وزارة التعليم العالي و البحث العلمي

UNIVERSITE BADJI MOKHTAR
ANNABA

جامعة باجي مختار
عنابة

Faculté des Sciences
Département de Mathématiques

Année 2008

THESE

Présentée en vue de l'obtention du diplôme de **DOCTORAT**

STUDY OF THE CONTROLLABILITY OF DIFFERENTIAL EQUATIONS UNDER IMPULSIVE CONDITIONS

Option :

E.D.P. et théorie des opérateurs

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Dedication

To my parents and all my family.

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Dedication Acknowledgements

This work has been made possible in great part by the discussions with a great deal of people. My deep graduate to my advisor Prof. Dr. Said Mazouzi who was always there to provide me with a lot of intuition and guidance which led to many of the results in this thesis. I would like to thank the members of my thesis committee Professors: A. S. Chibi, N. Hamri, S. Mesloub, H Sissaoui and L. Nisse, Special thanks are owed to the members of our Department of mathematics and to the colleagues of graduate mathematical studies at the University of Annaba. Finally, I thank everybody who assisted me in the preparation of this thesis.

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Résumé

L'objet de cette thèse est l'étude de la contrôlabilité impulsive de certaines équations d'évolution abstraites impulsives. On applique la méthode d'unicité hilbertienne **(HUM)** pour obtenir le contrôle impulsionnel dans le cas où l'espace d'état initial est un espace de Hilbert. Ce problème de contrôlabilité n'est pas simple, et en général, il ne peut pas être résolu explicitement. Par ailleurs, nous donnons une condition nécessaire et suffisante pour la résolution du problème de la contrôlabilité nulle ainsi considéré.

Finalement, on donne quelques applications d'équations aux dérivées partielles impulsives, à savoir l'exemple de l'équation des ondes et celle de Schrödinger.

Abstract

The aim of the present thesis is the study of the impulsive controllability of certain abstract evolution equations. We apply the Hilbert Uniqueness Method **(HUM)** to obtain the impulsive control in the case where the initial state space is a Hilbert space. This problem of controllability is not at all simple, and in general there is no universal method to get it explicitly.

On the other hand, we give a necessary and sufficient condition for the null controllability of such a problem.

Finally, we give some applications of impulsive Partial Differential Equations, namely the example of the Wave equation and that of Schrödinger.

Contents

ملخص

تتم في هذه الأطروحة دراسة مسائل التحكم للمعادلات التفاضلية النبضية المعرفة على فضاء هلبرت باستعمال طريقة الوحدانية الهلبرتية (HUM). حيث حصلنا على شرط لازم و كاف لإثبات التحكم و تعيين على الأقل إحدى هذه التحكمات للمسائل المعتبرة موضع الدراسة.

نختم دراستنا بإعطاء بعض التطبيقات والأمثلة لمسائل التحكم بالنسبة لبعض المعادلات التفاضلية الجزئية النبضية من الرتبة الأولى و من الرتبة الثانية و المتمثلة في معادلة الاهتزاز و معادلة شرودنغير.

INTRODUCTION

The theory of impulsive differential equations has become an important area of investigation in recent years. It has been the subject of mathematical research for almost fifty years. The first papers in this theory are related to the names of V. D. Milman and A. D. Mishkis in 1960, [85]. An impulsive system is a system of special kind of consisting of a differential system and a difference system that respectively describe continuous evolutions and discrete events occurring in a mathematical model of a physical system. Many evolutionary processes are characterized by the fact that at certain moments between intervals of continuous evolutions they undergo changes of state abruptly. The durations of these changes are often negligible when compared to the total duration of the process, so that these changes can be reasonably approximated as instantaneous changes of state, or impulses. These evolutionary processes are suitably modeled as impulsive differential systems, or simply impulsive systems. Generally, an impulsive system is characterized by a pair of equations, a system of ordinary evolution equations that describes a continuous evolutionary process and a difference equation defining discrete impulsive actions. Impulsive differential equations, meanwhile, are fundamental in most branches of applied mathematics. They have applications in various fields such as physical and engineering sciences, population dynamics, theoretical physics, radiophysics, mathematical economy, chemistery, metallurgy, ecology, industrial robotics and biotechnology.

