

BADJI MOKHTAR-ANNABA
UNIVERSITY
UNIVERSITE BADJI MOKHTAR
ANNABA



جامعة باجي مختار
- عنابة -

Faculty of Sciences
Department of Mathematics Year: 2025/2026



THESIS

Presented with a view to obtaining the doctorate degree

Multiphase Problems and Applications

Stream
Mathematics

Speciality
Applied Mathematics

By
Lakikza Hanane

SUPERVISOR: Frekh Taallah Prof. U.B.M.- Annaba

CO-SUPERVISOR: Gasmi Souad Prof. U.B.M.- Annaba

In front of the jury

PRESIDENT:	Bouras Mohamed Cherif	Prof. U.B.M.- Annaba
EXAMINER:	Zouyed Fairouz	Prof. U.B.M. - Annaba
EXAMINER:	Bouzettouta Lamine	Prof. Univ.-Skikda

وزارة التعليم العالي والبحث العلمي

Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

BADJI MOKHTAR-ANNABA

UNIVERSITY

UNIVERSITE BADJI MOKHTAR
ANNABA



جامعة باجي مختار

- عنابة -

Faculté des Sciences

Département de Mathématiques

Année : 2025/2026



THÈSE

Présentée en vue de l'obtention du diplôme de Doctorat

Problèmes multiphasiques et applications

Filière
Mathématiques

Spécialité
Mathématiques Appliquées

Par
Lakikza Hanane

DIRECTEUR DE THÈSE: Frekh Taallah

Prof. U.B.M. – Annaba

Co-DIRECTEUR DE THÈSE: Gasmi Souad

MCB. U.B.M. - Annaba

Devant le jury

PRESIDENT: Bouras Mohamed Cherif

Prof. U.B.M. - Annaba

EXAMINATEUR : Zouyed Fairouz

Prof. U.B.M. - Annaba

EXAMINATEUR : Bouzettouta Lamine

Prof. Univ.-Skikda

Acknowledgements

First and foremost, I would like to thank **Allah** for granting me the strength, patience, and perseverance to complete this work.

I thank my thesis supervisor, **Prof. Frekh Taallah**, and my co-supervisor, **Dr. Souad Gasmi**, for supervising this thesis.

I am deeply grateful to the members of the examination committee for their valuable time and consideration. I would like to express special appreciation to **Prof. Bouras Mohamed Cherif** for kindly agreeing to chair the defense committee, as well as **Prof. Zouyed Fairouz** and **Prof. Bouzettouta Lamine** for their generosity in serving as committee members and providing thoughtful feedback.

I would like to express my heartfelt gratitude to **my parents** for their unconditional love, encouragement, and sacrifices. Their constant support has been my source of strength throughout my academic journey. I am also thankful to my sisters, family members, classmates, and friends for their encouragement, companionship, and understanding during these years of study.

Dedication

I proudly dedicate this work, first and foremost, to **my beloved parents**. To my mother, for her endless love and prayers, and to my father, for his unwavering support, guidance, and sacrifices.

To **my dear family**, for always being my source of strength, encouragement, and motivation throughout my life.

To **all my teachers and professors** across all educational levels, who illuminated my path with knowledge, inspired my curiosity, and helped shape my future.

To **my friends and classmates**, for the support we shared during these years of study.

مسائل متعددة الأطوار وتطبيقاتها

ملخص

تتناول هذه الرسالة النمذجة الرياضية والعددية للأغشية الحيوية البكتيرية بوصفها أنظمة متعددة الأطوار، مع التركيز على تأثير الحرارة، والتواصل الجماعي، وتنظيم النمو في ديناميكياتها. وتعتمد الدراسة نماذج مستمرة تصف التفاعل بين الكتلة الحيوية، والمصفوفة خارج الخلية، والطور السائل، والمغذيات، باستخدام أدوات التحليل الوظيفي ونظرية شبه المجموعات وطرائق التقريب العددي.

تتمثل المساهمة الأولى في توسيع نموذج ثلاثي الأطوار بإدماج اقتران حراري--بيوفيزيائي من خلال معادلة تصف تأثير درجة الحرارة والحرارة الناتجة عن النشاط الأيضي. ويشمل النموذج تفاعل الطور الخلوي، والمصفوفة خارج الخلية، والطور المائي، مع انتقال المغذيات والتدفق الأسموزي على سطح صلب مسامي. وباستخدام تقريب الغشاء الرقيق، اشتق نموذج مختزل يتضمن حداً لتخميد النمو، قصد تمثيل المقاومة الميكانيكية والحد من التشنج غير الواقعي. كما اقترح مخطط بالفروق المنتهية، ودُرس استقراره بتقدير طاقي متقطع. وأظهرت المحاكاة المتماثلة محورياً أثر الحرارة، والتورم، والتدفق الأسموزي، وتخمد النمو في تطور الغشاء الحيوي وانتقال المغذيات.

أما المساهمة الثانية فتتمثل في بناء نموذج تفاعل--انتشار يدمج التواصل الجماعي، والاستجابة للمضادات الحيوية، وديناميكيات درجة الحرارة. ويصف النموذج تفاعل الخلايا منخفضة التنظيم ومرتفعة التنظيم مع المغذيات وجزيئات الإشارة والبروتينات المنظمة، ضمن اقتران حراري يربط النشاط الأيضي بالسرعات الحيوية وآليات التنظيم الجماعي. وقد أُثبت وجود الحل ووحدانيته باستخدام نظرية شبه المجموعات. وعددياً، استُخدمت العناصر المنتهية في الفضاء وطريقة أويلر العكسية في الزمن، مع إثبات الوجود والوحدانية المتقطعة وتقديرات طاقة ونتائج تقارب. كما أظهرت التجارب العددية تأثير درجة الحرارة في التواصل الجماعي والتطور الزمني والمكاني للغشاء الحيوي.

تخلص الرسالة إلى أن إدماج التأثيرات الحرارية والبنية متعددة الأطوار وتنظيم النمو ضروري لتحسين نمذجة الأغشية الحيوية وفهم ديناميكياتها، ويفتح المجال للمعايرة التجريبية وتطوير وسائل للتحكم فيها في التطبيقات الطبية والصناعية والبيئية.

الكلمات المفتاحية: الأغشية الحيوية، النمذجة متعددة الأطوار، تخمد النمو، التواصل الجماعي، التأثيرات الحرارية، معادلات التفاعل--الانتشار، الفروق المنتهية، العناصر المنتهية.

Problèmes multiphasiques et applications

Résumé

Cette thèse développe des cadres mathématiques et numériques pour la modélisation des films biologiques bactériens en tant que systèmes multiphasiques, avec un accent particulier sur les effets thermiques et les mécanismes de communication dépendante du quorum. Après avoir présenté le contexte biologique des films biologiques et leur organisation multiphasique, ce travail établit les principaux outils analytiques et numériques nécessaires aux développements ultérieurs, notamment les notions fonctionnelles, les concepts de convergence faible, les techniques fondées sur la théorie des semi-groupes et les approches classiques de discrétisation.

Une première contribution consiste à étendre une formulation triphasique d'un film biologique en y incorporant un couplage thermobiophysique à travers une équation de bilan thermique, permettant aux coefficients cinétiques et de transport de dépendre explicitement de la température. Une réduction de type film mince est introduite afin d'obtenir un système traitable, et un schéma numérique par différences finies est proposé, accompagné d'une analyse de stabilité. Des simulations réalisées dans un cadre axisymétrique illustrent l'impact des variations de température sur la croissance, le gonflement et les processus de transport des films biologiques.

Une deuxième contribution propose un cadre couplé de type réaction–diffusion intégrant la communication dépendante du quorum, la réponse aux antibiotiques et la dynamique de la température, y compris la génération de chaleur métabolique et ses rétroactions sur l'évolution biologique. Le caractère bien posé du système résultant est établi à l'aide de la théorie des semi-groupes, et une approximation numérique fondée sur les éléments finis en espace ainsi que sur l'intégration temporelle par la méthode d'Euler implicite est construite et analysée. Des expériences numériques mises en œuvre sous Matlab mettent en évidence l'impact de la température sur l'activité de la communication dépendante du quorum et sur l'évolution globale du film biologique. La thèse s'achève par une synthèse des principaux résultats obtenus et par la présentation de plusieurs perspectives de recherche, parmi lesquelles l'étalonnage des paramètres, la validation expérimentale, l'extension des modèles, ainsi que l'amélioration de l'efficacité des calculs et le développement d'applications orientées vers la commande.

Mots-clés : films biologiques, modélisation multiphasique, communication dépendante du quorum, effets thermiques, équations de réaction–diffusion, éléments finis.

Multiphase Problems and Applications

Abstract

This thesis develops mathematical and numerical frameworks for the modeling of bacterial biofilms as multiphase systems, with particular emphasis on thermal effects, quorum-sensing mechanisms, and growth regulation. After presenting the biological background of biofilms and their multiphase organization, the work establishes the main analytical and numerical tools required for the subsequent developments, including functional-analytic notions, concepts of weak convergence, semigroup-based techniques, and standard discretization methods.

A first contribution consists in extending a three-phase biofilm formulation by incorporating a thermobiophysical coupling through a heat-balance equation, allowing the main kinetic and transport coefficients to depend explicitly on temperature. A thin-film reduction is introduced in order to derive a tractable system, and a finite-difference numerical scheme is proposed together with a stability analysis. In addition, a growth-damping term is included in the reduced thickness equation to prevent unrealistically rapid thickening and to account, at the macroscopic level, for the mechanical resistance induced by biomass accumulation. Numerical simulations in an axisymmetric setting illustrate the influence of temperature variations on biofilm growth, swelling behavior, transport processes, and thickness regulation.

A second contribution proposes a coupled reaction–diffusion framework integrating quorum sensing, antibiotic response, and temperature dynamics, including metabolic heat generation and its feedback on biological evolution. The well-posedness of the resulting system is established through semigroup theory, and a numerical approximation based on finite elements in space and backward Euler time integration is constructed and analyzed. Numerical experiments implemented in Matlab highlight the effect of temperature on quorum-sensing activity and on the overall evolution of the biofilm.

The thesis concludes with a synthesis of the main results and outlines several perspectives for future research, including parameter calibration, experimental validation, model extensions, improvements in computational efficiency, and control-oriented applications.

Keywords: biofilms, multiphase modeling, growth damping, quorum sensing, thermal effects, reaction–diffusion equations, finite elements.

Table of Contents

Acknowledgements	i
Dedication	ii
Introduction	1
1 Preliminary	5
1.1 General Concepts on Biofilms	5
1.1.1 What is the Biofilms?	5
1.1.2 Biofilms as Multiphasic Systems	5
1.1.3 Formation of Biofilms on Surfaces	6
1.1.4 Extracellular Polymeric Matrix (EPS)	6
1.1.5 Quorum Sensing	7
1.1.6 Antibiotic Resistance of Biofilms	7
1.2 Preliminary Mathematical Concepts	7
1.2.1 Functional Spaces	7
1.2.2 The weak and weak-star topologies	8
1.2.3 Hilbert spaces	9
1.2.4 The $L^p(\Omega)$ spaces	10
1.2.5 Sobolev spaces	10
1.2.6 Inequalities	10
1.2.7 Basic theory of C_0 -semigroups	11
1.2.8 Lipschitz Perturbations of Linear Evolution Equations	12
1.2.9 Banach Fixed-Point Theorem	13
1.2.10 Basic numerical methods	13
2 Numerical Modeling of Multiphase Biofilm Formation on a Porous Solid Surface with Heat Equation Coupling and Growth Damping Term	16
2.1 Introduction	16
2.2 Proposed model	17
2.2.1 Model Description	17

2.2.2	Osmotic Influx Formulation	19
2.2.3	Boundary Conditions	19
2.2.4	Model Reduction with Growth Damping	21
2.2.5	Reduction to Two-Phase Models	22
2.2.6	Precursor Film Regularization	22
2.3	Numerical Approximation	23
2.3.1	Finite Difference Approximation	24
2.3.2	Stability Analysis and Discrete Energy Estimation	27
2.3.3	Numerical Results	32
2.4	Conclusion	38
3	Theoretical and Numerical Analysis of Thermally Coupled Quorum-Sensing Biofilm Model	39
3.1	Introduction	39
3.2	Proposed model	40
3.3	The well-posedness of the problem	42
3.3.1	Existence of Solutions	44
3.3.2	Regularity of the solution:	51
3.3.3	Uniqueness	51
3.4	Numerical Approximation	52
3.4.1	Existence and Uniqueness	53
3.4.2	Energy Estimate	56
3.4.3	Convergence	59
3.4.4	Numerical Results	61
3.5	Conclusion	65
	General conclusion	66
	References	69

List of Figures

1	Microscopy images of a multiphase bacterial biofilm composed of cells, EPS, and an aqueous phase.	2
2.1	Schematic representation of a bacterial biofilm on a substrate, illustrating the three primary phases. Adapted from [48].	17
2.2	Initial thickness profile $h(r)$	34
2.3	Thickness profile $h(r)$ at $t = 1$	34
2.4	Initial cell volume fraction $\varphi(r)$	35
2.5	Cell volume fraction $\varphi(r)$ at $t = 1$	35
2.6	Initial ECM volume fraction $\psi(r)$	35
2.7	ECM volume fraction $\psi(r)$ at $t = 1$	35
2.8	Initial aqueous-phase fraction $A(r)$	36
2.9	Aqueous-phase fraction $A(r)$ at $t = 1$	36
2.10	Initial nutrient concentration $C(r)$	36
2.11	Nutrient concentration $C(r)$ at $t = 1$	36
2.12	Initial substrate concentration $S(r)$	37
2.13	Substrate concentration $S(r)$ at $t = 1$	37
2.14	Initial temperature profile $T(r)$	37
2.15	Temperature profile $T(r)$ at $t = 1$	37
3.1	Representation of down-regulated and up-regulated cells at time $t = 10$ and $t = 15$	62
3.2	Total amounts of down-regulated and up-regulated cells with respect to time.	63
3.3	Evolution of total AHL molecule and LasR protein.	63
3.4	Temperature distribution at $t = 15$ under antibiotic and QSB treatment.	64

Introduction

Multiphase problems arise naturally in a wide range of physical, biological, and engineering systems in which several interacting phases coexist. Their analysis is particularly challenging due to the presence of strong nonlinear couplings, complex interfacial interactions, and, in many cases, evolving free boundaries. Within this broad class of problems, biological systems provide especially rich and intricate examples, where transport, reaction, and mechanical processes are tightly coupled across multiple spatial and temporal scales [32].

In this context, biofilms constitute a paradigmatic example of a multiphasic biological structure. A biofilm typically consists of microbial cells embedded in a self-produced extracellular polymeric substance (EPS) matrix, together with an aqueous phase occupying pores and channels within the structure. This heterogeneous composition gives rise to pronounced spatial variations in transport properties, growth rates, and mechanical behavior. Owing to their importance in medical, industrial, and environmental applications, the modeling of biofilms as multiphasic media has attracted increasing attention in recent years. In particular, three-phase biofilm models have emerged as a powerful framework for capturing the coupled dynamics of cells, EPS, and fluid phases with greater fidelity than simpler single- or two-phase descriptions.

From a broader fluid-mechanical perspective, multiphase flows refer to systems involving the simultaneous motion or interaction of more than one phase, provided that the constituents are not uniformly mixed at scales larger than the molecular level. Such flows encompass a wide spectrum of configurations, including gas–solid, liquid–solid, bubbly, and particle-laden regimes. A useful unifying classification distinguishes between *disperse* flows, in which particles, droplets, or bubbles are distributed within a continuous carrier phase, and *separated* flows, in which two or more continuous phases coexist and are separated by moving interfaces. Biofilms naturally fall within this multiphase framework, as they combine solid-like biological components with fluid transport through an interconnected pore network.

The relevance of multiphase modeling is underscored by its many applications. In chemical and process engineering, it underlies the description of suspensions, emulsions, filtration, deposition, and porous-media transport. In environmental science, it plays a central role in contaminant migration across coupled water–soil–air systems and in biological treatment

processes. In biomedicine, multiphase approaches are essential for understanding transport limitations, reaction kinetics, and mechanical effects in complex biological media. Within this general landscape, biofilms stand out as a particularly rich and challenging example due to their biological activity and adaptive behavior.

Biofilms are communities of microorganisms—including bacteria, fungi, algae, and protozoa—that adhere to surfaces and are embedded within a polymeric matrix produced by the cells themselves [28]. They are ubiquitous in daily life and can be either beneficial or harmful. On the beneficial side, biofilms contribute to water purification processes and may act as biological barriers limiting soil contamination. On the harmful side, they are implicated in industrial fouling, corrosion, and, most critically, hospital-acquired infections associated with medical devices and implanted materials. Pathogens such as *Pseudomonas aeruginosa* and *Staphylococcus aureus* are particularly notorious for forming persistent biofilms that exhibit enhanced resistance to antibiotics and immune responses.

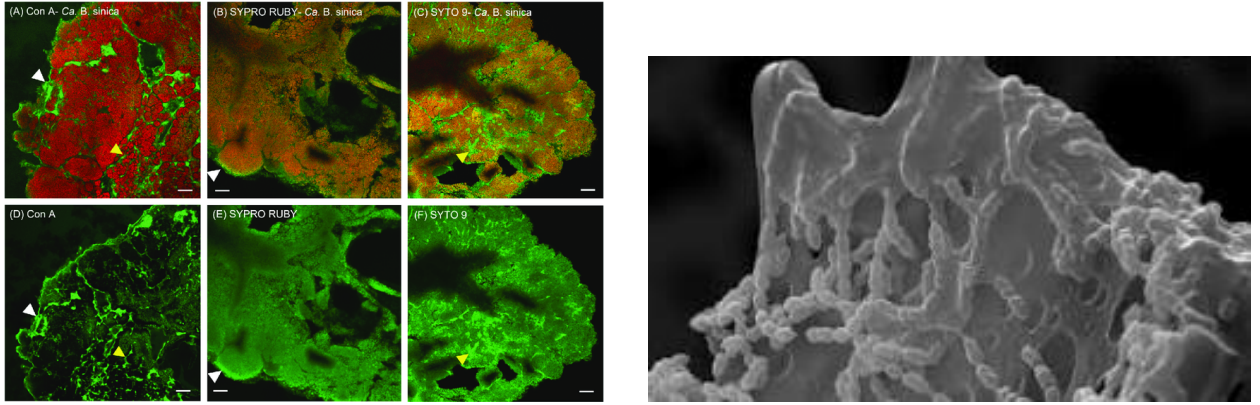


Figure 1: Microscopy images of a multiphase bacterial biofilm composed of cells, EPS, and an aqueous phase.

From the modeling viewpoint, biofilms are naturally described as heterogeneous media composed of interacting components whose local composition evolves in time and space. This motivates volume-fraction formulations governed by coupled balance laws for mass and momentum, together with transport equations for nutrients and signaling molecules. Such models are able to capture essential features of biofilm development, including spatial heterogeneity, nutrient gradients, growth-driven expansion, deformation, and, in some settings, detachment or erosion. Because both mechanical and transport properties depend on the local mixture, these systems exhibit strong feedback between composition, transport, and reaction kinetics.

A substantial body of work has been devoted to the mathematical modeling of multiphase biofilms. Klapper and Dockery [31] introduced a two-phase model focusing on the coupled dynamics of bacterial cells and EPS, with particular emphasis on detachment and structural stability. Eberl *et al.* [35] investigated a one-dimensional formulation highlighting substrate

diffusion and biomass growth. More recently, Trinschek *et al.* [67] developed a thin-film model for passive suspensions aimed at describing osmotic biofilm spreading. Of particular relevance to the present work is the three-phase framework proposed by Uttam Kumar and co-authors [48], which explicitly accounts for bacterial cells, EPS, and an aqueous phase. Their model describes the evolution of volume fractions and transport processes through coupled nonlinear partial differential equations and incorporates key mechanical effects such as Darcy-type flow and biomass compressibility. Three-phase models are especially attractive because they reconcile biological realism with analytical and numerical tractability.

Beyond mechanical and transport processes, bacterial behavior within biofilms is strongly influenced by biochemical communication mechanisms. In medical contexts, bacterial growth and virulence depend not only on nutrient availability but also on the ability of bacteria to communicate through chemical signaling molecules [40]. This process, known as quorum sensing (QS), enables bacteria to activate specific genes once the concentration of signaling molecules exceeds a threshold. In *Pseudomonas aeruginosa*, quorum sensing is mediated by N-acylated homoserine lactones (AHLs), such as 3-oxo-C12-HSL, which bind to the regulatory protein LasR and trigger the expression of virulence genes. At low population densities, bacterial activity remains relatively inconspicuous; however, once a critical density is reached, quorum sensing leads to a rapid and coordinated activation of virulence factors, often overwhelming host defenses. This mechanism contributes significantly to the high antibiotic resistance observed in biofilm-associated infections [57], [58], [69].

Motivated by these observations, Williams [72] proposed targeting quorum sensing pathways as a therapeutic strategy, aiming to disrupt bacterial communication and thereby inhibit the activation of virulence genes. Subsequent studies have developed mathematical models to investigate the combined effects of quorum-sensing inhibition, enzymatic degradation of signaling molecules, and antibiotic treatment. In particular, Blouza [10] proposed a system of partial differential equations in a two-dimensional setting to study quorum sensing in *Pseudomonas aeruginosa* under antibiotic pressure, supported by numerical simulations. This model serves as an important point of reference for the present work.

An additional factor that plays a crucial role in biofilm dynamics is temperature. Growth and decay kinetics, EPS production, transport coefficients, and signaling pathways are all sensitive to thermal conditions. Moreover, metabolic activity within biofilms can act as a local heat source, generating feedback between biological processes and the thermal field. Building on the three-phase framework of [48], this thesis incorporates a thermo-biophysical coupling by introducing a heat balance equation and allowing selected kinetic and transport coefficients to depend explicitly on temperature. This approach provides a systematic way to account for thermal sensitivity while preserving analytical structure and numerical feasibility. The work presented in this thesis develops mathematical and numerical frameworks for the modeling of bacterial biofilms as multiphase systems, with particular emphasis on thermal

effects, quorum-sensing mechanisms, and growth regulation. The thesis is organized as follows:

- **Chapter 1** introduces the biological background of biofilms and the fundamental concepts underlying their multiphase organization. It also presents the mathematical tools required throughout the thesis, including functional analysis, Banach and Hilbert spaces, C_0 -semigroup theory, classical inequalities, and basic numerical methods.
- **Chapter 2** develops a thermally coupled three-phase biofilm model on a porous solid surface. The formulation incorporates a heat balance equation together with a growth-damping term in the reduced thickness equation, in order to account for thermal feedback and to regulate biofilm expansion. A thin-film reduction is derived, followed by a finite-difference discretization, a stability analysis, and numerical simulations illustrating the influence of temperature on growth, swelling, transport processes, and biofilm-thickness regulation.
- **Chapter 3** develops a coupled reaction–diffusion framework for biofilm dynamics under quorum sensing, antibiotic response, and temperature variations. Existence and uniqueness results are established, and a finite-element spatial discretization combined with backward Euler time integration is proposed and analyzed, together with numerical experiments highlighting the impact of thermal effects on quorum-sensing activity and biofilm evolution.
- **Chapter 4** concludes the thesis by summarizing the main contributions and discussing perspectives for future work, including model extensions, parameter calibration, experimental validation, and numerical improvements.

Chapter 1

Preliminary

1.1 General Concepts on Biofilms

1.1.1 What is the Biofilms?

Biofilms are defined as structured communities of microorganisms that adhere to solid or semi-solid surfaces and are embedded within a self-produced extracellular polymeric matrix. Unlike planktonic (free-living) cells, microorganisms within a biofilm exhibit collective behavior and spatial organization, resulting in a heterogeneous and dynamic structure.

From a modeling perspective, biofilms can be viewed as evolving systems governed by coupled biological, chemical, and physical processes, including microbial growth, substrate consumption, and mass transport.

1.1.2 Biofilms as Multiphasic Systems

Biofilms are widely recognized as *multiphasic systems*, as they consist of several interacting components occupying the same spatial domain, each with distinct physical and functional properties. The principal phases include:

- **Biomass phase:** This phase corresponds to the microbial cells forming the biofilm. It is responsible for biological activity such as growth, division, and substrate consumption. In mathematical models, this phase is often represented by biomass density variables governed by nonlinear growth laws.
- **Extracellular polymeric substance (EPS) phase:** The EPS phase constitutes the polymeric matrix secreted by the microorganisms. It provides structural cohesion to the biofilm and significantly influences its mechanical properties and transport processes. In multiphase models, EPS is frequently treated as a distinct phase affecting diffusion and permeability.

- **Liquid phase:** This phase consists of water and dissolved nutrients, ensuring the transport of substrates and chemical species throughout the biofilm. It is typically described using diffusion equations.

In more advanced formulations, a gaseous phase may be included to account for oxygen transport. The multiphase nature of biofilms naturally motivates the use of continuum and multiphase mathematical models.

1.1.3 Formation of Biofilms on Surfaces

Biofilm formation on surfaces occurs through successive stages that can be summarized as follows:

1. Initial reversible attachment of planktonic cells to a surface,
2. Irreversible attachment through the production of adhesive substances,
3. Formation of microcolonies via cellular division,
4. Maturation into a three-dimensional structured biofilm,
5. Dispersion, during which cells detach and colonize new environments.

