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Sur les polynômes d-orthogonaux au sens de Sobolev

A Doctoral Thesis
By Ali Krelifa

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To My parents, My wife and childrens, Iyad and Maram

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Abstract

In this thesis, we study the d $-$orthogonal polynomials and the $d-$ orthogonality in the Sobolev sens. We present an algebraic theory of classical $d-$orthogonal polynomials and we want to fill in some gaps. We broaden and close some inclusions that exist and are known perhaps as consequences. Several characterizations of classical d-OPS are given in terms of $d+2$ term recurrence relation as well as in terms of functional and differential equations. An additional tool in determining the integral representation for such class is presented.

Furthermore, we study the inner products which generate the sequence of orthogonal polynomials in the sense of Sobolev.

Using the theory of Riordan group and $d-$ orthogonal polynomials, we shall show that sequences of $d-$ orthogonal polynomials, can be also generated by Riordan group. We interpret some families of $d-$ orthogonal polynomials in the frame of Riordan group.

keyword:

Orthogonal polynomials , $d-$Orthogonal polynomials , classical polynomials , Riordan Matrix

ملخص

في هده الرسالة قمنا بدراسة كثيرات الحدود د-المتعامدة بمفهوم سوبولاف. وبعد ذكر بعض خصائص تعامد سوبولاف ، قمنا بإنشاء جداءات داخلية التي تولد متتاليات كثيرات الحدود د- المتعامدة بمفهوم سوبولاف وأظهرنا أن كثيرات الحدود الكلاسيكية بالنسبة للشعاع دو البعد - د - للتطبيقات الخطية انها ليست كلاسيكية بمعنى سوبولاف

ثم درسنا العديد من خصائص كثيرات الحدود الكلاسيكية د-المتعامدة . أولا، نقدم برهان جديدا على معادلة بيرسون باستخدام د- ضعف التعامد و توصيف جديد لكثيرات الحدود الكلاسيكية د-المتعامدة باستعمال د- شبه التعامد. كما تم التعبير عن خصائص أخرى باستعمال د-شبه التعامد و د-ضعف التعامد. وبالإضافة إلى ذلك، نقدم أداة لتمثيل التكامل لهذه الفئة من كثيرات الحدود.

والموضوع الثاني في هذه الأطروحة مخصص لكثيرات الحدود د-المتعامدة ومجموعات ريوردان. وقد استطعنا أن نبين أن تسلسل كثيرات الحدود د-المتعامدة يمكن أن يتولد أيضا عن هذا النوع من المجموعات. وعلاوة على ذلك، عرضنا بعض كثيرات الحدود د-المتعامدة من معاملات هذه المصفوفات. النتيجة الرئيسية التي تم الحصول عليها في هذه الدراسة هي أن معكوس مصفوفة ريوردان الأسي هو جدول معاملات عائلة من كثيرات الحدود د- المتعامدة، إذا وفقط إذا كانت مصفوفة ستيلجس تساوي مصفوفة الإنتاج

الكلمات المفتاحية

كثيرات الحدود، كثيرات الحدود د-المتعامدة، كثيرات الحدود الكلاسيكية، مجموعات ريوردان

Résumé

Le thème principal de ce travail porte sur les polynômes $d-$ orthogonaux au sens de Sobolev. Après avoir cité quelques propriétés de l'orthogonalité de Sobolev, nous avons construit des produits Scalaires qui génèrent des suites $d-$ orthogonaux au sens de Sobolev et nous avons montré que les suites $2-$classiques par rapport au vecteur τ de dimension $2-$ des formes linéaires ne sont pas definie une suite $2-$ classique au sens de la généralisation de Sobolev

Nous avons ensuite étudié de nombreuses caractérisations de la $d-$SPO classique. Tout d'abord, nous donnons une nouvelle démonstration de l'équation de Pearson en utilisant l'orthogonalité faible ainsi que sur une nouvelle caractérisation du $d-$SPO classique en termes de la $d-$ quasi-orthogonalité. D'autres caractérisations ont également été exprimées en termes de la $d-$quasi orthogonalité ainsi que de l'orthogonalité faible. En outre, nous présentons un outil pour déterminer la représentation intégrale de telle classe de polynômes.

Le second thème dans cette thèse est consacré au polynômes $d-$orthogonaux et les groupes de Riordan. Nous avons pu montrer que les suites de polynômes $d-$orthogonaux peuvent être également générées par ce type de groupe. De plus, nous avons exhibé quelques familles $d-$orthogonaux à partir des coefficients de ces matrices. Le résultat principal obtenu dans cette étude est que l'inverse d'une matrice de Riordan exponentielle L est un tableau des coefficients d'une famille de polynômes $d-$orthogonaux si et seulement si la matrice Stieltjes est égale à la matrice Production.

Mots clé: polynômes orthogonaux , $d-$Orthogonalité , polynômes classique , Matrices de Riordan

Contents

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Chapter 1

Introduction

1.1 Orthogonality

The notion of orthogonal system of functions appeared through the study of certain problems of functional analysis (integral equations, strum-Liouville problem and more generally, boundary problems in partial differential equations). Orthogonal polynomials in general, and classical orthogonal polynomials in specific, have been the subject of extensive work. They are related to many problems of applied mathematics, theoretical physics, chemistry, theory of approximation, and several other mathematical branches. In particular, their applications are widely used in theories such as Padé approximants, continuous fractions, spectral study of Schrödinger, discrete operators, polynomial solutions of second-order differential equations, and others.

The first family of orthogonal polynomials known as "Legendre polynomials" appeared with the first work of A.M.Legendre on the planetary movements in 1784 . Actually, he established some common properties and he derived a second order differential equation which has as solutions this families. He also studied its zeros.

Subsequently, other families were introduced. For instance, Hermite (1864), constructed a new family, called Hermite polynomials, which was used in the interpolation theory.

N. H. Abel and V. L. Lagrange P .L. Chebyshev began with T.J Stieltjes the creation and development of general theory of orthogonal polynomials by the use of continuous fractions.

E. N .Laguerre introduced in 1897 a family of polynomials (called Laguerre polynomials) and showed the link between it and the continuous fractions.

In the middle of the nineteenth century, Jacobi introduced a new family which generalized the polynomials of Legendre and Chebyshev. These three families (Jacobi, Laguerre and Hermite), are known as the "classical polynomials".

Other authors such as T. J. Stieltjes, L. M. Humbureger, T. Carleman, F. Hausdorff, and M .H. Stone have shown that orthogonal polynomials solve also the problem of moments via continuous fractions.

Bochner in 1929, demonstrated that the sturm -Liouville differential equation, defined by

$$\alpha(x)y'' + \beta(x)y' + \gamma(x)y(x) = \lambda y(x) \tag{1.1}$$

has a polynomial solution if and only if $\deg \alpha(x) \leq 2$, $\deg \beta(x) \leq 1$ and $\gamma(x) \equiv 0$.

Furthermore, by using a linear change of variable, Bochner has also shown that there are only 4 families of classical polynomials orthogonal solutions of (1.1) which are Jacobi, Bessel, Laguerre and Hermite.

The second characterization of classical orthogonal polynomials is the Rodrigues formula which gives us the explicit expression of the classical orthogonal polynomials in terms of α and β coefficients in equation (1.1).

In 1938, Krall demonstrated that the classical polynomials satisfied in fact an even order differential equation of the type

$$\sum_{i=1}^{2r} \alpha_i(x)y^{(i)} = \lambda y. \tag{1.2}$$

The results of Bochner and Krall have been generalized by Louriero and Maroni in 2008 and 2010 by giving the link between the coefficients of (1.1) and (1.2).

Other criteria for the classification of the classical orthogonal polynomials have been recently given by Kil. H. Kwon and L. L. Littlejohn (1996) on a Sobolev class, i.e., a classification based on a symmetric bilinear form defined by

$$\phi(p,q) = \langle \sigma, p\ q \rangle + \langle \tau, p'q' \rangle, \quad p,q \in \mathcal{P},$$

In 2006, Jamei and Koepf gave a class of a differential equation which generalizes the Bochner equation and the Sturm-Liouville problem. More precisely, they considered the case when the degree of the coefficients α and β are ≥ 2 and also when they are rational functions. They showed that the solutions of this class are symmetric and orthogonal and they gave explicit expressions of their weight function.

The notion of orthogonality has undergone a great deal of generalization. It began with the notion of $(1/p), (p > 1)$ orthogonality (A.Boukhmis 1988), the d-orthogonality (P.Maroni 1989), the biotogonality (Brezinski 1992), and finally the multiple orthogonality (Aptekarev, van Assche ...). These generalizations intended to make the phenomena modeled by differential equations more comprehensible, to improve the approximation methods, and to increase the order of the differential equations having these polynomials as solution.

The notion of $d-$orthogonality appeared in the thesis of J.Van Isegham [82]. This notion was used in the study of the Padé approximation of simultaneous formal $d-$series. Subsequently, it was observed that $d-$orthogonal polynomials satisfied an recurrence relation of order $(d+1)$.

After two years, the first properties of a sequence of $d-$ orthogonal polynomials have been given in [54]. The author gave a new algebraic approach showing many interesting characterization for the $d-$orthogonality, based on the orthogonality's vectorial form. In addition, he introduced the notion of $d-$quasi orthogonality in the same article.

Many authors have attempted to improve this theory. Although the classical character has been approached by Douak and Maroni in numerous articles, other researchers (Y. Ben Cheikh, N. Ben Romdhane, K. Douak, I. Lamiri, A. Bokhemis, E. Zerouki, A. Saib, etc.) studied certain problems which lead to the construction of many d-analogues of the classical families and to the discovery of new ones.

In 2006, A. Boukhemis and E. Zerouki found an interesting result where they determined a linear differential equation which admits the 2-orthogonal classical polynomials as solution given by

$$R_{n,4}(x)P^{(3)}_{n+3}(x) + R_{n,3}(x)P^{(2)}_{n+3}(x) + R_{n,2}(x)P^{(1)}_{n+3}(x) + R_{n,1}(x)P_{n+3}(x) = 0,$$

with $R_{n,i}(x)$ for $i = 1, ..., 4$ are polynomials of degree less than or equal to i and

$$R_{n,4}(x) = F_{n,1}(x)S_3(x),$$

where $S_3(x)$ is a polynomial which does not depend on n and whose degree is less than 3.

The classical character, *i.e.* the sequence of orthogonal polynomials in which the derived sequence is orthogonal (OPS). This is known as the property of Hahn.

1.2 Riordan array

The concept of Riordan arrays has been introduced in 1991 by Shapiro et al [73] with the aim of generalizing the concept of Renewal Array defined by Rogers [67], and have pointed out its connection to the Umbral calculus [68]. Their basic idea was to define a class of infinite lower triangular arrays with properties analogous to those of the Pascal's triangle whose elements are the binomial coefficients.

The Riordan groups are particularly important in studying combinatorial identities and sums. This concept has been investigated by Sprugnoli [75] who pointed out the relevance of these matrices from the theoretical and practical point of view. In addition, his work verified that many combinatorial sums can be solved by transforming the generating functions. So, we can see that the Riordan array concept is particularly important. For example, identities involving Stirling numbers can not be treated by methods related to hypergeometric functions. In [76], Sprugnoli paid attention to the identities of Abel and Gould. Further applications in reciprocal functions are discussed in [22, 83], on subgroups of the Riordan group in Peart and Woan [66], on some characterizations of Riordan matrices in Rogers [67] and Merlini et al [60], and in many interesting related results in Cheon et al. [21], He et al [38], Nkwanta [42], Shapiro [74], Barry [7] and so forth.

Successively, some other aspects of the theory have been studied [60]. Further generalizations of these last characterizations are given in [39, 36].

The Sheffer sequence is a very general concept that includes many polynomial sequences as special cases. The concept of Sheffer Riordan group of Sheffer-type polynomials is defined by He et al. [38]. They prove the isomorphism between the Sheffer group and the Riordan group and present an equivalence between the Riordan array pair and the generalized Stirling number pair.

The lower triangular matrices and matrix factorization problems have catalyzed many investigations in recent years. Peart and Woodson [64] did some researches on this problem. They showed that some classical Riordan arrays have triple factorizations of the form $L = PCF$, where P, C, F are also Riordan arrays. We notice also that the inverse of the Stieltjes matrix represents a coefficient of the sequence polynomials [4, 3, 51, 81]. In [79], the authors present new factorizations, and provide extensive examples of families of classical polynomials.

The main results of this thesis

This thesis contains six chapters.

Chapter 3

In Chapter 3, we are interested in the study of orthogonal polynomials of Sobolev. To this aim, we first investigate properties of symmetric bilinear forms and in particular the one which is defined by

$$\phi(p, q) = \langle \sigma, p\ q \rangle + \langle \tau, p'q' \rangle, \quad p, q \in \mathcal{P}.$$

We give necessary and sufficient conditions for orthogonal polynomials with respect to the symmetric bilinear form ϕ given above, which satisfy the Bochner's second - order differential equation. A classification of the orthogonal polynomial sequences of Sobolev solutions of the Bochner differential equation will be investigated also.

Finally, we study the regularity of symmetric $d-$dimensional vector forms $\phi = (\phi_0, \phi_1, ..., \phi_{d-1})^T$ defined by

$$\phi_r(p, q) := (a\delta(x - c), p\ q\rangle + \langle \tau_r, p'q'\rangle$$

More precisely, we show in the case when, $a = 0$ and τ is 2-classical functional moment, the sequence of classical 2-orthogonal polynomials satisfies a non-homogeneous third-differential equation of Boukhemis type.

Chapter 4.

Many of the properties of the classical orthogonal polynomials have nice extensions in this d-orthogonality setting: there will be a higher order linear recurrence relation [16, 54], there are nice functional equation (Pearson equation) [29, 30], linear combination (finite-type relation) between the sequence and its derivatives [58] and differential equation [19]. In addition, if the same ideas are to be found, each aspect of the theory is more complicated and the links between the different aspects are not so clear.

This chapter deals with algebraic aspects of classical d-orthogonal polynomials. In the first section, we recall some definitions and characterizations of the $d-$orthogonality which we need in the sequel. In addition, it is well known that there is exactly 2^d sets of $d-$symmetric $d-$orthogonal polynomials. We discuss here again this result from the combinatorics point of view. The main results are listed in the third section. Many characterizations of the classical d-OPS are presented. First, we give a new proof of the Pearson equation in the algebraic aspects with the aid of the weak orthogonality as well as on a new characterization of the classical d-OPS in terms of $d-$quasi-orthogonality. Further characterizations also expressed in terms of d-quasi-orthogonality as well as on the weak orthogonality. Further, we present a set of tools help to determinate the integral representation for a such class of polynomials.

Chapter 5

In this chapter, we shall show that the sequence of d-orthogonal polynomials, i.e., a sequence of polynomial satisfying a linear recurrence relation of $d + 2$- terms, can be also generated by a Riordan array. General references for orthogonal and $d-$orthogonal polynomials can be found in [24, 54]. Links between Riordan arrays and orthogonal polynomials have been explored in [3, 4, 7]. Note that there are some examples in the literature of Riordan arrays that generate some families of polynomials sequence satisfying a recurrence relation in four terms and more as well as a recurrence with variable coefficients [**?**, 6, 23]. In this chapter, two examples of d-orthogonal polynomials are presented, for which an ordinary Riordan array exists. The first one is the d-Chebyshev polynomials of second kind. The second one presents in fact the co-recursive d-Chebyshev of second kind. The next sections deal with Stieltjes matrix. We generalize the result of Peart and

Barry where we show that the corresponding Stieltjes matrix characterizes the sequence of d-orthogonal polynomials and we obtain the explicit form of that Riordan array. For the exponential Riordan array, we show again that Stieltjes matrix characterizes the sequence of d-orthogonal polynomials. Some examples are discussed. Finally, we improve the Sheffer Riordan array by giving an example.

1.2. Riordan array

Chapter 2

Preliminaries and notations

The present chapter reminds some concepts about orthogonal polynomials of a real variable and their fundamental properties.

2.1 Orthogonality in general

Let $\mathcal{P}$ be the linear space of complex polynomials of one variable and $\mathcal{P}'$ its topological dual space. We denote by $\langle u, P\rangle$ the action of u and $P \in \mathcal{P}$.

2.1.1 Free and monic polynomials

Let $\{P_n\}_{n\geq 0}$ be a sequence of polynomials with coefficients in $\mathbb{C}$.

Definition 2.1 The sequence $\{P_n\}_{n\geq 0}$ is said free if and only if $\deg(P_n) = n, \forall n \geq 0$

Definition 2.2 We say that a free sequence $\{P_n\}_{n\geq 0}$ is monic if each polynomial P_n is written in the form :

$$P_n(x) = x^n + \sum_{k=0}^{n-1} a_{n-k} x^{n-k-1},\ for\ n \geq 0$$

Proposition 2.1 *If $\{P_n\}_{n\geq 0}$ is monic, then there exist a unique sequence $\{\beta_n\}$ and a unique array $\chi_{n,v}$, $0 \leq v \leq n$, such that*

$$\begin{array}{l} P_0(x) = 1,\ P_1(x) = x - \beta_0 \\ P_{n+2}(x) = (x-\beta)P_{n+1} - \sum_{v=0}^{n} \chi_{n,v} P_v(x) \quad , \quad n \geq 0 \end{array}$$

Definition 2.3 Let $\{P_n\}_{n\geq 0}$ be a sequence of free polynomials, the function $G(x;t)$ which can be developed into a power series of t is called the generating function of the sequence $\{P_n\}_{n\geq 0}$, if it is put in the form :

$$G(x,t) = \sum_{n\geq 0} c_n P_{n(x)} t^n, \quad \text{where } \ c_n \neq 0 \ , \forall n \geq 0.$$

2.1.2 Regular and definite positive forms

Definition 2.4 We call the dual sequence of the free sequence $\{P_n\}_{n\geq 0}$ the sequence of linear forms $\{u_n\}$ defined by

$$u_n(P_m) := \langle u_n, P_m \rangle = \delta_{nm}, \ n, m \geq 0,$$

where $\langle ., . \rangle$ is the duality hook. The linear form u_0 is called the canonical form of the sequence $\{P_n\}_{n\geq 0}$.

Proposition 2.2 i) *[24]For each monic sequence, the dual sequence exists and is unique.*

ii) *In this case, we have*

$$\begin{aligned} \beta_n &= \langle u_n, xP_n(x) \rangle, \quad n \geq 0, \\ \chi_{n,v} &= \langle u_v, xP_{n+1}(x) \rangle, \quad 0 \leq v \leq n. \end{aligned} \tag{2.1}$$

Definition 2.5 The form $\pounds$ is called regular (or admissible, or quasi-definite) if there exists a free sequence $\{P_n\}_{n\geq 0}$ such that

$$\begin{aligned} &\pounds(P_n P_m) = 0, n \neq m, \\ &\pounds(P_n^2) \neq 0, \quad n \geq 0. \end{aligned} \tag{2.2}$$

Such a sequence is called orthogonal (or regularly orthogonal) relative to $\pounds$. Such a sequence is free and unique up to a multiplicative constant.

Lemma 2.1 *For any $\pounds \in \mathcal{P}'$ and any integer $p \geq 1$, the following statements are equivalent [54, 55]:*

I) $\langle \pounds, P_{p-1} \rangle \neq 0$, $\langle \pounds, P_n \rangle = 0$, $n \geq p$,

II) $\exists \lambda_\nu \in \mathbb{C}$, $0 \leq \nu \leq p-1$, $\lambda_{p-1} \neq 0$ *such that*

$$\pounds = \sum\nolimits_{\nu=0}^{p-1} \lambda_\nu \pounds_\nu.$$

Lemma 2.2 *The sequence $\{P_n\}_{n\geq 0}$ is orthogonal with respect to $\pounds$. Necessarily, $\pounds = \lambda \pounds_0$, $\lambda \neq 0$. In this case, we have [58, 55]*

$$\pounds_n = \left(\langle \pounds_0, P_n^2 \rangle \right)^{-1} P_n u_0, \quad n \geq 0.$$

when $\pounds$ is regular, let A be a polynomial such that $A\pounds = 0$. Then $A = 0$.

Remark 2.1 For any linear form $\pounds$ and for any polynomial π, we can define the two forms $D\pounds = \pounds'$ and $\pi\pounds$ by

$$\langle \pounds', f\rangle := -\langle \pounds, f'\rangle, \quad \langle \pi\pounds, f\rangle := \langle \pounds, \pi f\rangle, \quad f \in \mathcal{P}.$$

and thus, the dual sequence $\{\tilde{u}_n\}_{n\geq 0}$ of $\{Q_n\}$, is given by

$$\tilde{u}_n' = -(n+1)\, u_{n+1}, \quad n \geq 0, \tag{2.3}$$

where $Q_n(x) = (n+1)^{-1} P_{n+1}'(x)\,,\, n \geq 0.$

Definition 2.6 The sequence $\{\pounds_n\}_{n\geq 0}$ defined by

$$\pounds_n := \pounds(x^n), \;\; n \geq 0$$

is called a sequence of moments of the form $\pounds$. We say that the form μ is real if the moments $\pounds_n, n \in \mathbb{N}$, are real.

Theorem 2.1 *[24, 49]The form $\pounds$ is regular if and only if the **Hankel** determinants defined by*

$$\Delta_n = \det(\pounds_{i+j})_{i,j=0}^n, n \geq 0$$

are all non null.

Proposition 2.3 *[24, 49] The following conditions are all equivalent*

1) $\{P_n\}_{n\geq 0}$ is orthogonal polynomials sequence relative to μ.

2)

$$\pounds(\pi(x)P_n(x)) = \begin{cases} 0 \;\; if \;\; \deg\pi < n, \forall n \geq 1 \\ \neq 0 \quad if \;\; \deg\pi = n \end{cases}$$

3)

$$\pounds(x^m P_n(x)) = A_n\delta_{nm}, A_n \neq 0, o \leq m \leq n$$

Definition 2.7 A form $\pounds$ is said positive definite if $\langle \pounds, \pi(x)\rangle \geq 0$ for any polynomial $\pi \equiv 0$ such that $\pi(x) \geq 0$ for all real x.

Proposition 2.4 *[24, 49] If a form $\pounds$ is positive definite, then it is regular and all moments of $\pounds$ are real.*

Theorem 2.2 *[24, 49] A form $\pounds$ is positive definite if and only if it is real and the determinants of Hankel Δ_n are strictly positive for all $n \geq 0$.*

2.1.3 Fundamental recurrence.

Theorem 2.3 *[24, 49]Let £ be an admissible linear form on $\mathcal{P}$ and $\{P_n\}_{n\geq 0}$ the orthogonal monic sequence relative to £. Then this sequence satisfies a linear recurrence of order two, i.e. there exist two series of numbers $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 0}$ such that*

$$\begin{cases} P_0(x) = 1, \;\; P_1(x) = x - \beta_0 \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1} - \alpha_{n+1}P_n(x) \quad , \quad n \geq 0 \end{cases} \tag{2.4}$$

with the regularity condition $\alpha_n \neq 0$ for all $n \geq 0$. Moreover, if we set $P_n(x) = x^n + b_n x^{n-1} + ...$ for all $n \geq 1$, we have, $\beta_n = b_{n+2} - b_{n+1}$ and $\alpha_n = \dfrac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n}$ where $\Delta_{-1} = 1$ (by convention).

Theorem 2.4 *[24] Let $\{P_n\}_{n\geq 0}$ be a monic sequence of polynomials satisfying the recurrence 2.4, where $\alpha_n \neq 0$ for any $n \geq 0$.Then there exists a single linear form £ such that $£(P_0) = \alpha_0$ and $£(P_m(x)P_n(x)) = 0$ for any $n \neq m$.*

Corollary 2.1 *Let $\{P_n\}_{n\geq 0}$ be a monic sequence of polynomials satisfying the recurrence 2.4. Then the form μ is positive definite if and only if $\beta_n \in \mathbb{R}$ and $\alpha_n > 0$ for any $n \geq 0$.*

Definition 2.8 Let $\{P_n\}_{n\geq 0}$ be a monic sequence of polynomials satisfying the recurrence 2.4. We call associated polynomial sequence of $\{P_n\}_{n\geq 0}$ the sequence denoted by $\{P_n(x,c)\}_{n\geq 0}$, for any $c \in \mathbb{R}$, defined by :

$$\begin{aligned} &P_0(x,c) = 1, \;\; P_1(x,c) = x - \beta_c \\ &P_{n+2}(x,c) = (x - \beta_{n+c+1})P_{n+1}(x,c) - \alpha_{n+c+1}P_n(x,c) \quad , \quad n \geq 0 \end{aligned}$$

Theorem 2.5 *[24] Let $\{P_n\}_{n\geq 0}$ be a sequence of orthogonal polynomials with respect to the positive definite form μ, then each polynomial P_n admits n real and simple roots for all $n \geq 1$. Moreover, the zeros of P_n and P_{n+1} are alternate, i.e.*

$$x_{n+1,v} < x_{n,v} < x_{n+1,v+1} < x_{n,v+1}, \;\; n \geq 1, v \geq 1,$$

where $\{x_{n,v}\}_{v=1}^{n}$ and $\{x_{n+1}, v\}_{v=1}^{n+1}$ are respectively the zeros of P_n and P_{n+1}.