Recall that the impulsive differential equation is described by three components: a continuous-time differential equation, which governs the state of the system between impulses, an impulse equation, which models an impulsive jump defined by a jump function at the instant an impulse occurs, and a jump criterion. Mathematically these equations take the form

$$\begin{cases} y'(t) + Ay(t) = 0, t_0 < t < t_{m+1} = T,\ t \neq t_k \\ y(t_0) = y^0, \\ y(t_k + 0) - y(t_k) = \Delta y(t_k) = I_k y(t_k),\ k = 1, 2, ..., m, \end{cases} \tag{0.1}$$

where $0 = t_0 < t_1 < t_2 < ... < t_m < T$, y^0 is an initial condition in a Banach space X.

Existence and uniqueness are the most fundamental qualitative properties of impulsive systems. Early research results on existence and uniqueness have been obtained. As a result, there are some works including books for the basic theory on impulsive evolution equation in Banach spaces by Lakshmikanthan, D. D. Bainov [67], D. D. Bainov, P.S. Simeonov [13], A. M. Samoilenko [95], Benchohra *et al.*[24] and Yangs book [115]. The nonlinear impulsive evolution equations on Banach spaces have been studied by using semigroup theory in the articles of E. Hernandez [53, 54], J. H. Liu [78], W. Zhang, R. P. Agarwal and E. Akin-Bohner [116], Chen Fangqi, chen Yushu [43], Benchohra [$20-22$, 24], N.U. Ahmed [3, 4], whereas Bainov *et al.* [$11-15$] discussed the same impulsive problem with finite impulses in Banach spaces.

In recent years, controllability and its applications to evolution equations has been extensively studied by several authors. Benchohra *et al.* in [18, 19, 23] and the papers of Balachandran [16], the authors studied exact controllability for non impulsive evolution equation by fixed point theory, and obtained some important results. There are different methods for investigation of both controllability for different types of evolution equations. The choice of the appropriate method depends on the type of evolution equations and the initial state of these equations. There are various fixed-point theorems available, the most popular being Schauder's fixed point theorem, Banach contraction theorem and Schaefer's fixed point theorem. Controllability of nonlinear systems represented by ordinary differential equations has been extended to infinite dimensional systems in Banach spaces with bounded operators by Triggiani [104, 105], Naito [89] established the approximate controllability of semilinear control systems using fundamental assumptions on the system components. Yamamoto and Park [114] established necessary and sufficient conditions for the approximate controllability of a parabolic equation in a Banach space with uniformly bounded nonlinear term by estimating solutions to the nonlinear parabolic systems. Nakagiri and Yamamoto [88] gave a number of criteria for controllability and observability for evolution systems in general Banach spaces. Zhou [117] derived a set of sufficient condition for the approximate controllability of semilinear abstract equation with distributed control. Exact controllability of abstract semilinear equations has been studied by Lasieka and Triggiani [104, 105]. Furthermore, many results have been extended to impulsive differential equations. Leela [71] studied the controllability for an impulsive evolution equation in finite dimensional spaces, R. K. George and A. K. Nandakumaran, and A. Arapostathis [47], Z.H. Guan, T.H. Qian, X. Yu [49], X. Z. Liu, *et al.* in [79, 81, 82], B. Lui [75, 76] and M. U. Akhmet *et al.* in [7], S.A. Belbas in [17] gave a necessary and sufficient conditions for controllability of impulsive control systems in Euclidean spaces. B. M. Miller [86] and Ahmed [3, 4] considered optimal problem of systems governed by impulsive evolution equations in finite dimensional Banach space. In the case of finite dimensional evolution systems these notions are quite simple and so are well mastered. However, in case of infinite dimensional, the study is more complex and the classical notions may be defined through various angles.

In general it is difficult to test the exact controllability of infinite-dimensional systems.