These stages correspond to spatial and temporal variations in biomass density and structural organization, which can be captured using time-dependent mathematical models.

1.1.4 Extracellular Polymeric Matrix (EPS)

The extracellular polymeric matrix is a fundamental component of biofilms, composed mainly of polysaccharides, proteins, and extracellular DNA. The EPS matrix plays a crucial role in:

- maintaining biofilm cohesion and adhesion to surfaces,
- regulating mass transport within the biofilm,
- protecting microbial cells from external stresses.

In mathematical modeling, EPS is often incorporated as an additional phase influencing transport coefficients and diffusion rates.

1.1.5 Quorum Sensing

Quorum sensing is a cell-to-cell communication mechanism based on the production and detection of signaling molecules. It enables microorganisms to coordinate collective behavior depending on cell density. This mechanism regulates several biofilm-related processes, including EPS production, biofilm maturation, and metabolic activity.

From a mathematical standpoint, quorum sensing can be modeled by introducing reaction–diffusion equations for signaling molecules coupled with biomass growth dynamics.

1.1.6 Antibiotic Resistance of Biofilms

Microorganisms embedded within biofilms exhibit significantly enhanced resistance to antimicrobial agents compared to planktonic cells. This resistance arises from multiple factors, including:

- limited penetration of antibiotics through the EPS matrix,
- reduced metabolic activity of certain cell populations,
- spatial heterogeneity within the biofilm structure.

Such resistance mechanisms highlight the importance of biofilms as complex systems and justify the incorporation of transport limitations and spatial effects in mathematical models.

1.2 Preliminary Mathematical Concepts

1.2.1 Functional Spaces

Let $\mathbf{R}^n$ denote the n -dimensional Euclidean space, and let $\Omega \subset \mathbf{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$. For $k \in \mathbb{N}$, we write $C^k(\Omega)$ for the class of functions on Ω that are k times continuously differentiable, whereas $C^\infty(\Omega)$ stands for smooth functions on Ω . We also set $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$ for the space of smooth functions with compact support in Ω .

Definition 1.1 (Semi-norms and norms). Let X be a vector space over K with $K \in \{\mathbb{R}, \mathbb{C}\}$. A function $p : X \rightarrow [0, \infty)$ is called a *semi-norm* if:

- (a) $p(x) \geq 0$ for all $x \in X$,
- (b) $p(\lambda x) = |\lambda| p(x)$ for every $x \in X$ and $\lambda \in K$,
- (c) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

If, in addition,

$$p(x) = 0 \implies x = 0,$$

then p is a *norm*. The pair (X, p) is then called a normed space.

Definition 1.2 (Convergence and Cauchy property). Let $(X, \|\cdot\|_X)$ be a normed space and let $(u_n)_{n \in \mathbb{N}}$ be a sequence in X .

(a) We say that $u_n \rightarrow u$ in X if

$$\lim_{n \rightarrow \infty} \|u_n - u\|_X = 0.$$

(b) The sequence (u_n) is said to be Cauchy if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|u_m - u_n\|_X < \varepsilon, \quad \forall m, n \geq N.$$

A normed space in which every Cauchy sequence converges is called *complete*.

Definition 1.3 (Banach space and dual). A complete normed space is called a *Banach space*. Its dual space X' is the set of all continuous linear functionals $f : X \rightarrow K$.

Proposition 1.2.1. X' equipped with the operator norm

$$\|f\|_{X'} = \sup_{\|v\|_X \leq 1} |f(v)|,$$

the dual space X' is a Banach space.

Remark 1.1. The bidual is $X'' := (X')'$. For $u \in X$, define $J(u) \in X''$ by

$$\langle J(u), f \rangle := f(u), \quad \forall f \in X'.$$

Then $\|J(u)\|_{X''} = \|u\|_X$, and the mapping $J : X \rightarrow X''$ is a linear isometry.

Definition 1.4 (Reflexivity). The space X is said to be *reflexive* if the canonical embedding $J : X \rightarrow X''$ is surjective.

1.2.2 The weak and weak-star topologies

Let X be a Banach space and $f \in X'$. Denote by

$$\varphi_f : X \rightarrow \mathbb{R}$$

defined by

$$x \longmapsto \varphi_f(x) = f(x).$$

When f cover X' , we obtain a family $(\varphi_f)_{f \in X'}$ of mappings from X into $\mathbb{R}$.

Definition 1.5. The weak topology on X , denoted by $\sigma(X, X')$, is the weakest topology on X for which every mapping $(\varphi_f)_{f \in X'}$ is continuous.

We now define a topology on X' , called the weak-star topology, denoted by $\sigma(X', X)$.

For all $x \in X$, denote by

$$\varphi_x : X' \rightarrow \mathbb{R}$$

defined by

$$f \mapsto \varphi_x(f) = \langle f, x \rangle_{X', X}.$$

Definition 1.6. The weak-star topology on X' is the weakest topology on X' for which every mapping $(\varphi_x)_{x \in X}$ is continuous.

Remark 1.2. Since $X \hookrightarrow X''$, it is clear that the weak-star topology $\sigma(X', X)$ is weaker than the topology $\sigma(X', X'')$, and the latter is weaker than the strong topology.

Definition 1.7. A sequence $(x_n)_{n \in \mathbb{N}}$ in X is said to converge weakly to $x \in X$ if and only if

$$\lim_{n \rightarrow \infty} f(x_n) = f(x)$$

for every $f \in X'$.

This is denoted by

$$x_n \rightharpoonup x.$$

Remark 1.3. 1. If the weak limit exists, then it is unique.

2. If $x_n \rightarrow x$ strongly in X , then $x_n \rightharpoonup x$ weakly in X .

3. If $\dim X < \infty$, then weak convergence implies strong convergence.

1.2.3 Hilbert spaces

Definition 1.8. A Hilbert space $\mathcal{H}$ is a complete vector space endowed with an inner product $(\cdot, \cdot)_{\mathcal{H}}$ such that the induced norm satisfies

$$\|u\|_{\mathcal{H}} = \sqrt{(u, u)_{\mathcal{H}}}.$$

Theorem 1.2.2. [1] Every bounded sequence in a Hilbert space admits a weakly convergent subsequence.

Theorem 1.2.3. [1] Every weakly convergent sequence in a Hilbert space is bounded.

Theorem 1.2.4. [1] If $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in a Hilbert space, then

$$\lim_{n \rightarrow \infty} (u_n, v_n)_{\mathcal{H}} = (u, v)_{\mathcal{H}}.$$

1.2.4 The $L^p(\Omega)$ spaces

Definition 1.9. Let $1 \leq p \leq \infty$. The Lebesgue space $L^p(\Omega)$ consists of (equivalence classes of) measurable functions $g : \Omega \rightarrow \mathbb{R}$ for which

$$\int_{\Omega} |g(x)|^p dx < \infty$$

(with the usual modification when $p = \infty$).

Theorem 1.2.5. For $1 \leq p \leq \infty$, the space $L^p(\Omega)$ endowed with its standard norm is a Banach space.

Remark 1.4. When $p = 2$, $L^2(\Omega)$ is a Hilbert space with inner product

$$(g, h)_{L^2(\Omega)} = \int_{\Omega} g(x) h(x) dx.$$

1.2.5 Sobolev spaces

Definition 1.10. Let $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. The Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \left\{ g \in L^p(\Omega) : D^{\alpha} g \in L^p(\Omega) \text{ for all multi-indices } \alpha \text{ with } |\alpha| \leq m \right\},$$

where D^{α} denotes the distributional derivative of order α .

Remark 1.5. For $p = 2$ we write $H^m(\Omega) := W^{m,2}(\Omega)$, which is a Hilbert space.

1.2.6 Inequalities

Young inequalities

Theorem 1.2.6. [1] Let $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $a, b \geq 0$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Theorem 1.2.7. [1] Let $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. For every $\varepsilon > 0$ and all $a, b \geq 0$,

$$ab \leq \frac{\varepsilon}{p} a^p + \frac{1}{q \varepsilon^{q/p}} b^q.$$

Cauchy–Schwarz Inequality

Theorem 1.2.8. Let V be an inner product space over the field of real (or complex) numbers, with the inner product denoted by $\langle \cdot, \cdot \rangle_V$ and the induced norm denoted by $\|u\|_V = \sqrt{\langle u, u \rangle_V}$.

For any vectors $u, v \in V$, the following inequality holds:

$$|\langle u, v \rangle_V| \leq \|u\|_V \|v\|_V \quad (1.1)$$

Furthermore, the equality holds if and only if u and v are linearly dependent.

Discrete Gronwall inequality

Theorem 1.2.9. Let $(E^n)_{n \geq 0}$ be a sequence of non-negative real numbers. Suppose that there exist non-negative constants C_1, C_3 , and $\Delta t > 0$ such that for all integers $n \geq 0$:

$$E^{n+1} \leq (1 + C_1 \Delta t) E^n + C_3 \Delta t \quad (1.2)$$

Then, the sequence is bounded by:

$$E^n \leq E^0 (1 + C_1 \Delta t)^n + C_3 \Delta t \sum_{k=0}^{n-1} (1 + C_1 \Delta t)^k \quad (1.3)$$

1.2.7 Basic theory of C_0 -semigroups

Let X be a Banach space and let $\mathcal{L}(X)$ denote the space of bounded linear operators on X , equipped with the operator norm

$$\|B\|_{\mathcal{L}(X)} := \sup_{\|x\|_X=1} \|Bx\|_X.$$

Definition 1.11 (Semigroup). [53] A family $(T(t))_{t \geq 0} \subset \mathcal{L}(X)$ is called a (one-parameter) semigroup if

$$T(0) = I, \quad T(t+s) = T(t)T(s) \quad \text{for all } t, s \geq 0.$$

Definition 1.12 (C_0 -semigroup). A semigroup $(T(t))_{t \geq 0}$ on X is called strongly continuous (or a C_0 -semigroup) if for every $x \in X$,

$$\lim_{t \rightarrow 0^+} \|T(t)x - x\|_X = 0.$$

Definition 1.13 (Infinitesimal generator). [53] Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup on X . Its infinitesimal generator is the (generally unbounded) operator $A : D(A) \subset X \rightarrow X$ defined by

$$D(A) = \left\{ x \in X : \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} \text{ exists in } X \right\}, \quad Ax = \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t}, \quad x \in D(A).$$

A basic property of C_0 -semigroups is the existence of an exponential growth estimate:

there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad \forall t \geq 0.$$

This bound is fundamental for estimating integral expressions involving $T(t)$.

Remark 1.6. The notions introduced above provide the functional-analytic framework for evolution equations of the form $u'(t) + Au(t) = \cdots$, where $-A$ generates a C_0 -semigroup. In the next section, these tools will be used to derive the integral (variation-of-constants) formulation and to state existence and uniqueness results under Lipschitz assumptions.

1.2.8 Lipschitz Perturbations of Linear Evolution Equations

In this subsection we consider a semilinear Cauchy problem on a Banach space X :

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t, u(t)), & t > t_0, \\ u(t_0) = u_0. \end{cases} \quad (1.4)$$

Here $-A$ is the infinitesimal generator of a strongly continuous (C_0) semigroup $\{T(t)\}_{t \geq 0}$ on X , and the mapping $f : [t_0, T] \times X \rightarrow X$ is continuous in t and satisfies a Lipschitz condition with respect to u .

Many statements below (and in subsequent parts where A is taken independent of t) remain valid for a time-dependent family $\{A(t)\}_{t \in [0, T]}$, provided one can associate an evolution system $U(t, s)$, $0 \leq s \leq t \leq T$, to that family. We do not pursue these extensions here; consequently, the later results in this part are understood independently of earlier chapters.

The initial value problem (1.4) need not possess a solution in a classical sense. Nevertheless, whenever a classical (or strong) solution exists, it also satisfies the variation-of-constants formula

$$u(t) = T(t - t_0)u_0 + \int_{t_0}^t T(t - s) f(s, u(s)) ds. \quad (1.5)$$

Definition 1.14 (Mild solution). [53] A continuous function u that satisfies the integral equation (1.5) is called a *mild solution* of the initial value problem (1.4).

We now state a standard existence–uniqueness statement for mild solutions under a uniform Lipschitz assumption on f .

Theorem 1.2.10. [53] Let $f : [t_0, T] \times X \rightarrow X$ be continuous in t on $[t_0, T]$ and uniformly Lipschitz continuous (with constant L) on X . If $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \geq 0$, on X then for every $u_0 \in X$ the initial value problem (1.4) has a unique mild solution $u \in C([t_0, T] : X)$. Moreover, the mapping $u_0 \rightarrow u$ is Lipschitz continuous from X into $C([t_0, T] : X)$.

1.2.9 Banach Fixed-Point Theorem

Definition 1.15 (Self-Mapping). Let S be a non-empty closed subset of a Banach space $(X, \|\cdot\|_X)$. An operator (or mapping) G is said to map S into itself if:

$$G(S) \subseteq S \quad (1.6)$$

Equivalently, this means that for every element $u \in S$, the image $G(u)$ also belongs to S .

Definition 1.16 (Contraction Mapping). Let $(X, \|\cdot\|_X)$ be a normed vector space, and let $S \subseteq X$. A mapping $G : S \rightarrow S$ is called a *contraction* (or a strictly contractive mapping) if there exists a constant $k \in [0, 1)$ such that for all $u, v \in S$, the following inequality holds:

$$\|G(u) - G(v)\|_X \leq k\|u - v\|_X \quad (1.7)$$

The constant k is referred to as the contraction constant.

Theorem 1.2.11 (Banach Fixed-Point Theorem). Let S be a non-empty closed subset of a Banach space $(X, \|\cdot\|_X)$, and let $G : S \rightarrow S$ be a contraction mapping. Then, G admits a unique fixed point in S . That is, there exists exactly one element $x^* \in S$ such that:

$$G(x^*) = x^* \quad (1.8)$$

Furthermore, for any arbitrary initial element $x_0 \in S$, the sequence defined by the iteration $x_{n+1} = G(x_n)$ converges to the unique fixed point x^* .

1.2.10 Basic numerical methods

This section briefly introduces the main numerical methods used for the approximation of partial differential equations. The presentation is limited to the underlying ideas and discrete formulations, without addressing convergence, stability, or error analysis.

Finite Difference Method

The finite difference method (FDM) is based on the approximation of derivatives by finite differences on a discrete grid in space and time [2].

Spatial discretization. Let $\Omega = (a, b) \subset \mathbb{R}$ be a one-dimensional spatial domain. We introduce a uniform grid

$$x_i = a + i \Delta x, \quad i = 0, 1, \dots, N,$$

where $\Delta x = (b - a)/N$.

For a sufficiently smooth function $u(x, t)$, the second-order central finite difference approximation of the second spatial derivative is given by

$$\frac{\partial^2 u}{\partial x^2}(x_i, t) \approx \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{(\Delta x)^2},$$

where $u_i(t) \approx u(x_i, t)$.

Temporal discretization. Let $t^n = t_0 + n \Delta t$ denote a uniform partition of the time interval, with time step $\Delta t > 0$. The explicit finite difference approximation of the time derivative is defined by

$$\frac{\partial u}{\partial t}(x, t^n) \approx \frac{u^{n+1}(x) - u^n(x)}{\Delta t},$$

where $u^n(x) \approx u(x, t^n)$.

Fully discrete scheme. Combining the explicit discretization in time with the second-order central approximation in space leads to a fully discrete scheme. Letting $u_i^n \approx u(x_i, t^n)$, the unknown solution at time t^{n+1} is computed directly from known values at time t^n . For a generic evolution equation $\frac{\partial u}{\partial t} = \mathcal{L}u$ (where $\mathcal{L}$ is a spatial differential operator), the explicit scheme takes the general form:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \mathcal{L}_h u_i^n,$$

where $\mathcal{L}_h$ represents the corresponding discrete spatial operator. This approach is straightforward to implement and widely used in time-dependent problems.

Finite Element Method

The finite element method (FEM) is a variational technique based on the weak formulation of differential equations. It is particularly well suited for problems defined on complex geometries [2].

Weak formulation. Consider a model elliptic or parabolic problem posed on a domain $\Omega \subset \mathbb{R}^d$. The finite element approach starts by rewriting the problem in weak (or variational) form: find u in a suitable function space V such that

$$a(u, v) = \ell(v) \quad \forall v \in V,$$

where $a(\cdot, \cdot)$ is a bilinear form associated with the differential operator, and $\ell(\cdot)$ is a linear functional representing the source terms.

Spatial discretization. The domain Ω is partitioned into a finite number of simple elements (such as intervals in 1D, triangles in 2D, or tetrahedra in 3D). A finite-dimensional subspace $V_h \subset V$ is constructed using locally supported basis functions $\{\varphi_j\}_{j=1}^{N_h}$ associated with the mesh nodes.

The approximate discrete solution is sought in the form

$$u_h(x) = \sum_{j=1}^{N_h} u_j \varphi_j(x),$$

where the coefficients u_j are the unknown nodal values.

Discrete problem. Substituting the finite-dimensional approximation u_h into the weak formulation and choosing the test functions $v = \varphi_i$ (Galerkin approach) leads to a linear system of algebraic equations:

$$\mathbf{A}\mathbf{u} = \mathbf{F},$$

where $\mathbf{A}_{ij} = a(\varphi_j, \varphi_i)$ is the stiffness matrix, $\mathbf{F}_i = \ell(\varphi_i)$ is the load vector, and $\mathbf{u} = (u_1, \dots, u_{N_h})^T$ is the vector of unknowns.

Time-dependent problems. For evolution equations, the nodal values become functions of time, i.e., $u_h(x, t) = \sum_{j=1}^{N_h} u_j(t) \varphi_j(x)$. The spatial discretization then yields a semi-discrete system of ordinary differential equations:

$$\mathbf{M} \frac{d\mathbf{u}(t)}{dt} + \mathbf{A}\mathbf{u}(t) = \mathbf{F}(t),$$

where $\mathbf{M}_{ij} = \int_{\Omega} \varphi_j \varphi_i dx$ is the mass matrix. This system is subsequently solved using standard time-stepping schemes.

Remark 1.7. The finite difference and finite element methods differ fundamentally in their construction: the former relies on local derivative approximations on structured grids, while the latter is based on variational principles and piecewise-defined basis functions. Both approaches provide flexible frameworks for the numerical approximation of evolution equations.

Chapter 2

Numerical Modeling of Multiphase Biofilm Formation on a Porous Solid Surface with Heat Equation Coupling and Growth Damping Term

2.1 Introduction

In this chapter, we develop a multiphase mathematical model for biofilm formation on a porous solid substrate. The biofilm is represented as a three-phase immiscible mixture of cellular, extracellular matrix (ECM), and aqueous phases. The model builds on the framework of Uttam Kumar et al. [48], which accounts for nutrient transport, biomass compressibility, and Darcy-type flow.

Our main contribution is the coupling of biofilm evolution with a dynamic heat equation, where metabolic activity acts as an internal heat source and temperature influences cell growth, ECM production, and cell death. We also introduce a growth damping term in the biofilm thickness equation to represent the mechanical resistance caused by cell crowding and ECM accumulation, thereby preventing unrealistic thickening.

The governing equations are reduced using a thin-film approximation in an axisymmetric framework and discretized by a finite-difference scheme. Stability is established through discrete energy estimates, while numerical simulations illustrate the effects of temperature, osmotic influx, and growth damping on biofilm evolution.

This chapter is organized as follows. Section 1 presents the multiphase model, Section 2 derives the reduced formulation with growth damping, Section 3 develops the numerical scheme and stability analysis, and Section 4 presents and discusses the numerical results.

2.2 Proposed model

2.2.1 Model Description

We investigate bacterial biofilm growth on a porous agar substrate and model the biofilm as a three-phase, immiscible, incompressible Newtonian mixture. The mixture comprises a cellular phase, an extracellular matrix (ECM) phase containing polymers and other compounds secreted by microbes, and a free aqueous phase that carries dissolved nutrients.

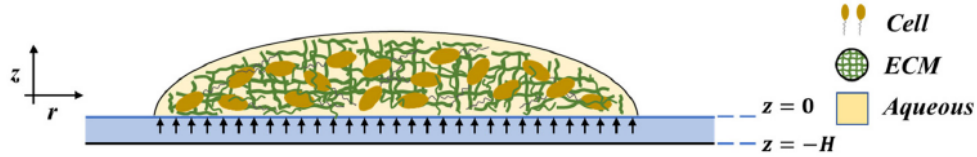


Figure 2.1: Schematic representation of a bacterial biofilm on a substrate, illustrating the three primary phases. Adapted from [48].

The primary unknowns (φ, ψ, A) represent the *volume fractions* of the cell, ECM, and aqueous phases, respectively. Water uptake from the underlying substrate is incorporated through an osmotic influx mechanism, driven by differences in osmotic pressure between the biofilm and the substrate. Nutrients are supplied from the substrate to the biofilm through a combination of osmotic transport and diffusion. Consequently, the phase volume fractions (φ, ψ, A) and the nutrient concentrations in the biofilm (C) and in the substrate (S) vary in both space and time (Figure. 2.1). The governing equations are given by:

Mass balance of the individual phases in the biofilm:

$$\varphi + \psi + A = 1, \quad (2.1)$$

$$\frac{\partial(\rho_c \varphi)}{\partial t} + \nabla \cdot (\rho_c \varphi u_c) = k_c(T) \rho_c \varphi \frac{C}{k_0 + C} - k_d(T) \rho_c \varphi, \quad (2.2)$$

$$\frac{\partial(\rho_e \psi)}{\partial t} + \nabla \cdot (\rho_e \psi u_e) = k_e(T) \rho_c \varphi \frac{C}{k_0 + C} + k_d(T) \rho_c \varphi, \quad (2.3)$$

$$\frac{\partial(\rho_a A)}{\partial t} + \nabla \cdot (\rho_a A u_a) = 0. \quad (2.4)$$

Nutrient balance in the biofilm:

$$\frac{\partial C}{\partial t} + \nabla \cdot (C u_b) = D_b \nabla^2 C - k_b \rho_c \varphi \frac{C}{k_0 + C}. \quad (2.5)$$

Nutrient balance in the substrate:

$$\frac{\partial S}{\partial t} + \nabla \cdot (S u'_b) = D_s \nabla^2 S. \quad (2.6)$$

Heat equation:

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k_T \nabla T) + Q \varphi \frac{C}{k_0 + C}. \quad (2.7)$$

Bacterial growth and ECM production are assumed to follow Monod kinetics, with k_0 being the half-saturation constant and k_b the maximum specific rate of nutrient consumption. Cell death is considered proportional to the cell concentration, and the resulting dead cells are incorporated into the ECM phase. The constants ρ_c , ρ_e , and ρ_a denote the intrinsic densities of the cell, ECM, and aqueous phases, respectively. The rate functions $k_c(T)$, $k_e(T)$, and $k_d(T)$ represent the temperature-dependent rates of cell growth, ECM production, and cell death. Nutrient diffusion in the biofilm and in the substrate is characterized by D_b and D_s , respectively. For the thermal dynamics described by the heat equation, T denotes the temperature, ρ and c_p are the effective density and specific heat capacity of the mixture, k_T is the effective thermal conductivity, and Q is the metabolic heat generation rate.

Let u_α denote the velocity of phase $\alpha \in \{c, e, a\}$. The mean biofilm advection velocity is defined as

$$u_b := \varphi u_c + \psi u_e + A u_a,$$

so that nutrients within the biofilm are transported with u_b . The aqueous-phase velocity in the agar substrate is denoted by u'_b .

Momentum balance for each phase:

$$\frac{\partial(\varphi_\alpha \rho_\alpha u_\alpha)}{\partial t} + \nabla \cdot (\varphi_\alpha \rho_\alpha u_\alpha \otimes u_\alpha) = \nabla \cdot (\tilde{\varphi} \sigma_\alpha) + F_\alpha, \quad (2.8)$$

with $\alpha = c, e, a$ and $\tilde{\varphi} = \varphi, \psi, A$, respectively. The corresponding stress tensor is

$$\sigma_\alpha = -p_\alpha I + \mu_\alpha \left(\nabla u_\alpha + \nabla u_\alpha^\top - \frac{2}{3} (\nabla \cdot u_\alpha) I \right),$$

and the interphase force densities are

$$\begin{aligned} F_c &= p_c \nabla \varphi - k_{ce}(u_c - u_e) - k_{ac}(u_c - u_a), \\ F_e &= p_e \nabla \psi - k_{ea}(u_e - u_a) - k_{ce}(u_e - u_c), \\ F_a &= p_a \nabla A - k_{ac}(u_a - u_c) - k_{ea}(u_a - u_e). \end{aligned}$$

Here, p_c , p_e , and p_a denote the hydrodynamic pressures of the cell, ECM, and aqueous phases, respectively. For the ECM phase, p_e accounts for both hydrodynamic and osmotic

contributions. The gradient terms represent interfacial force densities, while k_{ce} , k_{ac} , and k_{ea} are the viscous drag coefficients for the cell–ECM, aqueous–cell, and ECM–aqueous phase interactions, respectively.

