.9) of the operator Q_h we obtain (2.8.14).

Now we prove Statement (c). Assertion (b) allows to write

$$\int_{x_i}^{x_i+1} (Q_h\varphi)'(x)q(x)dx = \int_{x_i}^{x_{i+1}} \varphi'(x)q(x)dx, \forall q \in P_{k-1},$$

which proves that $(Q_h\varphi)' \mid \Omega_i$ is only dependent on $\varphi' \mid \Omega_i$, where $\Omega_i = (x_i, x_{i+1})$. Now let us introduce the mapping Π_{Ω_i} from $L^P(\Omega_i)$ onto P_{k-1} defined by

$$\int_{\Omega_i} (\Pi_{\Omega_i} v)(x)q(x)dx = \int_{\Omega} v(x)q(x)dx, \quad \forall q \in P_{k-1}.$$

Obviously $\pi_{\Omega_i}\left(\varphi'|_{\Omega_i}\right) = Q_h\varphi' \mid \Omega_i$, and, if $p \geqslant 2, \pi_{\Omega_i}$ is the L^2-projection onto $\mathrm{P_{k-1}}$. Consequently we have

$$\|Q_h\varphi'\|_{o,p,\Omega_i} \leqslant c_p \|\varphi'\|_{o,p,\Omega_i}$$

where

$$c_p = \|\Pi_{\Omega_i}\|_{L(L^p(\Omega_i),L^p(\Omega_i))}\,.$$

The constant c_p a priori depends on Ω_i, p and k, but the change of variables $x+s = (x-x_i)/h_i$ allows to prove that $c_p = \|\Pi_{\Omega}\|_{L(L^p(\Omega),L^p(\Omega))}$, that is to say c_p is not dependent on Ω_i. Finally, let us show Statement (d). We first remark that the mapping Π_{Ω_i} satisfies

$$\Pi_{\Omega_i q} = q, \quad \forall q \in P_{r-1}, \quad 1 \leqslant r \leqslant k.$$

Consequently, if we denotes the identity operator of $L^P(\Omega_i)$, we have

$$\left(\varphi' - (Q_h\varphi')\right)|_{\Omega_i} = (I - \Pi_{\Omega_i})\left(\varphi'|_{\Omega_i}{}^{-q}\right) \quad , \quad \forall q \in P_{r-1}.$$

Therefore

$$\left\|\varphi' - (Q_h\varphi)'\right\|_{o,p,\Omega_i} \leqslant (1+c_p)\left\|\varphi' - q\right\|_{o,p}, \Omega_i, \forall q \in P_{r-1},$$

since

$$\|I - \Pi_{\Omega_i}\|_{L(L^p(\Omega_i),L^p(\Omega_i))} \leqslant 1 + c_p.$$

On the other hand, one easily proves that if the $(r+1)^{\text{th}}$ derivatives $\varphi^{(r+1)}$ of φ belongs to $L^p(\Omega_i)$, then

$$\inf_{\mathrm{q \in P_{r-1}}} \left\|\varphi' - q\right\|_{o,p,\Omega_\mathrm{i}} < \mathrm{ch^r} \left\|\varphi^{(\mathrm{r+1})}\right\|_{o,p,\Omega_\mathrm{i}}, \tag{2.8.15}$$

where c is a constant independent of h and φ. Inequality (2.8.13) is a direct consequence of

(2.8.2) and (2.8.15). Note that, for all φ in $\mathrm{w}_0^{1,\mathrm{p}}(\Omega_i)$, the following inequality holds

$$\|\varphi\|_{0,p,\Omega_i} \leqslant \mathrm{h}_i \left\|\varphi'\right\|_{0,p,\Omega_i}.$$

According to Statement (a), $\varphi = Q_h\varphi$ belongs to $w_0^{1,p}(\Omega_i)$, whence

$$\|\varphi - Q_h\varphi\|_{0,p,\Omega_i} \leqslant h \left\|\varphi' - (Q_h\varphi)'\right\|_{0,p,\Omega_i}. \tag{2.8.16}$$

Then (2.8.12) is a direct consequence of (2.8.2), (2.8.15) and (2.8.16). ☐

2.8.3 Heat Equation

Let's consider the problem (2.8.2) with the introduction of the temporal derivative φ_t, i.e., the following model parabolic problem:

$$\begin{cases} \varphi_t(x,t) - \varphi_{xx}(x,t) + \varphi(x,t) = l(x,t), \forall (x,t) \in (0,1) \times [0,T], \\ \varphi(0,t) = \varphi(1,t) = 0, \forall t \in [0,T] \\ \varphi(x,0) = \varphi_0(x), \forall x \in (0,1), \end{cases} \tag{2.8.17}$$

where $l(x,t)$, and $\varphi_0(x)$ are given functions or data. This formulation introduces the temporal derivative and transforms the original problem into a parabolic partial differential equation. The solution $\varphi(x,t)$ models the temperature at each instant $t \in [0,T]$ of a thin and fully insulated wire, where the temperature at the ends is maintained constantly at zero degrees. We will continue using the notation introduced earlier, although the functions are now time-dependent. Thus, if $w(\cdot,t) \in V$, it means that, for each t, $w(\cdot,t)$ is a function defined in Ω. More precisely,

$$\begin{aligned} w :&[0,T] \to V, \\ &t \mapsto w(t) := w(\cdot,t). \end{aligned}$$

If V is a Banach space with the norm $||\cdot||_V$, we define $L^p(0,T;V)$ as the space of functions v such that, for almost every $t \in [0,T]$, $v(t) \in V$, and the real function $t \mapsto ||v(t)||_V$ belongs to $L^p(0,T)$. In this case,

$$||w||_{L^p(0,T;V)} = \left(\int_0^T ||w(t)||_V^p \, dt\right)^{1/p} \tag{2.8.18}$$

defines a norm in $L^p(0,T;V)$. Additionally, if V is Banach with the norm $||\cdot||_V$, then $L^p(0,T;V)$ is also Banach with the norm defined above.

In the context we are addressing, V will be replaced by the spaces $H_0^1(\Omega)$ or $L^2(\Omega)$, where Ω is the interval $(0,1)$, and $p \in [1,\infty)$ or $p = \infty$. For $p = \infty$, the space $L^\infty(0,T;V)$ denotes functions

w such that $||w(t)||_V$ is essentially bounded on $[0,T]$, and in this case, the norm is defined as

$$||w||_{L^\infty(0,T;V)} = \sup_{t\in[0,T]} ||w(t)||_V < \infty.$$

It is worth noting that for a force $f \in L^2(0,T;L^2(0,1))$, for example, the solution of the problem (2.8.17) can be obtained using the method of separation of variables (Fourier method), where the solution $\varphi(x,t)$ is represented by an infinite series. The numerical challenge in this case is the slow convergence of the Fourier series, even when using fast Fourier transform. This is one of the reasons for introducing the finite element method. As before, we define the bilinear form on $H^1(0,1)\times H^1(0,1)$:

$$a(\varphi,v) = \int_0^1 \varphi_x v_x\,dx + \int_0^1 uv\,dx, \tag{2.8.19}$$

associated with the norm:

$$\|\varphi\|_{2,a} = \sqrt{\|\varphi_x\|_0^2 + \|\varphi\|_0^2},$$

we have:

$$\nu_1\|\varphi\|_1^2 \leq \|\varphi\|_{2,a}^2 \leq \nu_2\|\varphi\|_1^2,$$

meaning that the norms $\|.\|_a$ and $\|.\|_1$ are equivalent.

Variational Formulation for the Heat Equation

Let $D(0,1)$ be the space of C^∞ functions with compact support in $(0,1)$, which we refer to as the space of test functions on $(0,1)$. Multiplying the equation (2.8.17) by $v \in D(0,1)$ and integrating over $(0,1)$, we obtain:

$$\int_0^1 \varphi_t(t)v\,dx - \int_0^1 \varphi_{xx}(t)v\,dx + \int_0^1 \varphi(t)v\,dx = \int_0^1 l(t)v\,dx, \quad \forall v \in D(0,1). \tag{2.8.20}$$

By integrating by parts, using (2.8.19), equation (2.8.20) becomes the variational form of (2.8.17), and the problem can be formulated as follows: determine a function $\varphi(t)$ satisfying

$$\begin{cases} (\varphi_t,v) + a(\varphi,v) = (l,v), & \forall v \in D(0,1), \\ (\varphi(0),v) = (\varphi_0,v), & \forall v \in D(0,1), \end{cases} \tag{2.8.21}$$

where, for simplicity, we omit the variable x, meaning $\varphi(t)$ denotes the function $\varphi(\cdot,t)$, and $\varphi(0) = \varphi(\cdot,0)$. Since $D(0,1)$ is dense in $H_0^1(0,1)$, the variational formulation (2.8.21) can be expressed as

$$\begin{cases} (\varphi_t,v) + a(\varphi,v) = (l,v), & \forall v \in H_0^1(0,1), \\ (\varphi(0),v) = (\varphi_0,v), & \forall v \in H_0^1(0,1), \end{cases} \tag{2.8.22}$$

In this case, under certain regularity conditions, it can be shown that a solution of (2.8.22) is a solution of (2.8.17).

Approximated Problem

Let $T > 0$ and $V_m = [w_1, w_2, ..., w_m] \subset V = H_0^1(0,1)$ be the vector subspace generated by the first m elements of the basis of the Hilbert space V (the existence of the basis is a consequence of V being a separable Hilbert space).i.e.,

$$V_m = \{v_h \in V \,|\, v_h|_{I_i} = v_i,\, v_i \in P_1(I_i)\},$$

where $P_1(I_i)$ is the set of first-degree polynomials on I_i, and V is a subspace of $H^1(0,1)$ (depending on the boundary conditions of the problem). We know that every function $\varphi_h(t) \in V_m$ can be written as a linear combination of the basis elements, or more precisely,

$$\varphi_m(t) = \sum_{i=1}^{m} d_i(t) w_i(x). \tag{2.8.23}$$

We can then establish the approximate problem by restricting (2.8.21) to the subspace V_m, transforming the original problem into a system of ordinary differential equations. More precisely,

Approximated Problem:

$$\begin{aligned} &\text{For } m \in \mathbb{N}, \text{ determine a function} \\ &\varphi_m : [0,T] \to V_m, \\ &\quad t \mapsto \varphi_m(t) \in V_m, \end{aligned}$$

solution of the following approximated equation:

$$\begin{cases} (\varphi_m'(t), v_m) + a(\varphi_m(t), v_m) = (l(t), v_m), \quad \forall v \in V_m, \\ \varphi_m(0) = \varphi_{0m} \in V_m. \end{cases} \tag{2.8.24}$$

For the solution of problem (2.8.24) (with $m \in \mathbb{N}$ fixed) to approximate the original problem, we need to choose the initial condition in such a way that the sequence $\{\varphi_{0m}\}_m$ converges to u_0 in V. In particular, we can choose this sequence as follows:

$$\varphi_{0m} = \sum_{i=1}^{m} (\varphi_0, w_i) w_i. \tag{2.8.25}$$

Note that φ_{0m} defined in this way is the orthogonal projection of φ_0 onto the subspace V_m, ensuring that we have, necessarily,

$$\lim_{m\to\infty} \varphi_{0m} = \sum_{i=1}^{\infty} (\varphi_0, w_i) w_i = \varphi_0 \text{ in } V.$$

As we will see later, the sequence $\{\varphi_m\}_{m\in\mathbb{N}}$ formed by the solutions of the approximate problems converges to a solution φ of the problem (2.8.21). However, our current interest is in developing methods for numerically solving the approximate problem (2.8.24).

Substituting (2.8.23) into (2.8.24), we obtain, from the properties of the inner product and the bilinear form a,

$$\sum_{i=1}^{m} d_i'(t)(w_i, v_m) + \sum_{i=1}^{m} d_i(t) a(w_i, v_m) = (l(t), v_m), \quad \forall v_m \in V. \tag{2.8.26}$$

$$\sum_{i=1}^{m} d_i'(t)(w_i, w_j) + \sum_{i=1}^{m} d_i(t) a(w_i, w_j) = (l(t), w_j).$$

Defining the symmetric matrices A and B, whose coefficients are defined respectively as:

$$A_{ij} = (w_i, w_j) \quad \text{and} \quad B_{ij} = a(w_i, w_j),$$

we can rewrite the equation in matrix form as:

$$\mathbf{d}'(t) \cdot \mathbf{A} + \mathbf{d}(t) \cdot \mathbf{B} = \mathbf{l}(t),$$

where $\mathbf{d}'(t)$ and $\mathbf{d}(t)$ are vectors with components $d_i'(t)$ and $d_i(t)$, $\mathbf{A}$ is the matrix with elements A_{ij}, $\mathbf{B}$ is the matrix with elements B_{ij}, and $\mathbf{l}(t)$ is a vector with components $(l(t), w_j)$.

$$A_{ij} = (w_i, w_j) = \int_0^1 w_i(x) w_j(x)\, dx,$$

$$B_{ij} = a(w_i, w_j) = \alpha \int_0^1 w_i'(x) w_j'(x)\, dx + \beta \int_0^1 w_i(x) w_j(x)\, dx,$$

This leads to the following system of m ordinary differential equations:

$$\sum_{i=1}^{m} d_i'(t) A_{ij} + \sum_{i=1}^{m} d_i(t) B_{ij} = L_j(t), \quad \text{for } j = 1, \ldots, m,$$

where $F(t)$ is the vector with components $L_j(t) = (l(t), w_j) = \int_0^1 l(x,t) w_j(x)\,dx$ for $j = 1, \ldots, m$. This system can be expressed in matrix form as:

$$\begin{cases} \mathbf{A}\mathbf{d}'(t) + \mathbf{B}\mathbf{d}(t) = \mathbf{L}(t), & \text{for } t \in (0,T) \\ \mathbf{d}(0) = \begin{bmatrix} (\varphi_0, w_1) \\ (\varphi_0, w_2) \\ \vdots \\ (\varphi_0, w_m) \end{bmatrix} = \mathbf{d}_0, \end{cases}$$

where $\mathbf{d}(0)$ is the initial condition, and $\mathbf{d}(t) = (d_1(t), d_2(t), \ldots, d_m(t))^T$ is the unknown vector. In the context of heat diffusion, matrix $\mathbf{A}$ is called the capacitance matrix, matrix $\mathbf{B}$ is the conductivity matrix, and $\mathbf{L}(t)$ is a heat source. Note that $\mathbf{A}$ is the same mass matrix M defined in (2.8.8) and $\mathbf{B}$ is the same stiffness matrix plus mass matrix $P+M$ defined in (2.8.8)and (2.8.9).

Furthermore, the matrices $\mathbf{A}$ and $\mathbf{B}$ are symmetric and positive definite. In particular, as matrix $\mathbf{A}$ is a Gram matrix for the linearly independent vectors $w_1, \ldots, w_m$, $\mathbf{A}$ is invertible. Thus, we can write the system of ordinary differential equations in the form

$$\mathbf{d}'(t) + \mathbf{A}^{-1}\mathbf{B}\mathbf{d}(t) = \mathbf{A}^{-1}\mathbf{L}(t), \quad \forall t > 0,$$

with $\mathbf{d}(0) = \mathbf{d}_0$, and therefore, it has a unique solution $\mathbf{d}(t)$ for $t \in [0,T]$. Thus, by (2.8.23), the approximate solution $\varphi_m(x,t)$ of the problem (2.8.24) can be calculated. The solution $\mathbf{d}(t)$ can be expressed in terms of the matrix exponential of $\mathbf{A}^{-1}\mathbf{B}$, which is diagonalizable. However, in practice, a numerically approximate solution is more suitable. Let's return to the system of ordinary differential equations (2.8.3), derived from the finite element method for spatial approximation. This system will then be solved for discrete times t_n, using the finite difference method. To achieve this, we consider the system of m ordinary differential equations at discrete times t_n, where $t_n = n\Delta t$ for $n = 1, 2, \ldots, N$. Thus, we have

$$\begin{cases} \mathbf{A}\mathbf{d}'(t_n) + \mathbf{B}\mathbf{d}(t_n) = \mathbf{F}(t_n), \\ \mathbf{d}(0) = \mathbf{d}_0. \end{cases}$$

Semi-discrete problems

We will consider error estimates in Sobolev spaces between the exact solution $\varphi(x,t)$ and the approximate solution $\varphi_h(x,t)$, analogous to the error analysis of the stationary problem but without discretization of the variable t. As we have seen earlier, the basis functions w_i of the subspace V_m of approximate solutions are polynomials of degree k defined on each finite element

Ω. For $k = 1,2,3$, we have linear, quadratic, and cubic spline basis functions, respectively. To explicitly consider the dependence on the degree k of the polynomial, we write the FE space V_m as:

$$V_k^m(\Omega) := \{v_h \in V;\, v_e^h \in P_k(\Omega_e)\},$$

where v_e^h denotes the restriction of v_h to the element e, and P_k is the set of polynomials defined in Ω_e with a degree less than or equal to k in the variable x.

The semi-discrete problem is formulated as follows: Ddetermine $\varphi_h : [0,T] \to V_m$, a solution of the system

$$(\varphi_h'(t), v_h) + a(\varphi_h(t), v_h) = (L(t), v_h), \quad \forall v_h \in V_k^m, \tag{2.8.27}$$

$$(\varphi_h(0), v_h) = (\varphi_{0h}, v_h). \tag{2.8.28}$$

Fully discrete problems

Backward Euler Method From the backward difference in time, we have

$$(d_i'(t))_{t=t_n} \approx \frac{1}{\Delta t}(d_{n,i} - d_{n-1,i}),$$

where each function d_i, as we know, depends only on the time variable t. Substituting this approximation into the system (6.13), we obtain

$$\frac{1}{\Delta t}A(d_n - d_{n-1}) + Bd_n = F_n, \quad n = 1,2,\ldots,N,$$

or equivalently

$$(A + \Delta t B)d_n = \Delta t F_n + Ad_{n-1} = b_n, \quad n = 1,2,\ldots,N.$$

The independent vector $b_n = (b_{n1}, b_{n2}, \ldots, b_{nm})^T$ is defined since A, B, F, and the time increment Δt are known, and furthermore, matrices A and B are independent of time.

Note that the Euler method is also an implicit method.(Trivially it is explicit if matrices A and B are diagonal, i.e., the basis vectors w_i form an orthogonal basis)

The unknown vector $d_n = (d_{n1}, d_{n2}, \ldots, d_{nm})^T$ can be uniquely determined because the matrix $A + \Delta t B$ is symmetric and positive definite, and thus nonsingular. Hence, the system can be solved for each fixed n using, for example, direct methods such as Cholesky (only when the matrix is symmetric and positive definite), Crout (LDL^T) for a symmetric matrix, or the Gauss elimination method.

Note that the sparsity pattern of the matrix depends directly on the choice of the basis $w_i(x)$. For example, if $w_i(x)$ is the linear or cubic basis, the matrix $A + \Delta t B$ will be tridiagonal or pentadiagonal, respectively.

Chapter 3

Bresse system with Green–Naghdi thermoelasticity

3.1 Introduction

3.1.1 Literature Overview

Initially, mathematicians and engineers focused mostly on classical thermoelasticity, in which the heat flux is determined by Fourier's law, this classical theory predicts an infinite speed of heat propagation. As a result, the classic thermoelasticity leads to an unrealistic property that a sudden disturbance will be sensed instantly somewhere else in the materials at some time. In an attempt to avoid this problem, Green and Naghdi presented three forms of thermoelastic theories in the 1990s, called thermoelasticity of types I, II, and III. These theories are based on entropy equality rather than the usual entropy inequality. Whereas type I is the same as the classical type, in type II the heat is permitted to move by thermal waves, also known as thermoelasticity without dissipation. Types I and II are thermoelasticity type III 's limiting cases (for more information, see [9]). In the context of the thermoelastic theories, several studies have been published on the thermoelastic Bresse system we highlight some of them briefly. To our best knowledge, the first study related to the well-posedness and stability of thermoelastic Bresse systems was proposed by Liu and Rao in [47], where they studied a system with two thermal dissipations given by the Fourier's law (type I thermoelasticity), acting in the shear and longitudinal motions, respectively. They considered the following system:

$$\begin{aligned}
&\rho_1\varphi_{tt}-k\left(\varphi_x+\psi+lw\right)_x-k_0 l\left(w_x-l\varphi\right)+\gamma l\theta=0\\
&\rho_2\psi_{tt}-b\psi_{xx}+k\left(\varphi_x+\psi+lw\right)+\gamma\Theta_x=0\\
&\rho_1 w_{tt}-k_0\left(w_x-l\varphi\right)_x+kl\left(\varphi_x+\psi+lw\right)+\gamma\theta_x=0\\
&\rho_3\theta_t-\kappa\theta_{xx}+m\left(w_x-l\varphi\right)_t=0\\
&\rho_3\Theta_t-\kappa\Theta_{xx}+m\psi_{xt}=0.
\end{aligned} \tag{3.1.1}$$

The authors proved that the system is exponentially stable if and only if $\lambda=0$ where

$$\lambda=\frac{\rho_1}{\rho_2}-\frac{k}{b}, \tag{3.1.2}$$

with two different boundary conditions, Dirichlet conditions for all functions or

$$\varphi=\psi_x=w_x=\theta=\Theta=0,\ \text{on}\ (t,x)\in(0,T)\times\{0,\ell\}.$$

They also proved that the system is polynomially stable with rates of decay that depends on the boundary conditions, these results were extended to many other Bresse systems with different kinds of dissipation and thermal dampings like memory or frictional dampings. One of them was given by Fatori et al. [48] where they considered the system (3.1.1) without θ and the fourth equation, they arrived to similar result as in [47]. The resultant system is,

$$\begin{aligned}&\rho_1\varphi_{tt}-k(\varphi_x+\psi+lw)_x-k_0l(w_x-l\varphi)=0\\&\rho_2\psi_{tt}-b\psi_{xx}+k(\varphi_x+\psi+lw)+\gamma\Theta_x=0\\&\rho_1w_{tt}-k_0(w_x-l\varphi)_x+kl(\varphi_x+\psi+lw)=0\\&\rho_3\Theta_t-\kappa\ \Theta_{xx}+m\psi_{xt}=0,\end{aligned}\tag{3.1.3}$$

In the case of thermal memory dampings, the same system (3.1.3) studied by Dell'Oro [49] where the term $\kappa\Theta_{xx}$ replaced by $\kappa\int_0^\infty g(s)\Theta_{xx}(x,t-s)ds$, for the exponential stability a necessary and sufficient condition

$$\chi_g=\left(\frac{\rho_1}{k\rho_3}-\frac{1}{g(0)\kappa}\right)\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)-\frac{1}{g(0)\kappa}\frac{\rho_1\gamma^2}{\rho_3bk},$$

was established for $\chi_g=0$.
Recently, Mukiawa et al. [50] considered the system (3.1.3) with viscoelastic damping acting on the shear force, i.e,

$$Q=k(\varphi_x+\psi+lw)-\int_0^t g(t-s)(\varphi_x+\psi+lw)(x,s)ds,$$

showing that with weaker conditions on the relaxation function g and physical parameters $(k=k_0)$, the solution energy has general and optimal decay rates. In [51] the authors investigate a type III thermoelasticity acting on shear force where the heat conduction is given by Green and Naghdi,

$$\begin{aligned}&\rho_1\varphi_{tt}-k(\varphi_x+\psi+lw)_x-k_0l(w_x-l\varphi)+\kappa\theta_{xt}=0,\\&\rho_2\psi_{tt}-b\psi_{xx}+k(\varphi_x+\psi+lw)-\kappa\theta_t=0,\\&\rho_1w_{tt}-k_0(w_x-l\varphi)_x+kl(\varphi_x+\psi+lw)+\kappa l\theta_t=0,\\&\rho_3\theta_{tt}-\delta\theta_{xx}-\gamma\theta_{txx}+\kappa(\varphi_x+\psi+lw)_t=0,\end{aligned}\tag{3.1.4}$$

and they showed that the system (3.1.4) is exponentially stable if $\lambda=0$ and $k=k_0$. Otherwise, the system will be polynomially stable with optimal rate. Keddi et al. [52] considered the one-dimensional thermoelastic Bresse system and studied the exponential and polynomial stability, where the heat conduction is given by Cattaneo's law effective in the shear angle displacements.

Alabau-Boussouira et al. [53] established the exponential stability of the Bresse systems with frictional dissipation working only on the angle displacement and they found some numerical result to verify their analytical results. Guesmia and Kirane [54] proved two decay estimates of the Bresse systems depending on the speeds of wave propagations with two infinite memories acting only on two equations. We refer the reader to [55–58], for some other results on Bresse system.

Additionally, we mention some numerical results. Santos and Almeida in [59] considered the Bresse system with frictional dissipative terms (i.e., the the first three equation of (3.1.16) without ϑ and $\left(F^1, F^2, F^3\right) = (-\sigma_1 \varphi_t, -\sigma_2 \psi_t, -\sigma_3 w_t)$. They introduced a fully discrete approximations based on the finite difference method element in time and space and showed the numerical exponential decay for system as well as same the numerical computations. For more results on the numerical study of the Bresse systems, we refer the reader to [60–64] .

3.1.2 Model derivation

In recent years, there is a growing interest in the modeling of thermal-mechanical behavior in curved beams. Notably, Silva al. 2023, in [65] conducted an extensive analysis, providing a comprehensive framework for the derivation of the Bresse governing model for arched beams with thermal couplings. Their work delves into the intricate interplay between axial load, shear force, and bending moment, while also considering the nuanced effects of temperature variations. Furthermore, their study contributes significantly by introducing new categories of problems that highlight specific thermal features. Building upon this foundation, the present study aims to further explore and expand upon the understanding of thermal-mechanical phenomena in curved beam systems. For those seeking a deeper understanding of this subject matter, I recommend delving into the aforementioned paper.

Presently, we examine a one-dimensional linear thermoelastic Bresse system wherein heat conduction influences the axial force. Let us begin with the following system[1]

$$\rho_1 \ddot{\boldsymbol{\varphi}} = \bar{Q}_x + l \aleph + F^1, \tag{3.1.5}$$

$$\rho_2 \ddot{\boldsymbol{\psi}} = \bar{M}_x - \bar{Q} + F^2, \tag{3.1.6}$$

$$\rho_1 \ddot{w} = \aleph_x - l\bar{Q} + F^3, \tag{3.1.7}$$

$$\rho_3 \dot{\vartheta} = -q_x - k_0 l \left(\dot{w}_x - l \dot{\boldsymbol{\varphi}}\right), \tag{3.1.8}$$

where the stress-strain relations for thermoelastic behavior are the axial force $\aleph$, the shear force $\bar{Q}$ and the bending moment $\bar{M}$. The functions $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$, w are the vertical, the shear angle and the longitudinal displacements, respectively, ϑ is the empirical temperature and F^1, F^2, F^3 denote

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[1] In literature, the common notations used are: $\boldsymbol{\varphi} := \varphi$, $w := w$, and $\boldsymbol{\psi} := \psi$.

external forces. We use the following the constitutive laws:

$$\begin{aligned} &\aleph = k_0\left(w_x - l\varphi\right) - k_0 l \vartheta, \\ &\bar{Q} = k\left(\varphi_x + \psi + lw\right), \\ &\bar{M} = b\psi_x. \end{aligned} \tag{3.1.9}$$

We consider the heat conduction from the Green and Naghdi theories (see Chapter 2 and [66]-[67]), known as type III thermoelasticity. It is given by the following equations:

$$\begin{aligned} &q = -\left(\kappa\alpha + \tau\dot{\alpha}\right)_x \\ &\vartheta = \dot{\alpha}. \end{aligned} \tag{3.1.10}$$

Here $\rho_1, \rho_2, \rho_3, k, k_0, l, b, \kappa, \tau$ are positive constant coefficients and α is the thermal displacement magnitude.
By inserting (3.1.10) into (3.1.8) and derivating with respect to t, we get

$$\rho_3\ddot{\vartheta} = \kappa\vartheta_{xx} + \tau\dot{\vartheta}_{xx} - k_0 l\left(\ddot{w}_x - l\ddot{\varphi}\right), \tag{3.1.11}$$

equation (3.1.11) may be viewed as a wave equation with a viscous damping term $\dot{\vartheta}_{xx}$. Under these settings, the system (3.1.5) $-$(3.1.8) can be written as

$$\left\{\begin{array}{ll} \rho_1\ddot{\varphi} - k\left(\varphi_x + \psi + lw\right)_x - k_0 l\left(w_x - l\varphi\right) + k_0 l^2\vartheta = F^1, & \text{in } (0,1)\times\mathbb{R}_+, \\ \rho_2\ddot{\psi} - b\psi_{xx} + k\left(\varphi_x + \psi + lw\right) = F^2, & \text{in } (0,1)\times\mathbb{R}_+, \\ \rho_1\ddot{w} - k_0\left(w_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + lw\right) + k_0 l\vartheta_x = F^3, & \text{in } (0,1)\times\mathbb{R}_+, \\ \rho_3\ddot{\vartheta} - \kappa\vartheta_{xx} - \tau\dot{\vartheta}_{xx} + k_0 l\left(\ddot{w}_x - l\ddot{\varphi}\right) = 0, & \text{in } (0,1)\times\mathbb{R}_+. \end{array}\right. \tag{3.1.12}$$

with initial and boundary conditions given by

$$\left\{\begin{array}{l} \varphi(x,0) = \varphi^0(x),\ \psi(x,0) = \psi^0(x),\ w(x,0) = w^0(x), \\ \vartheta(x,0) = \vartheta^0(x),\ \dot{\varphi}(x,0) = \varphi^1(x),\ \dot{\psi}(x,0) = \psi^1(x), \\ \dot{w}(x,0) = w^1(x),\ \dot{\vartheta}(x,0) = \vartheta^1(x),\ \forall x \in (0,1). \\ \varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = 0, \\ w_x(0,t) = w_x(1,t) = \vartheta(0,t) = \vartheta(1,t) = 0,\ \forall t \geq 0. \end{array}\right. \tag{3.1.13}$$

It is convenient to introduce the new function θ (see [68]), in order to exhibit the dissipative nature of system (3.1.12)–(3.1.13)

$$\theta(x,t) := \int_0^t \vartheta(x,s)ds + \nu(x), \tag{3.1.14}$$

where $\nu \in H^1(0,1)$ satisfies

$$\kappa \nu_{xx} = \rho_3 \vartheta^1 - \tau \vartheta^0_{xx} + k_0 l \left(w^1_x - l\varphi^1 \right). \tag{3.1.15}$$

Then, by using (3.1.14), (3.1.15) and taking $\left(F^1, F^2, F^3\right) = (-\sigma\dot{\varphi}, \alpha\dot{\psi}_{xx}, 0)$, the starting system (3.1.12)–(3.1.13) is transformed to

$$\begin{cases} \rho_1 \ddot{\varphi} - k\left(\varphi_x + \psi + lw\right)_x - k_0 l \left(w_x - l\varphi\right) + k_0 l^2 \dot{\theta} + \sigma\dot{\varphi} = 0, & \text{in } (0,1)\times\mathbb{R}_+, \\ \rho_2 \ddot{\psi} - b\psi_{xx} + k\left(\varphi_x + \psi + lw\right) - \alpha\dot{\psi}_{xx} = 0, & \text{in } (0,1)\times\mathbb{R}_+, \\ \rho_1 \ddot{w} - k_0 \left(w_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + lw\right) + k_0 l \dot{\theta}_x = 0, & \text{in } (0,1)\times\mathbb{R}_+, \\ \rho_3 \ddot{\theta} - \kappa\theta_{xx} - \tau\dot{\theta}_{xx} + k_0 l \left(\dot{w}_x - l\dot{\varphi}\right) = 0, & \text{in } (0,1)\times\mathbb{R}_+. \end{cases} \tag{3.1.16}$$

with initial conditions and boundary conditions given by

$$\begin{cases} \left(\varphi, \psi, w, \theta\right)(x,0) = \left(\varphi^0, \psi^0, w^0, \theta^0\right), \ \forall x \in (0,1), \\ \left(\dot{\varphi}, \dot{\psi}, \dot{w}, \dot{\theta}\right)(x,0) = \left(\varphi^1, \psi^1, w^1, \theta^1\right), \ \forall x \in (0,1), \\ \varphi = \theta = \psi_x = w_x = 0, x = \{0,1\}, \, t > 0. \end{cases} \tag{3.1.17}$$

where

$$\left(\theta^1(x), \theta^0(x)\right) := \left(\vartheta^0(x), \nu(x)\right).$$

Remark 3.1. *By setting* $\bar{y}(t) = \int_0^1 \psi(x,t)dx$, $\bar{z}(t) = \int_0^1 w(x,t)dx$ *and integrating on* $(0,1)$ *the 2st and 3th equations of the system* (3.1.16)*, and using the boundary conditions, we obtain*

$$\ddot{\bar{y}} = -\frac{k}{\rho_2}\bar{y} - \frac{kl}{\rho_2}\bar{z}, \tag{3.1.18}$$

and

$$\ddot{\bar{z}} = -\frac{kl^2}{\rho_1}\bar{z} - \frac{kl}{\rho_1}\bar{y}, \tag{3.1.19}$$

Therefore, (3.1.18) *implies that*

$$\bar{z} = -\frac{\rho_2}{kl}\ddot{\bar{y}} - \frac{1}{l}\bar{y}, \tag{3.1.20}$$

Substituting (3.1.20) *into* (3.1.19)*, we get the following ordinary differential equation*

$$\ddddot{\bar{y}}(t) + \left(\frac{k}{\rho_2} + \frac{kl^2}{\rho_1}\right)\ddot{\bar{y}}(t) = 0, \tag{3.1.21}$$

with

$$\begin{cases} \bar{y}(0) = y_0 \; \dot{\bar{y}}(0) = y_1, \\ \ddot{\bar{y}}(0) = y_2 \quad \dddot{\bar{y}}(0) = y_3 \end{cases} \tag{3.1.22}$$

where

$$\begin{cases} y_0 = \int_0^1 \psi^0 dx, \;\; y_1 = \int_0^1 \psi^1 dx \\ y_2 = -\frac{k}{\rho_2} y_0 - \frac{kl}{\rho_2} z_0 \\ y_3 = -\frac{k}{\rho_2} y_1 - \frac{kl}{\rho_2} z_1 \\ z_0 = \int_0^1 w^0 dx, \;\; z_1 = \int_0^1 w^1 dx, \end{cases}$$

Let $l_0 = \sqrt{\frac{k}{\rho_2} + \frac{kl^2}{\rho_1}}$*. Then, solving* (3.1.21)-(3.1.22)*, we find the solution*

$$\bar{y}(t) = C_1 + C_2 t + C_3 \cos(t l_0) + C_4 \sin(t l_0), \tag{3.1.23}$$

where C_i *are real constants*

$$\begin{cases} C_1 = y_0 + \frac{1}{l_0^2} y_2 \\ C_2 = y_1 + \frac{1}{l_0^2} y_3 \\ C_3 = -\frac{1}{l_0^2} y_2, \;\; C_4 = -\frac{1}{l_0^3} y_3. \end{cases}$$

Using (3.1.20) *to get*

$$\bar{z}(t) = \left(\frac{\rho_2 l_0^2}{kl} - \frac{1}{l}\right) C_3 \cos(t l_0) + \left(\frac{\rho_2 l_0^2}{kl} - \frac{1}{l}\right) C_4 \sin(t l_0) - \frac{C_1}{l} - \frac{C_2}{l} t. \tag{3.1.24}$$

Let

$$\widetilde{\psi}(x,t) = \psi(x,t) - \bar{y}(t) \tag{3.1.25}$$

and

$$\widetilde{w}(x,t) = w(x,t) - \bar{z}(t) \tag{3.1.26}$$

Then, from (3.1.23) *and* (3.1.24)*, we can check that*

$$\int_0^1 \widetilde{w} dx = \int_0^1 \widetilde{\psi} dx = 0$$

Therefore, Poincare's inequality is applicable for $\widetilde{w}$, $\widetilde{\psi}$*. So from now on we work with* $\widetilde{w}, \widetilde{\psi}$ *instead of* w, ψ *respectively, but, for simplicity of notation, we use* w, ψ*. i.e.,*

$$\exists c_0 > 0 : \|u\|_{L^2(0,1)} \leq c_0 \|u_x\|_{L^2(0,1)}, \quad \forall u \in H_*^1(0,1) \cup H_0^1(0,1),$$

such that

$$H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1),$$
$$L_*^2(0,1) = \left\{ u \in L^2(0,1) : \int_0^1 u(x)\, dx = 0 \right\}.$$

In addition, $(\widetilde{w}, \widetilde{\psi})$ *the initial data*

$$\psi_0 - (C_1 + C_3), \quad \psi_1 - (C_2 + C_4 l_0), \; w_0 - \left(\left(\frac{\rho_2 l_0^2}{kl} - \frac{1}{l} \right) C_3 - \frac{C_1}{l} \right),$$
$$w_1 - \left(C_4 l_0 \left(\frac{\rho_2 l_0^2}{kl} - \frac{1}{l} \right) - \frac{C_2}{l} \right).$$

In Section 3.2, we study the existence and uniqueness of solutions for the system (3.1.16)-(3.1.17) using the semigroup theory. In Section 3.3, we prove the exponential stability of the system (3.1.16)-(3.1.17) using the multipliers method. In Section 3.4, we propose a numerical scheme based on finite elements in space and finite difference in time of this problem. In addition, we obtain the decay of the discrete energy; then we perform a priori estimate analysis, which leads to the convergence of the scheme. Finally, some numerical simulations are obtained using MATLAB software.