The controllability of infinite-dimensional systems in Banach spaces has been studied extensively by virtue of the fixed point theorem. The essential part of this method is to transform the controllability problem into a fixed point problem for an appropriate operator in a function space. The question of the impulsive control has attracted the attention of many authors. Ahmed [3, 4], Peng *et al.* [93], X. Xiang. *et al.* [108 – 110], W. Wei, X. Xiang [107], and S. Hipang, X. Xiang [56], considered optimal problem of systems governed by impulsive evolution equations in infinite dimensional Banach space. M. Benchohra, L Górniewiez, S. Ntouyas, A. Ouahab in [22], and N. Abadaa, M. Benchohra, H. Hammouchec in [1, Article in press in Nonlinear Analysis 2007] studied the controllability results by fixed point theorem for the impulsive functional differential inclusions. Also, by using the fixed-point theorem, M. Guo, X. Xue, and R. Li [51] discussed a controlability of impulsive inclusions with nonlocal conditions.

Our technique is the method called **HUM** (Hilbert Uniqueness Method) which exists in the continuous case for finding a control that steers the solution to a given final state, which is based on the duality between a linear control system and its adjoint system. Actually, the HUM has been developed for evolution equations without impulses *i.e.* in the special case where (0.1) has no impulses, by Lions. The method first saw the light in 1988, see for instance Lions [74], Lagnese [65, 66], who treated certain first order systems, while Bensoussan [25] gave some abstract views on HUM. This method has been studied in the classical case by many authors; Zuazua [118] studied the observability and stability for the evolution equation without impulses and obtained a unique control by this method. Komornik [62, 63], G. Lebeau [70], D'ager [36], M. Milla Miranda, [86], Bensoussan [25], Lagnese [65, 66], obtained the same conclusion by applying the HUM when (0.1) does not contain the jump conditions. Haraux [53], studied the exact controllability of (0.1) without impulses by the HUM but he did not obtain a controllability result for (0.1) with impulsive condition. This method also applies for impulsive differential systems in a Hilbert space. The controllability of evolution equation with impulses by the HUM was considered by R. Boukhamla and S. Mazouzi [30]. In this thesis we generalize some results of controllability obtained for classical evolution equations to the impulsive evolution equations in a Hilbert space. Sufficient conditions are established for the controllability result by using semigroup theory and fixed point theorem. Our technique is based on fixed point theorems and the **HUM**. In fact, the HUM is one of the important techniques used to obtain controllability of differential equations and can be applied to impulsive partial differential equations such as impulsive wave equation and impulsive Schrödinger equation.

We fix the final time $T > 0$ for which we expect the solution of our problem to exist.

The questions one can ask are the following:

- What is a controllability ?
- What does controllability mean for a impulsive system in a Hilbert space with finite impulses?
- What is one possible test $T > 0$, $\{t_k\}_{k\in\mathbb{N}} \subset (0, T)$ for impulsive controllability?
- What is the possible initial data space which we want to control?
- What is the right control vector space in which we do have to control the system?
- Is the impulsive system null controllable ? Is it exactly controllable?
- What is the result about the exact controllability for impulsive evolution equations with infinite impulses?
- Which neccessary or sufficient conditions should be imposed on the operators B, D_k for the above impulsive controled system (0.2) to be controllable?

Finally,

- If the answers to the above questions are positive, how can we obtain the control that conduct to that aim?
- If the answers are negative, what can we say about the attainable states from some initial state y^0? Can we decompose this system in controllable part and non-controllable one or not?

The theory of impulsive differential equations is richer than the corresponding theory of differential equations without impulses. Controllability is concerned with the coupling between initial states and final states. The fundamental controllability problem is the following: given an initial state y^0 and a final state $y(T)$, find a control that "steers" the solution of

$$\begin{cases} y'(t) + Ay(t) = Bu(t), \quad t \in (0,T) \setminus \{t_k\}_{k\in\sigma_1^m}, \\ y(0) = y^0, \\ \Delta y(t_k) = I_k y(t_k) + D_k v_k, \ \ k \in \sigma_1^m, \end{cases} \tag{1}$$

from y^0 to $y(T)$ in time T. To solve this problem we should describe an appropriate function space X for which (0.2) has a unique solution which is piecewise continuous in time (so that it makes sense to speak of initial and final values). If such a control vector $\left(u(t), \{v_k\}_{k\in\sigma_1^m}\right)$ exists, for all possible $y^0 \in X$, we say that (0.2) is controllable in $(0,T)$.