2.1.4 Classical sequences

Definition 2.9 Let $\{P_n\}_{n\geq 0}$ be a sequence of orthogonal polynomials. This sequence is called classical if these sequences of successive derivatives are also orthogonal (**Hahn's** property).

Theorem 2.6 *[24, 49] The classical sequences are the only real polynomial sequences solution of the second-order differential equation of the hypergeometric type of Sturm - Liouville*

$$\alpha(x)\,y'' + \beta(x)\,y' + \lambda_n y = 0, \tag{2.5}$$

where α and β are polynomials of degree two and one respectively and where λ_n is the following constant:

$$\lambda_n = \alpha''(x)n(n-1) + n\beta'(x)$$

Theorem 2.7 *[24] The classical orthogonal polynomials are given by the formula of Rodriguez*

$$P_n(x) = \frac{1}{k_n w(x)} \frac{d^n}{dx^n}(\alpha^n(x)w(x)), \tag{2.6}$$

where k_n is a non-zero constant and w is a solution of the differential equation $(\alpha w)' = \beta w$.

Corollary 2.2 *If £ admits the following integral representation*

$$£(P) = \int_a^b p(x)w(x)dx \text{ , for any polynomial } P,$$

then the sequences of polynomials given by 2.6 are orthogonal to the weight function w.

2.1.5 Integrated Functional Representation

We know, from Duran's classical theorem concerning the problem of moments of Stieltjes, that every moment function can be represented by a measure in the form of an integral of Rieman-Stieltjes. Now it is interesting to establish the integral representations of the moment functionnelle.

Definition 2.10 [24] Let ϕ be a function defined on $I \subseteq \mathbb{R}$. ϕ is said to be a rapidly decreasing function on I if, for every pair of integers (n, m), we have

$$\left| x^n \frac{d^m}{dx^m}\phi(x) \right| < +\infty \quad \text{for any } \ x \in I. \tag{2.7}$$

Definition 2.11 A bounded non -decreasing ϕ whose moments

$$\mu_n := \int_{-\infty}^{+\infty} x^n d\phi(x).$$

are all finite is called a distribution function,

Theorem 2.8 *[34] For any sequence $\{\mu_n\}_{n\geq 0}$ of complex numbers there exists a fast decay function on $\mathbb{R}^+$ (respectively on $\mathbb{R}$) such that*

$$\mu_n := \int_0^{+\infty} x^n \phi(x)dx \quad \textit{respectively} \quad \mu_n := \int_{-\infty}^{+\infty} x^n \phi(x)dx. \tag{2.8}$$

Theorem 2.9 *[46] Let $\{P_n\}_{n\geq 0}$ be a regular orthogonal polynomial sequence satisfying the differential equation 2.5. If $w(x)$ is an orthogonalization weight function of $\{P_n\}_{n\geq 0}$ then $w(x)$ satisfies the following differential equation*

$$(\alpha(x)w(x))' - \beta(x)w(x) = g(x) \tag{2.9}$$

where $g(x)$ is a weight distribution with all its zero moments, i.e.

$$\langle g(x), x^n\rangle = 0, n \geq 0. \tag{2.10}$$

Conversely, if $w(x)$ is a distribution satisfying the following conditions

I) *$w(x)$ decreases rapidly when $|x|$ tends to ∞; then $\langle w(x), x^n\rangle$ exists and is finite for all $n \geq 0$;*

II) *$w(x)$ defines a non trivial moment function, i.e., its moments are not all null ;*

III) *$w(x)$ is a solution of equation 2.10 in the distribution space in $\mathbb{R}$;*

then $w(x)$ is said orthogonalization distribution of $\{P_n\}_{n\geq 0}$.

Definition 2.12 Relation 2.9 is called the non-homogeneous weight equation for the differential equation 2.5.

2.2 The $d-$Orthogonality

Let us consider linear forms $\Gamma_0, ..., \Gamma_{d-1}$ $(d \geq 1)$.

Definition 2.13 [54] The sequence $\{P_n\}_{n\geq 0}$ is called d-orthogonal polynomials sequence, in short a d-OPS, with respect to the $d-$dimensional vector of linear forms $\Gamma = (\Gamma_0, ..., \Gamma_{d-1})^T$, if it fulfills

$$\begin{aligned} &\langle \Gamma_\alpha, x^m P_n(x)\rangle = 0, \quad n \geq md + \alpha + 1, \quad m \geq 0, \\ &\langle \Gamma_\alpha, x^m P_{md+\alpha}(x)\rangle \neq 0, \quad m \geq 0, \end{aligned} \tag{2.11}$$

for each $0 \leq \alpha \leq d-1$.

In this case, the d-dimensional functional $-$ is called regular.

Remark 2.2 [18]1 If $\{P_n\}_n$ is a $d-$orthogonal polynomial sequence, then its polynomials

are exactly of degree n and can hence be normalized , thus it follows the uniqueness of the sequence.

2) We say that the functional Γ is regular of dimension d

3) When $d = 1$, we find the usual notion of regular orthogonality

Theorem 2.10 *[54]It is necessary and sufficient, for the functional Γ of dimension d to be regular, that the following determinants be non-zero :*

$$H_{md+v} = \begin{bmatrix} \Gamma_0 & \Gamma_1 & \cdots & \Gamma_{md+v-1} \\ \Gamma_1 & \Gamma_2 & \cdots & \Gamma_{md+v} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{m-1} & \Gamma_m & \cdots & \Gamma_{m-1+md+v-1} \\ \Gamma^1_m & \Gamma^1_{m+1} & \cdots & \Gamma^1_{(m+1)d+v-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma^v_m & \Gamma^v_{m+1} & \cdots & \Gamma^v_{(m+1)d+v-1} \end{bmatrix}, \ m \geq 0, \ \overline{v = 0, d-1} \tag{2.12}$$

We have posed $\Gamma_n = \Gamma(x^n) = (\Gamma^1(x^n), \Gamma^2(x^n), ..., \Gamma^d(x^n))^T, n \geq 0$. Represents actually d line.

Therefore, each determinant H_{md+v} is of order $md+v$ if $md+v \geq 1$.

By convention $H_0 = 1$. Each of the first m lines of the determinant 2.12 represents in fact d line.

we have (25) :

$$H_{md+v} = \prod_{u=0}^{m-1} \prod_{\alpha=1}^{d} U^\alpha(x^u P_{ud+\alpha-1}(x)) \prod_{\alpha=1}^{v} U^\alpha(x^m P_{md+\alpha-1}(x))$$

for any $m \geq 1$ and $1 \leq v \leq d-1$

$$H_{md} = \prod_{u=0}^{m-1} \prod_{\alpha=1}^{d} U^\alpha(x^u P_{ud+\alpha-1}(x)) \prod_{\alpha=1}^{v}, m \geq 1$$

$$H_v = \prod_{\alpha=1}^{v} U^\alpha(P_{\alpha-1}(x)) \ , 1 \leq v \leq d$$

and

$$\Gamma^\alpha(x^n P_{nd+\alpha-1}(x)) = \frac{H_{nd+\alpha}}{H_{nd+\alpha-1}}, n \geq 0, 1 \leq \alpha \leq d$$

Let us now recall some characterizations we need in the sequel.

Theorem 2.11 *[54] Let $\{P_n\}_{n\geq 0}$ be a monic sequence of polynomials, then the following statements are equivalent.*

(a) The sequence $\{P_n\}_{n\geq 0}$ is d-OPS with respect to $\Gamma = (\Gamma_0, ..., \Gamma_{d-1})^T$,

(b) The sequence $\{P_n\}_{n\geq 0}$ satisfies a recurrence relation of order $d+1$ $(d\geq 1)$:

$$P_{m+d+1}(x)=(x-\beta_{m+d})P_{m+d}(x)-\sum_{\nu=0}^{d-1}\gamma_{m+d-\nu}^{d-1-\nu}P_{m+d-1-\nu}(x), \quad m\geq 0, \tag{2.13}$$

with the initial data

$$\begin{aligned}&P_0(x)=1, \quad P_1(x)=x-\beta_0,\\ &P_m(x)=(x-\beta_{m-1})P_{m-1}(x)-\textstyle\sum_{\nu=0}^{m-2}\gamma_{m-1-\nu}^{d-1-\nu}P_{m-2-\nu}(x), \quad 2\leq m\leq d,\end{aligned} \tag{2.14}$$

and the regularity conditions $\gamma_{m+1}^{0}\neq 0, \quad m\geq 0.$

(c) For each (n,ν), $n\geq 0$, $0\leq \nu\leq d-1$, there exist d polynomials $\phi_{n,\nu}^{\mu}$, $0\leq\mu\leq d-1$ such that

$$u_{nd+\nu}=\sum_{\mu=0}^{d-1}\phi_{n,\nu}^{\mu}u_{\mu}, \quad n\geq 0, \quad 0\leq\nu\leq d-1, \tag{2.15}$$

and verifying

$$\begin{aligned}&\deg\phi_{n,\nu}^{\nu}=n, && 0\leq\nu\leq d-1, && \text{and if } \ d\geq 2,\\ &\deg\phi_{n,\nu}^{\mu}\leq n, && 0\leq\mu\leq\nu-1, && \text{if } \ 1\leq\nu\leq d-1,\\ &\deg\phi_{n,\nu}^{\mu}\leq n-1, && \nu+1\leq\mu\leq d-1, && \text{if } \ 0\leq\nu\leq d-2.\end{aligned} \tag{2.16}$$

Notice that the coefficients $\{\beta_m\}$and $\{\gamma_{m+1}\}$are given by [54]

$$\begin{aligned}\beta_v &= \langle u_v, xP_v\rangle, \ 0\leq v\leq d-1\\ \gamma_{n+1+v}^{v} &= \langle u_{n+v}, xP_{n+d}\rangle, \ 0\leq v\leq d-1, n\geq 0\end{aligned} \tag{2.17}$$

Proposition 2.5 *[69] For any $d-$OPS satisfying the recurrence relations 2.13 and 2.14, the first moments*

$(u_0)_1=\beta_0$
$(u_0)_1=\beta_0^2+\quad\gamma_1^{d-1}$
$(u_1)_2=\beta_0+\beta_1$
$(u_2)_3=\beta_0+\beta_1+\beta_2$
$(u_3)_4=\beta_0+\beta_1+\beta_2+\beta_3$
$(u_1)_3=\beta_0^2+\beta_1^2+\beta_2^2+\beta_0\beta_1++\gamma_1^{d-1}+\gamma_2^{d-1}$
$(u_0)_3=\beta_0^3+\gamma_1^{d-1}(2\beta_0+\beta_1)+\gamma_1^{d-2}$
$(u_2)_4=\beta_0^2+\beta_1^2+\beta_2^2+\beta_0\beta_1+\beta_0\beta_2+\beta_1\beta_2+\gamma_1^{d-1}+\gamma_2^{d-1}+\gamma_3^{d-1}$
$(u_1)_4=\beta_0^3+\beta_1^3+(\beta_0+\beta_1)(\beta_0\beta_1+2\gamma_1^{d-1})+\gamma_2^{d-1}(\beta_0+2\beta_1+\beta_2)+\gamma_1^{d-2}+\gamma_2^{d-2}$
$(u_1)_4=\beta_0^4+\gamma_1^{d-1}\left[2\beta_0^2+(\beta_0+\beta_1)^2+\gamma_1^{d-1}+\gamma_2^{d-1}\right]+\gamma_1^{d-2}(2\beta_0+\beta_1+\beta_2)+\gamma_1^{d-2}$

Proposition 2.6 *[56] The statements in theorem 2.11 are equivalent to the following*

(d) $\chi_{n+d-1,v} = 0, \quad n \geq v+1$ *and* $\chi_{v+d-1,v} \neq 0, \quad v \geq 0,$

(e)

$$\begin{aligned} xu_n &= u_{n-1} + \beta_n u_n + \sum_{\nu=0}^{d-1} \chi_{n+v,n} u_{n+1+\nu}, \quad n \geq 0 \quad (u_{-1} = 0) \\ \chi_{n+d-1,n} &\neq 0, \qquad n \geq 0. \end{aligned} \tag{2.18}$$

Corollary 2.3 *[72]The multiplicity of zeros of any* $d - OPS$ *is at most* d*. Moreover, any* $d+1$ *consecutive polynomials as well as an* $d+1$ *consecutive polynomials of* $r-$*associated sequence* $\{P_n\}_{n\geq 0}$ *have no common zero. And any* $r \geq 0$ *the polynomials* $P_n^{(r)}, P_n^{(r+1)}, .., P_n^{(r+d)}$ *have no common zero*

2.3 Quasi - monomiality

We give some results of quasi - monomiality in order to get the dual sequence of a given polynomial set by using the lowering operator associated with the involved polynomials. The results obtained will be applied to Boas -Buck polynomial sets. For an integer j, we denote by $\Lambda^{(j)}$ the space of operator acting on analytic function that augment (resp-reduce) the degree of every polynomial by exactly j if $j \in \mathbb{N}$ (resp $j \in \mathbb{Z}^-$), $\mathbb{Z}^-$ being the set of non positive integers.This tacitly includes the fact that, if $\sigma \in \Lambda^{(-1)}$, then $\sigma(1) = 0$.

Definition 2.14 [8] Let $\sigma \in \Lambda^{(-1)}$. A polynomial set $\{P_n\}_{n\geq 0}$ is called a sequence of basic polynomials for σ if :

(i) $P_0(x) = 1$,

(ii) $P_n(0) = 0$ whenever $n > 0$,

(iii) $\sigma P_n(x) = nP_{n-1}(x)$.

As a consequence of this definition, we mention the orthogonality relation

$$\sigma^m P_n(0) = n!\delta_{nm}, \quad n, m = 0, 1, ... \tag{2.19}$$

Starting from a polynomial set, We give a method of constructing its dual sequence in terms of its lowering operator

Lemma 2.3 *[8] Every* $\sigma \in \Lambda^{(-1)}$ *has a sequence of basic polynomials*

Lemma 2.4 *[8] Let $\{Q_n\}_{n\geq 0}$ be a polynomial set and let σ be its lowering operator. Let $\{P_n\}_{n\geq 0}$ be a sequence of basic polynomials for σ. Then there exists a power series*

$$\varphi(t) = \sum_{k=0}^{\infty} \alpha_n t^n, \alpha_0 \neq 0,$$

such that

$$\varphi(\sigma)(Q_n) = P_n, \quad n = 0, 1, ...$$

Let us combine the two preceding lemmas, gives us the following theorem

Theorem 2.12 *[8]Let $\{Q_n\}_{n\geq 0}$ be a polynomial set and let σ lowering operator , and let $\{\pounds_n\}_{n\geq 0}$ be its dual sequence. Then there exists a power series*

$$\varphi(t) = \sum_{k=0}^{\infty} \alpha_n t^n, \alpha_0 \neq 0,$$

such that

$$\langle \pounds_n, f \rangle = \frac{1}{n!} \left[\sigma^n \varphi(\sigma)(f)(x)\right]_{x=0} = \frac{\sigma^n \varphi(\sigma)}{n!} f(o), \ n = 0, 1, ..., \quad f \in \mathcal{P} \tag{2.20}$$

If we apply Theorem 2.12 we obtain a further method to construct the sequence $\{\pounds_n\}_{n\geq 0}$ for general case from which we deduce the following expansion theorem

Theorem 2.13 *[8] Let $\{Q_n\}_{n\geq 0}$ be a polynomial set and let σ be its lowering operator , and let $\{\pounds_n\}_{n\geq 0}$ be its dual sequence. Then there exists a power series*

$$\varphi(t) = \sum_{k=0}^{\infty} \alpha_n t^n, \alpha_0 \neq 0$$

such that every analytic function f has the expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{\sigma^n \varphi(\sigma) f(0)}{n!} Q_n(z) \tag{2.21}$$

If, morever, the translation operator T_a commutates with σ, then

$$f(z+a) = \sum_{n=0}^{\infty} \frac{\sigma^n \varphi(\sigma) f(a)}{n!} Q_n(z)$$

Corollary 2.4 *[8] Let $\{Q_n\}_{n\geq 0}$ be a Boas -Buck polynomial set generated by the formal relation*

$$G(x,t) = A(t)B(xC(t)) = \sum_{n=0}^{\infty} \frac{Q_n(t)}{n!} t^n \tag{2.22}$$

where

$$A(t) = \sum_{k=0}^{\infty} a_k t^k, \quad B(t) = \sum_{k=0}^{\infty} b_k t^k \text{ and } C(t) = \sum_{k=0}^{\infty} c_k t^{k+1} \tag{2.23}$$

are three formal power series with the condition $a_0 c_0 b_k \neq 0$, *for all* k. *Let* $\nu := \nu_x \in \Lambda^{(-1)}$ *such that*

$$\nu B(xt) = tB(xt)$$

Put $\sigma = C^*(\nu)$ *where* C^**is the inverse of* C *i.e.*

$$C^*(C(t)) = C(C^*(t)) = t, with\ C^*(t) = \sum_{k=0}^{\infty} c_k^* t^{k+1}, c_0^* \neq 0,$$

then every analytic function has the expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{\sigma^n}{A(z)} f(0) \frac{Q_n(z)}{n!}$$

If, moreover, the translation operator T_a *commutates with* σ, *then*

$$f(z+a) = \sum_{n=0}^{\infty} \frac{\sigma^n}{A(z)} f(a) \frac{Q_n(z)}{n!}$$

Definition 2.15 [9] A polynomial set $\{P_n\}_{n\geq 0}$ is called quasi -monomial if and only if it possible to define two operators σ and τ, independent of n, such

$$\begin{aligned} \sigma P_n(x) &= nP_{n-1}(x) \\ \tau P_n(x) &= P_{n+1}(x) \end{aligned} \tag{2.24}$$

Theorem 2.14 *[9] Every polynomial set is quasi-monomial*

Theorem 2.15 *[9] Let* $(\sigma, \tau) \in \Lambda^{(-1)} \times \Lambda^{(1)}$ *and let* $P := \{P_n\}_{n\geq 0}$ *be a polynomial set generated by*

$$G(x,t) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n \tag{2.25}$$

Then we have the equivalences:

$$\sigma G(x,t) = tG(x,t) \Longleftrightarrow \sigma P_n = nP_{n-1}$$

and

$$\tau G(x,t) = \frac{\partial}{\partial t} G(x,t) \Longleftrightarrow \tau P_n = P_{n+1}$$

where $\sigma := \sigma_x$ *and* $\tau := \tau_x$

Definition 2.16 [?] the operator σ is called lowering operator of $\{P_n(x)\}_{n\geq 0}$

Remark 2.3 According to Theorems 2.14 and 2.15 , we deduce that if a function of two variables $G(x,t)$ has an expansion of type 2.25, when $\deg P_n = n$, there exist two operators $(\sigma, \tau) \in \Lambda^{(-1)} \times \Lambda^{(1)}$ such that $G(x,t)$ is a eigenfunction of $\sigma := \sigma_x$,associated with the eigenvalue t and is also a solution of that equation associated with the operator τ.

2.4 Riordan arrays and orthogonal polynomials

Riordan arrays give us an intuitive method of solving combinatorial problems, and help to build an understanding of many number patterns. They provide an effective method of proving combinatorial identities and solving numerical puzzles as in [61] rather than using computer based approaches [62, 77]

In this section we review mainly known results related to integer sequences and Riordan arrays that will be referred to in the rest of the work.

2.4.1 Integer sequences and generating functions

Definition 2.17 The ordinary generating function ($o.g.f.$) of a sequence a_n is the formal power series

$$g(x) = \sum_{n=0}^{\infty} a_n x^n \tag{2.26}$$

Definition 2.18 The exponential generating function ($e.g.f.$) of a sequence a_n is the formal power series

$$h(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \tag{2.27}$$

Definition 2.19 The bivariate generating functions ($b.g.f.'s$), either ordinary or exponential of an array $(a_{n,k})$ are the formal power series in two variables defined by

$$a(x,y) = \sum_{n,k} a_{n,k} x^n y^k \qquad \text{(o,g,f)} \quad \text{and} \tag{2.28}$$

$$b(x,y) = \sum_{n,k} a_{n,k} \frac{x^n}{n!} y^k \qquad \text{(e,g,f)}$$

Notation 2.1 *The coefficient of x^n is denoted by $[x^n]g(x)$ and from the definition of the e.g.f., we have $n![x^n]h(x) = [\frac{x^n}{n!}]hx)$. We refer to the inverse of f as the series reversion $\bar{f}$.*

Lagrange inversion [35] provides a simple method to calculate the coefficients of the series reversion.

Theorem 2.16 *[35] Let $\phi(u) = \sum_{n=0}^{\infty} \phi_k u^k$ be a power series of $\mathbb{C}[[u]]$ with $\phi_0 \neq 0$. Then, the equation $y = z\phi(y)$ admits a unique solution in $\mathbb{C}[[u]]$ whose coefficients are given by*

$$y(z) = \sum_{n=0}^{\infty} y_n z^n, \qquad y_n = \frac{1}{n}\left[u^{n-1}\right]\phi(u)^n$$

The Lagrange Inversion Theorem may be written as

$$[x^n]\, G\left(\bar{f}(x)\right) = \frac{1}{n}\left[x^{n-1}\right] G'(x)\left(\frac{x}{f(x)}\right)^n$$

The simplest case is that of $G(x) = x$, in which we get

$$[x^n]\, \bar{f}(x) = \frac{1}{n}\left[x^{n-1}\right]\left(\frac{x}{f(x)}\right)^n$$

If the product of two power series f and g is 1 then f and g are termed reciprocal sequences and satisfy the following. For *o.g.f.'s* we have [161]

Definition 2.20 A reciprocal series $g(x) = \sum_{n=0}^{\infty} a_n x^n$ with $a_0 = 1$, of a series $f(x) = \sum_{n=0}^{\infty} b_n x^n$ with $b_0 = 1$, is a power series where $g(x)f(x) = 1$, which can be calculated as follows

$$\sum_{n=0}^{\infty} a_n x^n = -\sum_{n=0}^{\infty}\sum_{i=1}^{n} b_i a_{n-i} x^n, \qquad a_0 = 1 \tag{2.29}$$

and for e.g.f.'s we have

Definition 2.21 A reciprocal series $g(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ with $a_0 = 1$, of a series $f(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$ with $b_0 = 1$, is a power series where $g(x)f(x) = 1$, which can be calculated as follows

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = -\sum_{n=0}^{\infty}\sum_{i=1}^{n} b_i a_{n-i} \frac{x^n}{n!}, \qquad a_0 = 1 \tag{2.30}$$

2.4.2 The Riordan group

Definition 2.22 A Riordan array is a pair $(g(x), f(x))$ in which $g(x)$ and $f(x)$ are formal power series such that $g(0) \neq 0$ and $f(0) = 0$.

The pair defines an infinite, lower triangular array $L = (l_{n,k})_{n,k\in\mathbb{N}}$ where

$$l_{n,k} = [x^n]\, g(x)(f(x))^k$$

If $f'(0) \neq 0$ the Riordan array is called proper.

Remark 2.4 From this definition, it easily follows that $g(x)f(x)^k$ is the generating function of column k in array

Remark 2.5 A lower triangular matrix L is a Riordan Array if the generating function k^{th} column is

$$g(x)f(x)^k, for\ k = 0, 1, 2, 3, ...$$

with

$$\begin{aligned} g(x) &= 1 + g_1x + g_2x^2 + g_3x^3 + ... \\ f(x) &= x + f_2x^2 + f_3x^3 + ... \end{aligned}$$

Theorem 2.17 *The fundamental theorem of Riordan arrays*

$$(g(z), f(z)) \begin{bmatrix} a_0 \\ a_1 \\ . \\ . \\ . \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ . \\ . \\ . \end{bmatrix}$$

where the generating function of the two column vectors are $A(z) = \sum_{n\geq 0} a_n z^n$ *and* $B(z) = \sum_{n\geq 0} b_n z^n$ *respectively. The identity is true if and only if the following equation holds :*

$$g(z)A(f(z) = B(z)$$

Proof. *We look at the Riordan Array* $(g(z), f(z))$ *column by column, and multiply it by* *the column vector on the left-hand side*

$$\begin{bmatrix} \\ g & gf & gf^2 & . & . \\ \\ \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \end{bmatrix}$$

This yields

$$a_0 g + a_1 gf + a_2 gf^2 + ... = g(a_0 + a_1 f + a_2 f^2 + ...) = g(z).A(f(z)) = B(z)$$

and we have our result ■

Definition 2.23 The Riordan group

$$\mathcal{R} = \left\{ \begin{array}{l} (g(z), f(z)) \ / \ (g(z), f(z)) \text{ is a Riordan array and } f(x) = f_0 + f_1 z + f_2 z^2 + ... \ , \\ \text{where } f_0 = 0, f_1 = 1 \end{array} \right\}$$

,

i.e.,

each member of $\mathcal{R}$ is a lower triangular matrix with $1'$s on the main diagonal.The multiplication in $\mathcal{R}$ is

$$(g(z), f(z))(G(z), F(z)) = (g(z)G(f(z)), F(f(z))).$$

The identity is $I = (1, z)$.