2.2.2 Osmotic Influx Formulation

Water uptake by the ECM hydrogel drives the expansion of the biofilm. Newly synthesized ECM at the colony periphery is initially out of equilibrium and relaxes toward a swollen state as polymer chains reorganize and fluid is absorbed [48]. The swelling pressure per unit ECM is defined as

$$\Psi(\psi) = \frac{\delta(\psi)}{\psi}, \quad \delta(\psi) = \delta_0 \psi^3,$$

where $\delta_0 = K\bar{T}/b^3$ according to a simplified Flory–Huggins model, with K the Boltzmann constant, $\bar{T}$ the absolute temperature, and b the monomer size.

The osmotic water influx across the biofilm–agar interface is given by

$$v(r) = Q_0 (\psi^3 - \psi_q^3),$$

where Q_0 sets the influx scale, and

$$\psi_q = \left(\frac{\delta_{eq}}{\delta_0} \right)^{1/3}$$

is the equilibrium ECM fraction corresponding to the substrate osmotic pressure δ_{eq} [71].

2.2.3 Boundary Conditions

To close the system (2.1) – (2.6) and (2.7), the following boundary conditions are imposed.

Free-surface traction balance at $z = h$. The tangential traction at the free surface is zero, while the normal traction is balanced by capillarity:

$$\hat{t} \cdot (\tilde{\varphi} \sigma_\alpha \hat{n}) = 0, \quad \hat{n} \cdot (\tilde{\varphi} \sigma_\alpha \hat{n}) = -\gamma_\alpha k, \quad \text{on } z = h, \quad (2.9)$$

where $\hat{n}$ is the outward unit normal to the free surface $z - h(r, t) = 0$, and k denotes the mean curvature. Under axisymmetry:

$$\hat{n} = \frac{(-\partial h / \partial r, 0, 1)}{\sqrt{1 + (\partial h / \partial r)^2}}, \quad \text{on } z = h, \quad k = \frac{-\frac{1}{r} \partial_r (r \partial_r h) - \frac{1}{r} (\partial_r h)^3}{[1 + (\partial_r h)^2]^{3/2}}, \quad \text{on } z = h. \quad (2.10)$$

Kinematic condition at $z = h$. The free surface is transported with the local phase velocity:

$$\left(\frac{\partial}{\partial t} + u_\alpha \cdot \nabla \right) (z - h(r, t)) = 0, \quad \text{on } z = h. \quad (2.11)$$

Nutrient flux at the biofilm-air interface. No nutrient crosses the free surface:

$$(Cu_b - D_b \nabla C) \cdot \hat{n} = 0, \quad \text{on } z = h. \quad (2.12)$$

Biofilm-substrate mechanical coupling at $z = 0$. No slip is enforced along the substrate:

$$u_\alpha \cdot \hat{t} = 0, \quad \text{on } z = 0, \quad (2.13)$$

with normal velocities

$$u_a \cdot \hat{n} = v(r), \quad u_c \cdot \hat{n} = 0, \quad u_e \cdot \hat{n} = 0, \quad \text{on } z = 0, \quad (2.14)$$

where $v(r) = Q_0(\psi^3 - \psi_q^3)$.

Nutrient exchange at $z = 0$. Convective-diffusive fluxes across the biofilm-substrate interface are proportional to the concentration jump:

$$(Su'_b - D_s \nabla S) \cdot \hat{n} = k_m(S - C), \quad (2.15)$$

$$(Cu_b - D_b \nabla C) \cdot \hat{n} = k_m(C - S), \quad \text{on } z = 0, \quad (2.16)$$

where u_b and u'_b denote the average velocities in the biofilm and substrate, respectively.

Substrate base at $z = -H$. The bottom of the substrate is rigid and impermeable:

$$(Su'_b - D_s \nabla S) \cdot \hat{n} = 0, \quad u'_b \cdot \hat{n} = 0, \quad \text{on } z = -H. \quad (2.17)$$

Thermal boundary conditions. At the biofilm-air interface $z = h$, we assume an adiabatic boundary:

$$-k_T \nabla T \cdot \hat{n} = 0. \quad (2.18)$$

At the biofilm-substrate interface $z = 0$: temperature continuity and flux balance,

$$T_{\text{biofilm}} = T_{\text{substrate}}, \quad k_T \nabla T \cdot \hat{n} = k_T^{\text{sub}} \nabla T^{\text{sub}} \cdot \hat{n}. \quad (2.19)$$

At the bottom of the substrate $z = -H$: insulated base,

$$\nabla T \cdot \hat{n} = 0. \quad (2.20)$$

2.2.4 Model Reduction with Growth Damping

We focus on an axisymmetric biofilm and assume the phases are incompressible with identical densities. We employ a thin-film approximation with aspect ratio $\epsilon = h^*/R^* \ll 1$, where h^* and R^* denote the characteristic height and radius. The dependent variables are expanded in ϵ , and higher-order contributions are neglected. To eliminate the z -dependence, the volume fractions (φ, ψ, A) are assumed uniform in z and the governing equations are integrated over the film thickness. We further assume strong drag between cells and ECM, so that they are transported by a common velocity, with equal viscosities that differ from the aqueous-phase viscosity. Full details are provided in the Supplementary Material.

To improve the robustness of the reduced formulation and to prevent unrealistically fast thickening, we introduce a *growth damping* term acting directly on the biofilm thickness. Specifically, a linear sink term $-\lambda_h h$ is added to the thickness equation. Physically, this term represents an effective mechanical resistance that increases as the biofilm thickens, capturing the macroscopic effect of cell crowding and ECM accumulation in limiting further expansion. This modification regularizes the net growth of the layer without altering the internal phase volume fractions. The resulting reduced axisymmetric model is

$$\left\{ \begin{array}{l} \frac{\partial h}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \omega_b) = \varphi \frac{C}{k+C} h + Av(r) - \lambda_h h, \\ \frac{\partial(h\varphi)}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \varphi \omega_m) = \eta_c(T) \varphi \frac{C}{k+C} h - \eta_d(T) \varphi h, \\ \frac{\partial(h\psi)}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \psi \omega_m) = \eta_e(T) \varphi \frac{C}{k+C} h + \eta_d(T) \varphi h, \\ \frac{\partial(hA)}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r A \omega_a) = Av(r), \\ \text{Pe} h \frac{\partial C}{\partial t} = -\text{Pe} \frac{1}{r} \frac{\partial}{\partial r} (r C \omega_b) + \frac{1}{r} \frac{\partial}{\partial r} \left(r h \frac{\partial C}{\partial r} \right) - k_{mb}(C - S) - \eta_b \varphi h \frac{C}{k+C}, \\ \frac{\partial S}{\partial t} = D \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial S}{\partial r} \right) - D k_{ms}(S - C), \\ \rho c_p h \frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(r h k_T \frac{\partial T}{\partial r} \right) + Q \varphi h \frac{C}{k+C} - k_{mT}(T - T_{\text{sub}}). \end{array} \right. \quad (2.21)$$

The depth-averaged fluxes are defined as

$$\begin{aligned}\omega_m &= \frac{h^3}{3\varphi_m} \left(\gamma^* h_{3r} - \delta^* \frac{\partial \psi^3}{\partial r} \right), \\ \omega_a &= \frac{h^3}{3\varphi_m} \left(\gamma^* h_{3r} - \delta^* \frac{\partial \psi^3}{\partial r} \right) + \frac{A\gamma^*}{\beta^*\varphi_m} h h_{3r}, \\ \omega_b &= \varphi_m \omega_m + A \omega_a, \\ h_{3r} &= \frac{\partial}{\partial r} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial h}{\partial r} \right) \right\}, \quad v(r) = Q_0(\psi^3 - \psi_q^3).\end{aligned}$$

Here $\eta_c(T)$, $\eta_e(T)$, $\eta_d(T)$, and η_b are the dimensionless rate functions for cell production, ECM production, cell death, and nutrient consumption, respectively. The Peclet number Pe quantifies the relative importance of advection to diffusion in the biofilm, and D is the dimensionless nutrient diffusivity in the substrate. The parameters γ^* and β^* are the dimensionless surface-tension and interphase-friction coefficients, while φ_m denotes the maximal packing fraction. In the depth-averaged thermal equation, k_{mT} represents the effective thermal transfer coefficient between the biofilm and the substrate T_{sub} .

2.2.5 Reduction to Two-Phase Models

The three-phase framework reduces to standard two-phase biofilm models under specific limits. Grouping cells and ECM into a single biomass phase and neglecting interfacial mass-transfer resistance yields the two-phase model of Srinivasan *et al.* [59]. Further neglect of the aqueous phase and osmotic influx $v(r)$ collapses the system to the yeast-biofilm model of Tam *et al.* [65]. Details and parameter mappings are given in the Supplementary Material.

2.2.6 Precursor Film Regularization

Imposing no slip at the biofilm–substrate interface leads to a moving contact-line singularity. We regularize this behavior by introducing a thin *precursor film* ($h^* \ll 1$) of passive aqueous fluid, for which $\varphi = \psi = 0$ and nutrient consumption is absent.

To remove the singularity at $z = 0$ and to construct smooth initial conditions compatible

with the precursor layer, we prescribe

$$\begin{aligned}
h(r, 0) &= h^* + \frac{h_0 - h^*}{2} [1 - \tanh \{5(r - r_{in})\}], \\
\varphi(r, 0) &= \frac{\varphi_0}{2} [1 - \tanh \{5(r - r_{in})\}], \\
\psi(r, 0) &= \frac{\psi_0}{2} [1 - \tanh \{5(r - r_{in})\}], \\
A(r, 0) &= 1 - \varphi(r, 0) - \psi(r, 0), \\
T(r, 0) &= T_0 + \delta T \exp(-r^2), \\
C(r, 0) &= 0, \quad S(r, 0) = 1.
\end{aligned} \tag{2.22}$$

To suppress growth and osmotic influx inside the precursor film, we multiply the corresponding source terms by the Heaviside factor

$$\Theta(h) = \text{Heaviside}(h - h^*).$$

The precursor-film-regularized model then becomes

$$\left\{ \begin{aligned}
&\frac{\partial h}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \omega_b) = \Theta(h) \left[\varphi \frac{C}{k + C} h + Av(r) - \lambda_h h \right], \\
&\frac{\partial(h\varphi)}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \varphi \omega_m) = \eta_c(T) \varphi \frac{C}{k + C} h - \eta_d(T) \varphi h, \\
&\frac{\partial(h\psi)}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \psi \omega_m) = \eta_e(T) \varphi \frac{C}{k + C} h + \eta_d(T) \varphi h, \\
&\frac{\partial(hA)}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r A \omega_a) = \Theta(h) Av(r), \\
&\text{Pe} h \frac{\partial C}{\partial t} = \Theta(h) \left[-\text{Pe} \frac{1}{r} \frac{\partial}{\partial r} (r C \omega_b) + \frac{1}{r} \frac{\partial}{\partial r} \left(r h \frac{\partial C}{\partial r} \right) - k_{mb}(C - S) - \eta_b \varphi h \frac{C}{k + C} \right], \\
&\frac{\partial S}{\partial t} = D \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial S}{\partial r} \right) - \Theta(h) D k_{ms}(S - C), \\
&\rho c_p h \frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(r k_T h \frac{\partial T}{\partial r} \right) + \Theta(h) Q \varphi h \frac{C}{k + C} - k_{mT}(T - T_{\text{sub}}).
\end{aligned} \right. \tag{2.23}$$

2.3 Numerical Approximation

In this part, we discretize (2.23) and establish a stability condition for the proposed numerical method.

2.3.1 Finite Difference Approximation

The governing partial differential equations were discretized using a standard finite-difference scheme. Spatial derivatives were approximated with second-order central differences, while time advancement employed a first-order forward (explicit) method. One-dimensional radial profiles were computed under no-flux boundary conditions at both ends of the radial domain for the biofilm and substrate, enforced numerically using the ghost-node technique. The temporal and spatial derivatives are defined as follows:

$$\left. \frac{\partial h}{\partial t} \right|_i^n \approx \frac{h_i^{n+1} - h_i^n}{\Delta t}, \quad i = 0, 1, \dots, N. \quad (2.24)$$

$$\left. \frac{\partial h}{\partial r} \right|_i^n \approx \frac{h_{i+1}^n - h_{i-1}^n}{2\Delta r}, \quad (2.25)$$

$$\left. \frac{\partial(\psi^3)}{\partial r} \right|_i^n \approx \frac{(\psi_{i+1}^n)^3 - (\psi_{i-1}^n)^3}{2\Delta r}. \quad (2.26)$$

For the axisymmetric third-order radial operator (h_{3r}), the inner operator is first discretized at node i :

$$L_i^n \approx \frac{1}{r_i} \frac{1}{\Delta r} \left[r_{i+1/2} \frac{h_{i+1}^n - h_i^n}{\Delta r} - r_{i-1/2} \frac{h_i^n - h_{i-1}^n}{\Delta r} \right], \quad r_{i\pm 1/2} = r_i \pm \frac{\Delta r}{2}. \quad (2.27)$$

Then, the full operator is approximated using a second-order central difference:

$$(h_{3r})_i^n \approx \frac{L_{i+1}^n - L_{i-1}^n}{2\Delta r}. \quad (2.28)$$

Subsequently, the radial fluxes at integer nodes are computed using the discretized derivatives:

$$\omega_{m,i}^n \approx \frac{(h_i^n)^3}{3\varphi_m} \left[\gamma^* (h_{3r})_i^n - \delta^* \frac{(\psi_{i+1}^n)^3 - (\psi_{i-1}^n)^3}{2\Delta r} \right], \quad (2.29)$$

$$\omega_{a,i}^n \approx \frac{(h_i^n)^3}{3\varphi_m} \left[\gamma^* (h_{3r})_i^n - \delta^* \frac{(\psi_{i+1}^n)^3 - (\psi_{i-1}^n)^3}{2\Delta r} \right] + \frac{A_i^n \gamma^*}{\beta^* \varphi_m} h_i^n (h_{3r})_i^n, \quad (2.30)$$

$$\omega_{b,i}^n \approx \varphi_m \omega_{m,i}^n + A_i^n \omega_{a,i}^n. \quad (2.31)$$

To evaluate the fluxes at the half-integer cell faces (e.g., $i \pm 1/2$) required for the conservative divergence terms, arithmetic averaging is employed (e.g., $\omega_{i+1/2} = (\omega_i + \omega_{i+1})/2$). For purely advective terms, an upwind scheme can alternatively be used to enhance numerical stability.

The osmotic influx is evaluated pointwise as:

$$v_i^n = Q_0 \max \left(0, (\psi_i^n)^3 - (\psi_q)^3 \right). \quad (2.32)$$

Thus, the fully discretized system takes the following form:

$$\frac{h_i^{n+1} - h_i^n}{\Delta t} + \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} \omega_{b,i+\frac{1}{2}}^n - r_{i-\frac{1}{2}} \omega_{b,i-\frac{1}{2}}^n}{\Delta r} = \Theta_i^n \left[\varphi_i^n \frac{C_i^n}{k + C_i^n} h_i^n + A_i^n v_i^n - \lambda_h h_i^n \right], \quad (2.33)$$

$$\begin{aligned} h_i^n \frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} + \varphi_i^n \frac{h_i^{n+1} - h_i^n}{\Delta t} + \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} (\varphi \omega_m)_{i+\frac{1}{2}}^n - r_{i-\frac{1}{2}} (\varphi \omega_m)_{i-\frac{1}{2}}^n}{\Delta r} \\ = \eta_c(T_i^n) \varphi_i^n \frac{C_i^n}{k + C_i^n} h_i^n - \eta_d(T_i^n) \varphi_i^n h_i^n, \end{aligned} \quad (2.34)$$

$$\begin{aligned} h_i^n \frac{\psi_i^{n+1} - \psi_i^n}{\Delta t} + \psi_i^n \frac{h_i^{n+1} - h_i^n}{\Delta t} + \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} (\psi \omega_m)_{i+\frac{1}{2}}^n - r_{i-\frac{1}{2}} (\psi \omega_m)_{i-\frac{1}{2}}^n}{\Delta r} \\ = \eta_e(T_i^n) \varphi_i^n \frac{C_i^n}{k + C_i^n} h_i^n + \eta_d(T_i^n) \varphi_i^n h_i^n, \end{aligned} \quad (2.35)$$

$$h_i^n \frac{A_i^{n+1} - A_i^n}{\Delta t} + A_i^n \frac{h_i^{n+1} - h_i^n}{\Delta t} + \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} (A \omega_a)_{i+\frac{1}{2}}^n - r_{i-\frac{1}{2}} (A \omega_a)_{i-\frac{1}{2}}^n}{\Delta r} = \Theta_i^n A_i^n v_i^n, \quad (2.36)$$

$$\begin{aligned} \text{Pe} h_i^n \frac{C_i^{n+1} - C_i^n}{\Delta t} = \Theta_i^n \left[-\text{Pe} \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} (C \omega_b)_{i+\frac{1}{2}}^n - r_{i-\frac{1}{2}} (C \omega_b)_{i-\frac{1}{2}}^n}{\Delta r} \right. \\ + \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} h_{i+\frac{1}{2}}^n \frac{C_{i+1}^n - C_i^n}{\Delta r} - r_{i-\frac{1}{2}} h_{i-\frac{1}{2}}^n \frac{C_i^n - C_{i-1}^n}{\Delta r}}{\Delta r} \\ \left. - k_{mb} (C_i^n - S_i^n) - \eta_b \varphi_i^n h_i^n \frac{C_i^n}{k + C_i^n} \right], \end{aligned} \quad (2.37)$$

$$\frac{S_i^{n+1} - S_i^n}{\Delta t} = D \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} \frac{S_{i+1}^n - S_i^n}{\Delta r} - r_{i-\frac{1}{2}} \frac{S_i^n - S_{i-1}^n}{\Delta r}}{\Delta r} - \Theta_i^n D k_{ms} (S_i^n - C_i^n), \quad (2.38)$$

$$\begin{aligned} \rho C_p h_i^n \frac{T_i^{n+1} - T_i^n}{\Delta t} = & \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} k_T h_{i+\frac{1}{2}}^n \frac{T_{i+1}^n - T_i^n}{\Delta r} - r_{i-\frac{1}{2}} k_T h_{i-\frac{1}{2}}^n \frac{T_i^n - T_{i-1}^n}{\Delta r}}{\Delta r} \\ & + \Theta_i^n Q \varphi_i^n h_i^n \frac{C_i^n}{k + C_i^n} - k_{mT} (T_i^n - T_{\text{sub}}). \end{aligned} \quad (2.39)$$

Then, we obtain sequentially h_i^{n+1} , φ_i^{n+1} , ψ_i^{n+1} , A_i^{n+1} , C_i^{n+1} , S_i^{n+1} , and T_i^{n+1} as follows:

$$h_i^{n+1} = h_i^n + \Delta t \left[-\frac{1}{r_i} \frac{r_{i+\frac{1}{2}} \omega_{b,i+\frac{1}{2}}^n - r_{i-\frac{1}{2}} \omega_{b,i-\frac{1}{2}}^n}{\Delta r} + \Theta_i^n \left(\varphi_i^n \frac{C_i^n}{k + C_i^n} h_i^n + A_i^n v_i^n - \lambda_h h_i^n \right) \right]. \quad (2.40)$$

$$\varphi_i^{n+1} = \varphi_i^n + \frac{\Delta t}{h_i^n} \left[-\varphi_i^n \frac{h_i^{n+1} - h_i^n}{\Delta t} - \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} (\varphi \omega_m)_{i+\frac{1}{2}}^n - r_{i-\frac{1}{2}} (\varphi \omega_m)_{i-\frac{1}{2}}^n}{\Delta r} + \eta_c (T_i^n) \varphi_i^n \frac{C_i^n}{k + C_i^n} h_i^n - \eta_d (T_i^n) \varphi_i^n h_i^n \right]. \quad (2.41)$$

$$\psi_i^{n+1} = \psi_i^n + \frac{\Delta t}{h_i^n} \left[-\psi_i^n \frac{h_i^{n+1} - h_i^n}{\Delta t} - \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} (\psi \omega_m)_{i+\frac{1}{2}}^n - r_{i-\frac{1}{2}} (\psi \omega_m)_{i-\frac{1}{2}}^n}{\Delta r} + \eta_e (T_i^n) \varphi_i^n \frac{C_i^n}{k + C_i^n} h_i^n + \eta_d (T_i^n) \varphi_i^n h_i^n \right]. \quad (2.42)$$

$$A_i^{n+1} = A_i^n + \frac{\Delta t}{h_i^n} \left[-A_i^n \frac{h_i^{n+1} - h_i^n}{\Delta t} - \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} (A \omega_a)_{i+\frac{1}{2}}^n - r_{i-\frac{1}{2}} (A \omega_a)_{i-\frac{1}{2}}^n}{\Delta r} + \Theta_i^n A_i^n v_i^n \right]. \quad (2.43)$$

$$\begin{aligned} C_i^{n+1} = C_i^n + \frac{\Delta t}{\text{Pe} h_i^n} \left\{ \Theta_i^n \left[-\text{Pe} \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} (C \omega_b)_{i+\frac{1}{2}}^n - r_{i-\frac{1}{2}} (C \omega_b)_{i-\frac{1}{2}}^n}{\Delta r} \right. \right. \\ + \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} h_{i+\frac{1}{2}}^n \frac{C_{i+1}^n - C_i^n}{\Delta r} - r_{i-\frac{1}{2}} h_{i-\frac{1}{2}}^n \frac{C_i^n - C_{i-1}^n}{\Delta r}}{\Delta r} \\ \left. \left. - k_{mb} (C_i^n - S_i^n) - \eta_b \varphi_i^n h_i^n \frac{C_i^n}{k + C_i^n} \right] \right\}. \end{aligned} \quad (2.44)$$

$$S_i^{n+1} = S_i^n + \Delta t \left[D \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} \frac{S_{i+1}^n - S_i^n}{\Delta r} - r_{i-\frac{1}{2}} \frac{S_i^n - S_{i-1}^n}{\Delta r}}{\Delta r} - \Theta_i^n D k_{ms} (S_i^n - C_i^n) \right]. \quad (2.45)$$

$$\begin{aligned}
T_i^{n+1} = T_i^n + \frac{\Delta t}{\rho c_p h_i^n} & \left[\frac{1}{r_i} \frac{r_{i+\frac{1}{2}} k_T h_{i+\frac{1}{2}}^n \frac{T_{i+1}^n - T_i^n}{\Delta r} - r_{i-\frac{1}{2}} k_T h_{i-\frac{1}{2}}^n \frac{T_i^n - T_{i-1}^n}{\Delta r}}{\Delta r} \right. \\
& \left. + \Theta_i^n Q \varphi_i^n h_i^n \frac{C_i^n}{k + C_i^n} - k_{mT} (T_i^n - T_{\text{sub}}) \right].
\end{aligned} \tag{2.46}$$

2.3.2 Stability Analysis and Discrete Energy Estimation

In the framework of the discrete L^2 -stability analysis, our aim is to derive an L^2 -bound for the numerical solution. For each time level $n \in \mathbb{N}$, the discrete solution is identified with a piecewise constant function over the spatial mesh. More precisely, for any grid function $u^n = (u_i^n)$, the corresponding discrete L^2 -norm, adapted to the axisymmetric geometry of the problem, is defined by

$$\|u^n\|_r = \left(\sum_{i=1}^{N-1} r_i |u_i^n|^2 \Delta r \right)^{1/2}.$$

We also introduce the associated discrete inner product for any two grid functions u and v :

$$(u, v)_r = \sum_{i=1}^{N-1} r_i u_i v_i \Delta r.$$

The stability analysis is based on a discrete energy argument. More precisely, we show that the proposed numerical scheme satisfies a discrete energy estimate, which provides uniform control over the numerical solution and yields the conditional stability of the fully discrete scheme.

Definition 2.1. The total discrete energy of the system at time level n is defined by

$$E^n = \|h^n\|_r^2 + \|\varphi^n\|_r^2 + \|\psi^n\|_r^2 + \|A^n\|_r^2 + \|C^n\|_r^2 + \|S^n\|_r^2 + \|T^n\|_r^2. \tag{2.47}$$

Thanks to the precursor film regularization introduced previously, the discrete film thickness remains strictly positive, avoiding any contact-line singularity or division by zero. Thus, we can legitimately assume the existence of a strictly positive lower bound h_* and an upper bound H^* such that:

$$0 < h_* \leq h_i^n \leq H^*, \quad \forall i, n.$$

Furthermore, we denote the maximum absolute interfacial fluxes by:

$$W_b = \max_{i,n} |\omega_{b,i+\frac{1}{2}}^n|, \quad W_m = \max_{i,n} |\omega_{m,i+\frac{1}{2}}^n|, \quad W_a = \max_{i,n} |\omega_{a,i+\frac{1}{2}}^n|.$$

Proposition 2.3.1. Under the following CFL condition:

$$\Delta t \leq c_0 \min \left\{ \frac{\Delta r}{W_b + W_m + W_a}, \frac{\Delta r^2}{H^* + 1} \right\}, \quad (2.48)$$

where $c_0 > 0$ is a sufficiently small constant independent of Δr and Δt , the discrete energy (2.47) satisfies the recursive inequality:

$$E^{n+1} \leq (1 + C_1 \Delta t) E^n + C_3 \Delta t, \quad (2.49)$$

Consequently, by the discrete Gronwall lemma, the energy is uniformly bounded up to final time $T = n\Delta t$:

$$E^n \leq E^0 e^{C_1 T} + \frac{C_3}{C_1} (e^{C_1 T} - 1), \quad (2.50)$$

where $C_1 > 0$ and $C_3 > 0$ are constants independent of Δt .