3.2 Existence and Uniqueness

In this section, we briefly summarize the existence and uniqueness result for problem (3.1.16)-(3.1.17) using the semigroup theory. For more details, we refer the reader to [69–71]. We will use the following standard $L^2(0,L)$ space with the scalar product and norm denoted by

$$\langle u, v \rangle_{L^2} = \int_0^1 u.v dx, \; \|u\|_{L^2}^2 = \int_0^1 u^2 dx,$$

respectively. So, if we denote $U = (\varphi, \widehat{\varphi}, \psi, \widehat{\psi}, w, \widehat{w}, \theta, \widehat{\theta})^T$, where $\widehat{\varphi} = \dot{\varphi}$, $\widehat{\psi} = \dot{\psi}$, $\widehat{w} = \dot{w}$ and $\widehat{\theta} = \dot{\theta}$. Then, system (3.1.16)-(3.1.17) can be rewritten as follows:

$$\begin{cases} \dot{U} + \mathscr{A} U = 0, \; t > 0, \\ U(x,0) = U_0(x) = (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, \theta_0, \theta_1)^T, \end{cases}$$

where the operator $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \longrightarrow \mathscr{H}$ is defined by

$$\mathscr{A} U = \begin{pmatrix} -\widehat{\varphi} \\ -\frac{k}{\rho_1} (\varphi_x + \psi + lw)_x - \frac{k_0 l}{\rho_1} (w_x - l\varphi) + \frac{k_0 l^2}{\rho_1} \dot{\theta} + \frac{\sigma}{\rho_1} \dot{\varphi} \\ -\widehat{\psi} \\ -\frac{b}{\rho_2} \psi_{xx} + \frac{k}{\rho_2} (\varphi_x + \psi + lw) - \frac{\alpha}{\rho_2} \dot{\psi}_{xx} \\ - \widehat{w} \\ -\frac{k_0}{\rho_1} (w_x - l\varphi)_x + \frac{kl}{\rho_1} (\varphi_x + \psi + lw) + \frac{k_0 l}{\rho_1} \dot{\theta}_x \\ -\widehat{\theta} \\ -\frac{\kappa}{\rho_3} \theta_{xx} - \frac{\tau}{\rho_3} \dot{\theta}_{xx} + \frac{k_0 l}{\rho_3} (\dot{w}_x - l\dot{\varphi}) \end{pmatrix}, \tag{3.2.1}$$

such that $\mathscr{H}$ is the energy space given by

$$\mathscr{H}=H_0^1(0,1)\times L^2(0,1)\times H_*^1(0,1)\times L_*^2(0,1)\times H_*^1(0,1)\times L_*^2(0,1)\times H_0^1(0,1)\times L^2(0,1)$$

For any $U=(\varphi,\widehat{\varphi},\psi,\widehat{\psi},w,\widehat{w},\theta,\widehat{\theta})^T\in\mathscr{H}$, $\tilde{U}=(\varkappa,\widehat{\varkappa},\zeta,\widehat{\zeta},\omega,\widehat{\omega},\bar{\vartheta},\widehat{\bar{\vartheta}})^T\in\mathscr{H}$, we equip $\mathscr{H}$ with the inner product defined by

$$\begin{aligned}\left\langle U,\tilde{U}\right\rangle_{\mathscr{H}}=&\rho_1\int_0^1\widehat{\varphi}\widehat{\varkappa}dx+\rho_2\int_0^1\widehat{\psi}\widehat{\zeta}dx+\rho_1\int_0^1\widehat{w}\widehat{\omega}dx+\rho_3\int_0^1\widehat{\theta}\widehat{\bar{\vartheta}}dx+b\int_0^1\psi_x\zeta_xdx\\&+\kappa\int_0^1\theta_x\bar{\vartheta}_xdx+k\int_0^1(\varphi_x+\psi+lw)(\varkappa_x+\zeta+l\omega)dx\\&+k_0\int_0^1(w_x-l\varphi)(\omega_x-l\varkappa)dx.\end{aligned}\tag{3.2.2}$$

The domain of $\mathscr{A}$ for the system (3.1.16)-(3.1.17) is given by:

$$\begin{aligned}D(\mathscr{A})=\Big\{&U\in\mathscr{H}\mid\varphi\in H^2(0,1);\ \widehat{\varphi},\widehat{\theta},\theta\in H_0^1(0,1);\\&w\in H_*^2(0,1);\psi,\ \widehat{\psi},\widehat{w}\in H_*^1(0,1);\kappa\theta+\tau\widehat{\theta}\ \in H^2(0,1);\\&b\psi+\alpha\widehat{\psi}\in H_*^2(0,1)\Big\},\end{aligned}$$

where

$$H_*^2(0,1)=\left\{\Psi\in H^2(0,1):\Psi_x(0)=\Psi_x(1)=0\right\}.$$

Lemma 3.1. *The operator A is monotone and satisfies, for any $U\in D(\mathscr{A})$,*

$$\langle\mathscr{A}U,U\rangle_{\mathscr{H}}=\tau\left\|\widehat{\theta}_x\right\|^2+\sigma\left\|\widehat{\varphi}\right\|^2+\alpha\left\|\widehat{\psi}_x\right\|^2\geq 0.\tag{3.2.3}$$

Proof. From inner product (3.2.2), we have

$$\begin{aligned}\langle\mathscr{A}U,U\rangle_{\mathscr{H}}=&\rho_1\int_0^1\widehat{\varphi}\dot{\widehat{\varphi}}dx+\rho_2\int_0^1\widehat{\psi}\dot{\widehat{\psi}}dx+\rho_1\int_0^1\widehat{w}\dot{\widehat{w}}dx\\&+\kappa\int_0^1\theta_x\widehat{\theta}_xdx+\rho_3\int_0^1\widehat{\theta}\dot{\widehat{\theta}}dx+b\int_0^1\psi_x\widehat{\psi}_xdx\\&+k_0\int_0^1(w_x-l\varphi)(\widehat{w}_x-l\widehat{\varphi})dx\\&+k\int_0^1(\varphi_x+\psi+lw)(\widehat{\varphi}_x+\widehat{\psi}+l\widehat{w})dx,\end{aligned}$$

so

$$\langle\mathscr{A}U,U\rangle_{\mathscr{H}}=\frac{d}{2dt}(\rho_1\int_0^1\widehat{\varphi}^2dx+\rho_2\int_0^1\widehat{\psi}^2dx+\rho_1\int_0^1\widehat{w}^2dx+\rho_3\int_0^1\widehat{\theta}^2dx+b\int_0^1\psi_x^2dx$$

$$+\kappa\int_0^1 \theta_x^2 dx + k\int_0^1 (\varphi_x+\psi+lw)^2 x + k_0\int_0^1 (w_x - l\varphi)^2 dx),$$

we get

$$\langle \mathscr{A}U,U\rangle_{\mathscr{H}} = \tau\left\|\widehat{\theta}_x\right\|^2 + \sigma\left\|\widehat{\varphi}\right\|^2 + \alpha\left\|\widehat{\psi}_x\right\|^2 \geq 0.$$

We now prove that $\mathscr{A}$ is a maximal monotone operator. For this purpose we need the following two lemmas. □

Lemma 3.2. *The operator $I+\mathscr{A}$ is surjective.*

Proof. The dissipativity of $-\mathscr{A}$ is proved by (3.2.3) and clearly, $D(\mathscr{A})$ is dense in $\mathscr{H}$. It remains to prove that

$$U+\mathscr{A}U = F,\ \forall F\in\mathscr{H}, \tag{3.2.4}$$

where $F = \left(f^1,f^2,f^3,f^4,f^5,f^6,f^7,f^8\right)^T \in \mathscr{H}$, has a unique solution $U\in D(\mathscr{A})$.

From (4.2.6) we have

$$\begin{cases} \varphi-\widehat{\varphi} = f^1 \in H_0^1(0,1) \\ \rho_1\widehat{\varphi} - k(\varphi_x+\psi+lw)_x - k_0 l(w_x - l\varphi) + k_0 l^2\dot{\theta} + \sigma_1\dot{\varphi} = \rho_1 f^2 \in L^2(0,1) \\ \psi-\widehat{\psi} = f^3 \in H_*^1(0,1) \\ \rho_2\widehat{\psi} - b\psi_{xx} + k(\varphi_x+\psi+lw) - \sigma_2\dot{\psi}_{xx} = \rho_2 f^4 \in L_*^2(0,1) \\ w-\ \widehat{w}\ = f^5 \in H_*^1(0,1) \\ \rho_1\widehat{w} - k_0(w_x - l\varphi)_x + kl(\varphi_x+\psi+lw) + k_0 l\dot{\theta}_x \ \ = \rho_1 f^6 \in L_*^2(0,1) \\ \theta-\widehat{\theta}\ = f^7 \in H_0^1(0,1) \\ \rho_3\widehat{\theta} - \kappa\theta_{xx} - \tau\dot{\theta}_{xx} + k_0 l(\dot{w}_x - l\dot{\varphi}) = \rho_3 f^8 \in L^2(0,1), \end{cases} \quad , \tag{3.2.5}$$

Inserting $\widehat{\varphi} = \varphi - f^1$, $\widehat{\psi} = \psi - f^3$, $\widehat{w} = w - f^5$, $\widehat{\theta} = \theta - f^7$ we get

$$\begin{cases} (\rho_1+\sigma_1)\varphi - k(\varphi_x+\psi+lw)_x - k_0 l(w_x - l\varphi) + k_0 l^2\theta = g_1 \\ \rho_2\psi - (b+\sigma_2)\psi_{xx} + k(\varphi_x+\psi+lw) = g_2 \\ \rho_1 w - k_0(w_x - l\varphi)_x + kl(\varphi_x+\psi+lw) + k_0 l\theta_x = g_3 \\ \rho_3\theta - (\kappa+\tau)\theta_{xx} + k_0 l(w_x - l\varphi) = g_4 + g_5, \end{cases} \tag{3.2.6}$$

where

$$\begin{cases} g_1 = \rho_1 f^2 + (\rho_1+\sigma_1) f^1 + k_0 l^2 f^7 \\ g_2 = \rho_2 f^4 + \rho_2 f^3 - \sigma_2 f_{xx}^3 \\ g_3 = \rho_1 f^6 + \rho_1 f^5 + k_0 l f_x^7 \\ g_4 = \rho_3 f^8 + \rho_3 f^7 + k_0 l\left(f_x^5 - f^1\right) \\ g_5 = \tau f_x^7 \end{cases}$$

To solve (3.2.6), we consider the following variational formulation

$$B((\varphi,\psi,w,\theta),(\overline{\varphi},\overline{\psi},\overline{w},\overline{\theta}))=G\left(\overline{\varphi},\overline{\psi},\overline{w},\overline{\theta}\right)\quad\forall\left(\overline{\varphi},\overline{\psi},\overline{w},\overline{\theta}\right)\in V$$

where $V:=\left[H_0^1(0,1)\times H_*^1(0,1)\times H_*^1(0,1)\times H_0^1(0,1)\right]$, $B:V\times V\to\mathbb{R}$ is the bilinear form defined by

$$\begin{aligned}B((\varphi,\psi,w,\theta),(\overline{\varphi},\overline{\psi},\overline{w},\overline{\theta}))=&k\int_0^1(\varphi_x+\psi+lw)(\overline{\varphi}_x+\overline{\psi}+l\overline{w})\,dx\\&+k_0\int_0^1(w_x-l\varphi)(\overline{w}_x-l\overline{\varphi})\,dx+(\rho_1+\sigma_1)\int_0^1\varphi\overline{\varphi}dx\\&+\rho_2\int_0^1\psi\overline{\psi}dx+(b+\sigma_2)\int_0^1\psi_x\overline{\psi}_xdx+\rho_3\int_0^1\theta\overline{\theta}dx\\&+(\kappa+\tau)\int_0^1\theta_x\overline{\theta}_xdx+k_0l^2\int_0^1\theta\overline{\varphi}-\varphi\overline{\theta}dx\\&+k_0l\int_0^1\theta_x\overline{w}-w\overline{\theta}_xdx\end{aligned}\tag{3.2.7}$$

and the Linear form $G:V\to\mathbb{R}$ defined by

$$G\left(\overline{\varphi},\overline{\psi},\overline{w},\overline{\theta}\right)=\int_0^1g_1\overline{\varphi}dx+\int_0^1g_2\overline{\psi}dx+\int_0^1g_3\overline{w}dx+\int_0^1g_4\overline{\theta}dx+\int_0^1g_{5,x}\overline{\theta}_xdx.\tag{3.2.8}$$

It is clear that V is Hilbert space equipped with the norm

$$\|(\varphi,\psi,w,\theta)\|_V^2=\|\varphi_x+\psi+lw\|_{L^2}^2+\|w_x-l\varphi\|_{L^2}^2+\|\psi_x\|_{L^2}^2+\|\theta\|_{L^2}^2+\|\theta_x\|_{L^2}^2,$$

we can easily show, using Cauchy–Schwarz inequality, that B and G are continuous.

$$B((\varphi,\psi,w,\theta),(\overline{\varphi},\overline{\psi},\overline{w},\overline{\theta}))\leq\xi_1\|(\varphi,\psi,w,\theta)\|_V\left\|\left(\overline{\varphi},\overline{\psi},\overline{w},\overline{\theta}\right)\right\|_V$$

Similarly

$$G\left(\overline{\varphi},\overline{\psi},\overline{w},\overline{\theta}\right)\leq\xi_1\left\|\left(\overline{\varphi},\overline{\psi},\overline{w},\overline{\theta}\right)\right\|_V$$

On the other hand,

$$\begin{aligned}B((\varphi,\psi,w,\theta),(\overline{\varphi},\overline{\psi},\overline{w},\overline{\theta}))=&k\int_0^1(\varphi_x+\psi+lw)^2dx+k_0\int_0^1(w_x-l\varphi)^2dx\\&+(\rho_1+\sigma_1)\int_0^1\varphi^2dx+\sigma_2\int_0^1\psi^2dx\\&+(b+\sigma_2)\int_0^1\psi_x^2dx+\rho_3\int_0^1\theta^2dx+(\kappa+\tau)\int_0^1\theta_x^2dx\end{aligned}$$

$$\geq c \left\| \left(\overline{\varphi}, \overline{\psi}, \overline{w}, \overline{\theta}\right) \right\|_V$$

for some $c > 0$. Hence, B is coercive. Consequently, Lax–Milgram lemma guarantees the existence of a unique solution $(\varphi, \psi, w, \theta)$ in V satisfying (3.2.6).

Hence, there exists a unique $U \in D(A)$ satisfies (4.2.6). Finally, using Lumer-Phillips theorem we deduce that A is an infinitesimal generator of a contraction semigroup in $\mathscr{H}$, and this completes the proof. □

Now, we can give the the following well-posedness result.

Theorem 3.1. *Let $U_0 \in \mathscr{H}$, then there exists a unique solution $U \in C(\mathbb{R}_+, \mathscr{H})$ of problem* (3.1.16)-(3.1.17). *Moreover, if $U_0 \in D(\mathscr{A})$, then*

$$U \in C(\mathbb{R}_+, D(\mathscr{A})) \cap C^1(\mathbb{R}_+, \mathscr{H}).$$

3.3 Exponential stability

In this section, we use energy method to prove that system (3.1.16)-(3.1.17) is exponentially stable.

Theorem 3.2. *Let $(\varphi, \psi, \theta, w)$ be a solution of the problem determined by system* (3.1.16), *initial conditions and boundary conditions* (3.1.17). *Then, there exist two positive constants λ_1, λ_2 such that*

$$E(t) \leq \lambda_1 E(0) e^{-\lambda_2 t}, \ \forall t \geq 0. \tag{3.3.1}$$

First, we state and prove some technical lemmas needed for the proof of our result.

Lemma 3.3. *Let $(\varphi, \psi, \theta, w)$ be a solution of* (3.1.16)-(3.1.17). *Then, the energy functional $E(t)$, defined by*

$$\begin{aligned} E(t) = \frac{1}{2} \int_0^1 \Big[& \rho_1 \dot{\varphi}^2 + \rho_2 \dot{\psi}^2 + \rho_1^2 \dot{w} + \rho_3 \dot{\theta}^2 + b \psi_x^2 + \kappa \theta_x^2 + k (\varphi_x + \psi + l w)^2 \\ & + k_0 (w_x - l \varphi)^2 \Big] dx, \ t > 0, \end{aligned} \tag{3.3.2}$$

satisfies

$$\frac{d}{dt} E(t) = -\sigma \int_0^1 \dot{\varphi}^2 dx - \tau \int_0^1 \dot{\theta}_x^2 dx - \alpha \int_0^1 \dot{\psi}_x^2 dx \leq 0, \ t > 0 \tag{3.3.3}$$

Proof. Multiplying $(3.1.16)_1$, $(3.1.16)_2$, $(3.1.16)_3$, $(3.1.16)_4$ by $\dot{\varphi}$, $\dot{\psi}$, $\dot{w}$, $\dot{\theta}$ respectively, integrating by parts over $(0,1)$, using the boundary conditions and summing them up, we obtain (3.3.3). □

Lemma 3.4. *Let* (φ,ψ,w,θ) *be a solution of* (3.1.16) – (3.1.17), *then, for any* $\varepsilon_1>0$ *the functional* Y_1 *defined by*

$$Y_1(t)=-\rho_1\rho_3\int_0^1\dot\theta\left(\int_0^x\dot w(y)dy\right)dx-\rho_1\kappa\int_0^1 w\theta_x dx,\ t>0,\tag{3.3.4}$$

satisfies , the estimate

$$\begin{aligned}Y_1'(t)\le&-\frac{\rho_1k_0l}{2}\int_0^1\dot w^2dx+c_0\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1\dot\theta_x^2dx+\int_0^1\dot\varphi^2dx\\&+\varepsilon_1\int_0^1(w_x-l\varphi)^2dx+\varepsilon_1\int_0^1\psi_x^2dx\\&+\varepsilon_1\int_0^1(\varphi_x+\psi+lw)^2dx,\ t>0,\end{aligned}\tag{3.3.5}$$

Proof. By differentiating the functional $Y_1(t)$ and integrating by parts, we have

$$\begin{aligned}Y_1'(t)=&-\rho_1k_0l\int_0^1\dot w^2dx+k_0\rho_3\int_0^1\dot\theta(w_x-l\varphi)dx-k_0l^2\rho_1\int_0^1\dot\varphi\left(\int_0^x\dot w(y)dy\right)dx\\&-\rho_3k_0l\int_0^1\dot\theta^2dx+\rho_3kl\int_0^1\dot\theta\left(\int_0^x(\varphi_x+\psi+lw)(y)dy\right)dx\\&+\rho_1\tau\int_0^1\dot\theta_x\dot wdx-\rho_1\kappa\int_0^1 w\dot\theta_xdx.\end{aligned}$$

Using Young's inequality and Poincaré's inequalities with

$$\int_0^1 w^2dx\le\int_0^1 w_x^2dx\le c_0\left(\int_0^1(w_x-l\varphi)^2dx+\int_0^1(\varphi_x+\psi+lw)^2dx+\int_0^1\psi_x^2dx\right),$$

we obtain (3.3.5). □

Lemma 3.5. *Let* (φ,ψ,w,θ) *be a solution of* (3.1.16) – (3.1.17). *Then, the functional* Y_2 *defined by*

$$Y_2(t)=\int_0^1\left(\rho_2\dot\psi\psi dx+\frac{\alpha}{2}(\psi_x)^2\right)dx+\int_0^1\left(\rho_1\dot\varphi\varphi+\frac{\sigma}{2}\varphi^2\right)dx+\rho_1\int_0^1\dot wwdx,\quad t>0,\tag{3.3.6}$$

satifies

$$\begin{aligned}Y_2'(t)\le&-k\int_0^1(\varphi_x+\psi+lw)^2dx-\frac{k_0}{2}\int_0^1(w_x-l\varphi)^2dx\\&-b\int_0^1\psi_x^2dx+\rho_2c_0\int_0^1\dot\psi_x^2dx+\rho_1\int_0^1\dot\varphi^2dx\\&+\rho_1\int_0^1\dot w^2dx+c_0\int_0^1\dot\theta_x^2dx.\end{aligned}\tag{3.3.7}$$

Proof. By differentiating the Y_2, we get

$$Y_2'(t) = -b\int_0^1 \psi_x^2 dx - k\int_0^1 (\varphi_x+\psi+lw)^2 dx - k_0\int_0^1 (w_x - l\varphi)^2 dx$$
$$+k_0 l\int_0^1 \dot{\theta}(w_x - l\varphi)\,dx\ +\rho_1\int_0^1 \dot{\varphi}^2 dx + \rho_2\int_0^1 \dot{\psi}^2 dx + \rho_2\int_0^1 \dot{w}^2 dx.$$

Using, Young's and Poincaré inequalities, we arrive at (3.3.7). ☐

Lemma 3.6. *Let* (φ,ψ,w,θ) *be a solution of* (3.1.16) – (3.1.17). *Then, the functional* Y_3 *defined by*

$$Y_3(t) = \int_0^1 \left(\rho_3\dot{\theta}\theta + \frac{\tau}{2}(\theta_x)^2\right)dx + k_0 l\int_0^1 (w_x - l\varphi)\,\theta dx,\ t>0,$$

satisfties

$$Y_3'(t) \le -\kappa\int_0^1 \theta_x^2 dx + c_0\int_0^1 \dot{\theta}_x^2 dx + c_0\int_0^1 (w_x - l\varphi)^2 dx,\ t>0 \tag{3.3.8}$$

Proof. By differentiating the functional $Y_3(t)$, we get

$$Y_3'(t) = -\kappa\int_0^1 \theta_x^2 dx + \rho_3\int_0^1 \dot{\theta}^2 dx + kl\int_0^1 (w_x - l\varphi)\,\dot{\theta} dx.$$

By using Young and Poincaré inequalities, we arrive at (3.3.8). ☐

We are now ready to state and prove the following exponential stability result. We define for N, N_1 and $N_2>0$ the following functional

$$\mathscr{L}(t) = NE(t) + N_1Y_1(t) + N_2Y_2(t) + Y_3(t). \tag{3.3.9}$$

Lemma 3.7. *Let* (φ,ψ,w,θ) *be a solution of* (3.1.16) – (3.1.17) *Then, there exist two positive constants* β, μ_1 *and* μ_2, *such that the Lyapunov functional* (3.3.9) *satisfies*

$$\mu_1E(t) \le \mathscr{L}(t) \le \mu_2E(t),\ \forall t\ge 0, \tag{3.3.10}$$

and

$$\mathscr{L}'(t) \le -\beta E(t). \tag{3.3.11}$$

Proof. By using Young's, Cauchy–Schwarz, Poincaré's inequalities and from (3.3.9), we get

$$|\mathscr{L}(t) - NE(t)| \le cE(t),$$

then

$$(N-c)E(t) \le \mathscr{L}(t) \le (N+c)E(t),\ \forall t\ge 0,$$

choosing N sufficiently large, we obtain (3.3.10).
By the previous lemmas we get for $t>0$,

$$\begin{aligned}\mathscr{L}'(t) \leq & -[\sigma N - N_1 - \rho_1 N_2]\int_0^1 \dot{\varphi}^2 dx - [kN_2 - \varepsilon_1 N_1]\int_0^1 (\varphi_x + \psi + lw)^2 dx \\ & - \left[\frac{k_0}{2}N_2 - \varepsilon_1 N_1 - c_0\right]\int_0^1 (w_x - l\varphi)^2 dx - [bN_2 - \varepsilon_1 N_1]\int_0^1 \dot{\psi}^2 dx \\ & - \left[\tau N - c_0\left(1+\frac{1}{\varepsilon_1}\right)N_1 - N_2 c_0 - c_0\right]\int_0^1 \dot{\theta}_x^2 dx - \kappa\int_0^1 \theta_x^2 dx \\ & - \left[\frac{\rho_1 k_0 l}{2}N_1 - \rho_1 N_2\right]\int_0^1 \dot{w}^2 dx - [N_2 b - \varepsilon_1 N_1]\int_0^1 \psi_x^2 dx. \end{aligned} \tag{3.3.12}$$

Now, by setting $\varepsilon_1 = \frac{1}{N_1}$, we arrive at

$$\begin{aligned}\mathscr{L}'(t) \leq & -[\sigma N - N_1 - \rho_1 N_2]\int_0^1 \dot{\varphi}^2 dx - [kN_2 - c_0]\int_0^1 (\varphi_x + \psi + lw)^2 dx \\ & - \left[\frac{k}{2}N_2 - c_0\right]\int_0^1 (w_x - l\varphi)^2 dx - [bN_2 - c_0]\int_0^1 \dot{\psi}^2 dx \\ & - [\tau N - c_0(1+N_1)N_1 - N_2 c_0 - c_0]\int_0^1 \dot{\theta}_x^2 dx - \kappa\int_0^1 \theta_x^2 dx \\ & - \left[\frac{\rho_1 k_0 l}{2}N_1 - \rho_1 N_2\right]\int_0^1 \dot{w}^2 dx - [N_2 b - c_0]\int_0^1 \psi_x^2 dx. \end{aligned} \tag{3.3.13}$$

Now, all these terms (on the right-hand side of (3.3.13)) become negative if we select our parameters appropriately. First, we choose N_2 large enough so that

$$N_2 > c_0 \max\left(\frac{1}{b}, \frac{1}{k}\right),$$

and N_1 large enough so that

$$N_1 > \frac{2}{k_0 l}N_2.$$

Finally, we select N large enough so that

$$\begin{cases} \tau N - c_0(1+N_1)N_1 - N_2 c_0 - c_0 > 0, \\ \sigma N - N_1 - \rho_1 N_2 > 0, \\ \alpha N - \rho_2 c_0 N_2 > 0. \end{cases}$$

So, we arrive at

$$\mathscr{L}'(t) < -\beta E(t), \; t > 0.$$

Having in mind the remark on the equivalence of $E(t)$ and $\mathscr{L}(t)$, we infer that for some positive constant d

$$\mathscr{L}'(t) \leq -\frac{d}{\mu_2}\mathscr{L}(t),\ t>0, \tag{3.3.14}$$

A simple integration of (3.3.14) gives

$$\mathscr{L}(t) \leq \mathscr{L}(0)e^{-\gamma t},\ t>0, \tag{3.3.15}$$

where $\gamma = \frac{d}{\mu_2} > 0$, which yields the desired result (3.3.1) by using the other side of the equivalence relation again. □

3.4 The Discrete Problem

In this section, we consider a numerical approximation proposed by Tébou [72] based on the auxiliary problem that includes artificial viscosity, for $\varepsilon_1, \varepsilon_2 \in (0,1)$,

$$\begin{cases} \rho_1\ddot{\varphi} - k(\varphi_x+\psi+lw)_x - k_0 l(w_x - l\varphi) + k_0 l^2\dot{\theta} + \sigma\dot{\varphi} \\ -\varepsilon_1 k(\dot{\varphi}_x + \dot{\psi} + l\dot{w})_x - \varepsilon_2 k_0 l(\dot{w}_x - l\dot{\varphi}) = 0, \\ \\ \rho_2\ddot{\psi} - b\psi_{xx} + k(\varphi_x+\psi+lw) - \alpha\dot{\psi}_{xx} + \varepsilon_1 k(\dot{\varphi}_x + \dot{\psi} + l\dot{w}) = 0, \\ \\ \rho_1\ddot{w} - k_0(w_x - l\varphi)_x - k_0\varepsilon_2(\dot{w}_x - l\dot{\varphi})_x + kl(\varphi_x+\psi+lw) \\ +\varepsilon_1 kl(\dot{\varphi}_x + \dot{\psi} + l\dot{w}) + k_0 l\dot{\theta}_x = 0, \\ \\ \rho_3\ddot{\theta} - \kappa\theta_{xx} - \tau\dot{\theta}_{xx} + k_0 l(\dot{w}_x - l\dot{\varphi}) = 0, \end{cases} \tag{3.4.1}$$

with the same initial and boundary conditions as (3.1.17). Then we introduce a discretization by finite elements in space and backward Euler scheme in time.

Remark 3.2. *Regarding the existence and uniqueness of solutions to the auxiliary system* (3.4.1)-(3.1.17) *it is quite similar to the exact system.*

Problem VP: To obtain the week formulation, we multiply the system (3.4.1)-(3.1.17) by test functions $z,\, r \in H_0^1(0,1)$ and $\ell,\, p \in H_*^1(0,1)$, then we integrate by parts to obtain

$$\begin{cases} \rho_1(\dot{\widehat{\varphi}},z)+k(\varphi_x+\psi+lw,z_x)+\varepsilon_1 k(\widehat{\varphi}_x+\widehat{\psi}+l\widehat{w},z_x) \\ -k_0 l(w_x-l\varphi,z)-\varepsilon_2 k_0 l(\widehat{w}_x-l\widehat{\varphi},z)+k_0 l^2(\widehat{\theta},z)+\sigma(\widehat{\varphi},z)=0, \\ \\ \rho_2(\dot{\widehat{\psi}},l)+b(\psi_x,l_x)+\alpha(\widehat{\psi}_x,l_x)+k(\varphi_x+\psi+lw,l) \\ +\varepsilon_1 k(\widehat{\varphi}_x+\widehat{\psi}+l\widehat{w},l)=0, \\ \\ \rho_1(\dot{\widehat{w}},p)+k_0(w_x-l\varphi,p_x)+\varepsilon_2 k_0(\widehat{w}_x-l\widehat{\varphi},p_x) \\ +kl(\varphi_x+\psi+lw,p)+\varepsilon_1 kl(\widehat{\varphi}_x+\widehat{\psi}+l\widehat{w},p)+k_0 l(\widehat{\theta}_x,p)=0, \\ \\ \rho_3(\dot{\widehat{\theta}},r)+\kappa(\theta_x,r_x)+\tau(\widehat{\theta}_x,r_x)+k_0 l(\widehat{w}_x-l\widehat{\varphi},r)=0, \end{cases} \tag{3.4.2}$$

where $(.,.)$ is the inner product in $L_2(0,1)$. We consider a uniform partition of the space interval $\Omega=(0,1)$ into subintervals $\Omega_i=(x_{i-1},x_i)$, $i=1,...,J$, of length $h=\frac{L}{J}$ with $0=x_0<x_1<\cdot\cdot\cdot<x_J=1$ and define the finite element spaces

$$S_h=\left\{\psi\in H_*^1(0,1)\,\middle|\,\psi\in C\left(\overline{\Omega}\right),\ \psi|_{\Omega_i}\text{ is a linear polynomial}\right\},$$

$$Y_h=\left\{\psi\in H_0^1(0,1)\,\middle|\,\psi\in C\left(\overline{\Omega}\right),\ \psi|_{\Omega_i}\text{ is a linear polynomial}\right\}.$$

For the partition of the time interval $[0,T]$, denoted by $0=t_0<t_1<\cdot\cdot\cdot<t_N=T$, we use a uniform partition with step size $\Delta t=\frac{T}{N}$ and nodes $t_n=n\Delta t$, $n=0,1,...,N$. Using the backward Euler scheme, the fully discrete approximations are considered as follows. Next the numerical approximation to the variational problems (3.4.2).

Problem VP^h: Finding $(\widehat{\varphi}_h^n,\widehat{\psi}_h^n,\widehat{w}_h^n,\widehat{\theta}_h^n)\in Y_h\times(S_h)^2\times Y_h$, $n=1,...,N$, such that for all $(z_h,l_h,p_h,r_h)\in Y_h\times(S_h)^2\times Y_h$,

$$\frac{\rho_1}{\Delta t}(\widehat{\varphi}_h^n-\widehat{\varphi}_h^{n-1},z_h)+k(\varphi_{hx}^n+\psi_h^n+w_h^n,z_{hx})+\varepsilon_1 k(\widehat{\varphi}_{hx}^n+\widehat{\psi}_h^n+l\widehat{w}_h^n,z_{hx})$$

$$-k_0 l(w_{hx}^n-l\varphi_h^n,z_h)-\varepsilon_2 k_0 l(\widehat{w}_{hx}^n-l\widehat{\varphi}_h^n,z_h)+k_0 l^2(\widehat{\theta}_h^n,z_h)+\sigma(\widehat{\varphi}_h^n,z_h)=0, \tag{3.4.3}$$

$$\frac{\rho_2}{\Delta t}(\widehat{\psi}_h^n-\widehat{\psi}_h^{n-1},l_h)+b(\psi_{hx}^n,l_{hx})+\alpha(\widehat{\psi}_{hx}^n,l_{hx})$$

$$+k(\varphi_{hx}^n+\psi_h^n+lw_h^n,l_h)+\varepsilon_1 k(\widehat{\varphi}_{hx}^n+\widehat{\psi}_h^n+l\widehat{w}_h^n,l_h)=0, \tag{3.4.4}$$

$$\frac{\rho_1}{\Delta t}(\widehat{w}_h^n-\widehat{w}_h^{n-1},p_h)+k_0(w_{hx}^n-l\varphi_h^n,p_{hx})+\varepsilon_2 k_0(\widehat{w}_{hx}^n-l\widehat{\varphi}_h^n,p_{hx})$$

$$+kl(\varphi_{hx}^n+\psi_h^n+lw_h^n,p_h)+\varepsilon_1 kl(\widehat{\varphi}_{hx}^n+\widehat{\psi}_h^n+l\widehat{w}_h^n,p_h)+k_0 l(\widehat{\theta}_{hx}^n,p_h)=0, \tag{3.4.5}$$

$$\frac{\rho_3}{\Delta t}(\widehat{\theta}_h^n-\widehat{\theta}_h^{n-1},r_h)+\kappa(\theta_{hx}^n,r_{hx})+\tau(\widehat{\theta}_{hx}^n,r_{hx})+k_0 l(\widehat{w}_{hx}^n-l\widehat{\varphi}_h^n,r_h)=0, \tag{3.4.6}$$

where φ_h^n, ψ_h^n, w_h^n, θ_h^n are recovered from the relations

$$\begin{aligned} \varphi_h^n &= \Delta t \widehat{\varphi}_h^n + \varphi_h^{n-1}, \quad & \psi_h^n &= \Delta t \widehat{\psi}_h^n + \psi_h^{n-1}, \\ w_h^n &= \Delta t \widehat{w}_h^n + w_h^{n-1}, \quad & \theta_h^n &= \Delta t \widehat{\theta}_h^n + \theta_h^{n-1}, \end{aligned} \tag{3.4.7}$$

and $\widehat{\varphi}_h^n$ is the approximation to the velocity $\widehat{\varphi}(t_n) = \dot{\varphi}(t_n)$ and $\widehat{\psi}_h^n$ to $\widehat{\psi}(t_n) = \dot{\psi}(t_n)$. Here, φ_h^0, $\widehat{\varphi}_h^0$, ψ_h^0, $\widehat{\psi}_h^0$, w_h^0, $\widehat{w}_h^0$, θ_h^0, $\widehat{\theta}_h^0$ are approximations to the initial data φ^0, $\widehat{\varphi}^0$, ψ^0, $\widehat{\psi}^0$,w^0, $\widehat{w}^0$, θ^0 and $\widehat{\theta}^0$ respectively.

Let us introduce the discrete energy for $n = 0,...,N$.

$$\begin{aligned} E^n = \frac{1}{2}(&\rho_1 \|\widehat{\varphi}_h^n\|^2 + \rho_2 \|\widehat{\psi}_h^n\|^2 + \rho_1 \|\widehat{w}_h^n\|^2 + \rho_3 \left\|\widehat{\theta}_h^n\right\|^2 + b\|\psi_{hx}^n\|^2 \\ &+ \kappa\|\theta_{hx}^n\|^2 + k\|\varphi_{hx}^n + \psi_h^n + l w_h^n\|^2 + k_0\|w_{hx}^n - l\varphi_h^n\|^2), \end{aligned} \tag{3.4.8}$$

where$\|.\|$denotes $\|.\|_{L^2(0,1)}$. The decay of the energy is presented in the following theorem.

Theorem 3.3. *The discrete energy E^n defined by* (4.6.5) *satisfies*

$$E^n \leq E^{n-1}, \; n = 1,...,N.$$

Proof. Choosing $z_h = \widehat{\varphi}_h^n$, $l_h = \widehat{\psi}_h^n$, $p_h = \widehat{w}_h^n$ and $r_h = \widehat{\theta}_h^n$ in (3.4.3)-(3.4.6) to get

$$\begin{aligned} &\frac{\rho_1}{\Delta t}(\widehat{\varphi}_h^n - \widehat{\varphi}_h^{n-1}, \widehat{\varphi}_h^n) + k(\varphi_{hx}^n + \psi_h^n + w_h^n, \widehat{\varphi}_{hx}^n) + \varepsilon_1 k(\widehat{\varphi}_{hx}^n + \widehat{\psi}_h^n + l\widehat{w}_h^n, \widehat{\varphi}_{hx}^n) \\ &-k_0 l(w_{hx}^n - l\varphi_h^n, \widehat{\varphi}_h^n) - \varepsilon_2 k_0 l(\widehat{w}_{hx}^n - l\widehat{\varphi}_h^n, \widehat{\varphi}_h^n) + k_0 l^2(\widehat{\theta}_h^n, \widehat{\varphi}_h^n) + \sigma\|\widehat{\varphi}_h^n\|^2 = 0, \end{aligned}$$

$$\begin{aligned} &\frac{\rho_2}{\Delta t}(\widehat{\psi}_h^n - \widehat{\psi}_h^{n-1}, \widehat{\psi}_h^n) + b(\psi_{hx}^n, \widehat{\psi}_{hx}^n) + \alpha\|\widehat{\psi}_{hx}^n\|^2 \\ &+k(\varphi_{hx}^n + \psi_h^n + l w_h^n, \widehat{\psi}_h^n) + \varepsilon_1 k(\widehat{\varphi}_{hx}^n + \widehat{\psi}_h^n + l\widehat{w}_h^n, \widehat{\psi}_h^n) = 0, \end{aligned}$$

$$\begin{aligned} &\frac{\rho_1}{\Delta t}(\widehat{w}_h^n - \widehat{w}_h^{n-1}, \widehat{w}_h^n) + k_0(w_{hx}^n - l\varphi_h^n, \widehat{w}_{hx}^n) + \varepsilon_2 k_0(\widehat{w}_{hx}^n - l\widehat{\varphi}_h^n, \widehat{w}_{hx}^n) \\ &+kl(\varphi_{hx}^n + \psi_h^n + l w_h^n, \widehat{w}_h^n) + \varepsilon_1 kl(\widehat{\varphi}_{hx}^n + \widehat{\psi}_h^n + l\widehat{w}_h^n, \widehat{w}_h^n) + k_0 l(\widehat{\theta}_{hx}^n, \widehat{w}_h^n) = 0, \end{aligned}$$

$$\frac{\rho_3}{\Delta t}(\widehat{\theta}_h^n - \widehat{\theta}_h^{n-1}, \widehat{\theta}_h^n) + \kappa(\theta_{hx}^n, \widehat{\theta}_{hx}^n) + \tau\left\|\widehat{\theta}_{hx}^n\right\| + k_0 l(\widehat{w}_{hx}^n - l\widehat{\varphi}_h^n, \widehat{\theta}_h^n) = 0,$$

then adding the four equations and using $(x-y,x) = \frac{1}{2}\left(\|x-y\|^2 + \|x\|^2 - \|y\|^2\right)$ and (4.6.4), we get

$$\frac{\rho_1}{2\Delta t}\|\widehat{\varphi}_h^n\|^2 + \frac{\rho_2}{2\Delta t}\|\widehat{\psi}_h^n\|^2 + \frac{\rho_1}{2\Delta t}\|\widehat{w}_h^n\|^2 + \frac{\rho_3}{\Delta t}\left\|\widehat{\theta}_h^n\right\|^2 + \frac{b}{2\Delta t}\|\psi_{hx}^n\|^2$$

$$
\begin{aligned}
&+\frac{\kappa}{2\Delta t}\left\|\theta_{hx}^{n}\right\|^{2}+\frac{k}{2\Delta t}\left\|\varphi_{hx}^{n}+\psi_{h}^{n}+w_{h}^{n}\right\|^{2}+\frac{k_0}{2\Delta t}\left\|w_{hx}^{n}-l\varphi_{h}^{n}\right\|^{2}\\
&-\frac{\rho_1}{2\Delta t}\left\|\widehat{\varphi}_{h}^{n-1}\right\|^{2}-\frac{\rho_2}{2\Delta t}\left\|\widehat{\psi}_{h}^{n-1}\right\|^{2}-\frac{\rho_1}{2\Delta t}\left\|\widehat{w}_{h}^{n-1}\right\|^{2}-\frac{\rho_3}{\Delta t}\left\|\widehat{\theta}_{h}^{n-1}\right\|^{2}-\frac{b}{2\Delta t}\left\|\psi_{hx}^{n-1}\right\|^{2}\\
&-\frac{\kappa}{2\Delta t}\left\|\theta_{hx}^{n-1}\right\|^{2}-\frac{k}{2\Delta t}\left\|\varphi_{hx}^{n-1}+\psi_{h}^{n-1}+w_{h}^{n-1}\right\|^{2}-\frac{k_0}{2\Delta t}\left\|\left(w_{hx}^{n-1}-l\varphi_{h}^{n-1}\right)\right\|^{2}\\
&=-\tau\left\|\widehat{\theta}_{hx}^{n}\right\|^{2}-\sigma\left\|\widehat{\varphi}_{h}^{n}\right\|^{2}-\alpha\left\|\widehat{\psi}_{hx}^{n}\right\|^{2}-\frac{\rho_1}{2\Delta t}\left\|\widehat{w}_{h}^{n}-\widehat{w}_{h}^{n-1}\right\|^{2}-\frac{\rho_3}{2\Delta t}\left\|\widehat{\theta}_{h}^{n}-\widehat{\theta}_{h}^{n-1}\right\|^{2}\\
&-\frac{\kappa}{2\Delta t}\left\|\theta_{hx}^{n}-\theta_{hx}^{n-1}\right\|^{2}-\frac{\rho_1}{2\Delta t}\left\|\widehat{\varphi}_{h}^{n}\ -\widehat{\varphi}_{h}^{n-1}\right\|^{2}\\
&-\frac{k}{2\Delta t}\left\|\varphi_{hx}^{n}+\psi_{h}^{n}+w_{h}^{n}-\left(\varphi_{hx}^{n-1}+\psi_{h}^{n-1}+w_{h}^{n-1}\right)\right\|^{2}\\
&-\frac{\rho_2}{2\Delta t}\left\|\widehat{\psi}_{h}^{n}\ -\widehat{\psi}_{h}^{n-1}\right\|^{2}-\frac{b}{2\Delta t}\left\|\psi_{hx}^{n}-\psi_{hx}^{n-1}\right\|^{2}-\varepsilon_1 k\left\|\widehat{\varphi}_{hx}^{n}+\widehat{\psi}_{h}^{n}+l\widehat{w}_{h}^{n}\right\|^{2}\\
&-\varepsilon_2 k_0\left\|\widehat{w}_{hx}^{n}-l\widehat{\varphi}_{h}^{n}\right\|^{2}-\frac{k_0}{2\Delta t}\left\|w_{hx}^{n}-l\varphi_{h}^{n}-\left(w_{hx}^{n-1}-l\varphi_{h}^{n-1}\right)\right\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\tfrac{E^{n}-E^{n-1}}{\Delta t}=&-\tau\left\|\widehat{\theta}_{hx}^{n}\right\|^{2}-\sigma\left\|\widehat{\varphi}_{h}^{n}\right\|^{2}-\alpha\left\|\widehat{\psi}_{hx}^{n}\right\|^{2}\\
&-\varepsilon_1 k\left\|\widehat{\varphi}_{hx}^{n}+\widehat{\psi}_{h}^{n}+l\widehat{w}_{h}^{n}\right\|^{2}-\varepsilon_2 k_0\left\|\widehat{w}_{hx}^{n}-l\widehat{\varphi}_{h}^{n}\right\|^{2}\\
&-\tfrac{\rho_3}{2\Delta t}\left\|\widehat{\theta}_{h}^{n}-\widehat{\theta}_{h}^{n-1}\right\|^{2}-\tfrac{\kappa}{2\Delta t}\left\|\theta_{hx}^{n}-\theta_{hx}^{n-1}\right\|^{2}\\
&-\tfrac{\rho_1}{2\Delta t}\left\|\widehat{\varphi}_{h}^{n}\ -\widehat{\varphi}_{h}^{n-1}\right\|^{2}-\tfrac{\rho_1}{2\Delta t}\left\|\widehat{w}_{h}^{n}-\widehat{w}_{h}^{n-1}\right\|^{2}\\
&-\tfrac{k}{2\Delta t}\left\|\varphi_{hx}^{n}+\psi_{h}^{n}+lw_{h}^{n}-\left(\varphi_{hx}^{n-1}+\psi_{h}^{n-1}+lw_{h}^{n-1}\right)\right\|^{2}\\
&-\tfrac{k_0}{2\Delta t}\left\|w_{hx}^{n}-l\varphi_{h}^{n}-\left(w_{hx}^{n-1}-l\varphi_{h}^{n-1}\right)\right\|^{2}\\
&-\tfrac{\rho_2}{2\Delta t}\left\|\widehat{\psi}_{h}^{n}\ -\widehat{\psi}_{h}^{n-1}\right\|^{2}-\tfrac{b}{2\Delta t}\left\|\psi_{hx}^{n}-\psi_{hx}^{n-1}\right\|^{2}\leq 0.
\end{aligned}
$$

□

3.4.1 Priori error estimate

In this subsection we obtain a priori error estimate on the numerical approximations, in which we obtain the convergence of the error. **The exact and auxiliary problems** The following Theorem provide an estimate of the difference between the solutions of the exact and auxiliary problems in term of $\varepsilon_1 and\, \varepsilon_2$.