The inverse of $(g(z), f(z))$ is $\left(\dfrac{1}{g(\bar{f}(z)}, \bar{f}(z)\right)$, where $\bar{f}(z)$ is the compositional inverse of $f(z), i.e.$

$$f(\ \bar{f}(z)) = \bar{f}(f(z) = z$$

To check the inverse property, we compute

$$\left(\frac{1}{g(\bar{f}(z)}, \bar{f}(z)\right)(g(z), f(z)) = \left(\frac{1}{g(\bar{f}(z)} g(\bar{f}(z)\right), f(\ \bar{f}(z))) = (1, z)$$

2.4.3 The $A-$sequence

The $A-$sequence introduced by Rogers (1978) characterizes the column elements of after the first column

Theorem 2.18 *An infinite lower array $D = (d_{n,k})_{n,k\in\mathbb{N}_0}$ is a riordan array if and only if a sequence $A = \{a_n\}_{n\in\mathbb{N}_0}$ existe such that for every $n, k \in \mathbb{N}_0$ it is true*

$$d_{n+1,k+1} = \sum_{i=0}^{n-k} a_i d_{n,k+i}$$

even more, if $D = (d(x), h(x))$, then the generating function of the $A-$sequence is such that $A(x) = \sum_{i=0}^{\infty} a_i x^i$ satisfies the equation

$$h(x) = xA(h(x)) \Rightarrow A(x) = \frac{x}{\bar{h}(x)}$$

2.4.4 The $Z-$sequence

The $Z-$sequence introduced by Merlini et al characterizes the elements of the first column of a proper Riordan as follows

Theorem 2.19 *Let $D = (d(x), h(x)) = (d_{n,k})_{n,k\in\mathbb{N}}$. Then a unique sequence $Z = ((z_0, z_1, z_2, ...)$ can be determined such that every element in column 0 excluding the element in the first row can be expressed as a linear combination of all the elementin the preceeding row with the coefficients identified as the elements of the sequence Z satisfying the relation*

$$d_{n+1,0} = \sum_{i=0}^{n} z_i d_{n,i} \quad n \in \mathbb{N}_0$$

the generating function of the $Z-$sequence satisfies the equation

$$d(x) = \frac{d_0}{1 - x(Z(h(x))} \Rightarrow Z(x) = \frac{x}{\bar{h}(x)}\left(1 - \frac{1}{d(\bar{h}(x))}\right)$$

2.4.5 Stieltjes matrix

In the context of Riordan arrays, We see the Stieltjes expansion Theorem in [65], defined as follows.

Definition 2.24 Let $L = (l_{n,k})_{n,k}$ be a lower triangular matrix with $l_{i,i} = 1$ for all $i \geq 0$. The Stieltjes matrix S_L associated with L is given by $S_L = L^{-1}\bar{L}$ where $\bar{L}$ is obtained from L by deleting the first row of L, that is, the element in the n^{th} row and k^{th} column of $\bar{L}$ is given by $l_{n,k} = l_{n+1,k}$

Using the definition of the Stieltjes matrix above [65] leads to the following theorem relating the Riordan matrix to a Hankel matrix with a particular decomposition

Theorem 2.20 *[65] Let $H = \{h_{n,k} = a_{n+k}\}_{n,k\geq 0}$ be the Hankel matrix generated by the sequence $1, a_1, a_2, a_3, \ldots$. Assume that $H = LDU$ where*

$$L = (l_{n,k})_{n,k\geq 0} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ l_{1,0} & 1 & 0 & 0 & 0 & \cdots \\ l_{2,0} & l_{2,1} & 1 & 0 & 0 & \cdots \\ l_{3,0} & l_{3,1} & l_{3,2} & 1 & 0 & \cdots \\ l_{4,0} & l_{4,1} & l_{4,2} & l_{4,3} & 1 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$D = \begin{bmatrix} d_0 & 0 & 0 & 0 & \cdots \\ 0 & d_1 & 0 & 0 & \cdots \\ 0 & 0 & d_2 & 0 & \cdots \\ 0 & 0 & 0 & d_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}, d_i \neq 0,\ U = L^T$$

Then the Stieltjes matrix S_L is tridiagonal, with the form

$$\begin{bmatrix} \beta_0 & 1 & 0 & 0 & \cdots \\ \alpha_1 & \beta_1 & 1 & 0 & \cdots \\ 0 & \alpha_2 & \beta_2 & 1 & \cdots \\ 0 & 0 & \alpha_3 & \beta_3 & \cdots \\ 0 & 0 & 0 & \alpha_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where

$$\beta_0 = a_1, \alpha_1 = d_1, \beta_k = l_{k+1,k} - l_{k,k+1}, \alpha_{k+1} = \frac{d_{k+1}}{d_k},\ k \geq 0$$

Now, we state other relevant results from this section, relating to generating functions which satisfy particular Stieltjes matrices. The first result relates to o.g.f.s

Theorem 2.21 *[65] Let H be the Hankel matrix generating by the sequence $1, a_1, a_2, a_3, \ldots$ and let $H = LDL^T$.Then S_L has form*

$$\begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots \\ \alpha_1 & \beta & 1 & 0 & \cdots \\ 0 & \alpha & \beta & 1 & \cdots \\ 0 & 0 & \alpha & \beta & \cdots \\ 0 & 0 & 0 & \alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

if and only if the o.g.f. $g(x)$ *of the sequence* $1, a_1, a_2, a_3, ..$ *is given by*

$$g(x) = \frac{1}{1 - a_1 x - \alpha_1 x f}$$

where

$$f(x) = x(1 + \beta f + \alpha f^2)$$

Solving both equations above give us the required result. Similarly for e.g.f.s we have the following result.

Theorem 2.22 *[65]Let* H *be the Hankel matrix generating by the sequence* $1, a_1, a_2, a_3, ...$ *and let* $H = LDL^T$*.Then* S_L *has form*

$$\begin{bmatrix} \beta_0 & 1 & 0 & 0 & \cdots \\ \alpha_1 & \beta_1 & 1 & 0 & \cdots \\ 0 & \alpha_2 & \beta_2 & 1 & \cdots \\ 0 & 0 & \alpha_3 & \beta_3 & \cdots \\ 0 & 0 & 0 & \alpha_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

if and only if the o.g.f. $g(x)$ *of the sequence* $1, a_1, a_2, a_3, ..$*is given by*

$$g(x) = \int (a_1 - \alpha_1 f)dx, \quad with \ \ g(0) = 1$$

where f *is the solution of*

$$\frac{df}{dx} = 1 + \beta f + \alpha f^2, \ \ f(0) = 0$$

The proof again in [65] involves looking at the form of the n^{th} column of the Riordan array. However, intuitively this result can be from looking at the form of the matrix equation $\bar{L} = LS$. In the case that $L = [g(x), f(x)]$ is an exponential Riordan array, we have the following

Proposition 2.7 *[?]* $\bar{L} = \frac{d}{dx}L$

Proof.

$$\begin{aligned} \frac{d}{dx}\left(\sum_{n=0}^{\infty} g_n \frac{x^n}{n!}\right) &= \sum_{n=1}^{\infty} g_n \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} g_{n+1} \frac{x^n}{n!} \end{aligned}$$

Equating the first column of matrix $\bar{L}$and LS we have

$$\frac{d}{dx}(g(x) = \beta_0 g(x) + \alpha_1 g(x)f(x)$$

and second column equate to

$$\frac{d}{dx}(f(x) = \beta_1 f(x) + \alpha_2 g(x)f(x)^2$$

which gives the required result. ■

The Stieltjes matrix as we have seen above is a tridiagonal infinite matrix which is associated with orthogonal polynomials, and therefore have a generalization of the Stieltjes matrix to the Riordan group. Referred to as a production matrix ,it is defined in the following terms

Let P be an infinite matrix (most often it will have integer entries). Letting r_0 be through throw vector $r_0 = (1, 0, 0, 0, ...)$, we define $r_i = r_{i-1}P$, where $i \geq 1$.

Stacking these leads to another infinite matrix which we denote by A_P .Then P is said production matrix for A_P . If we let $U^T = (1, 0, 0, 0, ...)$ then we have

$$A_P = \begin{bmatrix} U^T \\ U^T P \\ U^T P^2 \\ . \\ . \\ . \end{bmatrix}$$

and $DA_P = A_p P$ where $D = (\delta i_{i,j+1})_{i,j\geq 0}$ in [65] , P is called the Stieltjes matrix associated to A_P

2.4.6 the production matrix

In the context of ordinary Riordan arrays, the production matrix associated to a proper Riordan array takes on special form:

Proposition 2.8 *[26] Let P be an infinite production matrix and let A_p be the matrix induced by P. Then A_P is an (ordinary) Riordan matrix if and only if P is of the form*

$$P = \begin{bmatrix} \xi_0 & \alpha_0 & 0 & 0 & 0 & 0 & \cdots \\ \xi_1 & \alpha_1 & \alpha_0 & 0 & 0 & 0 & \cdots \\ \xi_2 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 & \cdots \\ \xi_3 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \cdots \\ \xi_4 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha & \cdots \\ \xi_5 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Moreover, columns 0 *and* 1 *of the matrix* P *are the* $Z-$ *and* $A-$ *sequences, respectively of the Riordan array* A_P

Chapter 3

Sequences of the orthogonal polynomials of Sobolev

3.1 Definitions and properties

We are given 2 functional moments σ and τ and the bilinear form ϕ defined by

$$\phi(p,q) = \langle \sigma, p\ q\rangle + \langle \tau, p'q'\rangle, \quad p,q \in \mathcal{P}. \tag{3.1}$$

Here, we prove that, when ϕ is regular, we can classify the polynomials solution of the Bochner's differential equation. We point out that ϕ is symmetric $(\phi(p,q) = \phi(q,p))$.

Definition 3.1 [41] The numbers

$$\phi_{n,m} := \phi(x^n, x^m), \quad m,n \geq 0.$$

are called the moments of $\phi\,(.,.)$, for any symmetric bilinear form $\phi(.,.)$.

Definition 3.2 [41, 37] A symmetric bilinear form ϕ is said to be quasi-definite (resp defined positive) if it satisfies the Hamburger condition

$$\Delta_n(\phi) =: \det\left[\phi_{i,j}\right]_{i,j=0}^{n} \neq 0 \ \ (\text{resp. } \Delta_n(\phi) > 0,\ n \geq 0). \tag{3.2}$$

Lemma 3.1 *[37] Let ϕ be a symmetric bilinear form. For any normalized sequence of polynomials $\{\Psi_n(x)\}_{n\geq 0}$ we have*

$$\Delta_n(\phi) = \det\left\{\phi(\Psi_i, \Psi_j)\right\}_{i,j=0}^{n} \ , n \geq 0. \tag{3.3}$$

Proof. For any $n \geq 0$, let $A = (a_{i,j})_{i,j=0}^{n}$ be the square matrix of order $(n+1)$ and the sequence $\left\{\tilde{\Psi}_j(x)\right\}_{j=0}^{n}$ defined by

$$\tilde{\Psi}_j(x) = \sum_{k=0}^{n} a_{j,k}\Psi_k(x)\ , 0 \leq j \leq n.$$

Then, we have

$$\begin{cases} \{\phi(\Psi_i(x), \Psi_j(x))\}_{i,j=0}^n A^t = \left\{\phi(\Psi_i(x), \tilde{\Psi}_j(x))\right\}_{i,j=0}^n \\ \text{and} \\ A\left\{\phi(\Psi_i(x), \tilde{\Psi}_j(x))\right\}_{i,j=0}^n = \left\{\phi(\tilde{\Psi}_i(x), \tilde{\Psi}_j(x))\right\}_{i,j=0}^n. \end{cases}$$

So,

$$\left[\phi(\tilde{\Psi}_i(x), \tilde{\Psi}_j(x))\right]_{i,j=0}^n = A\left[\phi(\Psi_i(x), \Psi_j(x))\right]_{i,j=0}^n A^t$$

i, e

$$\det\left[\phi(\tilde{\Psi}_i, \tilde{\Psi}_j)\right]_{i,j=0} = (\det A)^2 \det\left[\phi(\Psi_i, \Psi_j)\right]_{i,j=0}.$$

If we choose a lower triangular matrix such that $a_{j,j} = 1$ and take $\tilde{\Psi}_j(x) = x^j$, for $j = 0, 1, 2, ..., n$, then

$$\Delta_n(\phi) = \det\left[\phi(x^i, x^j)\right]_{i,j=0}^n := \det\left[\phi_{i,j}\right]_{i,j=0}^n = \det\left[\phi(\Psi_i, \Psi_j)\right]_{i,j=0}^n.$$

Hence, 3.3 holds. ■

Lemma 3.2 *[41] A symmetric bilinear form is quasi-defined (resp defined positive) if and only if there exist a sequence of polynomials $\{P_n(x)\}_{n\geq 0}$ and real $k_n \neq 0$ (resp $k_n > 0$) for $n > 0$ such that*

$$\phi(P_n, P_m) = k_n \delta_{n,m} \quad n, m \geq 0. \tag{3.4}$$

In this case, an SP $\{P_n(x)\}_{n\geq 0}$ satisfying 3.4 is uniquely (to a non-zero multiplicative factor) determined.

Corollary 3.1 *The following statements are equivalent.*

a) $\{P_n(x)\}_{n\geq 0}$ is a sequence of orthogonal polynomials with respect to ϕ

$$\begin{cases} \phi(x^m, P_n(x)) = 0 \quad \text{if } 0 \leq m \leq n-1, \\ \phi(x^n, P_n(x)) \neq 0. \end{cases}$$

b) For every polynomial $\pi(x)$ of degree m,

$$\begin{cases} \phi(\pi(x), P_n(x)) = 0 \quad \text{if} \quad 0 \leq m \leq n-1, \\ \phi(\pi(x), P_n(x)) \neq 0 \quad \text{if} \quad m = n. \end{cases}$$

We now investigate the regularity of the form ϕ defined by 3.1.

Remark 3.1 The integral representation of a quasi-definite moment function is always possible according to Duran's classical theorem. It is convenient to define the orthogonality following in the sense of Sobolev.

Definition 3.3 We say that $\{P_n(x)\}_{n\geq 0}$ is a sequence of orthogonal polynomials of Sobolev ($SOPS$) if it is orthogonal to ϕ defined by 3.1, i.e.,

$$\phi(P_n, P_m) := \langle \sigma, P_n P_m \rangle + \langle \tau, P_n' P_m' \rangle = k_n \delta_{n,m}, \quad \text{with } k_n \neq 0,\ n, m \geq 0. \tag{3.5}$$

Lemma 3.3 *[41, 37, 49] Let ϕ be a quasi-definite bilinear form defined by 3.1 and let $\{P_n(x)\}_{n=0}^{\infty}$ be an $SPOS$ with respect to ϕ. Then, σ is the canonical functional of $\{P_n(x)\}_{n=0}^{\infty}$, i.e.,*

$$\langle \sigma, 1 \rangle \neq 0. and \ \ \langle \sigma, P_n(x) \rangle = 0, \qquad n \geq 1.$$

We give necessary and sufficient conditions for the form ϕ to be regular. To this end, we investigate three cases.

Case 1: σ is regular.

Case 2: τ is regular.

Case 3: σ and τ are not regular.

Definition 3.4 The sequence of derivatives of the sequence of polynomials $\{K_n(x,y)\}_{n\geq 0}$ is defined by

$$K_n^{(r,s)}(x,y) = \frac{d^{r+s}}{dx^r dy^s} K_n(x,y), \quad n \geq 0.$$

Let us denote by$\{Q_n(x)\}$, $\{R_n(x)\}_{n=0}^{\infty}$ and $\{P_n(x)\}_{n=0}^{\infty}$ the corresponding $SPON$ verifying the orthogonality conditions

$$\begin{cases} \langle \tau, Q_m(x) Q_n(x) \rangle & = k_n(\tau)\, \delta_{m,n} & n, m \geq 0 \\ \\ \langle \phi, R_m(x) R_n(x) \rangle & = k_n(\phi)\, \delta_{m,n} & n, m \geq 0 \\ \\ \langle \sigma, P_n(x) P_m(x) \rangle & = k_n(\sigma)\, \delta_{m,n} & n, m \geq 0. \end{cases}$$

Now, we begin with the first case (σ is regular). where $\langle\ \delta_c, p\ \rangle := p(c)$, $c \in \mathbb{R}$

Lemma 3.4 *[49] The bilinear and symmetric form ϕ defined by*

$$\phi(p,q) := \langle \sigma, pq \rangle + a \langle \delta_c, p'q' \rangle \quad p, q \in \mathcal{P}$$

is quasi-definite if and only if

$$1 + a K_{n-1}^{(1,1)}(c,c) \neq 0, \quad n \geq 1. \tag{3.6}$$

If ϕ is quasi-definite then

$$R_n(x) = P_n(x) - \frac{a P_n'(x) K_{n-1}^{(0,1)}(x,c)}{1 + a K_{n-1}^{(1,1)}(c,c)} \quad for \quad n \geq 0. \tag{3.7}$$

Moreover, with $K_{-1}(x,y)=0$

$$k_n(\phi)=\frac{a_n}{a_{n-1}}k_n(\sigma) \qquad n\geq 0$$

or

$$\begin{cases} a_n = 1 + aK^{(1,1)}_{n-1}(c,c) & n \geq 0 \\ \\ R'_n(x)=\dfrac{P'_n(c)}{a_{n-1}} & n\geq 0 \\ \\ P_n(x)\neq R_n(x), & \forall n\geq 0 \\ \\ P_0(x)=Q_0(x) \ \ et \ P_1(x)=Q_1(x). \end{cases}$$

Now, we consider the second case ($\sigma=\mathbf{a}\delta_c$).

Lemma 3.5 *[49] Let ϕ be the bilinear and symmetric form defined by*

$$\phi(p,q):=a\langle\delta_c,pq\rangle+\langle\tau,p'q'\rangle.$$

Then, ϕ is quasi-defined (positive definite) if and only if $a\neq 0$ and τ is quasi-defined (resp. $a>0$ and τ is positive definite).

In the case where ϕ is quasi-defined, we denote by$\{Q_n(x)\}$ and $\{R_n(x)\}_{n=0}^{\infty}$ respectively the corresponding SPON verifying the orthogonality

$$\begin{cases} \langle\tau,Q_m(x)Q_n(x)\rangle=k_n\left(\tau\right)\delta_{m,n} \\ \\ \langle\phi,R_m(x)R_n(x)\rangle=k_n\left(\phi\right)\delta_{m,n} \end{cases}$$

then

$$\begin{cases} R_n(c) & =0, \ n\geq 1 \ \ \textit{where } \delta_c(x) \textit{ is a canonical functional of } \{R_n(x)\}_{n=0}^{\infty} \\ \\ k_{n+1}(\phi) & =(n+1)^2k_n(\tau), \quad n\geq 0 \\ \\ Q_n(x) & =\dfrac{1}{n+1}P'_{n+1}(x), \quad n\geq 0 \\ \\ R_n(x) & =\int_c^x nQ_{n-1}(x)dx, \quad n\geq 1, \quad R_0(x)=1 \end{cases}$$

Proposition 3.1 *[49] Suppose that $a\neq 0$ and τ is a moment classical functional verifying Pearson functional equation*

$$(l_2(x)\tau)'=l_1(x)\tau,$$

or

$$l_i(x) = \sum_{j=0}^{i} l_{i,j} x^j, \quad i = 1, 2.$$

Then $\{R_n(x)\}$*satisfies the non-homogeneous equation of order two*

$$l_2(x)y'' + (l_1(x) - l_2(x))y' + (nl_2''(x) - \lambda_n y(x)) = \upsilon_n, \tag{3.8}$$

with

$$\begin{cases} \lambda_n & = l_{22} n(n-1) + l_{11} n \\ \\ \upsilon_n & = l_2(c) R_n''(c) + (l_1(c) + l_2'(c) R_n'(c) \quad \textit{for } n \geq 0. \end{cases}$$

3 Case σ**,** τ are not quasi-definite and ϕ quasi-definite.

Example 3.1 [49] Consider the 2 functional moments σ and τ, defined by

$$\begin{cases} \tau = \dfrac{1}{8}\delta_0 = \dfrac{1}{8}\delta(x), \\ \\ \text{and } \sigma_0 = 3, \quad \sigma_1 = 1 \ \text{ et } \ \sigma_n = \dfrac{1}{n+1}, \quad n \geq 1, \end{cases}$$

or $\{\sigma_n\}_{n \geq 0}$ is the sequence of moments of FM σ.

As

$$\tau(x^n) = \begin{cases} \dfrac{1}{8}, & n = 0 \\ \\ 0, & n \geq 1, \end{cases}$$

we deduce that τ is not quasi-definite.

In addition, we have $\Delta_1(\sigma) = 0$, therefore σ is not quasi-definite.

Let us now define the bilinear form ϕ by

$$\phi(p, q) := \langle \sigma, p\ q \rangle + \frac{1}{8}\langle \delta_0, p'q' \rangle, \quad p, \ q \in \mathcal{P}.$$

We know that

$$\phi(p, p) = \int_0^1 p^2(x)dx + (p'(0)^2 > 0;$$

then, ϕ is positive definite $\Rightarrow$ ϕ is quasi-defined.

We can construct more such examples via

Lemma 3.6 *[49] For any symmetric bilinear form* Ψ*, there is a positive-definite moment functional* σ *such that the symmetric bilinear form* ϕ *defined by*

$$\phi(p, q) := \Psi(p', \ q') + \langle \sigma, pq \rangle$$

is positive-definite.

In particular, for any FM τ, there exists a positive- defined FM σ such that

$$\phi(p,q) := (\sigma, p\ q\rangle + \langle \tau, p'q'\rangle$$

is positive-definite.

3.2 Sobolev Orthogonal polynomials and Second order Differential Equations

Remark 3.2 Let ϕ be a quasi-definite bilinear form given by the relation 3.1 where σ and τ are two FM.

Our aim here is to determine necessary conditions for the $SOPS$ with respect to ϕ, satisfy a second order differential equation having the form

$$L\left[y\right](x) = l_2(x)y" + l_1(x)y' = \lambda_n y, \quad \text{with} \quad L\left[.\right] \quad \text{the differential operator} \tag{3.9}$$

where the coefficients l_i, $i = 1, 2$ and λ_n, $n \geq 0$ are as follow

$$\begin{cases} l_i(x) = \sum\limits_{j=0}^{i} l_{ij}x^j, \quad i = 1, 2 \\ \\ \lambda_n = n(n-1)l_{22} + nl_{11}, \ n \geq 0 \\ \\ l_{11}^2 + l_{22}^2 \neq 0. \end{cases}$$

Theorem 3.1 *[48, 49] Consider the symmetric bilinear form ϕ defined by 3.1. The following statements are equivalent.*

a) *The differential operator $L\left[.\right]$, defined by 3.9 satisfies the following relation*

$$\phi(L\left[p\right], q) = \phi(p, L\left[q\right]), \quad p, q \in \mathcal{P}. \tag{3.10}$$

b) *The two functional moments σ and τ satisfy the following functional equations*

$$(l_2(x)\sigma)' - l_1(x)\sigma = 0 \tag{3.11}$$

$$l_2(x)\tau' - l_1(x)\tau = 0 \tag{3.12}$$

c) *The moments of ϕ, σ and τ are given by*

$$\begin{cases} \phi_{mn} = \sigma_{m+n} + mn\tau_{m+n-2}, \quad m, \ n \geq 0 \\ \\ \tau_{-1} = \tau_{-2} = 0, \end{cases} \tag{3.13}$$

$$\begin{cases} (nl_{22}+l_{11})\sigma_{n+1}+(nl_{21}+l_{10})\sigma_n+nl_{20}\sigma_{n-1}=0, & n\geq 0, \\ \sigma_{-1}=0 \end{cases} \tag{3.14}$$

and

$$\begin{cases} \left[(n+2)l_{22}+l_{11}\right]\tau_{n+1}+\left[(n+1)l_{21}+l_{10}\right]\tau_n+nl_{20}\tau_{n-1}=0, \ n\geq 0 \\ \tau_{-1}=0 \end{cases}. \tag{3.15}$$

Moreover, if $\phi(.,.)$ is quasi-definite and $\{P_n(x)\}_{n=0}^{\infty}$ is a SOPS, then the conditions a), b) and c) are equivalent to condition d), given below

d) $\{P_n(x)\}_{n=0}^{\infty}$ *satisfies the differential equation 3.9.*

Definition 3.5 The functional equations 3.11 and 3.12 are the Sobolev weight equations of the operator $L[.]$ with respect to the form $\phi(.,.)$. Equations 3.14 and 3.15 are the moments of Sobolev for $L[.]$ with respect to $\phi(.,.)$.

Remark 3.3 Theorem 3.1 is a generalization of the Krall theorem concerning the differential equations of the 2^{nd} order of the series of orthogonal polynomials.

Corollary 3.2 *Let $\phi(.,.)$ be a symmetric bilinear form defined by 3.1. Then there exists a sequence of orthogonal polynomials of Sobolev with respect to $\phi(.,.)$ satisfying a differential equation of type 3.9 if and only if both of the following are satisfied*

1) $\phi(.,.)$ *is quasi-definite*

2) *The moments $\{\sigma_n\}_{n=0}^{\infty}$and $\{\tau_n\}_{n=0}^{\infty}$ of σ and τ respectively satisfy the moment equation 3.14 and 3.15.*

Remark 3.4 By deriving 3.9 with respect to x, and setting $y'(x)=Z(x)$, one obtains the following second–order differential equation

$$M[Z](x)=l_2(x)Z''(x)+(l_2'(x)+l_1''(x))Z'(x)=(\lambda_{n+1}-l_1''(x))Z(x). \tag{3.16}$$

This leads us to denote 3.12 and 3.15 respectively by the weight and moment equations of $M[Z](x)$.