Proof. For simplicity, we define the discrete flux-divergence and spatial operators:

$$\begin{aligned} (\mathcal{D}_b^n)_i &= \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} \omega_{b,i+\frac{1}{2}}^n - r_{i-\frac{1}{2}} \omega_{b,i-\frac{1}{2}}^n}{\Delta r}, \\ (\mathcal{D}_\varphi^n)_i &= \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} (\varphi \omega_m)_{i+\frac{1}{2}}^n - r_{i-\frac{1}{2}} (\varphi \omega_m)_{i-\frac{1}{2}}^n}{\Delta r}, \\ (\mathcal{D}_\psi^n)_i &= \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} (\psi \omega_m)_{i+\frac{1}{2}}^n - r_{i-\frac{1}{2}} (\psi \omega_m)_{i-\frac{1}{2}}^n}{\Delta r}, \\ (\mathcal{D}_A^n)_i &= \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} (A \omega_a)_{i+\frac{1}{2}}^n - r_{i-\frac{1}{2}} (A \omega_a)_{i-\frac{1}{2}}^n}{\Delta r}, \\ (\mathcal{D}_{C,b}^n)_i &= \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} (C \omega_b)_{i+\frac{1}{2}}^n - r_{i-\frac{1}{2}} (C \omega_b)_{i-\frac{1}{2}}^n}{\Delta r}, \\ (\mathcal{L}_C^n)_i &= \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} h_{i+\frac{1}{2}}^n \frac{C_{i+1}^n - C_i^n}{\Delta r} - r_{i-\frac{1}{2}} h_{i-\frac{1}{2}}^n \frac{C_i^n - C_{i-1}^n}{\Delta r}}{\Delta r}, \\ (\mathcal{L}_S^n)_i &= \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} \frac{S_{i+1}^n - S_i^n}{\Delta r} - r_{i-\frac{1}{2}} \frac{S_i^n - S_{i-1}^n}{\Delta r}}{\Delta r}, \\ (\mathcal{L}_T^n)_i &= \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} h_{i+\frac{1}{2}}^n \frac{T_{i+1}^n - T_i^n}{\Delta r} - r_{i-\frac{1}{2}} h_{i-\frac{1}{2}}^n \frac{T_i^n - T_{i-1}^n}{\Delta r}}{\Delta r}. \end{aligned}$$

From (2.40), we have:

$$|h_i^{n+1}|^2 = \left| h_i^n + \Delta t \left[-(\mathcal{D}_b^n)_i + \Theta_i^n \left(\varphi_i^n \frac{C_i^n}{k + C_i^n} h_i^n + A_i^n v_i^n - \lambda_h h_i^n \right) \right] \right|^2.$$

Applying Young's inequality in the form $|a+b|^2 \leq (1+\Delta t)|a|^2 + (1+\frac{1}{\Delta t})|b|^2$, and noting that $(1+\frac{1}{\Delta t})\Delta t^2 = \Delta t(1+\Delta t) \leq C\Delta t$ for small Δt , we find:

$$|h_i^{n+1}|^2 \leq (1+\Delta t)|h_i^n|^2 + C\Delta t \left| -(\mathcal{D}_b^n)_i + \Theta_i^n \left(\varphi_i^n \frac{C_i^n}{k + C_i^n} h_i^n + A_i^n v_i^n - \lambda_h h_i^n \right) \right|^2.$$

Since $0 \leq \frac{C_i^n}{k+C_i^n} \leq 1$, $0 \leq \Theta_i^n \leq 1$, $|h_i^n| \leq H^*$, and $|v_i^n|$ is bounded, we get:

$$|h_i^{n+1}|^2 \leq (1+\Delta t)|h_i^n|^2 + C_1\Delta t (|(\mathcal{D}_b^n)_i|^2 + |\varphi_i^n|^2 + |A_i^n|^2 + |h_i^n|^2).$$

Multiplying by $r_i\Delta r$ and summing over i , we obtain:

$$\|h^{n+1}\|_r^2 \leq (1+\Delta t)\|h^n\|_r^2 + C_1\Delta t \left(\|\mathcal{D}_b^n\|_r^2 + \|\varphi^n\|_r^2 + \|A^n\|_r^2 + \|h^n\|_r^2 \right). \quad (2.51)$$

From (2.41), we have

$$|\varphi_i^{n+1}|^2 = \left| \varphi_i^n + \frac{\Delta t}{h_i^n} \left[-\varphi_i^n \frac{h_i^{n+1} - h_i^n}{\Delta t} - (\mathcal{D}_\varphi^n)_i + \eta_c(T_i^n)\varphi_i^n \frac{C_i^n}{k + C_i^n} h_i^n - \eta_d(T_i^n)\varphi_i^n h_i^n \right] \right|^2,$$

since $h_i^n \geq h_* > 0$, and $|\eta_c(T_i^n)|, |\eta_d(T_i^n)|$ are bounded, we get

$$|\varphi_i^{n+1}|^2 \leq (1+\Delta t)|\varphi_i^n|^2 + C_2\Delta t \left(\left| \varphi_i^n \frac{h_i^{n+1} - h_i^n}{\Delta t} \right|^2 + |(\mathcal{D}_\varphi^n)_i|^2 + |\varphi_i^n|^2 \right),$$

where $C_2 > 0$.

Multiplying by $r_i\Delta r$ and summing over i yields

$$\|\varphi^{n+1}\|_r^2 \leq (1+\Delta t)\|\varphi^n\|_r^2 + C_2\Delta t \left(\|\varphi^n\|_r^2 \left\| \frac{h^{n+1} - h^n}{\Delta t} \right\|_r^2 + \|\mathcal{D}_\varphi^n\|_r^2 + \|\varphi^n\|_r^2 \right). \quad (2.52)$$

From (2.42), we have

$$|\psi_i^{n+1}|^2 \leq (1+\Delta t)|\psi_i^n|^2 + C_3\Delta t \left(|\psi_i^n|^2 \left| \frac{h_i^{n+1} - h_i^n}{\Delta t} \right|^2 + |(\mathcal{D}_\psi^n)_i|^2 + |\varphi_i^n|^2 \right),$$

hence

$$\|\psi^{n+1}\|_r^2 \leq (1 + \Delta t)\|\psi^n\|_r^2 + C_3\Delta t\left(\|\psi^n\|_r^2\left\|\frac{h^{n+1} - h^n}{\Delta t}\right\|_r^2 + \|\mathcal{D}_\psi^n\|_r^2 + \|\varphi^n\|_r^2\right). \quad (2.53)$$

From (2.43), we have

$$|A_i^{n+1}|^2 \leq (1 + \Delta t)|A_i^n|^2 + C_4\Delta t\left(|A_i^n|^2\left|\frac{h_i^{n+1} - h_i^n}{\Delta t}\right|^2 + |(\mathcal{D}_A^n)_i|^2 + |A_i^n|^2\right),$$

hence

$$\|A^{n+1}\|_r^2 \leq (1 + \Delta t)\|A^n\|_r^2 + C_4\Delta t\left(\|A^n\|_r^2\left\|\frac{h^{n+1} - h^n}{\Delta t}\right\|_r^2 + \|\mathcal{D}_A^n\|_r^2 + \|A^n\|_r^2\right). \quad (2.54)$$

From (2.44), and using $h_i^n \geq h_*$, we have

$$|C_i^{n+1}|^2 \leq (1 + \Delta t)|C_i^n|^2 + C_5\Delta t\left(|(\mathcal{D}_{C,b}^n)_i|^2 + |(\mathcal{L}_C^n)_i|^2 + |C_i^n - S_i^n|^2 + |\varphi_i^n|^2\right),$$

and summation gives

$$\|C^{n+1}\|_r^2 \leq (1 + C\Delta t)\|C^n\|_r^2 + C_5\Delta t\left(\|\mathcal{D}_{C,b}^n\|_r^2 + \|\mathcal{L}_C^n\|_r^2 + \|C^n\|_r^2 + \|S^n\|_r^2 + \|\varphi^n\|_r^2\right). \quad (2.55)$$

From (2.45), we have

$$|S_i^{n+1}|^2 = \left|S_i^n + \Delta t\left[D(\mathcal{L}_S^n)_i - \Theta_i^n Dk_{ms}(S_i^n - C_i^n)\right]\right|^2 \leq (1 + \Delta t)|S_i^n|^2 + C_6\Delta t\left(|(\mathcal{L}_S^n)_i|^2 + |S_i^n - C_i^n|^2\right),$$

and after summation,

$$\|S^{n+1}\|_r^2 \leq (1 + \Delta t)\|S^n\|_r^2 + C_6\Delta t\left(\|\mathcal{L}_S^n\|_r^2 + \|S^n\|_r^2 + \|C^n\|_r^2\right). \quad (2.56)$$

From (2.46), we have

$$|T_i^{n+1}|^2 \leq (1 + \Delta t)|T_i^n|^2 + C_7\Delta t\left(|(\mathcal{L}_T^n)_i|^2 + |\varphi_i^n|^2\right),$$

hence

$$\|T^{n+1}\|_r^2 \leq (1 + \Delta t)\|T^n\|_r^2 + C_7\Delta t\left(\|\mathcal{L}_T^n\|_r^2 + \|\varphi^n\|_r^2\right). \quad (2.57)$$

Adding equations (2.51) through (2.57), the total energy satisfies:

$$\begin{aligned}
E^{n+1} &\leq (1 + C\Delta t)E^n \\
&\quad + C_8\Delta t \left(\|\mathcal{D}_b^n\|_r^2 + \|\mathcal{D}_\varphi^n\|_r^2 + \|\mathcal{D}_\psi^n\|_r^2 + \|\mathcal{D}_A^n\|_r^2 + \|\mathcal{D}_{C,b}^n\|_r^2 \right. \\
&\quad \left. + \|\mathcal{L}_C^n\|_r^2 + \|\mathcal{L}_S^n\|_r^2 + \|\mathcal{L}_T^n\|_r^2 + \left\| \frac{h^{n+1} - h^n}{\Delta t} \right\|_r^2 \right). \tag{2.58}
\end{aligned}$$

By virtue of the precursor film regularization introduced earlier, the discrete solution remains within a regular regime where contact-line singularities are successfully avoided. Consequently, the discrete spatial gradients, advective flux-divergences, and second-order diffusive operators remain uniformly bounded in the discrete L^2 -norm. Crucially, by invoking the standard CFL condition (2.48), the time step Δt is explicitly constrained by the maximum interfacial fluxes W_b, W_m, W_a and the upper film bound H^* . This condition mathematically neutralizes the mesh-dependent denominators (Δr^{-1} and Δr^{-2}) arising from the spatial discretization. Furthermore, under this CFL tracking, the discrete time derivative of the film thickness is also well-controlled, such that:

$$\begin{aligned}
\left\| \frac{h^{n+1} - h^n}{\Delta t} \right\|_r^2 &\leq C_*, \\
\|\mathcal{D}_b^n\|_r^2 + \|\mathcal{D}_\varphi^n\|_r^2 + \|\mathcal{D}_\psi^n\|_r^2 + \|\mathcal{D}_A^n\|_r^2 + \|\mathcal{D}_{C,b}^n\|_r^2 &\leq C_{**}, \\
\|\mathcal{L}_C^n\|_r^2 + \|\mathcal{L}_S^n\|_r^2 + \|\mathcal{L}_T^n\|_r^2 &\leq C_{***}.
\end{aligned}$$

By inserting these uniform bounds into (2.58), all the non-homogeneous operator contributions are absorbed into a single global positive constant $C_3 = C_8(C_* + C_{**} + C_{***})$. As a result, the total energy inequality directly reduces to the desired one-step recursive estimate:

$$E^{n+1} \leq (1 + C_1\Delta t)E^n + C_3\Delta t. \tag{2.59}$$

Applying the discrete Gronwall inequality, we obtain:

$$E^n \leq E^0 (1 + C_1\Delta t)^n + C_3\Delta t \sum_{k=0}^{n-1} (1 + C_1\Delta t)^k.$$

Using the inequality $(1 + x)^n \leq e^{nx}$ for $x \geq 0$, and summing the geometric series, we finally get:

$$E^n \leq E^0 e^{C_1 T_1} + \frac{C_3}{C_1} (e^{C_1 T_1} - 1),$$

where $T_1 = n\Delta t$. This yields the conditional stability of the explicit finite-difference scheme. $\square$

2.3.3 Numerical Results

Numerical simulations are performed on an axisymmetric radial domain $\Omega = (0, 10)$ over the time interval $t \in [0, 1]$. The governing system is discretized in space using a finite-difference scheme on a uniform grid composed of $N + 1$ nodes. Time integration is carried out with an explicit forward Euler method, and the time step is adaptively selected in order to satisfy CFL-type stability constraints associated with transport and diffusion processes.

Numerical parameters

In order to account for thermal effects on biological activity, the kinetic coefficients are updated pointwise at each time step according to the local temperature field T . The dimensional rates for cell growth, extracellular matrix (ECM) production, and cell death follow Arrhenius-type laws:

$$\begin{aligned} k_c(T) &= k_{c0} \exp\left(-\frac{E_a}{RT}\right), \\ k_e(T) &= k_{e0} \exp\left(-\frac{E_e}{RT}\right), \\ k_d(T) &= k_{d0} \exp\left(-\frac{E_d}{RT}\right). \end{aligned} \tag{2.60}$$

These dimensional rates are then scaled by the characteristic time to yield the dimensionless temperature-dependent functions $\eta_c(T)$, $\eta_e(T)$, and $\eta_d(T)$ used in the reduced model. This formulation allows the kinetics to be computed dynamically during the simulation and analyzed a posteriori.

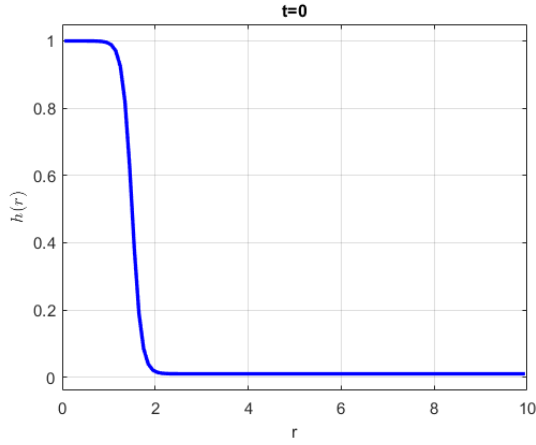
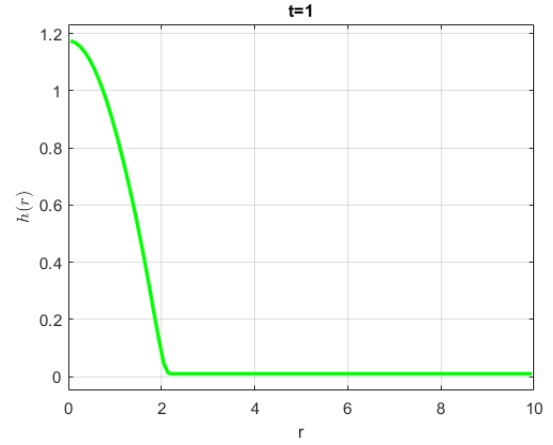
No-flux and symmetry boundary conditions are enforced through the use of ghost nodes, ensuring mass and energy conservation at the domain boundaries. Unless otherwise stated, the dimensional and nondimensional parameter values used in the simulations are listed in Tables 2.1 and 2.2, respectively. Most values are consistent with those reported in the literature (see, e.g., [59, 15]).

Table 2.1: Dimensional parameters used in the model.

Symbol	Description	Value (SI)
k_{c0}	Pre-exponential factor for cell growth	$2.5 \times 10^5 \text{ min}^{-1}$
k_{e0}	Pre-exponential factor for ECM prod.	$2.0 \times 10^5 \text{ min}^{-1}$
k_{d0}	Pre-exponential factor for cell death	$1.0 \times 10^{11} \text{ min}^{-1}$
E_a	Activation energy for cell growth	$4.0 \times 10^4 \text{ J mol}^{-1}$
E_e	Activation energy for ECM prod.	$4.0 \times 10^4 \text{ J mol}^{-1}$
E_d	Activation energy for cell death	$8.0 \times 10^4 \text{ J mol}^{-1}$
R	Ideal gas constant	$8.314 \text{ J mol}^{-1} \text{ K}^{-1}$
k_b	Nutrient consumption rate	$1/25 \text{ min}^{-1}$
D_s	Nutrient diffusivity in agar	$5 \times 10^{-10} \text{ m}^2 \text{ s}^{-1}$
D_b	Nutrient diffusivity in biofilm	$1.25 \times 10^{-10} \text{ m}^2 \text{ s}^{-1}$
k_m	Mass-transfer coefficient (biofilm)	$2 \times 10^{-8} \text{ m s}^{-1}$
k_{mT}	Thermal transfer coefficient	$100 \text{ W m}^{-2} \text{ K}^{-1}$
ψ_0	Osmotic pressure scale	2100 Pa
Q_0	Swelling velocity	$0.4 \text{ } \mu\text{m s}^{-1}$
δ_{eq}	Equilibrium volume fraction of ECM	4×10^{-4}
μ_m	Biofilm viscosity	10^5 Pa s
μ_a	Aqueous viscosity	10^{-3} Pa s
β	Interphase friction	$2 \times 10^{14} \text{ Pa s m}^{-2}$
H_b	Biofilm vertical length scale	$400 \text{ } \mu\text{m}$
R_b	Biofilm horizontal length scale	$1100 \text{ } \mu\text{m}$
ρc_p	Volumetric heat capacity (biofilm)	$4.18 \times 10^6 \text{ J m}^{-3} \text{ K}^{-1}$
k_T	Thermal conductivity (biofilm)	$0.6 \text{ W m}^{-1} \text{ K}^{-1}$
k_T^{sub}	Thermal conductivity (substrate)	$0.6 \text{ W m}^{-1} \text{ K}^{-1}$
Q	Metabolic heat source	-10^4 W m^{-3}
T_0	Initial ambient temperature	298.15 K
δT	Initial temperature perturbation	1.0 K

Table 2.2: Dimensionless parameters used after nondimensionalization.

Symbol	Description	Value
k_m	Mass-transfer coefficient (nondim.)	0.30
k_{mb}	Interfacial exchange (biofilm)	1.21
Q_{os}	Swelling scale in influx law	0.24
δ^*	Scaled osmotic coefficient	6.66
ϵ	Initial aspect ratio	0.36
Pe	Péclet number (biofilm)	4.03
D	Substrate diffusivity (nondim.)	0.99
k	Monod half-saturation (nondim.)	0.5
η_b	Biomass consumption factor	6
γ^*	Surface-tension ratio	1
β^*	Biomass-water friction ratio	320
h^*	Precursor-film thickness	0.01
λ_h	Biomass growth damping coefficient	0.05

Figure 2.2: Initial thickness profile $h(r)$.Figure 2.3: Thickness profile $h(r)$ at $t = 1$.

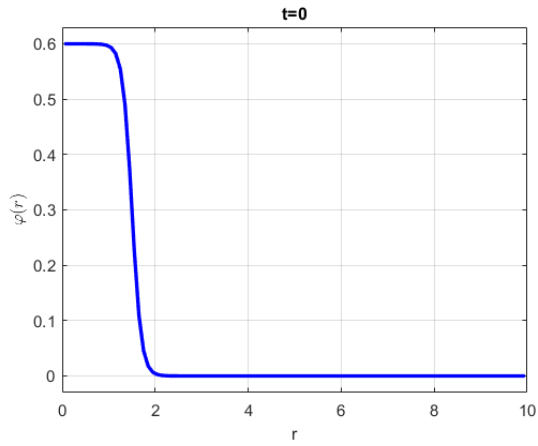


Figure 2.4: Initial cell volume fraction $\phi(r)$.

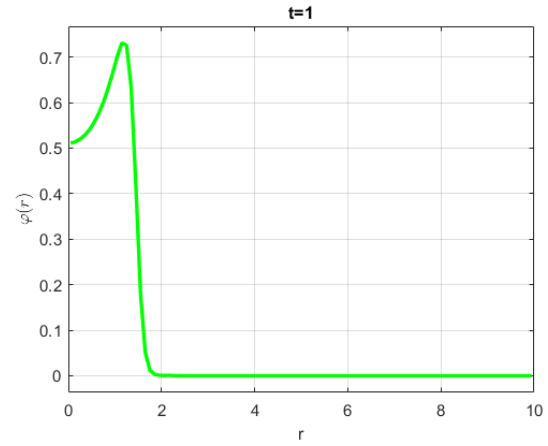


Figure 2.5: Cell volume fraction $\phi(r)$ at $t = 1$.

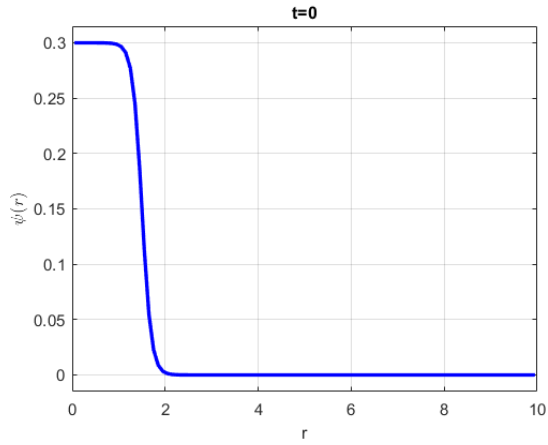


Figure 2.6: Initial ECM volume fraction $\psi(r)$.

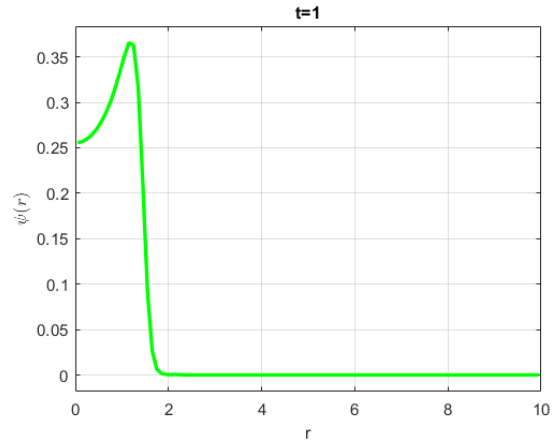


Figure 2.7: ECM volume fraction $\psi(r)$ at $t = 1$.

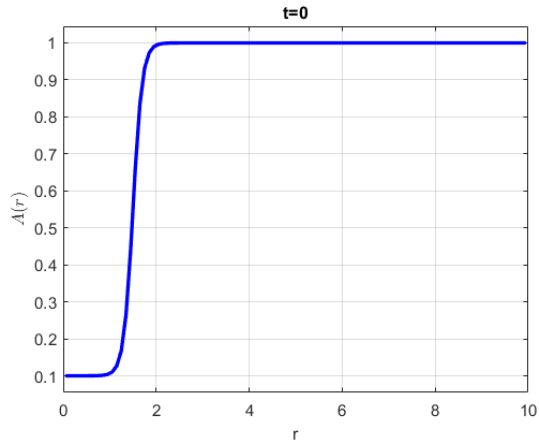


Figure 2.8: Initial aqueous-phase fraction $A(r)$.

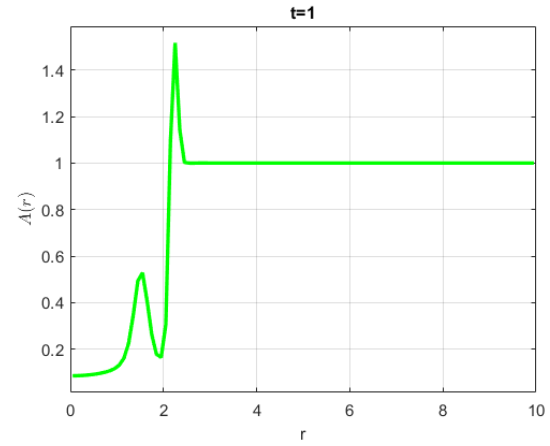


Figure 2.9: Aqueous-phase fraction $A(r)$ at $t = 1$.

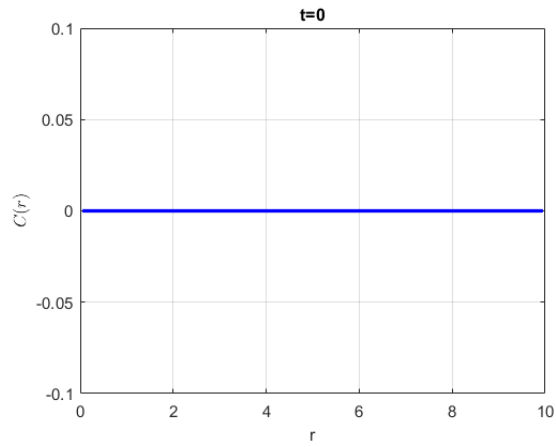


Figure 2.10: Initial nutrient concentration $C(r)$.