Theorem 3.4. *Suppose that* (φ,ψ,w,θ) *the solution of the exact system* (3.1.16)-(3.1.17) *and* (ϕ,χ,ω,ν) *the solution of auxiliary system* (3.4.1)-(3.1.17) *and the energy difference* $E_d(t)$ *define by*

$$
\begin{aligned}
E_d(t)=(\rho_1&\left\|\widehat{\varphi}-\widehat{\phi}\right\|^{2}+\rho_2\left\|\widehat{\psi}-\widehat{\chi}\right\|^{2}+\rho_1\left\|\widehat{w}-\widehat{\omega}\right\|^{2}\\
&+\rho_3\left\|\widehat{\theta}-\widehat{\nu}\right\|^{2}+b\left\|\psi_x-\chi_x\right\|^{2}+\kappa\left\|\theta_x-\nu_x\right\|^{2}
\end{aligned}
$$

$$
\begin{aligned}
&+k\left\|\varphi_x+\psi+lw-(\phi_x+\chi+l\omega)\right\|^2\\
&+k_0\left\|w_x-l\varphi-(\omega_x-l\phi)\right\|^2\big)(t),
\end{aligned}
$$

then there exists positive constants $c_1,\ c_2$ such that the following estimate holds,

$$E_d(t)\le c_1\varepsilon_1+c_2\varepsilon_2,\ 0\le t\le T.$$

Proof. By combining the exact and auxiliary systems, we have for all $z,\, r\in H^1_*(0,1)$ and $\ell,\, p\in H^1_0(0,1)$,

$$
\begin{cases}
\rho_1(\dot{\widehat{\varphi}}-\dot{\widehat{\phi}},z\ )+k\left(\varphi_x+\psi+lw-(\phi_x+\chi+l\omega),z_x\right)\\
-k_0l\left(w_x-l\varphi-(\omega_x-l\phi),z\right)+k_0l^2(\widehat{\theta}-\widehat{\nu},z)+\sigma(\widehat{\varphi}-\widehat{\phi},z)\\
-\varepsilon_1k\left(\widehat{\phi}_x+\widehat{\chi}+l\widehat{\omega},z_x\right)+\varepsilon_2k_0l\left(\widehat{\omega}_x-l\widehat{\phi},z\right)=0,\\
\\
\rho_2(\dot{\widehat{\psi}}-\dot{\widehat{\chi}},l)+b(\psi_x-\chi_x,l_x)+\alpha(\widehat{\psi}_x-\widehat{\chi}_x,l_x)\\
+k(\varphi_x+\psi+lw-(\phi_x+\chi+l\omega),l)-\varepsilon_1k\left(\widehat{\phi}_x+\widehat{\chi}+l\widehat{\omega},l\right)=0,\\
\\
\rho_1(\dot{\widehat{w}}-\dot{\widehat{\omega}},p)+k_0(w_x-l\varphi-(\omega_x-l\phi),p_x)\\
+kl(\varphi_x+\psi+lw-(\phi_x+\chi+l\omega),p)+k_0l(\widehat{\theta}_x-\widehat{\nu}_x,p)\\
-\varepsilon_2k_0\left(\widehat{\omega}_x-l\widehat{\phi},p_x\right)-\varepsilon_1kl\left(\widehat{\phi}_x+\widehat{\chi}+l\widehat{\omega},p\right)=0,\\
\\
\rho_3(\dot{\widehat{\theta}}-\dot{\widehat{\nu}},r)+\kappa(\theta_x-\nu_x,r_x)+\tau(\widehat{\theta}_x-\widehat{\nu}_x,r_x)\\
+k_0l(\widehat{w}_x-l\widehat{\varphi}-\left(\widehat{\omega}_x-l\widehat{\phi}\right),r)=0.
\end{cases}
\tag{3.4.9}
$$

Choosing $z=\widehat{\varphi}-\widehat{\phi}$, $l=\widehat{\psi}-\widehat{\chi}$, $p=\widehat{w}-\widehat{\omega}$ and $r=\widehat{\theta}-\widehat{\nu}$, then we sum the equations of (3.4.9),

$$
\begin{gathered}
\rho_1(\dot{\widehat{\varphi}}-\dot{\widehat{\phi}},\widehat{\varphi}-\widehat{\phi}\ )+\rho_2(\dot{\widehat{\psi}}-\dot{\widehat{\chi}},\widehat{\psi}-\widehat{\chi})+\rho_1(\dot{\widehat{w}}-\dot{\widehat{\omega}},\widehat{w}-\widehat{\omega})\\
+\rho_3(\dot{\widehat{\theta}}-\dot{\widehat{\nu}},\widehat{\theta}-\widehat{\nu})+b(\psi_x-\chi_x,(\widehat{\psi}-\widehat{\chi})_x)+\kappa(\theta_x-\nu_x,\left(\widehat{\theta}-\widehat{\nu}\right)_x)\\
+k\left(\varphi_x+\psi+lw-(\phi_x+\chi+l\omega),\left(\widehat{\varphi}-\widehat{\phi}\right)_x+(\widehat{\psi}-\widehat{\chi})+l\,(\widehat{w}-\widehat{\omega})\right)\\
-\varepsilon_1k\left(\widehat{\varphi}_x+\widehat{\psi}+l\widehat{w}-\left(\widehat{\phi}_x+\widehat{\chi}+l\widehat{\omega}\right),\left(\widehat{\varphi}-\widehat{\phi}\right)_x+(\widehat{\psi}-\widehat{\chi})+l\,(\widehat{w}-\widehat{\omega})\right)\\
+k_0(w_x-l\varphi-(\omega_x-l\phi),(\widehat{w}-\widehat{\omega})_x-l\left(\widehat{\varphi}-\widehat{\phi}\right))\\
-\varepsilon_2k_0(\widehat{w}_x-l\widehat{\varphi}-\left(\widehat{\omega}_x-l\widehat{\phi}\right),(\widehat{w}-\widehat{\omega})_x-l\left(\widehat{\varphi}-\widehat{\phi}\right))\\
+\sigma\left\|\widehat{\varphi}-\widehat{\phi}\right\|^2+\alpha\left\|\widehat{\psi}_x-\widehat{\chi}_x\right\|^2+\tau\left\|\widehat{\theta}_x-\widehat{\nu}_x\right\|^2=0
\end{gathered}
$$

Therefore,

$$\begin{aligned}\frac{d}{dt}E_d(t) \leq \varepsilon_1 k \left\| \widehat{\varphi}_x + \widehat{\psi} + l\widehat{w} - \left(\widehat{\phi}_x + \widehat{\chi} + l\widehat{\omega}\right) \right\|^2 \\ + \varepsilon_2 k_0 \left\| \widehat{w}_x - l\widehat{\varphi} - \left(\widehat{\omega}_x - l\widehat{\phi}\right) \right\|^2\end{aligned}$$

Integrating from 0 to t, we get

$$\begin{aligned}E_d(t) &\leq \int_0^t (\varepsilon_1 k \left\| \widehat{\varphi}_x + \widehat{\psi} + l\widehat{w} - \left(\widehat{\phi}_x + \widehat{\chi} + l\widehat{\omega}\right) \right\|^2 \\ &+ \varepsilon_2 k_0 \left\| \widehat{w}_x - l\widehat{\varphi} - \left(\widehat{\omega}_x - l\widehat{\phi}\right) \right\|^2) ds \\ &\leq \int_0^T (\varepsilon_1 k \left\| \widehat{\varphi}_x + \widehat{\psi} + l\widehat{w} - \left(\widehat{\phi}_x + \widehat{\chi} + l\widehat{\omega}\right) \right\|^2 \\ &+ \varepsilon_2 k_0 \left\| \widehat{w}_x - l\widehat{\varphi} - \left(\widehat{\omega}_x - l\widehat{\phi}\right) \right\|^2) ds \\ &\leq c_1\varepsilon_1 + c_2\varepsilon_2\end{aligned}$$

□

Theorem 3.5. *Suppose that the solution* $(\varphi, \psi, w, \theta)$ *of the system* (3.4.1)-(3.1.17) *belongs to the space*

$$\left(H^4\left(0,T,L^2(0,1)\right) \cap H^3\left(0,T,H^1(0,1)\right)\right)^4$$

then the following priori error estimate holds:

$$\left\{\begin{aligned}&\left\|\widehat{\varphi}_h^n - \dot{\varphi}(t_n)\right\|^2 + \left\|\widehat{\psi}_h^n - \dot{\psi}(t_n)\right\|^2 + \left\|\widehat{w}_h^n - \dot{w}(t_n)\right\|^2 \\ &+ \left\|\widehat{\theta}_h^n - \dot{\theta}(t_n)\right\|^2 + \left\|\psi_{hx}^n - (\psi(t_n))_x\right\|^2 + \left\|\theta_{hx}^n - (\theta(t_n))_x\right\|^2 \\ &+ \left\|\varphi_{hx}^n + \psi_h^n + lw_h^n - \left((\varphi(t_n))_x + \psi(t_n) + lw(t_n)\right)\right\|^2 \\ &+ \left\|\left(w_{hx}^n - l\varphi_h^n\right)^2 - \left((w(t_n))_x - l\varphi(t_n)\right)\right\|^2 \leq c\left(h^2 + \Delta t^2\right).\end{aligned}\right. \tag{3.4.10}$$

Proof. Let

$$\begin{aligned}e^n &= \varphi_h^n - P_h^0\varphi(t_n), \quad \widehat{e}^n = \widehat{\varphi}_h^n - P_h^0\dot{\varphi}(t_n), \\ \eta^n &= \psi_h^n - P_h^*\psi(t_n), \ \widehat{\eta}^n = \widehat{\psi}_h^n - P_h^*\dot{\psi}(t_n), \\ \varsigma^n &= w_h^n - P_h^* w(t_n), \ \widehat{\varsigma}^n = \widehat{w}_h^n - P_h^*\dot{w}(t_n), \\ \rho^n &= \theta_h^n - P_h^0\theta(t_n), \quad \widehat{\rho}^n = \widehat{\theta}_h^n - P_h^0\dot{\theta}(t_n),\end{aligned} \tag{3.4.11}$$

where $P_h^0 : H_0^1(\Omega) \to Y_h,\ P_h^* : H_*^1(\Omega) \to S_h$ are respectively defined by

$$\begin{aligned}
&\forall \upsilon_h \in Y_h, \left(\left(P_h^0 \upsilon - \upsilon\right)_x, \upsilon_{hx}\right) = 0,\\
&\forall \upsilon_h \in S_h, \left((P_h^* \upsilon - \upsilon)_x, \upsilon_{hx}\right) = 0.
\end{aligned}$$

The projection operators defined above satisfy the following estimates (see lemma 2.7 and [46]).

$$\begin{aligned}
&\forall \upsilon \in H_0^1(\Omega); \left\|P_h^0 \upsilon - \upsilon\right\| \leq C_p \left\|\upsilon_x\right\| \qquad (3.4.12)\\
&\forall \upsilon \in H_*^1(\Omega); \left\|P_h^* \upsilon - \upsilon\right\| \leq C_* \left\|\upsilon_x\right\|.
\end{aligned}$$

The proof of this theorem is quite technical, we split it into several steps to make it easier to read (see [60]).

Step 1: choosing $z_h = \widehat{e}^n$, $l_h = \widehat{\eta}^n$, $p_h = \widehat{\varsigma}^n$ and $r_h = \widehat{\rho}^n$ in (3.4.3)-(3.4.6). Next, we take $z = \widehat{e}^n$, $l = \widehat{\eta}^n$, $p = \widehat{\varsigma}^n$ and $r = \widehat{\rho}^n$ in (3.4.2), then combining the result and summing the equations. All this gives

$$\begin{aligned}
&\frac{\rho_1}{2\Delta t}\left(\left\|\widehat{e}^n - \widehat{e}^{n-1}\right\|^2 + \left\|\widehat{e}^n\right\|^2 - \left\|\widehat{e}^{n-1}\right\|^2\right) + \alpha \left\|\widehat{\eta}_x^n\right\|^2\\
&+ \sigma \left\|\widehat{e}^n\right\|^2 + \tau \left\|\widehat{\rho}_x^n\right\|^2 + \varepsilon_1 k \left\|\widehat{e}_x^n + \widehat{\eta}^n + l\widehat{\varsigma}^n\right\|^2 + \varepsilon_2 k_0 \left\|\widehat{\varsigma}_x^n + l\widehat{e}^n\right\|^2\\
&+ \frac{\rho_2}{2\Delta t}\left(\left\|\widehat{\eta}^n - \widehat{\eta}^{n-1}\right\|^2 + \left\|\widehat{\eta}^n\right\|^2 - \left\|\widehat{\eta}^{n-1}\right\|^2\right)\\
&+ \frac{\rho_1}{2\Delta t}\left(\left\|\widehat{\varsigma}^n - \widehat{\varsigma}^{n-1}\right\|^2 + \left\|\widehat{\varsigma}^n\right\|^2 - \left\|\widehat{\varsigma}^{n-1}\right\|^2\right)\\
&+ \frac{\rho_3}{2\Delta t}\left(\left\|\widehat{\rho}^n - \widehat{\rho}^{n-1}\right\|^2 + \left\|\widehat{\rho}^n\right\|^2 - \left\|\widehat{\rho}^{n-1}\right\|^2\right)\\
&+ k\left(e_x^n + \eta^n + l\varsigma^n, \widehat{e}_x^n + \widehat{\eta}^n + l\widehat{\varsigma}^n\right) + b\left(\eta_x^n, \widehat{\eta}_x^n\right)\\
&+ k_0\left(\varsigma_x^n - le^n, \widehat{\varsigma}_x^n + l\widehat{e}^n\right) + \kappa\left(\rho_x^n, \widehat{\rho}_x^n\right)\\
&= \beta^1 + \beta^2 + \beta^3\\
&+ \rho_1\left(\dot{\widehat{\varphi}} - \frac{P_h^0 \dot{\varphi}(t_n) - P_h^0 \dot{\varphi}(t_{n-1})}{\Delta t}, \widehat{e}^n\right) + \rho_2\left(\dot{\widehat{\psi}} - \frac{P_h^* \dot{\psi}(t_n) - P_h^* \dot{\psi}(t_{n-1})}{\Delta t}, \widehat{\eta}^n\right)\\
&+ \rho_1\left(\dot{\widehat{w}} - \frac{P_h^* \dot{w}(t_n) - P_h^* \dot{w}(t_{n-1})}{\Delta t}, \widehat{\varsigma}^n\right) + \rho_3\left(\dot{\widehat{\theta}} - \frac{P_h^0 \dot{\theta}(t_n) - P_h^0 \dot{\theta}(t_{n-1})}{\Delta t}, \widehat{\rho}^n\right)\\
&+ k_0 l^2\left(\widehat{\theta} - P_h^0 \dot{\theta}(t_n), \widehat{e}^n\right) + k_0 l\left(\widehat{\theta}_x - \left(P_h^0 \dot{\theta}(t_n)\right)_x, \widehat{\varsigma}^n\right)\\
&+ k_0 l\left(\widehat{w}_x - l\widehat{\varphi} - \left(\left(P_h^* \dot{w}(t_n)\right)_x - lP_h^0 \dot{\varphi}(t_n)\right), \widehat{\rho}^n\right),
\end{aligned}$$

where β^1, β^2, β^3 define by

$$\begin{aligned}\beta^1 &= k\left(\varphi_x+\psi+lw-\left(\left(P_h^0\varphi(t_n)\right)_x+P_h^*\psi(t_n)+lP_h^*w(t_n)\right),\widehat{e}_x^n\ +\widehat{\eta}^n+l\widehat{\varsigma}^n\right)\\ &\quad+k\varepsilon_1\left(\widehat{\varphi}_x+\widehat{\psi}+l\widehat{w}-\left(\left(P_h^0\dot{\varphi}(t_n)\right)_x+P_h^*\dot{\psi}(t_n)+lP_h^*\dot{w}(t_n)\right),\widehat{e}_x^n\ +\widehat{\eta}^n+l\widehat{\varsigma}^n\right),\end{aligned}$$

$$\begin{aligned}\beta^2 &= k_0\left(w_x-l\varphi-\left(\left(P_h^*w(t_n)\right)_x-lP_h^0\varphi(t_n)\right),\widehat{\varsigma}_x^n+l\widehat{e}^n\right)\\ &\quad+\varepsilon_2k_0\left(\widehat{w}_x-l\widehat{\varphi}-\left(\left(P_h^*\dot{w}(t_n)\right)_x-lP_h^0\dot{\varphi}(t_n)\right),\widehat{\varsigma}_x^n+l\widehat{e}^n\right),\end{aligned}$$

$$\begin{aligned}\beta^3 &= b(\psi_x-(P_h^*\psi(t_n))_x,\widehat{\eta}_x^n)+\alpha(\widehat{\psi}_x-(P_h^*\dot{\psi}(t_n))_x,\widehat{\eta}_x^n)\\ &\quad+\kappa(\theta_x-\left(P_h^0\theta(t_n)\right)_x,\widehat{\rho}_x^n)+\tau(\widehat{\theta}_x-\left(P_h^0\dot{\theta}(t_n)\right)_x,\widehat{\rho}_x^n)\\ &\quad+\sigma(\widehat{\varphi}-\left(P_h^0\dot{\varphi}(t_n)\right),\widehat{e}^n),\end{aligned}$$

by using the Young's inequality, for $\varepsilon_i>0$ we have

$$\begin{aligned}\beta^1 &\le \frac{k}{2\varepsilon_1}\left\|\varphi_x+\psi+lw-\left(\left(P_h^0\varphi(t_n)\right)_x+P_h^*\psi(t_n)+lP_h^*w(t_n)\right)\right\|^2\\ &\quad+\frac{k}{2}\left\|\widehat{\varphi}_x+\widehat{\psi}+l\widehat{w}-\left(\left(P_h^0\dot{\varphi}(t_n)\right)_x+P_h^*\dot{\psi}(t_n)+lP_h^*\dot{w}(t_n)\right)\right\|^2\\ &\quad+k\left(\frac{\varepsilon_1}{2}+\frac{\varepsilon_1^2}{2}\right)\|\widehat{e}_x^n\ +\widehat{\eta}^n+l\widehat{\varsigma}^n\|^2,\end{aligned}\tag{3.4.13}$$

$$\begin{aligned}\beta^2 &\le \frac{k_0}{2\varepsilon_2}\left\|w_x-l\varphi-\left(\left(P_h^*w(t_n)\right)_x-lP_h^0\varphi(t_n)\right)\right\|^2\\ &\quad+\frac{k_0}{2}\left\|\widehat{w}_x-l\widehat{\varphi}-\left(\left(P_h^*\dot{w}(t_n)\right)_x-lP_h^0\dot{\varphi}(t_n)\right)\right\|^2\\ &\quad+k_0\left(\frac{\varepsilon_2}{2}+\frac{\varepsilon_2^2}{2}\right)\|\widehat{\varsigma}_x^n+l\widehat{e}^n\|^2,\end{aligned}\tag{3.4.14}$$

$$\begin{aligned}\beta^3 &\le \frac{b}{2\varepsilon_3}\left\|\psi_x-(P_h^*\psi(t_n))_x\right\|^2+\ \frac{a}{2\varepsilon_4}\left\|\widehat{\psi}_x-(P_h^*\dot{\psi}(t_n))_x\right\|^2\\ &\quad+\left(\frac{b\varepsilon_3+a\varepsilon_4}{2}\right)\|\widehat{\eta}_x^n\|^2+\frac{\kappa}{2\varepsilon_5}\left\|\theta_x-\left(P_h^0\theta(t_n)\right)_x\right\|^2\\ &\quad+\frac{\tau}{2\varepsilon_6}\left\|\widehat{\theta}_x-\left(P_h^0\dot{\theta}(t_n)\right)_x\right\|^2+\frac{\kappa\varepsilon_5+\tau\varepsilon_6}{2}\|\widehat{\rho}_x^n\|^2\\ &\quad+\frac{\sigma}{2}\left\|\widehat{\varphi}-P_h^0\dot{\varphi}(t_n)\right\|^2+\frac{\sigma}{2}\|\widehat{e}^n\|^2,\end{aligned}\tag{3.4.15}$$

we choose $\varepsilon_i, i=1,..,6$ small enough such that the following inequalities hold

$$\begin{cases} \left(\varepsilon_1 - \frac{\varepsilon_1}{2} - \frac{\varepsilon_1^2}{2}\right) \geq 0, \left(\varepsilon_2 - \frac{\varepsilon_2}{2} - \frac{\varepsilon_2^2}{2}\right) \geq 0 \\ \alpha \geq \frac{b\varepsilon_3}{2-\varepsilon_4}, \ \tau \geq \frac{\kappa\varepsilon_5}{2-\varepsilon_6}. \end{cases}$$

Step 2: Let

$$\begin{cases} a_n^1 = (e_x^n + \eta^n + l\varsigma^n, \widehat{e}_x^n \ + \widehat{\eta}^n + l\widehat{\varsigma}^n) \\ a_n^2 = (\varsigma_x^n - le^n, \widehat{\varsigma}_x^n + l\widehat{e}^n) \\ a_n^3 = b(\eta_x^n, \ \widehat{\eta}_x^n) + \kappa(\rho_x^n, \widehat{\rho}_x^n), \end{cases}$$

using (3.4.11) we get

$$\begin{aligned} a_n^1 &= \frac{1}{2\Delta t}\left(\|e_x^n + \eta^n + l\varsigma^n\|^2 - \|e_x^{n-1} + \eta^{n-1} + l\varsigma^{n-1}\|^2\right) \\ &+ \frac{1}{2\Delta t}\left\|e_x^n + \eta^n + l\varsigma^n - \left(e_x^{n-1} + \eta^{n-1} + l\varsigma^{n-1}\right)\right\|^2 \\ &+ \left(e_x^n + \eta^n + l\varsigma^n, \frac{\left(P_h^0\varphi(t_n)\right)_x - \left(P_h^0\varphi(t_{n-1})\right)_x}{\Delta t} - \left(P_h^0\dot{\varphi}(t_n)\right)_x\right) \\ &+ \left(e_x^n + \eta^n + l\varsigma^n, \frac{P_h^*\psi(t_n) - P_h^*\psi(t_{n-1})}{\Delta t} - P_h^*\dot{\psi}(t_n)\right) \\ &+ l\left(e_x^n + \eta^n + l\varsigma^n, \frac{P_h^*w(t_n) - P_h^*w(t_{n-1})}{\Delta t} - P_h^*\dot{w}(t_n)\right). \end{aligned} \tag{3.4.16}$$

Similarly, we have

$$\begin{aligned} a_n^2 &= \frac{1}{2\Delta t}\left(\left\|\varsigma_x^n - le^n - \left(\varsigma_x^{n-1} - le^{n-1}\right)\right\|^2 + \|\varsigma_x^n - le^n\|^2 - \left\|\varsigma_x^{n-1} - le^{n-1}\right\|^2\right) \\ &+ \left(\varsigma_x^n - le^n, \frac{\left(P_h^*w(t_n)\right)_x - \left(P_h^*w(t_{n-1})\right)_x}{\Delta t} - \left(P_h^*\dot{w}(t_n)\right)_x\right) \\ &+ l\left(\varsigma_x^n - le^n, \frac{P_h^0\varphi(t_n) - P_h^0\varphi(t_{n-1})}{\Delta t} - P_h^0\dot{\varphi}(t_n)\right), \end{aligned} \tag{3.4.17}$$

and

$$\begin{aligned} a_n^3 &= b\left(\eta_x^n, \frac{\left(P_h^*\psi(t_n)\right)_x - \left(P_h^*\psi(t_{n-1})\right)_x}{\Delta t} - \left(P_h^*\dot{\psi}(t_n)\right)_x\right) \\ &+ \kappa\left(\rho_x^n, \frac{\left(P_h^0\theta(t_n)\right)_x - \left(P_h^0\theta(t_{n-1})\right)_x}{\Delta t} - \left(P_h^0\dot{\theta}(t_n)\right)_x\right) \\ &+ \frac{b}{2\Delta t}\left(\left\|\eta_x^n - \eta_x^{n-1}\right\|^2 + \|\eta_x^n\|^2 - \left\|\eta_x^{n-1}\right\|^2\right) \\ &+ \frac{\kappa}{2\Delta t}\left(\left\|\rho_x^n - \rho_x^{n-1}\right\|^2 + \|\rho_x^n\|^2 - \left\|\rho_x^{n-1}\right\|^2\right). \end{aligned} \tag{3.4.18}$$

Inserting (3.4.16), (3.4.17) and (3.4.18), we get

$$\begin{aligned}
&\frac{I^n-I^{n-1}}{\Delta t}+\frac{\rho_1}{2\Delta t}\left\|\widehat{e}^n\ -\widehat{e}^{n-1}\right\|^2\ +\sigma\left\|\widehat{e}^n\right\|^2\\
&+\frac{\rho_3}{2\Delta t}\ \left\|\widehat{\rho}^n-\ \widehat{\rho}^{n-1}\right\|+\frac{\rho_2}{2\Delta t}\left\|\widehat{\eta}^n-\widehat{\eta}^{n-1}\right\|\\
&+\frac{k}{2\Delta t}\left\|e_x^n+\eta^n+l\varsigma^n-\left(e_x^{n-1}+\eta^{n-1}+l\varsigma^{n-1}\right)\right\|^2\\
&+\frac{k_0}{2\Delta t}\left\|\varsigma_x^n-le^n-\left(\varsigma_x^{n-1}-le^{n-1}\right)\right\|^2+\frac{\rho_1}{2\Delta t}\left\|\widehat{\varsigma}^n-\widehat{\varsigma}^{n-1}\right\|^2\\
&\frac{b}{2\Delta t}\left\|\eta_x^n-\eta_x^{n-1}\right\|^2+\frac{\kappa}{2\Delta t}\left\|\rho_x^n-\rho_x^{n-1}\right\|^2\\
&=\lambda_n+k\left(e_x^n+\eta^n+l\varsigma^n,\left(P_h^0\dot{\varphi}(t_n)\right)_x-\frac{\left(P_h^0\varphi(t_n)\right)_x-\left(P_h^0\varphi(t_{n-1})\right)_x}{\Delta t}\right)\\
&+k\left(e_x^n+\eta^n+l\varsigma^n,P_h^*\dot{\psi}(t_n)-\frac{P_h^*\psi(t_n)-P_h^*\psi(t_{n-1})}{\Delta t}\right)\\
&+kl\left(e_x^n+\eta^n+l\varsigma^n,P_h^*\dot{w}(t_n)-\frac{P_h^*w(t_n)-P_h^*w(t_{n-1})}{\Delta t}\right)\\
&+k_0\left(\varsigma_x^n-le^n,(P_h^*\dot{w}(t_n))_x-\frac{\left(P_h^*w(t_n)\right)_x-\left(P_h^*w(t_{n-1})\right)_x}{\Delta t}\right)\\
&+k_0l\left(\varsigma_x^n-le^n,P_h^0\dot{\varphi}(t_n)-\frac{P_h^0\varphi(t_n)-P_h^0\varphi(t_{n-1})}{\Delta t}\right)\\
&+b\left(\eta_x^n,(P_h^*\dot{\psi}(t_n))_x-\frac{\left(P_h^*\psi(t_n)\right)_x-\left(P_h^*\psi(t_{n-1})\right)_x}{\Delta t}\right)\\
&+\kappa\left(\rho_x^n,\left(P_h^0\dot{\theta}(t_n)\right)_x-\frac{\left(P_h^0\theta(t_n)\right)_x-\left(P_h^0\theta(t_{n-1})\right)_x}{\Delta t}\right)\\
&+\rho_1(\dot{\widehat{\varphi}}-\frac{P_h^0\dot{\varphi}(t_n)\ -P_h^0\dot{\varphi}(t_{n-1})}{\Delta t},\widehat{e}^n)+\rho_2(\dot{\widehat{\psi}}-\frac{P_h^*\dot{\psi}(t_n)-P_h^*\dot{\psi}(t_{n-1})}{\Delta t},\widehat{\eta}^n)\\
&+\rho_1(\dot{\widehat{w}}-\frac{P_h^*\dot{w}(t_n)-P_h^*\dot{w}(t_{n-1})}{\Delta t},\widehat{\varsigma}^n)+\rho_3(\dot{\widehat{\theta}}-\frac{P_h^0\dot{\theta}(t_n)-P_h^0\dot{\theta}(t_{n-1})}{\Delta t},\widehat{\rho}^n)\\
&+k_0l^2(\widehat{\theta}-P_h^0\dot{\theta}(t_n)\ ,\widehat{e}^n)+k_0l(\widehat{\theta}_x-\left(P_h^0\dot{\theta}(t_n)\right)_x,\widehat{\varsigma}^n)\\
&+k_0l(\widehat{w}_x-l\widehat{\varphi}-\left((P_h^*\dot{w}(t_n))_x-lP_h^0\dot{\varphi}(t_n)\right),\widehat{\rho}^n),
\end{aligned}\tag{3.4.19}$$

where I^n and λ_n define by

$$\begin{aligned}
2I^n&=\rho_1\left\|e^n\right\|^2+\rho_2\left\|\widehat{\eta}^n\right\|^2+\rho_1\left\|\widehat{\varsigma}^n\right\|^2+\rho_3\left\|\widehat{\rho}^n\right\|+k\left\|e_x^n+\eta^n+l\varsigma^n\right\|^2\\
&\quad+k_0\left\|\varsigma_x^n-le^n\right\|^2+b\left\|\eta_x^n\right\|^2+\kappa\left\|\rho_x^n\right\|^2,
\end{aligned}$$

$$\lambda_n=-\sigma\left\|\widehat{e}^n\right\|^2-\alpha\left\|\widehat{\eta}_x^n\right\|^2-\tau\left\|\widehat{\rho}_x^n\right\|^2-\varepsilon_1k\left\|\widehat{e}_x^n\ +\widehat{\eta}^n+l\widehat{\varsigma}^n\right\|^2-\varepsilon_2k_0\left\|\widehat{\varsigma}_x^n+l\widehat{e}^n\right\|^2+\beta^1+\beta^2+\beta^3,$$

using (3.4.13)-(3.4.15) and Young's inequality, we conclude that $\exists c > 0:$

$$\begin{aligned}\lambda_n \leq c\big(&\big\|\varphi_x + \psi + lw - \big(\big(P_h^0\varphi(t_n)\big)_x + P_h^*\psi(t_n) + lP_h^*w(t_n)\big)\big\|^2 \\ &+ \big\|\widehat{\varphi}_x + \widehat{\psi} + l\widehat{w} - \big(\big(P_h^0\dot{\varphi}(t_n)\big)_x + P_h^*\dot{\psi}(t_n) + lP_h^*\dot{w}(t_n)\big)\big\|^2 \\ &+ \big\|w_x - l\varphi - \big((P_h^*w(t_n))_x - lP_h^0\varphi(t_n)\big)\big\|^2 + \big\|\widehat{\varphi} - P_h^0\dot{\varphi}(t_n)\big\|^2 \\ &+ \big\|\widehat{w}_x - l\widehat{\varphi} - \big((P_h^*\dot{w}(t_n))_x - lP_h^0\dot{\varphi}(t_n)\big)\big\|^2 \\ &+ \big\|\theta_x - \big(P_h^0\theta(t_n)\big)_x\big\|^2 + \Big\|\widehat{\theta}_x - \big(P_h^0\dot{\theta}(t_n)\big)_x\Big\|^2 \\ &+ \big\|\psi_x - (P_h^*\psi(t_n))_x\big\|^2 + \big\|\widehat{\psi}_x - (P_h^*\dot{\psi}(t_n))_x\big\|^2\big). \end{aligned} \tag{3.4.20}$$

Step 3: Applying Young's inequality to (3.4.19) and using all these estimates, we observe that

$$(1 - c\Delta t)I^n \leq I^{n-1} + c\Delta t X^n, \tag{3.4.21}$$

where the residual X^n is the sum of some approximation on errors terms. All these terms is estimated using Taylor expansion in time of the solution $(\varphi, \psi, w, \theta)$ and space error which can be bounded from (3.4.12) plus the linearity of P_h^0 and P_h^*. where X^n

$$
\begin{aligned}
&X^n = \\
&\underbrace{\left\|\ddot{\varphi}\left(t_n\right)-\frac{P_h^0 \dot{\varphi}(t_n)\ -P_h^0 \dot{\varphi}(t_{n-1})}{\Delta t}\right\|^2}_{X_1}+\underbrace{\left\|\ddot{\psi}\left(t_n\right)-\frac{P_h^* \dot{\psi}(t_n)-P_h^* \dot{\psi}(t_{n-1})}{\Delta t}\right\|^2}_{X_2} \\
&+\underbrace{\left\|\ddot{w}\left(t_n\right)-\frac{P_h^* \dot{w}(t_n)-P_h^* \dot{w}(t_{n-1})}{\Delta t}\right\|^2}_{X_3}+\underbrace{\left\|\ddot{\theta}\left(t_n\right)-\frac{P_h^0 \dot{\theta}(t_n)-P_h^0 \dot{\theta}(t_{n-1})}{\Delta t}\right\|^2}_{X_4} \\
&\underbrace{\left\|P_h^* \dot{\psi}(t_n)-\frac{P_h^* \psi(t_n)-P_h^* \psi(t_{n-1})}{\Delta t}\right\|^2}_{M_1}+\underbrace{\left\|\left(P_h^0 \dot{\varphi}(t_n)\right)_x-\frac{\left(P_h^0 \varphi(t_n)\right)_x-\left(P_h^0 \varphi(t_{n-1})\right)_x}{\Delta t}\right\|^2}_{M_2} \\
&+\underbrace{\left\|P_h^* \dot{w}(t_n)-\frac{P_h^* w(t_n)-P_h^* w(t_{n-1})}{\Delta t}\right\|^2}_{M_3}+\underbrace{\left\|(P_h^* \dot{w}(t_n))_x-\frac{\left(P_h^* w(t_n)\right)_x-\left(P_h^* w(t_{n-1})\right)_x}{\Delta t}\right\|^2}_{M_4} \\
&+\underbrace{\left\|P_h^0 \dot{\varphi}(t_n)-\frac{P_h^0 \varphi(t_n)-P_h^0 \varphi(t_{n-1})}{\Delta t}\right\|^2}_{M_5}+\underbrace{\left\|(P_h^* \dot{\psi}(t_n))_x-\frac{\left(P_h^* \psi(t_n)\right)_x-\left(P_h^* \psi(t_{n-1})\right)_x}{\Delta t}\right\|^2}_{M_6} \\
&+\underbrace{\left\|\left(P_h^0 \dot{\theta}(t_n)\right)_x-\frac{\left(P_h^0 \theta(t_n)\right)_x-\left(P_h^0 \theta(t_{n-1})\right)_x}{\Delta t}\right\|^2}_{M_7} \\
&+\underbrace{\left\|\dot{\varphi}(t_n)-P_h^0 \dot{\varphi}(t_n)\right\|^2}_{R_1}+\underbrace{\left\|\dot{\theta}(t_n)-P_h^0 \dot{\theta}(t_n)\right\|^2}_{R_2}+\underbrace{\left\|\dot{\psi}_x(t_n)-(P_h^* \dot{\psi}(t_n))_x\right\|^2}_{R_3} \\
&+\underbrace{\left\|(\dot{\theta}_x(t_n)-\left(P_h^0 \dot{\theta}(t_n)\right)_x\right\|^2}_{R_4}+\underbrace{\left\|\theta_x(t_n)-\left(P_h^0 \theta(t_n)\right)_x\right\|^2}_{R_5} \\
&+\underbrace{\left\|\dot{w}_x(t_n)-l\dot{\varphi}(t_n)-\left((P_h^* \dot{w}(t_n))_x-lP_h^0 \dot{\varphi}(t_n)\right)\right\|^2)}_{R_6} \\
&+\underbrace{\left\|\psi_x(t_n)-(P_h^* \psi(t_n))_x\right\|^2}_{R_7}+\underbrace{\left\|w_x(t_n)-l\varphi(t_n)-\left((P_h^* w(t_n))_x-lP_h^0 \varphi(t_n)\right)\right\|^2}_{R_8} \\
&+\underbrace{\left\|\varphi_x(t_n)+\psi(t_n)+lw(t_n)-\left(\left(P_h^0 \varphi(t_n)\right)_x+P_h^* \psi(t_n)+lP_h^* w(t_n)\right)\right\|^2)}_{R_9}
\end{aligned}
$$

Step 4: By using Taylor exapnsion and the linearity of P_h^0 and P_h^* we deduce :

$$
\begin{aligned}
M_1 =& \left\|P_h^* \dot{\psi}(t_n)-\frac{P_h^* \psi(t_n)-P_h^* \psi(t_{n-1})}{\Delta t}\right\|^2 \\
\leq & c\left\|\dot{\psi}\left(t_n\right)-P_h^* \dot{\psi}(t_n)\right\|^2+c\left\|\dot{\psi}\left(t_n\right)-\frac{\left(P_h^* \psi(t_n)-P_h^* \psi(t_{n-1})\right)}{\Delta t}\right\|^2
\end{aligned}
$$

$$\begin{aligned}
&\leq ch^2 \left\| \dot{\psi}_x(t_n) \right\|^2 + c \left\| \dot{\psi}(t_n) - \frac{P_h^* \psi(t_n) - P_h^* \psi(t_n) + \Delta t P_h^* \dot{\psi}(t_n) - \frac{1}{2}\Delta t^2 P_h^* \ddot{\psi}(\nu_n)}{\Delta t} \right\|^2 \\
&\leq ch^2 \left\| \dot{\psi}_x(t_n) \right\|^2 + c \left\| \dot{\psi}(t_n) - P_h^* \dot{\psi}(t_n) + \frac{1}{2}\Delta t P_h^* \ddot{\psi}(\psi_n) \right\|^2 \\
&\leq ch^2 \left\| \dot{\psi}_x(t_n) \right\|^2 + c \left\| \dot{\psi}(t_n) - P_h^* \dot{\psi}(t_n) \right\|^2 + c\frac{1}{4}\Delta t^2 \left\| P_h^* \ddot{\psi}(\nu_n) \right\|^2 \\
&\quad + c\Delta t \left\| P_h^* \ddot{\psi}(\nu_n) \right\| \left\| \dot{\psi}(t_n) - P_h^* \dot{\psi}(t_n) \right\| \\
&\leq ch^2 \left\| \dot{\psi}_x(t_n) \right\|^2 + c\frac{1}{4}\Delta t^2 \left\| \ddot{\psi}^n(\nu_n) \right\|^2 + c\Delta t h \left\| \ddot{\psi}(\nu_n) \right\| \left\| \dot{\psi}_x(t_n) \right\|
\end{aligned}$$