Proposition 3.2 *Let $L[.]$ be the differential expression defined by 3.9) and $\phi(.,.)$ is the bilinear form given in 3.1. If $L(p)=\lambda p$ and $L(q)=\upsilon q$ for each $p,q\in\mathcal{P}$ and $\lambda,\ \upsilon\in\mathbb{R}$, with $\lambda\neq\upsilon$, then $\phi(p,q)=0$.*

For any solutions σ and τ of the Sobolev weight equations 3.11 and 3.12, respectively.

Proof. This following from Theorem 3.1, one has

$$\begin{aligned}(\lambda-\upsilon)\phi(p,q) &= \phi(\lambda p,q)-\phi(p,\upsilon q)\\ &= \phi(L(p),q)-\phi(p,l(q))=0.\end{aligned}$$

3.2. Sobolev Orthogonal polynomials and Second order Differential Equations

i.e.

$$\phi(p, q) = 0.$$

■

3.3 Classification of the Sequence Orthogonal Polynomial of Sobolev

In this section, we consider the bilinear form ϕ defined by relation 3.1. Knowing that ϕ is a linear combination of two functional moments, we will try to classify the sequences of polynomials that are solution of the differential equation 3.9. For this purpose, it is assumed initially that one of the two FM is quasi-definite and in a second time that the two are not quasi-definite.

3.3.1 σ quasi-definite

Theorem 3.2 *[48, 49] Suppose that there exists an orthogonal SPOS $\{P_n(x)\}_{n\geq 0}$ with respect to ϕ defined by 3.1 solution of equation 3.9. If σ is quasi-definite, then $\{P_n(x)\}_{n\geq 0}$ is necessarily classical with respect to σ and $\tau = kl_2(x)\sigma$ for some real constant k.*

In addition, we have $\tau = 0$ if $k = 0$, or else τ is also quasi-definite.

Remark 3.5 [49] If σ is quasi-definite and $\tau = c\delta_a$ for all $c \in \mathbb{R}^*$ and for all $a \in \mathbb{R}$, then any OPS with respect to ϕ given by 3.1 is called $SPOS$, but it does not satisfy the differential equation of Bochner's form 3.9.

Moreover, this system satisfies another differential equation of order 2 with the coefficient dependent on parameter n.

Theorem 3.3 *[47] The classical orthogonal polynomials are also characterized by the fact that they are the only OPS satisfying the following condition:*

There exist 2 functional moments τ_1 not identically zero and τ_0 such that

$$\langle \tau_1, P'_m P'_n \rangle + \langle \tau_0, P_m P_n \rangle = 0, \ \textit{for } m \neq n \textit{ and } n = 0, 1. \tag{3.17}$$

3.3.2 τ quasi-definite but σ is not quasi-definite

Theorem 3.4 *[48, 49] Suppose that there exists an orthogonal SPOS $\{P_n(x)\}_{n\geq 0}$ with respect to ϕ defined by 3.1 solution of equation 3.9. If τ is quasi-definite then*

i) $\{P'_n(x)\}_{n\geq 1}$ is a COPS with respect to τ verifies the differential equation 3.16.

ii) $\{P_n(x)\}_{n\geq 0}$ is a $WOPS$ with respect to σ.

iii) $l_2(x)\sigma = k\tau$, for some $k \in \mathbb{R}$. Furthermore, either $l_2(x)\sigma = 0$, or $l_2(x)\sigma$ is quasi-definite.

3.3.3 σ and τ are not quasi-definite

Theorem 3.5 *[48, 49] Let $\{P_n(x)\}_{n\geq 0}$ be a sequence of orthogonal polynomials with respect to ϕ defined by 3.1. If σ and τ are not quasi-defined, then the polynomial sequence $\{P_n(x)\}_{n\geq 0}$ does not satisfy the Bochner equation.*

3.4 The $d-$Orthogonality of Sobolev

We give a generalization of the orthogonality of Sobolev, starting with the study of the regularity of the bilinear forms and then giving the conditions on the polynomial sequences $d-$orthogonal in the sense of Sobolev. We also show that the classical sequences are not classical in the sense of the generalization of the Sobolev orthogonality.

Proposition 3.3 *Let $\{P_n(x)\}_{n\geq 0}$ $d-$OPS with respect to regular $U = (u_0, u_1, ...u_{d-1})^T$. Let $V = (v_0, v_1, ...v_{d-1})$ such as*

$$v_r = u_r + \alpha_r\delta_c \ , \qquad 0 \leq r \leq d-1 \tag{3.18}$$

V regular if and only if

$$1 + L^*_{dn+k-1}(c, c) \neq 0 \tag{3.19}$$

where $L_n(x, c)$ is the polynomial defined as

$$L^*_{dn+k}(x, c) = \sum_{j=0}^{n-1} P_j(c) \left\{ \sum_{r=0}^{d-1} \frac{\alpha_r P_{dj+r}}{\langle u_r, P_{dj+r}P_j\rangle} \right\} + P_n(c) \sum_{r=0}^{k} \frac{\alpha_r P_{dn+r}}{\langle u_r, P_{dn+r}P_n\rangle} \tag{3.20}$$

if $\alpha_r = \alpha$ then $L^*_{dn+k}(x, c) = \alpha L_{dn+k}(x, c)$ where $L_{dn+k}(x, c)$ defined in [72]

Proof. Suppose that V is regular and let

$$Q_m = P_m + \sum_{i=0}^{m-1} \alpha_{m,i}P_i. \ \alpha_{m,i} \in \mathbb{R} \tag{3.21}$$

If we let $m = dn + k$ with $0 \leq k \leq d-1$, then we get

$$\langle u_r, Q_{dn+r}P_j\rangle = \begin{cases} \langle u_r, P_{dn+r}P_n\rangle = \Pi_{v=0}^{n}\gamma^0_{d(v-1)+r+1} & \text{for } \ j = n \\ \alpha_{dn+r,dj+r} \langle u_r, P_{dj+r}P_j\rangle & \text{for } \ 0 \leq j \leq n-1 \end{cases} \tag{3.22}$$

and by 3.18 we also have

$$\langle u_r, Q_{dn+r}P_j\rangle = \begin{cases} \langle v_r, Q_{dn+r}P_n\rangle - \alpha_r Q_{dn+r}(c)P_n(c) & \text{for } \ j = n \\ -aQ_{dn+r}(c)P_n(c) & \text{for } \ 0 \leq j \leq n-1 \end{cases} \tag{3.23}$$

Accordingly, we obtain

$$Q_{dn+k}(x) = P_{dn+k}(x) - Q_{dn+k}(c)L^{*}_{dn+k-1}(x;c) \tag{3.24}$$

Set $x = c$ in 3.24 to get

$$Q_{dn+k}(c)\left[1 + L^{*}_{dn+k-1}(c;c)\right] = P_{dn+k}(c) \tag{3.25}$$

where necessarily $1 + L^{*}_{dn+k-1}(c;c) \neq 0$,otherwise we also have $P_{dn+k}(c) = 0$. It now follows by induction that c should be a common zero for mor than d consecutive polynomials $\{P_n\}$ wich is impossible (corollary 2.3) ■

Let us consider $d-$ bilinear forms $\phi_0, \phi_1, ..., \phi_{d-1}$ $(d \geq 1)$.

Consider a symmetric $d-$ dimensional vector bilinear forms $\phi = (\phi_0, \phi_1, ..., \phi_{d-1})^T$ defined by

$$\phi_r(p,q) := (\sigma_r, p\ q\rangle + \langle \tau_r, p'q' \rangle, \tag{3.26}$$

where $\sigma = (\sigma_0, \sigma_1, ..., \sigma_{d-1})$ and $\tau = (\tau_0, \tau_1, ..., \tau_{d-1})$ are $d-$dimensional functionals.

The $d-$orthogonality in the sense of Sobolev is a generalization of the orthogonality in the sense of Sobolev. On the basis of the $d-$orthogonality definition, we give the following definition.

Definition 3.6 We say that the sequence $\{R_n(x)\}_{n\geq 0}$ is a sequence of $d-$orthogonal polynomials of Sobolev ($d - SOPS$ in short) if it is orthogonal to ϕ which is defined by 3.26, *i.e.*

$$\begin{aligned} &\phi_\alpha(R_m, R_n) = 0, \quad n \geq md + \alpha + 1, \quad m \geq 0, \\ &\langle \phi_\alpha, R_m R_{md+\alpha} \rangle \neq 0, \quad m \geq 0. \end{aligned} \tag{3.27}$$

We give necessary and sufficient conditions for the vector form ϕ defined by 3.26 to be regular. To this end, we study the following cases.

case 1: $\sigma = (\sigma_0, \sigma_1, ..., \sigma_{d-1})$ is regular. In this case, let $\{P_n\}_{n\geq 0}$ be the $d-$COPS with respect to σ.

We notice that if $L_n(x,y)$ is an orthogonal polynomial for some $n \geq 0$, then its derivative is given by

$$L_n^{(r,s)}(x,y) = \frac{d^{r+s}}{dx^r dy^s} L_n(x,y), \quad n \geq 0.$$

Theorem 3.6 *The symmetric $d-$dimensional vector bilinear form $\phi = (\phi_0, \phi_1, ..., \phi_{d-1})$ defined by*

$$\phi_r(p,q) := \langle \sigma_r, pq \rangle + a\langle \delta_c, p'q' \rangle \quad p, q \in \mathcal{P} \quad \textit{and} \quad 0 \leq r \leq d-1 \tag{3.28}$$

is quasi-definite if and only if

$$1 + aL_{m-1}^{(1,1)}(c,c) \neq 0, \quad m \geq 1. \tag{3.29}$$

If ϕ is quasi-definite then

$$R_m(x) = P_m(x) - \frac{aP'_m(x)L_{dn+k-1}^{(0,1)}(x,c)}{1 + aL_{m-1}^{(1,1)}(c,c)} \quad \textit{for} \quad n \geq 0. \tag{3.30}$$

Proof. For the first part , we assume that σ is quasi definite. Then we can write

$$R_m = P_m + \sum_{i=0}^{m-1} \alpha_{m,i} P_i. \tag{3.31}$$

If we assume that $m = dn + k$ with $0 \leq k \leq d-1$, then we get

$$\langle \sigma_r, R_{dn+r} P_j \rangle = \begin{cases} \langle \sigma_r, P_{dn+r} P_n \rangle & \text{for } j = n \\ a_{dn+r,dj+r} \langle \sigma_r, P_{dj+r} P_j \rangle & \text{for } 0 \leq j \leq n-1 \end{cases}. \tag{3.32}$$

From 3.28 , one has

$$\langle \sigma_r, R_{dn+r} P_j \rangle = \begin{cases} \phi_r(R_{dn+r}, P_n) - aR'_{dn+r}(c)P'_n(c) & \text{for } j = n \\ -aR'_{dn+r}(c)P'_n(c) & \text{for } 0 \leq j \leq n-1 \end{cases} \tag{3.33}$$

Accordingly, we obtain

$$R_{dn+k}(x) = P_{dn+k}(x) - aR'_{dn+k}(c)L_{dn+k-1}^{(0,1)}(x;c). \tag{3.34}$$

Differentiating 3.34 and setting $x = c$, then we get

$$R'_{dn+k}(c)\left[1 + L_{dn+k-1}^{(1,1)}(c;c)\right] = P'_{dn+k}(c),$$

where necessarily $1 + L_{dn+k-1}^{(1,1)}(c;c) \neq 0$, otherwise we also have $P'_{dn+k}(c) = 0$. It now follows by induction that c should be a common zero for more than d consecutive polynomials $\{P'_m\}_{m\geq 1}$ which is impossible. Hence, 3.30 follows immediately. ■

case 2: $\tau = (\tau_0, \tau_1, ..., \tau_{d-1})$ is regular. In this case, let $\{Q_n\}_{n\geq 0}$ be the $d-$OPS with respect to to τ.

Now consider the following symmetric $d-$dimensional vector bilinear forms $\phi = (\phi_0, \phi_1, ..., \phi_{d-1})$ defined by

$$\phi_r(p,q) := (a\delta_c, p\ q\rangle + \langle \tau_r, p'q' \rangle. \quad 0 \leq r \leq d-1\ , c \in \mathbb{R} \tag{3.35}$$

Now we let $\{Q_n(x)\}_{n\geq 0}$ be d-OPS with respect to τ satisfying

$$\langle \tau_r, Q_n Q_m \rangle = \begin{cases} 0 & \text{if } n \geq dm + r + 1 \\ k_n(\tau_r) & \text{if } n = dm + r, m \geq 0 \end{cases}$$

and $\{R_n(x)\}_{n\geq 0}$ be d- OPS with respect to ϕ satisfying

$$\langle \tau_r, R_n R_m \rangle = \begin{cases} 0 \ \text{ if } \ n \geq dm+r+1 \\ k_n(\phi_r) \ \text{ if } \ n = dm+r \quad ,m \geq 0 \end{cases},$$

The next proposition will prove useful later on.

Proposition 3.4 *The vector ϕ is quasi-definite if and only if $a \neq 0$ and τ is quasi definite. Then,*

$$\begin{aligned} &R_n(c) = 0 \text{ for } n \geq r+1, \\ &k_{n+1}(\phi) = (dm+r)mk_{dm+1}(\tau), \\ &Q_n = \frac{1}{n+1} R'_{n+1}. \end{aligned}$$

Proof. Assume that ϕ is quasi definite

$$\phi_r(R_m, R_n) = \begin{cases} 0 \quad \text{if } \ n \geq dm+r+1 \quad ,m \geq 0 \\ k_n(\phi) \ \text{ if } \ n = dm+r \quad ,m \geq 0. \end{cases}$$

On the other hand

$$\phi_r(R_m, R_n) := \begin{cases} (a\delta(x-c), R_m R_n\rangle + \langle \tau_r, R'_m R'_n \rangle \\ aR_m(c)R_n(c) + \langle \tau_r, R'_m R'_n \rangle \end{cases}$$

In particular, for $m = 0$ we have

$$\phi_r(R_0, R_n) = aR_n(c) = \begin{cases} 0 \quad \text{if } \ n \geq r+1 \\ k_n(\phi_r) \ \text{ if } \ n = r \end{cases},$$

i.e.,

$$\begin{cases} a = \dfrac{k_r(\phi_r)}{R_r(c)} \neq 0 \\ R_n(c) = 0 \quad \text{ for } \ n \geq r+1 \end{cases}.$$

Thus for m and $n \geq 1$ we have

$$\phi_r(R_m, R_n) = \langle \tau_r, R'_m R'_n \rangle = \begin{cases} 0 \quad \text{if } \ n \geq dm+r+1 \quad ,m \geq 0 \\ k_n(\phi_r) \ \text{ if } \ n = dm+r \quad ,m \geq 0 \end{cases}.$$

Then, τ is quasi definite and $\{R'_n(x)\}$ is $d-$OPS relative to τ. In this case we get

$$R'_n(x) = nQ_{n-1}, \quad n \geq 1$$

and

$$\langle \tau_r, R'_m R'_n \rangle = \langle \tau_r mnQ_{m-1}Q_{n-1} \rangle = \begin{cases} 0 \quad \text{ if } \quad n \geq dm+r+1 \\ (dm+r)mk_{dm+r}(\tau_r) \qquad \text{if } \quad n = dm+r, m \geq 0 \end{cases}$$

Now, we assume that $a \neq 0$, τ is quasi definite and $\{H_n\}_{n\geq 0}$ is MPS, such that $H_n' = nQ_{n-1}$. This leads to

$$\phi_r(H_m, H_n) = aH_m(c)H_n(c) + \langle \tau_r, H_m' H_n' \rangle = \begin{cases} a \text{ if } m = n = 0 \\ 0 \text{ if } m = 0 \text{ and } n \geq r+1 \\ 0 \text{ if } n \geq dm + r + 1 \\ (dm+r)mk_{dm+r-1}(\tau_r) \text{ if } n = dm + r, m \geq 0 \end{cases}$$

i.e. $\{H_n\}_{n\geq 0}$ is a d- OPS with respect to ϕ and by unicity, we obtain $H_n = R_n$. This completes the proof.

In this cas, we shall prove that the 2-classical orthogonal polynomials with respect to τ, satisfy homogeneous third order linear differential equation of Boukhemis type.

$$\alpha_{n,4}(x)Q^{(3)}_{n+3}(x) + \alpha_{n,3}(x)Q^{(2)}_{n+3}(x) + \alpha_{n,2}(x)Q^{(1)}_{n+3}(x) + \alpha_{n,1}(x)Q_{n+3}(x) = 0, \tag{3.36}$$

with $\alpha_{n,i}(x)$ for $i = 1, ..., 4$ are polynomials of degree less than or equal to i and ■

$$\alpha_{n,4}(x) = F_{n,1}(x)S_3(x),$$

is not define a 2−classical OPS in the sense of Sobolev

Theorem 3.7 *Assume that $a \neq 0$ and τ is 2-classic moment functional vector.*

If $\deg \alpha_{n,i}(x) \leq i-1$ then the sequence $\{R_n(x)\}_{n\geq 0}$ satisfies the following non-homogeneous third order linear differential equation

$$\begin{aligned} &\alpha_{n-1,4}R^{(3)}_{n+3} + (\alpha_{n-1,3} - \alpha'_{n-1,4})R^{(2)}_{n+3} + (\alpha''_{n-1,4} + \alpha_{n-1,2} - \alpha'_{n-1,3})R^{(1)}_{n+3} + \\ &(\alpha_{n-1,1} - \alpha'_{n-1,2} - \alpha^{(3)}_{n-1,4} + \alpha''_{n-1,3})R_{n+3} = v_n, \end{aligned} \tag{3.37}$$

where

$$v_n = \alpha_{n-1,4}R^{(3)}_{n+3} + (\alpha_{n-1,3} - \alpha'_{n-1,4})R^{(2)}_{n+3} + (\alpha''_{n-1,4} + \alpha_{n-1,2} - \alpha'_{n-1,3})R^{(1)}_{n+3}\backslash_{x=c},\ n \geq o \tag{3.38}$$

If $\deg \alpha_{n,i}(x) = i$ then the sequence $\{R_n(x)\}_{n\geq 0}$ satisfies the following homogeneous four order linear differential equation:

$$\alpha_{n-1,4}R^{(4)}_{n+3} + \alpha_{n-1,3}R^{(3)}_{n+3} + \alpha_{n-1,2}R^{(2)}_{n+3} + \alpha_{n-1,1}R'_{n+3} = 0. \tag{3.39}$$

Proof. Let $\{Q_n\}_{n\geq 0}$ be a classical orthogonal polynomial sequence with respect to τ satisfying 3.36 and $R_n'(x) = nQ_{n-1}$, $n \geq 1$.

For $n = 0$, R_0 satisfies 3.37.

For $n \geq 1$ and $\deg \alpha_{n,i}(x) \leq i-1$ one has

$$\begin{aligned}
&(\alpha_{n-1,4}R^{(3)}_{n+3} + (\alpha_{n-1,3} - \alpha'_{n-1,4})R^{(2)}_{n+3} + (\alpha''_{n-1,4} + \alpha_{n-1,2} - \alpha'_{n-1,3})R^{(1)}_{n+3})' \\
&= (n+4)(\alpha_{n-1,4}Q^{(2)}_{n+2} + (\alpha_{n-1,3} - \alpha'_{n-1,4})Q^{(1)}_{n+2} + (\alpha_{n-1,2} + \alpha''_{n-1,4} - \alpha'_{n-1,3})Q_{n+2})' \\
&= (n+4)(\alpha_{n-1,4}Q^{(3)}_{n+2} + \alpha_{n-1,3}Q^{(2)}_{n+2} + \alpha_{n-1,2}Q^{(1)}_{n+2} + (\alpha'_{n-1,2} + \alpha^{(3)}_{n-1,4} - \alpha''_{n-1,3})Q_{n+2}) \\
&= (n+4)(-\alpha_{n-1,1}Q_{n-1} + (\alpha'_{n-1,2} + \alpha^{(3)}_{n-1,4} - \alpha''_{n-1,3})Q_{n+2}) \\
&= (-\alpha_{n-1,1} + \alpha'_{n-1,2} + \alpha^{(3)}_{n-1,4} - \alpha''_{n-1,3})R'_{n+3}
\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}
&\alpha_{n-1,4}R^{(3)}_{n+3} + (\alpha_{n-1,3} - \alpha'_{n-1,4})R^{(2)}_{n+3} + (\alpha''_{n-1,4} + \alpha_{n-1,2} - \alpha'_{n-1,3})R^{(1)}_{n+3} \\
&= \int(-\alpha_{n-1,1} + \alpha'_{n-1,2} + \alpha^{(3)}_{n-1,4} - \alpha''_{n-1,3})R'_{n+3}dx \\
&= (-\alpha_{n-1,1} + \alpha'_{n-1,2} + \alpha^{(3)}_{n-1,4} - \alpha''_{n-1,3})R_{n+3} + v_n
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\alpha_{n-1,4}R^{(3)}_{n+3} + (\alpha_{n-1,3} - \alpha'_{n-1,4})R^{(2)}_{n+3} + (\alpha''_{n-1,4} + \alpha_{n-1,2} - \alpha'_{n-1,3})R^{(1)}_{n+3} + \\
&(\alpha_{n-1,1} - \alpha'_{n-1,2} - \alpha^{(3)}_{n-1,4} + \alpha''_{n-1,3})R_{n+3} = v_n
\end{aligned}$$

for some constant v_n. Since $R_n(c) = 0$, $n \geq 1$ we have 3.38.

If $\deg \alpha_{n,i}(x) = i$ and according to 3.40 we obtain 3.39.

Thus the proof is complete. ■

Chapter 4

Characterizations of classical d-OPS

4.1 Introduction

Polynomials are special functions that we can differentiate and integrate many times. Any polynomial may be written as a linear combination of all other polynomials of lower degree, i.e., each polynomial of degree n must satisfy a linear recurrence of order n ($n+1$ terms). If the recurrence is finite, i.e., there exists $m > 0$ such that

$$P_n\left(x\right) = \left(a_n x + b_n\right) P_{n-1} + \sum_{k=m}^{n-2} a_{n,k} P_k,$$

we say that the sequence of polynomials $\left\{P_n\right\}_n$ verifies an orthogonality of such type well defined.

All notions of generalization of the orthogonality introduced by the authors, i.e., multivariate orthogonality, multiple orthogonality, bi-orthogonality, d-orthogonality, ..., are obviously for searching a solution of some problems like Padé approximation, rationality or irrationality solution of higher order ordinary or partial differential equations, ...

Classical orthogonal polynomials play an important role among orthogonal polynomials. The concept of d-orthogonality, d being a positive integer, appears as a particular case of the general multiple orthogonality. In particular, they satisfy orthogonality conditions with respect to several positive measures. Some recent works were focused on the analysis of properties of d-orthogonal polynomials generalizing the classical orthogonal polynomials and some characterization theorems were derived.

Many examples of multiple orthogonal polynomials as well as of d-orthogonal polynomials were derived. They satisfy a linear differential equation of order $d+1$, i.e., when $d = 1$ we meet the standard orthogonality. In this case, we know that there exist only four families of continuous and four families of discrete classical orthogonal polynomials solution of the Bochner's equation, i.e., a linear equation of second order of the form

$$\alpha_2(x)\,y'' + \beta_1(x)\,y' + \gamma y = 0, \tag{4.1}$$

where α_2 and β_1 are polynomials of degree two and one respectively and where γ is constant.

The solution of the equation (4.1) are known as classical orthogonal polynomials. Starting from this point of view, many characterizations of the classical orthogonal polynomials are given.

In this chapter, we wish to generalize most of these well-known characterizations for standard orthogonality in the case of $d-$orthogonality. Most of the results are given in the case d, but according to the complexity of the calculations we give some results in the case $d = 2$.

Notice that the application of the classical polynomials is very large and in various domain. This is our wishes for this study. We hope that our results enlighten such a way for the applications in a new domain that not possible in the case $d = 1$, at least at the moment, we know that the classical 2 -orthogonal polynomials satisfy a third order differential equation which is impossible by the standard orthogonality.

4.2 Preliminaries and notations

4.2.1 m-Symmetric sequences

Problems related to symmetrization of sequences of orthogonal polynomials on the real line play an important role [24]. In general, we have

Definition 4.1 [16] Let m be a nonnegative integer.