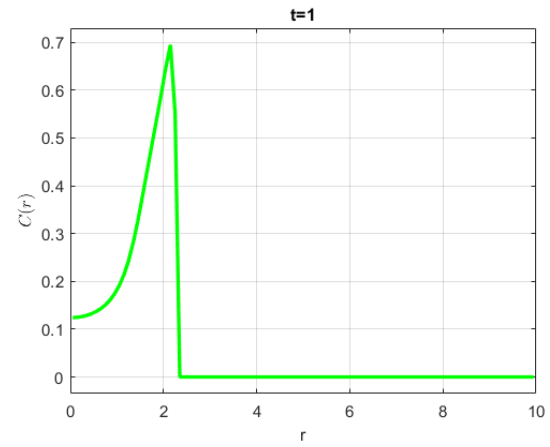


Figure 2.11: Nutrient concentration $C(r)$ at $t = 1$.

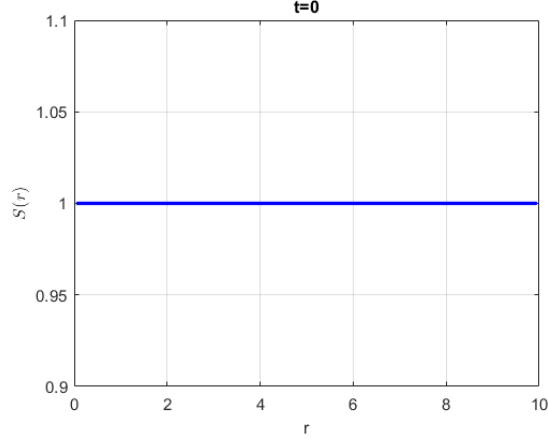


Figure 2.12: Initial substrate concentration $S(r)$.

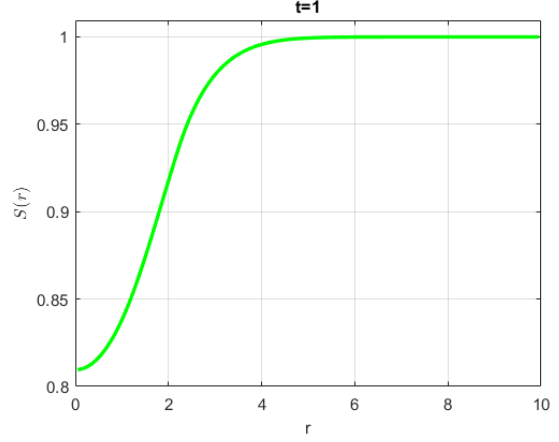


Figure 2.13: Substrate concentration $S(r)$ at $t = 1$.

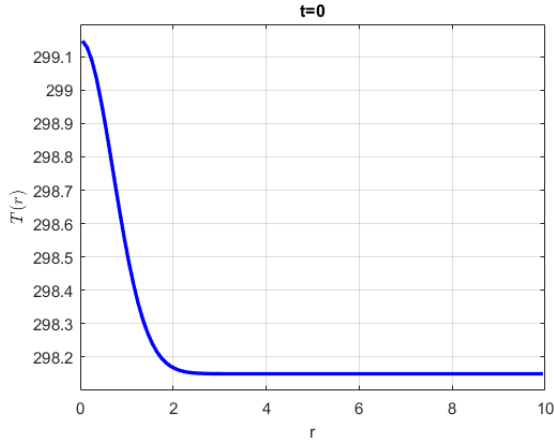


Figure 2.14: Initial temperature profile $T(r)$.

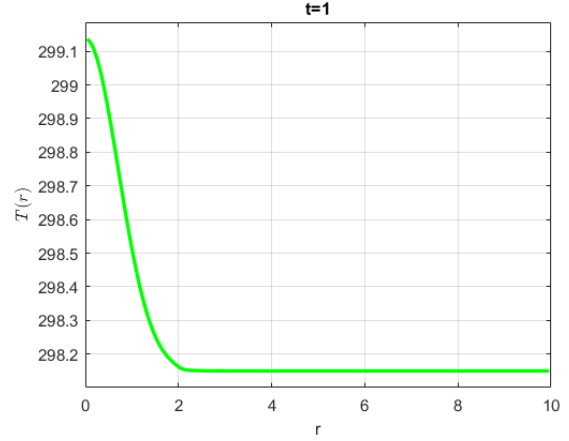


Figure 2.15: Temperature profile $T(r)$ at $t = 1$.

Discussion of the numerical results. The initial profiles shown in Figures 2.2–2.14 describe a localized biofilm with smooth interfaces, consistent with the imposed initial conditions. At $t = 1$ (Figures 2.3–2.15), the thickness $h(r)$ exhibits a moderate increase near the colony center, indicating early-stage biofilm expansion. The biomass and ECM volume fractions (φ, ψ) remain spatially localized with a slight increase in their peak values, while the aqueous-phase fraction $A(r)$ decreases within the biofilm region. The nutrient concentration $C(r)$ displays a sharp transition near the biofilm edge due to consumption, whereas the substrate $S(r)$ shows a noticeable depletion gradient towards the center. The temperature profile $T(r)$ maintains a mild elevation near the center, reflecting metabolic heat production. Overall, the numerical solutions remain smooth and physically consistent, supporting the stability of the proposed numerical scheme.

2.4 Conclusion

In this chapter, we have developed a thermally coupled multiphase model for biofilm formation on a porous solid surface. The model accounts for the interactions between the cellular phase, the extracellular matrix, and the aqueous phase, as well as nutrient transport and osmotic water influx.

The coupling with the heat equation allows us to capture the influence of temperature on biological processes, while incorporating metabolic heat generation as a feedback mechanism. This thermally coupled framework provides a more realistic representation of biofilm behavior under varying environmental conditions.

The introduction of a growth damping term in the thickness equation plays a crucial role in regulating biofilm expansion. It prevents unbounded growth and reflects the mechanical limitations arising from biomass accumulation and structural constraints within the biofilm.

A finite-difference numerical scheme has been proposed and analyzed. Under suitable CFL-type conditions, the scheme satisfies a discrete energy estimate ensuring conditional stability. Numerical simulations confirm the robustness of the method and illustrate the complex interplay between multiphase transport, thermal effects, and growth regulation.

Overall, this work highlights the importance of combining multiphase modeling, thermal coupling, and growth damping mechanisms to better understand biofilm formation. The proposed framework opens perspectives for further investigations, including experimental validation, parameter identification, and the study of control strategies for biofilm growth.

Chapter 3

Theoretical and Numerical Analysis of Thermally Coupled Quorum-Sensing Biofilm Model

3.1 Introduction

In this chapter, we develop a system of nonlinear coupled reaction–diffusion equations to describe biofilm dynamics under the combined effects of quorum sensing, antibiotic response, and temperature variations. The model includes bacterial populations, nutrients, signaling molecules, and temperature governed by a heat equation. Temperature affects bacterial growth and quorum-sensing mechanisms, while bacterial metabolism acts as a heat source, creating a bidirectional thermo-biological coupling.

This work extends the two-dimensional PDE model proposed by Blouza [10] for quorum sensing in *Pseudomonas aeruginosa* under antibiotic treatment.

Our main contributions of this chapter are as follows. We introduce a novel quorum-sensing–heat model for bacterial biofilms with temperature-dependent reaction kinetics and metabolic heat generation. We establish the existence and uniqueness of solutions for the coupled system using semigroup theory. A numerical scheme based on finite elements in space and a backward Euler discretization in time is then proposed, and its stability properties are rigorously analyzed. Finally, the numerical method is implemented in Matlab, and simulations are presented to illustrate the influence of thermal fluctuations on quorum sensing activity and biofilm dynamics.

The chapter is organized as follows. Section 2 introduces the mathematical model and states the underlying assumptions. Section 3 addresses the well-posedness of the coupled system by means of semigroup theory. In Section 4, a numerical approximation based on finite elements and backward Euler time integration is presented, and Section 5 presents the numerical simulations and discussion.

3.2 Proposed model

We consider a mathematical model describing the dynamics of a bacterial biofilm regulated by quorum sensing, antibiotic action, and temperature effects. The model consists of a coupled system of nonlinear reaction–diffusion equations defined on the space–time domain $\Omega \times (0, T)$, where $T > 0$ and $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial\Omega$ decomposed as $\partial\Omega = \Gamma_D \cup \Gamma_N$, with $\Gamma_D \cap \Gamma_N = \emptyset$. The nutrient substrate is supplied through the portion Γ_D of the boundary.

Nutrient dynamics

The nutrient concentration C diffuses within the biofilm and is consumed by the total biomass composed of down-regulated cells ψ and up-regulated cells φ . The consumption process follows a Monod-type kinetics. Accordingly, the nutrient evolution is governed by

$$\frac{\partial C}{\partial t} = d_C \Delta C - k_1 \frac{C}{k_2 + C} (\psi + \varphi), \quad (3.1)$$

where d_C denotes the nutrient diffusion coefficient, k_1 the maximum consumption rate, and k_2 the half-saturation constant.

Biomass evolution

Let $M = \psi + \varphi$ denote the total biomass concentration. Based on mass conservation principles, its evolution is described by

$$\frac{\partial M}{\partial t} = \nabla \cdot (D \nabla M) + G(u, M), \quad (3.2)$$

where $G(u, M)$ represents the biomass growth term and D is an effective diffusion coefficient. For simplicity, and since the biofilm shape is not the focus of this study, D is assumed to be constant.

Cell population dynamics

The biofilm consists of two bacterial phenotypes: down-regulated cells ψ and up-regulated cells φ . Transitions between these two states are controlled by quorum sensing mechanisms. Their dynamics are described by

$$\begin{cases} \frac{\partial \psi}{\partial t} = d \Delta \psi + k_3 \frac{C}{k_3 + C} \psi - k_5 A^m \psi + k_5 \varphi - k_4 \psi, \\ \frac{\partial \varphi}{\partial t} = d \Delta \varphi + k_3 \frac{C}{k_3 + C} \varphi + k_5 A^m \psi - k_5 \psi - k_4 \varphi, \end{cases} \quad (3.3)$$

where k_3 is the maximum growth rate, k_4 the lysis rate, k_5 the quorum-sensing regulation rate, A the signaling molecule concentration, and m a Hill coefficient.

The biofilm domain is defined as

$$\{x \in \Omega : \psi(x, t) + \varphi(x, t) > 0\}. \quad (3.4)$$

Quorum sensing signaling molecules

The signaling molecules (AHL) diffuse throughout the domain and undergo abiotic degradation at rate γ . Down-regulated cells produce AHL at a basal rate α , while up-regulated cells produce it at an enhanced rate $\alpha + \beta$, with $\alpha + \beta > \gamma$. The AHL dynamics are given by

$$\frac{\partial A}{\partial t} = d_A \Delta A - \gamma A + \alpha \frac{RA}{K_L + RA} \psi + (\alpha + \beta) \frac{RA}{K_L + RA} \varphi. \quad (3.5)$$

LasR protein and chemical agents

The LasR protein concentration R diffuses, degrades naturally, and interacts with both the signaling molecules and quorum-sensing blockers. Its evolution reads

$$\frac{\partial R}{\partial t} = d_R \Delta R - k_R R + k_3 \frac{RA}{k_R + RA} - k_3 \frac{\psi + \varphi}{k_2 + \psi + \varphi} R. \quad (3.6)$$

Thermal dynamics

Temperature evolution within the biofilm is governed by a heat equation accounting for thermal diffusion and metabolic heat generation:

$$\frac{\partial T}{\partial t} = k \Delta T + q, \quad (3.7)$$

where k is the thermal diffusivity and q denotes the heat source term. We assume that bacterial metabolic activity acts as a heat source proportional to the biomass concentration:

$$q = q_0(\psi + \varphi). \quad (3.8)$$

Temperature-dependent parameters

Reaction and regulation rates are assumed to depend on temperature according to an Arrhenius-type law

$$\tilde{k}(T) = k_0 \exp\left(-\frac{E_a}{R_g T}\right), \quad (3.9)$$

where k_0 is a pre-exponential factor, E_a the activation energy, and R_g the gas constant.

Coupled thermo-biophysical system

The complete coupled system reads

$$\left\{ \begin{array}{l} \frac{\partial C}{\partial t} = d_C \Delta C - k_1(T) \frac{C}{k_2 + C} (\psi + \varphi), \\ \frac{\partial A}{\partial t} = d_A \Delta A - \gamma(T) A + \alpha(T) \frac{RA}{K_L + RA} \psi + (\alpha(T) + \beta(T)) \frac{RA}{K_L + RA} \varphi, \\ \frac{\partial \psi}{\partial t} = d \Delta \psi + k_3(T) \frac{C}{k_3 + C} \psi - k_5(T) A^m \psi + k_5(T) \varphi - k_4(T) \psi, \\ \frac{\partial \varphi}{\partial t} = d \Delta \varphi + k_3(T) \frac{C}{k_3 + C} \varphi + k_5(T) A^m \psi - k_5(T) \psi - k_4(T) \varphi, \\ \frac{\partial R}{\partial t} = d_R \Delta R - k_R(T) R + k_3(T) \frac{RA}{k_R + RA} - k_3(T) \frac{\psi + \varphi}{k_2 + \psi + \varphi} R, \\ \frac{\partial T}{\partial t} = k \Delta T + q_0 (\psi + \varphi). \end{array} \right. \quad (3.10)$$

Initial and boundary conditions

The system is supplemented with the following boundary conditions

$$\nabla \psi \cdot n = \nabla \varphi \cdot n = \nabla A \cdot n = \nabla R \cdot n = \nabla T \cdot n = 0, \quad (x, t) \in \partial \Omega \times (0, T), \quad (3.11)$$

$$\nabla C \cdot n = 0, \quad (x, t) \in \Gamma_N \times (0, T), \quad (3.12)$$

$$C(x, t) = C_D, \quad (x, t) \in \Gamma_D \times (0, T). \quad (3.13)$$

And initial conditions

$$\begin{aligned} C(x, 0) &= C_0, \quad \psi(x, 0) = \psi_0, \quad \varphi(x, 0) = \varphi_0, \\ A(x, 0) &= A_0, \quad R(x, 0) = R_0, \quad T(x, 0) = T_0, \quad x \in \Omega. \end{aligned} \quad (3.14)$$

3.3 The well-posedness of the problem

Assumptions

In what follows, we adopt the assumptions below.

(H_1) The functions $C(x, t)$, $A(x, t)$, $\psi(x, t)$, $\varphi(x, t)$, $R(x, t)$, and $T(x, t)$ are bounded. More precisely, there exists a constant $M > 0$ such that

$$|C(x, t)|, |A(x, t)|, |\psi(x, t)|, |\varphi(x, t)|, |R(x, t)|, |T(x, t)| < M.$$

(H_2) The functions $k_1(T)$, $k_2(T)$, $k_3(T)$, $k_4(T)$, $k_5(T)$, $\alpha(T)$, $\beta(T)$, and $\gamma(T)$ are continuously differentiable; in particular, they are Lipschitz continuous on bounded intervals.

In this section, we establish existence and uniqueness for problem (3.10). Define the vector-valued unknown $\Phi = (C, A, \psi, \varphi, R, T)^\top$. Then the system (3.10) can be recast in the abstract

form

$$\begin{cases} \Phi_t = A\Phi + F(\Phi) & ; \quad t > 0 \\ \Phi(0) = \Phi_0 = (C_0, A_0, \psi_0, \varphi_0, R_0, T_0)^\top & , \end{cases} \quad (3.15)$$

where

$$A\Phi = \begin{pmatrix} d_C \Delta C \\ d_2 \Delta A \\ d \Delta \psi \\ d \Delta \varphi \\ d_R \Delta R \\ k \Delta T \end{pmatrix},$$

and

$$F(\Phi) = \begin{pmatrix} F_1(\Phi) \\ F_2(\Phi) \\ F_3(\Phi) \\ F_4(\Phi) \\ F_5(\Phi) \\ F_6(\Phi) \end{pmatrix} = \begin{pmatrix} -k_1(T) \frac{C}{k_2+C} (\psi + \varphi) \\ -\gamma(T) A + \alpha(T) \frac{RA}{K_L+RA} \psi + (\alpha(T) + \beta(T)) \frac{RA}{K_L+RA} \varphi \\ k_3(T) \frac{C}{k_3+S} \psi - k_5(T) A^m \psi + k_5(T) \varphi - k_4(T) \psi \\ k_3(T) \frac{C}{k_3+C} \varphi + k_5(T) A^m \psi - k_5(T) \psi - k_4(T) \varphi \\ -k_R(T) R + k_3(T) \frac{RA}{k_R+RA} - k_3(T) \frac{\psi+\varphi}{k_2+\psi+\varphi} R \\ q_0(\psi + \varphi) \end{pmatrix}.$$

Accordingly, the linear operator $A : D(A) \subset H \rightarrow H$ is given by

$$A = \begin{pmatrix} d_C \Delta & 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 \Delta & 0 & 0 & 0 & 0 \\ 0 & 0 & d \Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & d \Delta & 0 & 0 \\ 0 & 0 & 0 & 0 & d_R \Delta & 0 \\ 0 & 0 & 0 & 0 & 0 & k \Delta \end{pmatrix},$$

where the Hilbert space H is defined by

$$H = H_*^1(\Omega) \times H_*^1(\Omega) \times H_*^1(\Omega) \times H_*^1(\Omega) \times H_*^1(\Omega) \times H_*^1(\Omega).$$

Moreover,

$$\begin{aligned} H_*^1(\Omega) &= \{w \in H^1(\Omega) : \nabla w \cdot n|_{\partial\Omega} = 0\} \\ L^2(\Omega) &= \{w \in L^2(\Omega)\} \\ X &= L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \\ H_*^2(\Omega) &= \{w \in H^2(\Omega) : \nabla w \cdot n|_{\partial\Omega} = 0\} \end{aligned}$$

We endow H with the inner product: for any $\Phi = (C, A, \psi, \varphi, R, T)^\top \in H$ and $\tilde{\Phi} =$

$$(\tilde{C}, \tilde{A}, \tilde{\psi}, \tilde{\varphi}, \tilde{R}, \tilde{T})^\top \in H,$$

$$\langle \Phi, \tilde{\Phi} \rangle = \int_{\Omega} C \tilde{C} dx + \int_{\Omega} A \tilde{A} dx + \int_{\Omega} \psi \tilde{\psi} dx + \int_{\Omega} \varphi \tilde{\varphi} dx + \int_{\Omega} R \tilde{R} dx + \int_{\Omega} T \tilde{T} dx.$$

and the induced norm $\|\cdot\|_H$ is equivalent to the standard one.

Finally, the domain of A is

$$D(A) = \left\{ \begin{array}{l} \Phi \in H; \ C \in H_*^1(\Omega) \cap H_*^2(\Omega), \ A \in H_*^1(\Omega) \cap H_*^2(\Omega), \ \psi \in H_*^1(\Omega) \cap H_*^2(\Omega), \\ \varphi \in H_*^1(\Omega) \cap H_*^2(\Omega), \ R \in H_*^1(\Omega) \cap H_*^2(\Omega), \ T \in H_*^1(\Omega) \cap H_*^2(\Omega) \end{array} \right\}.$$

3.3.1 Existence of Solutions

Clearly, $D(A)$ is dense in H .

Moreover, we compute

$$\begin{aligned} \langle A\Phi, \Phi \rangle &= -d_C \int_{\Omega} |\nabla C|^2 dx - d_2 \int_{\Omega} |\nabla A|^2 dx - d \int_{\Omega} |\nabla \psi|^2 dx - d \int_{\Omega} |\nabla \varphi|^2 dx \\ &\quad - d_R \int_{\Omega} |\nabla R|^2 dx - k \int_{\Omega} |\nabla T|^2 dx \leq 0. \end{aligned} \quad (3.16)$$

Hence, the operator A is dissipative.

Next, we show that $(I - A)$ is surjective. Let $f = (f_1, f_2, f_3, f_4, f_5, f_6)^\top \in H$. We prove that there exists a unique $\Phi \in D(A)$ satisfying

$$(I - A)\Phi = f,$$

that is,

$$\left\{ \begin{array}{l} C - d_C \Delta C = f_1 \\ A - d_A \Delta A = f_2 \\ \psi - d \Delta \psi = f_3 \\ \varphi - d \Delta \varphi = f_4 \\ R - d_R \Delta R = f_5 \\ T - k \Delta T = f_6 \end{array} \right. \quad (3.17)$$

We now work on $V \times V$, where

$$V = H_*^1(\Omega) \times H_*^1(\Omega) \times H_*^1(\Omega) \times H_*^1(\Omega) \times H_*^1(\Omega) \times H_*^1(\Omega).$$

Define the bilinear form

$$\begin{aligned}
b((C, A, \psi, \varphi, R, T); (C^1, A^1, \psi^1, \varphi^1, R^1, T^1)) &= \int_{\Omega} C C^1 dx + d_C \int_{\Omega} \nabla C \cdot \nabla C^1 dx \\
&+ \int_{\Omega} A A^1 dx + d_2 \int_{\Omega} \nabla A \cdot \nabla A^1 dx \\
&+ \int_{\Omega} \psi \psi^1 dx + d \int_{\Omega} \nabla \psi \cdot \nabla \psi^1 dx \\
&+ \int_{\Omega} \varphi \varphi^1 dx + d \int_{\Omega} \nabla \varphi \cdot \nabla \varphi^1 dx \\
&+ \int_{\Omega} R R^1 dx + d_R \int_{\Omega} \nabla R \cdot \nabla R^1 dx \\
&+ \int_{\Omega} T T^1 dx + k \int_{\Omega} \nabla T \cdot \nabla T^1 dx,
\end{aligned}$$

and the linear functional

$$\begin{aligned}
l(C^1, A^1, \psi^1, \varphi^1, R^1, T^1) &= \int_{\Omega} f_1 C^1 dx + \int_{\Omega} f_2 A^1 dx + \int_{\Omega} f_3 \psi^1 dx \\
&+ \int_{\Omega} f_4 \varphi^1 dx + \int_{\Omega} f_5 R^1 dx + \int_{\Omega} f_6 T^1 dx. \tag{3.18}
\end{aligned}$$

We equip V with the norm

$$\begin{aligned}
\|(C, A, \psi, \varphi, R, T)\|_V^2 &= \|C\|_2^2 + \|\nabla C\|_2^2 + \|A\|_2^2 + \|\nabla A\|_2^2 \\
&+ \|\psi\|_2^2 + \|\nabla \psi\|_2^2 + \|\varphi\|_2^2 + \|\nabla \varphi\|_2^2 \\
&+ \|R\|_2^2 + \|\nabla R\|_2^2 + \|T\|_2^2 + \|\nabla T\|_2^2.
\end{aligned}$$

By the Cauchy–Schwarz inequality, there exists $M_2 > 0$ such that

$$|b((C, A, \psi, \varphi, R, T); (C^1, A^1, \psi^1, \varphi^1, R^1, T^1))| \leq M_2 \|(C, A, \psi, \varphi, R, T)\|_V \|(C^1, A^1, \psi^1, \varphi^1, R^1, T^1)\|_V.$$

Similarly, by Cauchy–Schwarz there exists $M_3 > 0$ such that

$$|l(C^1, A^1, \psi^1, \varphi^1, R^1, T^1)| \leq M_3 \|(C^1, A^1, \psi^1, \varphi^1, R^1, T^1)\|_V. \tag{3.19}$$

Therefore, b and l are continuous.

Moreover, for any $(C, A, \psi, \varphi, R, T) \in V$,

$$\begin{aligned}
b((C, A, \psi, \varphi, R, T); (C, A, \psi, \varphi, R, T)) &= \int_{\Omega} C^2 dx + d_C \int_{\Omega} |\nabla C|^2 dx \\
&+ \int_{\Omega} A^2 dx + d_2 \int_{\Omega} |\nabla A|^2 dx \\
&+ \int_{\Omega} \psi^2 dx + d \int_{\Omega} |\nabla \psi|^2 dx \\
&+ \int_{\Omega} \varphi^2 dx + d \int_{\Omega} |\nabla \varphi|^2 dx \\
&+ \int_{\Omega} R^2 dx + d_R \int_{\Omega} |\nabla R|^2 dx \\
&+ \int_{\Omega} T^2 dx + k \int_{\Omega} |\nabla T|^2 dx.
\end{aligned}$$

In particular, there exists $M_4 > 0$ such that

$$b((C, A, \psi, \varphi, R, T); (C, A, \psi, \varphi, R, T)) \geq M_4 \|(C, A, \psi, \varphi, R, T)\|_V^2, \quad M_4 > 0.$$

Hence, b is coercive.