Applying Gagliardo-Neirenberg ineqaulity leads to

$$\begin{aligned}
\left\| \ddot{\psi}(\nu_n) \right\| &\leq C \left\| \left\| \ddot{\psi}(\nu_n) \right\|_{L^2(0,1)} \right\|_{L^2(0,T)} \left\| \left\| \dddot{\psi}(\nu_n) \right\|_{L^2(0,1)} \right\|_{L^2(0,T)} \\
&\leq C \left\| \psi \right\|_{H^2(0,T,L^2(\Omega))} \left\| \psi \right\|_{H^3(0,T,L^2(\Omega))}
\end{aligned}$$

and

$$\left\| \dot{\psi}(t_n) \right\| \leq C \left\| \psi \right\|_{H^1(0,T,L^2(\Omega))} \left\| \psi \right\|_{H^2(0,T,L^2(\Omega))}$$

Consequently

$$M_1 \leq C\left(h^2 + (\Delta t)^2\right)\left(\|\psi\|^2_{H^3(0,T,H^1(\Omega))} + \|\psi\|^2_{H^2(0,T,L^2(\Omega))} + \|\psi\|^2_{H^1(0,T,H^1(\Omega))}\right).$$

Similarly, we have

$$\begin{aligned}
M_2 &\leq C\left(h^2 + (\Delta t)^2\right)\left(\|\varphi\|^2_{H^3(0,T,H^2(\Omega))} + \|\varphi\|^2_{H^2(0,T,H^1(\Omega))} + \|\varphi\|^2_{H^1(0,T,H^2(\Omega))}\right). \\
M_3 &\leq C\left(h^2 + (\Delta t)^2\right)\left(\|w\|^2_{H^3(0,T,H^1(\Omega))} + \|w\|^2_{H^2(0,T,L^2(\Omega))} + \|w\|^2_{H^1(0,T,H^1(\Omega))}\right). \\
M_4 &\leq C\left(h^2 + (\Delta t)^2\right)\left(\|w\|^2_{H^3(0,T,H^2(\Omega))} + \|w\|^2_{H^2(0,T,H^1(\Omega))} + \|w\|^2_{H^1(0,T,H^2(\Omega))}\right). \\
M_5 &\leq C\left(h^2 + (\Delta t)^2\right)\left(\|\varphi\|^2_{H^3(0,T,H^1(\Omega))} + \|\varphi\|^2_{H^2(0,T,L^2(\Omega))} + \|\varphi\|^2_{H^1(0,T,H^1(\Omega))}\right). \\
M_6 &\leq C\left(h^2 + (\Delta t)^2\right)\left(\|\psi\|^2_{H^3(0,T,H^2(\Omega))} + \|\psi\|^2_{H^2(0,T,H^1(\Omega))} + \|\psi\|^2_{H^1(0,T,H^2(\Omega))}\right). \\
M_7 &\leq C\left(h^2 + (\Delta t)^2\right)\left(\|\theta\|^2_{H^3(0,T,H^2(\Omega))} + \|\theta\|^2_{H^2(0,T,H^1(\Omega))} + \|\theta\|^2_{H^1(0,T,H^2(\Omega))}\right).
\end{aligned}$$

and

$$\begin{aligned}
X_1 &\leq C\left(h^2 + (\Delta t)^2\right)\left(\|\varphi\|^2_{H^4(0,T,L^2(\Omega))} + \|\varphi\|^2_{H^3(0,T,H^1(\Omega))}\right). \\
X_2 &\leq C\left(h^2 + (\Delta t)^2\right)\left(\|\psi\|^2_{H^4(0,T,L^2(\Omega))} + \|\psi\|^2_{H^3(0,T,H^1(\Omega))}\right).
\end{aligned}$$

$$
\begin{aligned}
X_3 \leq & C\left(h^2+(\Delta t)^2\right)\left(\|w\|^2_{H^4(0,T,L^2(\Omega))}+\|w\|^2_{H^3(0,T,H^1(\Omega))}\right).\\
X_4 \leq & C\left(h^2+(\Delta t)^2\right)\left(\|\theta\|^2_{H^4(0,T,L^2(\Omega))}+\|\theta\|^2_{H^3(0,T,H^1(\Omega))}\right).\\
R_1 \leq & Ch^2\left(\|\varphi\|^2_{H^1(0,T,H^1(\Omega))}\right).\\
R_2 \leq & Ch^2\left(\|\theta\|^2_{H^1(0,T,H^1(\Omega))}\right).\\
R_3 \leq & Ch^2\left(\|\psi\|^2_{H^1(0,T,H^1(\Omega))}\right).\\
R_4 \leq & Ch^2\left(\|w\|^2_{H^1(0,T,H^2(\Omega))}\right).\\
R_5 \leq & Ch^2\left(\|\theta\|^2_{L^2(0,T,H^2(\Omega))}\right).\\
R_6 \leq & Ch^2\left(\|w\|^2_{H^1(0,T,H^2(\Omega))}+\|\varphi\|^2_{H^1(0,T,H^1(\Omega))}\right).\\
R_7 \leq & Ch^2\,\|\psi\|^2_{L^2(0,T,H^2(\Omega))}.\\
R_8 \leq & Ch^2\left(\|\varphi\|^2_{L^2(0,T,H^2(\Omega))}+\|w\|^2_{L^2(0,T,H^1(\Omega))}\right).\\
R_9 \leq & Ch^2\left(\|\varphi\|^2_{L^2(0,T,H^2(\Omega))}+\|\psi\|^2_{L^2(0,T,H^1(\Omega)}+\|w\|^2_{L^2(0,T,H^1(\Omega)}\right).
\end{aligned}
$$

We deduce from the estimates of the residual term X^n that

$$\Delta t\sum_{j=1}^{n} X^j \leq c(h^2+\Delta t^2+\frac{h^2}{(\Delta t)^2})$$

This yields

$$\Delta t\sum_{j=1}^{n} X^j \leq c(h^2+\Delta t^2)$$

where c is constant depends only on appropriate norms of the solution (φ,ψ,w,θ). Applying the discrete Gronwall lemma (backward difference form) to (3.4.21),

$$I^n \leq e^{nc\Delta t}\left(I^0+\Delta t\sum_{j=1}^{n} e^{-jc\Delta t}X^j\right).$$

Since $I^0=0$ and $n\Delta t \leq T$, this provides the appropriate bound for each X^n and applying a triangle inequality and (3.4.12) gives the desired result. □

In the next theorem we provide an estimate of the difference between the true solution and the solution of the full-discrete problem.

Theorem 3.6. *Let* (φ,ψ,w,θ) *be the solution of the exact system* (3.1.16)-(3.1.17) *and* $\left(\phi_h^n,\chi_h^n,\omega_h^n,\nu_h^n\right)$ *the solution of the full discrete problem* (3.4.3)-(3.4.6), *Under the regularity of Theorem 4.2 the* *following order of convergence is obtained:*

$$E_q(t_n) \leq C(h^2 + (\Delta t)^2),$$

where

$$\begin{aligned}
E_q(t_n) = {} & \rho_1 \left\| \dot{\varphi}(t_n) - \widehat{\phi}_h^n \right\|^2 + \rho_2 \| \dot{\psi}(t_n) - \widehat{\chi}_h^n \|^2 + \rho_1 \| \dot{w}(t_n) - \widehat{\omega}_h^n \|^2 \\
& + \rho_3 \left\| \dot{\theta}(t_n) - \widehat{\nu}_h^n \right\|^2 + b \| \psi_x(t_n) - \chi_{hx}^n \|^2 + \kappa \| \theta_x(t_n) - \nu_{hx}^n \|^2 \\
& + k \| (\varphi_x + \psi + lw)(t_n) - (\phi_{hx}^n + \chi_h^n + l\omega_h^n) \|^2 \\
& + k_0 \| (w_x - l\varphi)(t_n) - (\omega_{hx}^n - l\phi_h^n) \|^2 \leq C(h^2 + (\Delta t)^2).
\end{aligned}$$

Proof. Using that $\|a+b\|^2 = \|a-c+c+b\|^2 \leq 2\|a-c\|^2 + 2\|c+b\|^2$, we get

$$\begin{aligned}
E_q(t_n) \leq c(& \left\| \dot{\varphi}(t_n) - \dot{\phi}(t_n) \right\|^2 + \left\| \dot{\phi}(t_n) - \widehat{\phi}_h^n \right\|^2 \\
& + \| \dot{\psi}(t_n) - \dot{\chi}(t_n) \|^2 + \| \dot{\chi}(t_n) - \widehat{\chi}_h^n \|^2 \\
& + \| \dot{w}(t_n) - \dot{\omega}(t_n) \|^2 + \| \dot{\omega}(t_n) - \widehat{\omega}_h^n \|^2 \\
& + \left\| \dot{\theta}(t_n) - \dot{\nu}(t_n) \right\|^2 + \| \dot{\nu}(t_n) - \widehat{\nu}_h^n \|^2 \\
& + \| \psi_x(t_n) - \chi_x(t_n) \|^2 + \| \chi_x(t_n) - \chi_{hx}^n \|^2 \\
& + \| \theta_x(t_n) - \nu_x(t_n) \|^2 + \| \nu_x(t_n) - \nu_{hx}^n \|^2 \\
& + \| (w_x - l\varphi)(t_n) - (\omega_x - l\phi)(t_n) \|^2 \\
& + \| (\omega_x - l\phi)(t_n) - (\omega_{hx}^n - l\phi_h^n) \|^2 \\
& + \| (\varphi_x + \psi + lw)(t_n) - (\phi_x + \chi + l\omega)(t_n) \|^2 \\
& + \| (\phi_x + \chi + l\omega)(t_n) - (\phi_{hx}^n + \chi_h^n + l\omega_h^n) \|^2),
\end{aligned}$$

and for $t = t_n$, the result follows from Theorem 3.4 (we put $\varepsilon_1 = \varepsilon_2 = h^2$) and Theorem 4.2. □

3.4.2 Simulation

In the first experiment, we investigated the discrete energy decay. We consider the following physical parameters,

$$\begin{aligned}
G = {} & E/(2+2r), \; I = ae^3/12, \;\; A = ae, \; b = EI, \; k = K'GA \\
\rho_1 = {} & \rho A, \; \rho_2 = \rho I, \; \rho_3 = \rho_1, \; k_0 = EA,
\end{aligned}$$

and

$$\begin{aligned}\Delta t =&0.005,\, h=0.01,\, E=22\times 10^4,\ \rho=8850,\ K^{'}=5/6,\\ r =&0.29,\ e=0.025,\ a=0.004,\ \sigma=\tau=10^{-2},\ \kappa=5,\ \alpha=1,\end{aligned} \tag{3.4.22}$$

with the initial conditions

$$\begin{aligned}&\varphi^0(x)=\widehat{\varphi}^0(x)=5\sin(\pi x), && \theta^0(x)=\widehat{\theta}^0(x)=x^2(x-1),\\ &\psi^0(x)=\widehat{\psi}^0(x)=\tfrac{1}{10}(\cos(\pi x)+1), && w^0(x)=\widehat{w}^0(x)=\tfrac{1}{20}(\cos(\pi x)+1).\end{aligned}$$

The Figure 3.1 shows that the energy E(t) of the system (3.1.16)-(3.1.17) decreases exponentially

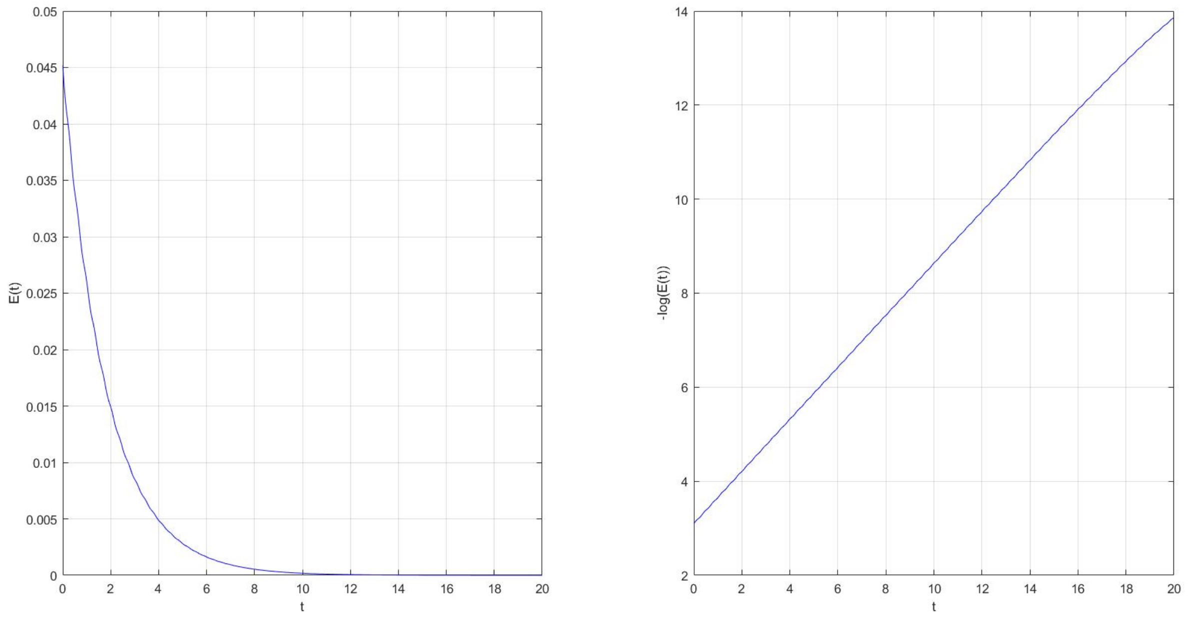

Figure 3.1: The evolution of the discrete energy in natural and log.

which is confirmed by Figure 3.2.

Next, in the Figures 3.3, 3.4, 3.5, 3.6, we show the behavior of the solutions by taking $\Delta t = 0.001$, $h=0.001$ and the following initial condition,

$$\begin{aligned}&\varphi^0(x)=\widehat{\varphi}^0(x)=\theta^0(x)=\widehat{\theta}^0(x)=x^2(x-1),\\ &\psi^0(x)=\widehat{\psi}^0(x)=w^0(x)=\widehat{w}^0(x)=0.\end{aligned}$$

Example 3.1 (Numerical errors)**.** *The same problem* $(3.1.16)-(3.1.17)$ *we used as an academic*

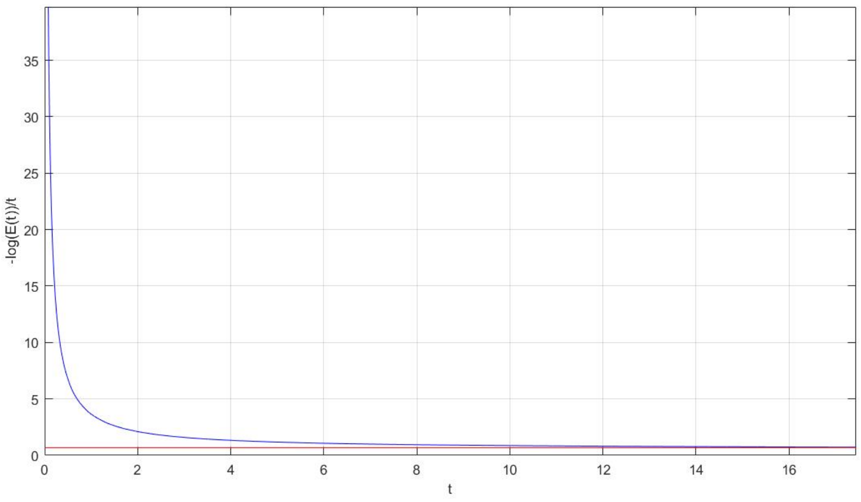

Figure 3.2: The -Log(E(t))/t development in time.

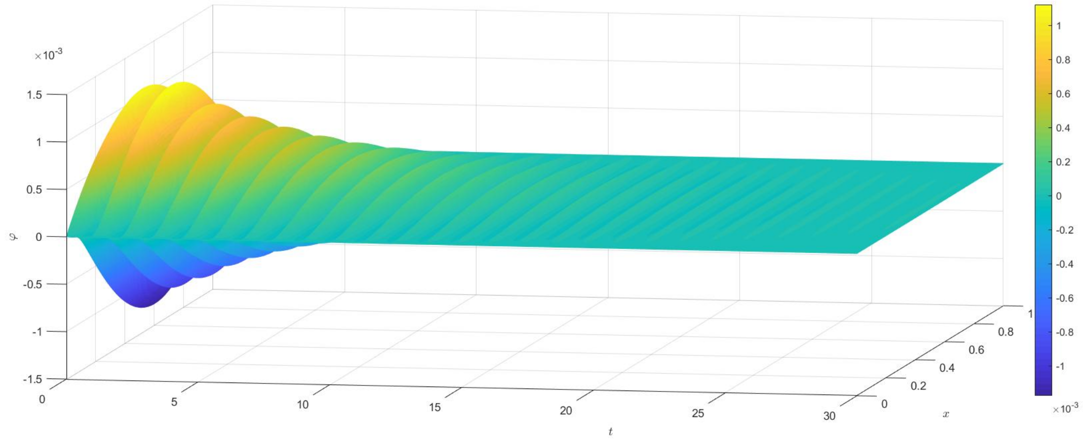

Figure 3.3: Numerical solution for the vertical displacement.

example to demonstrate the accuracy of approximations for

$$\begin{cases} F^1 = -\sigma\dot{\varphi} + e^t\left(\left(\rho_1+\sigma+2k_0l^2-k-l\,(k+k_0)\right)x(x-1)-2k\right) \\ F^2 = \alpha\psi_{xx} + e^t\left((\rho_2+k+kl)\,x^2(\tfrac{x}{3}-\tfrac{1}{2})+(k-\alpha-b)\,(2x-1)\right) \\ F^3 = e^t\left(\left(\rho_1+kl+kl^2\right)x^2(\tfrac{x}{3}-\tfrac{1}{2})+\left(l\,(k+k_0)-k_0+k_0l\right)(2x-1)\right) \\ F^4 = e^t\left(\left(\rho_3+k_0l-k_0l^2\right)x(x-1)-2\,(\kappa+\tau)\right) \end{cases} \tag{3.4.23}$$

with the following data,

$$\Delta t = 0.01,\ \ \kappa = l = b = k = \sigma = \tau = 1,$$
$$\rho_1 = 10,\ \ \rho_2 = 8,\ \ \rho_3 = \rho_1,\ \ k_0 = 2,$$

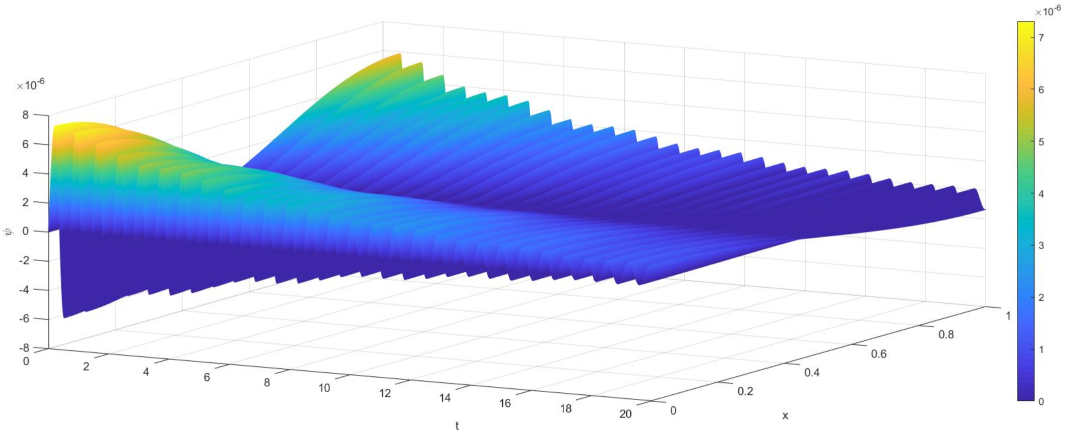

Figure 3.4: Numerical solution for the shear angle.

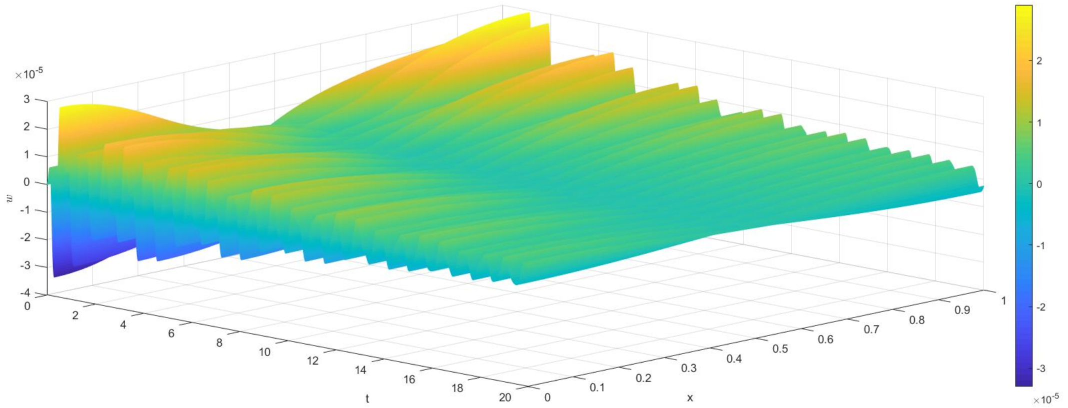

Figure 3.5: Numerical solution for londitudinal displacement.

and initial conditions

$$\varphi^0 = \widehat{\varphi}^0 = \theta^0 = \widehat{\theta}^0 = x(x-1)$$

$$\psi^0 = \widehat{\psi}^0 = w^0 = \widehat{w}^0 = \frac{x^2}{6}(2x-3).$$

The exact solution to Problem $(3.1.16)-(3.1.17)-(3.4.23)$ *it can easily determined which are*

$$\begin{cases} \varphi(x,t) = \theta(x,t) = e^t x(x-1), \\ \psi(x,t) = w(x,t) = \frac{e^t x^2}{6}(2x-3). \end{cases}$$

The numerical errors are presented in Table 1 were it is shown that the error of the energy

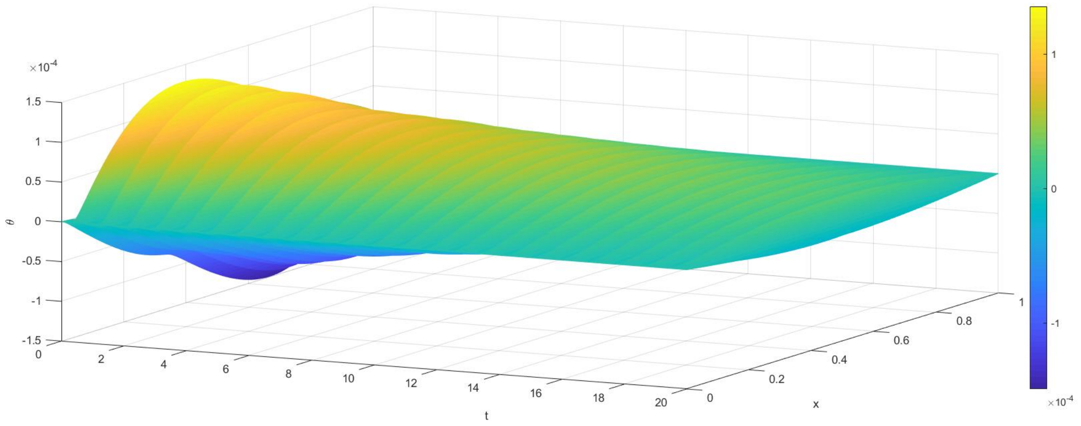

Figure 3.6: Numerical solution for temperature.

elements,

$$\begin{cases} e_1 = \left\|\widehat{\varphi}_h^n - \dot{\varphi}(t_n)\right\|^2, \ e_2 = \left\|\widehat{\psi}_h^n - \dot{\psi}(t_n)\right\|^2, \ e_3 = \left\|\widehat{w}_h^n - \dot{w}(t_n)\right\|^2 \\ e_4 = \left\|\widehat{\theta}_h^n - \dot{\theta}(t_n)\right\|^2, e_5 = \left\|\psi_{hx}^n - (\psi(t_n))_x\right\|^2, \ e_6 = \left\|\theta_{hx}^n - (\theta(t_n))_x\right\|^2 \\ e_7 = \left\|\varphi_{hx}^n + \psi_h^n + lw_h^n - ((\varphi(t_n))_x + \psi(t_n) + lw(t_n))\right\|^2 \\ e_8 = \left\|\left(w_{hx}^n - l\varphi_h^n\right)^2 - ((w(t_n))_x - l\varphi(t_n))\right\|^2 \end{cases}$$

Table 1: the numerical errors

J	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	$\sum_{i=1}^{8} e_i$
500	*0.0481*	*0.0165*	*0.0139*	*0.0523*	*0.0009*	*0.0026*	*0.0093*	*0.0174*	*0.161*
1000	*0.0240*	*0.0083*	*0.0070*	*0.0261*	*0.0004*	*0.0013*	*0.0047*	*0.0087*	*0.0805*
1500	*0.0160*	*0.0055*	*0.0046*	*0.0174*	*0.0003*	*0.0009*	*0.0031*	*0.0087*	*0.0537*
2000	*0.0120*	*0.0041*	*0.0035*	*0.0131*	*0.0002*	*0.0007*	*0.0023*	*0.0043*	*0.0403*
2500	*0.0096*	*0.0033*	*0.0028*	*0.0105*	*0.0002*	*0.0007*	*0.0019*	*0.0035*	*0.0322*

Resolution Method

Let $\{\bar{\chi}_i\}_{i=1}^J$ and $\{\bar{\eta}_i\}_{i=1}^J$ be the basis functions of Y^h and S^h, respectively. We set the following functions:

$$\psi^n = \sum_{i=1}^{J} \breve{\psi}_i^n \bar{\chi}_i, \quad \widehat{\psi}^n = \sum_{i=1}^{J} \tilde{\psi}_i^n \bar{\chi}_i,$$

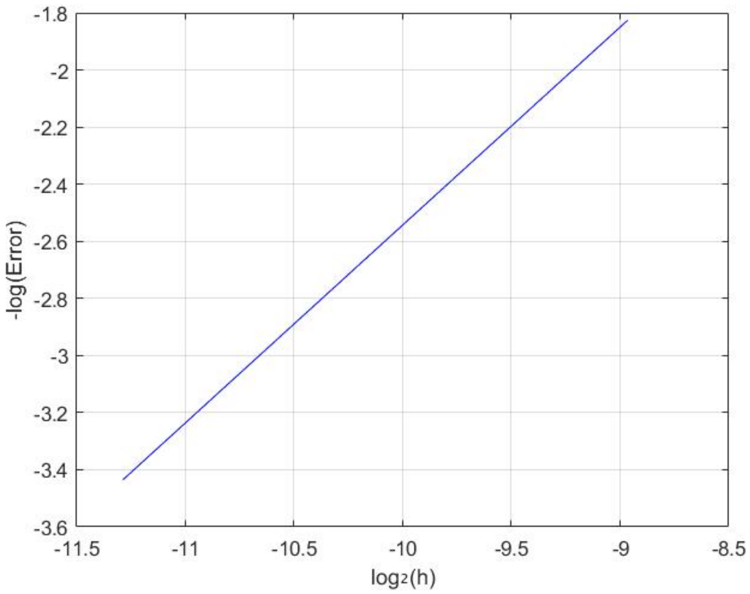

Figure 3.7: the rate of convergence

$$\varphi^n = \sum_{i=1}^{J} \breve{\varphi}_i^n \bar{\eta}_i, \quad \widehat{\varphi}^n = \sum_{i=1}^{J} \tilde{\varphi}_i^n \bar{\eta}_i,$$
$$w^n = \sum_{i=1}^{J} \breve{w}_i^n \bar{\chi}_i, \quad \widehat{w}^n = \sum_{i=1}^{J} \tilde{w}_i^n \bar{\chi}_i,$$
$$\theta^n = \sum_{i=1}^{J} \breve{\theta}_i^n \bar{\eta}_i, \quad \widehat{\theta}^n = \sum_{i=1}^{J} \tilde{\theta}_i^n \bar{\eta}_i.$$

Taking $(z_h, l_h, p_h, r_h) = (\bar{\eta}_i, \bar{\chi}_i, \bar{\chi}_i, \bar{\eta}_i)$, we find that the finite element problems (3.4.3)-(3.4.6) can be written in matrix form as:

$$\bar{A}\hat{y}^n = \bar{B}\hat{y}^{n-1} + \bar{C}y^{n-1}$$

with

$$\hat{y}^n = (\widehat{\psi}^n, \widehat{\varphi}^n, \widehat{w}^n, \widehat{\theta}^n)^T,$$
$$\hat{y}^{n-1} = (\widehat{\psi}^{n-1}, \widehat{\varphi}^{n-1}, \widehat{w}^{n-1}, \widehat{\theta}^{n-1})^T,$$
$$y^{n-1} = (\psi^{n-1}, \varphi^{n-1}, w^{n-1}, \theta^{n-1})^T,$$

Here, $\bar{A}$, $\bar{B}$, $\bar{C}$ are block matrices ($\bar{B}$ is a diagonal block matrix) given by:

$$\bar{A} = \begin{pmatrix} A_1 & A_5 & A_8 & Z \\ \acute{A}_5 & A_2 & A_6 & A_9 \\ \acute{A}_8 & \acute{A}_6 & A_3 & A_7 \\ Z & -A_9 & A_7 & A_4 \end{pmatrix},$$

$$\bar{C} = \begin{pmatrix} C_1 & C_5 & C_7 & Z \\ \acute{C}_5 & C_2 & C_6 & Z \\ \acute{C}_7 & \acute{C}_6 & C_3 & Z \\ Z & Z & Z & C_4 \end{pmatrix},$$

$$\bar{B} = \operatorname{diag}(B_1, B_2, B_3, B_4),$$

where $Z = 0_{J+1\times N}$. and the matrices A_i, B_i, C_i are defined as:

$$\begin{cases} A_1 = \left(\left(\frac{\rho_2}{\triangle t} + k\triangle t\right)\bar{M} + (b\triangle t + \sigma_2)\bar{P}\right) \\ A_2 = \left(\left(\frac{\rho_1}{\triangle t} + \sigma_1 + k_0 l^2 \triangle t\right) M + k\triangle t P\right) \\ A_3 = \left(\left(\frac{\rho_1}{\triangle t} + kl^2\triangle t\right)\bar{M} + k_0\triangle t\bar{P}\right) \\ A_4 = \left(\frac{\rho_3}{\triangle t} M + (\kappa\triangle t + \tau)P\right) \\ A_5 = k\triangle t\bar{O}, \quad \acute{A}_5 = -k\triangle t O \\ A_6 = -l(k+k_0)dtO, \quad \acute{A}_6 = l(k+k_0)\triangle t\bar{O} \\ A_7 = k_0 l\bar{O}, \quad \acute{A}_7 = k_0 l O \\ A_8 = \triangle t k l\bar{M}, \quad \acute{A}_8 = \triangle t k l M \\ A_9 = \triangle k_0 l^2\bar{M}, \quad \acute{A}_9 = -k_0 l^2 M \end{cases},$$

$$\begin{cases} B_1 = \frac{\rho_2}{\triangle t}\bar{M}, \quad B_2 = \frac{\rho_1}{\triangle t}M, \\ B_3 = \frac{\rho_1}{\triangle t}\bar{M}, \quad B_4 = \frac{\rho_3}{\triangle t}M \end{cases},$$

$$\begin{cases} C_1 = -(b\bar{P} + k\bar{M}), \quad C_2 = -(kP + k_0 l^2 M) \\ C_3 = -(k_0\bar{P} + kl^2\bar{M}), \quad C_4 = -\kappa P \\ C_5 = -k\bar{O}, \quad \acute{C}_5 = kO \\ \acute{C}_7 = -kl\bar{M}, \quad C_7 = -kl\bar{M} \\ C_6 = l(k+k_0)O, \quad \acute{C}_6 = -l(k+k_0)\bar{O} \end{cases}$$

where:

$$\bar{M} = (\bar{\chi}_i, \bar{\chi}_j), \quad \bar{P} = (\bar{\chi}_{ix}, \bar{\chi}_{jx}), \quad \bar{O} = (\bar{\chi}_{ix}, \bar{\chi}_j),$$

$$M=(\bar{\eta}_i,\bar{\eta}_i),\quad P=(\bar{\eta}_{ix},\bar{\eta}_{ix}),\quad O=(\bar{\eta}_{ix},\bar{\eta}_j).$$

With a variable change using the **Galerkin method**

$$N^1(x)=1-\frac{x}{h},N^2(x)=\frac{x}{h},x\in(0,h), \tag{3.4.24}$$

and denoting $N=\begin{pmatrix}N^1\\N^2\end{pmatrix}$, we have $V^e=\begin{pmatrix}N^1 & N^2\end{pmatrix}\begin{pmatrix}v_1^n\\v_2^n\end{pmatrix}$ between two nodes. and

$$M^e=\langle N,N^t\rangle=\frac{h}{6}\begin{pmatrix}2&1\\1&2\end{pmatrix},\quad P^e=\langle N_x,N_x^t\rangle=\frac{1}{h}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\quad ,O^e=\langle N,N_x^t\rangle=\frac{1}{2}\begin{pmatrix}-1&1\\-1&1\end{pmatrix}$$

Finally, we perform the assembly of M^e, P^e, O^e to obtain $\bar{M}$, $\bar{P}$, $\bar{O}$.

Algorithm

Algorithm 3.1.

```
1: Initialize parameters:
    • Set values for $\rho_1$, $\rho_2$, $\sigma_1$, $\sigma_2$, $k$, $k_0$, $l$, $\kappa$, $\tau$.
    • Define time and space steps $\Delta t,h,N,J,T$.
    • Set initial condition $y^0,\hat{y}^0$.
    • Define the matrices $M^e,P^e,O^e$.
    • Perform the assembly of $M^e$, $P^e$, $O^e$ to obtain $\bar{M},M$, $\bar{P},P$, $\bar{O},O$.
    • Define the block matrices $\bar{A}$, $\bar{B}$, $\bar{C}$ using the provided definition.
2: Compute $\hat{y}^n$
3: for $i=1$ to $N$ do compute $\hat{y}^n$:
        $\hat{y}^n=\bar{A}^{-1}\left(\bar{B}\hat{y}^{n-1}+\bar{C}y^{n-1}\right)$
    Compute $y^n$
4: end for
```

Chapter 4

Timoshenko System with Dual-Phase-Lag Thermoelasticity

4.1 Introduction

The Timoshenko beam, a pivotal theory in applied mathematics and mechanics, has attracted considerable research attention since its inception in the early 20th century. Various papers delve into the stability analysis of the Timoshenko system under different damping scenarios, encompassing strong and weak frictional damping, viscoelastic, and thermal dampings (see [25, 73–79]). However, the current literature on linear thermoelasticity of solids, adopting the Fourier model for heat conduction, faces criticism due to the paradox of infinite thermal signal propagation speeds.

To address this critique, researchers have shifted focus towards developing theories of linear thermoelasticity with finite wave speeds. Notable theories in this regard include the Lord-Shulman theory [6], characterized by one relaxation time; the Green-Lindsay theory [8], featuring two relaxation times; and the dual-phase-lag thermoelasticity (DPL) proposed by Tzou [12] in 1997, aiming to model ultrafast thermoelastic processes. Additionally, Choudhuri's generalized theory of three-phase-lag thermoelasticity was presented in [18].

The dual-phase-lag (DPL) heat conduction equation [13, 14] was formulated by introducing two phase-lags into the Fourier law of heat conduction, accounting for microstructural changes during high-rate heat transfer. It also describes the impact of material defects and thermo-mechanical coupling induced by ultrafast heating (see [12, 15, 16, 80, 81]), where experimental data from Tzou [17] confirm the physical significance of the dual-phase-lag model. Mathematically, Fourier's law is approximated by an equation, where τ_θ represents the phase lag of the temperature gradient and τ_q is the phase lag of the heat flux. In [82], Quintanilla established stability conditions for solutions in the DPL theory, emphasizing that τ_θ and τ_q must be sufficiently small and satisfy the condition $2\tau_\theta \geq \tau_q$.