(a) A polynomials sequence $\{P_n\}_{n\geq 0}$ is called m-symmetric if

$$P_n(\omega_k x) = \omega_k^m P_n(x), \quad n \geq 0, \quad \text{where} \ \ \omega_k = \exp\{2k\pi i/(m+1)\}, \ 1 \leq k \leq m.$$

(b) A form $\Gamma = (\Gamma_0, ..., \Gamma_{d-1})^T$ is called m-symmetric if

$$\left\langle \Gamma_\upsilon, x^{(m+1)n+\mu} \right\rangle = 0, \quad 0 \leq \mu \leq m, \quad \upsilon \neq \mu, \quad n \geq 0,$$

for each $0 \leq \upsilon \leq d-1$. In this case,

$$\langle \Gamma_\mu, P(x) \rangle = \omega_k^\mu \langle u_\mu, P(\omega_k x) \rangle, \quad \text{where} \ \ \omega_k = \exp\{2k\pi i/(m+1)\}.$$

For the particular case: $m = 1$, we meet the well known notion of symmetric PS [24]. The components of the sequence $\{P_n\}_{n\geq 0}$ are $(m+1)$ sequences $\{P_n^\mu\}_{n\geq 0}$, $0 \leq \mu \leq m$, such that

$$P_{(m+1)n+\mu} = x^\mu P_n^\mu \left(x^{m+1}\right), \quad 0 \leq \mu \leq m, \quad n \geq 0. \tag{4.2}$$

Definition 4.2 [29] A d-OPS $\{P_n\}_{n\geq 0}$ is called classical according to the Hahn property if $Q_n = (n+1)^{-1} P'_{n+1}$ is also a d-OPS.

Since the derivative operator takes each d-symmetric PS into a d-symmetric PS, then a d-OPS $\{P_n\}_{n\geq 0}$ is called d-symmetric classical, if Q_n is d-symmetric d-OPS.

A characteristic property of m-symmetric PS is given by the following

Proposition 4.1 *[29, 16] For each d-OPS $\{P_n\}_{n\geq 0}$ with respect to $\mathcal{U}$, the following statements are equivalent.*

(i) The form $\mathcal{U}$ is d-symmetric,

(ii) The sequence $\{P_n\}_{n\geq 0}$ is d-symmetric,

(iii) The sequence $\{P_n\}_{n\geq 0}$ satisfying the recurrence

$$\begin{aligned} P_{n+d+1}(x) &= xP_{n+d}(x) - \gamma_{n+1}^0 P_n(x), \quad n \geq 0, \\ P_n(x) = & \quad x^n, \quad 0 \leq n \leq d. \end{aligned} \tag{4.3}$$

Note here that if $\{P_n\}_{n\geq 0}$ a $m-$symmetric $d-$OPS, all the components $\{P_n^\mu\}_{n\geq 0}$, $0 \leq \mu \leq m$, are $d-$orthogonal and if moreover $\{P_n\}_{n\geq 0}$ is classical, then all the $(d+1)$ components are classical.

4.2.2 Characterization of $m-$ symmetric

Let d be a positive integer and m de a nonnegative integer satisfying $m \leq d$. Next, we give a necessary condition on m and d to have an $m-$ symmetric $d-$OPS and two characterizations of $m-$ symmetric $d-$OPS. We denote by $\tilde{X}^k$ the multiplication operator by X^k in $\mathcal{P}$

Theorem 4.1 *[16]Let $\{P_n\}_{n\geq 0}$ be a $d-$OPS .Then the following properties are equivalent:*

1. The PS $\{P_n\}_{n\geq 0}$ is $m-$symmetric
2. $d+1$ is a multiple of $m+1$, say $d+1 = p(m+1)$, and the PS $\{P_n\}_{n\geq 0}$ satisfies a $(d+1)-$order recurrence relation of type

$$\tilde{X}P_n = P_{n+1} + \sum_{j=1}^{p} \gamma_{n,j} P_{n-j(m+1)+1}$$

with $\gamma_{n,p} \neq 0$ and the convention $P_{-n} = 0$ for all $n \geq 1$

Theorem 4.2 *[16] Let $\{P_n\}_{n\geq 0}$ be a $d-$OPS .Then its components $\left\{P_n^k\right\}_{n\geq 0}$ $k=0,...,m$, are $d-$ orthogonal*

Classical $d-$OPS

Theorem 4.3 *[16] If $\{P_n\}_{n\geq 0}$ is a $d-$ symmetric classical $d-$OPS, then its the components $\left\{P_n^k\right\}_{n\geq 0}$ $k=0,...,d$, are classical $d-$ orthogonal*

Theorem 4.4 *[16] Let $\{P_n\}_{n\geq 0}$ be an $m-$symmetric classical $d-$OPS then its first component $\{P_n^0\}_{n\geq 0}$ is a classical $d-$ OPS*

4.2.3 d-quasi-orthogonality

In the rest of this section, we recall the notion of d-quasi-orthogonality and some it characterizations.

Definition 4.3 [54] A sequence $\{P_n\}_{n\geq 0}$ is said d-quasi-orthogonal of order s with respect to $\Gamma = (\Gamma_0, ..., \Gamma_{d-1})^T$, if for every $0 \leq \alpha \leq d-1$, there exist $s_\alpha \geq 0$ and $\sigma_\alpha \geq s_\alpha$ integers such that

$$\begin{aligned} &\langle \Gamma_\alpha, P_m P_n \rangle = 0, \quad n \geq (m + s_\alpha)\, d + \alpha + 1, \quad m \geq 0, \\ &\left\langle \Gamma_\alpha, P_{\sigma_\alpha} P_{(\sigma_\alpha + s_\alpha)d+\alpha} \right\rangle \neq 0, \quad m \geq 0, \end{aligned} \tag{4.4}$$

for every $0 \leq \alpha \leq d-1$. We put $s = \max s_\alpha, \quad 0 \leq \alpha \leq d-1$.

Definition 4.4 [54]A sequence $\{P_n\}_{n\geq 0}$ is said strictly $d-$ quasi orthogonal of order s with respect to $\Gamma = (\Gamma_0, ..., \Gamma_{d-1})^T$ if it satisfies

$$\begin{aligned} &\langle \Gamma_v, P_m P_n \rangle = 0, \quad n \geq (m+s)d + v + 1, \quad m \geq 0, \\ &\left\langle \Gamma_\alpha, P_m P_{(m+s)d+v} \right\rangle \neq 0, \quad m \geq 0, \end{aligned} \tag{4.5}$$

for every $0 \leq v \leq d-1$ with $s = \max_{0\leq v\leq d-1} s_\alpha$

Definition 4.5 [54] Let $v \in \mathcal{P}'$ and $d, s \geq 1$ integers. The sequence $\{P_n\}_{n\geq 0}$ is strictly s/d-orthogonal with respect to v if it fulfills:

$$\begin{aligned} &\langle v, P_m P_n \rangle = 0, \quad n \geq md + s, \quad m \geq 0, \\ &\langle v, P_m P_{md+s-1} \rangle \neq 0, \quad m \geq 0, \end{aligned} \tag{4.6}$$

With this definition we can express: $\{P_n\}_{n\geq 0}$ is d-OPS with respect to $\Gamma = (\Gamma_0, ..., \Gamma_{d-1})^T$ if and only if is strictly $(\alpha+1)/d$-orthogonal with respect to Γ_α, $0 \leq \alpha \leq d-1$.

A $\tilde{d}$ -OPS $\{P_n\}_{n\geq 0}$ with respect to V,can be $d-$quasi orthogonal of order s with respect to $-$ of dimension d . We have the following result.

Proposition 4.2 *[54] Let* $\{P_n\}_{n\geq 0}$ *a* $\tilde{d}$ *-OPS with respect* V *. Suppose that exist* Γ *of dimension* d *such that* $\{P_n\}_{n\geq 0}$ *is quasi orthogonal of ordre* s *with respect to* Γ*. Then necessarily* $d \geq \tilde{d}$

We consider only the case $d = \tilde{d}$, where we need the following characterization .

Theorem 4.5 *[54] For any sequence* $\{P_n\}_{n\geq 0}$ *a* d *-OPS with respect* V *, the following statements are equivalent*

(a) *There exist* Γ *of dimension* d *such that* $\{P_n\}_{n\geq 0}$ *is quasi -orthogonal of order* s *with respect to* Γ

(b) *There exist* Γ *of dimension* d *and* s_α*,* $\sigma_\alpha \geq s_\alpha$ *integers,* $0 \leq \alpha \leq d-1$ *such that*

$$\begin{cases} \langle \Gamma_\alpha, P_n \rangle = 0, \quad n \geq s_\alpha d + \alpha + 1, \\ \left\langle \Gamma_\alpha, P_{\sigma_\alpha} P_{d(\sigma_\alpha + s_\alpha)+\alpha} \right\rangle \neq 0, \end{cases} \tag{4.7}$$

for each $0 \leq \alpha \leq d-1$

(c) *There exist* $-$ *of dimension* d *and* d^2 *polynomials* ϕ_α^u*,*$0 \leq u, \alpha \leq d-1$

$$\Gamma_\alpha = \sum_{\alpha=0}^{d-1} \phi_\alpha^u v_u, 0 \leq \alpha \leq d-1 \tag{4.8}$$

where

$$\begin{aligned} &\deg \phi_\alpha^\alpha = s_\alpha, && 0 \leq \alpha \leq d-1, && \textit{and if } \; d \geq 2, \\ &\deg \phi_\alpha^u \leq s_\alpha, && 0 \leq \alpha \leq u-1, && \textit{if } \; 1 \leq \alpha \leq d-1 \\ &\deg \phi_\alpha^u \leq q-1, && u+1 \leq u \leq d-1 && \textit{if } \; 0 \leq \alpha \leq d-2, \end{aligned} \tag{4.9}$$

(d) *There exist* Γ *of dimension* d *such that is strictly quasi orthogonal of order* s *with respect to* $\Gamma = (\Gamma_0, ..., \Gamma_{d-1})^T$

(e) *There exist* $-$*of dimension* d *and* $s_\alpha \geq 0$*,integers,* $0 \leq \alpha \leq d-1$ *such that*

$$\begin{aligned} &\langle \Gamma_\alpha, P_n \rangle = 0, \quad n \geq s_\alpha d + \alpha + 1, \\ &\langle \Gamma_\alpha, P_{s_\alpha d + \alpha} \rangle \neq 0, \end{aligned} \tag{4.10}$$

for each $0 \leq \alpha \leq d-1$

Lemma 4.1 *[54] A sequence $\{P_n\}_{n\geq 0}$satisfy 4.4 can not be d−quasi-orthogonal of order $s-1$ with respect to* $\Gamma = (\Gamma_0, ..., \Gamma_{d-1})^T$

Definition 4.6 [20] A sequence $\{P_n\}_{n\geq 0}$ is said to be weakly d-orthogonal of index (k, l) with respect to $\Gamma = (\Gamma_0, ..., \Gamma_{d-1})^T$ if there exist integers, $k, l \geq 1$ such that

$$\begin{cases} \langle \Gamma_\alpha, P_n \rangle = 0, \quad n \geq d(k_\alpha - 1) + \alpha + 1, \\ \left\langle \Gamma_\alpha, P_{d(k_\alpha - 1)+\alpha} \right\rangle \neq 0, \end{cases} \tag{4.11}$$

$$\begin{cases} \langle \Gamma_\alpha, xP_n \rangle = 0, \quad n \geq d(l_\alpha - 1) + \alpha + 1, \\ \left\langle \Gamma_\alpha, xP_{d(l_\alpha - 1)+\alpha} \right\rangle \neq 0, \end{cases} \tag{4.12}$$

with $k = \max k_\alpha$ and $l = \max l_\alpha$ for every $0 \leq \alpha \leq d-1$.

Then, a strictly d-quasi-orthogonal sequence of order s with respect to Γ, is weakly d-orthogonal of index $(s+1, s+2)$ with respect to $\mathcal{U}$.

Theorem 4.6 *[54] For each sequence $\{P_n\}_{n\geq 0}$ d-OPS with respect to $-$, then the following statements are equivalent.*

(i) There exists $\mathcal{L} \in \mathcal{P}'$ and an integer $s \geq 1$ such that

$$\langle \mathcal{L}, P_n \rangle = 0, \quad n \geq s \quad \text{and} \quad \langle \mathcal{L}, P_{s-1} \rangle \neq 0, \tag{4.13}$$

(ii) There exist $\mathcal{L} \in \mathcal{P}'$, an integer $s \geq 1$ and d polynomials ϕ^α, $0 \leq \alpha \leq d-1$ such that $\mathcal{L} = \sum_{\alpha=0}^{d-1} \phi^\alpha u_\alpha$ with the following properties: if $s-1 = qd + r$, $0 \leq r \leq d-1$, we have

$$\begin{aligned} &\deg \phi^r = q, && 0 \leq r \leq d-1, && \text{and if } d \geq 2, \\ &\deg \phi^\alpha \leq q, && 0 \leq \alpha \leq r-1, && \text{if } 1 \leq r \leq d-1, \\ &\deg \phi^\alpha \leq q-1, && r+1 \leq \alpha \leq d-1, && \text{if } 0 \leq r \leq d-2, \end{aligned} \tag{4.14}$$

Another characterization of d-quasi orthogonality in terms of sequences is the following

Proposition 4.3 *[70]For any two d−OPS,s '$\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ relative to Γ and $\mathcal{V}$ respectively , the following are equivalent*

a Γ and $\mathcal{V}$ satisfy the relations 2.15 and 2.16 for some non zero $d \times d$ matrix polynomials $\oplus = (\phi_\alpha^\mu)$

b there is nonnegative integer l such that

$$P_n(x) = Q_n(x) + \sum_{i=1}^{dl} \alpha_{n,i} Q_{n-i} \quad \text{where} \quad \alpha_{n,dl} \neq 0 \tag{4.15}$$

4.3 Classical d-orthogonal polynomials

The characterizations of classical orthogonal polynomials in the algebraic aspect made by Maroni in [57], can be broaden here in the sens of d–orthogonality. We point out that the results given in the theorem below were the first characterization of d-classical OPS pioneered by Maroni and Douak (see, e.g., [30]). Here we first aim to simplify their proof and we shall give another characterization of $d-$orthogonality.

Theorem 4.7 *[45] For any d-OPS $\{P_n\}_{n\geq 0}$ with respect to $\Gamma = (\Gamma_0, ..., \Gamma_{d-1})^T$, the next statements are equivalent:*

(i) The d-OPS $\{P_n\}_{n\geq 0}$ possesses the Hahn's property.

(ii) There exist two $d \times d$ polynomials matrix Φ and Ψ such that

$$(\Phi\Gamma)' + \Psi\Gamma = 0, \tag{4.16}$$

where $\Phi = (\phi_\alpha^v)$ and Ψ are in the following forms [30]

$$\Psi = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & d-1 \\ \psi(x) & \xi_1 & \cdots & \xi_{d-1} \end{bmatrix} \tag{4.17}$$

with $\psi(x)$ is a polynomial of degree 1, $\{\xi_\mu\}_{\mu=1}^{d-1}$ are constants and

$$\Phi = \begin{bmatrix} \phi_0^0(x) & \cdots & \phi_0^{d-1}(x) \\ \phi_1^0(x) & \cdots & \phi_1^{d-1}(x) \\ \vdots & \ddots & \vdots \\ \phi_{d-1}^0(x) & \cdots & \phi_{d-1}^{d-1}(x) \end{bmatrix} \tag{4.18}$$

where ϕ_α^v are polynomials such that:

$$\begin{aligned} &\deg \phi_\alpha^v \leq 1, \quad 0 \leq v \leq \alpha+1, \qquad \textit{if } \ 0 \leq \alpha \leq d-2, \\ &\deg \phi_\alpha^v = 0, \quad \alpha+2 \leq v \leq d-1, \quad \textit{if } \ 0 \leq \alpha \leq d-3, \\ &\deg \phi_{d-1}^0 \leq 2 \ \ \textit{and} \deg \phi_{d-1}^v \leq 1, \qquad 1 \leq v \leq d-1. \end{aligned} \tag{4.19}$$

(iii) $\{P_n\}_{n\geq 0}$ is d-quasi-orthogonal of order 2 at most with respect to $\mathcal{V} = \Phi\Gamma$ and $\{Q_n\}$ is d-OPS with respect to $\mathcal{V} = \Phi\Gamma$.

Proof. we prove the implication (**i**) $\Rightarrow$ (**ii**) . Suppose that $\{P_n\}_{n\geq 0}$ is classical d-OPS and $\{Q_n\}$ d-OPS with respect to $\mathcal{V}=(v_0,v_1,...,v_{d-1})^T$ where $Q_n=(n+1)^{-1}P'_{n+1}$. By theorem (2.11) we have

$$v_{dn+\mu}=\phi^0_{n,\mu}v_0+\phi^1_{n,\mu}v_1+...+\phi^{d-1}_{n,\mu}v_{d-1}$$

then, taking the derivative and using again the system (2.16) and using also (2.3) we obtain

$$\begin{aligned}&-(dn+\mu+1)\,\Gamma_{dn+\mu+1}=\left(\phi^0_{n,\mu}\right)'v_0+\left(\phi^1_{n,\mu}\right)'v_1+...+\left(\phi^{d-1}_{n,\mu}\right)'v_{d-1}\\&-d\phi^{d-1}_{n,\mu}\psi\Gamma_0-\textstyle\sum_{i=0}^{d-2}\left[(i+1)\phi^i_{n,\mu}+d\xi_{i+1}\phi^{d-1}_{n,\mu}\right]\Gamma_{i+1},\\&\Gamma_{dn+\mu+1}=\varphi^0_{n,\mu+1}\Gamma_0+\varphi^1_{n,\mu+1}\Gamma_1+...+\varphi^{d-1}_{n,\mu+1}\Gamma_{d-1}:=\varphi^0_{\mu+1}\Gamma_0+\varphi^1_{\mu+1}\Gamma_1+...+\varphi^{d-1}_{\mu+1}\Gamma_{d-1}\end{aligned}\tag{4.20}$$

which implies

$$\begin{aligned}\left(\phi^0_{n,\mu}\right)'v_0+\left(\phi^1_{n,\mu}\right)'v_1+...+\left(\phi^{d-1}_{n,\mu}\right)'v_{d-1}=\left[d\phi^{d-1}_{n,\mu}\psi-(dn+\mu+1)\,\varphi^0_{\mu+1}\right]u_0+\\\textstyle\sum_{i=0}^{d-2}\left[(i+1)\phi^i_{n,\mu}+d\xi_{i+1}\phi^{d-1}_{n,\mu}-(dn+\mu+1)\,\varphi^{i+1}_{\mu+1}\right]u_{i+1},\end{aligned}$$

note by

$$\begin{aligned}k_0(n,\mu)\ &:\ =d\phi^{d-1}_{n,\mu}\psi-(dn+\mu+1)\,\varphi^0_{\mu+1}\\&\quad\text{and}\\k_\alpha(n,\mu)\ &:\ =\alpha\phi^{\alpha-1}_{n,\mu}+d\xi_\alpha\phi^{d-1}_{n,\mu}-(dn+\mu+1)\,\varphi^\alpha_{\mu+1}\end{aligned}$$

with

$$\deg k_0(n,\mu)=\begin{cases}\leq n & \text{if}\quad 0\leq\mu\leq d-2\\\leq n+1 & \text{if}\quad \mu=d-1\end{cases}$$

and

$$\deg\left[k_\alpha(n,\mu)\right]=\begin{cases}\leq n-1 \ \text{ if }\ \mu+2\leq\alpha\leq d-1 \text{ and } 0\leq\mu\leq d-3\\\leq n \quad \text{if} \quad 1\leq\alpha\leq\mu+1 \ \text{ and } 0\leq\mu\leq d-2\\=n \ \text{ if } \mu=d-1 \text{ and } \ 1\leq\alpha\leq d-1\end{cases}$$

If we take $n=1$, $\mu=0$ and $\left(\phi^j_{1,j}\right)'=1$, we have

$$v_0=k_0(1,0)u_0+\sum_{\alpha=1}^{d-1}k_\alpha(1,0)u_\alpha.\tag{4.21}$$

And for $n = 1$, $\mu = 1$, we obtain

$$(\phi^0_{1,1})' v_0 + v_1 = k_0(1,1)u_0 + \sum_{\alpha=1}^{d-1} k_\alpha(1,1)u_\alpha., \tag{4.22}$$

to simplify calculations take $(\phi^i_{1,j})' = 1$ for $i \leq j$
i.e.

$$v_1 = \left[k_0(1,1) - k_0(1,0)\right] u_0 + \sum_{\alpha=1}^{d-1} \left[k_\alpha(1,1) - k_\alpha(1,0)\right] u_\alpha \tag{4.23}$$

put $n = 1$, $1 \leq \mu \leq d-1$, we obtain

$$v_\mu = \left[k_0(1,\mu) - k_0(1,\mu-1)\right] u_0 + \sum_{\alpha=1}^{d-1} \left[k_\alpha(1,\mu) - k_\alpha(1,\mu-1)\right] u_\alpha \tag{4.24}$$

Note by

$$k_\alpha(1,0) := \phi^\alpha_0 \qquad \alpha := 0, ..., d-1$$

$$\left[k_\alpha(1,\mu) - k_\alpha(1,\mu-1)\right] := \phi^\alpha_\mu \quad \alpha := 0, ..., d-1 \text{ and } \mu = 1, ...d-1$$

with

$$\begin{array}{c} \deg \phi^\alpha_\mu \ \leq 1 \text{ , if } \ 0 \leq \alpha \leq \mu + 1 \ \text{ and } \ 0 \leq \mu \leq d-2 \\ \deg \phi^\alpha_\mu \ = 0 \ \text{ if } \mu + 2 \leq \alpha \leq d-1 \text{ and } \ 0 \leq \mu \leq d-3 \\ \deg \phi^\alpha_{d-1} \ \leq 1 \text{ if } \ 1 \leq \alpha \leq d-1 \text{ and } \ \deg \phi^0_{d-1} \ \leq 2 \end{array}$$

hence we have obtain

$$\mathcal{V} = \Phi\left(x\right)\Gamma.$$

No, by taking $n = 0$, $\mu = d-1$ in (4.20) and using also (2.3) we obtain

$$\begin{aligned} -d\Gamma_d &= c_0\Gamma_1 + c_1\Gamma_2 + ... + c_{d-2}\Gamma_{d-1} + c_{d-1}v'_{d-1} \\ &= -d\left[\psi_1\Gamma_0 + \lambda_1\Gamma_1 + \lambda_2\Gamma_2 + ... + \lambda_{d-1}\Gamma_{d-1}\right], \qquad \deg \psi_1 = 1 \end{aligned}$$

i.e

$$\begin{aligned} v'_{d-1} &= \frac{1}{c_{d-1}}\left[-d\psi_1\Gamma_0 - (c_0 + d\lambda_1)\Gamma_1 - .... - (c_{d-2} + d\lambda_{d-1})\Gamma_{d-1}\right] \\ : &= -\left[\psi\Gamma_0 + \xi_1\Gamma_1 + ... - +\zeta_{d-1}\Gamma_{d-1}\right] \end{aligned}$$

whence (4.16) and (4.17).

4.3. Classical d-orthogonal polynomials

(ii)⇒(iii) Since $\mathcal{V} = \Phi\Gamma$, then

$$\langle v_\alpha, P_n\rangle = \sum_{\mu=0}^{d-1} \langle \Gamma_\mu, \phi_\alpha^\mu P_n\rangle = 0, \quad n \geq d+\mu+1,$$

$$\langle v_\alpha, P_{d+\alpha}\rangle = \langle \Gamma_\alpha, \phi_\alpha^\alpha P_{d+\alpha}\rangle \neq 0,$$

for $0 \leq \alpha \leq d-2$ and

$$\begin{array}{l} \langle v_{d-1}, P_n\rangle = \left\langle \Gamma_0, \phi_{d-1}^0 P_n\right\rangle + \sum_{\mu=1}^{d-1} \left\langle \Gamma_\mu, \phi_{d-1}^\mu P_n\right\rangle = 0, \quad n \geq 2d+\mu+1, \\ \langle v_{d-1}, P_{2d}\rangle = \left\langle \Gamma_0, \phi_{d-1}^0 P_{2d+\mu}\right\rangle \neq 0. \end{array}$$

Then there exists $0 \leq t \leq 2$ such that $\{P_n\}$ is d-quasi-orthogonal of order $2-t$ with respect to $\mathcal{V} = \Phi-$.

(iii)⇒(i) Trivial since by hypothesis both $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are d-OPS. ■

Unlike in the case $d = 1$, if $\{P_n\}$ is d-classical ($d \geq 2$), we can't conclude from Pearson equation that $\{Q_n\}$ is also d-classical. Indeed, since $\{Q_n\}$ is d-OPS with respect to $\mathcal{V} = \Phi\Gamma$, then

$$(\Phi\mathcal{V})' = \Phi'\mathcal{V} + \Phi\mathcal{V}' = \Phi'\mathcal{V} + \Phi\Psi\Gamma := \Psi_1\mathcal{V} \Leftrightarrow \Phi\Psi = \Psi\Phi.$$

Then, the following problem is open: the derivative of classical d-OPS is again classical d-OPS?

We try to give an answer to this question in the present section.

Fairly, it seems as the first look that the answer is negative! In fact, we have an example [17, p.86, Tableau II]. In spite of that the family (E) satisfies a second order linear differential equation, meanwhile this family isn't classical at all.

Our results show that the solution of Boukhemis and Zerouki differential equation [19] are the only 2 -classical OPS for which any derivative of that sequence is again $2-$OPS. For this end, we set the following definition

Definition 4.7 A sequence of d-orthogonal polynomials $\{P_n\}$ is called very classical d-OPS if $\{P_n\}$ as well as it's derivative of any order are classical d-OPS.