Therefore, by the Lax–Milgram theorem, there exists a unique solution $\Phi \in V$ such that

$$\begin{aligned}
b((C, A, \psi, \varphi, R, T); (C^1, A^1, \psi^1, \varphi^1, R^1, T^1)) &= l(C^1, A^1, \psi^1, \varphi^1, R^1, T^1), \\
&\forall (C^1, A^1, \psi^1, \varphi^1, R^1, T^1) \in V.
\end{aligned}$$

Regularity. Once the weak solution $\Phi \in V$ is obtained, we invoke elliptic regularity for systems. The operator $I - A$ is elliptic, and we have $f \in X$. Hence, by standard elliptic regularity results, $\Phi \in [H_*^2(\Omega)]^6$, provided that Ω is of class $C(\Omega)^2$ and the boundary conditions are sufficiently smooth. In particular, there exists a constant $c_1 > 0$ such that

$$\|\Phi\|_{[H_*^2(\Omega)]^6} \leq c_1 \|f\|_X,$$

where c_1 depends on Ω and the coefficients d_C, d_2, d, d_R , and k .

Consequently, $(I - A)$ is surjective. By the Lumer–Phillips theorem, it follows that A is the infinitesimal generator of a linear C_0 -semigroup on H .

We next show that F is Lipschitz continuous. Specifically, we prove the existence of a constant $L > 0$ such that for any $\zeta, \chi \in X$, where $\zeta = (C, A, \psi, \varphi, R, T)$ and $\chi =$

$$(\tilde{C}, \tilde{A}, \tilde{\psi}, \tilde{\varphi}, \tilde{R}, \tilde{T}),$$

$$\|F(\zeta) - F(\chi)\|_X \leq L \|\zeta - \chi\|_X.$$

We write $F(\zeta) - F(\chi)$ componentwise as follows:

$$F_1(\zeta) - F_1(\chi) = - \left[k_1(T) \frac{C}{k_2 + C} (\psi + \varphi) - k_1(\tilde{T}) \frac{\tilde{C}}{k_2 + \tilde{C}} (\tilde{\psi} + \tilde{\varphi}) \right],$$

$$\begin{aligned} F_2(\zeta) - F_2(\chi) &= - \left[\gamma(T) A - \gamma(\tilde{T}) \tilde{A} \right] + \left[\alpha(T) \frac{RA}{K_L + RA} \psi - \alpha(\tilde{T}) \frac{\tilde{R}\tilde{A}}{K_L + \tilde{R}\tilde{A}} \tilde{\psi} \right] \\ &\quad + \left[(\alpha(T) + \beta(T)) \frac{RA}{K_L + RA} \varphi - (\alpha(\tilde{T}) + \beta(\tilde{T})) \frac{\tilde{R}\tilde{A}}{K_L + \tilde{R}\tilde{A}} \tilde{\varphi} \right], \end{aligned}$$

$$\begin{aligned} F_3(\zeta) - F_3(\chi) &= k_3(T) \frac{C}{k_3 + C} \psi - k_3(\tilde{T}) \frac{\tilde{C}}{k_3 + \tilde{C}} \tilde{\psi} + \left[k_5(T) A^m \psi - k_5(\tilde{T}) \tilde{A}^m \tilde{\psi} \right] \\ &\quad + \left[k_5(T) \varphi - k_5(\tilde{T}) \tilde{\varphi} \right] - \left[k_4(T) \psi - k_4(\tilde{T}) \tilde{\psi} \right], \end{aligned}$$

$$\begin{aligned} F_4(\zeta) - F_4(\chi) &= k_3(T) \frac{C}{k_3 + C} \varphi - k_3(\tilde{T}) \frac{\tilde{C}}{k_3 + \tilde{C}} \tilde{\varphi} + \left[k_5(T) A^m \psi - k_5(\tilde{T}) \tilde{A}^m \tilde{\psi} \right] \\ &\quad - \left[k_5(T) \psi - k_5(\tilde{T}) \tilde{\psi} \right] + \left[k_4(T) \varphi - k_4(\tilde{T}) \tilde{\varphi} \right], \end{aligned}$$

$$\begin{aligned} F_5(\zeta) - F_5(\chi) &= - \left[k_R(T) R - k_R(\tilde{T}) \tilde{R} \right] + \left[k_3(T) \frac{RA}{k_R + RA} - k_3(\tilde{T}) \frac{\tilde{R}\tilde{A}}{k_R + \tilde{R}\tilde{A}} \right] \\ &\quad - \left[k_3(T) \frac{\psi + \varphi}{k_2 + \psi + \varphi} R - k_3(\tilde{T}) \frac{\tilde{\psi} + \tilde{\varphi}}{k_2 + \tilde{\psi} + \tilde{\varphi}} \tilde{R} \right], \end{aligned}$$

$$F_6(\zeta) - F_6(\chi) = Q_0 \left[\psi + \varphi - \tilde{\psi} - \tilde{\varphi} \right].$$

For $F_1(\zeta) - F_1(\chi)$, we estimate

$$\begin{aligned} |F_1(\zeta) - F_1(\chi)| &\leq \left| k_1(T) - k_1(\tilde{T}) \right| \left| \frac{C}{k_2 + C} (\psi + \varphi) \right| \\ &\quad + \left| k_1(\tilde{T}) \right| \left| \frac{C}{k_2 + C} (\psi + \varphi) - \frac{\tilde{C}}{k_2 + \tilde{C}} (\tilde{\psi} + \tilde{\varphi}) \right|. \end{aligned}$$

Since k_1 is Lipschitz on bounded sets, there exists $L_{k_1} > 0$ such that

$$\left| k_1(T) - k_1(\tilde{T}) \right| \leq L_{k_1} |T - \tilde{T}|.$$

Moreover, on bounded sets, the mapping $g(C) = \frac{C}{k_2 + C}$ is of class C^1 . Hence, it is Lipschitz continuous and bounded. Therefore, there exists $L_1 > 0$ such that

$$\left| \frac{C}{k_2 + C}(\psi + \varphi) - \frac{\tilde{C}}{k_2 + \tilde{C}}(\tilde{\psi} + \tilde{\varphi}) \right| \leq L_1 \left(|C - \tilde{C}| + |\psi - \tilde{\psi}| + |\varphi - \tilde{\varphi}| \right).$$

Since $\frac{C}{k_2 + C}(\psi + \varphi)$ is bounded by a constant C_1 on the bounded set under consideration, we obtain

$$\begin{aligned} |F_1(\zeta) - F_1(\chi)| &\leq L_{k_1} C_1 |T - \tilde{T}| + L_1 |k_1(\tilde{T})| \left(|C - \tilde{C}| + |\psi - \tilde{\psi}| + |\varphi - \tilde{\varphi}| \right) \\ &\leq L_{f_1} \left(|T - \tilde{T}| + |C - \tilde{C}| + |\psi - \tilde{\psi}| + |\varphi - \tilde{\varphi}| \right), \end{aligned} \quad (3.20)$$

for some $L_{f_1} > 0$.

Proceeding analogously, we obtain the estimates

$$|F_2(\zeta) - F_2(\chi)| \leq L_{f_2} \left(|T - \tilde{T}| + \|A - \tilde{A}\|_{L^\infty} + \|\psi - \tilde{\psi}\|_{L^\infty} + \|\varphi - \tilde{\varphi}\|_{L^\infty} \right), \quad (3.21)$$

where $L_{f_2} > 0$ depends on the Lipschitz constants of $\alpha(T)$, $\beta(T)$, and $\gamma(T)$, as well as on the term $\frac{RA}{K_L + RA}$ on bounded sets. Similarly,

$$|F_3(\zeta) - F_3(\chi)| \leq L_{f_3} \left(|T - \tilde{T}| + \|A - \tilde{A}\|_{L^\infty} + \|\psi - \tilde{\psi}\|_{L^\infty} + \|\varphi - \tilde{\varphi}\|_{L^\infty} \right), \quad (3.22)$$

$$|F_4(\zeta) - F_4(\chi)| \leq L_{f_4} \left(|T - \tilde{T}| + \|A - \tilde{A}\|_{L^\infty} + \|\psi - \tilde{\psi}\|_{L^\infty} + \|\varphi - \tilde{\varphi}\|_{L^\infty} \right), \quad (3.23)$$

$$|F_5(\zeta) - F_5(\chi)| \leq L_{f_5} \left(|T - \tilde{T}| + \|A - \tilde{A}\|_{L^\infty} + \|\psi - \tilde{\psi}\|_{L^\infty} + \|\varphi - \tilde{\varphi}\|_{L^\infty} \right), \quad (3.24)$$

and finally,

$$|F_6(\zeta) - F_6(\chi)| \leq L_{f_6} \left(\|\psi - \tilde{\psi}\|_{L^\infty} + \|\varphi - \tilde{\varphi}\|_{L^\infty} \right), \quad (3.25)$$

for some $L_{f_6} > 0$.

Combining (3.20)–(3.25), we deduce that there exists $L > 0$ such that

$$\|F(\zeta) - F(\chi)\|_X \leq L \|\zeta - \chi\|_X, \quad (3.26)$$

where L depends on the Lipschitz constants of $\alpha(T)$, $\beta(T)$, $\gamma(T)$, $k_3(T)$, $k_4(T)$, $k_5(T)$, $k_R(T)$, and on the bounded terms A^m , $\frac{C}{k_2+C}$, and $\frac{RA}{K_L+RA}$. Hence, F is Lipschitz on X . and Since we have already proved that F is Lipschitz on X hence continuous, there exists $L > 0$ such that, for any sequence $t_n \rightarrow t_0$, and for any function $v \in C([0, \bar{T}]; X)$

$$\|F(v(t_n)) - F(v(t_0))\|_X \leq L \|v(t_n) - v(t_0)\|_X \rightarrow 0.$$

Therefore,

$$F(v(t_n)) \rightarrow F(v(t_0)) \quad \text{in } X,$$

which proves that the mapping $t \mapsto F(v(t))$ is continuous on t . Hence,

$$F(v) \in C([0, \bar{T}]; X).$$

As a consequence, problem (3.10)–(3.11) admits a unique mild solution Φ of the form

$$\Phi(t) = U(t)\Phi_0 + \int_0^t U(t-y)F(\Phi(y))dy,$$

where $U(t)$ is the C_0 -semigroup generated by A .

We now address the well-posedness of the integral term by a fixed-point argument. Define the operator G by

$$G(\delta(t)) = U(t)\Phi_0 + \int_0^{\bar{T}} U(t-y)F(\delta(y))dy.$$

We will show that G maps a suitable function space into itself.

Consider the space $C([0, \bar{T}]; X)$ defined by

$$C([0, \bar{T}], X) = \{v : [0, \bar{T}] \rightarrow X \text{ such that } v(t) \text{ is continuous in } t\}.$$

We prove that if $v \in C([0, \bar{T}], X)$, then $G(v) \in C([0, \bar{T}], X)$.

Boundedness of $U(t)$. The C_0 -semigroup $U(t)$ satisfies

$$\|U(t)\|_{L_*^2(\Omega) \rightarrow L_*^2(\Omega)} \leq M e^{wt}, \quad M > 0.$$

Hence, $U(t)$ is bounded for $t > 0$. Since F is Lipschitz on X , we also have the growth estimate below.

Boundedness of $G(v)$. For $v \in C([0, \bar{T}], X)$, we have

$$\|U(t)\Phi_0\|_X \leq M e^{wt} \|\Phi_0\|_X.$$

Moreover,

$$\left\| \int_0^t U(t-y) F(\delta(y)) dy \right\|_X \leq \int_0^t \|U(t-y)\|_X \|F(\delta(y))\|_X dy,$$

and, since

$$\|F(\delta(y))\|_X \leq C(1 + \|\delta(y)\|_X),$$

we obtain

$$\|U(t-y)F(\delta(y))\|_X \leq M e^{w(t-y)} (1 + \|\delta(y)\|_X).$$

Consequently,

$$\left\| \int_0^{\bar{T}} U(t-y) F(\delta(y)) dy \right\|_X \leq MC \int_0^{\bar{T}} e^{w(t-y)} (1 + \|\delta(y)\|_X) dy.$$

Thus, $G(\delta)$ is bounded in X . Since $U(t)\Phi_0$ is continuous in t (by strong continuity of $U(t)$) and the integral term is continuous (by continuity of δ and F), we conclude that $G(\delta) \in C([0, \bar{T}], X)$.

Next, we show that G is a contraction. Let $\delta_1, \delta_2 \in C([0, \bar{T}], X)$. We aim to prove that

$$\|G(\delta_1)(t) - G(\delta_2)(t)\|_X \leq ML \|\delta_1 - \delta_2\|_{C([0, \bar{T}], X)}. \quad (3.27)$$

Indeed,

$$\begin{aligned} \|G(\delta_1)(t) - G(\delta_2)(t)\|_X &\leq \int_0^{\bar{T}} \|U(t-y)\| \|F(\delta_1(y)) - F(\delta_2(y))\|_X dy \\ &\leq M \int_0^{\bar{T}} e^{w(t-y)} \|F(\delta_1(y)) - F(\delta_2(y))\|_X dy. \end{aligned}$$

However, since F is Lipschitz, we find:

$$\begin{aligned} \|G(\delta_1)(t) - G(\delta_2)(t)\|_X &\leq ML \int_0^{\bar{T}} e^{w(t-y)} \|\delta_1(y) - \delta_2(y)\|_X dy \\ &\leq ML \|\delta_1 - \delta_2\|_{C([0, \bar{T}], X)} \int_0^{\bar{T}} e^{w(t-y)} dy, \end{aligned}$$

where

$$\int_0^{\bar{T}} e^{w(t-y)} dy = \begin{cases} \int_0^{\bar{T}} 1 dy = \bar{T}, & \text{if } w = 0 \\ \frac{1 - e^{wt}}{w}, & \text{if } w \neq 0 \end{cases},$$

Thus,

$$\|G(\delta_1)(t) - G(\delta_2)(t)\|_X \leq ML \|\delta_1 - \delta_2\|_{C([0, \bar{T}], X)}.$$

For a sufficiently small $\bar{T} > 0$ such that $ML\bar{T} < 1$, G is a contraction, ensuring a unique fixed point by the Banach Fixed-Point Theorem.

Remark 3.1. Let $\bar{T} > 0$ be the small time for which the contraction $ML \cdot \bar{T} < 1$ holds. Denote the unique solution on $[0, \bar{T}]$ as $\Phi(t)$. Since $\Phi(\bar{T})$ lies in the same Banach space, we can restart the contraction argument on $[\bar{T}, 2\bar{T}]$ with initial data $\Phi(\bar{T})$. By iterating this stepwise continuation, the solution extends uniquely to any finite time interval $[0, T_{\text{final}}]$.

3.3.2 Regularity of the solution:

If the initial condition $\Phi_0 \in D(A)$, the solution satisfies:

$$\frac{d\Phi}{dt} \in X; \quad \Phi(t) \in D(A).$$

This makes $\Phi(t)$ a strong solution.

If $\Phi_0 \in H^k(\Omega)$ and F is smooth, the solution $\Phi(t)$ inherits higher regularity:

$$\Phi(t) \in H^k(\Omega), \quad \text{for } t > 0.$$

3.3.3 Uniqueness

Uniqueness of the solution follows directly from the contraction mapping argument.

If there were two solutions $\Phi_1(t)$ and $\Phi_2(t)$ that satisfy:

$$\frac{d}{dt} (\Phi_1 - \Phi_2) = A(\Phi_1 - \Phi_2) + F(\Phi_1) - F(\Phi_2),$$

Using the Lipschitz property of F , we get:

$$\|\Phi_1(t) - \Phi_2(t)\|_X \leq L\bar{T} \|\Phi_1 - \Phi_2\|_{C([0,\bar{T}],X)},$$

where $\bar{T}L < 1$, hence $\Phi_1(t) = \Phi_2(t)$.

This completes the proof of existence, uniqueness, and regularity of the solution to (3.10)-(3.11).

3.4 Numerical Approximation

In this section, we present a numerical approximation of the nonlinear system of partial differential equations introduced in the previous section. The spatial discretization is carried out using finite elements, while the temporal discretization is performed by means of an implicit Euler scheme.

We begin with the weak formulation. To this end, we multiply equations (3.10)₁, (3.10)₂, (3.10)₃, (3.10)₄, (3.10)₅, and (3.10)₆ by $v^C, v^A, v^\psi, v^\varphi, v^R$, and v^T , respectively. Then, integrating over Ω and applying integration by parts, we obtain the following weak formulation:

Find $(C, A, \psi, \varphi, R, T) \in (H_*^1(\Omega))^6$, such that $\forall (v^C, v^A, v^\psi, v^\varphi, v^R, v^T) \in (H_*^1(\Omega))^6$,

$$\left\{ \begin{array}{l} \int_{\Omega} \frac{\partial C}{\partial t} v^C dx + d_C \int_{\Omega} \nabla C \cdot \nabla v^C dx = \int_{\Omega} k_1(T) g(C, \psi + \varphi) v^C dx \\ \int_{\Omega} \frac{\partial A}{\partial t} v^A dx + d_A \int_{\Omega} \nabla A \cdot \nabla v^A dx + \int_{\Omega} \gamma(T) A v^A dx = \int_{\Omega} \alpha(T) \frac{RA}{K_L + RA} \psi v^A dx \\ \quad + \int_{\Omega} (\alpha(T) + \beta(T)) \frac{RA}{K_L + RA} \varphi v^A dx \\ \int_{\Omega} \frac{\partial \psi}{\partial t} v^\psi dx + d \int_{\Omega} \nabla \psi \cdot \nabla v^\psi dx + \int_{\Omega} k_5(T) |A|^m \psi v^\psi dx = \int_{\Omega} (k_3(T) g(C, \psi) + k_5(T) \varphi) v^\psi dx \\ \int_{\Omega} \frac{\partial \varphi}{\partial t} v^\varphi dx + d \int_{\Omega} \nabla \varphi \cdot \nabla v^\varphi dx + \int_{\Omega} (k_4(T) + k_5(T)) \varphi v^\varphi dx = \int_{\Omega} (k_3(T) g(C, \varphi) + k_5(T) |A|^m \psi) v^\varphi dx \\ \int_{\Omega} \frac{\partial R}{\partial t} v^R dx + d_R \int_{\Omega} \nabla R \cdot \nabla v^R dx + \int_{\Omega} k_R(T) R v^R dx + \int_{\Omega} k_3(T) g(\psi + \varphi, R) v^R dx \\ \quad = \int_{\Omega} k_3(T) \frac{RA}{k_R + RA} v^R dx \\ \int_{\Omega} \frac{\partial T}{\partial t} v^T dx + k \int_{\Omega} \nabla T \cdot \nabla v^T dx = \int_{\Omega} q_0(\psi + \varphi) v^T dx \end{array} \right. \quad (3.28)$$

Next, let $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = \bar{T}$ be a partition of $[0, \bar{T}]$ into variable time steps $\tau_n = t_n - t_{n-1}$, for $n = 1, \dots, N$. We set $\tau = \max_{1 \leq n \leq N} \tau_n$. We introduce $\mathcal{T}_h$ as a shape-regular family of triangulations of Ω . For a triangle $K \in \mathcal{T}_h$, let h_K denote its diameter, and set $h = \max_{K \in \mathcal{T}_h} h_K$.

We define $P^1(\mathcal{T}_h)$ as the space of continuous, piecewise linear finite elements:

$$P^1(\mathcal{T}_h) = \{v_h \in C^0(\bar{\Omega}) : \forall K \in \mathcal{T}_h, v_h|_K \in P^1(K)\},$$

$$P_{1,C_D}(\mathcal{T}_h) = \{v_h \in P^1(\mathcal{T}_h) : v_h|_{\Gamma_D} = C_D\},$$

and the non-negative subsets:

$$w_h = \{v_h \in P^1(\mathcal{T}_h) : v_h \geq 0\}, \quad w_h^{C_D} = w_h \cap P_{1,C_D}(\mathcal{T}_h).$$

We consider the problem as follows: For $(C_h^n, A_h^n, \psi_h^n, \varphi_h^n, R_h^n, T_h^n) \in P^1(\lambda_h) \times P_{C_D}^1(\lambda_h) \times P^1(\lambda_h) \times P^1(\lambda_h) \times P^1(\lambda_h) \times P^1(\lambda_h)$, we seek

$$\begin{aligned} (C_h^{n+1}, A_h^{n+1}, \psi_h^{n+1}, \varphi_h^{n+1}, R_h^{n+1}, T_h^{n+1}) &\in P_{C_D}^1(\mathcal{T}_h) \times P^1(\mathcal{T}_h) \\ &\times P^1(\mathcal{T}_h) \times P^1(\mathcal{T}_h) \times P^1(\mathcal{T}_h) \times P^1(\mathcal{T}_h) \end{aligned}$$

such that, for all $(v_h^C, v_h^A, v_h^\psi, v_h^\varphi, v_h^R, v_h^T) \in P_0^1(\mathcal{T}_h) \times P^1(\mathcal{T}_h) \times P^1(\mathcal{T}_h) \times P^1(\mathcal{T}_h) \times P^1(\mathcal{T}_h) \times P^1(\mathcal{T}_h)$,

$$\left\{ \begin{aligned} &\int_{\Omega} \frac{(C_h^{n+1} - C_h^n)}{\tau_n} v_h^C dx + \int_{\Omega} [d_C \nabla C_h^{n+1} \cdot \nabla v_h^C + k_1(T_h^n) g(C_h^{n+1}, \psi_h^n + \varphi_h^n) v_h^C] dx = 0 \\ &\int_{\Omega} \frac{(A_h^{n+1} - A_h^n)}{\tau_n} v_h^A dx + \int_{\Omega} [d_A \nabla A_h^{n+1} \cdot \nabla v_h^A + \gamma(T_h^n) A_h^{n+1} v_h^A] dx = \int_{\Omega} \alpha(T_h^n) \frac{R_h^n A_h^{n+1}}{K_L + R_h^n A_h^{n+1}} \psi_h^n v_h^A dx \\ &\quad + \int_{\Omega} (\alpha(T_h^n) + \beta(T_h^n)) \frac{R_h^n A_h^{n+1}}{K_L + R_h^n A_h^{n+1}} \varphi_h^n v_h^A dx \\ &\int_{\Omega} \frac{(\psi_h^{n+1} - \psi_h^n)}{\tau_n} v_h^\psi dx + \int_{\Omega} [d \nabla \psi_h^{n+1} \cdot \nabla v_h^\psi + k_5(T_h^n) |A_h^{n+1}|^m \psi_h^{n+1} v_h^\psi] dx \\ &\quad = \int_{\Omega} (k_3(T_h^n) g(C_h^{n+1}, \psi_h^n) + k_5(T_h^n) \varphi_h^n) v_h^\psi dx \\ &\int_{\Omega} \frac{(\varphi_h^{n+1} - \varphi_h^n)}{\tau_n} v_h^\varphi dx + \int_{\Omega} [d \nabla \varphi_h^{n+1} \cdot \nabla v_h^\varphi + (k_4(T_h^n) + k_5(T_h^n)) \varphi_h^{n+1} v_h^\varphi] dx \\ &\quad = \int_{\Omega} (k_3(T_h^n) g(C_h^{n+1}, \varphi_h^{n+1}) + k_5(T_h^n) |A_h^{n+1}|^m \psi_h^{n+1}) v_h^\varphi dx \\ &\int_{\Omega} \frac{(R_h^{n+1} - R_h^n)}{\tau_n} v_h^R dx + \int_{\Omega} [d_R \nabla R_h^{n+1} \cdot \nabla v_h^R + k_R(T_h^n) R_h^{n+1} v_h^R + k_3(T_h^n) g(\psi_h^{n+1} + \varphi_h^{n+1}, R_h^{n+1}) v_h^R] dx \\ &\quad = \int_{\Omega} k_3(T_h^n) \frac{R_h^{n+1} A_h^{n+1}}{k_R + R_h^{n+1} A_h^{n+1}} v_h^R dx \\ &\int_{\Omega} \frac{(T_h^{n+1} - T_h^n)}{\tau_n} v_h^T dx + \int_{\Omega} [k \nabla T_h^{n+1} \cdot \nabla v_h^T] dx = \int_{\Omega} q_0(\psi_h^{n+1} + \varphi_h^{n+1}) v_h^T dx \end{aligned} \right. , \quad (3.29)$$

3.4.1 Existence and Uniqueness

Theorem 3.4.1. Let

$$(C_h^n, A_h^n, \psi_h^n, \varphi_h^n, R_h^n, T_h^n) \in w_h^{C_D} \times w_h \times w_h \times w_h \times w_h \times w_h.$$

Then, for every $h > 0$ and every $\tau_n > 0$, there exists a unique solution

$$(C_h^{n+1}, A_h^{n+1}, \psi_h^{n+1}, \varphi_h^{n+1}, R_h^{n+1}, T_h^{n+1}) \in w_h^{C_D} \times w_h \times w_h \times w_h \times w_h \times w_h.$$

of 3.29.

Proof. Let $\{\gamma_i\}_{1 \leq i \leq N}$ be the standard basis of $P_0^1(\mathcal{T}_h)$. For any

$$w = \sum_{j=1}^N w_j \gamma_j \in P_0^1(\mathcal{T}_h),$$

we associate the vector

$$\tilde{w} = (w_1, \dots, w_N)^T \in \mathbb{R}^N.$$

In the spirit of [6], the proof relies on rewriting each discrete equation in terms of an operator of the form

$$F(\tilde{w}) = A\tilde{w} + \phi(\tilde{w}),$$

where A is an M -matrix and ϕ is a continuous isotone diagonal mapping. It then follows from the classical theory of monotone mappings that such an operator is an inverse isotone homeomorphism from $\mathbb{R}^N$ into itself.