In light of this, there is a growing interest in applying various thermoelasticity models. For instance, in [20], Z. Liu and R. Quintanilla examined a one-dimensional thermo-porous-elastic problem with microtemperatures in the context of dual-phase-lag heat conduction, the system is given by

$$\begin{aligned}
&\rho \ddot{u} = \mu^* u_{xx} + \mu_0 \varphi_x - \beta_0 \left(\tfrac{\tau_q^2}{2} \ddot{\theta} + \tau_q \dot{\theta} + \theta \right)_x , \\
&J \ddot{\varphi} = a_0 \varphi_{xx} - \mu_0 u_x - \mu_2 \left(\tfrac{\tau_q^2}{2} \ddot{T} + \tau_q \dot{T} + T \right)_x + \beta_1 \left(\tfrac{\tau_q^2}{2} \ddot{\theta} + \tau_q \dot{\theta} + \theta \right) - \xi \varphi, \\
&a \left(\tfrac{\tau_q^2}{2} \dddot{\theta} + \tau_q \ddot{\theta} + \dot{\theta} \right) = -\beta_0 \dot{u}_x + k \left(\tau_\theta \dot{\theta}_x + \theta_x \right)_x - \beta_1 \dot{\varphi}_x + k_1 \left(\tau_\theta \dot{T}_x + T_x \right), \\
&b \left(\tfrac{\tau_q^2}{2} \dddot{T} + \tau_q \ddot{T} + \dot{T} \right) = k_4 \left(\tau_\theta \dot{T}_x + T_x \right)_x - \mu_2 \dot{\varphi}_x - k_2 \left(T + \tau_\theta \dot{T} \right) - k_1 \left(\tau_\theta \dot{\theta}_x + \theta_x \right).
\end{aligned} \tag{4.1.1}$$

Wherein the research showcased the demonstration of the existence and uniqueness of solutions, establishing polynomial stability in the scenario $2\tau_\theta = \tau_q$ and exponential decay when $2\tau_\theta > \tau_q$. Bazarra et al. in [83] delved into the exploration of a thermoelastic Bresse system incorporating a dual-phase-lag model. Their investigation yielded numerical results pertaining to energy decay, solution behavior, and numerical convergence. Specifically, they scrutinized the following system,

$$\begin{aligned}
&\rho_1 \ddot{\varphi} - k (\varphi_x + \psi + lw)_x - k_0 l (w_x - l\varphi) = 0, \\
&\rho_2 \ddot{\psi} - b \psi_{xx} + k (\varphi_x + \psi + lw) + \tfrac{\tau_q^2 \gamma}{2} \ddot{\theta}_x + \tau_q \gamma \dot{\theta}_x + \gamma \theta_x = 0, \\
&\rho_1 \ddot{w} - k_0 (w_x - l\varphi)_x + kl (\varphi_x + \psi + lw) = 0, \\
&\left(\tfrac{\tau_q^2}{2} \dddot{\theta} + \tau_q \ddot{\theta} + \ddot{\theta} \right) + \gamma \theta^0 \dot{\psi}_x - \kappa \left(\tau_\theta \dot{\theta}_x + \theta_x \right)_x = 0.
\end{aligned} \tag{4.1.2}$$

4.1.1 Model derivation

In Chapter 2, specifically in section 2.5, we explored [29] various constitutive laws in the mathematical-physics context of the classic Timoshenko system without thermal effects or temperature considerations. To justify the inclusion of thermal effects in the Timoshenko model, we posit that the beam experiences an unknown temperature difference denoted as $\widehat{\theta}(x,y,z,t)$. This temperature difference significantly influences the deformation of the beam, with its deviation measured from a reference state characterized by a uniform temperature distribution $\widehat{\theta}_0$ when the beam is in a rest position (without stresses or strains). Applying thermoelastic principles based on references [26, 84, 85], specifically focusing on the (x, z)-plane. The temperature distribution is simplified by assuming $\widehat{\theta}(x,0,z,t) = \widehat{\theta}(x,z,t)$, indicating no temperature variation in the y-direction. To accommodate the thinness of the beam/plate and guided by assumptions from [[85], Chapt. III] and [26], we utilize Taylor's expansion in the (x, z)-plane at y = 0:

$$\widehat{\theta}(x,z,t) = \widehat{\theta}(x,0,z,t) = z\theta(x,t). \tag{4.1.3}$$

Here, θ denote temperature component (functions) that may represent temperature deviations from the reference temperature Θ along the longitudinal and vertical directions. Now we will derive a linear model for thermoelastic Timoshenko beams by taking into account the displacements and the temperature distribution in the plane of reference, see Fig.2 again. This will be done in the same steps as in section 2.5 where the thermal strains is given by

$$\varepsilon_{1j}^{T} = \delta_{1j}\varepsilon^{T}, \quad j = 1,3, \tag{4.1.4}$$

In addition, it is assumed that the change of temperature $\widehat{\theta}$ is small when compared to the reference temperature θ_0 (that is, $\left|\widehat{\theta}/\theta_0\right| << 1$) and, consequently, one gets the relation

$$\varepsilon^{T} = \gamma\widehat{\theta}, \tag{4.1.5}$$

where $\alpha > 0$ is a constant called coefficient of thermal expansion. Therefore, combining (4.1.4) and (4.1.5) , we obtain the next expressions for the thermal strains

$$\varepsilon_{11}^{T}(x,z,t) = \gamma\delta_{11}\widehat{\theta}(x,z,t) = \gamma\delta_{11}\left[z\theta(x,t)\right], \tag{4.1.6}$$

$$\varepsilon_{13}^{T}(x,z,t) = \gamma\delta_{13}\widehat{\theta}(x,z,t) = \gamma\delta_{13}\left[z\theta(x,t)\right]. \tag{4.1.7}$$

Going back to (2.5.1) and (2.5.4) and following the same steppes as in section 2.5 the conventional formulas to express the bending moment and the shear force are given by

$$\bar{S}(x,t) = 2k'GA\left[\frac{1}{2}(\psi + \varphi_x)(x,t)\right], \quad x \in [0,L], t \geq 0. \tag{4.1.8}$$

$$\bar{M}(x,t) = EI\left[\psi_x(x,t) - \gamma\delta_{11}\theta(x,t)\right], \quad x \in [0,L], t \geq 0. \tag{4.1.9}$$

To establish the thermoelastic relation, we need to introduce a heat conduction motion equation for the temperature deviation $\widehat{\theta}(x,z,t)$. Utilizing the general Newton's law for heat flux (refer to, e.g., Eq. (30) in [84] or (2.12) in [85]), we consider

$$\rho_0 c_\theta \dot{\widehat{\theta}} = -\mathbf{q}_x - \gamma\theta_0\left(\dot{\varepsilon}_{13}\right), \tag{4.1.10}$$

where heat flux vector $\mathbf{q}(x,0,z,t)$ is also taken as

$$\mathbf{q}(x,0,z,t) = zq(x,t),$$

and c_v denotes the heat capacity, and ρ_0 represents the density per unit of reference. Incorporating the linearized expansion for $\widehat{\theta}$ and expressions (2.5.1) for the strains, (2.5.2) and (2.5.3), the heat conduction Eq. (4.1.10) simplifies to

$$\rho_0 c_v \dot{\theta} = -q_x - \gamma\theta_0\dot{\psi}_x. \tag{4.1.11}$$

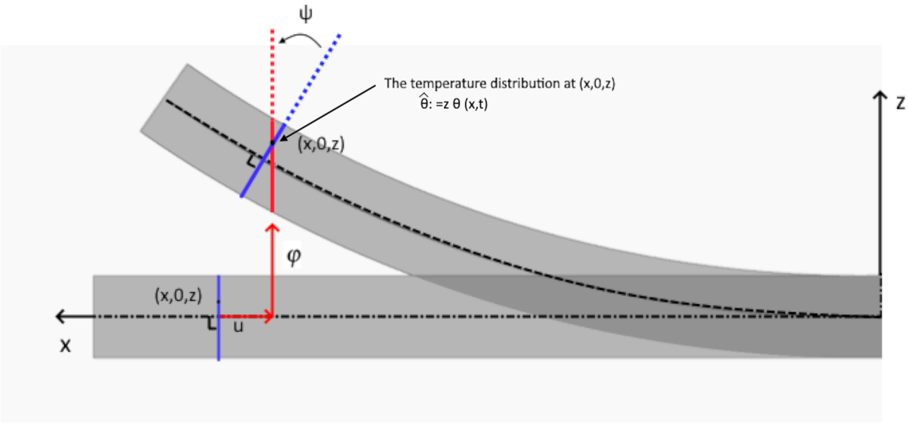

Figure 4.1: Displacements and temperature distribution in the (x , z)-plane

In this investigation, we employ the concept of coupling a Timoshenko system with dual-phase-lag heat conduction to analyze its influence on system stability. So the thermoelastic Timoshenko system, with effective thermoelastic dissipation on bending moments is given by this basic evolution equations

$$\rho A\ddot{\varphi} - \bar{S}_x = 0 \ , \tag{4.1.12}$$

$$\rho I\ddot{\psi} - \bar{M}_x + \bar{S} = 0, \tag{4.1.13}$$

$$\rho_0 c_\upsilon \dot{\theta} + q_x + \gamma\theta^0 \dot{\psi}_x = 0, \tag{4.1.14}$$

where t is the time, x is the distance along the center line of the beam structure, ρ is the density, c_υ being the heat capacity, A is the cross-sectional area, I is a cross-section's moment of inertia, $\bar{M}$ is the bending moment, $\bar{S}$ is the shear force, q denotes the heat flow, θ is the temperature difference and the positive constants θ^0, γ represent the reference temperature and the coupling coefficient. The variables φ and ψ are the transversal displacement and the rotational angle of the beam, respectively. From (4.1.8) the constitutive equations are:

$$\bar{S} = k(\varphi_x + \psi), \quad \bar{M} = b\psi_x - \gamma\theta. \tag{4.1.15}$$

For simplicity, we note ρ_1 $=\rho A$, $\rho_2 = \rho I$, $\rho_3 = \rho c_\upsilon$, $k = 2k'AG$, $b = EI$, in which k'_0 is a constant which depends upon the shape of the cross-section, E is the Young's modulus of elasticity, and G is the modulus of rigidity. Using Fourier's law of heat conduction

$$q = -\kappa\theta_x, \tag{4.1.16}$$

we get the following system by substituting the constitutive Eqs. (4.1.15)-(4.1.16) into the evolution Eqs. (4.1.12) and (4.1.14)

$$\begin{cases} \rho_1 \ddot{\varphi} - k\left(\varphi_x + \psi\right)_x = 0, \\ \rho_2 \ddot{\psi} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \gamma\theta_x = 0, \\ \rho_3 \dot{\theta} - \kappa\theta_{xx} + \gamma\theta^0 \dot{\psi}_x = 0, \end{cases} \tag{4.1.17}$$

this system has been studied by Muñoz Rivera and Reinhard Racke in [86]. They proved that the system Eqs. (4.1.17) is exponentially stable if and only if $\lambda = 0$,

$$\lambda = \frac{\rho_1}{k} - \frac{\rho_2}{b}. \tag{4.1.18}$$

In this study, we consider the second order dual phase-lag heat conduction

$$q + \tau_q \dot{q} + \frac{\tau_q^2}{2}\ddot{q} = -\kappa\left(\tau_\theta \dot{\theta} + \theta\right)_x. \tag{4.1.19}$$

Now, by substituting the constitutive Eqs. (4.1.15) and (4.1.19) into the evolution Eqs. (4.1.12) and (4.1.14), we get the following linear system:

$$\begin{cases} \rho_1 \ddot{\varphi} - k\left(\varphi_x + \psi\right)_x = 0, \\ \rho_2 \ddot{\psi} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \gamma\theta_x = 0, \\ \left(\frac{\tau_q^2}{2}\dddot{\theta} + \tau_q \ddot{\theta} + \dot{\theta}\right) - \kappa\left(\tau_\theta \dot{\theta}_x + \theta_x\right)_x \quad + \gamma\theta^0\left(\frac{\tau_q^2}{2}\dddot{\psi} + \tau_q \ddot{\psi} + \dot{\psi}\right)_x = 0. \end{cases} \tag{4.1.20}$$

Here we take $\rho_3 = 1$, then to establish the dissipative nature of system (4.1.20), we use the notation $\hat{f} = f + \tau_q \dot{f} + \frac{\tau_q^2}{2}\ddot{f}$, to rewrite it as

$$\begin{cases} \rho_1 \ddot{\hat{\varphi}} - k\left(\hat{\varphi}_x + \hat{\psi}\right)_x = 0, \\ \rho_2 \ddot{\hat{\psi}} - b\hat{\psi}_{xx} + k\left(\hat{\varphi}_x + \hat{\psi}\right) + \gamma\left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q \dot{\theta} + \theta\right)_x = 0, \\ \left(\frac{\tau_q^2}{2}\dddot{\theta} + \tau_q \ddot{\theta} + \dot{\theta}\right) - \kappa\left(\tau_\theta \dot{\theta}_x + \theta_x\right)_x + \gamma\theta^0 \dot{\hat{\psi}}_x = 0. \end{cases} \tag{4.1.21}$$

For simplicity of notation, we continue to use the notation (φ, ψ) instead of $(\hat{\varphi}, \hat{\psi})$. We will work with the following system:

$$\begin{cases} \rho_1 \ddot{\varphi} - k\left(\varphi_x + \psi\right)_x = 0, \\ \rho_2 \ddot{\psi} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \gamma\left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q \dot{\theta} + \theta\right)_x = 0, \\ \left(\frac{\tau_q^2}{2}\dddot{\theta} + \tau_q \ddot{\theta} + \dot{\theta}\right) - \kappa\left(\tau_\theta \dot{\theta}_x + \theta_x\right)_x + \gamma\theta^0 \dot{\psi}_x = 0, \end{cases} \tag{4.1.22}$$

where $x \in [0,1]$, $t \in \mathbb{R}_+$.

We consider homogeneous Dirichlet boundary conditions for φ, θ and Neumann for ψ,

$$\begin{aligned} &\varphi(0,t) = \varphi(1,t) = 0, \ \psi_x(0,t) = \psi_x(1,t) = 0, \\ &\theta(0,t) = \theta(1,t) = 0, \ t > 0, \end{aligned} \tag{4.1.23}$$

and the initial data,

$$\begin{aligned} &\varphi(x,0) = \varphi_0(x), \ \dot{\varphi}(x,0) = \varphi_1(x), \ \psi(x,0) = \psi_0(x), \\ &\dot{\psi}(x,0) = \psi_1(x), \theta(x,0) = \theta_0(x), \ \dot{\theta}(x,0) = \theta_1(x), \\ &\ddot{\theta}(x,0) = \theta_2(x), \ x \in [0,1]. \end{aligned} \tag{4.1.24}$$

We note that our system (4.1.22) is composed of three equations. Tow are conservative, while only one is dissipative. In which γ is the coupling parameter between the thermal and mechanical parts and the two mechanical equations are also coupled between them. Besides that, using the second equation of (4.1.22) and the boundary conditions (4.1.23), it follows that

$$\rho_2 \frac{d^2}{dt^2} \int_0^1 \ddot{\psi} dx + k \int_0^1 \psi dx = 0. \tag{4.1.25}$$

Solving Eq. (4.1.25) and using the initial data (Eq. (4.1.24)) of ψ, we obtain

$$\int_0^1 \psi dx = \left(\int_0^1 \psi_0 dx \right) \cos\left(\sqrt{\frac{k}{\rho_2}} t \right) + \sqrt{\frac{\rho_2}{k}} \left(\int_0^1 \psi_1 dx \right) \left(\sin \sqrt{\frac{k}{\rho_2}} t \right).$$

Consequently, if we set

$$\overline{\psi} = \psi - \left(\int_0^1 \psi_0 dx \right) \cos\left(\sqrt{\frac{k}{\rho_2}} t \right) - \sqrt{\frac{\rho_2}{k}} \left(\int_0^1 \psi_1 dx \right) \left(\sin \sqrt{\frac{k}{\rho_2}} t \right),$$

we get

$$\int_0^1 \overline{\psi} dx = 0, \forall t \geq 0,$$

with initial data

$$\overline{\psi}_0(x) = \psi_0(x) - \int_0^1 \psi_0(x) dx \text{ and } \overline{\psi}_1(x) = \psi_1(x) - \int_0^1 \psi_1(x) dx.$$

Therefore, we can use Poincaré's inequality for $\overline{\psi}$. Henceforth, we work with $\overline{\psi}$ instead of ψ but write ψ.

Chapter structure

In the next section 4.2, we establish the existence and uniqueness of the solutions. Section 4.3 is devoted to stating and proving the technical lemmas needed for the proof of our exponential stability result, while in Section 4.4 we present the necessary and sufficient conditions over the system's coefficients in order to achieve exponential stability. In particular, when $2\tau_\theta > \tau_q$ and $\varkappa = 0$, where $\varkappa$ is a stability number defined by:

$$\varkappa = (\rho_1 b - \rho_2 k)\left(\kappa - \frac{\tau_q^2 k}{2\tau_\theta \rho_1}\right) - \frac{k\gamma^2 \tau_q^2 \theta^0}{2\tau_\theta}. \tag{4.1.26}$$

Section 4.5 is dedicated to the critical case of stability, where we prove the lack of exponential stability of the system (4.1.22)-(4.1.24) when the phase-lag τ_θ and τ_q verifies $2\tau_\theta = \tau_q$ and $\varkappa \neq 0$. In section 4.6 we present some numerical simulations obtained using MATLAB software. We finish the paper by given some conclusions. It is worth noting that when $\tau_q = 0$ the new stability number $\varkappa$ reduces to the classical stability number λ, i.e., we arrive at the same results obtained for the Timoshenko systems with the Fourier law of heat conduction [86].

4.2 Existence and Uniqueness

We denote $U = (\varphi, \widehat{\varphi}, \psi, \widehat{\psi}, \theta, \vartheta, \widehat{\theta})^T$, where $\widehat{\varphi} = \dot{\varphi}$, $\widehat{\psi} = \dot{\psi}$, $\vartheta = \dot{\theta}$ and $\widehat{\theta} = \dot{\vartheta}$. Then, system (4.1.22)-(4.1.24) can be rewritten as follows:

$$\begin{cases} \dot{U} = \mathscr{A}U, \ t > 0, \\ U(x,0) = U_0(x) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \theta_1, \theta_2)^T, \end{cases} \tag{4.2.1}$$

where the operator $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \longrightarrow \mathscr{H}$ is defined by

$$\mathscr{A}U = \begin{pmatrix} \widehat{\varphi} \\ \frac{k}{\rho_1}(\varphi_x + \psi)_x \\ \widehat{\psi} \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi) - \frac{\tau_q^2 \gamma}{2\rho_2}\widehat{\theta}_x - \frac{\tau_q \gamma}{\rho_2}\vartheta_x - \frac{\gamma}{\rho_2}\theta_x \\ \vartheta \\ \widehat{\theta} \\ -\frac{2}{\tau_q}\widehat{\theta} - \frac{2}{\tau_q^2}\vartheta - \frac{2\gamma\theta^0}{\tau_q^2}\widehat{\psi}_x + \frac{2\kappa\tau_\theta}{\tau_q^2}\vartheta_{xx} + \frac{2\kappa}{\tau_q^2}\theta_{xx} \end{pmatrix}, \tag{4.2.2}$$

and $\mathscr{H}$ is the energy space given by

$$\mathscr{H} = H_0^1(0,1) \times L^2(0,1) \times H_*^1(0,1) \times L_*^2(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \times L^2(0,1),$$

such that

$$\begin{aligned} &H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1), \\ &L_*^2(0,1) = \left\{ \varphi \in L^2(0,1) : \int_0^1 \varphi(x)\, dx = 0 \right\}. \end{aligned}$$

For any $U = (\varphi, \widehat{\varphi}, \psi, \widehat{\psi}, \theta, \vartheta, \widehat{\theta})^T \in \mathscr{H}$, $\tilde{U} = (\widetilde{\varphi}, \widetilde{\widehat{\varphi}}, \widetilde{\psi}, \widetilde{\widehat{\psi}}, \widetilde{\theta}, \widetilde{\vartheta}, \widetilde{\widehat{\theta}})^T \in \mathscr{H}$, we equip the space $\mathscr{H}$ with the inner product defined by

$$\begin{aligned} \langle U, \tilde{U} \rangle_{\mathscr{H}} = \int_0^1 &\Big[\rho_1 \theta^0 \widehat{\varphi} \widetilde{\widehat{\varphi}} + \rho_2 \theta^0 \widehat{\psi} \widetilde{\widehat{\psi}} + b\theta^0 \psi_x \widetilde{\psi}_x dx + k\theta^0 (\varphi_x + \psi)(\widetilde{\varphi}_x + \widetilde{\psi}) + \\ &+ \left(\frac{\tau_q^2}{2} \widehat{\theta} + \tau_q \vartheta + \theta \right) \left(\frac{\tau_q^2}{2} \widetilde{\widehat{\theta}} + \tau_q \widetilde{\vartheta} + \widetilde{\theta} \right), \\ &+ \frac{\kappa \tau_\theta \tau_q^2}{2} \vartheta_x \widetilde{\vartheta}_x + \kappa (\tau_\theta + \tau_q) \theta_x \widetilde{\theta}_x + \frac{\kappa \tau_q^2}{2} \left(\theta_x \widetilde{\vartheta}_x + \widetilde{\theta}_x \vartheta_x \right) \Big] dx. \end{aligned} \tag{4.2.3}$$

Remark 4.1. *With the assumption $2\tau_\theta \geq \tau_q$, it is not difficult to see that the symmetric matrix Z,*

$$Z := \begin{pmatrix} \frac{\tau_\theta \tau_q^2}{2} & \frac{\tau_q^2}{2} \\ \frac{\tau_q^2}{2} & (\tau_\theta + \tau_q) \end{pmatrix},$$

is positive definite $(i.e., \det(Z) = \frac{1}{4} \tau_q^2 \left(2\tau_\theta^2 + \tau_q (2\tau_\theta - \tau_q) \right) > 0)$, hence, the following inequality holds

$$(\tau_\theta + \tau_q) \theta_x^2 + \frac{\tau_\theta \tau_q^2}{2} \dot{\theta}_x^2 + \tau_q^2 \theta_x \dot{\theta}_x \geq 0. \tag{4.2.4}$$

The space $\mathscr{H}$ is a Hilbert space with the norm,

$$\begin{aligned} \|U\|_{\mathscr{H}}^2 = \int_0^1 &\Big[\rho_1 \theta^0 \widehat{\varphi}^2 + \rho_2 \theta^0 \widehat{\psi}^2 + b\theta^0 \psi_x^2 dx + k\theta^0 (\varphi_x + \psi)^2 \\ &+ \frac{\kappa \tau_\theta \tau_q^2}{2} \vartheta_x^2 + \kappa (\tau_\theta + \tau_q) \theta_x^2 + \kappa \tau_q^2 \theta_x \vartheta_x \\ &+ \left(\frac{\tau_q^2}{2} \widehat{\theta} + \tau_q \vartheta + \theta \right)^2 \Big] dx, \end{aligned} \tag{4.2.5}$$

imposing Eq. (4.2.4), we can conclude that $\|U\|_{\mathscr{H}}^2$ is positive definite.

The domain of the operator $\mathscr{A}$ is given by:

$$D(\mathscr{A}) = \left\{ U \in \mathscr{H} \mid \begin{array}{c} \varphi \in H^2(0,1);\ \widehat{\varphi}, \vartheta, \theta, \widehat{\theta} \in H_0^1(0,1); \\ \theta + \tau_\theta \vartheta \in H^2(0,1);\ \psi \in H_*^2(0,1);\ \widehat{\psi} \in H_*^1(0,1); \end{array} \right\},$$

where

$$H_*^2(0,1) = \left\{ \Psi \in H^2(0,1) : \Psi_x(0) = \Psi_x(1) = 0 \right\}.$$

Clearly, the domain $D(\mathscr{A})$ is dense in $\mathscr{H}$. To investigate the well-posedness of the above problem (Eq. (4.2.1)), we begin by demonstrating that the operator $\mathscr{A}$ creates a C_0-semigroup of contractions over $\mathscr{H}$. First, imposing $2\tau_\theta \geq \tau_q$ shows that $\mathscr{A}$ is a dissipative operator in the space $\mathscr{H}$. To be more specific, from inner product Eq. (4.2.3), we have

$$\begin{aligned}\operatorname{Re}\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = \int_0^1 \Big[& \rho_1 \theta^0 \widehat{\varphi} \dot{\widehat{\varphi}} + \rho_2 \theta^0 \widehat{\psi} \dot{\widehat{\psi}} + b\theta^0 \psi_x \widehat{\psi}_x + k\theta^0 (\varphi_x + \psi)(\widehat{\varphi}_x + \widehat{\psi}) \\ & + \left(\frac{\tau_q^2}{2} \widehat{\theta} + \tau_q \vartheta + \theta \right) \left(\frac{\tau_q^2}{2} \dot{\widehat{\theta}} + \tau_q \dot{\vartheta} + \gamma \dot{\theta} \right) \\ & + \kappa (\tau_\theta + \tau_q) \theta_x \vartheta_x + \kappa \tau_q^2 \left(\theta_x \widehat{\theta}_x + \vartheta_x \vartheta_x \right) + \frac{\kappa \tau_\theta \tau_q^2}{2} \widehat{\theta}_x \vartheta_x \Big] dx,\end{aligned}$$

by using the equations of the system (4.1.22), we obtain

$$\begin{aligned}\operatorname{Re}\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = \frac{d}{2dt} \int_0^1 \Big[& \rho_1 \theta^0 \widehat{\varphi}^2 + \rho_2 \theta^0 \widehat{\psi}^2 + b\theta^0 \psi_x^2 \\ & + k\theta^0 (\varphi_x + \psi)^2 + \kappa \tau_q^2 \theta_x \vartheta_x + \frac{\kappa \tau_\theta \tau_q^2}{2} \vartheta_x^2 \\ & + \left(\frac{\tau_q^2}{2} \widehat{\theta} + \tau_q \vartheta + \theta \right)^2 + \kappa (\tau_\theta + \tau_q) \theta_x^2 \Big] dx\end{aligned}$$

we arrived at

$$\operatorname{Re}\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = -\kappa \|\theta_x\|^2 - \kappa \tau_q \left(\tau_\theta - \frac{\tau_q}{2} \right) \|\vartheta_x\|^2 \leq 0,$$

with the assumption $2\tau_\theta \geq \tau_q$, we conclude that $\mathscr{A}$ is a dissipative operator in the space $\mathscr{H}$. It remains to prove that $Id - \mathscr{A}$ is surjective i,e,.

$$U - \mathscr{A}U = F, \ \forall F \in \mathscr{H} \tag{4.2.6}$$

has a unique solution $U \in D(\mathscr{A})$.

Let $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T \in \mathscr{H}$, then by simplifying (4.2.6), we have

$$\begin{cases} \varphi - \widehat{\varphi} = f_1 \\ \rho_1 \widehat{\varphi} - k\left(\varphi_x + \psi\right)_x = \rho_1 f_2 \\ \psi - \widehat{\psi} = f_3 \\ \rho_2 \widehat{\psi} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \frac{\tau_q^2 \gamma}{2} \widehat{\theta}_x + \tau_q \gamma \vartheta_x + \gamma \theta_x = f_4 \\ \theta - \vartheta = f_5 \\ \vartheta - \widehat{\theta} = f_6 \\ \frac{\tau_q^2}{2} \widehat{\theta} + \tau_q \widehat{\theta} + \vartheta + \gamma \theta^0 \widehat{\psi}_x - \tau_\theta \kappa \vartheta_{xx} - \kappa \theta_{xx} = \frac{\tau_q^2}{2} f_7 \end{cases}$$

consequently, we get

$$\widehat{\varphi} = \varphi - f_1,\ \widehat{\psi} = \psi - f_3,\ \vartheta = \theta - f_5, \widehat{\theta} = \theta - \left(f_5 + f_6\right), \tag{4.2.7}$$

$$\begin{cases} \rho_1 \varphi - k\varphi_{xx} - k\psi_x = g_1 \\ \left(\rho_2 + k\right)\psi - b\psi_{xx} + k\varphi_x + \left(\frac{\tau_q^2}{2} + \tau_q + 1\right)\gamma\theta_x = g_2 \\ \left(\frac{\tau_q^2}{2} + \tau_q + 1\right)\theta - \kappa\left(\tau_\theta + 1\right)\theta_{xx} + \gamma\theta^0\psi_x = g_3 + g_{4,xx} \end{cases} \tag{4.2.8}$$

Multiplying $(4.2.8)_1$–$(4.2.8)_3$ by $\theta^0\ \widehat{\varphi}$, $\theta^0 \widehat{\psi}$ and $\left(\frac{\tau_q^2}{2} + \tau_q + 1\right)\theta$, respectively and integrating them over$(0,1)$. Then adding all equations, we transform (4.2.6) to the following variational problem,

$$B(\left(\varphi, \psi, \theta\right), \left(\widehat{\varphi}, \widehat{\psi}, \widehat{\theta}\right)) = G\left(\widehat{\varphi}, \widehat{\psi}, \widehat{\theta}\right), \quad \forall \left(\widehat{\varphi}, \widehat{\psi}, \widehat{\theta}\right) \in V \tag{4.2.9}$$

where, $B : V \times V \to \mathbb{R}$ is the bilinear form defined on

$$V := \left[H_0^1\left(0,1\right) \times H_*^1\left(0,1\right) \times H_0^1\left(0,1\right)\right]$$

by

$$\begin{aligned} B(\left(\varphi, \psi, \theta\right), \left(\widehat{\varphi}, \widehat{\psi}, \widehat{\theta}\right)) = & \rho_1\theta^0 \int_0^1 \varphi\widehat{\varphi} dx + k\theta^0 \int_0^1 \varphi_x \widehat{\varphi}_x dx + k\theta^0 \int_0^1 \left(\varphi_x \widehat{\psi} - \psi_x \widehat{\varphi}\right) dx \\ & + \left(\rho_2 + k\right)\theta^0 \int_0^1 \psi\widehat{\psi} dx + b\theta^0 \int_0^1 \psi_x \widehat{\psi}_x dx \\ & + \gamma\theta^0 \left(\frac{\tau_q^2}{2} + \tau_q + 1\right) \int_0^1 \left(\theta_x \widehat{\psi} + \psi_x \widehat{\theta}\right) dx \\ & + \kappa\left(\tau_\theta + 1\right)\left(\frac{\tau_q^2}{2} + \tau_q + 1\right) \int_0^1 \theta_x \widehat{\theta}_x dx \\ & + \left(\frac{\tau_q^2}{2} + \tau_q + 1\right)^2 \int_0^1 \theta\widehat{\theta} dx, \end{aligned}$$

and the linear form $G : V \to \mathbb{R}$ defined by

$$G\left(\widehat{\varphi}, \widehat{\psi}, \widehat{\theta}\right) = \theta^0 \int_0^1 g_1 \widehat{\varphi} dx + \theta^0 \int_0^1 g_2 \widehat{\psi} dx + \left(\frac{\tau_q^2}{2} + \tau_q + 1\right)\left[\int_0^1 g_3 \widehat{\theta} dx + \int_0^1 g_{4,x} \widehat{\theta}_x dx\right],$$

where

$$\begin{cases} g_1 = \rho_1 (f_2 + f_1) \\ g_2 = \rho_2 (f_4 + f_3) + \frac{\tau_q^2 \gamma}{2} (f_5 + f_6)_x + \tau_q \gamma (f_5)_x \\ g_3 = \frac{\tau_q^2}{2} f_7 + \left(\frac{\tau_q^2}{2} + \tau_q\right)(f_5 + f_6) + f_5 + \gamma \theta^0 (f_3)_x \\ g_4 = \tau_\theta \kappa f_{5,x}. \end{cases} \tag{4.2.10}$$

It is obvious that V is a Hilbert space equipped with the L^2 norm.

$$\|(\varphi, \psi, \theta)\|_V^2 = \|\varphi\|_{L^2}^2 + \|\varphi_x\|_{L^2}^2 + \|\psi\|_{L^2}^2 + \|\psi_x\|_{L^2}^2 + \|\theta\|_{L^2}^2 + \|\theta_x\|_{L^2}^2,$$

Using the Cauchy-Schwarz inequality, we can easily demonstrate that B is continuous.

$$B((\varphi, \psi, \theta), \left(\widehat{\varphi}, \widehat{\psi}, \widehat{\theta}\right)) \leq \xi_1 \|(\varphi, \psi, \theta)\|_V \left\|\left(\widehat{\varphi}, \widehat{\psi}, \widehat{\theta}\right)\right\|_V.$$

Similarly

$$G\left(\widehat{\varphi}, \widehat{\psi}, \widehat{\theta}\right) \leq \xi_2 \left\|\left(\widehat{\varphi}, \widehat{\psi}, \widehat{\theta}\right)\right\|_V.$$

On the other hand,

$$\begin{aligned} B((\varphi, \psi, \theta), (\varphi, \psi, \theta)) = & \rho_1 \theta^0 \int_0^1 \varphi^2 dx + k\theta^0 \int_0^1 \varphi_x^2 dx + 2k\theta^0 \int_0^1 \varphi_x \psi dx \\ & + (\rho_2 + k)\theta^0 \int_0^1 \psi^2 dx + b\theta^0 \int_0^1 \psi_x^2 dx \\ & + (\tau_\theta \kappa + \kappa)\left(\frac{\tau_q^2}{2} + \tau_q + 1\right)\int_0^1 \theta_x^2 dx \\ & + \left(\frac{\tau_q^2}{2} + \tau_q + 1\right)^2 \int_0^1 \theta^2 dx \geq \|(\varphi, \psi, \theta)\|_V. \end{aligned}$$

Now, by using the Lax–Milgram Lemma we conclude that (4.2.9) has a unique solution $(\varphi, \psi, \theta) \in V$. Consequently, the following claim is true:

Proposition 4.1. *The operator $\mathscr{A}$ generates a C_0-semigroup $S(t)$ of contractions on the space $\mathscr{H}$.*

Now, we give the well-posedness result.

Theorem 4.1. *Let* $U_0 \in D(\mathscr{A}^n)$*,* $n \in \mathbb{N}$*. Then the problem* (4.1.22)-(4.1.24) *has a unique solution*

$$U \in \cap_{i=0}^{n} C^i\left(\mathbb{R}_+, D(\mathscr{A}^{n-i})\right).$$

Moreover, if $U_0 \in \mathscr{H}$*, then* $U \in C\left(\mathbb{R}_+, \mathscr{H}\right)$*.*

4.3 Technical lemmas

In this section, we state and prove several lemmas needed for the proof of our stability result. Throughout this section, we refer to c_0 as a generic positive constant that will vary from line to line depending on the stated numbers in an increasing manner.

Lemma 4.1. *Let* (φ, ψ, θ) *be a solution of Eqs.* (4.1.22)-(4.1.24) *and* $2\tau_\theta \geq \tau_q$ *. Then, the energy functional* $E(t)$

$$\begin{aligned} E(t) = {} & \frac{1}{2}\int_0^1 \left(\rho\theta^0\dot{\varphi}^2 + \rho_2\theta^0\dot{\psi}^2 + k\theta^0(\varphi_x+\psi)^2 + b\theta^0\psi_x^2 + \left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta} + \theta\right)^2\right)dx \\ & + \frac{\kappa}{2}\int_0^1 \left(\frac{\tau_\theta\tau_q^2}{2}\dot{\theta}_x^2 + \tau_q^2\theta_x\dot{\theta}_x + (\tau_\theta+\tau_q)\theta_x^2\right)dx, \end{aligned} \tag{4.3.1}$$

satisfies

$$E'(t) = -\int_0^1 \left(\kappa\theta_x^2 + \kappa\tau_q\left(\tau_\theta - \frac{\tau_q}{2}\right)\dot{\theta}_x^2\right)dx \leq 0, \ t > 0. \tag{4.3.2}$$

Proof. The Eq. (4.3.2) is obtained by multiplying the first three equations of (4.1.22) by $\dot{\varphi}$, $\dot{\psi}$ and $\left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta} + \theta\right)$ respectively, integrating by parts over $(0,1)$ and using the boundary conditions Eq. (4.1.22). ☐

In the following lammas, we consider $(\varphi,\ \psi,\ \theta)$ to be the solution of the system (4.1.22)-(4.1.24).

Lemma 4.2. *Let the functional* F_1 *be defined as*

$$F_1(t) := -\rho_1\int_0^1 \dot{\varphi}\varphi dx + \rho_2\int_0^1 \dot{\psi}\psi dx, \ t > 0. \tag{4.3.3}$$

Then, F_1 *satisfies the following estimate*

$$\begin{aligned} F_1'(t) \leq {} & -\rho_1\int_0^1 \dot{\varphi}^2 dx - \frac{b}{2}\int_0^1 \psi_x^2 dx + \frac{k}{2}\mu\int_0^1 (\varphi_x+\psi)^2 dx \\ & + \rho_2\int_0^1 \dot{\psi}^2 dx + \frac{\gamma^2}{b}\int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta} + \theta\right)^2 dx, \ t > 0, \end{aligned} \tag{4.3.4}$$

where $\mu = 2\left(\frac{2k}{b}+1\right)$.

Proof. Multiplying Eq. (4.1.22)$_1$ by φ and Eq. (4.1.22)$_2$ by ψ, then integrating by parts over $(0,1)$, we arrive at

$$-\rho_1 \frac{d}{dt}\int_0^1 \dot{\varphi}\varphi dx = -\rho_1 \int_0^1 \dot{\varphi}^2 dx + k\int_0^1 (\varphi_x+\psi)^2 dx - k\int_0^1 (\varphi_x+\psi)\psi dx, \tag{4.3.5}$$

and

$$\begin{aligned}\rho_2 \frac{d}{dt}\int_0^1 \dot{\psi}\psi dx = \rho_2 \int_0^1 \dot{\psi}^2 dx - b\int_0^1 \psi_x^2 dx - k\int_0^1 (\varphi_x+\psi)\psi dx \\ +\gamma\int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta}+\tau_q\dot{\theta}+\theta\right)\psi_x dx.\end{aligned} \tag{4.3.6}$$

Adding (4.3.5) and (4.3.6), we get

$$\begin{aligned}F_1'(t) = &-\rho_1\int_0^1 \dot{\varphi}^2 dx - b\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \dot{\psi}^2 dx + k\int_0^1 (\varphi_x+\psi)^2 dx - 2k\int_0^1 (\varphi_x+\psi)\psi dx \\ &\underbrace{+\gamma\int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta}+\tau_q\dot{\theta}+\theta\right)\psi_x dx.}_{=I_1}\end{aligned}$$

By applying Young's and Poincaré–Wirtinger inequalities, we have

$$-2k\int_0^1 (\varphi_x+\psi)\psi dx \le \frac{2k^2}{b}\int_0^1 (\varphi_x+\psi)^2 dx + \frac{b}{4}\int_0^1 \psi_x^2 dx, \tag{4.3.7}$$

$$I_1 \le \frac{\gamma^2}{b}\int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta}+\tau_q\dot{\theta}+\theta\right)^2 dx + \frac{b}{4}\int_0^1 \psi_x^2 dx, \tag{4.3.8}$$

By using the estimates (4.3.7) and (4.3.8), we obtain (4.3.4). □

Lemma 4.3. *Let the functional* F_2 *be defined as,*

$$F_2(t) := -\frac{\rho_2}{\gamma}\int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta}+\tau_q\dot{\theta}+\theta\right)\left(\int_0^x \dot{\psi}(y,t)dy\right)dx + \frac{b\tau_q^2}{2\gamma}\int_0^1 \dot{\theta}\psi_x dx,\ t>0. \tag{4.3.9}$$

Then for any $\varepsilon_1 > 0$, *the functional* F_2 *satisfies the following estimate*

$$\begin{aligned}F_2'(t) < &-\frac{\rho_2\theta^0}{2}\int_0^1 \dot{\psi}^2 dx + b\varepsilon_0\int_0^1 \psi_x^2 dx + k\varepsilon_1\int_0^1 (\varphi_x+\psi)^2 dx + c_0(1+\frac{1}{\varepsilon_0})\int_0^1 \left(\dot{\theta}_x^2+\theta_x^2\right)dx \\ &+c_0(1+\frac{1}{\varepsilon_1})\int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta}+\tau_q\dot{\theta}+\theta\right)^2 dx,\ t>0.\end{aligned} \tag{4.3.10}$$

Proof. By differentiating the expression of F_2 (Eq.(4.3.9)) and exploiting the Eqs. $(4.1.22)_2$, $(4.1.23)_3$ with the fact $\int_0^1 \psi dy = 0$, we obtain

$$F_2'(t) = -\rho_2\theta^0 \int_0^1 \dot{\psi}^2 dx + \underbrace{\left(\frac{\kappa\tau_\theta\rho_2}{\gamma} - \frac{b\tau_q^2}{2\gamma}\right)\int_0^1 \dot{\theta}_x \dot{\psi} dx}_{=I_2} + \frac{\kappa\rho_2}{\gamma}\int_0^1 \theta_x \dot{\psi} dx$$

$$\underbrace{-\frac{b}{\gamma}\int_0^1 \left(\tau_q\dot{\theta} + \theta\right)\psi_x dx}_{=I_3} + \underbrace{\frac{k}{\gamma}\int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta} + \theta\right)\int_0^x (\varphi_x + \psi)\, dydx}_{=I_4}$$

$$+ \int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta} + \theta\right)^2 dx.$$

Applying Young's, Poincaré–Wirtinger inequalities, we arrived at

$$\kappa\frac{\rho_2}{\gamma}\int_0^1 \theta_x \dot{\psi} dx \leq \frac{\rho_2\theta^0}{4}\int_0^1 \dot{\psi}^2 dx + c_0\int_0^1 \theta_x^2 dx, \tag{4.3.11}$$

$$I_2 \leq \frac{\rho_2\theta^0}{4}\int_0^1 \dot{\psi}^2 dx + c_0\int_0^1 \dot{\theta}_x^2 dx, \tag{4.3.12}$$

and

$$\begin{aligned} I_3 &\leq b\varepsilon_0\int_0^1 \psi_x^2 dx + \frac{b}{4\varepsilon_0\gamma^2}\int_0^1 \left(\tau_q\dot{\theta}_x + \theta_x\right)^2 dx \\ &\leq b\varepsilon_0\int_0^1 \psi_x^2 dx + \frac{b}{2\varepsilon_0\gamma^2}\int_0^1 \left(\tau_q^2\dot{\theta}_x^2 + \theta_x^2\right) dx, \end{aligned} \tag{4.3.13}$$

then using Cauchy Schwarz, we have

$$I_4 \leq k\varepsilon_1\int_0^1 (\varphi_x + \psi)^2 dx + \frac{k}{4\varepsilon_1\gamma^2}\int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta} + \theta\right)^2 dx. \tag{4.3.14}$$

Utilizing (4.3.11), (4.3.12), (4.3.13), and (4.3.14) we conclude (4.3.10). □

Lemma 4.4. *The functional* F_3 *defined by*

$$F_3(t) := -\int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta}\right)\left(\frac{\tau_q^2}{2}\dot{\theta} + \tau_q\theta\right) dx - \frac{\tau_q}{2}\int_0^1 \theta^2 dx - \frac{1}{2}\left(\kappa\tau_q\tau_\theta + \kappa\frac{\tau_q^2}{2}\right)\int_0^1 \theta_x^2 dx,\ t > 0, \tag{4.3.15}$$

satisfies, for any $\varepsilon_3 > 0$*, the following estimate*

$$
\begin{aligned}
F_3'(t) \leq & -\frac{1}{2}\int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta} + \theta\right)^2 dx + \rho_2\varepsilon_2\int_0^1 \dot{\psi}^2 dx \\
& + c_0\left(1 - \frac{1}{\varepsilon_2}\right)\int_0^1 \left(\dot{\theta}_x^2 + \theta_x^2\right) dx,\ t > 0. \qquad (4.3.16)
\end{aligned}
$$