So now, the question is: determine all the very classical $d-$OPS.

On the other hand, note here also that we can extract a simple characterization from proposition 2.6 as follows (see [57])

Proposition 4.4 *Let $\{P_n\}_{n\geq 0}$ monic sequence verifying (4.4) such that*

(a) $\chi_{n+d-1,v} = 0, \quad 0 \leq v \leq n-1$ *and* $\chi_{v+d-1,v} \neq 0, \quad v \geq 0,$

(b) $\{Q_n\}_{n\geq 0}$ *is d-OPS with respect to* $\mathcal{V}$.

Then, the sequence $\{P_n\}_{n\geq 0}$ is d-classical OPS.

According to the last assertion of Proposition 4.3, since $\{P_n\}_{n\geq 0}$ is d-classical and $Q_n = (n+1)^{-1} P_{n+1}$ is also d-OPS, and from theorem 4.7 we know that $\{P_n\}_{n\geq 0}$ is d-quasi-orthogonal of order two at most, then we have the following characterization of d-classical OPS

Corollary 4.1 *Let $\{P_n\}_{n\geq 0}$ be d-OPS with respect to $\Gamma = (\Gamma_0, ..., \Gamma_{d-1})^T$. Then $\{P_n\}_{n\geq 0}$ is d-classical if and only if there exists an integer $\rho \leq 2d$ such that*

$$P_n(x) = \sum_{\nu=n-\rho}^{n} \lambda_{n,v} Q_\nu(x), \quad n \geq \rho. \tag{4.25}$$

Note that when $d = 1$, this characterization is given in [52, 53]. And for $d = 2$, this result proved first by Maroni in [58] using another approach.

The next characterization is presented for the semi-classical case in [71], we give here a direct proof for the classical character and it can be also used to prove a similar result in the semi-classical case[71] .

Theorem 4.8 *[71] Let $\{P_n\}_{n\geq 0}$ be d-OPS with respect to $\Gamma = (\Gamma_0, ..., \Gamma_{d-1})^T$. Then $\{P_n\}_{n\geq 0}$ is d-classical if and only if $\{Q_n\}_{n\geq 0}$ is weakly d-orthogonal of index $(1, 2)$.*

Proof. Suppose $\{P_n\}_{n\geq 0}$ d-classical, then $\{Q_n\}_{n\geq 0}$ is d-OPS with respect to $\mathcal{V} = \Phi-$. Afterwards, we have

$$\langle v_\mu, x^m Q_n \rangle = 0, \quad n \geq dm + \mu + 1,$$
$$\langle v_\mu, x^m Q_{dm+\mu} \rangle \neq 0.$$

In particular, respectively for $m = 0$ and $m = 1$, we have

$$\begin{cases} \langle v_\mu, Q_n \rangle = 0, \quad n \geq \mu + 1, \\ \langle v_\mu, Q_\mu \rangle \neq 0, \end{cases}$$

and from the $(d+2)$ order recurrence of $\{Q_n\}$, we have

$$\begin{cases} \langle v_\mu, x Q_n \rangle = 0, \quad n \geq d + \mu + 1, \\ \langle v_\mu, x Q_{d+\mu} \rangle = \tilde{\gamma}^0_{\mu+1} \langle v_\mu, Q_\mu \rangle \neq 0. \end{cases}$$

Comparing with the Definition 4.6, we conclude that $(k, l) = (1, 2)$.

Conversely, suppose that $\{Q_n\}$ is weakly d-orthogonal of index $(1, 2)$ with respect to $\mathcal{V}$ and we show that $\{P_n\}$ is d-classical with respect to Γ, i.e., we will prove that Γ satisfy Pearson equation (4.16). Let $\mathcal{W} = (w_0, ..., w_{d-1})^T$ such that $\mathcal{W} = \mathcal{V}'$, i.e. $\langle w_\alpha, \pi \rangle = \langle v'_\alpha, \pi \rangle$, $\forall \pi \in \mathcal{P}$. Then

$$\langle w_\alpha, P_n \rangle = -n \langle v_\alpha, Q_{n-1} \rangle = 0, \quad n \geq \alpha + 2,$$
$$\langle w_\alpha, P_{\alpha+1} \rangle = -(\alpha + 1) \langle v_\alpha, Q_\alpha \rangle \neq 0.$$

Then, there exists $\alpha \leq t_\alpha \leq \alpha+1$ such that

$$\begin{aligned} \langle w_\alpha, P_n\rangle &= 0, \quad n \geq t_\alpha+1, \\ \langle w_\alpha, P_{t_\alpha}\rangle &\neq 0. \end{aligned}$$

Hence, by theorem 4.6, there exist d polynomials ψ_α^μ, $\alpha \leq \mu \leq d-1$, such that

$$w_\alpha = \sum_{\mu=0}^{d-1} \psi_\alpha^\mu \Gamma_\mu$$

with, if $t_\alpha = q_\alpha d + r_\alpha$, $0 \leq r_\alpha \leq d-1$, we have

$$\begin{aligned} &\deg \psi_\alpha^{r_\alpha} = q_\alpha, && 0 \leq r_\alpha \leq d-1, && \text{and if } d \geq 2, \\ &\deg \psi_\alpha^\mu \leq q_\alpha, && 0 \leq \mu \leq r_\alpha - 1, && \text{if } 1 \leq r_\alpha \leq d-1, \\ &\deg \psi_\alpha^\mu \leq q_\alpha - 1, && r_\alpha + 1 \leq \mu \leq d-1, && \text{if } 0 \leq r_\alpha \leq d-2. \end{aligned}$$

Since $\alpha \leq t_\alpha \leq \alpha+1$, we distinguish two cases:

(1) If $0 \leq \alpha \leq d-2$, whence $\alpha \leq t_\alpha \leq \alpha+1 < d$. Consequently, $q_\alpha \leq 0$.

(2) If $\alpha = d-1$, then $d-1 \leq t_{d-1} \leq d$, whence $q_{d-1} \leq 1$.

If $q_{d-1} = 1$, then $r_{d-1} = 0$ and $t_{d-1} = d$, and if $q_{d-1} = 0$, then $r_{d-1} \leq d-1$. Which prove that all polynomials are constant except $\psi_{d-1}^0 = ax + b$.

Now, it remain to show that the matrix Ψ is of the form (4.17). Indeed, from (2.3), we have

$$\mathcal{V}' = \begin{bmatrix} v_0' \\ v_1' \\ \vdots \\ v_{d-1}' \end{bmatrix} = -\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & & d \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_d \end{bmatrix},$$

and from (2.15) and (2.16), we obtain

$$\Gamma_d = \phi_{1,0}^0(x)\, u_0 + \sum_{\mu=1}^{d-1} \phi_{1,0}^\mu(x)\, \Gamma_\mu$$

where $\phi_{1,0}^0(x) = cx + d$ and where $\phi_{1,0}^\mu(x) = \xi$ for $1 \leq \mu \leq d-1$. Hence

$$\begin{bmatrix} v_0' \\ v_1' \\ \vdots \\ v_{d-1}' \end{bmatrix} = -\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & d-1 \\ \psi(x) & \xi_1 & \xi_2 & \cdots & \xi_{d-1} \end{bmatrix} \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \\ \vdots \\ \Gamma_{d-1} \end{bmatrix}$$

Now, we let the form $\mathcal{W} = (w_0, ..., w_{d-1})^T$ such that $\mathcal{W} = (x\mathcal{V})'$, i.e. $\langle w_\alpha, \pi \rangle = \left\langle (xv_\alpha)', \pi \right\rangle$, $\forall \pi \in \mathcal{P}$. So, since $\{Q_n\}$ is weakly d-orthogonal of index $(1,2)$ with respect to $\mathcal{V}$, then

$$\begin{aligned}
\langle w_\alpha, P_n \rangle &= \left\langle (xv_\alpha)', P_n \right\rangle = -n \langle v_\alpha, xQ_{n-1} \rangle = 0, \quad n \geq d+\alpha+2, \\
\langle w_\alpha, P_{d+\alpha+1} \rangle &= -(d+\alpha+1) \langle v_\alpha, xQ_{d+\alpha} \rangle \neq 0.
\end{aligned}$$

Then, there exists $\alpha \leq t_\alpha \leq d+\alpha+1$ such that

$$\begin{aligned}
&\langle w_\alpha, P_n \rangle = 0, \quad n \geq t_\alpha + 1, \\
&\langle w_\alpha, P_{t_\alpha} \rangle \neq 0,
\end{aligned}$$

In addition, there exist d polynomials ϑ_α^μ, $\alpha \leq \mu \leq d-1$, such that

$$w_\alpha = \sum_{\mu=0}^{d-1} \vartheta_\alpha^\mu u_\mu$$

put $t_\alpha = q_\alpha d + r_\alpha, \ \ 0 \leq r_\alpha \leq d-1$, we have

$$\begin{aligned}
&\deg \vartheta_\alpha^{r_\alpha} = q_\alpha, && 0 \leq r_\alpha \leq d-1, && \text{and if } \ d \geq 2, \\
&\deg \vartheta_\alpha^\mu \leq q_\alpha, && 0 \leq \mu \leq r_\alpha - 1, && \text{if } \ 1 \leq r_\alpha \leq d-1 \\
&\deg \vartheta_\alpha^\mu \leq q_\alpha - 1, && r_\alpha + 1 \leq \mu \leq d-1, && \text{if } \ 0 \leq r_\alpha \leq d-2.
\end{aligned}$$

Since $\alpha \leq t_\alpha \leq d+\alpha+1$, we distinguish several cases:

(1) If $0 \leq \alpha \leq d-2$, whence $\alpha \leq t_\alpha \leq d+\alpha+1 < 2d$. Consequently, $q_\alpha \leq 1$.

(2) If $q_\alpha = 1$, then $t_\alpha = d + r_\alpha \leq d+\alpha+1$ hence $r_\alpha \leq \alpha+1$. If $q_\alpha = 0$, then necessary $\alpha \leq r_\alpha$.

(3) If $\alpha = d-1$, then $d-1 \leq t_{d-1} \leq 2d$, whence $q_{d-1} \leq 2$.

If $q_{d-1} = 2$, then $r_{d-1} = 0$, and if $r_{d-1} \leq d-1$, then $q_\alpha \leq 1$.
If $q_{d-1} = 0$, then $r_{d-1} = d-1$. This show that only ϑ_{d-1}^0 is of degree two.
Now, we have $\mathcal{V}' = \Psi\Gamma$ and $(x\mathcal{V})' = \Theta\Gamma$. If we let $\mathcal{V} = \Phi\Gamma$, then

$$\Theta - = (x\mathcal{V})' = x\mathcal{V}' + \mathcal{V} = x\Psi\Gamma + \Phi\Gamma.$$

Hence $\Phi = \Theta - x\Psi$ with $\mathcal{V} = \Phi\Gamma$ and $(\Phi\Gamma)' = \Psi\Gamma$. ■

Now for any sequence of OPS it's necessary to know their explicit expression of the measure for which the orthogonality is satisfied. It is however good enough to give a such tool for determining the orthogonalization measure. The quasi-monomiality principle is a very useful tool as well. Because it's not required any information about the classical character. Our approach deals only with the classical OPS and it will be possible to generalized for semi classical case. In the following, we prove a new characterization of 2−classical OPS, which will be without a doubt the pillar in this context. Note here that the next results are used before by Maroni and Douak to give an integral representation for some particular class studied starting from the corresponding generating functions.

Theorem 4.9 *For any 2-OPS $\{P_n\}_{n\geq 0}$ with respect to $\Gamma = (\Gamma_0, \Gamma_1)^T$, the next statements are equivalent:*

(i) The functional moment Γ satisfy the Pearson equation (4.16) with (4.17), (i.e. Γ is classical),

(ii) There exist $\{\alpha_n^i\}_{i=1}^3$ and $\{\beta_n^j\}_{j=1}^3$ polynomials of degree n, and μ_1, H_2, K_2, M_2 and N_3, L_3 of degree ≤ 1, 2, 3 respectively, such that

$$\begin{cases} \alpha_4^3(x)\,\Gamma_0'' + \alpha_3^2(x)\,\Gamma_0' + \alpha_2^1(x)\,\Gamma_0 = 0, \\ \beta_5^3(x)\,\Gamma_1'' + \beta_4^2(x)\,\Gamma_1' + \beta_3^1(x)\,\Gamma_1 = 0, \end{cases} \tag{4.26}$$

with

$$\begin{cases} H_2(x)\,\Gamma_0 = K_2(x)\,\Gamma_1 + L_3(x)\,\Gamma_1', \\ \mu_1(x)\,\Gamma_1 = M_2(x)\,\Gamma_0 + N_3(x)\,\Gamma_0'. \end{cases} \tag{4.27}$$

and

$$\begin{cases} H_2 = \omega'\phi - \omega(\phi' - \psi), \quad K_2 = \omega\theta_1 - \phi\pi_1, \quad L_3 = \omega\theta - \phi\pi, \\ \mu_1 = \theta\pi_1 - \pi\theta_1, \quad M_2 = \pi(\phi' - \psi) - \theta\omega', \quad N_3 = \pi\phi - \omega\theta, \end{cases}$$

Proof. Write the matrices in (4.16) when $d = 2$ in the form

$$\Psi = \begin{bmatrix} 0 & 1 \\ \psi(x) & \xi \end{bmatrix} \quad \text{and} \quad \Phi = \begin{bmatrix} \omega(x) & \pi(x) \\ \phi(x) & \theta(x) \end{bmatrix}$$

with ξ is constant and the polynomials ψ, ω, π and θ are of degree 1 at most and $\deg\phi \leq 2$. In this case, the system (4.16) is equivalent to

$$\begin{aligned} &\omega'\Gamma_0 + \omega\Gamma_0' + \pi_1\Gamma_1 + \pi\Gamma_1' = 0, \\ &(\phi' - \psi)\,\Gamma_0 + \phi\Gamma_0' + \theta_1\Gamma_1 + \theta\Gamma_1' = 0, \end{aligned} \tag{4.28}$$

with $\pi_1 = \pi' - 1 = C^{te}$ and $\theta_1 = \theta' - \xi = C^{te}$.

First we establish all possible cases of the second equation in (4.26). So, eliminate Γ_0' from (4.28) we get

$$H_2(x)\,\Gamma_0 = K_2(x)\,\Gamma_1 + L_3(x)\,\Gamma_1', \tag{4.29}$$

where

$$H_2 = \omega'\phi - \omega\left(\phi' - \psi\right), \quad K_2 = \omega\theta_1 - \phi\pi_1, \quad L_3 = \omega\theta - \phi\pi, \tag{4.30}$$

with $\deg H_2 \leq 2$, $\deg K_2 \leq 2$ and $\deg L_3 \leq 3$.

We distinguish three possible cases according to the degree of the polynomial $H_2(x)$.

1) $\deg H_2 = 0$, i.e. $H_2(x) = C^{te} := C$. In this case, replacing Γ_0 and Γ_0' obtained from (4.29) in the second equation of (4.28) we find

$$A_5(x)\,\Gamma_1'' + B_4(x)\,\Gamma_1' + D_3(x)\,\Gamma_1 = 0, \tag{4.31}$$

with

$$\begin{aligned} D_3(x) &= \tfrac{1}{C}\left(\phi' - \psi\right) K_2 + \tfrac{1}{C}\phi K_2' + \theta_1 \\ B_4(x) &= \tfrac{1}{C}\left(\phi' - \psi\right) L_3 + \tfrac{1}{C}\phi\left(K_2 + L_3'\right) + \theta \\ A_5(x) &= \tfrac{1}{C}\phi L_3. \end{aligned} \tag{4.32}$$

Now it is necessary to search a common factor to reduce the degree of the polynomials A_5, B_4 and D_3. In fact, we have

$$D_3(x) = \phi(x)\,\beta_3^1(x), \quad B_4(x) = \phi(x)\,\beta_4^2(x) \quad \text{and} \quad A_5(x) = \phi(x)\,\beta_5^3(x)$$

with

$$\begin{aligned} \beta_3^1 &= \tfrac{1}{C}\left[\left(\psi - 2\phi'\right)\pi_1 + \omega'\theta\right] \\ \beta_4^2 &= \tfrac{1}{C}\left[\left(\psi - 2\phi'\right)\pi + K_2 + 2\omega'\theta + \omega\theta' - \phi\pi'\right]. \end{aligned} \tag{4.33}$$

Hence, we have (4.26) with $\deg \beta_5^3 \leq 3$, $\deg \beta_4^2 \leq 2$ and $\deg \beta_3^1 \leq 1$.

2) $\deg H_2 = 1$, put $H_2'(x) = t$. In this case, we have

$$t\Gamma_0 = \left(H_2\Gamma_0\right)' - H_2\Gamma_0'.$$

Eliminating Γ_0 from the system (4.28), we obtain

$$H_2(x)\,\Gamma_0' = R_1(x)\,\Gamma_1 + S_2(x)\,\Gamma_1', \tag{4.34}$$

then

$$t\Gamma_0 = \left(K_2' - R_1\right)\Gamma_1 + \left(K_2 + K_3' - S_2\right)\Gamma_1' + K_3\Gamma_1''. \tag{4.35}$$

Using again the second equation of the system (4.28) to get

$$\eta^1(x)\,\Gamma_1'' + \eta^2(x)\,\Gamma_1' + \eta^3(x)\,\Gamma_1 = 0,$$

where

$$\begin{aligned} \eta^3 &= \left(\phi' - \psi\right)\left[\tfrac{1}{t}\left(K_2' - R_1\right)H_2 + \phi\pi_1 - \omega\theta_1\right] = \left(\phi' - \psi\right)\beta_3^1 \\ \eta^2 &= \left(\phi' - \psi\right)\left[\tfrac{1}{t}\left(K_2 + L_3' - R_2\right)H_2 + \phi\pi - \omega\theta\right] = \left(\phi' - \psi\right)\beta_4^2 \\ \eta^1 &= \tfrac{1}{t}\left(\phi' - \psi\right)H_2L_3 = \left(\phi' - \psi\right)\beta_5^3 \end{aligned}$$

with $\deg \beta_5^3 \leq 4$, $\deg \beta_4^2 \leq 3$ and $\deg \beta_3^1 \leq 2$.

3) $\deg H_2 = 2$. In this case, we have by using (4.34)

$$H_2' \Gamma_0 = (K_2 \Gamma_1 + L_3 \Gamma_1')' - R_1 \Gamma_1 - S_2 \Gamma_1'.$$

Multiplying the first equation of the system (4.28) by $H_2 H_2'$ we obtain the second equation of (4.26) where

$$\begin{aligned}
\beta_3^1 &= \omega' (\phi' - \psi) (K_2' - R_1) H_2 + \omega H_2' R_1 + \pi_1 H_2 H_2' \\
\beta_4^2 &= \omega' (\phi' - \psi) (K_2 + L_3' - S_2) H_2 + \omega H_2' S_2 + \pi H_2 H_2' \\
\beta_5^3 &= \omega' H_2 L_3,
\end{aligned}$$

with $\deg \beta_5^3 \leq 5$, $\deg \beta_4^2 \leq 4$ and $\deg \beta_3^1 \leq 3$.

For the first equation of (4.26), eliminate u_1' from from (4.28) we get

$$\mu_1 (x) \Gamma_1 = M_2 (x) \Gamma_0 + N_3 (x) \Gamma_0', \tag{4.36}$$

where

$$\mu_1 = \theta \pi_1 - \pi \theta_1, \;\; M_2 = \pi (\phi' - \psi) - \theta \omega', \;\; N_3 = \pi \phi - \omega \theta, \tag{4.37}$$

with $\deg \mu_1 \leq 1$, $\deg M_2 \leq 2$ and $\deg N_3 \leq 3$.

We distinguish two possible cases according to the degree of the polynomial $\mu_1 (x)$.

1) $\deg \mu_1 = 0$, i.e. $\mu_1 (x) = C^{te} := C$. As in the first part, we obtain after simplification the first equation of (4.26) with

$$\begin{aligned}
\alpha_2^1 &= \pi_1 (\phi' - \psi) - \omega_1' \theta + M_2' \\
\alpha_3^2 &= \phi \pi_1 - \omega \theta_1 + M_2 + N_3' \\
\alpha_4^3 &= N_3,
\end{aligned}$$

where $\deg \alpha_4^3 \leq 3$, $\deg \alpha_3^2 \leq 2$ and $\deg \leq 1$.

2) $\deg \mu_1 = 1$, i.e. $\mu_1' (x) = k$ in the same way we have the first equation of (4.26) with

$$\begin{aligned}
\alpha_2^1 &= \omega' \mu_1 + \tfrac{1}{k} \pi_1 \mu_1 + \pi \theta_1 \\
\alpha_3^2 &= \omega \mu_1 + \tfrac{1}{k} \pi_1 \mu_1 (M_2 + N_3' + \phi \pi_1 - \omega \theta_1) + \pi (\omega \theta_1 - \phi \pi_1) \\
\alpha_4^3 &= \tfrac{1}{k} \mu_1 N_3.
\end{aligned}$$

where $\deg \alpha_4^3 \leq 4$, $\deg \alpha_3^2 \leq 3$ and $\deg \leq 2$.

Conversely, it is enough to prove that we can deduce the system (4.28) from (4.27). Indeed, on account of (4.29)-(4.30) and (4.36)-(4.37), the system (4.27) may be written

$$\begin{aligned}
\phi \left[\omega' \Gamma_0 + \pi_1 \Gamma_1 + \pi \Gamma_1' \right] &= \omega \left[(\phi' - \psi) \Gamma_0 + \theta_1 \Gamma_1 + \theta' \Gamma_1' \right], \\
\theta \left[\pi_1 \Gamma_1 + \omega' \Gamma_0 + \omega \Gamma_0' \right] &= \pi \left[\theta_1 \Gamma_1 + (\phi' - \psi) \Gamma_0 + \phi \Gamma_0' \right].
\end{aligned} \tag{4.38}$$

4.3. Classical d-orthogonal polynomials

After adding $\omega\phi\Gamma_0'$ and $\theta\pi\Gamma_1'$ respectively, to both sides of the first and second equality of (4.38), we obtain

$$\phi X = \omega Y$$
$$\theta X = \pi Y,$$

with

$$\begin{aligned} X &= \left[\omega'\Gamma_0 + \omega\Gamma_0' + \pi_1\Gamma_1 + \pi\Gamma_1'\right] \\ Y &= \left[\left(\phi' - \psi\right)\Gamma_0 + \phi\Gamma_0' + \theta_1\Gamma_1 + \theta'\Gamma_1'\right]. \end{aligned}$$

Hence eliminating X and Y, respectively, we deduce

$$\left(\omega\theta - \pi\phi\right) Y = \left(\omega\theta - \pi\phi\right) X = 0,$$

whence the result since $\omega\theta - \pi\phi \not\equiv 0$. ■

4.4 Application

Furthermore, in [12], the authors showed that there are exactly 2^d sets of d-symmetric d-classical polynomials (see also [19]). The method used to obtain the differential equation of classical 2-OPS in [19], can be used again for the d-symmetric d-classical polynomials to give the explicit form of solutions. Indeed, we have the system

$$\gamma^0_{n+d+1}\left[\gamma^0_{n+1} - \tilde{\gamma}^0_{n+1}\right] = \tilde{\gamma}^0_{n+1}\left[\gamma^0_{n+d+1} - \tilde{\gamma}^0_{n+d+1}\right],$$

in which we distinguish $d+1$ cases in each case we have $\begin{pmatrix} d \\ k \end{pmatrix}$ solutions, hence we have

$$\sum_{k=0}^{d} \begin{pmatrix} d \\ k \end{pmatrix} = 2^d$$

solutions in total. In other words, solutions are

$S_0)$ $\tilde{\gamma}^0_{n+1} = \gamma^0_{n+1}$ (trivial solution)

$S_1)$ $\tilde{\gamma}^0_{dn+\alpha_0} = \left(\frac{n+1+\rho_{\alpha_0}}{n+\rho_{\alpha_0}}\right)\gamma^0_{dn+\alpha_0}$ and $\tilde{\gamma}^0_{dn+\alpha} = \gamma^0_{dn+\alpha}$ for $\alpha \neq \alpha_0$ and $1 \leq \alpha \leq d$

$S_2)$ $\tilde{\gamma}^0_{dn+\alpha_0} = \left(\frac{n+1+\rho_{\alpha_0}}{n+\rho_{\alpha_0}}\right)\gamma^0_{dn+\alpha_0}$ and $\tilde{\gamma}^0_{dn+\alpha_1} = \left(\frac{n+1+\rho_{\alpha 1}}{n+\rho_{\alpha_1}}\right)\gamma^0_{dn+\alpha_1}$ and

$\tilde{\gamma}^0_{dn+\alpha} = \gamma^0_{dn+\alpha}$ for $\alpha \neq \alpha_0, \alpha_1$ and $1 \leq \alpha \leq d$

$\vdots$

$S_d)$ $\tilde{\gamma}^0_{dn+\alpha} = \gamma^0_{dn+\alpha}$ for $1 \leq \alpha \leq d$.