For the nutrient concentration C , we introduce the mapping

$$F_C^n : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$$

defined componentwise by

$$[F_C^n(\tilde{w}, \tilde{z})]_j = (w, \gamma_j) + \tau_n d_C(\nabla w, \nabla \gamma_j) + \tau_n (k_1(T_h^n)g(w, z), \gamma_j), \quad j = 1, \dots, N.$$

For given ψ_h^n and φ_h^n , the first discrete equation of (P_h) may be written as

$$F_C^n(\tilde{C}^{n+1}, \tilde{\psi}^n + \tilde{\varphi}^n) = \tilde{G}_C^n,$$

where

$$[\tilde{G}_C^n]_j = (C^n, \gamma_j).$$

For every fixed $z \in \mathbb{R}^N$, the mapping $F_C^n(\cdot, z)$ can be decomposed as

$$F_C^n(\tilde{w}, z) = A_C \tilde{w} + \phi_C(\tilde{w}),$$

with A_C an M -matrix and ϕ_C a continuous isotone diagonal mapping. Hence $F_C^n(\cdot, z)$ is an inverse isotone homeomorphism. Therefore, the first equation admits a unique solution C_h^{n+1} .

We proceed analogously for the AHL concentration A . For fixed R_h^n , ψ_h^n , and φ_h^n , the corresponding discrete equation may be recast in the form

$$F_A^n(\tilde{A}) = \tilde{G}_A^n,$$

where F_A^n can be expressed as

$$F_A^n(\tilde{A}) = A_A \tilde{A} + \phi_A(\tilde{A}),$$

with A_A an M -matrix and ϕ_A a continuous isotone diagonal mapping. Consequently, F_A^n is an inverse isotone homeomorphism, and the existence and uniqueness of A_h^{n+1} follow.

The same argument applies to the equation satisfied by ψ_h^{n+1} . For fixed values of C_h^{n+1} , φ_h^n , and A_h^{n+1} , one defines an operator

$$F_\psi^n(\tilde{\psi}) = A_\psi \tilde{\psi} + \phi_\psi(\tilde{\psi}),$$

where the matrix part A_ψ is of type M and the nonlinear contribution ϕ_ψ is diagonal, continuous, and isotone. It follows that F_ψ^n is an inverse isotone homeomorphism, which yields the existence and uniqueness of ψ_h^{n+1} .

Similarly, for φ_h^{n+1} , with C_h^{n+1} , ψ_h^{n+1} , and A_h^{n+1} regarded as known quantities, the corresponding operator may be written as

$$F_\varphi^n(\tilde{\varphi}) = A_\varphi \tilde{\varphi} + \phi_\varphi(\tilde{\varphi}),$$

where A_φ is an M -matrix and ϕ_φ is a continuous isotone diagonal mapping. Thus, F_φ^n is again an inverse isotone homeomorphism, and the discrete problem admits a unique solution φ_h^{n+1} .

For the LasR concentration R , once ψ_h^{n+1} , φ_h^{n+1} , and A_h^{n+1} are fixed, the fifth equation can be rewritten as

$$F_R^n(\tilde{R}) = A_R \tilde{R} + \phi_R(\tilde{R}) = \tilde{G}_R^n,$$

with A_R an M -matrix and ϕ_R a continuous isotone diagonal mapping. Therefore, the operator F_R^n is an inverse isotone homeomorphism, and this ensures the existence and uniqueness of R_h^{n+1} .

Finally, for the temperature T , the discrete equation takes the form

$$F_T^n(\tilde{T}) = A_T \tilde{T} + \phi_T(\tilde{T}) = \tilde{G}_T^n,$$

where A_T is an M -matrix and ϕ_T is a continuous isotone diagonal mapping. Hence F_T^n is also an inverse isotone homeomorphism, which implies the existence and uniqueness of T_h^{n+1} .

By solving successively the six discrete equations, we obtain a unique vector

$$(C_h^{n+1}, A_h^{n+1}, \psi_h^{n+1}, \varphi_h^{n+1}, R_h^{n+1}, T_h^{n+1}) \in w_h^{C_D} \times w_h \times w_h \times w_h \times w_h \times w_h.$$

This completes the proof. $\square$

3.4.2 Energy Estimate

For any discrete sequence w_h , we define the discrete energy functional

$$F(w_h^{n+1}) = \sum_{n=0}^{N-1} \tau_n \left\| \frac{w_h^{n+1} - w_h^n}{\tau_n} \right\|_0^2 + \max_{0 \leq n \leq N-1} |w_h^{n+1}|_1^2 + \sum_{n=0}^{N-1} |w_h^{n+1} - w_h^n|_1^2,$$

where $\|\cdot\|_0$ denotes the $L^2(\Omega)$ norm, and $|\cdot|_1$ denotes the $H^1(\Omega)$ semi-norm.

Lemma 3.4.2. For all $h > 0$ and for every time partition $\{\tau_n\}_{n=1}^N$, the solution

$$(C_h^n, A_h^n, \psi_h^n, \varphi_h^n, R_h^n, T_h^n)$$

of the discrete problem (P_h) satisfies

$$\begin{aligned} & F(C_h^{n+1}) + F(A_h^{n+1}) + F(\psi_h^{n+1}) + F(\varphi_h^{n+1}) + F(R_h^{n+1}) + F(T_h^{n+1}) \\ & \leq K_1 \left(|C_h^0|_1^2 + |A_h^0|_1^2 + |\psi_h^0|_1^2 + |\varphi_h^0|_1^2 + |R_h^0|_1^2 + |T_h^0|_1^2 + K_2 \right). \end{aligned}$$

Proof. The estimate is derived by testing each equation of the discrete scheme with the corresponding time increment.

We begin with the first equation of (P_h) and choose the test function

$$v_h^C = C_h^{n+1} - C_h^n.$$

This yields

$$\begin{aligned} & \int_{\Omega} \frac{C_h^{n+1} - C_h^n}{\tau_n} (C_h^{n+1} - C_h^n) dx \\ & + d_C \int_{\Omega} \nabla C_h^{n+1} \cdot \nabla (C_h^{n+1} - C_h^n) dx \\ & + \int_{\Omega} k_1(T_h^n) g(C_h^{n+1}, \psi_h^n + \varphi_h^n) (C_h^{n+1} - C_h^n) dx = 0. \end{aligned}$$

Using the algebraic identity

$$2a(a-b) = a^2 - b^2 + (a-b)^2,$$

we obtain

$$\begin{aligned} & \tau_n \left\| \frac{C_h^{n+1} - C_h^n}{\tau_n} \right\|_0^2 + \frac{d_C}{2} \left(|C_h^{n+1}|_1^2 - |C_h^n|_1^2 + |C_h^{n+1} - C_h^n|_1^2 \right) \\ & \leq \left| \int_{\Omega} k_1(T_h^n) g(C_h^{n+1}, \psi_h^n + \varphi_h^n) (C_h^{n+1} - C_h^n) dx \right|. \end{aligned}$$

By the boundedness of the coefficients and the nonlinear term, and by using Young's inequality ($ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$) to absorb terms into the left-hand side, there exists a constant $K > 0$ such that

$$\tau_n \left\| \frac{C_h^{n+1} - C_h^n}{\tau_n} \right\|_0^2 + |C_h^{n+1}|_1^2 - |C_h^n|_1^2 + |C_h^{n+1} - C_h^n|_1^2 \leq K\tau_n. \quad (3.30)$$

Next, in the second equation, we take

$$v_h^A = A_h^{n+1} - A_h^n.$$

Proceeding in the same manner, we find

$$\begin{aligned} & \tau_n \left\| \frac{A_h^{n+1} - A_h^n}{\tau_n} \right\|_0^2 + |A_h^{n+1}|_1^2 - |A_h^n|_1^2 + |A_h^{n+1} - A_h^n|_1^2 \\ & \leq K\tau_n + K\tau_n \left(\|A_h^{n+1}\|_0^2 + \|\psi_h^n\|_0^2 + \|\varphi_h^n\|_0^2 + \|R_h^n\|_0^2 \right). \end{aligned}$$

Using again the boundedness assumptions and Young's inequality, this leads to

$$\tau_n \left\| \frac{A_h^{n+1} - A_h^n}{\tau_n} \right\|_0^2 + |A_h^{n+1}|_1^2 - |A_h^n|_1^2 + |A_h^{n+1} - A_h^n|_1^2 \leq K\tau_n. \quad (3.31)$$

For the third equation, we choose

$$v_h^\psi = \psi_h^{n+1} - \psi_h^n.$$

Then

$$\begin{aligned} & \tau_n \left\| \frac{\psi_h^{n+1} - \psi_h^n}{\tau_n} \right\|_0^2 + |\psi_h^{n+1}|_1^2 - |\psi_h^n|_1^2 + |\psi_h^{n+1} - \psi_h^n|_1^2 \\ & \leq K\tau_n + K\tau_n \left(\|A_h^{n+1}\|_0^{2m} + \|C_h^{n+1}\|_0^2 + \|\varphi_h^n\|_0^2 \right), \end{aligned}$$

and, by the boundedness of the discrete unknowns and Young's inequality,

$$\tau_n \left\| \frac{\psi_h^{n+1} - \psi_h^n}{\tau_n} \right\|_0^2 + |\psi_h^{n+1}|_1^2 - |\psi_h^n|_1^2 + |\psi_h^{n+1} - \psi_h^n|_1^2 \leq K\tau_n. \quad (3.32)$$

For the fourth equation, taking

$$v_h^\varphi = \varphi_h^{n+1} - \varphi_h^n,$$

we similarly obtain

$$\begin{aligned} & \tau_n \left\| \frac{\varphi_h^{n+1} - \varphi_h^n}{\tau_n} \right\|_0^2 + |\varphi_h^{n+1}|_1^2 - |\varphi_h^n|_1^2 + |\varphi_h^{n+1} - \varphi_h^n|_1^2 \\ & \leq K\tau_n + K\tau_n \left(\|C_h^{n+1}\|_0^2 + \|A_h^{n+1}\|_0^{2m} + \|\psi_h^{n+1}\|_0^2 \right), \end{aligned}$$

whence

$$\tau_n \left\| \frac{\varphi_h^{n+1} - \varphi_h^n}{\tau_n} \right\|_0^2 + |\varphi_h^{n+1}|_1^2 - |\varphi_h^n|_1^2 + |\varphi_h^{n+1} - \varphi_h^n|_1^2 \leq K\tau_n. \quad (3.33)$$

For the fifth equation, we take

$$v_h^R = R_h^{n+1} - R_h^n.$$

Arguing as before yields

$$\begin{aligned} & \tau_n \left\| \frac{R_h^{n+1} - R_h^n}{\tau_n} \right\|_0^2 + |R_h^{n+1}|_1^2 - |R_h^n|_1^2 + |R_h^{n+1} - R_h^n|_1^2 \\ & \leq K\tau_n + K\tau_n \left(\|\psi_h^{n+1}\|_0^2 + \|\varphi_h^{n+1}\|_0^2 + \|A_h^{n+1}\|_0^2 + \|R_h^{n+1}\|_0^2 \right), \end{aligned}$$

and therefore

$$\tau_n \left\| \frac{R_h^{n+1} - R_h^n}{\tau_n} \right\|_0^2 + |R_h^{n+1}|_1^2 - |R_h^n|_1^2 + |R_h^{n+1} - R_h^n|_1^2 \leq K\tau_n. \quad (3.34)$$

Finally, for the temperature equation, we choose

$$v_h^T = T_h^{n+1} - T_h^n.$$

This gives

$$\begin{aligned} & \tau_n \left\| \frac{T_h^{n+1} - T_h^n}{\tau_n} \right\|_0^2 + |T_h^{n+1}|_1^2 - |T_h^n|_1^2 + |T_h^{n+1} - T_h^n|_1^2 \\ & \leq K\tau_n + K\tau_n \left(\|\psi_h^{n+1}\|_0^2 + \|\varphi_h^{n+1}\|_0^2 \right), \end{aligned}$$

and hence

$$\tau_n \left\| \frac{T_h^{n+1} - T_h^n}{\tau_n} \right\|_0^2 + |T_h^{n+1}|_1^2 - |T_h^n|_1^2 + |T_h^{n+1} - T_h^n|_1^2 \leq K\tau_n. \quad (3.35)$$

Summing the inequalities (3.30)-(3.35) over $n = 0, \dots, N - 1$, we obtain

$$\begin{aligned} & F(C_h^{n+1}) + F(A_h^{n+1}) + F(\psi_h^{n+1}) + F(\varphi_h^{n+1}) + F(R_h^{n+1}) + F(T_h^{n+1}) \\ & \leq K_1 \left(|C_h^0|_1^2 + |A_h^0|_1^2 + |\psi_h^0|_1^2 + |\varphi_h^0|_1^2 + |R_h^0|_1^2 + |T_h^0|_1^2 + K_2 \right), \end{aligned}$$

which concludes the proof. $\square$

3.4.3 Convergence

Let us now state that the solution of the numerical scheme converges to a weak solution of (P_h) . We set $\Omega_T = \Omega \times (0, T)$ and introduce the following piecewise interpolations in time. For any discrete sequence $\{z_h^n\}_{n=0}^N$, we define the continuous piecewise linear interpolant:

$$z_h(t) = \frac{t - t^n}{\tau_n} z_h^{n+1} + \frac{t^{n+1} - t}{\tau_n} z_h^n, \quad t \in (t^n, t^{n+1}], \quad n = 0, \dots, N - 1,$$

and the forward and backward piecewise constant interpolants:

$$z_h^+(t) = z_h^{n+1}, \quad z_h^-(t) = z_h^n, \quad t \in (t^n, t^{n+1}], \quad n = 0, \dots, N - 1.$$

Using this notation, the discrete problem (P_h) can be rewritten in a global continuous-in-time integral form over Ω_T . For any test functions $v_h \in L^2(0, T; W_h)$, we seek the continuous interpolants

$$(C_h, A_h, \psi_h, \varphi_h, R_h, T_h) \in [C^0([0, T]; W_h)]^6,$$

such that

$$\left\{ \begin{array}{l} \int_{\Omega_T} \frac{\partial C_h}{\partial t} v_h^C dxdt + d_C \int_{\Omega_T} \nabla C_h^+ \cdot \nabla v_h^C dxdt = \int_{\Omega_T} k_1 (T_h^+) g(C_h^+, \psi_h^+ + \varphi_h^+) v_h^C dxdt, \\ \int_{\Omega_T} \frac{\partial A_h}{\partial t} v_h^A dxdt + d_2 \int_{\Omega_T} \nabla A_h^+ \cdot \nabla v_h^A dxdt + \int_{\Omega_T} \gamma (T_h^+) A_h^+ v_h^A dxdt \\ \quad = \int_{\Omega_T} \alpha (T_h^+) \frac{R_h^+ A_h^+}{K_L + R_h^+ A_h^+} \psi_h^+ v_h^A dxdt + \int_{\Omega_T} (\alpha (T_h^+) + \beta (T_h^+)) \frac{R_h^+ A_h^+}{K_L + R_h^+ A_h^+} \varphi_h^+ v_h^A dxdt, \\ \int_{\Omega_T} \frac{\partial \psi_h}{\partial t} v_h^\psi dxdt + d \int_{\Omega_T} \nabla \psi_h^+ \cdot \nabla v_h^\psi dxdt + \int_{\Omega_T} k_5 (T_h^+) |A_h^+|^m \psi_h^+ v_h^\psi dxdt \\ \quad = \int_{\Omega_T} (k_3 (T_h^+) g(C_h^+, \psi_h^+) + k_5 (T_h^+) \varphi_h^+) v_h^\psi dxdt, \\ \int_{\Omega_T} \frac{\partial \varphi_h}{\partial t} v_h^\varphi dxdt + d \int_{\Omega_T} \nabla \varphi_h^+ \cdot \nabla v_h^\varphi dxdt + \int_{\Omega_T} (k_4 (T_h^+) + k_5 (T_h^+)) \varphi_h^+ v_h^\varphi dxdt \\ \quad = \int_{\Omega_T} (k_3 (T_h^+) g(C_h^+, \varphi_h^+) + k_5 (T_h^+) |A_h^+|^m \psi_h^+) v_h^\varphi dxdt, \\ \int_{\Omega_T} \frac{\partial R_h}{\partial t} v_h^R dxdt + d_R \int_{\Omega_T} \nabla R_h^+ \cdot \nabla v_h^R dxdt + \int_{\Omega_T} k_R (T_h^+) R_h^+ v_h^R dxdt + \int_{\Omega_T} k_3 (T_h^+) g(\psi_h^+ + \varphi_h^+, R_h^+) v_h^R dxdt \\ \quad = \int_{\Omega_T} k_3 (T_h^+) \frac{R_h^+ A_h^+}{k_R + R_h^+ A_h^+} v_h^R dxdt, \\ \int_{\Omega_T} \frac{\partial T_h}{\partial t} v_h^T dxdt + k \int_{\Omega_T} \nabla T_h^+ \cdot \nabla v_h^T dxdt = \int_{\Omega_T} Q_0(\psi_h^+ + \varphi_h^+) v_h^T dxdt. \end{array} \right.$$

Introducing the functional

$$\chi(z_h) = \|z_h^\pm\|_{L^\infty(\Omega_T)}^2 + \|z_h^\pm\|_{L^\infty(0,T;H^1(\Omega))}^2 + \left\| \frac{\partial z_h}{\partial t} \right\|_{L^2(\Omega_T)}^2 + \tau^{-1} \|z_h^+ - z_h^-\|_{L^2(0,T;H^1(\Omega))}^2,$$

the energy estimate from the previous lemma yields that for $z_h \in \{C_h, A_h, \psi_h, \varphi_h, R_h, T_h\}$, we have

$$\chi(C_h) + \chi(A_h) + \chi(\psi_h) + \chi(\varphi_h) + \chi(R_h) + \chi(T_h) \leq K,$$

where K is a positive constant independent of h and τ .

From the above uniform bound, we infer the compactness results stated in the following lemma, which rely on the Aubin-Lions lemma.

Lemma 3.4.3. As $h \rightarrow 0$ and $\tau \rightarrow 0$, there exists a subsequence, still denoted by $\{z_h\}$, and a limit function $z \in L^\infty(\Omega_T) \cap H^1(0, T; L^2(\Omega))$ such that:

$$\begin{aligned} z_h, z_h^\pm &\rightarrow z \text{ weak-* in } L^\infty(0, T; L^\infty(\Omega)), \\ \frac{\partial z_h}{\partial t} &\rightarrow \frac{\partial z}{\partial t} \text{ weakly in } L^2(\Omega_T), \\ z_h, z_h^\pm &\rightarrow z \text{ weak-* in } L^\infty(0, T; H^1(\Omega)), \\ z_h, z_h^\pm &\rightarrow z \text{ strongly in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T. \end{aligned}$$

Theorem 3.4.4. Under the assumptions of the previous lemma, and assuming appropriate

regularity for the initial data, the solution of the discrete scheme (3.29) converges to the solution of the weak form (3.28) of the system (3.10)–(3.11) as $h, \tau \rightarrow 0$.

3.4.4 Numerical Results

We consider the two-dimensional computational spatial domain $\Omega = [0, 1] \times [0, 0.5]$ and the time interval $[0, 20]$. The spatio-temporal domain is therefore denoted by $\Omega_T = \Omega \times (0, 20]$. The boundary portion Γ_D corresponds to the top part on which the nutrient substrate and the chemical agents are applied.

At $t = 0$, the initial biofilm domain is defined by

$$\Omega_b(0) = \{(x_1, x_2) \in \Omega : x_2 \geq 0, x_1^2 + x_2^2 \leq 0.01\},$$

and contains only down-regulated biomass.

The dimensionless model parameters used in the system, adapted from [6], are listed in Table 3.1.

Table 3.1: Dimensionless model parameters used in the simulations.

Parameter	Value
m	2.5
d	10^{-7}
d_C	1.67
d_R	0.1
d_2	12.9
Q_0	0.1
k	0.1

Figure 3.1 illustrates the spatial distribution of down-regulated and up-regulated cells at times $t = 10$ and $t = 15$. The top row corresponds to the case without antibiotic or QSB agent, the middle row to the case where only the antibiotic is applied, and the bottom row to the case where both the antibiotic and the QSB agent are applied. These results show that the antibiotic and the QSB agent reduce the bacterial population and alter its spatial expansion in the presence of thermal effects.

The numerical results presented below describe the temporal evolution of the main vari-

ables. To this end, we define the spatial averages by

$$\begin{aligned}\psi_{\text{Total}}(t) &= \frac{1}{|\Omega|} \int_{\Omega} \psi(x, t) dx, & \varphi_{\text{Total}}(t) &= \frac{1}{|\Omega|} \int_{\Omega} \varphi(x, t) dx, \\ A_{\text{Total}}(t) &= \frac{1}{|\Omega|} \int_{\Omega} A(x, t) dx, & R_{\text{Total}}(t) &= \frac{1}{|\Omega|} \int_{\Omega} R(x, t) dx, \\ T_{\text{Total}}(t) &= \frac{1}{|\Omega|} \int_{\Omega} T(x, t) dx.\end{aligned}$$

The quantity of antibiotic applied on the boundary Γ_D is taken as $V_D = 10$, while the quantity of QSB agent imposed on Γ_D is set to $Q_D = 10$.

Figure 3.2 shows the temporal evolution of the down-regulated (ψ) and up-regulated (φ) biomass. Figure 3.3 presents the total amounts of the AHL molecule (A) and LasR protein (R) over time in four different cases: without antibiotic and QSB, with antibiotic only, with QSB only, and with both antibiotic and QSB. Finally, Figure 3.4 displays the temperature distribution at $t = 15$ under the combined action of the antibiotic and the QSB agent.

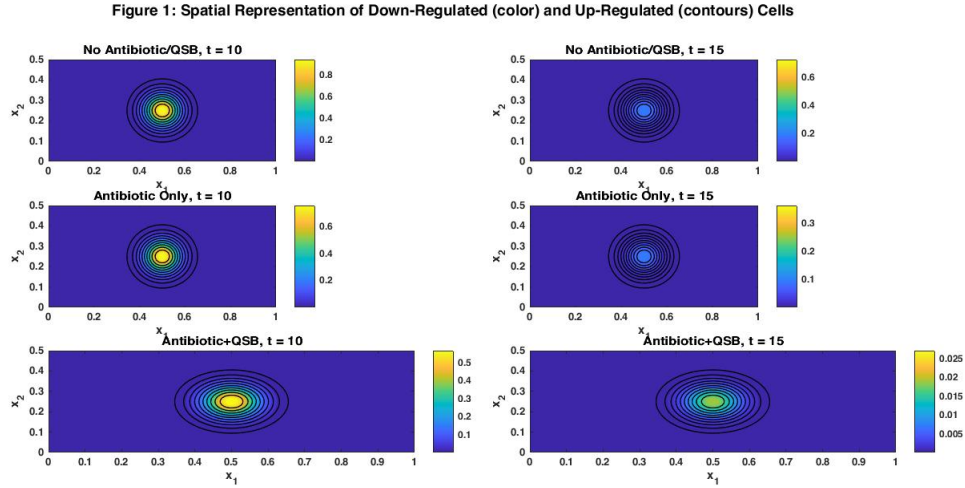


Figure 3.1: Representation of down-regulated and up-regulated cells at time $t = 10$ and $t = 15$.

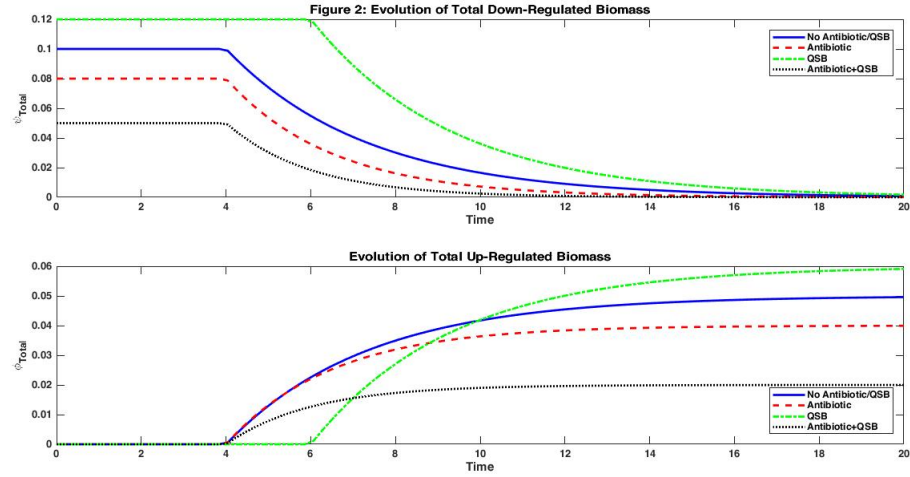


Figure 3.2: Total amounts of down-regulated and up-regulated cells with respect to time.