Proof. By differentiating the expression of F_3 (with the use of Eq. $(4.1.22)_3$) and integrating by parts, we obtain

$$
\begin{aligned}
F_3'(t) = & -\int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta}\right)^2 dx + \kappa\frac{\tau_q^2}{2}\tau_\theta\int_0^1 \dot{\theta}_x^2 dx \\
& + \kappa\tau_q\int_0^1 \theta_x^2 dx + \frac{\tau_q^2}{2}\int_0^1 \dot{\theta}^2 dx \underbrace{- \gamma\theta^0\int_0^1 \dot{\psi}\left(\frac{\tau_q^2}{2}\dot{\theta}_x + \tau_q\theta_x\right) dx}_{=I_5}.
\end{aligned}
$$

Using Young and Poincaré inequalities, we get

$$
I_5 \leq \varepsilon_2\rho_2\int_0^1 \dot{\psi}^2 dx + \frac{\gamma^2\left(\theta^0\right)^2}{2\rho_2\varepsilon_2}\int_0^1 \left(\frac{\tau_q^4}{4}\dot{\theta}_x^2 + \tau_q^2\theta_x^2\right) dx,
$$

and the fact that

$$
-\int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta}\right)^2 dx \leq -\frac{1}{2}\int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta} + \theta\right)^2 dx + \int_0^1 \theta_x^2 dx,
$$

we conclude the estimate (4.3.16). ☐

To prove the essential Lemma that contains the condition on the coefficients (4.1.26), we define the following two functionals G_1 and G_2 by:

$$
G_1(t) := \rho_1\frac{b}{k}\int_0^1 \dot{\varphi}\psi_x dx + \rho_2\int_0^1 \dot{\psi}\left(\varphi_x + \psi\right) dx + \gamma\frac{\tau_q^2}{2}\int_0^1 \dot{\theta}_x\left(\varphi_x + \psi\right) dx, t > 0 \qquad (4.3.17)
$$

and

$$
G_2(t) := \rho_1\int_0^1 \dot{\varphi}\left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta} + \theta\right) dx + \kappa\rho_1\int_0^1 \theta_x\varphi_x dx + k\frac{\tau_q^2}{2}\int_0^1 \left(\varphi_x + \psi\right)\dot{\theta}_x dx, t > 0. \qquad (4.3.18)
$$

Then, we have the following lemma:

Lemma 4.5. *The functional* F_4 *defined by*

$$F_4(t) := G_1(t) + \frac{1}{\gamma\theta^0\rho_1}\left(\frac{b\rho_1}{k}-\rho_2\right)G_2(t),\ t>0,$$

satisfies for any $\varepsilon_3 > 0$, *the following estimate*

$$\begin{aligned} F_4'(t) \le & -\frac{k}{2}\int_0^1(\varphi_x+\psi)^2\,dx + 2\rho_2\int_0^1\dot\psi^2dx + \varepsilon_3 b\int_0^1\psi_x^2dx \\ & + c_0(1+\frac{1}{\varepsilon_3})\int_0^1\dot\theta_x^2dx + c_0\int_0^1\theta_x^2dx - \frac{\tau_\theta\varkappa}{\gamma\theta^0k}\int_0^1\dot\theta_x\dot\varphi_xdx,\ t>0. \end{aligned} \tag{4.3.19}$$

Proof. By differentiating the expression of G_1 and G_2, then using Eqts. $(4.1.22)_1-(4.1.22)_3$ and integrating by parts, we arrive at

$$\begin{aligned} G_1'(t) = & -k\int_0^1(\varphi_x+\psi)^2\,dx - \gamma\int_0^1\left(\tau_q\dot\theta+\theta\right)_x(\varphi_x+\psi)\,dx + \gamma\frac{\tau_q^2}{2}\int_0^1\dot\theta_x\dot\varphi_xdx \\ & + \left(\rho_2-\frac{b\rho_1}{k}\right)\int_0^1\dot\varphi_x\dot\psi dx + \gamma\frac{\tau_q^2}{2}\int_0^1\dot\theta_x\dot\psi dx + \rho_2\int_0^1\dot\psi^2dx, \end{aligned} \tag{4.3.20}$$

and

$$\begin{aligned} G_2'(t) = & -k\int_0^1(\varphi_x+\psi)\left(\tau_q\dot\theta+\theta\right)_xdx + k\frac{\tau_q^2}{2}\int_0^1\dot\psi\dot\theta_xdx \\ & + \left(\frac{\tau_q^2}{2}k-\kappa\tau_\theta\rho_1\right)\int_0^1\dot\theta_x\dot\varphi_xdx - \gamma\theta^0\rho_1\int_0^1\dot\psi\dot\varphi_xdx \\ & + \kappa\rho_1\int_0^1\dot\theta_x\varphi_xdx. \end{aligned} \tag{4.3.21}$$

Then, by multiplying Eq. (4.3.21) by $\frac{1}{\gamma\theta^0\rho_1}\left(\frac{b\rho_1}{k}-\rho_2\right)$ and adding (4.3.20), we obtain

$$\begin{aligned} F_4'(t) = & -k\int_0^1(\varphi_x+\psi)^2\,dx + \rho_2\int_0^1\dot\psi^2dx - \gamma\int_0^1\left(\tau_q\dot\theta+\theta\right)_x(\varphi_x+\psi)\,dx \\ & - k\frac{1}{\gamma\theta^0\rho_1}\left(\frac{b\rho_1}{k}-\rho_2\right)\int_0^1(\varphi_x+\psi)\left(\tau_q\dot\theta+\theta\right)_xdx \\ & + \kappa\rho_1\frac{1}{\gamma\theta^0\rho_1}\left(\frac{b\rho_1}{k}-\rho_2\right)\int_0^1\dot\theta_x(\varphi_x+\psi)\,dx \\ & - \frac{\tau_\theta}{\gamma\theta^0k}\left(\underbrace{(\rho_1b-\rho_2k)\left(\kappa-\frac{\tau_q^2k}{2\tau_\theta\rho_1}\right)-\frac{\tau_q^2\gamma^2\theta^0k}{2\tau_\theta}}_{=\varkappa}\right)\int_0^1\dot\theta_x\dot\varphi_xdx \end{aligned}$$

$$
+\underbrace{\left(k\frac{\tau_q^2}{2}\frac{1}{\gamma\theta^0\rho_1}\left(\frac{b\rho_1}{k}-\rho_2\right)+\gamma\frac{\tau_q^2}{2}\right)\int_0^1 \psi\dot{\theta}_x dx}_{=I_7}
$$

$$
\underbrace{-\kappa\rho_1\frac{1}{\gamma\theta^0\rho_1}\left(\frac{b\rho_1}{k}-\rho_2\right)\int_0^1 \dot{\theta}_x\psi dx.}_{=I_8} \tag{4.3.22}
$$

Let

$$
\begin{aligned}
I_6 = & -k\frac{1}{\gamma\theta^0\rho_1}\left(\frac{b\rho_1}{k}-\rho_2\right)\int_0^1 (\varphi_x+\psi)\left(\tau_q\dot{\theta}+\theta\right)_x dx \\
& +\kappa\rho_1\frac{1}{\gamma\theta^0\rho_1}\left(\frac{b\rho_1}{k}-\rho_2\right)\int_0^1 \dot{\theta}_x(\varphi_x+\psi)\,dx \\
& -\gamma\int_0^1 \left(\tau_q\dot{\theta}+\theta\right)_x(\varphi_x+\psi)\,dx.
\end{aligned} \tag{4.3.23}
$$

Now, using Young's and Poincaré–Wirtinger inequalities on (4.3.22) and (4.3.23), we get

$$
I_6 \leq \frac{k}{2}\int_0^1 (\varphi_x+\psi)^2 dx + c_0\int_0^1 \left(\dot{\theta}_x^2+\theta_x^2\right)dx, \tag{4.3.24}
$$

$$
I_7 \leq \rho_2\int_0^1 \psi^2 dx + c_0\int_0^1 \dot{\theta}_x^2 dx, \tag{4.3.25}
$$

$$
I_8 \leq b\varepsilon_3\int_0^1 \psi_x^2 dx + \frac{c_0}{\varepsilon_3}\int_0^1 \dot{\theta}_x^2 dx. \tag{4.3.26}
$$

Utilizing the previous estimates (4.3.24), (4.3.25) and (4.3.26), we conclude (4.3.19). □

4.4 Exponential decay

In this section, we use the energy method to prove that the system (4.1.22)-(4.1.24) is exponentially stable when $2\tau_\theta > \tau_q$ *and* $\varkappa = 0$.

Theorem 4.2. *Suppose that* $2\tau_\theta > \tau_q$ *and* $\varkappa = 0$. *Then there exist positive constants* a_1, a_2 *such that the energy* $E(t)$ *satisfies*

$$
E(t) \leq a_1 e^{-a_2 t}, \tag{4.4.1}
$$

for any $t > 0$.

To prove this theorem, we need the following Lemma.

Lemma 4.6. *Let lyapunov functional* $\mathscr{F}$ *defined by*

$$
\mathscr{F} := NE(t) + N_1F_4(t) + F_1(t) + N_2F_2(t) + N_3F_3(t), \tag{4.4.2}
$$

where N, N_1, N_2, N_3 *and* N_4 *are positives constants. Then* $\mathscr{F}$ *verifies the following inequalities:*

$$\alpha E(t) \leq \mathscr{F}(t) \leq \beta E(t), \tag{4.4.3}$$

$$\mathscr{F}(t) \leq -\eta E(t), \tag{4.4.4}$$

where η, α *and* β *are positives constant.*

Proof. From Eq. (4.4.2) we have

$$|\mathscr{F}(t) - NE(t)| \leq N_1 |F_4(t)| + |F_1(t)| + N_2 |F_2(t)| + N_3 |F_3(t)|.$$

By exploiting the expression of F_4, F_1, F_2 and F_3 then using Young's and Cauchy Schwarz inequalities we get

$$\begin{aligned}
|\mathscr{F}(t) - NE(t)| \leq \frac{C}{2} \Bigg(& \rho_1 \theta^0 \int_0^1 \dot{\varphi}^2 dx + \rho_2 \theta^0 \int_0^1 \dot{\psi}^2 dx + b\theta^0 \int_0^1 \psi_x^2 dx \\
& + k\theta^0 \int_0^1 (\varphi_x + \psi)^2 dx + \int_0^1 \left(\frac{\tau_q^2}{2} \ddot{\theta} + \tau_q \dot{\theta} + \theta \right)^2 dx \\
& + \int_0^1 \left(\kappa (\tau_\theta + \tau_q) \theta_x^2 + \frac{\kappa \tau_\theta \tau_q^2}{2} \dot{\theta}_x^2 + \kappa \tau_q^2 \theta_x \dot{\theta}_x \right) dx \Bigg) \\
& - C\kappa \tau_q^2 \int_0^1 \theta_x \dot{\theta}_x dx,
\end{aligned}$$

and the fact that (see (4.2.4))

$$-\kappa \tau_q^2 \int_0^1 \theta_x \dot{\theta}_x dx \leq \kappa (\tau_\theta + \tau_q) \int_0^1 \theta_x^2 dx + \frac{\kappa \tau_\theta \tau_q^2}{2} \int_0^1 \dot{\theta}_x^2 dx$$

therefore there exist positive constant $\widetilde{C}$ such that

$$|\mathscr{F}(t) - NE(t)| \leq \widetilde{C} E(t)$$

which implies that

$$\Bigg(\underbrace{N - \widetilde{C}}_{\alpha} \Bigg) E(t) \leq \mathscr{F}(t) \leq \Bigg(\underbrace{\widetilde{C} + N}_{\beta} \Bigg) E(t) \tag{4.4.5}$$

we conclude that $\mathscr{F}(t) \sim E(t)$ for N large enough such that $N - \widetilde{C} > 0$. □

Proof. **of Theorem 4.2** By differentiating $\mathscr{F}$ with respect to t and using the estimates (4.3.2), (4.3.19), (4.3.4), (4.3.10) and (4.3.16) in previous lemmas and the assumption $\varkappa = 0$, we obtain

$$\begin{aligned}\frac{d}{dt}\mathscr{F}(t) \leq -\int_0^1 \big(& d_{\dot\varphi}\dot\varphi^2 + d_{\dot\psi}\dot\psi^2 + d_{\psi_x}\psi_x^2 + d_{\dot\theta_x}\dot\theta_x^2 \\ &+ d_{(\varphi_x+\psi)}(\varphi_x+\psi)^2 + d_{\theta_x}\theta_x^2 \\ &+ d_\theta\left(\frac{\tau_q^2}{2}\ddot\theta + \tau_q\dot\theta + \theta\right)^2\Bigg) dx,\end{aligned}$$

where

$$\begin{cases} d_{\dot\varphi} = \rho_1 \\ d_{(\varphi_x+\psi)} = k\left(\frac{N_1}{2} - \frac{\mu}{2} - N_2\varepsilon_1\right) \\ d_{\psi_x} = b\left(\frac{1}{2} - \varepsilon_3 N_1 - N_2\varepsilon_0\right) \\ d_\theta = \frac{1}{2}N_3 - \frac{\gamma^2}{b} - N_2 c_0(1+\frac{1}{\varepsilon_1}) \\ d_{\dot\psi} = \rho_2\left(\frac{\theta^0}{2}N_2 - 2N_1 - 1 - \varepsilon_2 N_3\right) \\ d_{\theta_x} = N\kappa - c_0N_1 - N_2c_0(1+\frac{1}{\varepsilon_0}) - N_3c_0\left(1 - \frac{1}{\varepsilon_2}\right) \\ d_{\dot\theta_x} = \kappa\tau_q\left(\tau_\theta - \frac{\tau_q}{2}\right)N - c_0(1+\frac{1}{\varepsilon_3})N_1 - N_2c_0(1+\frac{1}{\varepsilon_1}) - N_3c_0\left(1-\frac{1}{\varepsilon_2}\right). \end{cases}$$

By choosing

$$\begin{aligned} &N_1 = 4\mu,\ \varepsilon_0 = \frac{1}{8N_2},\ \varepsilon_1 = \frac{\mu}{2N_2}, \\ &\varepsilon_2 = \frac{N_1}{2N_3},\ \varepsilon_3 = \frac{1}{8N_1}, \end{aligned}$$

we get

$$\begin{cases} d_{\dot\varphi} = \rho_1 \\ d_{(\varphi_x+\psi)} = k\mu \\ d_{\psi_x} = \frac{b}{4} \\ d_\theta = \frac{1}{2}N_3 - \frac{\gamma^2}{b} - c_0N_2(1+\frac{2N_2}{\mu}) \\ d_{\dot\psi} = \rho_2\left(\frac{\theta^0}{2}N_2 - 10\mu - 1\right) \\ d_{\theta_x} = N\kappa - c_0N_1 - N_2c_0(1+8N_2) - N_3c_0\left(1-\frac{2N_3}{N_1}\right), \\ d_{\dot\theta_x} = \kappa\tau_q\left(\tau_\theta - \frac{\tau_q}{2}\right)N - c_0N_1(1+8N_1) - N_2c_0(1+\frac{2N_2}{\mu}) - N_3c_0\left(1-\frac{2N_3}{N_1}\right). \end{cases}$$

Then, we select N_2 and N_3 respectively, such that the following inequalities hold

$$\begin{aligned} &N_2 > \frac{2}{\theta^0}(10\mu+1), \\ &N_3 > 2\left[\frac{\gamma^2}{b} + c_0N_2(1+\frac{2N_2}{\mu})\right]. \end{aligned}$$

Also, we select N large enough, such that

$$d_{\theta_x} > 0,\ d_{\dot{\theta}_x} > 0, \text{ and } N > \widetilde{C}.$$

Using previous selections, we get

$$\mathscr{L}(t) \leq -\eta E(t),\ \forall t > 0,$$

where η is a positive constant. And, by using the inequality (4.4.5) we get

$$\mathscr{L}(t) \leq -\frac{\eta}{\beta}\mathscr{L}(t),\ \forall t > 0. \tag{4.4.6}$$

A simple integration of inequality (5.4.6) over $(0,t)$ yields

$$\mathscr{L}(t) \leq \mathscr{L}(0)e^{-\frac{\eta}{\beta}t},\ \forall t > 0. \tag{4.4.7}$$

Recalling the inequality (4.4.5) and the estimate (4.4.7) we arrive at the desired result (4.4.1).

□

4.5 The lack of exponential decay

Now, we prove the lack of exponential decay under the assumption $2\tau_\theta = \tau_q$ and $\varkappa \neq 0$ by using Gearhart-Prüss theorem [87]. We write $\varkappa$ as

$$\varkappa = (\rho_1 b - \rho_2 k)\rho - \gamma^2\tau_q k\theta^0,$$

where

$$\rho = \kappa - \frac{\tau_q k}{\rho_1}.$$

Proposition 4.2. *The C_0-semigroup of contractions associated with the operator $\mathscr{A}$ on the Hilbert space $\mathscr{H}$ is exponentially stable, if and only if*

$$i\mathbb{R} \equiv \{i\alpha, \alpha \in \mathbb{R}\} \subset \rho(A)$$

and

$$\overline{\lim_{|\alpha|\to\infty}} \left\|(i\alpha I - \mathscr{A})^{-1}\right\|_{\mathscr{H}} < +\infty.$$

The section's main result is

Theorem 4.3. *Let $2\tau_\theta = \tau_q$ and $\mathscr{A}$ the operator defined by* (4.2.2) *and we assume that $\varkappa \neq 0$. Then the semigroup associated with $\mathscr{A}$ is not exponentially stable.*

Proof. It is sufficient to demonstrate the existence of $F \in \mathbb{H}$, with $\|F\|_{\mathscr{H}} < \infty$ such that

$$\|U_\alpha\|_{\mathscr{H}} = \left\|(i\alpha I - \mathscr{A})^{-1} F\right\|_{\mathscr{H}} \to \infty, as\,|\alpha| \to \infty.$$

Let's assume that there exists $U \in \mathbb{H}$ such that

$$(i\alpha I - \mathscr{A})\,U{=}F, \tag{4.5.1}$$

where $\|U\|_{\mathscr{H}} \neq 0$. Simplifying (4.5.1), for a fixed $F{=}\left(0, \frac{\sin(n\pi x)}{\rho}, 0,0,0,0,0\right) \in \mathscr{H}$ and using (4.2.2), we get

$$\widehat{\varphi} = i\alpha\varphi,\ \ \widehat{\psi} = i\alpha\psi,\ \ \vartheta = i\alpha\theta,\ \ \widehat{\theta} = i\alpha\vartheta = -\alpha^2\theta, \tag{4.5.2}$$

$$\begin{cases} i\alpha\rho_1\widehat{\varphi} - k\varphi_{xx} - k\psi_x = \sin(n\pi x), \\ i\rho_2\alpha\widehat{\psi} - b\psi_{xx} + k\varphi_x + k\psi + \frac{\tau_q^2\gamma}{2}\widehat{\theta}_x + \tau_q\gamma\vartheta_x + \gamma\theta_x = 0, \\ i\frac{\tau_q^2}{2}\alpha\widehat{\theta} + \tau_q\widehat{\theta} + \vartheta + \gamma\theta^0\widehat{\psi}_x - \kappa\tau_\theta\vartheta_{xx} - \kappa\theta_{xx} \ = 0, \end{cases} \tag{4.5.3}$$

By substituting (4.5.2) in (4.5.3), we have

$$\begin{cases} -\alpha^2\rho_1\varphi - k\varphi_{xx} - k\psi_x = \sin(n\pi x), \\ \left(k - \alpha^2\rho_2\right)\psi - b\psi_{xx} + k\varphi_x + \left(\tau_q\gamma i\alpha - \frac{\tau_q^2\gamma}{2}\alpha^2 + \gamma\right)\theta_x = 0. \\ \left(-i\alpha^3\frac{\tau_q^2}{2} - \alpha^2\tau_q + i\alpha\right)\theta + i\alpha\gamma\theta^0\psi_x - (i\alpha\kappa\tau_\theta + \kappa)\,\theta_{xx} \ = 0. \end{cases} \tag{4.5.4}$$

Taking into consideration the boundary conditions (4.1.23), we can take

$$\varphi = X\sin(n\pi x),\ \ \psi = Y\cos(n\pi x),\ \ \theta = Z\sin(n\pi x).$$

By replacing φ, ψ, θ in (4.5.4), we get the following system

$$\begin{pmatrix} D_1 & kn\pi & 0 \\ k\pi n & D_2 & D_4 \\ 0 & -i\alpha\gamma\theta_0 n\pi & D_3 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{4.5.5}$$

where

$$\begin{cases} D_1 = k\,(n\pi)^2 - \alpha^2\rho_1,\ D_2 = \left(k - \alpha^2\rho_2\right) + b\,(n\pi)^2 \\ D_3 = (n\pi)^2\,\kappa\,(i\alpha\tau_\theta + 1)\ \ - i\alpha^3\frac{\tau_q^2}{2} - \alpha^2\tau_q + i\alpha \\ D_4 = \left(1 + \tau_q i\alpha - \frac{\tau_q^2}{2}\alpha^2\right)\gamma n\pi. \end{cases} \tag{4.5.6}$$

By solving (4.5.5), we obtain

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \frac{D_2D_3+i\pi n\alpha\gamma\theta^0 D_4}{-\pi^2k^2n^2D_3+D_1D_2D_3+i\pi n\alpha\gamma\theta^0 D_1D_4} \\ -\pi kn\frac{D_3}{-\pi^2D_3k^2n^2+i\pi\alpha\gamma\theta^0D_1D_4n+D_1D_2D_3} \\ -i\pi^2kn^2\alpha\gamma\frac{\theta^0}{-\pi^2D_3k^2n^2+i\pi\alpha\gamma\theta^0D_1D_4n+D_1D_2D_3} \end{pmatrix}.$$

The solution X can be written in the following form

$$X = \frac{Q}{D_1Q - \pi^2k^2n^2}, \; Q = D_2 + i\pi n\alpha\gamma\theta^0\frac{D_4}{D_3}.$$

Then, for selected $r \in \mathbb{R}$, we take α such that

$$D_1(\alpha) = r. \tag{4.5.7}$$

Consequently, we have

$$-\alpha^2 = \frac{r}{\rho_1} - \frac{k}{\rho_1}(n\pi)^2, \tag{4.5.8}$$

by using (4.5.8) and $2\tau_\theta = \tau_q$, we arrive at

$$\begin{aligned} Q &= D_2 + \theta^0 (n\pi\gamma)^2 \frac{-\alpha^2\tau_q + i\alpha\left(1 - \frac{\tau_q^2}{2}\alpha^2\right)}{(n\pi)^2\kappa - \alpha^2\tau_q + i\alpha\left(\kappa\tau_\theta (n\pi)^2 - \alpha^2\frac{\tau_q^2}{2} + 1\right)} \\ &= D_2 + \theta^0 (n\pi\gamma)^2 \frac{-\alpha^2\tau_q + i\alpha\left(1 - \frac{\tau_q^2}{2}\alpha^2\right)}{(n\pi)^2\left(\kappa - \frac{k\tau_q}{\rho_1}\right) + \frac{\tau_q r}{\rho_1} + i\alpha\left(\left(\kappa\tau_\theta - \frac{k}{\rho_1}\frac{\tau_q^2}{2}\right)(n\pi)^2 + \frac{\tau_q^2}{2}\frac{r}{\rho_1} + 1\right)} \\ &= D_2 + \theta^0 (n\pi\gamma)^2 \frac{-\alpha^2\tau_q + i\alpha\left(1 - \frac{\tau_q^2}{2}\alpha^2\right)}{(n\pi)^2\rho + \frac{\tau_q r}{\rho_1} + i\alpha\left(\rho (n\pi)^2 + \frac{\tau_q^2}{2}\frac{r}{\rho_1} + 1\right)}. \end{aligned} \tag{4.5.9}$$

Now, we discuss two cases:

First case: let $\rho = 0$ and by choosing $r = 0$, Q take the form

$$Q = k + (b\rho_1 - k\rho_2)\frac{(n\pi)^2}{\rho_1} + \theta^0 (n\pi\gamma)^2 \left(-i\alpha\tau_q + \left(1 - \frac{\tau_q^2}{2}\alpha^2\right)\right).$$

Since $r = 0$ then $D_1 = 0$ and consequently

$$\begin{aligned} X &= \frac{Q}{-\pi^2k^2n^2} \\ &= -\frac{k}{\pi^2k^2n^2} - \frac{\theta^0\gamma^2}{\rho_1 k^2}(b\rho_1 - k\rho_2) + \frac{\gamma^2}{k^2}\left(\frac{\tau_q^2}{2}\alpha^2 - i\alpha\tau_q - 1\right) \approx c\alpha^2 \end{aligned}$$

for α large. Therefore,

$$\begin{aligned}\|U_n\|_{\mathscr{H}}^2 &\geq \rho_1\theta^0 \left\|\widehat{\varphi}_n\right\|^2 = \rho_1\theta^0\alpha_n^2X^2\int_0^1 \left(\sin(\pi n x)\right)^2 dx\\ &= \frac{\rho_1\theta^0\alpha_n^2X^2}{2} \geq \frac{\rho_1\theta^0}{2}\alpha_n^4,\end{aligned}$$

from (4.5.8), we have $\alpha_n^2 = \frac{k}{\rho_1}\left(n\pi\right)^2$. This implies:

$$\|U_n\|_{\mathscr{H}} \to \infty$$

as $n \to \infty$. Second case: Let $\rho \neq 0$. Then, from (4.5.9) we get

$$\begin{aligned}Q &= D_2 + \theta^0\left(n\pi\gamma\right)^2 \frac{-\alpha^2\tau_q + i\alpha\left(1-\frac{\tau_q^2}{2}\alpha^2\right)}{\left(n\pi\right)^2\rho + \frac{\tau_q r}{\rho_1} + i\alpha\left(\rho\left(n\pi\right)^2 + \frac{\tau_q^2}{2}\frac{r}{\rho_1}+1\right)}\\ &= D_2 - \left(n\pi\right)^2\frac{k\tau_q\gamma^2\theta^0}{\rho_1\rho}\\ &\quad + \left(n\pi\gamma\right)^2\frac{i\alpha\left(1-\frac{\tau_q^2}{2}\alpha^2\right)+\frac{k\tau_q}{\rho_1\rho}\left(\frac{r}{k}-\frac{r}{\rho_1}-i\frac{\alpha}{\tau_q}\left(\rho\left(n\pi\right)^2+\frac{\tau_q^2}{2}\frac{r}{\rho_1}+1\right)\right)}{\left(n\pi\right)^2\rho + \frac{\tau_q r}{\rho_1} + i\alpha\left(\rho\left(n\pi\right)^2 + \frac{\tau_q^2}{2}\frac{r}{\rho_1}+1\right)},\end{aligned}$$

Using (4.5.6) and (4.5.8), we obtain

$$\begin{aligned}Q =& k + \frac{r}{\rho_1}\rho_2 + \left(n\pi\right)^2\left(b - \frac{k}{\rho_1}\rho_2 - \frac{k\tau_q\gamma^2\theta^0}{\rho_1\rho}\right)\\ &+ \left(n\pi\gamma\right)^2\frac{i\alpha\left(1-\frac{\tau_q^2}{2}\alpha^2\right)+\frac{k\tau_q}{\rho_1\rho}\left(\frac{r}{k}-\frac{r}{\rho_1}-i\frac{\alpha}{\tau_q}\left(\rho\left(n\pi\right)^2+\frac{\tau_q^2}{2}\frac{r}{\rho_1}+1\right)\right)}{\left(n\pi\right)^2\rho + \frac{\tau_q r}{\rho_1} + i\alpha\left(\rho\left(n\pi\right)^2 + \frac{\tau_q^2}{2}\frac{r}{\rho_1}+1\right)}.\end{aligned}$$

Then, we choose r such that

$$\left(b - \frac{k}{\rho_1}\rho_2 - \frac{k\tau_q\gamma^2\theta^0}{\rho_1\rho}\right)r = k^2, \tag{4.5.10}$$

From (4.5.10), we write r as

$$r = \frac{\rho_1\rho k^2}{\varkappa}. \tag{4.5.11}$$

By using (4.5.9), we arrived at

$$\begin{aligned}Q &= \left(n\pi\right)^2\frac{k^2}{r} + k + \frac{r}{\rho_1}\rho_2\\ &\quad + \left(n\pi\gamma\right)^2\frac{i\alpha\left(1-\frac{\tau_q^2}{2}\alpha^2\right)+\frac{k\tau_q}{\rho_1\rho}\left(\frac{r}{k}-\frac{r}{\rho_1}-i\frac{\alpha}{\tau_q}\left(c\alpha^2+c'\right)\right)}{\left(n\pi\right)^2\rho + \frac{\tau_q r}{\rho_1} + i\alpha\left(c\alpha^2+c'\right)}\end{aligned}$$

$$\approx (n\pi)^2 \frac{k^2}{r} + k + \frac{r}{\rho_1}\rho_2 + \frac{\tau_q r \gamma^2}{\rho_1^2 \rho^2}(\rho_1 - k) + i\alpha\left(c\alpha^2 + c'\right). \tag{4.5.12}$$

Now, for the solution X we have

$$X = \frac{Q}{D_1 Q - \pi^2 k^2 n^2} = \frac{1}{D_1}\left(1 + \frac{\pi^2 k^2 n^2}{D_1 Q - \pi^2 k^2 n^2}\right),$$

then using (4.5.12) and $D_1(\alpha) = r = \frac{\rho_1 \rho k^2}{\varkappa}$,

$$\begin{aligned} X &= \frac{1}{r}\left(1 + k\frac{r + \rho_1\alpha^2}{rk + \frac{r^2}{\rho_1}\rho_2 + \frac{\tau_q r^2 \gamma^2}{\rho_1^2 \rho^2}(\rho_1 - k) + ri\alpha\left(c\alpha^2 + c'\right)}\right) \\ &= \frac{\varkappa}{\rho_1 \rho k^2}\left(1 + k\frac{r + \rho_1\alpha^2}{rk + \frac{r^2}{\rho_1}\rho_2 + \frac{\tau_q r^2 \gamma^2}{\rho_1^2 \rho^2}(\rho_1 - k) + ri\alpha\left(c\alpha^2 + c'\right)}\right). \end{aligned}$$

Therefore

$$\begin{aligned} \|U_n\|^2_{\mathscr{H}} \geq \rho_1 \theta^0 \|\widehat{\varphi}_n\|^2 &= \rho_1 \theta^0 \alpha_n^2 X^2 \int_0^1 (\sin(\pi n x))^2 \, dx \\ &= \frac{\rho_1 \theta^0 \alpha_n^2 X^2}{2} \approx c\varkappa\alpha_n^2 \approx c\varkappa\left(\frac{k}{\rho_1}(n\pi)^2 - \frac{r}{\rho_1}\right). \end{aligned}$$

This means that if $\varkappa \neq 0$, then

$$\|U_n\|_{\mathscr{H}} \to \infty$$

as $n \to \infty$. As a result, our conclusion follows. ☐

4.6 The Discrete Problem

We introduce a discretization by finite elements in space and backward Euler scheme in time, see [89].

4.6.1 Problem VP

To obtain the weak formulation, we multiply Eqs.(4.1.22)-(4.1.24) by test functions $z, r \in H_0^1(0,1)$ and $\ell \in H_*^1(0,1)$, then integrating by parts, to obtain

$$\begin{cases} \rho_1(\dot{\widehat{\varphi}}, z) + k(\varphi_x + \psi, z_x) = 0, \\ \rho_2\left(\dot{\widehat{\psi}}, l\right) + b(\psi_x, l_x) + k(\varphi_x + \psi, l) + \gamma(\frac{\tau_q^2}{2}\widehat{\theta}_x + \tau_q \vartheta_x + \theta_x, l) = 0, \\ \frac{\tau_q^2}{2}(\dot{\theta}, r) + \tau_q(\widehat{\theta}, r) + (\vartheta, r) + \gamma\theta^0(\widehat{\psi}_x, r) + \kappa(\tau_\theta \vartheta_x + \theta_x, r_x) = 0, \end{cases} \tag{4.6.1}$$

where $(.,.)$ is the inner product of $L_2(0,1)$.

We consider a uniform partition of the space interval $\Omega = (0,1)$ into sub-intervals $\Omega_i = (x_{i-1}, x_i)$, $i = 1, ..., J$, of length $h = \frac{L}{J}$ with $0 = x_0 < x_1 < \cdot\cdot\cdot < x_J = 1$ and define the finite element spaces

$$S_h = \left\{ \psi \in H_*^1(0,1) \,\middle|\, \psi \in C\left(\overline{\Omega}\right),\ \psi|_{\Omega_i} \text{ is a linear polynomial} \right\},$$

$$Y_h = \left\{ \psi \in H_0^1(0,1) \,\middle|\, \psi \in C\left(\overline{\Omega}\right),\ \psi|_{\Omega_i} \text{ is a linear polynomial} \right\}.$$

For the partition of the time interval $[0,T]$, denoted by $0 = t_0 < t_1 < \cdot\cdot\cdot < t_N = T$, we use a uniform partition with step size $\Delta_t = \frac{T}{N}$ and nodes $t_n = n\Delta t$, $n = 0, 1, ..., N$. Using the backward Euler scheme, the fully discrete approximations are considered as follows. Next, the numerical approximation to the variational problem (Eq. (4.6.1)).

4.6.2 Problem VP^h

Finding $(\widehat{\varphi}_h^n, \widehat{\psi}_h^n, \widehat{\theta}_h^n) \in Y_h \times S_h \times Y_h$, $n = 1, ..., N$, such that for all $(z_h, l_h, r_h) \in Y_h \times S_h \times Y_h$,

$$\frac{\rho_1}{\Delta t}(\widehat{\varphi}_h^n - \widehat{\varphi}_h^{n-1}, z_h) + k(\varphi_{hx}^n + \psi_h^n, z_{hx}) = 0, \tag{4.6.2}$$

$$\begin{aligned} &\frac{\rho_2}{\Delta t}\left(\widehat{\psi}_h^n - \widehat{\psi}_h^{n-1}, l_h\right) + b(\psi_{hx}^n, l_{hx}) + k(\varphi_{hx}^n + \psi_h^n, l_h) \\ &+ \frac{\gamma\tau_q^2}{2}(\widehat{\theta}_{hx}^n, l_h) + \gamma\tau_q(\vartheta_{hx}^n, l_h) + \gamma(\theta_{hx}^n, l_h) = 0, \end{aligned} \tag{4.6.3}$$

$$\frac{\tau_q^2}{2\Delta t}(\widehat{\theta}_h^n - \widehat{\theta}_h^{n-1}, r_h) + \tau_q(\widehat{\theta}_h^n, r_h) + (\vartheta_h^n, r_h) + \gamma\theta^0(\widehat{\psi}_{hx}^n, r_h) + \kappa(\tau_\theta\vartheta_{hx}^n + \theta_{hx}^n, r_{hx}) = 0,$$

where

$$\begin{array}{ll} \varphi_h^n = \Delta t\widehat{\varphi}_h^n + \varphi_h^{n-1}, & \psi_h^n = \Delta t\widehat{\psi}_h^n + \psi_h^{n-1}, \\ \theta_h^n = \Delta t\vartheta_h^n + \theta_h^{n-1}, & \vartheta_h^n = \Delta t\widehat{\theta}_h^n + \vartheta_h^{n-1}, \end{array} \tag{4.6.4}$$

with $\widehat{\varphi}_h^n$ is the approximation to the velocity $\widehat{\varphi}(t_n) = \dot{\varphi}(t_n)$ and $\widehat{\psi}_h^n$ to $\widehat{\psi}(t_n) = \dot{\psi}(t_n)$. Here, φ_h^0, $\widehat{\varphi}_h^0$, ψ_h^0, $\widehat{\psi}_h^0$, θ_h^0, ϑ_h^0 and $\widehat{\theta}_h^0$ are approximations to the initial data φ^0, $\widehat{\varphi}^0$, ψ^0, $\widehat{\psi}^0$, θ^0, $\vartheta^0$and $\widehat{\theta}^0$ respectively.

Let us introduce the discrete energy, for $n = 0, ..., N$.

$$\begin{aligned} E^n = \frac{1}{2}\Big(&\rho_1\theta^0\|\widehat{\varphi}_h^n\|^2 + \rho_2\theta^0\|\widehat{\psi}_h^n\|^2 + b\theta^0\|\psi_{hx}^n\|^2 + k\theta^0\|\varphi_{hx}^n + \psi_h^n\|^2 \\ &+ \left\|\frac{\tau_q^2}{2}\widehat{\theta}_h^n + \tau_q\vartheta_h^n + \theta_h^n\right\|^2 + \kappa\frac{\tau_q^2}{2}\tau_\theta\|\vartheta_{hx}^n\|^2 \\ &+ \kappa\tau_q^2(\vartheta_{hx}^n, \theta_{hx}^n) + \kappa(\tau_q + \tau_\theta)\|\theta_{hx}^n\|^2\Big). \end{aligned} \tag{4.6.5}$$

With the assumption $\tau_q \le 2\tau_\theta$, it is clear that the energy E^n is positive. The decay of the energy is presented in the following theorem.