Chapter 5

Riordan Arrays and d-Orthogonality

5.1 Introduction

5.2 Riordan arrays

The Riordan group introduced by Shapiro et al [73] is a set of infinite lower-triangular integer matrices where each matrix is defined by a pairs of formal power series $g(z) = \sum_{n=0}^{\infty} g_n z^n$ and $f(z) = \sum_{n=1}^{\infty} f_n z^n$ with $g_0 \neq 0$ and $f_1 \neq 0$. An infinite lower triangular matrix $D = [d_{n,k}]_{n,k\geq 0}$ is called a Riordan array if its i^{th} column generating function is $g(x)[f(x)]^i$ for $i \geq 0$ (the first column being indexed by 0). With little loss of generality we also assume $d_{0,0} = g_0 = 1$. The matrix corresponding to the pairs g, f is denoted by (g, f). One example of a Riordan matrix is the Pascal matrix

$$P = \left(\frac{1}{1-z}, \frac{z}{1-z}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ & \cdots & & \cdots \end{bmatrix} \tag{5.1}$$

for which we have

$$\binom{n}{k} = [x^n] \frac{1}{1-z}\left(\frac{z}{1-z}\right)^k.$$

The group law is then given by

$$(g, f) . (h, l) = (g (h \circ f), l \circ f).$$

The identity of this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = \left(1/\left(g \circ \bar{f}\right), \bar{f}\right)$ where $\bar{f}$ is the reversion or compositional inverse of f. The reversion of f is the power series $\bar{f}$ such that $\left(f \circ \bar{f}\right)(x) = x$. We shall sometimes write this as $\bar{f} = \mathrm{Rev} f$.

A Riordan array of the form $(g(x), x)$, where $g(x)$ is the generating function of the sequence a_n, is called the *sequence array* of the sequence a_n. Its general term is a_{n-k}. Such arrays are also called *Appell* arrays as they form the elements of the so-called Appell subgroup.

For a Riordan matrix $D = [d_{n,k}]_{n,k\geq 0}$, Rogers [67] has found that every element $d_{n+1,k+1}$ can be expressed as a linear combination of the elements in the preceding row starting from the preceding column. Merlini et al [60] have found that every element in the 0 column can be expressed as a linear combination of all the elements of the preceding row also see [75]. That is, there exist unique sequences $A = (a_0, a_1, ...)$ and $Z = (z_0, z_1, ...)$ with $a_0 \neq 0$, $z_0 \neq 0$ such that

(1) $d_{n+1,k+1} = \sum\limits_{j=0}^{\infty} a_j d_{n,k+j}, \quad (k, n = 0, 1, ...),$

(2) $d_{n+1,0} = \sum\limits_{j=0}^{\infty} z_j d_{n,j}, \quad (n = 0, 1, ...).$

The coefficients a_0, $a_1, ...$ and z_0, $z_1, ...$ appearing in (1) and (2) are called the A-sequence and Z-sequence of the Riordan matrix $D = (g(z), f(z))$, respectively. Letting $A(z)$ and $Z(z)$ be the generating functions of the corresponding sequences, we have

$$f(z) = zA(f(z)),$$
$$g(z) = g(0)/(1 - Z(f(z))).$$

We therefore deduce that

$$A(z) = z/\bar{f}(z),$$

and

$$Z(z) = 1/\bar{f}(z)\left(1 - 1/\left(g \circ \bar{f}\right)(z)\right).$$

A consequence of this is the following result which was originally established [51] by Luzón:

Lemma 5.1 *Let $D = (g, f)$ be a Riordan array whose A-sequence, respectively Z-sequence have generating functions $A(x)$ and $Z(x)$. Then*

$$D^{-1} = \left(\frac{A - xZ}{d_{0,0}A}, \frac{x}{A}\right).$$

If $(a_0, a_1, \ldots)^T$ is a column vector with generating function $\mathcal{A}(x)$, then multiplying $D = (g, f)$ on the right by this column vector yields a column vector with generating function $g(x)\mathcal{A}(f(x))$.

5.3 Riordan arrays and d-orthogonal polynomials

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials in $\mathcal{P}$ for which there exist complex sequences $\{\beta_n\}_{n\geq 0}$, $(\chi_{n,v})$, $0 \leq v \leq n$ such that

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ &P_{n+1}(x) = (x - \beta_{n+1}) P_n(x) - \sum_{v=0}^{n-1} \chi_{n,v} P_v(x), \quad n \geq 0. \end{aligned} \tag{5.2}$$

The dual sequence $\{u_n\}_{n\geq 0}$, $u_n \in \mathcal{P}'$ of $\{P_n\}_{n\geq 0}$ is defined by $\langle u_n, P_m \rangle := \delta_{n,m}$, $n, m \geq 0$. In this case, we have

$$\begin{aligned} \beta_n &= \langle u_n, xP_n(x) \rangle, \quad n \geq 0, \\ \chi_{n,v} &= \langle u_v, xP_{n+1}(x) \rangle, \quad 0 \leq v \leq n. \end{aligned} \tag{5.3}$$

Let us consider d forms $u_0, ..., u_{d-1}$ $(d \geq 1)$. Let us now recall the following characterization which we need in the sequel.

Theorem 5.1 *[54] Let $\{P_n\}_{n\geq 0}$ be a monic sequence of polynomials, then the following statements are equivalent.*

(a) The sequence $\{P_n\}_{n\geq 0}$ is d-OPS with respect to $\mathcal{U} = (u_0, ..., u_{d-1})^T$.

(b) The sequence $\{P_n\}_{n\geq 0}$ satisfies a recurrence relation of order $d+1$ $(d \geq 1)$:

$$P_{m+d+1}(x) = (x - \beta_{m+d}) P_{m+d}(x) - \sum\nolimits_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} P_{m+d-1-\nu}(x), \quad m \geq 0, \tag{5.4}$$

with the initial data

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ &P_m(x) = (x - \beta_{m-1}) P_{m-1}(x) - \sum\nolimits_{\nu=0}^{m-2} \gamma_{m-1-\nu}^{d-1-\nu} P_{m-2-\nu}(x), \quad 2 \leq m \leq d, \end{aligned} \tag{5.5}$$

and the regularity conditions $\gamma_{m+1}^{0} \neq 0, \quad m \geq 0$.

It is well known that we can express (5.4) as $xP = J_d P$ where $P = (P_0(x), P_1(x), ...,)^T$ and $J_d = (a_{i,j})_{i,j=0}^{\infty}$ is a $(d+2)$-banded lower Hessenberg semi-infinite matrix, i.e.,

$$\begin{cases} a_{ij} = 0, & \text{for } j > i+1 \text{ and } i > j+d, \\ a_{i,i+1} = 1, & i \geq 0. \end{cases} \tag{5.6}$$

More precisely, the matrix J_d may be written

$$J_d = \begin{pmatrix} \beta_0 & 1 & 0 & \cdots & 0 & & & \\ \gamma_1^{d-1} & \beta_1 & 1 & 0 & \cdots & 0 & & \\ \gamma_1^{d-2} & \gamma_2^{d-1} & \beta_2 & 1 & 0 & \cdots & 0 & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ \gamma_1^0 & \gamma_2^1 & \cdots & \gamma_d^{d-1} & \beta_d & 1 & 0 & \cdots \\ 0 & \gamma_2^0 & \gamma_3^1 & \cdots & \gamma_{d+1}^{d-1} & \beta_{d+1} & 1 & 0 \\ \vdots & 0 & \ddots & \ddots & & \ddots & \ddots & \ddots \end{pmatrix} \tag{5.7}$$

and is called the monic Jacobi matrix of the monic d-OPS $\{P_n\}_{n\geq 0}$.

Now, when $\{P_n\}_{n\geq 0}$ is not monic, then the linear recurrence (5.4) become

$$P_n(x) = (\alpha_n x - \beta_{n-1}) P_{n-1}(x) - \sum\nolimits_{\nu=1}^{d} \gamma_{n-\nu}^{d-\nu} P_{n-1-\nu}(x), \quad n \geq 0.$$

Write

$$p_n(x) = \sum\nolimits_{j=0}^{n} a_{n,j} x^j \ldots \tag{5.8}$$

then

$$\begin{aligned} a_{n,0} &= \alpha_n a_{n-1,0}, \\ a_{n,i} &= \alpha_n a_{n-1,0} - \beta_{n-1} a_{n-1,i-1} - \sum\nolimits_{\nu=1}^{d} \gamma_{n-\nu}^{d-\nu} a_{n-1,i-1-v}, \quad 1 \leq i \leq n. \end{aligned} \tag{5.9}$$

On the other hand, for a family of monic d-OPS given by (5.4), and by using (5.8), we can have

$$\sum_{k=0}^{n+1} a_{n+1,k} x^k = (x - \beta_n) \sum_{k=0}^{n} a_{n,k} x^k - \gamma_n^{d-1} \sum_{k=0}^{n-1} a_{n-1,k} x^k - \ldots - \gamma_{n-d+1}^{0} \sum_{k=0}^{n-d} a_{n-d,k} x^k$$

from which we deduce

$$a_{n+1,0} = -\beta_n a_{n,0} - \sum_{k=0}^{d-1} \gamma_{n-v}^{d-1-v} a_{n-1-v,0} \tag{5.10}$$

and

$$a_{n+1,k} = a_{n,k-1} - \beta_n a_{n,k} - \sum_{k=0}^{d-1} \gamma_{n-v}^{d-1-v} a_{n-1-v,k}. \tag{5.11}$$

We note that if β_n and γ_n^{d-i}, $1 \leq i \leq d$ are constant, equal to β and γ^{d-i}, $1 \leq i \leq d$, respectively, then the sequence $(1, -\beta, -\gamma^{d-1}, -\gamma^{d-2}, ..., -\gamma^0, 0, 0, \ldots)$ forms an A-sequence of the coefficient array. The question immediately arises as to the conditions under which a Riordan array (g, f) can be the coefficient array of a family of d-OPS. A partial answer is given by the following proposition.

5.3. Riordan arrays and d-orthogonal polynomials

Proposition 5.1 *[44] Every Riordan array of the form*

$$\left(\frac{1}{1+\sum_{k=1}^{d+1}\theta_k x^k}, \frac{x}{1+\sum_{k=1}^{d+1}\theta_k x^k}\right) \tag{5.12}$$

is the coefficient array of a family of monic d-OPS.

Proof. The array $\left(1/\left(1+\sum_{k=1}^{d+1}\theta_k x^k\right), x/\left(1+\sum_{k=1}^{d+1}\theta_k x^k\right)\right)$ [40] has a C-sequence $C(x)=\sum_{n\geq 0} c_n x^n$ given by

$$\frac{x}{1+\sum_{k=1}^{d+1}\theta_k x^k} = \frac{x}{1-xC(x)},$$

and thus

$$C(x) = -\sum_{k=1}^{d+1}\theta_k x^{k-1}.$$

This means that the Riordan array $\left(1/\left(1+\sum_{k=1}^{d+1}\theta_k x^k\right), x/\left(1+\sum_{k=1}^{d+1}\theta_k x^k\right)\right)$ is determined by the fact that

$$a_{n+1,k} = a_{n,k-1} + \sum_{i\geq 0} c_i a_{n-i,k} \quad \text{for } n,k=0,1,2,\ldots$$

where $a_{n,-1}=0$. In the case of $\left(1/\left(1+\sum_{k=1}^{d+1}\theta_k x^k\right), x/\left(1+\sum_{k=1}^{d+1}\theta_k x^k\right)\right)$ we have

$$a_{n+1,k} = a_{n,k-1} - \sum_{v=0}^{d}\theta_{v+1}a_{n-v,k}. \tag{5.13}$$

Working backwards, this now ensures that

$$P_{n+1}(x) = (x-\theta_1)P_n(x) - \sum_{v=1}^{d}\theta_{v+1}P_{n-v},$$

where $P_n(x)=\sum_{k=0}^{n} a_{n,k}x^n$. ■

We note that in this case the $(d+2)$-term recurrence coefficients β_n and γ_n^{d-i}, $1\leq i\leq d$, are constants.

As an example, we have the following result

Proposition 5.2 *[44] The Riordan array (5.12) is the coefficient array of the modified d-orthogonal Chebyshev polynomials of the second kind given by*

$$P_n(x) = U_n\left(x-\theta_1\right), \quad n=0,1,2,\ldots$$

Proof. We have [31, (2.4)]

$$\frac{1}{1-xt+\sum_{k=1}^{d}\theta_{k+1}t^{k+1}}=\sum_{n=0}^{\infty}U_n(x)t^n.$$

Thus

$$\frac{1}{1-(x-\theta_1)\,t+\sum_{k=1}^{d}\theta_{k+1}t^{k+1}}=\sum_{n=0}^{\infty}U_n\,(x-\theta_1)\,t^n.$$

Now

$$\frac{1}{1-(x-\theta_1)\,t+\sum_{k=1}^{d}\theta_{k+1}t^{k+1}}=\left(\frac{1}{1+\sum_{k=0}^{d}\theta_{k+1}t^{k+1}},\frac{t}{1+\sum_{k=0}^{d}\theta_{k+1}t^{k+1}}\right)\cdot\frac{1}{1-xt}.$$

Thus

$$\left(\frac{1}{1+\sum_{k=0}^{d}\theta_{k+1}t^{k+1}},\frac{t}{1+\sum_{k=0}^{d}\theta_{k+1}t^{k+1}}\right)\cdot\frac{1}{1-xt}=\sum_{n=0}^{\infty}U_n\,(x-\theta_1)\,t^n.$$

■

Also we have the following motivating result

Proposition 5.3 *[44] Every Riordan array of the form*

$$\left(\frac{1-\sum_{k=1}^{d+1}\lambda_k x^k}{1+\sum_{k=1}^{d+1}\theta_k x^k},\frac{x}{1+\sum_{k=1}^{d+1}\theta_k x^k}\right) \tag{5.14}$$

is the coefficient array of a family of monic d-OPS.

Proof. We have

$$B=(1-\sum_{k=1}^{d+1}\lambda_k x^k,x)\cdot A$$

where

$$\begin{aligned} B &= (b_{n,k})=\left(\tfrac{1-\sum_{k=1}^{d+1}\lambda_k x^k}{1+\sum_{k=1}^{d+1}\theta_k x^k},\tfrac{x}{1+\sum_{k=1}^{d+1}\theta_k x^k}\right),\\ A &= (a_{n,k})=\left(\tfrac{1}{1+\sum_{k=1}^{d+1}\theta_k x^k},\tfrac{x}{1+\sum_{k=1}^{d+1}\theta_k x^k}\right)\end{aligned}$$

and $(1-\sum_{k=1}^{d+1}\lambda_k x^k,x)$ is the array with elements

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -\lambda_1 & 1 & 0 & 0 & 0 & 0 & \dots \\ -\lambda_2 & -\lambda_1 & 1 & 0 & 0 & 0 & \dots \\ \vdots & -\lambda_2 & -\lambda_1 & 1 & \ddots & & \dots \\ -\lambda_{d+1} & \vdots & \ddots & \ddots & \ddots & 0 & \dots \\ 0 & -\lambda_{d+1} & \cdots & -\lambda_2 & -\lambda_1 & 1 & 0 \\ \vdots & 0 & \ddots & & & \ddots & \ddots \end{pmatrix}. \tag{5.15}$$

5.3. Riordan arrays and d-orthogonal polynomials

Now, by using the recurrence (5.13), we deduce

$$\begin{aligned} b_{n+1,k} &= a_{n+1,k} - \textstyle\sum_{v=0}^{d} \lambda_{v+1} a_{n-v,k}, \\ &= a_{n,k-1} - \textstyle\sum_{v=0}^{d} \theta_{v+1} a_{n-v,k} \\ &-\lambda_1 \left[a_{n-1,k-1} - \textstyle\sum_{v=0}^{d} \theta_{v+1} a_{n-1-v,k} \right] \\ &-\ldots - \lambda_{d+1} \left[a_{n-1-d,k-1} - \textstyle\sum_{v=0}^{d} \theta_{v+1} a_{n-d-1-v,k} \right] \\ &= b_{n,k-1} - \textstyle\sum_{v=0}^{d} \lambda_{v+1} b_{n-v,k}. \end{aligned}$$

Hence, since $b_{n,k}$ is expressed in terms of $a_{n,k}$, then as in proof of the previous proposition, B is a Riordan array of a family of d-OPS $\{Q_n\}$, i.e.

$$Q_0(x) = 1, \quad Q_1(x) = x - \theta_1 - \lambda_1, \quad Q_2(x) = x^2 - (2\theta_1 + \lambda_1)x + \lambda_1\theta_2 - \lambda_2 + \theta_1^2 - \theta_2, \ldots$$

■

Proposition 5.4 *The Riordan array (5.14) is the coefficient array of the modified d-orthogonal Faber polynomials given by*

$$P_n(x) = F_n\left(x - \theta_1\right), \quad n = 0, 1, 2, \ldots$$

Proof. The d-orthogonal Faber polynomials $\{F_n\left(x\right)\}_{n\geq 0}$ are studied by Douak and Maroni [32], and in another way by Ben Romdhane [14]. This family of polynomials is defined with the help of the generating function as follows

$$\frac{1 - \sum_{k=1}^{d+1} \lambda_k t^k}{1 - xt + \sum_{k=1}^{d} \theta_{k+1} t^{k+1}} = \sum_{n=0}^{\infty} F_n(x) t^n.$$

Thus

$$\frac{1 - \sum_{k=1}^{d+1} \lambda_k t^k}{1 - \left(x - \theta_1\right) t + \sum_{k=1}^{d} \theta_{k+1} t^{k+1}} = \sum_{n=0}^{\infty} F_n\left(x - \theta_1\right) t^n,$$

i.e.,

$$\left(\frac{1 - \sum_{k=1}^{d+1} \lambda_k t^k}{1 + \sum_{k=0}^{d} \theta_{k+1} t^{k+1}}, \frac{t}{1 + \sum_{k=0}^{d} \theta_{k+1} t^{k+1}} \right) \cdot \frac{1}{1 - xt} = \sum_{n=0}^{\infty} F_n\left(x - \theta_1\right) t^n.$$

Also note that, we can express the d-Faber polynomials in terms of co-recursive d-OPS. In fact, Ben Cheikh and Ben Romdhane [11] have shown that the σ-Appell d-OPS are expressed in terms of d-orthogonal Chebyshev polynomials of second kind as follows

$$F_n\left(x\right) = U_n\left(x\right) + \sum_{k=1}^{d+1} \lambda_k U_{n-k}\left(x\right).$$

In this case, since $U_n^{(r)}(x) = U_n(x)$, it follows that the Riordan array (5.14) is the coefficient array of the co-recursive d-orthogonal Chebyshev polynomials of second kind given by [69]

$$P_n(x) = U_n(x-\theta_1) + \sum_{k=1}^{d+1} \lambda_k U_{n-k}(x-\theta_1).$$

■

5.4 Stieltjes matrix

Let $L = (l_{n,k})_{n,k\geq 0}$ be Riordan and $\bar{L}$ be the matrix obtained from L by deleting the first row, i.e., $\bar{l}_{nk} = l_{n+1,k}$. For example, if I is the identity, then

$$\bar{I} = \begin{bmatrix} 0 & 1 & & & 0 \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \ddots \\ & & \cdots & & \end{bmatrix}$$

Observe that $\bar{L} = \bar{I}L$. There exists a unique matrix $S_L = (s_{nk})_{n,k\geq 0}$ such that $LS_L = \bar{L}$. That is,

$$l_{n,k} = \sum_{i\geq 0} s_{ik} l_{n-1,i}, \quad \text{for} \quad n \geq 1. \tag{5.16}$$

We call this matrix the Stieltjes matrix of L. For example, the Stieltjes matrix corresponding to the Pascal matrix (5.1) is

$$S_L = \begin{bmatrix} 1 & 1 & & & 0 \\ 0 & 1 & 1 & & \\ 0 & 0 & 1 & 1 & \ddots \\ & & \cdots & & \end{bmatrix}.$$

We note also that the S_L is unique, i.e., $S_L = S_K \Leftrightarrow L = K$.

Theorem 5.2 *[44] If $L = (g, f)$ is Riordan and S_L is $(d+2)$-banded lower matrix as in (5.7), then*

(a) S_L is in the following form

$$S_L = \begin{pmatrix} \beta_0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \gamma_1^{d-1} & \beta & 1 & 0 & 0 & 0 & \cdots \\ \gamma_1^{d-2} & \gamma^{d-1} & \beta & 1 & 0 & 0 & \cdots \\ \vdots & \gamma^{d-2} & \gamma^{d-1} & \beta & \ddots & & \cdots \\ \gamma_1^0 & \vdots & \ddots & \ddots & \ddots & 1 & \cdots \\ 0 & \gamma^0 & \cdots & \gamma^{d-2} & \gamma^{d-1} & \beta & 1 \\ \vdots & 0 & \ddots & & & \ddots & \ddots \end{pmatrix} \tag{5.17}$$

(b) $f = x\left(1+\beta f+\gamma^{d-1}f^2+...+\gamma^0 f^{d+1}\right)$ *and* $g = \left(1-x\beta_0-\gamma_1^{d-1}xf-...-\gamma_1^0 xf^d\right)^{-1}$.

Proof. Write S_L in the form (5.7) with $C_i(x)$ the generating function of the i$^{\text{th}}$ column of L, $i\geq 0$. We have: $C_i = gf^i$. By looking at the first column of LS_L and $\bar{L}$, we obtain

$$C_0(x) = 1+g_1x+g_2x^2+... = 1+l_{10}x+l_{20}x^2+...$$

and

$$\begin{aligned}
\beta_0 &= l_{10} = g_1 \\
l_{10}\beta_0+\gamma_1^{d-1} &= l_{20} = g_2 \\
l_{20}\beta_0+l_{21}\gamma_1^{d-1}+\gamma_1^{d-2} &= l_{30} = g_3 \\
\\
l_{d0}\beta_0+l_{d1}\gamma_1^{d-1}+...l_{d,d-1}\gamma_1^1+\gamma_1^0 &= l_{d+1,0} = g_{d+1}.
\end{aligned}$$

Multiplying the first equation by x, the second one by x^2,... and the last by x^{d+1} and summing we get

$$x\beta_0 g+x\gamma_1^{d-1}gf+...+x\gamma_1^0 gf^d = g-1.$$

Whence $g(x) = \left(1-x\beta_0-\gamma_1^{d-1}xf-...-\gamma_1^0 xf^d\right)^{-1}$.

Repeat the procedure for $i\geq 1$, and in each case we try to find a system that contain all coefficient $\beta, \gamma^{d-1}, ..., \gamma^0$. One can see that we have

$$gf^i = xg\left[f^{i-1}+\beta_i f^i+\gamma_{i+1}^{d-1}f^{i+2}+...+\gamma_{i+1}^0 f^{d+i}\right], \tag{5.18}$$

that is

$$C_i = x\left(C_{i-1}+\beta_i C_i+\gamma_{i+1}^{d-1}C_{i+2}+...+\gamma_{i+1}^0 C_{i+d}\right). \tag{5.19}$$

Hence after simplification we find the expression of f.

For instance, rewrite the equality (5.18) in terms of j and subtract, we get

$$(\beta_i-\beta_j)f+\sum_{v=0}^{d-1}\left(\gamma_{i+1}^{d-1-v}-\gamma_{j+1}^{d-1-v}\right)f^{2+v} = 0$$

which is equivalent to

$$\beta_i = \beta_j = \beta, \quad \gamma_i^{d-1-v} = \gamma^{d-1-v}, \quad 0\leq v\leq d-1.$$

■

This leads to the important corollary which is an analogue of [81].