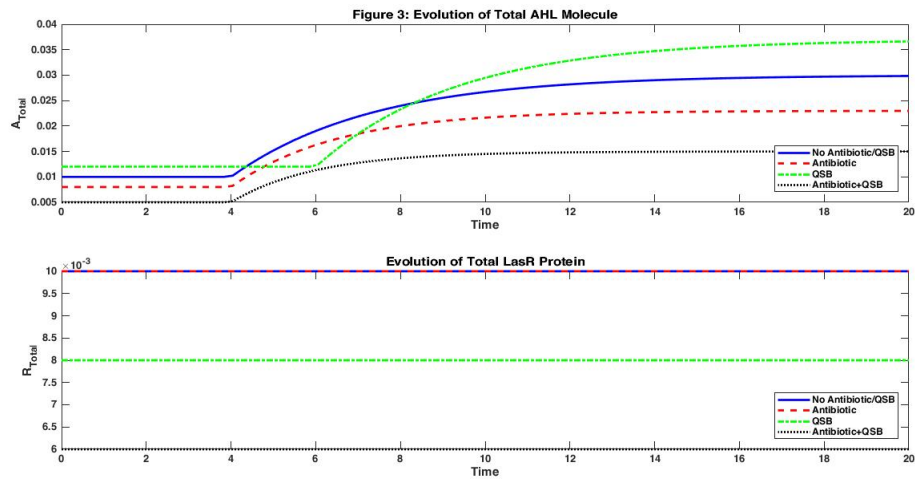


Figure 3.3: Evolution of total AHL molecule and LasR protein.

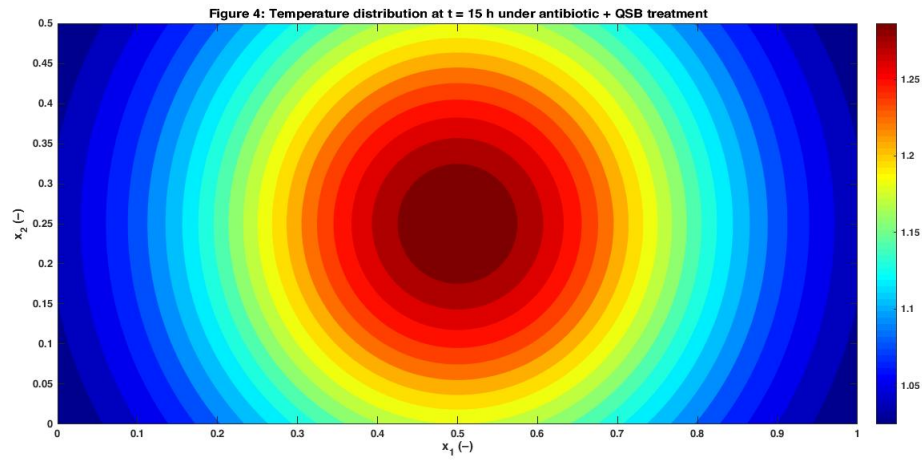


Figure 3.4: Temperature distribution at $t = 15$ under antibiotic and QSB treatment.

3.5 Conclusion

In this chapter, we proposed and analyzed a thermally coupled quorum-sensing biofilm model formulated as a system of nonlinear reaction–diffusion equations complemented by a heat equation. The model incorporates the main biological and physical mechanisms governing biofilm development, namely nutrient consumption, the interaction between down-regulated and up-regulated bacterial populations, quorum-sensing signaling, and temperature-dependent reaction kinetics. In addition, metabolic heat production was introduced as a source term in the thermal balance, allowing the model to capture the mutual interaction between biological activity and temperature variations.

From the theoretical point of view, the chapter established the well-posedness of the continuous problem and provided a mathematical basis for the study of the coupled system. A fully discrete approximation was then constructed by combining a finite-element spatial discretization with a backward Euler scheme in time. The corresponding discrete formulation was analyzed, and the main qualitative properties of the approximation were derived through existence, uniqueness, energy estimates, and convergence arguments. These results ensure the consistency and robustness of the numerical procedure adopted for the simulations.

From the numerical point of view, the simulations illustrated the influence of temperature on the evolution of the biofilm and on the regulation of quorum-sensing activity. Furthermore, the numerical results highlighted the combined effects of antibiotic treatment and Quorum Sensing Blockers (QSB) on reducing the bacterial population. The obtained results show that thermal effects modify the temporal evolution and spatial distribution of the interacting variables, and therefore play a significant role in the overall biofilm dynamics. In particular, the coupling between metabolic activity and temperature provides an additional mechanism capable of affecting bacterial growth, signaling intensity, and the structure of the biofilm.

Overall, this chapter highlights the relevance of incorporating thermal effects into quorum-sensing biofilm models, both from a mathematical and from a biological perspective. The proposed framework may serve as a basis for more refined studies involving heterogeneous environments, more realistic thermal boundary conditions, or experimentally calibrated parameter values. It also opens the way to further investigations on control strategies combining antibiotic treatment, quorum-sensing inhibition, and thermal regulation in order to better understand and limit biofilm persistence.

General conclusion

This thesis has been devoted to the mathematical and numerical modeling of bacterial biofilms viewed as complex multiphasic systems, with particular emphasis on the role of temperature, quorum sensing, and growth regulation in shaping biofilm formation, expansion, and persistence. The work combines biological motivation, rigorous mathematical analysis, and reliable numerical approximation in order to construct models that are both mechanistically meaningful and computationally tractable. Throughout the thesis, biofilms have been treated not as homogeneous structures, but as organized and evolving systems in which several interacting phases, transport mechanisms, and regulatory effects coexist and must be described in a coupled manner.

In Chapter 1, we established the biological and conceptual background required for the study of biofilms, emphasizing their structured organization, heterogeneous composition, and multiphasic nature. This description naturally motivated continuum formulations in which biomass, extracellular polymeric substance (EPS), and liquid phases coexist and interact through coupled transport, reaction, and mechanical effects. Particular attention was also given to the need for growth regulation mechanisms in reduced biofilm models, since unrestricted growth may lead to unrealistic thickening and loss of physical relevance. In this perspective, the idea of introducing a growth-damping contribution was part of the conceptual motivation underlying the subsequent modeling developments. Chapter 1 also gathered the core mathematical material used throughout the thesis, including functional spaces, weak convergence, compactness tools, and fundamental inequalities, together with essential notions from C_0 -semigroup theory and basic numerical methodologies. It therefore served not only as a biological introduction, but also as the analytical foundation on which the rest of the thesis was built.

In Chapter 2, we extended a three-phase biofilm model by incorporating a thermobiophysical coupling through a heat balance equation. This first contribution was developed in the framework of a multiphase description in which the biofilm is represented by a cellular phase, an extracellular matrix phase, and an aqueous phase. Such a formulation makes it possible to distinguish between the principal structural and functional components of the biofilm and to account for their respective interactions. The introduction of thermal effects allowed kinetic and transport coefficients to depend explicitly on temperature, while also

taking into account metabolic heat production and conductive heat transfer scaled by the local film thickness. This coupling produced a feedback loop between biological activity and thermal evolution, thereby enriching the physical content of the model.

A second important feature of Chapter 2 was the introduction of a growth-damping term in the reduced thickness equation. This contribution was motivated by the need to prevent unrealistically rapid thickening in the thin-film approximation and to represent, at a macroscopic level, the progressive limitation of biofilm expansion caused by cell crowding, matrix accumulation, and mechanical resistance within the colony. In this sense, the growth-damping term plays a dual role: it regularizes the reduced dynamics and improves the physical realism of the model. A thin-film reduction was then derived in order to preserve the main thermo-biophysical couplings while maintaining analytical coherence and computational efficiency. A finite-difference discretization was proposed and analyzed, and a discrete stability framework was established for the resulting numerical method. The numerical experiments highlighted how temperature variations may significantly modulate growth, swelling behavior, nutrient transport, and overall expansion, while the damping mechanism provides an effective control on the evolution of the film thickness. Chapter 2 therefore showed that thermal coupling and growth regulation are both essential ingredients for obtaining realistic reduced multiphase biofilm models.

In Chapter 3, we adopted a different but complementary perspective by developing a coupled reaction–diffusion framework integrating quorum sensing, antibiotic response, quorum-sensing inhibitors (QSB), and temperature dynamics governed by a heat equation. In this setting, the focus shifted from the mechanical multiphase structure of the biofilm to its biochemical and regulatory organization. The model distinguishes nutrient concentration, signaling molecules, a regulatory protein, and two interacting bacterial populations, namely down-regulated and up-regulated cells. The proposed system captures a feedback mechanism in which temperature influences biological reaction rates and quorum-sensing activity, while bacterial metabolism acts as a source term in the thermal equation. This provided a thermally coupled quorum-sensing model capable of describing the interaction between environmental conditions and internal biofilm regulation.

From the theoretical point of view, Chapter 3 established existence and uniqueness within a semigroup framework, thereby ensuring the well-posedness of the continuous problem under suitable assumptions. On the numerical side, a finite-element approximation in space combined with backward Euler time integration provided a robust discretization strategy. Existence and uniqueness at the discrete level, energy estimates, and convergence results were investigated in order to support the consistency of the numerical approximation. Simulations implemented in Matlab illustrated the influence of thermal conditions on quorum-sensing regulation, signaling activity, and biofilm evolution. In this way, Chapter 3 complemented Chapter 2: while the latter emphasized multiphase transport, swelling, and growth control, the former highlighted biochemical regulation, antibiotic effects, and thermal feedback in a

reaction–diffusion setting.

Taken together, the results obtained in this thesis demonstrate that temperature-dependent mechanisms should not be regarded as secondary refinements, but rather as central components of realistic biofilm modeling. Temperature affects growth kinetics, transport coefficients, swelling processes, signaling activity, and the balance between proliferation and stabilization. Likewise, the multiphasic viewpoint proves essential whenever one seeks to distinguish the roles of biomass, matrix, and liquid phases and to capture the interaction between transport, mechanics, and chemical regulation. In the same spirit, the growth-damping mechanism introduced in the reduced multiphase model shows that additional regulatory terms may be necessary to prevent oversimplified or physically unrealistic behavior and to improve the predictive quality of reduced formulations.

From a mathematical perspective, this thesis confirms the usefulness of combining several analytical approaches depending on the structure of the model under consideration. Semi-group theory provides a natural framework for nonlinear reaction–diffusion systems with temperature-dependent coefficients, while discrete energy methods remain essential for the study of stability and convergence of numerical schemes associated with multiphase transport models. From a numerical viewpoint, the work demonstrates that both finite-difference and finite-element methods can be successfully adapted to biofilm systems, provided that the coupling structure of the equations is carefully respected and that appropriate stability conditions are enforced.

Several perspectives naturally arise from this work. First, the proposed models could be refined by introducing more elaborate constitutive laws, more detailed temperature dependencies, or additional biochemical species involved in regulation and inhibition. Second, the numerical values of the parameters should ultimately be calibrated against experimental data in order to strengthen the predictive power of the models. Third, the framework could be extended to more complex geometries, heterogeneous substrates, or fully three-dimensional settings. Finally, the combined consideration of temperature, quorum sensing, multiphase transport, and growth regulation opens promising directions for the analysis of biofilm control strategies in biomedical, industrial, and environmental applications.

In conclusion, this thesis provides a coherent mathematical and numerical contribution to the modeling of bacterial biofilms by emphasizing three major ideas: the intrinsic multiphase nature of biofilm organization, the significant influence of thermal effects on biological and transport processes, and the importance of regulation mechanisms such as growth damping for realistic evolution laws. The proposed models and approximation tools constitute a consistent basis for further theoretical developments and for future confrontation with experimental observations.

References

- [1] Adams, R. A., Fournier, J. J. F. (2003). *Sobolev Spaces*, 2nd ed., Academic Press.
- [2] Allaire, G. (2005). *Numerical Analysis and Optimization*, Paris.
- [3] Al-Smadi, M., Zurigat, M. (2024). Reaction-Diffusion Modeling in PPO Micro Planes: Analysis of Akbari-Ganji and Homotopy Perturbation Methods, *Palestine Journal of Mathematics*, Vol. 13 (Special Issue I), pp. 247–260.
- [4] Amara, M. A. (2015). *Theory of C_0 -Semigroups and Applications*, Dr. Tahar Moulay University, Saïda.
- [5] Anguige, K., King, J. R., Ward, J. P., Williams, P. (2004). Mathematical modelling of therapies targeted at bacterial quorum sensing, *Mathematical Biosciences*, Vol. 192 (1), pp. 39–83.
- [6] Barrett, J. W., Nürnberg, R. (2002). Finite-element approximation of a nonlinear degenerate parabolic system describing bacterial pattern formation, *Interfaces and Free Boundaries*, Vol. 4, pp. 277–307.
- [7] Bénéilan, P., Brézis, H. (1972). Solutions faibles d'équations d'évolution dans les espaces de Hilbert, *Annales de l'Institut Fourier*, Vol. 22, pp. 311–329.
- [8] Berg, H. C. (2003). The rotary motor of bacterial flagella, *Annual Review of Biochemistry*, 72(1), 19–54.
- [9] Belytschko, T., Liu, W. K., Moran, B. (2013). *Nonlinear Finite Elements for Continua and Structures*, John Wiley & Sons.
- [10] Blouza, A., El Alaoui, L. (2018). Numerical modeling of the quorum sensing in a bacterial biofilm, *ESAIM: Proceedings and Surveys*, Vol. 62, pp. 17–29.
- [11] Boudarel, H., Mathias, J.-D., Blaysat, B., Grédiac, M. (2018). Towards standardized mechanical characterization of microbial biofilms: Analysis and critical review, *npj Biofilms and Microbiomes*, 4(1), 17.

- [12] Brézis, H. (1983). *Analyse fonctionnelle: théorie et applications*, Dunod, Paris.
- [13] Brézis, H. (1987). *Functional Analysis: Theory and Applications*, Masson, Paris–New York–Barcelona–Milan–Mexico–São Paulo.
- [14] Burman, E., Hansbo, P. (2012). Fictitious domain finite element methods using cut elements: II. A stabilized Nitsche method, *Applied Numerical Mathematics*, Vol. 62 (4), pp. 328–341.
- [15] Byrd, M. S., Sadovskaya, I., Vinogradov, E., Lu, H., Sprinkle, A. B., Richardson, S. H., Ma, L., Ralston, B., Parsek, M. R., Anderson, E. M., Lam, J. S., Wozniak, D. J. (2009). Genetic and biochemical analyses of the *Pseudomonas aeruginosa* psl exopolysaccharide reveal overlapping roles for polysaccharide synthesis enzymes in psl and lps production, *Molecular Microbiology*, 73(4), 622–638.
- [16] Cauchy, A. L. (1821). *Course of Analysis of the Royal Polytechnic School, First Part: Algebraic Analysis*, Paris.
- [17] Cazenave, T., Haraux, A. (1990). *Introduction to Semilinear Evolution Problems*, Ellipses.
- [18] Chen, B. (2011). Existence of solutions for quasilinear parabolic equations with nonlocal boundary conditions, *Nonlinear Analysis*, Vol. 74, pp. 6543–6552.
- [19] Chen, X., Friedman, A. (2003). A free boundary problem for an elliptic-hyperbolic system: An application to tumor growth, *SIAM Journal on Mathematical Analysis*, 35(4), 974–986.
- [20] Chorfi, S.-E. (2017). *Semigroup Theory: Abstract Cauchy Problem and Evolution Equations*, Master’s Thesis, Faculty of Sciences Semlalia, Marrakech.
- [21] Chopp, D. L., Kirisits, J., Moran, B. (2003). The dependence of quorum sensing on the depth of a growing biofilm, *Bulletin of Mathematical Biology*, Vol. 65 (6), pp. 1053–1079.
- [22] Cogan, N. G., Keener, J. P. (2004). The role of the biofilm matrix in structural development, *Mathematical Medicine and Biology*, Vol. 21 (2), pp. 147–166.
- [23] Cogan, N. G. (2006). Effects of persister formation on bacterial response to dosing, *Journal of Theoretical Biology*, Vol. 238 (3), pp. 694–706.
- [24] Cogan, N. G. (2007). Incorporating toxin hypothesis into a mathematical model of persister formation and dynamics, *Journal of Theoretical Biology*, Vol. 248 (2), pp. 340–349.

- [25] Cogan, N. G. (2008). Two-fluid model of biofilm disinfection, *Bulletin of Mathematical Biology*, Vol. 70 (3), pp. 800–819.
- [26] DiBenedetto, E., Friedman, A. (1983). Conduction–convection problems with change of phase, *Journal of Differential Equations*, 62, 129–185.
- [27] DiBenedetto, E., Elliott, C. M., Friedman, A. (1986). The free boundary of a flow in a porous body heated from its boundary, *Nonlinear Analysis: Theory, Methods & Applications*, 10, 879–900.
- [28] Dinh, A. T. (2018). *Finite Element Methods for Nonlinear Interface Problems: Application to a Biofilm Growth Model*, PhD Thesis.
- [29] Doi, M. (1996). *Introduction to Polymer Physics*, Oxford University Press.
- [30] Dockery, J., Keener, J. P. (2001). A mathematical model for quorum sensing in *P. aeruginosa*, *Bulletin of Mathematical Biology*, Vol. 63 (1), pp. 95–116.
- [31] Dockery, J., Klapper, I. (2002). Finger formation in biofilm layers, *SIAM Journal on Applied Mathematics*, Vol. 62 (3), pp. 853–869.
- [32] Donlan, R. M. (2002). *Biofilms: Microbial Life on Surfaces*, Emerging Infectious Diseases, 8(9), 881–890, September. doi: 10.3201/eid0809.020063.
- [33] Drew, D. A., Passman, S. L. (1999). *Theory of Multicomponent Fluids*, Springer, New York.
- [34] Duddu, R., Chopp, D. L., Moran, B. (2008). A two-dimensional continuum model of biofilm growth incorporating fluid flow and shear stress-based detachment, *Biotechnology and Bioengineering*, Vol. 103 (1), pp. 92–104.
- [35] Eberl, H. J., Parker, D. F., van Loosdrecht, M. C. M. (2001). A new deterministic spatio-temporal continuum model for biofilm development, *Journal of Theoretical Medicine*, Vol. 3 (3), pp. 161–175.
- [36] Engel, K.-J., Nagel, R. (2000). *One-Parameter Semigroups for Linear Evolution Equations*, Springer, New York.
- [37] A. Falah, A. Odeh. (2024). Analytical Modeling of Glucose-Sensitive Membranes in Closed-Loop Insulin Delivery Systems, *Palestine Journal of Mathematics*, Vol. 13 (Special Issue I), pp. 204–217.
- [38] Flemming, H.-C., Wingender, J., Szewzyk, U., Steinberg, P., Rice, S. A., Kjelleberg, S. (2016). Biofilms: An emergent form of bacterial life, *Nature Reviews Microbiology*, Vol. 14 (9), pp. 563–575.

- [39] Frederick, M. R., Kuttler, C., Hense, B. A., Eberl, H. J. (2011). A mathematical model of quorum sensing regulated EPS production in biofilm communities, *Theoretical Biology and Medical Modelling*, Vol. 8, Article 8.
- [40] Furuya, E. Y., Lowy, F. D. (2006). Antimicrobial-resistant bacteria in the community setting, *Nature Reviews Microbiology*, 4(1), 36–45.
- [41] Goldstein, J. A. (1985). *Semigroups of Operators and Applications*, Oxford University Press.
- [42] Hadamard, J. (1924). The principle of Huygens, *Bulletin de la Société Mathématique de France*, 52, 610–640.
- [43] Hall-Stoodley, L., Costerton, J. W., Stoodley, P. (2004). Bacterial biofilms: From the natural environment to infectious diseases, *Nature Reviews Microbiology*, 2(2), 95–108.
- [44] Hille, E. (1948). *Functional Analysis and Semigroups*, American Mathematical Society.
- [45] Kantaoui, H. (2016). *Properties of the Semigroup Generated by an m -Dissipative Operator*, Master's Thesis, Kasdi Merbah University, Ouargla.
- [46] Kesavan, S. (1989). *Topics in Functional Analysis and Applications*, Wiley Eastern, New Delhi.
- [47] Klapper, I., Rupp, C. J., Cargo, R., Purvedorj, B., Stoodley, P. (2002). Viscoelastic fluid description of bacterial biofilm material properties, *Biotechnology and Bioengineering*, Vol. 80 (3), pp. 289–296.
- [48] Kumar, U., Pushpavanam, S. (2024). A three-phase model for biofilm formation on a porous solid surface, *Physics of Fluids*, Vol. 36 (2), 021906.
- [49] Dam Lemle, L. (2001). *The Lie–Trotter Formula for Strongly Continuous Semigroups*.
- [50] Lakikza, H., Gasmi, S., Taallah, F.(2025) Numerical modeling of multiphase biofilm formation on a porous solid surface with heat equation coupling and temperature dependent parameters, *Gulf Journal of Mathematics*, Vol. 21, No. 1.
- [51] Lakikza, H., Gasmi, S., Taallah, F.(2025) Theoretical and numerical results of the quorum sensing model in a bacterial biofilm coupled with the heat equation, *Palestine Journal of Mathematics*, Vol. 14, No. 4, pp. 808–825.
- [52] Lubkin, S.(2003) *C_0 -Semigroups and Applications*, North-Holland.
- [53] Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer.

- [54] Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York–Berlin.
- [55] Rana, N., Ghosh, P., Perlekar, P. (2017). Spreading of nonmotile bacteria on a hard agar plate: Comparison between agent-based and stochastic simulations, *Physical Review E*, 96(5), 052403.
- [56] Raymond, J.-P.(2001). *Evolution Equations*, Paul Sabatier University.
- [57] Shrout, J. D., Chopp, D. L., Just, C. L., Hentzer, M., Givskov, M., Parsek, M. R. (2006). The impact of quorum sensing and swarming motility on *Pseudomonas aeruginosa* biofilm formation is nutritionally conditional, *Molecular Microbiology*, 62(5), 1264–1277.
- [58] Singh, R., Paul, D., Jain, R. K. (2006). Biofilms: Implications in bioremediation, *Trends in Microbiology*, 14(9), 389–397.
- [59] Srinivasan, S., Kaplan, C. N., & Mahadevan, L. (2019). A multiphase theory for spreading microbial swarms and films.
- [60] Solonnikov, V. (2001). Lp-estimates for solutions to the initial boundary-value problem for the generalized Stokes system in a bounded domain, *Journal of Mathematical Sciences*, 105, 2448–2484.
- [61] Solonnikov, V. (2002). Estimates of solutions of the second initial-boundary problem for the Stokes system in spaces of functions having Hölder continuous derivatives, *Journal of Mathematical Sciences*, 109, 1997–2017.
- [62] Solonnikov, V. (2006). Weighted Schauder estimates for evolution Stokes problem, *Annali dell'Università di Ferrara*, 52, 137–172.
- [63] Solonnikov, V. (2007). On Schauder estimates for the evolution generalized Stokes problem, in *Hyperbolic Problems and Regularity Questions*, Trends in Mathematics, Birkhäuser, Basel, pp. 197–205.
- [64] Sonner, S., Efendiev, M. A., Eberl, H. J. (2009). Existence and longtime behavior of a biofilm model, *Communications on Pure and Applied Analysis*, Vol. 8 (2), pp. 509–531.
- [65] Tam, A. K. Y., Harding, B., Green, J. E. F., Balasuriya, S., Binder, B. J. (2022). Thin-film lubrication model for biofilm expansion under strong adhesion, *Physical Review E*, 105(1), 014408.
- [66] Temam, R. (1979). *Navier–Stokes Equations: Theory and Numerical Analysis*, 2nd ed., North-Holland, Amsterdam.

- [67] Trinschek, S., John, K., Thiele, U. (2016). From a thin film model for passive suspensions towards the description of osmotic biofilm spreading, *AIMS Materials Science*, Vol. 3 (3), pp. 1138–1159.
- [68] Trinschek, S., John, K., Lécuyer, S., Thiele, U. (2016). Modelling of surfactant-driven front instabilities in spreading bacterial colonies, *AIMS Materials Science*, Vol. 3 (3), pp. 1138–1157.
- [69] Wang, Q., Zhang, T. (2010). Review of mathematical models for biofilms, *Solid State Communications*, 150(21–22), 1009–1022.
- [70] Ward, J. P., King, J. R. (2012). Thin-film modelling of biofilm growth and quorum sensing, *Journal of Engineering Mathematics*, 73(1), 71–92.
- [71] Winstanley, H. F., Chapwanya, M., McGuinness, M. J., Fowler, A. C. (2011). A polymer–solvent model of biofilm growth, *Proceedings of the Royal Society A*, Vol. 467 (2129), pp. 1449–1467.
- [72] P. Williams. (2002). Quorum sensing: an emerging target for antibacterial chemotherapy, *Expert Opinion on Therapeutic Targets*, Vol. 6, pp. 257–274.
- [73] Yan, J., Nadell, C. D., Stone, H. A., Wingreen, N. S., Bassler, B. L. (2017). Extracellular-matrix-mediated osmotic pressure drives *Vibrio cholerae* biofilm expansion and cheater exclusion, *Nature Communications*, Vol. 8 (1), 327.
- [74] Yosida, K. (1948). On the differentiability and representation of one-parameter semi-groups of linear operators, *Journal of the Mathematical Society of Japan*, 1, 15–21.