Theorem 4.4. *If* $\tau_q \le 2\tau_\theta$ *then the discrete energy* E^n *defined by Eq.* (4.6.5) *satisfies*

$$E^n \le E^{n-1},\ n = 1,...,N. \tag{4.6.6}$$

Proof. Choosing $z_h = \theta^0 \widehat{\varphi}_h^n$, $l_h = \theta^0 \widehat{\psi}_h^n$, and $r_h = \frac{\tau_q^2}{2}\widehat{\theta}_h^n + \tau_q \vartheta_h^n + \theta_h^n$ in Eqs. (4.6.2)-(4.6.2) we get

$$\begin{aligned}
&\frac{\rho_1\theta^0}{\Delta t}(\widehat{\varphi}_h^n - \widehat{\varphi}_h^{n-1}, \widehat{\varphi}_h^n) + k\theta^0(\varphi_{hx}^n + \psi_h^n, \widehat{\varphi}_{hx}^n) = 0,\\
&\frac{\rho_2\theta^0}{\Delta t}\left(\widehat{\psi}_h^n - \widehat{\psi}_h^{n-1}, \widehat{\psi}_h^n\right) + b\theta^0(\psi_{hx}^n, \widehat{\psi}_{hx}^n) + k\theta^0(\varphi_{hx}^n + \psi_h^n, \widehat{\psi}_h^n)\\
&\qquad + \gamma\theta^0\left(\frac{\tau_q^2}{2}\widehat{\theta}_{hx}^n + \tau_q\vartheta_{hx}^n + \gamma\theta_{hx}^n, \widehat{\psi}_h^n\right) = 0,\\
&\left(\frac{\tau_q^2}{2\Delta t}\left(\widehat{\theta}_h^n - \widehat{\theta}_h^{n-1}\right) + \tau_q\left(\vartheta_h^n - \vartheta_h^{n-1}\right) + \left(\theta_h^n - \theta_h^{n-1}\right), \frac{\tau_q^2}{2}\widehat{\theta}_h^n + \tau_q\vartheta_h^n + \theta_h^n\right)\\
&+ \kappa\left(\tau_\theta\vartheta_{hx}^n + \theta_{hx}^n, \frac{\tau_q^2}{2}\widehat{\theta}_{hx}^n + \tau_q\vartheta_{hx}^n + \theta_{hx}^n\right) + \gamma\theta^0\left(\widehat{\psi}_{hx}^n, \frac{\tau_q^2}{2}\widehat{\theta}_h^n + \tau_q\vartheta_h^n + \theta_h^n\right) = 0.
\end{aligned}$$

Then adding the four equations and using $(x-y,x) = \frac{1}{2}\left(\|x-y\|^2 + \|x\|^2 - \|y\|^2\right)$ and Eq. (4.6.4) we get

$$\begin{aligned}
&\frac{\rho_1\theta^0}{2\Delta t}\|\widehat{\varphi}_h^n\|^2 - \frac{\rho_1\theta^0}{2\Delta t}\left\|\widehat{\varphi}_h^{n-1}\right\|^2 + \frac{k\theta^0}{2\Delta t}\|\varphi_{hx}^n + \psi_h^n\|^2 - \frac{k\theta^0}{2\Delta t}\left\|\varphi_{hx}^{n-1} + \psi_h^{n-1}\right\|^2\\
&+ \frac{\rho_2\theta^0}{2\Delta t}\|\widehat{\psi}_h^n\|^2 - \frac{\rho_2\theta^0}{2\Delta t}\left\|\widehat{\psi}_h^{n-1}\right\|^2 + \frac{b\theta^0}{2\Delta t}\|\psi_{hx}^n\|^2 - \frac{b\theta^0}{2\Delta t}\left\|\psi_{hx}^{n-1}\right\|^2\\
&+ \frac{1}{2\Delta t}\left\|\frac{\tau_q^2}{2}\widehat{\theta}_h^n + \tau_q\vartheta_h^n + \theta_h^n\right\|^2 - \frac{1}{2\Delta t}\left\|\frac{\tau_q^2}{2}\widehat{\theta}_h^{n-1} + \tau_q\vartheta_h^{n-1} + \theta_h^{n-1}\right\|^2\\
&+ \kappa\left(\tau_\theta\vartheta_{hx}^n + \theta_{hx}^n, \frac{\tau_q^2}{2}\widehat{\theta}_{hx}^n + \tau_q\vartheta_{hx}^n + \theta_{hx}^n\right)\\
&= -\frac{\rho_1\theta^0}{2\Delta t}\|\widehat{\varphi}_h^n - \widehat{\varphi}_h^{n-1}\|^2 - \frac{\rho_2\theta^0}{2\Delta t}\|\widehat{\psi}_h^n - \widehat{\psi}_h^{n-1}\|^2 - \frac{b\theta^0}{2\Delta t}\|\psi_{hx}^n - \psi_{hx}^{n-1}\|^2\\
&- \frac{1}{2\Delta t}\left\|\left(\frac{\tau_q^2}{2}\widehat{\theta}_h^n + \tau_q\vartheta_h^n + \theta_h^n\right) - \left(\frac{\tau_q^2}{2}\widehat{\theta}_h^{n-1} + \tau_q\vartheta_h^{n-1} + \theta_h^{n-1}\right)\right\|^2\\
&- \frac{k\theta^0}{2\Delta t}\|\varphi_{hx}^n + \psi_h^n - (\varphi_{hx}^{n-1} + \psi_h^{n-1})\|^2.
\end{aligned}$$

Also, by using Eq. (4.6.4), we obtain

$$\kappa\left(\tau_\theta\vartheta_{hx}^n + \theta_{hx}^n, \frac{\tau_q^2}{2}\widehat{\theta}_{hx}^n + \tau_q\vartheta_{hx}^n + \theta_{hx}^n\right) = \kappa\frac{\tau_q^2}{2\Delta t}\tau_\theta(\vartheta_{hx}^n, \vartheta_{hx}^n - \vartheta_{hx}^{n-1})$$

$$
\begin{aligned}
&+\kappa\frac{\tau_q^2}{2\Delta t}\left((\theta_{hx}^n-\theta_{hx}^{n-1},\vartheta_{hx}^n)+(\theta_{hx}^n,\vartheta_{hx}^n-\vartheta_{hx}^{n-1})\right)\\
&+\frac{(\tau_q+\tau_\theta)\,\kappa}{\Delta t}(\theta_{hx}^n-\theta_{hx}^{n-1},\theta_{hx}^n)\\
&+\kappa\tau_q\left(\tau_\theta-\frac{\tau_q}{2}\right)\|\vartheta_{hx}^n\|^2+\kappa\|\theta_{hx}^n\|^2,
\end{aligned}
$$

with

$$
\begin{aligned}
(\theta_{hx}^n-\theta_{hx}^{n-1},\vartheta_{hx}^n)+(\theta_{hx}^n,\vartheta_{hx}^n-\vartheta_{hx}^{n-1})=&(\vartheta_{hx}^n,\theta_{hx}^n)-(\vartheta_{hx}^{n-1},\theta_{hx}^{n-1})\\
&+(\vartheta_{hx}^n-\vartheta_{hx}^{n-1},\theta_{hx}^n-\theta_{hx}^{n-1})
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{E^n-E^{n-1}}{\Delta t}\leq&-\kappa\tau_q\left(\tau_\theta-\frac{\tau_q}{2}\right)\|\vartheta_{hx}^n\|^2-\kappa\|\theta_{hx}^n\|^2\\
&-\frac{\rho_1\theta^0}{2\Delta t}\left\|\widehat{\varphi}_h^n-\widehat{\varphi}_h^{n-1}\right\|^2-\frac{\rho_2\theta^0}{2\Delta t}\left\|\widehat{\psi}_h^n-\widehat{\psi}_h^{n-1}\right\|^2\\
&-\frac{b\theta^0}{2\Delta t}\left\|\psi_{hx}^n-\psi_{hx}^{n-1}\right\|^2-\frac{k\theta^0}{2\Delta t}\left\|\varphi_{hx}^n+\psi_h^n-(\varphi_{hx}^{n-1}+\psi_h^{n-1})\right\|^2\\
&-\frac{1}{2\Delta t}\left\|\left(\frac{\tau_q^2}{2}\widehat{\theta}_h^n+\tau_q\vartheta_h^n+\theta_h^n\right)-\left(\frac{\tau_q^2}{2}\widehat{\theta}_h^{n-1}+\tau_q\vartheta_h^{n-1}+\theta_h^{n-1}\right)\right\|^2\\
&-\kappa\frac{\tau_q^2}{4\Delta t}\tau_\theta\left\|\vartheta_{hx}^n-\vartheta_{hx}^{n-1}\right\|^2-\frac{\kappa}{2\Delta t}(\tau_q+\tau_\theta)\left\|\theta_{hx}^n-\theta_{hx}^{n-1}\right\|^2\\
&-\kappa\frac{\tau_q^2}{2\Delta t}(\vartheta_{hx}^n-\vartheta_{hx}^{n-1},\theta_{hx}^n-\theta_{hx}^{n-1}).
\end{aligned}
$$

Recalling the inequality (4.2.4), we deduce the estimate (4.6.6). ☐

4.6.3 Numerical simulation

In the first experiment, we investigated the decay of discrete energy, by considering the following data

$$
\begin{aligned}
&\Delta t=0.01,\ h=0.002,\ \tau_q=0.4,\\
&\gamma=0.5,\ \tau_\theta=2\tau_q+0.2,\ \theta^0=1,\\
&k=30,\ b=20,\ \kappa=0.5,\\
&\rho_1=\frac{\tau_q^2\left(k+\gamma\theta^0\right)}{2\kappa\tau_\theta}=12.2,\\
&\rho_2=\rho_1\left(\frac{b}{k}-\gamma\right)=2.0333,
\end{aligned}
\tag{4.6.7}
$$

Here the assumption $\varkappa = 0$ is verified. The initial conditions are as follows:

$$\begin{aligned}
&\varphi_0(x) = \varphi_1(x) = 2x(x-1), \\
&\psi_0(x) = \psi_1(x) = 3\left(x^2(3-2x) + \tfrac{1}{10}\right), \\
&\theta_0(x) = \theta_1(x) = \theta_2(x) = x^2(x-1)^2,
\end{aligned} \tag{4.6.8}$$

Figure 4.2 shows that the energy of the system (Eqs. (4.1.22)-(4.1.24)) decreases exponentially,

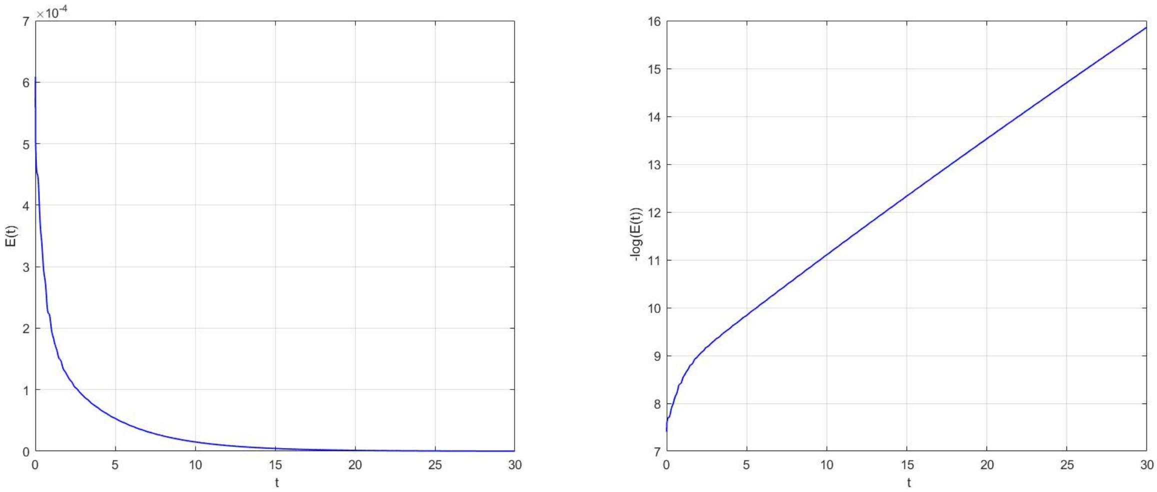

Figure 4.2: The evolution of the discrete energy in natural and log.

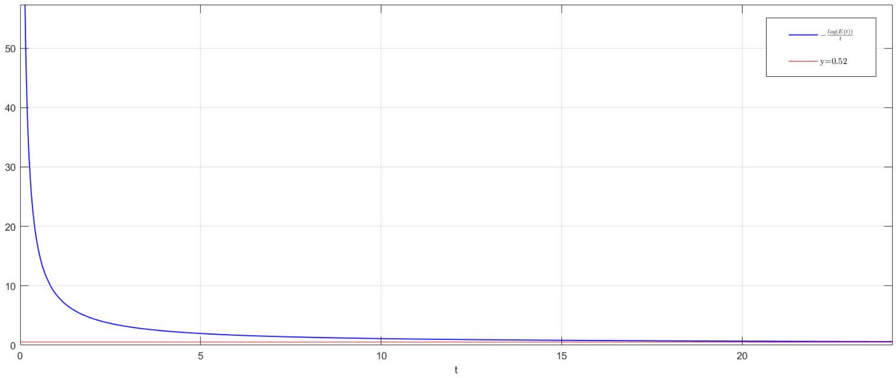

Figure 4.3: The $-\frac{Log(E(t))}{t}$ development in time.

which is confirmed also by Figure 4.3. Next, in the Figures 4.4, 4.5, 4.6, we show the behavior of the solutions by taking the same data Eq. (4.6.7) and the initial conditions (4.6.8). For the second experiment, we show the time cross-section of the solutions φ, ψ and θ at $x = \{0.3, 0.6, 0.9\}$ using the same data (4.6.7) with $\Delta t = 0.025$, $h = 0.005$, and the following initial

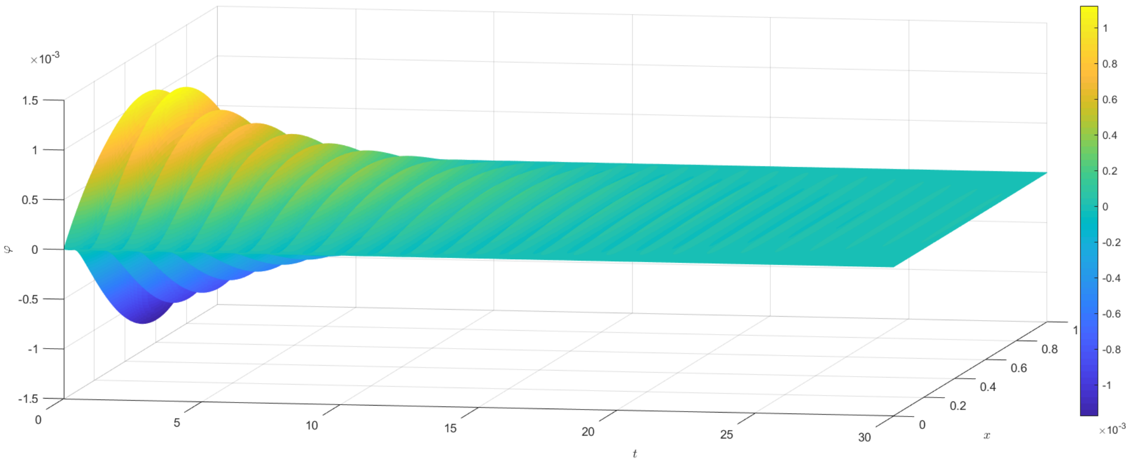

Figure 4.4: Numerical solution for the transversal displacement.

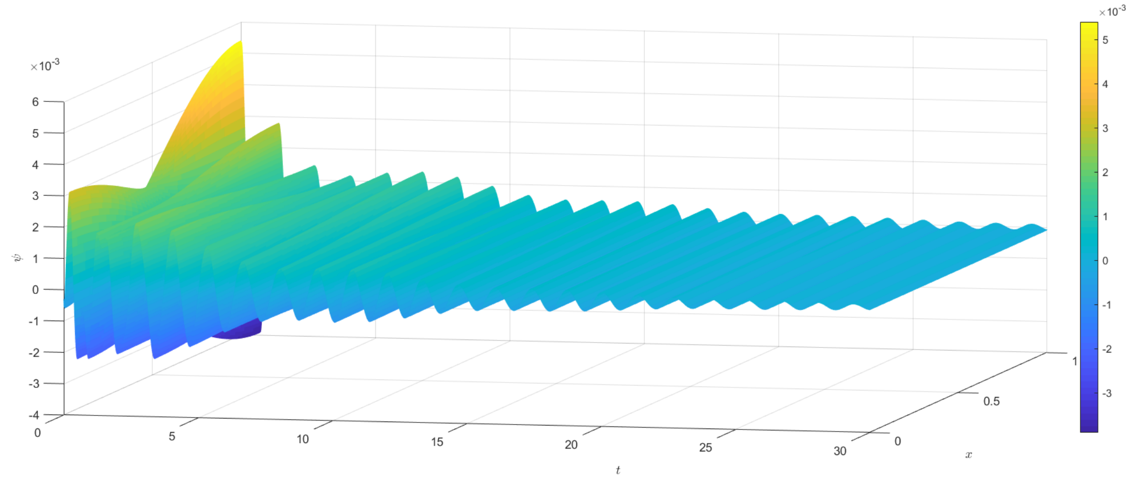

Figure 4.5: Numerical solution for the rotational angle.

conditions

$$\begin{aligned} \varphi_0(x) &= \varphi_1(x) = 2sin(\pi x), \\ \psi_0(x) &= \psi_1(x) = 100\left(x^2(3-2x)\right), \\ \theta_0(x) &= \theta_1(x) = \theta_2(x) = x(x-1). \end{aligned} \tag{4.6.9}$$

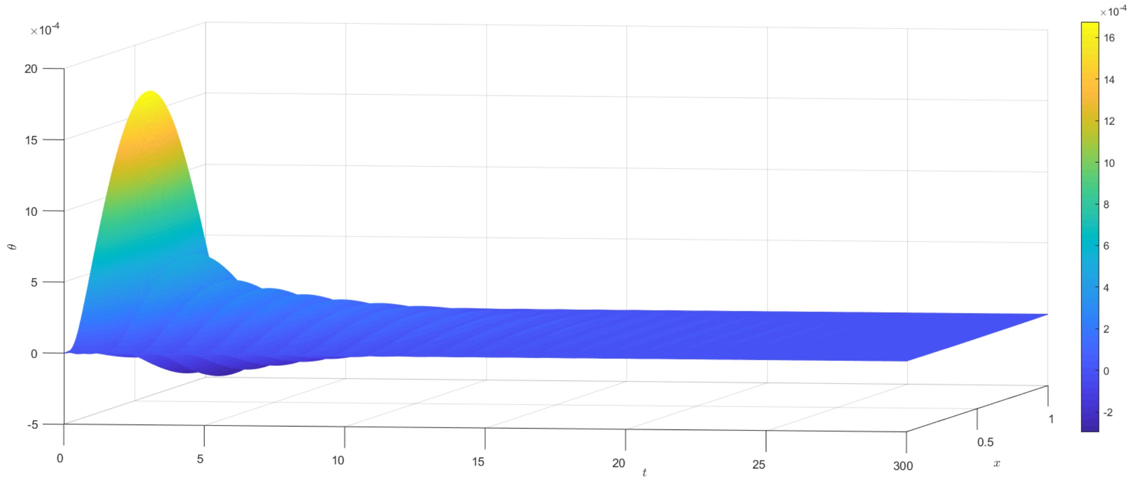

Figure 4.6: Numerical solution for temperature.

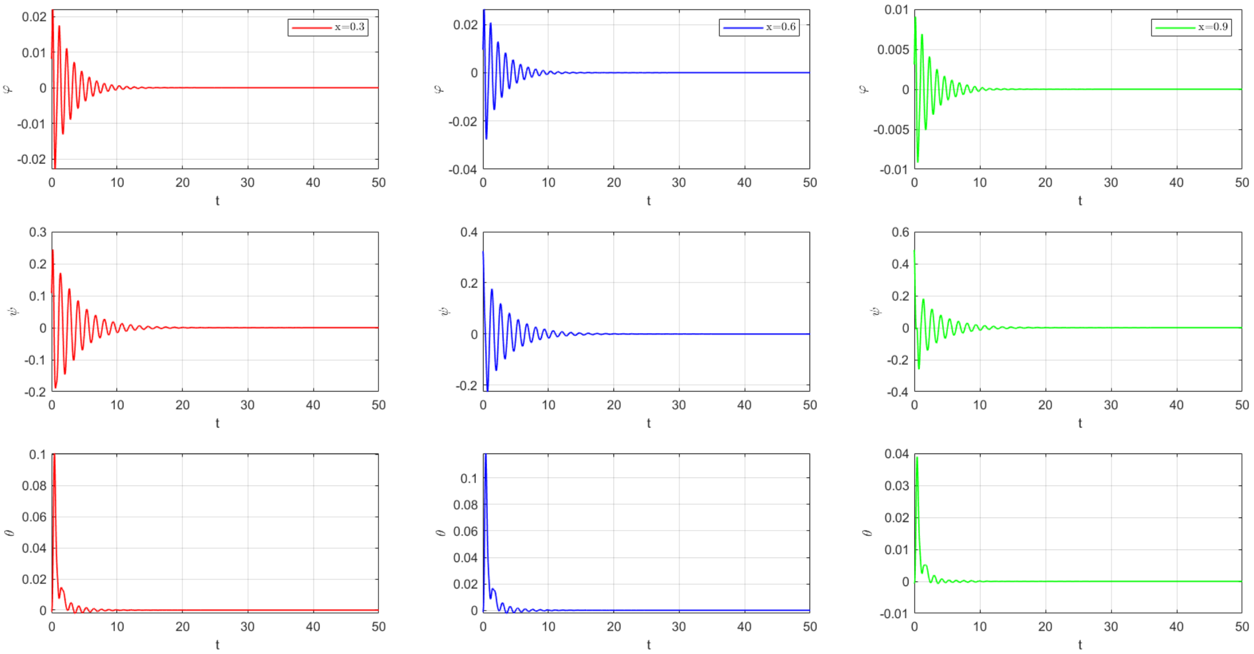

Figure 4.7: Time cross-section of the solutions

4.7 Conclusions

In this paper, we present a new dissipative number for a thermoelastic Timoshenko system with a dual-phase-lag thermal conductivity model. This analysis revealed that the thermal relaxation times τ_θ, τ_q and $\varkappa$ control the exponential stability state for ultrafast thermoelasticity in the Timoshenko beam, in which we have

Exponential stability	Lac of exponentially stability
$if\ 2\tau_\theta > \tau_q\ and\ \varkappa = 0$	$if\ 2\tau_\theta = \tau_q\ and\ \rho = 0$ or $if\ 2\tau_\theta = \tau_q\ and\ \ \varkappa \neq 0$

(4.7.1)

We note that if $\rho = 0$ then $\varkappa \neq 0$. Based on these results, the system's polynomial stability in the case of $2\tau_\theta = \tau_q$ is an interesting subject to explore.

CHAPTER 5

Bresse System with Dual-Phase-Lag Thermoelasticity

5.1 Introduction

In 1856, Bresse [29] introduced a system modelling longitudinal, vertical and angular motion, the so-called Bresse system. In fact, this system describes the behaviour of a thin curved beam of length l. Note that in the case where longitudinal displacement is not considered and assuming zero initial curvature, this system represents a generalisation of the Timoshenko system. Thus, in the last decades, there has been a growing interest in determining the numerical solution or the asymptotic behaviour of several Bresse-type thermoelastic systems (see [94]- [95]). The thermoelastic Bresse system model can be given by the following evolution equations:

$$\begin{aligned} &\rho_1\ddot{\varphi} = Q_x + lN, \quad &\rho_2\ddot{\psi} = M_x - Q, \\ &\rho_1\ddot{w} = N_x - lQ, \quad &\dot{\theta} = -q_x - \delta\theta^0\psi_x, \end{aligned} \tag{5.1.1}$$

where

$$\begin{aligned} N &= k_0\left(w_x - l\varphi\right) - \delta\theta, \\ Q &= k\left(\varphi_x + \psi + lw\right), \\ M &= b\psi_x. \end{aligned} \tag{5.1.2}$$

Here, the function φ denotes the transverse displacement, ψ the angle of rotation of the cross section, w the horizontal displacement and θ the thermal moment of the beam. The positive parameters associated with system (5.1.1)-(5.1.2) are as follows:
ρ_1 the mass density of the beam
ρ_2 the moment of mass inertia of the beam
k the shear modulus of elasticity
$k_0 = Eh$, where E is the Young's modulus and h is the cross sectional area
l is the initial curvature
G is the rigidity coefficient of the cross section
γ represents the coupling coefficient.
θ^0 is a constant reference temperature assumed to be positive.

Concerning the thermoelastic Bresse system, it's worth highlighting several observations related to the model's structure. Over recent years, numerous scholars have investigated the well-posedness and stability of Bresse systems with varying forms of damping. Consequently, diverse stability outcomes have been achieved, generally contingent on the characteristics of the control mechanism and the stability number

$$\chi_0 = \rho_1 b - \rho_2 k, \tag{5.1.3}$$

The condition $\chi_0 = 0$ is a necessary and sufficient condition for exponential stability, while asymptotic results are possible when this condition holds along with $k = k_0$. In which the condition indicates that shear and longitudinal motions share the same wave speeds from a physical perspective.

Note that, in the results mentioned earlier, the classical linear Fourier's theory of thermoelasticity is applied and this classic heat conduction based in this theory is acceptable for engineering applications at macrotemporal and spatial scales. However, this law may not be applicable in thermal non-equilibrium and/or nonlocal situations. Fourier's law shows infinitesimal heat disturbances that propagate at an infinite speed. Precisely,

$$q = -\kappa \theta_x, \tag{5.1.4}$$

where $\kappa > 0$ is the thermal conductivity which depends on the properties of the material. However, several attempts have been proposed to overcome this physical paradox such as Cattaneo's law [96] in which the heat is assumed to be transported by a wave propagation process rather than diffusion,

$$\tau_0 \dot{q} + q = -\kappa \theta_x. \tag{5.1.5}$$

In this work, we are interested in the proposal suggested by Tzou [97] in which he proposes a modification of Fourier's law where the delay parameters are present and therefore, the Cattaneo and Maxwell law can be seen as a particular case. More precisely, we consider the following Bresse system with the second order dual phase-lag heat conduction law,

$$q + \tau_q \dot{q} + \frac{\tau_q^2}{2} \ddot{q} = -\kappa_1 \left(\tau_\theta \dot{\theta}_x + \theta_x \right), \tag{5.1.6}$$

where τ_q and τ_θ are the thermal phase lags of heat flux vector and temperature gradient, respectively. Directly from (5.1.6), one can easily derive the Cattaneo's law (5.1.5) for the special case $\tau_q = 0$. Quintanilla in [98] showed that to ensure the stability of solutions in the Chandrasekharaiah-Tzou thermoelastic theory, it is necessary to impose certain conditions on the parameters τ_θ and τ_q. More precisely, τ_θ and τ_q must be small enough and satisfy

$$2\tau_\theta > \tau_q. \tag{5.1.7}$$

Now, by inserting (5.1.2) and (5.1.6) into (5.1.1), we obtain the following Bresse system:

$$\begin{aligned}
&\rho_1 \ddot{\varphi} = k\left(\varphi_x + \psi + lw\right)_x + k_0 l\left(w_x - l\varphi\right),\\
&\rho_2 \ddot{\psi} = b\psi_{xx} - k\left(\varphi_x + \psi + lw\right) - \theta_x,\\
&\rho_1 \ddot{w} = k_0\left(w_x - l\varphi\right)_x - kl\left(\varphi_x + \psi + lw\right),\\
&\left(\tfrac{\tau_q^2}{2}\dddot{\theta} + \tau_q\ddot{\theta} + \dot{\theta}\right) = -\gamma\theta^0\left(\tfrac{\tau_q^2}{2}\dddot{\psi} + \tau_q\ddot{\psi} + \dot{\psi}\right)_x + \kappa_1\left(\tau_\theta\dot{\theta}_x + \theta_x\right)_x,
\end{aligned} \tag{5.1.8}$$

here $x \in [0,1]$ and $t > 0$. As in [99], we introduce the following notation

$$\hat{f} = f + \tau_q\dot{f} + \tfrac{\tau_q^2}{2}\ddot{f},$$

and system (5.1.8) takes the form

$$\begin{aligned}
&\rho_1 \ddot{\hat{\varphi}} = k\left(\hat{\varphi}_x + \hat{\psi} + l\hat{w}\right)_x + k_0 l\left(\hat{w}_x - l\hat{\varphi}\right),\\
&\rho_2 \ddot{\hat{\psi}} = b\hat{\psi}_{xx} - k\left(\hat{\varphi}_x + \hat{\psi} + l\hat{w}\right) - \gamma\left(\tfrac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta} + \theta\right),\\
&\rho_1 \ddot{\hat{w}} = k_0\left(\hat{w}_x - l\hat{\varphi}\right)_x - kl\left(\hat{\varphi}_x + \hat{\psi} + l\hat{w}\right),\\
&\left(\tfrac{\tau_q^2}{2}\dddot{\theta} + \tau_q\ddot{\theta} + \dot{\theta}\right) = -\gamma\theta^0\dot{\hat{\psi}}_x + \kappa_1\left(\tau_\theta\dot{\theta}_x + \theta_x\right)_x.
\end{aligned} \tag{5.1.9}$$

Now, to simplify the notation, we will use (φ, ψ, w) instead of $(\hat{\varphi}, \hat{\psi}, \hat{w})$ from now on, i.e., we will work with

$$\begin{aligned}
&\rho_1 \ddot{\varphi} = k\left(\varphi_x + \psi + lw\right)_x + k_0 l\left(w_x - l\varphi\right),\\
&\rho_2 \ddot{\psi} = b\psi_{xx} - k\left(\varphi_x + \psi + lw\right) - \gamma\hat{\theta},\\
&\rho_1 \ddot{w} = k_0\left(w_x - l\varphi\right)_x - kl\left(\varphi_x + \psi + lw\right),\\
&\dot{\hat{\theta}} = -\gamma\theta^0\dot{\psi}_x + \kappa_1\left(\tau_\theta\dot{\theta}_x + \theta_x\right)_x.
\end{aligned} \tag{5.1.10}$$

We complete the system (5.1.10) by the following boundary conditions:

$$\begin{aligned}
\varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = 0, \qquad (5.1.11)\\
w_x(0,t) = w_x(1,t) = \theta(0,t) = \theta(1,t) = 0
\end{aligned}$$

and the initial data

$$\begin{aligned}
&\varphi(x,0) = \varphi_0(x),\ \dot{\varphi}(x,0) = \varphi_1(x),\ (x,0) = \psi_0(x), \qquad (5.1.12)\\
&\dot{\psi}(x,0) = \psi_1(x),\ w(x,0) = w_0(x),\ \dot{w}(x,0) = w_1(x),\\
&\theta(x,0) = \theta_0(x),\ \dot{\theta}(x,0) = \theta_1(x),\ \ddot{\theta}(x,0) = \theta_2(x).
\end{aligned}$$

We point out that Bazarra et al [?] studied the system (5.1.10)-(5.1.12) with Dirichlet boundary condition for ψ and w. They conducted a numerical study where the condition $\chi_0 = 0$ was mathematically ignored and without any restriction on k_0 and k, therefore the general stability results obtained from the numerical approach were also valid in $\chi_0 = 0$ as a special case. The current work introduces original results for the thermal Bresse system, considering thermal dissipation within the dual-phase-lag theory. Our objective is to mathematically assess whether it is possible to ignore the previously imposed stability conditions, and their adequacy in ensuring system stability, and to establish the prerequisites for achieving the exponential stability of system (5.1.10)-(5.1.12).

We aim to extend this finding by examining the system's stability using the multiplier method and providing a sufficient conditions on the coefficients that lead the system to the equilibrium state in an exponential manner. More precisely, if the coefficients satisfy the following conditions:

$$\chi_1 = 0 \text{ and } k = k_0 \tag{5.1.13}$$

where

$$\chi_1 = (\rho_1 b - \rho_2 k)\left(\kappa_1 - \frac{k\tau_q^2}{2\tau_\theta \rho_1}\right) - \frac{\gamma^2 \theta^0 k \tau_q^2}{2\tau_\theta}. \tag{5.1.14}$$

We note that when $\tau_q = 0$, the stability number (5.1.14) reduces to the classical stability number (5.1.3). I.e., we obtain the same results as the Bresse systems with Cattaneo's law of heat conduction.

5.2 Existence and Uniqueness

In this section, we give a brief summary of the existence and uniqueness result for problem (5.1.10)-(5.1.12) using the semigroup theory.

Let $U = (\varphi, \phi,\ \psi,\ \zeta,\ w,\ \omega,\ \theta,\ \vartheta,\ \widehat{\theta})^T$, where $\phi = \dot{\varphi}$, $\zeta = \dot{\psi}$, $\omega = \dot{w}$, $\vartheta = \dot{\theta}$ and $\widehat{\theta} = \dot{\vartheta}$. Then, system (5.1.10)-(5.1.12) can be rewritten as follows:

$$\begin{cases} \dot{U} = \mathscr{A}U,\ t > 0, \\ U(x,0) = U_0(x) = (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, \theta_0, \theta_1, \theta_2)^T, \end{cases} \tag{5.2.1}$$

where the operator $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \longrightarrow \mathscr{H}$ is given by

$$\mathscr{A}U=\begin{pmatrix} \phi \\ \frac{k}{\rho_1}\left(\varphi_x+\psi+lw\right)_x+\frac{k_0 l}{\rho_1}\left(w_x-l\varphi\right) \\ \zeta \\ \frac{b}{\rho_2}\psi_{xx}-\frac{k}{\rho_2}\left(\varphi_x+\psi+lw\right)-\frac{\tau_q^2\gamma}{2\rho_2}\widehat{\theta}_x-\frac{\tau_q\gamma}{\rho_2}\vartheta_x-\frac{\gamma}{\rho_2}\theta_x \\ \omega \\ \frac{k_0}{\rho_1}\left(w_x-l\varphi\right)_x-\frac{kl}{\rho_1}\left(\varphi_x+\psi+lw\right) \\ \vartheta \\ \widehat{\theta} \\ -\frac{2}{\tau_q}\widehat{\theta}-\frac{2}{\tau_q^2}\vartheta-\frac{2\gamma\theta^0}{\tau_q^2}\zeta_x+\frac{2\kappa_1\tau_\theta}{\tau_q^2}\vartheta_{xx}+\frac{2\kappa_1}{\tau_q^2}\theta_{xx} \end{pmatrix}. \tag{5.2.2}$$

Now, we introduce the energy space

$$\mathscr{H}=H_0^1(0,1)\times L^2(0,1)\times\left(H_*^1(0,1)\times L_*^2(0,1)\right)^2\times H_0^1(0,1)\times L^2(0,1)\times L^2(0,1),$$

where

$$H_*^1(0,1)=H^1(0,1)\cap L_*^2(0,1),$$

and

$$L_*^2(0,1)=\left\{\varphi\in L^2(0,1):\int_0^1\varphi(x)\,dx=0\right\}.$$

Note that, $\mathscr{H}$ is Hilbert space with the following norm

$$\begin{aligned}\|U\|_{\mathscr{H}}^2=\int_0^1\Big[&\rho_1\theta^0|\phi|^2+\rho_2\theta^0|\zeta|^2+\rho_1\theta^0|\omega|^2dx+b\theta^0|\psi_x|^2+k_0\theta^0|(w_x-l\varphi)|^2\\&+k\theta^0|(\varphi_x+\psi+lw)|^2+\left|\left(\frac{\tau_q^2}{2}\widehat{\theta}+\tau_q\vartheta+\theta\right)\right|^2\\&+\frac{\kappa_1\tau_\theta\tau_q^2}{2}|\vartheta_x|^2+\kappa_1(\tau_\theta+\tau_q)|\theta_x|^2+\kappa_1\tau_q^2|\theta_x\vartheta_x|\Big]\,dx.\end{aligned} \tag{5.2.3}$$

Remark 5.1. *Under the hypothesis* $2\tau_\theta>\tau_q$, *it is easy to see that the relation* (5.2.3) *defines a positive norm, see* (4.2.4).

The domain of $\mathscr{A}$ is given by:

$$\begin{aligned}D(\mathscr{A})=\Big\{U\in\mathscr{H}\mid\ &\varphi\in H^2(0,1)\cap H_0^1(0,1);\ \phi,\vartheta,\theta,\widehat{\theta}\in H_0^1(0,1);\\&w,\psi\in H_*^2(0,1)\cap H_*^1(0,1);\ \zeta,\omega\in H_*^1(0,1);\theta+\tau_\theta\vartheta\in H^2(0,1)\Big\},\end{aligned}$$

where

$$H_*^2(0,1)=\left\{\Psi\in H^2(0,1):\Psi_x(0)=\Psi_x(1)=0\right\}.$$

To study the well-posedness of the problem given by (5.2.1), we start by showing that the operator $\mathscr{A}$ generates a C_0-semigroup of contractions over the space $\mathscr{H}$. From the relation (5.2.3), it is not difficult to see that $\mathscr{A}$ is a dissipative operator in $\mathscr{H}$-space. Precisely, we have

$$\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = -\kappa_1 \|\theta_x\|^2 - \kappa_1 \tau_q \left(\tau_\theta - \frac{\tau_q}{2}\right) \|\vartheta_x\|^2 \leq 0.$$

Furthermore, it is simple to demonstrate that the resolvent set $\rho(\mathscr{A})$ satisfies,

$$0 \in \rho(\mathscr{A}).$$

Consequently, the following proposition holds

Proposition 5.1. *[?] The operator $\mathscr{A}$ generates a C_0-semigroup $S(t)$ of contractions on the space $\mathscr{H}$.*

And now, we can give, the well-posedness result.

Theorem 5.1. *Let $U_0 \in \mathscr{H}$, then there exists a unique solution $U \in C(\mathbb{R}_+, \mathscr{H})$ of problem* (5.1.10)-(5.1.12)*. Moreover, if $U_0 \in D(\mathscr{A})$, then*

$$U \in C(\mathbb{R}_+, D(\mathscr{A})) \cap C^1(\mathbb{R}_+, \mathscr{H}).$$

5.3 Technical lemmas

In this section, we state and prove several lemmas needed for the proof of our stability result and we use c_0 throughout this work to denote a generic positive constant.