Corollary 5.1 *[44] If $L = (g(x), f(x))$ is a Riordan array and $P = S_L$ is $(d+2)$-banded lower given by (5.17), then L^{-1} is the coefficient array of the family of d-OPS.*

5.4. Stieltjes matrix

Proof. By Favard's theorem, it suffices to show that L^{-1} defines a family of polynomials $\{P_n(x)\}$ that obey a $(d+2)$-term recurrence. Now L is lower-triangular and so L^{-1} is the coefficient array of a family of d-OPS $P_n(x)$, where

$$L^{-1}\begin{pmatrix}1\\x\\x^2\\x^3\\\vdots\end{pmatrix}=\begin{pmatrix}P_0(x)\\P_1(x)\\P_2(x)\\P_3(x)\\\vdots\end{pmatrix}.$$

We have

$$S_L\cdot L^{-1}=L^{-1}\cdot\bar{L}\cdot L^{-1}=L^{-1}\cdot\bar{I}\cdot L\cdot L^{-1}=L^{-1}\cdot\bar{I}.$$

Thus

$$S_L\cdot L^{-1}\cdot(1,x,x^2,\ldots)^T=L^{-1}\cdot\bar{I}\cdot(1,x,x^2,\ldots)^T=L^{-1}\cdot(x,x^2,x^3,\ldots)^T.$$

We therefore obtain

$$\begin{pmatrix}\beta_0 & 1 & 0 & 0 & 0 & \\ \gamma_1^{d-1} & \beta & 1 & 0 & 0 & \\ \gamma_1^{d-2} & \gamma^{d-1} & \beta & 1 & 0 & \\ \vdots & \gamma^{d-2} & \gamma^{d-1} & \ddots & \ddots & \\ \gamma_1^0 & \vdots & \ddots & \gamma^{d-1} & \beta & 1\\ 0 & \gamma^0 & \cdots & & & \\ \vdots & 0 & \ddots & & & \end{pmatrix}\begin{pmatrix}P_0(x)\\P_1(x)\\P_2(x)\\P_3(x)\\\vdots\end{pmatrix}=\begin{pmatrix}xP_0(x)\\xP_1(x)\\xP_2(x)\\xP_3(x)\\\vdots\end{pmatrix},$$

from which we infer that

$$\begin{aligned}P_1(x)&=x-\beta_0,\\P_2(x)&=(x-\beta)\,P_1\,(x)-\gamma_1^{d-1}\\&\vdots\\P_{d+1}\,(x)&=(x-\beta)\,P_d\,(x)-\gamma^{d-1}P_{d-1}-...\gamma^1P_1\,(x)-\gamma_1^0.\end{aligned}$$

and

$$P_{n+1}(x)=(x-\beta)P_n(x)-\sum_{v=0}^{d-1}\gamma^{d-1-v}P_{n+d-1-v}(x),\quad n\geq 1.$$

■

Combining these result with previous proposition, we have

Theorem 5.3 *[44] A Riordan array $L=(g(x),f(x))$ is the inverse of the coefficient array of a family of d-OPS if and only if its Stieltjes matrix S_L is $(d+2)$-banded lower matrix.*

5.4. Stieltjes matrix

We give the following example

Proposition 5.5 *[44] The Stieltjes matrix of the inverse Riordan array*

$$\left(\frac{1}{1+\sum_{k=1}^{d+1}\theta_k x^k}, \frac{x}{1+\sum_{k=1}^{d+1}\theta_k x^k}\right)$$

left-multiplied by the k^{th} binomial array

$$\left(\frac{1}{1-kx}, \frac{x}{1-kx}\right) = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)^k$$

is given by

$$P = \begin{pmatrix} \theta_1+k & 1 & 0 & 0 \\ \theta_2 & \theta_1+k & 1 & 0 \\ \theta_3 & \theta_2 & \theta_1+k & 1 \\ \vdots & \theta_3 & \theta_2 & \ddots \\ \theta_{d+1} & \vdots & & \\ 0 & \theta_{d+1} & & \ddots \\ \vdots & 0 & \ddots & \end{pmatrix}$$

Proof. We have

$$\begin{aligned} &\left(\frac{1}{1+\sum_{k=1}^{d+1}\theta_k x^k}, \frac{x}{1+\sum_{k=1}^{d+1}\theta_k x^k}\right)\cdot\left(\frac{1}{1+kx}, \frac{x}{1+kx}\right) \\ &= \left(\frac{1}{1+(\theta_1+k)x+\sum_{k=1}^{d}\theta_{k+1}x^{k+1}}, \frac{x}{1+(\theta_1+k)x+\sum_{k=1}^{d}\theta_{k+1}x^{k+1}}\right). \end{aligned}$$

And more generally,

$$\begin{aligned} &\left(\frac{1-\sum_{k=1}^{d+1}\lambda_k x^k}{1+\sum_{k=1}^{d+1}\theta_k x^k}, \frac{x}{1+\sum_{k=1}^{d+1}\theta_k x^k}\right)\cdot\left(\frac{1}{1+kx}, \frac{x}{1+kx}\right) \\ &= \left(\frac{1-\sum_{k=1}^{d+1}\lambda_k x^k}{1+(\theta_1+k)x+\sum_{k=1}^{d}\theta_{k+1}x^{k+1}}, \frac{x}{1+(\theta_1+k)x+\sum_{k=1}^{d}\theta_{k+1}x^{k+1}}\right). \end{aligned} \tag{5.20}$$

Therefore, the inverse of the last matrix has the Stieltjes array

$$\begin{pmatrix} \theta_1+k-\lambda_1 & 1 & 0 & 0 \\ \theta_2-\lambda_2 & \theta_1+k & 1 & 0 \\ \theta_3-\lambda_3 & \theta_2 & \theta_1+k & 1 \\ \vdots & \theta_3 & \theta_2 & \ddots \\ \theta_{d+1}-\lambda_{d+1} & \vdots & & \\ 0 & \theta_{d+1} & & \ddots \\ \vdots & 0 & \ddots & \end{pmatrix}.$$

■

5.4. Stieltjes matrix

5.5 Exponential Riordan arrays

The exponential Riordan group is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = g_0 + g_1 x + g_2 \frac{x^2}{2!} + \cdots$ and $f(x) = f_1 x + f_2 \frac{x^2}{2!} + \cdots$ where $g_0 \neq 0$ and $f_1 \neq 0$. In what follows, we shall assume

$$g_0 = f_1 = 1.$$

The associated matrix is the matrix whose i-th column has exponential generating function $g(x)f(x)^i/i!$ (the first column being indexed by 0). The matrix corresponding to the pair f, g is denoted by $[g, f]$. The group law is given by

$$[g, f] \cdot [h, l] = [g(h \circ f), l \circ f].$$

The identity for this law is $I = [1, x]$ and the inverse of $[g, f]$ is $[g, f]^{-1} = [1/(g \circ \bar{f}), \bar{f}]$ where $\bar{f}$ is the compositional inverse of f. We use the notation $e\mathcal{R}$ to denote this group.

If $\mathbf{M}$ is the matrix $[g, f]$, and $\mathbf{u} = (u_n)_{n \geq 0}$ is an integer sequence with exponential generating function $\mathcal{U}\ (x)$, then the sequence $\mathbf{Mu}$ has exponential generating function $g(x)\mathcal{U}(f(x))$. Thus the row sums of the array $[g, f]$ have exponential generating function given by $g(x)e^{f(x)}$ since the sequence $1, 1, 1, \ldots$ has exponential generating function e^x.

We will use the following [25, 26], important result concerning matrices that are production matrices for exponential Riordan arrays.

Proposition 5.6 *Let $A = (a_{n,k})_{n,k \geq 0} = [g(x), f(x)]$ be an exponential Riordan array and let*

$$c(y) = c_0 + c_1 y + c_2 y^2 + \ldots, \qquad r(y) = r_0 + r_1 y + r_2 y^2 + \ldots \tag{5.21}$$

be two formal power series that

$$r(f(x)) = f'(x) \tag{5.22}$$

$$c(f(x)) = \frac{g'(x)}{g(x)}. \tag{5.23}$$

Then

$$(i) \qquad a_{n+1,0} = \sum_i i! c_i a_{n,i} \tag{5.24}$$

$$(ii) \qquad a_{n+1,k} = r_0 a_{n,k-1} + \frac{1}{k!} \sum_{i \geq k} i!(c_{i-k} + k r_{i-k+1}) a_{n,i} \tag{5.25}$$

or, assuming $c_k = 0$ for $k < 0$ and $r_k = 0$ for $k < 0$,

$$a_{n+1,k} = \frac{1}{k!} \sum_{i \geq k-1} i!(c_{i-k} + k r_{i-k+1}) a_{n,i}. \tag{5.26}$$

Conversely, starting from the sequences defined by (5.21), the infinite array $(a_{n,k})_{n,k \geq 0}$ defined by (5.26) is an exponential Riordan array.

A consequence of this proposition is that the production matrix $P = (p_{i,j})_{i,j\geq 0}$ for an exponential Riordan array obtained as in the proposition satisfies [25, 26]

$$p_{i,j} = \frac{i!}{j!}(c_{i-j} + jr_{i-j+1}) \qquad (c_{-1} = 0).$$

Furthermore, the bivariate exponential function

$$\phi_P(t, z) = \sum_{n,k} p_{n,k} t^k \frac{z^n}{n!}$$

of the matrix P is given by

$$\phi_P(t, z) = e^{tz}(c(z) + tr(z)).$$

Note in particular that we have

$$r(x) = f'(\bar{f}(x)), \tag{5.27}$$

and

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))}. \tag{5.28}$$

Now, an important property of the matrix $\bar{L}$ which has led to the proof of various results in this context is the following

Proposition 5.7 *[44] We have $\bar{L} = \frac{d}{dx}L$.*

Proof. Since

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} g_n \frac{x^n}{n!}\right) = \left(\sum_{n=0}^{\infty} g_{n+1} \frac{x^n}{n!}\right),$$

then equating the first columns of the matrices $\bar{L}$ and LS, we obtain

$$\begin{aligned} \beta_0 &= l_{10} \\ l_{10}\beta_0 + \gamma_1^{d-1} &= l_{20} \\ l_{20}\beta_0 + l_{21}\gamma_1^{d-1} + \gamma_1^{d-2} &= l_{30} \\ \\ l_{d0}\beta_0 + l_{d1}\gamma_1^{d-1} + ...l_{d,d-1}\gamma_1^1 + \gamma_1^0 &= l_{d+1,0}, \end{aligned} \tag{5.29}$$

and for $n \geq d + 1$, we see that one can has

$$l_{n0}\beta_0 + l_{n1}\gamma_1^{d-1} + ...l_{n,d-1}\gamma_1^1 + \gamma_1^0 l_{n,d} \quad = l_{n+1,0}. \tag{5.30}$$

Multiplying the second equation of (5.29) by x, the third one by x^2,... and the last by x^d and (5.30) by x^n, and summing we get

$$\beta_0 \left(1 + l_{10}x + l_{20}x^2 + ...\right) + \gamma_1^{d-1}\left(x + l_{21}x^2 + ...\right) + ... + \gamma_1^0\left(x^d + l_{d+1,d}x^{d+1} + ...\right) = g'(x),$$

i.e.

$$\beta_0 g(x) + \gamma_1^{d-1} g(x) f(x) + ... + \gamma_1^0 g(x) [f(x)]^d = g'(x).$$

Now, by equating the second columns of the same matrices we get

$$\begin{aligned} 1 &= 1 \\ l_{10} + \beta_1 &= l_{21} \\ l_{20} + l_{21}\beta_1 + \gamma_2^{d-1} &= l_{31} \\ l_{30} + l_{31}\beta_1 + l_{32}\gamma_2^{d-1} + \gamma_2^{d-2} &= l_{41} \\ \\ l_{d+10} + l_{d+1,1}\beta_1 + l_{d+1,2}\gamma_2^{d-1} + ... + l_{d+1,d}\gamma_2^1 + \gamma_2^0 &= l_{d+2,1}, \end{aligned} \tag{5.31}$$

and for $n \geq d+2$, we see that one can has

$$l_{n0} + l_{n,1}\beta_1 + l_{n,2}\gamma_2^{d-1} + ... + l_{n,d}\gamma_2^1 + l_{n,d+1}\gamma_2^0 \quad = l_{n+1,1}, \tag{5.32}$$

Multiplying the second equation of (5.31) by x, the third one by x^2,... and the last by x^d and (5.32) by x^n, and summing we get

$$\begin{aligned} &(1 + l_{10}x + l_{20}x^2 + ...) + \beta_1 (x + l_{21}x^2 + ...) \\ &+\gamma_2^{d-1} (x^2 + l_{31}x^3 + ...) + ... + \gamma_2^0 \left(x^{d+1} + l_{d+2,d+1}x^{d+2} + ...\right) \\ &= (x + l_{21}x^2 + ...), \end{aligned}$$

i.e.,

$$g(x) + \beta_1 g(x) f(x) + \gamma_2^{d-1} g(x) [f(x)]^2 + ... + \gamma_2^0 g(x) [f(x)]^{d+1} = g(x) f'(x),$$

whence

$$1 + \beta_1 f(x) + \gamma_2^{d-1} [f(x)]^2 + ... + \gamma_2^0 [f(x)]^{d+1} = f'(x).$$

■

Theorem 5.4 *[44] If $L = [g(x), f(x)]$ is an exponential Riordan array and S_L is $(d+2)$-banded lower matrix, then necessarily*

$$S_L = \begin{pmatrix} \alpha_0 & 1 & 0 & \cdots & 0 & & \\ \mu_0^{d-1} & \alpha_1 & 1 & 0 & \cdots & 0 & \\ \mu_0^{d-2} & \mu_1^{d-1} & \alpha_2 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ \mu_0^0 & \mu_1^1 & \cdots & \mu_d^{d-1} & \alpha_d & 1 & 0 \\ 0 & \mu_1^0 & \mu_2^1 & \cdots & \mu_d^{d-1} & \alpha_{d+1} & 1 \\ \vdots & 0 & \ddots & \ddots & & \ddots & \ddots \end{pmatrix} \tag{5.33}$$

where $\{\alpha_k\}_{k\geq 0}$ *is an arithmetic sequence with common difference* α, *and* $\left\{\frac{\mu_k^{d-i}}{(k+1)_i}\right\}_{k\geq 1}$, $1 \leq i \leq d$, *is an arithmetic sequence with common difference* γ^{d-i} *and*

$$\ln(g) = \int \left(\alpha_0 + \mu_0^{d-1} f(x) + ... + \mu_0^0 [f(x)]^d\right) dx, \quad g(0) = 1, \tag{5.34}$$

where $(\alpha)_k$ *is the Pochhammer symbol and* f *is given by*

$$f' = 1 + \alpha f + \sum_{i=1}^{d} \gamma^{d-i} f^{i+1}, \quad f(0) = 0, \tag{5.35}$$

and vice-versa.

Proof. We consider the lower triangular matrix $\hat{L}$ with $\frac{1}{k!} g(x) [f(x)]^k$ for the exponential generating functions of the k$^{\text{th}}$ column, $k \geq 0$. We note that $\hat{L}$ is a Riordan matrix with exponential generating functions.

From (5.35), we see that

$$[x^n]\, g' = \beta_0 [x^n]\, g + \sum_{i=1}^{d} \gamma_2^{d-i} [x^n]\, g f^i,$$

i.e.

$$l_{n+1,0} = \beta_0 l_{n,0} + \gamma_1^{d-1} l_{n,1} + ... + \gamma_1^0 l_{n,d}.$$

Hence, for $k \geq 1$, we have

$$\begin{aligned}\left(\frac{1}{k!} g f^k\right)' &= \frac{1}{k!} g' f^k + \frac{1}{(k-1)!} g f^{k-1} f' \\ &= \frac{1}{k!}\left(\beta_0 f + \sum_{i=1}^{d} \gamma_1^{d-i} f^i\right) g f^k + \frac{1}{(k-1)!}\left(1 + \beta_1 f + \sum_{i=1}^{d} \gamma_2^{d-i} f^{i+1}\right) g f^{k-1},\end{aligned}$$

then

$$\begin{aligned}[x^n]\left(\frac{1}{k!} g f^k\right)' &= [x^n]\, \frac{1}{(k-1)!} g f^{k-1} + [x^n]\, \frac{(\beta_0 + k\beta_1)}{k!} g f^k \\ &+ [x^n]\, \frac{\left(\gamma_1^{d-1} + k\gamma_2^{d-1}\right)}{k!} g f^{k+1} + ... + [x^n]\, \frac{\left(\gamma_1^0 + k\gamma_2^0\right)}{k!} g f^{k+d} \\ &= [x^n]\, \frac{1}{(k-1)!} g f^{k-1} + [x^n]\, \frac{\alpha_k}{k!} g f^k \\ &+ [x^n]\, \frac{\mu_k^{d-1}}{(k+1)!} g f^{k+1} + ... + [x^n]\, \frac{\mu_k^0}{(k+d)!} g f^{k+d},\end{aligned}$$

where

$$\alpha_k = \beta_0 + k\beta_1 \quad \text{and} \quad \mu_k^{d-i} = (k+1)_i \left(\gamma_1^{d-i} + k\gamma_2^{d-i}\right) \quad \text{for} \quad 1 \leq i \leq d,$$

whence

$$l_{n+1,k} = l_{n,k-1} + \alpha_k l_{n,k} + \mu_k^{d-1} l_{n,1} + ... + \mu_k^0 l_{n,k+d}. \tag{5.36}$$

5.5. Exponential Riordan arrays

Finally, we easy observe that

$$\alpha_k - \alpha_{k-1} = \beta_1 := \alpha \;\text{ and }\; \frac{\mu_k^{d-i}}{(k+1)_i} - \frac{\mu_{k-1}^{d-i}}{(k)_i} = \gamma_2^{d-i} \;\text{ for }\; 1 \le i \le d.$$

■

We give now an example for a d-OPS defined by a generating function which can be also generated by a Riordan array.

Let us consider a family of Hermite polynomials generated by the following generating function

$$G(x,t) = \exp\left\{(x-b)\,t - \sum_{i=1}^{d} \frac{a_i}{i+1} t^{i+1}\right\} = \sum\nolimits_{n\ge 0} \frac{1}{n!} P_n(x)\, t^n. \tag{5.37}$$

This generating function satisfies the following first order differential equation

$$G_t'(x,t) = \left(x - b - \sum_{i=1}^{d} a_i t^i\right) G(x,t)\,. \tag{5.38}$$

Now replacing $G(x,t)$ and $G'(x,t)$ by the right hand side of (5.37), we can easily obtain the linear recurrence satisfied by the sequence of polynomials $\{P_n\}_{n\ge 0}$ as follows

$$P_{n+d+1}(x) = (x-b)\,P_{n+d}(x) - \sum_{i=1}^{d} a_i \langle n+d\rangle_i\, P_{n+d-i}(x)\,, \quad n \ge 0 \tag{5.39}$$

which means that $\{P_n\}_{n\ge 0}$ is 2-OPS where $\langle n+d\rangle_i = (n+d-i+1)_i$.

By translation, we can take $a = 0$. In this case, we have $G_x'(x,t) = tG(x,t)$, i.e.,

$$P_{n+1}'(x) = (n+1)\,P_n(x)\,, \tag{5.40}$$

that is, $\{P_n\}_{n\ge 0}$ is Appell sequence, and then it is classical d-OPS.

Now by choosing

$$b = d \;\text{ and }\; a_i = i+1 \;\text{ for }\; 1 \le i \le d,$$

the corresponding Stieltjes matrix (5.7) may be written

$$\begin{pmatrix}
b & 1 & 0 & \cdots & 0 & & \\
a_1 & b & 1 & 0 & \cdots & 0 & \\
2a_2 & 2a_1 & b & 1 & 0 & \cdots & 0 \\
\vdots & 6a_2 & \ddots & \ddots & \ddots & \ddots & \\
d!a_d & & \ddots & da_1 & b & 1 & 0 \\
0 & a_d\langle 2+d\rangle_d & \cdots & d(d+1)\,a_2 & (d+1)\,a_1 & b & 1 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}$$

5.5. Exponential Riordan arrays

where the common difference is 0 in all case, i.e.,

$$\frac{\mu_k^{d-i}}{\langle k+d\rangle_i} - \frac{\mu_{k-1}^{d-i}}{\langle k+d-1\rangle_i} = a_i - a_i = 0.$$

Therefore, from (5.34) and (5.35) we obtain

$$f(x) = x \quad \text{and} \quad \ln(g) = \int (b + a_1 x + 2a_2 x^2 ... + d! a_d x^d) dx,$$

i.e.,

$$g(x) = \exp\left\{ bx + \sum_{i=1}^{d} \delta_i x^{i+1} \right\}.$$

Corollary 5.2 *[44]If $L = [g(x), f(x)]$ is an exponential Riordan array and S_L in the form (5.7), then L^{-1} is the coefficient array of the family of monic d-OPS.*

Clearly, L^{-1} is then the coefficient array of the family of polynomials $\{P_n(x)\}$. Gathering these results, we have

Theorem 5.5 *[44] An exponential Riordan array $L = [g(x), f(x)]$ is the inverse of the coefficient array of a family of d-orthogonal polynomials if and only if its production matrix $P = S_L$ is in the form (5.33).*

Proposition 5.8 *[44] Let $L = [g(x), f(x)]$ be an exponential Riordan array with $(d+2)$-banded lower Stieltjes matrix S_L. Then*

$$n![x^n]g(x)\left(1, f(x), ..., [f(x)]^{d-1}\right)^T = \mathcal{U}(x^n) := \mu_n,$$

where $\mathcal{U} = (u_0, ..., u_{d-1})^T$ is the linear functional that defines the corresponding family of d-OPS.

Proof. Let $L = (l_{i,j})_{i,j\geq 0}$. We have

$$x^n = \sum_{i=0}^{n} l_{n,i} P_i(x).$$

Applying $\mathcal{U}$, we get for each $0 \leq \alpha \leq d-1$,

$$\begin{aligned} u_\alpha(x^n) &= u_\alpha\left(\sum_{i=0}^n l_{n,i} P_i(x)\right) = \sum_{i=0}^n l_{n,i} u_\alpha(P_i(x)) \\ &= \sum_{i=0}^n l_{n,i} \delta_{i,\alpha} = l_{n,\alpha} = n![x^n]g(x)\left[f(x)\right]^\alpha. \end{aligned}$$

■

Corollary 5.3 *[44]Let $L = [g(x), f(x)]$ be an exponential Riordan array with $(d+2)$-banded lower Stieltjes matrix S_L. Then the moments μ_n of the associated family of d-OPS are given by the terms of the first column of L.*

5.5. Exponential Riordan arrays

5.6 Sheffer Riordan array

Let $g(x)$ and $f(x)$ defined as above. Then the polynomials $P_n(x)$, $n = 0, 1, ...$ defined by the generating function

$$g(x)\, e^{xf(x)} = \sum_{n\geq 0} P_n(x) t^n,$$

are called Sheffer type polynomials with $P_0(x) = 1$.

Therefore, the set of all Sheffer type polynomials $\left\{P_n(x) = [x^n]\, g(x)\, e^{xf(x)}\right\}$ with an operation of umbral composition, forms a group called the Sheffer group.

In the following paragraph, we give an example of Sheffer type Riordan array. Let us now consider a family of d-OPS of Charlier type defined by the following generating function [10]

$$G(x,t) = (1+\tau t)^{(x/\tau)} \exp\left\{\sum_{i=1}^{d} \theta_i x^i\right\} = \sum\nolimits_{n\geq 0} \frac{1}{n!} P_n(x)\, t^n. \tag{5.41}$$

This generating function is of Sheffer A-type zero. Note also that when $\tau \to 0$, we get the previous example.

The family of polynomials generated by (5.41) satisfy the following recurrence relation

$$P_{n+1}(x) = (x + b - \tau n)\, P_n(x) - \sum_{i=1}^{d} a_i \langle n\rangle_i\, P_{n-i}(x), \quad n \geq 0 \tag{5.42}$$

which means that $\{P_n\}_{n\geq 0}$ is d-OPS where $a_i = (n+1)\,\theta_{i+1} + n\theta_i$ and $\langle n\rangle_i = (n-i+1)_i$. In this case, the corresponding Stieltjes matrix take the following form

$$\begin{pmatrix}
b & 1 & 0 & \cdots & 0 & \\
a_1 & b-\tau & 1 & 0 & \cdots & 0 \\
2a_2 & 2a_1 & b-2\tau & 1 & 0 & \cdots \\
6a_3 & 6a_2 & 3a_1 & b-3\tau & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & & \\
d!a_d & & \ddots & da_1 & b-d\tau & \\
0 & a_d \langle 2+d\rangle_d & \cdots & d(d+1)\, a_2 & (d+1)\, a_1 & \ddots \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}$$

where the common difference of $\{\alpha_k = b\}_{k\geq 0}$ is τ and it is 0 for $\left\{\frac{\mu_k^{d-i}}{\langle k+d\rangle_i}\right\}_{k\geq 0}$.

Therefore, from (5.34) and (5.35) we obtain

$$\begin{aligned}
f(x) &= b + xe^{\tau x} \quad \text{and} \\
\ln(g) &= \int \left[b + a_1 (b + xe^{\tau x}) + 2a_2 (b + xe^{\tau x})^2 + ... + d!a_d (b + xe^{\tau x})^d\right] dx,
\end{aligned}$$

i.e.,

$$\ln g\left(x\right) = \sum_{i=1}^{d} \delta_i x^i \exp\left\{i\tau x\right\}.$$

Chapter 6

Conclusions and future directions

In this thesis we are concerned with Riordan arrays and d–orthogonal polynomials. After giving some preliminaries results in the two first Chapters we focused in Chapter 3, on the construction of the inner products which generate the sequence of d–orthogonal polynomials in the sense of Sobolev. Our perspective in the future is to study d–classical of Sobolev and d–semi classical orthogonal polynomials of Sobolev.

In chapter 4, we characterize the $d-$ classical orthogonal polynomials. . We present an algebraic theory of classical $d-$orthogonal polynomials and we want to fill in some gaps. We broaden and close some inclusions that exist and are known perhaps as consequences. In the future we aim to extend the combinatorial theory of orthogonal polynomials created by Vienno to d–orthogonal polynomials. Indeed, a second method of calculating moments has been discovered by using a non-trivial combinatorial model from a generalization of Pollaczek polynomials that can be generalized to all Appell polynomials.

In Chapter 5, we pioneer the study of d–orthogonal polynomials and Riordan arrays. More precisely, we have extended the results of Riordan arrays and orthogonal polynomials in such a way to give a significant generalization from the orthogonal case to the d–orthogonal case. We show that the Riordan array, in the constant coefficient case, is a good vehicle for such studies. A principal discovery is that the d–orthogonal polynomials are characterized by the Rordan arrays. Our perspectives are then to extend this idea to algebraic structures in d–Hankel matrices and d– Hankel-plus-Toeplitz matrices relating to d–classical orthogonal polynomials. We aim also to extend our research to $d-$continued fraction and d–semi classical orthogonal polynomials.

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