Lemma 5.1. *Let $(\varphi,\ \psi,\ w,\ \theta)$ be a solution of Eqs.* (5.1.10)-(5.1.12)*. Then, the energy functional $E(t)$, defined by*

$$\begin{aligned} E(t) =& \frac{1}{2}\int_0^1 \Big[\rho_1\theta^0\dot{\varphi}^2 + \rho_2\theta^0\dot{\psi}^2 + \rho_1\theta^0\dot{w}^2 + b\theta^0\psi_x^2 \\ &+ k\theta^0(\varphi_x+\psi+lw)^2 + k_0\theta^0(w_x - l\varphi)^2 \\ &+ \frac{\kappa_1\tau_\theta\tau_q^2}{2}\dot{\theta}_x^2 + \kappa_1(\tau_\theta+\tau_q)\theta_x^2 + \kappa_1\tau_q^2\theta_x\dot{\theta}_x + \hat{\theta}^2 dx\Big] \end{aligned} \tag{5.3.1}$$

satisfies

$$E'(t) = -\kappa_1\int_0^1 \theta_x^2 dx - \kappa_1\tau_q\left(\tau_\theta - \frac{\tau_q}{2}\right)\int_0^1 \dot{\theta}_x^2 dx \leq 0,\ t > 0. \tag{5.3.2}$$

Proof. The Eq. (5.1.10) is obtained by multiplying the first three equations of (4.1.22) by $\theta^0\dot{\varphi}$, $\theta^0\dot{\psi}$, $\theta^0\dot{w}$ respectively, integrating by parts over $(0,1)$ and using the boundary conditions Eq. (5.1.11). ☐

Lemma 5.2. *Let* $(\varphi, \psi, w, \theta)$ *be a solution of the system* (5.1.10)-(5.1.12). *Then, the functional* $\mathscr{F}_1$ *defined by*

$$\mathscr{F}_1(t) : \ = -\rho_1 \int_0^1 (\dot{\varphi}\varphi + \dot{w}w)\,dx, \ t > 0,$$

satisfies for all $\varepsilon_0 > 0$, *the estimate*

$$\begin{aligned}\mathscr{F}_1'(t) \leq & -\rho_1 \int_0^1 \dot{w}^2 dx - \rho_1 \int_0^1 \dot{\varphi}^2 dx + \frac{c_0}{\varepsilon_0} \int_0^1 (\varphi_x + \psi + lw)^2\,dx \\ & + k_0 \int_0^1 (w_x - l\varphi)^2\,dx \ + \varepsilon_0 \int_0^1 \psi_x^2 dx, \ t > 0. \end{aligned} \tag{5.3.3}$$

Proof. By differentiating the expression of $\mathscr{F}_1$, using Eqts $(5.1.10)_1$, $(5.1.10)_3$ and integrating by parts, we get

$$\begin{aligned}\mathscr{F}_1'(t) = & -\rho_1 \int_0^1 \dot{w}^2 dx - \rho_1 \int_0^1 \dot{\varphi}^2 dx + k \int_0^1 (\varphi_x + \psi + lw)^2\,dx \\ & + k_0 \int_0^1 (w_x - l\varphi)^2\,dx - k \int_0^1 (\varphi_x + \psi + lw)\,\psi dx, \ \forall t > 0. \end{aligned} \tag{5.3.4}$$

By Young's inequality, we have

$$-k \int_0^1 (\varphi_x + \psi + lw)\,\psi dx \leq \varepsilon_0 \int_0^1 \psi_x^2 dx + \frac{k^2}{\varepsilon_0} \int_0^1 (\varphi_x + \psi + lw)^2\,dx. \tag{5.3.5}$$

Consequently, Estimate (5.3.3) follows by substituting (5.3.5) into (5.3.4). □

Lemma 5.3. *Let* $(\varphi, \psi, w, \theta)$ *be a solution of the system* (5.1.10)-(5.1.12) *and let* $k = k_0$. *Then, the functional* $\mathscr{F}_2$ *defined by*

$$\begin{aligned}\mathscr{F}_2(t) := & -\rho_1 \int_0^1 \dot{\varphi} \int_0^x (\varphi_x + \psi + lw)(y)\,dydx \\ & -\rho_1 \int_0^1 (w_x - l\varphi) \int_0^x \dot{w}(y)\,dydx, \ t > 0, \end{aligned} \tag{5.3.6}$$

satisfies

$$\begin{aligned}\mathscr{F}_2'(t) \leq & -\frac{\rho_1}{2} \int_0^1 \dot{\varphi}^2 dx - k_0 \int_0^1 (w_x - l\varphi)^2\,dx + \rho_1 \int_0^1 \dot{w}^2 dx \\ & + k \int_0^1 (\varphi_x + \psi + lw)^2\,dx + c_0 \int_0^1 \dot{\psi}^2 dx, \ t > 0. \end{aligned} \tag{5.3.7}$$

Proof. By differentiating the expression of $\mathscr{F}_2$, using Eqts $(5.1.10)_1$, $(5.1.10)_3$ and integrating by parts (with the fact that $k = k_0$), we get

$$\mathscr{F}_2'(t) = -k_0 \int_0^1 (w_x - l\varphi)^2\,dx - \rho_1 \int_0^1 \dot{\varphi}^2 dx + k \int_0^1 (\varphi_x + \psi + lw)^2\,dx$$

$$+\rho_1 \int_0^1 \dot{w}^2 dx - \rho_1 \int_0^1 \dot{\varphi} \int_0^x \dot{\psi}(y) dy dx. \tag{5.3.8}$$

Using Young's and Cauchy Schwarz inequalities, we obtain

$$\begin{aligned} -\rho_1 \int_0^1 \dot{\varphi} \int_0^x \dot{\psi}(y) dy dx \leq & \frac{\rho_1}{2} \int_0^1 \dot{\varphi}^2 dx + \frac{\rho_1}{2} \left(\int_0^x \dot{\psi}(y,t) dy \right)^2 \\ \leq & \frac{\rho_1}{2} \int_0^1 \dot{\varphi}^2 dx + \frac{\rho_1}{2} \int_0^1 \dot{\psi}^2 dx. \end{aligned} \tag{5.3.9}$$

By substituting (5.3.9) into (5.3.8), we conclude (5.3.7). □

Now, we define the following functional $\mathscr{F}_3$ by

$$\mathscr{F}_3(t) := \mathscr{F}_1(t) + \mathscr{F}_2(t), \ t > 0. \tag{5.3.10}$$

and we have

Lemma 5.4. *The functional $\mathscr{F}_3$ satisfies for all $\varepsilon_0 > 0$, the estimate*

$$\begin{aligned} \mathscr{F}_3'(t) \leq & -\frac{3\rho_1}{2} \int_0^1 \dot{\varphi}^2 dx + \varepsilon_0 \int_0^1 \psi_x^2 dx + c_0 \int_0^1 \dot{\psi}^2 dx \\ & + c_0 \left(1 + \frac{1}{\varepsilon_0} \right) \int_0^1 (\varphi_x + \psi + lw)^2 dx, \ t > 0 \end{aligned} \tag{5.3.11}$$

Proof. Using (5.3.3) and (5.3.7) the estimate (5.3.11) easly follows. □

Lemma 5.5. *Let $(\varphi, \psi, w, \theta)$ be a solution of the system* (5.1.10)-(5.1.12) *and let $k = k_0$. Then the functional $\mathscr{F}_4$ defined by*

$$\mathscr{F}_4(t) := -\rho_1 \int_0^1 \dot{\varphi} (w_x - l\varphi) dx - \rho_1 \int_0^1 \dot{w} (\varphi_x + \psi + lw) dx, \ t > 0 \tag{5.3.12}$$

satisfies the estimate

$$\begin{aligned} \mathscr{F}_4'(t) \leq & -\frac{\rho_1 l}{2} \int_0^1 \dot{w}^2 dx - k_0 l \int_0^1 (w_x - l\varphi)^2 dx + kl \int_0^1 (\varphi_x + \psi + lw)^2 dx \\ & + c_0 \int_0^1 \dot{\psi}^2 dx + \rho_1 l \int_0^1 \dot{\varphi}^2 dx, \ t > 0 \end{aligned} \tag{5.3.13}$$

Proof. By differentiating the expression of $\mathscr{F}_4$, using Eqts $(5.1.10)_1$, $(5.1.10)_3$ and integrating by parts (with the fact that $k = k_0$) , we get

$$\mathscr{F}_4'(t) = - k_0 l \int_0^1 (w_x - l\varphi)^2 dx + \rho_1 l \int_0^1 \dot{\varphi}^2 dx + kl \int_0^1 (\varphi_x + \psi + lw)^2 dx$$

$$-\rho_1\int_0^1 \dot{w}\psi dx-\rho_1 l\int_0^1 \dot{w}^2 dx.$$

Estimate (5.3.13) follows thanks to Young's inequality. □

As in [100], we introduce the multiplier p given by the solution of the Dirichlet

$$-p_{xx}=\psi_x,\ p(0)=p(1)=0, \tag{5.3.14}$$

then we can easily obtain the following inequality

$$\begin{aligned}&\int_0^1 \dot{p}^2 dx\leq\int_0^1 \dot{p}_x^2 dx\leq\int_0^1 \dot{\psi}^2 dx,\\ &\int_0^1 p^2 dx\leq\int_0^1 p_x^2 dx\leq\int_0^1 \psi^2 dx\leq\int_0^1 \psi_x^2 dx.\end{aligned} \tag{5.3.15}$$

Lemma 5.6. *Let* (φ,ψ,w,θ) *be a solution of the system* (5.1.10)-(5.1.12)*. Then we have, for all* $\varepsilon_1>0$*, the functional* $\mathscr{F}_5$ *defined by*

$$\begin{aligned}\mathscr{F}_5(t):=&-\rho_2\int_0^1 \dot{\psi}p_x dx-l\int_0^1(\rho_2 l\dot{\psi}-\rho_1\dot{w})\left(\int_0^x p(y)\right)dydx\\ &+\rho_1\int_0^1\dot{\varphi}pdx,\ \forall t>0\ ,\end{aligned} \tag{5.3.16}$$

verifies

$$\begin{aligned}\mathscr{F}_5'(t)\leq&-\frac{b}{2}\left(1-2l^2\right)\int_0^1\psi_x^2 dx+\varepsilon_1\int_0^1\dot{\varphi}^2 dx+\varepsilon_1\int_0^1\dot{w}^2 dx\\ &+c_0\int_0^1\hat{\theta}^2 dx+c_0\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1\dot{\psi}^2 dx,\ t>0\end{aligned} \tag{5.3.17}$$

.

Proof. By differentiating the expression of $\mathscr{F}_5$, exploiting Eqts $(5.1.10)_1-(5.1.10)_3$, we get

$$\begin{aligned}\mathscr{F}_5'(t):=&-b\int_0^1\psi_{xx}p_x dx+k\int_0^1(\varphi_x+\psi+lw)p_x dx\\ &+\gamma\int_0^1\hat{\theta}_x p_x dx-l^2 b\int_0^1\psi_{xx}\left(\int_0^x p(y)dy\right)dx+l^2\int_0^1\gamma\hat{\theta}_x\left(\int_0^x p(y)dy\right)dx\\ &+k_0 l\int_0^1(w_x-l\varphi)_x\left(\int_0^x p(y)dy\right)dx+k\int_0^1(\varphi_x+\psi+lw)_x pdx\\ &+\rho_1\int_0^1\dot{\varphi}\dot{p}dx+k_0 l\int_0^1(w_x-l\varphi)pdx-\rho_2\int_0^1\dot{\psi}\dot{p}_x dx\\ &-\rho_2 l^2\int_0^1\dot{\psi}\left(\int_0^x\dot{p}(y)\right)dydx+l\rho_1\int_0^1\dot{w}\left(\int_0^x\dot{p}(y)\right)dydx\end{aligned}$$

integrating by parts and using the fact that $-p_{xx}=\psi_x$

$$\begin{aligned}\mathscr{F}_5'(t):=&-b\int_0^1\psi_x^2dx+\gamma\int_0^1\hat{\theta}\psi_xdx+l^2b\int_0^1p_x^2dx-l^2\gamma\int_0^1\hat{\theta}pdx\\&+\rho_1\int_0^1\dot{\varphi}\dot{p}dx-\rho_2\int_0^1\dot{\psi}\dot{p}_xdx\\&-\rho_2l^2\int_0^1\dot{\psi}\left(\int_0^x\dot{p}(y)dy\right)dx+l\rho_1\int_0^1\dot{w}\left(\int_0^x\dot{p}(y)dy\right)dx\end{aligned}\tag{5.3.18}$$

By applying Young's, Poincaré and Cauchy Schwarz inequalities the relation (5.3.15), we arrive at

$$\begin{aligned}\gamma\int_0^1\hat{\theta}\psi_xdx&\le\frac{b}{4}\int_0^1\psi_x^2dx+\frac{\gamma^2}{b}\int_0^1\hat{\theta}^2dx,\\-l^2\gamma\int_0^1\hat{\theta}pdx&\le\frac{b}{4}\int_0^1\psi_x^2dx+\frac{l^4\gamma^2}{b}\int_0^1\hat{\theta}^2dx,\\l\rho_1\int_0^1\dot{w}\left(\int_0^x\dot{p}(y)\right)dydx&\le\varepsilon_1\int_0^1\dot{w}^2dx+\frac{\rho_1^2l^2}{4\varepsilon_1}\int_0^1\dot{\psi}^2dx,\\\rho_1\int_0^1\dot{\varphi}\dot{p}dx&\le\varepsilon_1\int_0^1\dot{\varphi}^2dx+\frac{\rho_1^2}{4\varepsilon_1}\int_0^1\dot{\psi}^2dx,\\-\rho_2l^2\int_0^1\dot{\psi}\left(\int_0^x\dot{p}(y)\right)dydx&\le\rho_2l^2\int_0^1\dot{\psi}^2dx,\\-\rho_2\int_0^1\dot{\psi}\dot{p}_xdx&\le\rho_2\int_0^1\dot{\psi}^2dx,\end{aligned}\tag{5.3.19}$$

we conclude (5.3.17) by by substituting (5.3.19) into (5.3.18).

□

Now, to prove the essential lemma which contains the condition on the coefficients (5.1.14), we introduce the following two functionals

$$\begin{aligned}G_1(t):=&\rho_1\frac{b}{k}\int_0^1\dot{\varphi}\psi_xdx+\rho_2\int_0^1\dot{\psi}(\varphi_x+\psi+lw)dx\\&+\gamma\frac{\tau_q^2}{2}\int_0^1\dot{\theta}_x(\varphi_x+\psi+lw)dx,\ t>0\end{aligned}\tag{5.3.20}$$

and

$$G_2(t):=\rho_1\int_0^1\dot{\varphi}\hat{\theta}dx+k\frac{\tau_q^2}{2}\int_0^1(\varphi_x+\psi+lw)\dot{\theta}_xdx+\kappa_1\rho_1\int_0^1\theta_x\varphi_xdx,\ t>0\tag{5.3.21}$$

So, we have

Lemma 5.7. *Let* (φ,ψ,w,θ) *be a solution of the system* (5.1.10)-(5.1.12)*. Then, the functional*

$\mathscr{F}_6$ *defined by*

$$\mathscr{F}_6(t) := G_1(t) + \frac{1}{\gamma\theta^0\rho_1}\left(\frac{\rho_1 b}{k} - \rho_2\right) G_2(t), \ t > 0, \tag{5.3.22}$$

satisfies for all $\varepsilon_2 > 0$, *the estimate*

$$\begin{aligned} \mathscr{F}_6'(t) \leq & -\left(\frac{k}{2} - \varepsilon_2\right)\int_0^1 (\varphi_x + \psi + lw)^2 dx + \varepsilon_2 \int_0^1 (w_x - l\varphi)^2 dx \\ & + \varepsilon_2 \int_0^1 \dot{w}^2 dx + c_0\left(1 + \frac{1}{\varepsilon_2}\right)\int_0^1 \dot{\theta}_x^2 dx \\ & + c_0\left(1 + \frac{1}{\varepsilon_2}\right)\int_0^1 \dot{\psi}^2 dx + c_0\left(1 + \frac{1}{\varepsilon_2}\right)\int_0^1 \theta_x^2 dx \\ & + \frac{c_0}{\varepsilon_2}\int_0^1 \hat{\theta}^2 dx + c_0\left(\frac{1}{\varepsilon_2} + 1\right)\int_0^1 \psi_x^2 dx - \frac{\tau_\theta}{k\gamma\theta^0}\chi_1 \int_0^1 \dot{\theta}_x \dot{\varphi}_x dx, \ t > 0 \end{aligned} \tag{5.3.23}$$

Proof. By differentiating the functionals G_1 and G_1 and integrating by parts, we get

$$\begin{aligned} G_1'(t) = & -k\int_0^1 (\varphi_x + \psi + lw)^2 dx + \frac{bk_0 l}{k}\int_0^1 (w_x - l\varphi)\,\psi_x dx \\ & - \gamma\tau_q \int_0^1 \dot{\theta}_x (\varphi_x + \psi + lw)\, dx - \gamma\int_0^1 \theta_x (\varphi_x + \psi + lw)\, dx \\ & + \rho_2 \int_0^1 \dot{\psi}^2 dx + l\rho_2 \int_0^1 \dot{\psi}\dot{w} dx + \gamma\frac{\tau_q^2}{2}\int_0^1 \dot{\theta}_x \dot{\psi} dx \\ & + \gamma\frac{\tau_q^2 l}{2}\int_0^1 \dot{\theta}_x \dot{w} dx + \gamma\frac{\tau_q^2}{2}\int_0^1 \dot{\theta}_x \dot{\varphi}_x dx + \left(\frac{\rho_1 b}{k} - \rho_2\right)\int_0^1 \dot{\varphi}\dot{\psi}_x dx, \end{aligned} \tag{5.3.24}$$

and

$$\begin{aligned} G_2'(t) = & -k\int_0^1 \left(\tau_q \dot{\theta}_x + \theta_x\right)(\varphi_x + \psi + lw)\, dx + k_0 l \int_0^1 (w_x - l\varphi)\,\hat{\theta} dx \\ & - \gamma\theta^0\rho_1 \int_0^1 \dot{\varphi}\dot{\psi}_x dx + \left(\frac{k\tau_q^2}{2} - \kappa_1\rho_1\tau_\theta\right)\int_0^1 \dot{\varphi}_x \dot{\theta}_x dx \\ & + \frac{k\tau_q^2}{2}\int_0^1 \dot{\psi}\dot{\theta}_x dx + \frac{lk\tau_q^2}{2}\int_0^1 \dot{w}\dot{\theta}_x dx + \kappa_1\rho_1 \int_0^1 \dot{\theta}_x \varphi_x dx. \end{aligned} \tag{5.3.25}$$

Now, Multiplying the relation (5.3.25) by $\eta = \frac{1}{\gamma\theta^0\rho_1}\left(\frac{\rho_1 b}{k} - \rho_2\right)$ and adding (5.3.24), we get,

$$\begin{aligned} G_1'(t) + \eta G_2'(t) \ = \ & -k\int_0^1 (\varphi_x + \psi + lw)^2 dx + \rho_2 \int_0^1 \dot{\psi}^2 dx \\ & + \frac{bk_0 l}{k}\int_0^1 (w_x - l\varphi)\,\psi_x dx - (k\tau_q\eta + \gamma\tau_q)\int_0^1 \dot{\theta}_x (\varphi_x + \psi + lw)\, dx \\ & - (\gamma + k\eta)\int_0^1 \theta_x (\varphi_x + \psi + lw)\, dx + l\rho_2 \int_0^1 \dot{\psi}\dot{w} dx \end{aligned} \tag{5.3.26}$$

$$
\begin{aligned}
&+\frac{\tau_q^2}{2}(\gamma+k\eta)\int_0^1 \dot{\theta}_x\dot{\psi}dx+\kappa_1\rho_1\eta\int_0^1 \dot{\theta}_x\varphi_x dx \\
&+k_0 l\eta\int_0^1 (w_x-l\varphi)\hat{\theta}dx+\frac{\tau_q^2 l}{2}(k\eta+1)\int_0^1 \dot{\theta}_x\dot{w}dx \\
&-\frac{\tau_\theta}{k\gamma\theta^0}\left(\underbrace{(\rho_1 b-\rho_2 k)\left(\kappa_1-\frac{k\tau_q^2}{2\tau_\theta\rho_1}\right)-\frac{\gamma^2\theta^0 k\tau_q^2}{2\tau_\theta}}_{=\chi_1}\right)\int_0^1 \dot{\theta}_x\dot{\varphi}_x dx.
\end{aligned}
$$

Applying Young's and Poincaré inequalities, we arrive at

$$
\begin{aligned}
k_0 l\eta\int_0^1 (w_x-l\varphi)\hat{\theta}dx \leq{}& \frac{\varepsilon_2}{4}\int_0^1 (w_x-l\varphi)^2 dx \\
&+\frac{c_0}{\varepsilon_2}\int_0^1 \hat{\theta}^2 dx,
\end{aligned}
\tag{5.3.27}
$$

$$
\begin{aligned}
-(k\tau_q\eta+\gamma\tau_q)\int_0^1 \dot{\theta}_x(\varphi_x+\psi+lw)dx \leq{}& \frac{k}{4}\int_0^1 (\varphi_x+\psi+lw)^2 dx \\
&+c_0\int_0^1 \dot{\theta}_x^2 dx,
\end{aligned}
\tag{5.3.28a}
$$

$$
-(\gamma+k\eta)\int_0^1 \theta_x(\varphi_x+\psi+lw)dx \leq \frac{k}{4}\int_0^1 (\varphi_x+\psi+lw)^2 dx+c_0\int_0^1 \theta_x^2 dx, \tag{5.3.29}
$$

$$
\frac{bk_0 l}{k}\int_0^1 \psi_x(w_x-l\varphi)dx \leq \frac{\varepsilon_2}{4}\int_0^1 (w_x-l\varphi)^2 dx+\frac{c_0}{\varepsilon_2}\int_0^1 \psi_x^2 dx, \tag{5.3.30}
$$

$$
\frac{\tau_q^2 l}{2}(k\eta+1)\int_0^1 \dot{\theta}_x\dot{w}dx \leq \frac{\varepsilon_2}{2}\int_0^1 \dot{w}^2 dx+\frac{c_0}{\varepsilon_2}\int_0^1 \dot{\theta}_x^2 dx, \tag{5.3.31}
$$

$$
\frac{\tau_q^2}{2}(\gamma+k\eta)\int_0^1 \dot{\theta}_x\dot{\psi}dx \leq c_0\int_0^1 \dot{\theta}_x^2 dx+c_0\int_0^1 \dot{\psi}^2 dx, \tag{5.3.32}
$$

$$
l\rho_2\int_0^1 \dot{\psi}\dot{w}dx \leq \frac{\varepsilon_2}{2}\int_0^1 \dot{w}^2 dx+\frac{c_0}{\varepsilon_2}\int_0^1 \dot{\psi}^2 dx. \tag{5.3.33}
$$

Now, as in [101], it is well known that there is a positive constant C_l such that

$$
\int_0^1 \varphi_x^2 dx \leq C_l\int_0^1 (\varphi_x+\psi+lw)^2 dx+C_l\int_0^1 \psi_x^2 dx+C_l\int_0^1 (w_x-l\varphi)^2 dx, \tag{5.3.34}
$$

so, we end up with

$$
\kappa_1\rho_1\eta\int_0^1 \dot{\theta}_x\varphi_x dx \leq \frac{\varepsilon_2}{2}\int_0^1 (\varphi_x+\psi+lw)^2 dx+\frac{\varepsilon_2}{2}\int_0^1 \psi_x^2 dx \tag{5.3.35}
$$

$$+\frac{\varepsilon_2}{2}\int_0^1 (w_x - l\varphi)^2 dx + \frac{c_0}{\varepsilon_2}\int_0^1 \dot{\theta}_x^2 dx.$$

Substituting (5.3.27)–(5.3.35) into (5.3.26), we conclude (5.3.23). □

Lemma 5.8. *Let* $(\varphi, \psi, w, \theta)$ *be a solution of the system* (5.1.10)-(5.1.12)*. Then we have, for all* $\varepsilon_3 > 0$*, the functional* $\mathscr{F}_7$ *defined by*

$$\begin{aligned}\mathscr{F}_7(t) :&= -\int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta}\right)\left(\frac{\tau_q^2}{2}\dot{\theta} + \tau_q\theta\right) dx \\ &\quad - \frac{\tau_q}{2}\int_0^1 \theta^2 dx,\ t > 0,\end{aligned} \tag{5.3.36}$$

Verifies

$$\begin{aligned}\mathscr{F}_7'(t) \leq &-\frac{1}{2}\int_0^1 \hat{\theta}^2 dx + \varepsilon_3\int_0^1 \psi^2 dx \\ &+ c_0\left(1 + \frac{1}{\varepsilon_3}\right)\int_0^1 \left(\dot{\theta}_x^2 + \theta_x^2\right) dx,\ t > 0,\end{aligned} \tag{5.3.37}$$

Proof. By differentiating the expression of $\mathscr{F}_7$, exploiting Eq. $(5.1.10)_4$ and integrating by parts, we get

$$\begin{aligned}\mathscr{F}_7'(t) = &-\int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta}\right)^2 dx + \frac{\tau_q^2}{2}\int_0^1 \dot{\theta}^2 dx + \frac{\kappa_1\tau_\theta\tau_q^2}{2}\int_0^1 \dot{\theta}_x^2 dx \\ &+ \kappa_1\tau_q\int_0^1 \theta_x^2 dx + \left(\kappa_1\tau_\theta + \frac{\tau_q}{2}\right)\tau_q\kappa_1\int_0^1 \theta_x\dot{\theta}_x dx \\ &- \frac{\gamma\theta^0\tau_q^2}{2}\int_0^1 \psi\dot{\theta}_x dx - \tau_q\gamma\theta^0\int_0^1 \psi\theta_x dx.\end{aligned}$$

Then applying Young's, Poincaré inequalities and using the fact that

$$-\int_0^1 \left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta}\right)^2 dx \leq -\frac{1}{2}\int_0^1 \underbrace{\left(\frac{\tau_q^2}{2}\ddot{\theta} + \tau_q\dot{\theta} + \theta\right)^2}_{=\hat{\theta}^2} dx + \int_0^1 \theta_x^2 dx,$$

we end up with (5.3.37). □

Lemma 5.9. *Let* $(\varphi, \psi, w, \theta)$ *be a solution of the system* (5.1.10)-(5.1.12)*. Then we have, for all* $\varepsilon_4 > 0$*, the functional* $\mathscr{F}_8$ *defined by*

$$\mathscr{F}_8(t) := -\rho_2\int_0^1 \hat{\theta}\left(\int_0^x \psi dy\right) dx,\ t > 0,$$

satisfies

$$\begin{aligned}\mathscr{F}_8'(t) \leq & -\frac{\gamma_1\theta^0\rho_2}{2}\int_0^1 \dot{\psi}^2 dx + \varepsilon_4\int_0^1 \psi_x^2 dx + \varepsilon_4\int_0^1 (\varphi_x+\psi+lw)^2 dx \\ & + c_0\int_0^1 \theta_x^2 dx + c_0\int_0^1 \dot{\theta}_x^2 dx + c_0\left(1+\frac{1}{\varepsilon_4}\right)\int_0^1 \hat{\theta}^2 dx,\ t\ >\ 0 \end{aligned} \tag{5.3.38}$$

Proof. By differentiating the expression of $\mathscr{F}_7$ and exploiting the Eqts $(1)_2$, $(1)_4$, then using integration by parts with the fact that $\int_0^1 \psi dy = 0$, we get

$$\begin{aligned}\mathscr{F}_8'(t) = & -\gamma_1\theta^0\rho_2\int_0^1 \dot{\psi}^2 dx + \kappa_1\rho_2\int_0^1 \left(\tau_\theta\dot{\theta}_x+\theta_x\right)\dot{\psi} dx \\ & - b\int_0^1 \hat{\theta}\psi_x dx + k\int_0^1\int_0^x (\varphi_y+\psi+lw)\, dy\hat{\theta} dx + \gamma\int_0^1 \hat{\theta}^2 dx, \end{aligned} \tag{5.3.39}$$

then applying Young's, Poincaré, Cauchy Schwarz inequalities, we have

$$\kappa_1\rho_2\int_0^1 \left(\tau_\theta\dot{\theta}_x+\theta_x\right)\dot{\psi} dx \leq \frac{\gamma_1\theta^0\rho_2}{2}\int_0^1 \dot{\psi}^2 dx + c_0\int_0^1 \left(\dot{\theta}_x^2+\theta_x^2\right) dx \tag{5.3.40}$$

$$-b\int_0^1 \hat{\theta}\psi_x dx \leq \varepsilon_4\int_0^1 \psi_x^2 dx + \frac{c_0}{\varepsilon_4}\int_0^1 \hat{\theta}^2 dx, \tag{5.3.41}$$

$$\begin{aligned}&\int_0^1\int_0^x (\varphi_x+\psi+lw)\, dy\hat{\theta} dx \\ &\leq \varepsilon_4\int_0^1\left(\int_0^x (\varphi_y+\psi+lw)\, dy\right)^2 dx + \frac{c_0}{\varepsilon_4}\int_0^1 \hat{\theta}^2 dx \\ &\leq \varepsilon_4\int_0^1 (\varphi_x+\psi+lw)\, dx + \frac{c_0}{\varepsilon_4}\int_0^1 \hat{\theta}^2 dx \end{aligned} \tag{5.3.42}$$

Estimate (5.3.38) follows by substituting (5.3.40)-(5.3.42) into (5.3.39). □

Now, we present our stability result.

5.4 Exponential decay

Theorem 5.2. *Assume* (5.1.7) *and* (5.1.13) *holds. Then, for small l, there exist two positive constants m_0 and m_1 such that the energy functional given by* (5.1.7) *satisfies*

$$E(t) \leq m_0 e^{-m_1 t},\ t > 0 \tag{5.4.1}$$

Proof. Let the Lyapunov functional $\mathscr{L}$ define as

$$\begin{aligned}\mathscr{L}(t) &= ME(t)+N_1\mathscr{F}_3(t)+N_2\mathscr{F}_4(t)+N_3\mathscr{F}_5(t)\\ &\quad +N_4\mathscr{F}_6(t)+N_5\mathscr{F}_7(t)+N_6\mathscr{F}_8(t), \forall t>0,\end{aligned} \tag{5.4.2}$$

where M and N_i, $i=1-8$ are positive constants to be properly chosen later. By simple routine computations, applying Young's, Cauchy–Schwarz, and Poincaré inequalities, it follows that $\mathscr{L}\sim E$ in the sense that there exist two positive constants α_1 and α_2 such that:

$$\alpha_1 E(t)\le \mathscr{L}(t)\le \alpha_2 E(t) \tag{5.4.3}$$

Now, by recalling (5.3.2), (5.3.3), (5.3.7), (5.3.11), (5.3.13), (5.3.17), (5.3.23), (5.3.37), (5.3.38) and taking

$$N_1=2l,\ N_2=2,\ \varepsilon_0=\rho_1, \varepsilon_1=\frac{\rho_1 l}{2N_3}\ ,\ \varepsilon_2=\frac{l}{N_4}, \varepsilon_3=\frac{2c_0}{N_5}, \varepsilon_4=\frac{l}{N_6}, \tag{5.4.4}$$

we end up with

$$\begin{aligned}\mathscr{L}'(t) \le &-d_{\dot\varphi}\int_0^1 \dot\varphi^2 dx - d_{(w_x-l\varphi)}\int_0^1 (w_x-l\varphi)^2\,dx - d_{\dot w}\int_0^1 \dot w^2 dx\\ &-d_{(\varphi_x+\psi+lw)}\int_0^1 (\varphi_x+\psi+lw)^2\,dx - d_{\hat\theta}\int_0^1 \hat\theta^2 dx\\ &-d_{\psi_x}\int_0^1 \psi_x^2 dx - d_{\theta_x}\int_0^1 \theta_x^2 dx - d_{\dot\theta_x}\int_0^1 \dot\theta_x^2 dx - d_{\dot\psi}\int_0^1 \dot\psi^2 dx,\end{aligned} \tag{5.4.5}$$

where

$$\begin{aligned}
&d_{\dot\varphi}=\tfrac{l\rho_1}{2}\\
&d_{(w_x-l\varphi)}=k_0 l\\
&d_{\dot w}=\tfrac{\rho_1 l}{4}\\
&d_{(\varphi_x+\psi+lw)}=\tfrac{k}{2}N_4-2lc_0\left(1+\tfrac{1}{\rho_1}\right)-2kl-2l\\
&d_{\hat\theta}=\tfrac{1}{2}N_5-c_0N_3-\tfrac{c_0}{l}N_4^2-c_0\left(1+N_6\right)\tfrac{N_6}{l}\\
&d_{\psi_x}=\tfrac{b}{2}N_3\left(1-2l^2\right)-\tfrac{2l}{\rho_1}-c_0N_4\left(\tfrac{N_4}{l}+1\right)-l\\
&d_{\theta_x}=\kappa_1 M-c_0\left(1+\tfrac{N_4}{l}\right)N_4-c_0\left(1+\tfrac{N_5}{2c_0}\right)N_5-c_0N_6\\
&d_{\dot\theta_x}=\left(\tfrac{\tau_q\kappa_1}{2}\right)(2\tau_\theta-\tau_q)M-c_0\left(1+\tfrac{N_4}{l}\right)N_4-c_0\left(1+\tfrac{N_5}{2c_0}\right)N_5-c_0N_6\\
&d_{\dot\psi}=\tfrac{\gamma_1\theta^0\rho_2}{2}N_6-2c_0l-4c_0-c_0\left(1+\tfrac{2N_3}{\rho_1 l}\right)N_3-c_0\left(1+\tfrac{N_4}{l}\right)N_4.
\end{aligned}$$

Now, we select N_4 large enough such that

$$N_4>\frac{2l}{k}\left[2c_0\left(1+\frac{1}{\rho_1}\right)+2k+2\right],$$

then we select N_3 large enough such that

$$N_3 > \frac{2}{b(1-2l^2)}\left[\frac{2l}{\rho_1}+c_0N_4\left(\frac{N_4}{l}+1\right)+l\right].$$

Similarly, we choose N_6 large enough such that

$$N_6 > \frac{2}{\gamma_1\theta^0\rho_2}\left[c_0\left(1+\frac{2N_3}{\rho_1 l}\right)N_3+c_0\left(1+\frac{N_4}{l}\right)N_4+2c_0l+4c_0\right],$$

and we pick N_5 large enough such that

$$N_5 > 2\left[c_0N_3+\frac{c_0}{l}N_4^2+c_0\left(1+N_6\right)\frac{N_6}{l}\right].$$

Finally, we select M large enough such that (5.4.3) remains valid and

$$M > \frac{m}{\kappa_1}\left[c_0\left(1+\frac{N_4}{l}\right)N_4+c_0\left(1+\frac{N_5}{2c_0}\right)N_5+c_0N_6\right],$$

where

$$m = \max\{1, \frac{2}{(2\tau_\theta-\tau_q)\tau_q}\}.$$

Therefore, there is a positive constant η_0 such that

$$\mathscr{L}'(t) \leq -\eta_0 E(t),\ \forall t>0,$$

using (5.4.3) we arrive at

$$\mathscr{L}'(t) \leq -\frac{\eta_0}{\beta}\mathscr{L}(t),\ \forall t>0. \tag{5.4.6}$$

A simple integration of (5.4.6) over $(0,t)$ yields

$$\mathscr{L}(t) \leq \mathscr{L}(0)e^{-\frac{\eta_0}{\beta}t},\ \forall t>0. \tag{5.4.7}$$

Consequently, (5.4.1) is established thanks to (5.4.3) and (5.4.7). □

Conclusions and perspectives

Drawing upon the aforementioned studies and research, we derive several significant conclusions. In Chapter 3, our comprehensive study has provided valuable insights into the behavior of the thermoelastic Bresse system. The theoretical analysis establishes crucial properties such as the existence and uniqueness of solutions, as well as exponential stability irrespective of system coefficients. This stability, notably unaffected by coefficient of system, can be attributed to the role of adding damping and the thermal dissipation caused by the Green and Naghdi theories in the system, which enhances its overall stability characteristics. On the numerical front, a finite element approximation is presented, demonstrating discrete energy decay, which enhances our understanding of the system's behavior. Furthermore, the study offers valuable insights into the practical implementation of these findings through numerical results, accompanied by an error estimate based on additional solution regularity.

In Chapter 4, we present new sufficient conditions for the exponential stability of a thermoelastic Timoshenko system using a dual-phase-lag thermal conductivity model. This work shows that for ultrafast thermoelasticity processes, the assumption of equal wave propagation velocities is insufficient to push the system to the stability state.

Except for the imposed requirements, improving this result is an intriguing open problem. Furthermore, the system's polynomial stability in the case of $2\tau_\theta = \tau_q$ is an interesting subject to explore (we predict that the system's polynomial stability is held with the same condition on the coefficient but with higher regularity for the solution).

Regarding Timoshenko and Bresse systems and the findings highlighted in Chapter 3, we suggested in future research could explore the time decay behavior of the one-dimensional thermoelastic Bresse system with the Three-phase-lag heat conduction model. An intriguing avenue for future research lies in extending our theoretical and numerical analysis to encompass a one-dimensional thermoelastic Timoshenko system incorporating the three-phase-lag (TPL) heat conduction model, i.e., the following system

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma \left(\frac{\tau_q^2}{2} \theta_{tt} + \tau_q \theta_t + \theta \right)_x = 0, \\ \left(\frac{\tau_q^2}{2} \theta_{tt} + \tau_q \theta_t + \theta \right)_t - (k_0^* v_x + \tau_v^* \theta_x + \tau_\theta k_0 \theta_{tx})_x + \gamma \theta^0 \psi_{tx} = 0, \end{cases} \tag{5.4.8}$$

Another problem related to our system worth studying in the case of ultrafast thermoelasticity is the laminated Timoshenko beam, it can be written as follows

$$\begin{cases} \rho_1\varphi_{tt}+G(\psi-\varphi_x)_x=0, \\ I_\rho\,(3w-\psi)_{tt}-D\,(3w-\psi)_{xx}-G\,(\psi-\varphi_x)-\delta\left(\frac{\tau_q^2}{2}\theta_{ttx}+\tau_q\theta_{tx}+\theta_x\right)=0, \\ I_\rho w_{tt}-Dw_{xx}+G\,(\psi-\varphi_x)+\frac{4}{3}\gamma w+\frac{4}{3}\alpha w_t=0, \\ \left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)_t-\kappa_1\,(\tau_\theta\theta_{tx}+\theta_x)_x+\gamma\theta^0\,(3w-\psi)_{tx}=0, \end{cases} \tag{5.4.9}$$

where φ, ψ, w are the transverse displacement, the rotation angle and proportional to the amount of slip, respectively see [102].

Appendix A

Some codes

A.1 Matlab script for algorithm 3.1

```
clear
clc
% Setting values constants
e=0.025;a=0.004;l=1/20;E=22*10^(4);p=8850;
Kp=5/6;r=0.29;G=E/(2+2*r);cpsi=sqrt(Kp*G/p);
I=(a*e^(3))/12; A=a*e ; k=Kp*G*A;
p1=p*A; p2=p*I; p3=p1; R=1/l; k0=E*A;b=k*p2/p1;
gma=pi;  kt=5;   to=0.01; sgma=0.01;  sgma2=5;
%Defining time and space steps
L=1;T=20;dt=0.01;dx=dt;
x=0:dx:L;t=0:dt:T;
m_x=size(x,2);
n_t=size(t,2);
%Setting initial condition
q1=0*x';
q2=(x.*(x-1))';
q3=(x.^2.*(x-1))';
%Define the matrices Me Pe Oe
Me=dx*[ 1/3 1/6 ;1/6 1/3];
Pe=(1/dx)*[ 1 -1 ;-1 1];
Oe=(1/2)*[ -1 1 ;-1 1];
%The assembly of the matrices M,P,O and defining
%the block matrices  $\bar{A}$, using the function MatrixBM.
[Abar,M,P,O,M2,P2,O2]=MatrixBM(Me,Pe,Oe,m_x,dt,p1,p2,p3,k,b,k0,l,
   sgma,sgma2,to,kt);
Z=zeros(size(M));
%Defining the block matrices B C using the function MatrixBM2
[Bbar,Cbar]=MatrixBM2(Me,Pe,Oe,m_x,dt,p1,p2,p3,k,b,k0,l,sgma,sgma2,
   to,kt);
```

```
%% solution calculations
A0=blkdiag(M,M2,M,M2);%blkdiag a function create a diagonal matrix
    by blocks
U=A0*[q1;q2;q1;q3];%initialisation
BNpart2and3(:,1)=Cbar*U(:,1);
%Using LU decompostion
[Ltri,Utri,Per]=lu(Abar);
for i=2:n_t
    BNpart1=Bbar*U(:,i-1);
    BNU=BNpart1+BNpart2and3(:,i-1);
    S = Ltri\(Per*BNU);
    U(:,i) = Utri\S;
    BNpart2and3(:,i)=BNpart2and3(:,i-1)+dt*Cbar*U(:,i);
end

%% Extraction of the solution
V0=A0*[q1;q2;q1;q3];
[KCIp,EXap,OMGp,Thetap]=extract_U(U,m_x,n_t);
V=[zeros(4*m_x,1),dt*cumsum(U(:,2:n_t),2)];
V=V+V0;
[KCI,EXa,OMG,Theta]=extract_U(V,m_x,n_t);

%% Energy calculation
Energy=calc_ENERGY2(EXap,KCIp,OMGp,Thetap,EXa,KCI,OMG,Theta,p1,p2,p3
    ,k,b,k0,l,kt,m_x,dx,n_t);

%% plotting energy
subplot(1,2,1)
plot(t,Energy,'b')
ylabel('E(t)');xlabel('t');
grid on
subplot(1,2,2)
plot(t,-log(Energy),'b')
ylabel('-log(E(t))');xlabel('t');
grid on
```

```
function [DA,B,C,M,P,O,M2,P2,O2]=MatrixBM(Me,Pe,Oe,m_x,dt,p1,p2,p3,k
```

```
   ,b,k0,l,sgma,sgma2,to,kt)
M=assombli(Me,m_x,0);M2=assombli(Me,m_x,1);
P=assombli(Pe,m_x,0);P2=assombli(Pe,m_x,1);
O=assombli(Oe,m_x,0);O2=assombli(Oe,m_x,1);
Z=zeros(size(M));
DM=[((p2/dt)+k*dt)*M,  Z,                    dt*k*l*M,           Z
   ;
    Z,        (p1/dt+sgma+dt*k0*(l^2))*M2,        Z,        -k0*l
       ^2*M2;
    dt*k*l*M,         Z,               (p1/dt+k*l^2*dt)*M,      Z;
    Z,           -k0*l^2*M2,                      Z,
       (p3/dt)*M2];

DO=[Z,         k*dt*O,            Z,                    Z;
    -k*dt*O2,      Z,       -(k+k0)*dt*l*O2,             Z;
     Z,      (k+k0)*dt*l*O,           Z,               k0*l*O;
    Z,          Z,                k0*l*O2,           Z];
%revoir k1 k3 et ki

DP=[(dt*b+sgma2)*P,               Z,           Z,                 Z
   ;
    Z,                     k*(dt)*P2,     Z,                Z;
    Z,                      Z,          k0*(dt)*P,       Z;
    Z,                      Z,          Z,        (dt*kt+to)*P2];
DA=DM+DP+DO;
B=blkdiag((p2/dt)*M,(p1/dt)*M2,(p1/dt)*M,(p3/dt)*M2);

Bp1=[-k*M,      Z,             -k*l*M,          Z;
     Z,       -k0*l^2*M2,         Z,             Z ;
    -k*l*M,        Z ,      -k*l^2*M,          Z;
    Z,                  Z,             Z,        Z];

Bp2=[Z,           -k*O,               Z,             Z;
    k*O2,          Z,     (k+k0)*l*O2,             Z ;
    Z,      -(k+k0)*l*O,             Z,              Z;
    Z,               Z,              Z,              Z];
```

```


Bp3=[-b*P,    Z,              Z,         Z;
    Z,              -k*P2,            Z,         Z ;
    Z,              Z,              -k0*P,     Z;
    Z,              Z,               Z,         -kt*P2];
C=Bp1+Bp2+Bp3;
end
```

```
function M=assombli(Me,m_x,choix)
% If choix=1 Dirichlet condition else Neumann condition
M=zeros(m_x);
a=Me(1,1)+Me(2,2);
b=Me(2,1);
c=Me(1,2);
M1=diag(a*ones(1,m_x-2))+ diag(c*ones(1,m_x-3),1)+diag(b*ones(1,m_x
   -3),-1);
M(2:m_x-1,2:m_x-1)=M1;
M(1:2,1)=Me(1:2,1);M(1,2)=Me(1,2);
M(m_x,m_x-1:m_x)=Me(2,1:2);M(m_x-1,m_x)=Me(1,2);
if choix==1
M(1,1:2)=[Me(1,1) 0];
M(m_x,m_x-1:m_x)=[0 Me(2,2)];
end

end
```

```
function [V,E,Theta,W]=extract_U(U,m_x,n_t)
V=U(1:m_x,:);
E=U(m_x+1:2*m_x,:);
Theta=U(2*m_x+1:3*m_x,:);
W=U(3*m_x+1:4*m_x,:);
end
```

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