We shall use the HUM to analyze impulsive system for null controllability and exact controllability in the subsequent chapters.

In the preliminary chapter, we introduce some fundamental notions and preliminaries which will be needed in the proof of existence and controllability results. It is ended with mathematical models of some important examples of impulsive evolution systems.

In the second chapter, some existence and uniqueness results for impulsive systems are presented. There, we generalize certain fundamental properties such as the existence of mild and classical solution for some abstract impulsive evolution equations, and we give an explicit form of these solutions.

In the third and forth chapters, we discuss exact controllability of impulsive evolution equation of first and second orders by the HUM; on the other hand, we obtain the control function by this method. The third chapter is based on the results of R. Boukhamla and S. Mazouzi [30], as well as those of A. Haraux [53]. We establish some controllability result for an impulsive evolution equation with finite fixed impulses. We also consider different types of controllability, such as null-and exact controllability. Also in this chapter we shall separately treat systems dealing with finite impulses as well as systems with infinite impulses. The case of the impulsive systems with infinite impulses is more complicated, but it is still easily analyzed whether one has controllability or not. In the last sections of chapter 3 and 4, we will apply the theory of impulsive controllability to some impulsive partial differential equations, (IPDE), such as the impulsive Schrödinger equation and the impulsive wave equation.

Finally, we present some conclusions for the controllability of an impulsive equation in

Hilbert spaces by the HUM.

Chapter 1

Preliminaries, Modelling and Applications

1.1 Preliminaries

This section summarizes some basic general information on impulsive evolution equations in Banach spaces and introduces fundamental theory and preliminary results that will be needed in the rest of this thesis.

We first introduce and define certain fundamental suitable function spaces which are very important for the study of impulsive differential equations.

The space of absolutely continuous functions $AC\left([a,b];X\right)$:

An absolutely continuous function plays a fundamental role in the theory of differential equations although it may not be differentiable at all points, it still can be recovered by integration from its derivative. In fact, it is characterized by this property, and in some sense, is the weakest acceptable kind of solution one can seek in a discontinuous (impulsive) differential equation. Let $(X;\|.\|)$ be a given Banach space.

We shall denote by $C\left([a,b];X\right)$ the set of all functions $y:[a,b]\rightarrow X$ which are continuous on the closed interval $[a,b]$, and let $C^1\left([a,b];X\right)$ be the set of all functions $y\in C\left([a,b];X\right)$ which are continuously differentiable in the open interval (a,b) and with left limit and right limit, respectively,

$$y_{-}^{'}(b) = \lim_{h\to 0^{-}} \frac{y(b+h)-y(b)}{h}, \quad y_{+}^{'}(a) = \lim_{h\to 0^{+}} \frac{y(a+h)-y(a)}{h},$$

exists in b, respectively, in a. We define similarly the higher order left and right derivatives of such functions, respectively, as follows:

$$\begin{aligned} y_{-}^{(n)}(b) &= \lim_{h\to 0^{-}} \frac{y^{(n-1)}(b+h)-y^{(n-1)}(b)}{h}, \\ y_{+}^{(n)}(a) &= \lim_{h\to 0^{+}} \frac{y^{(n-1)}(a+h)-y^{(n-1)}(a)}{h}, \end{aligned}$$

recursively for every $n \geq 1$.

On the other hand, let $C^{n}([a,b];X)$ denote the set of all functions $y \in C([a,b];X)$ such that $y^{(k)} \in C([a,b];X)$, for $0 \leq k \leq n$ and $y_{-}^{(n)}(b)$, $y_{+}^{(n)}(a)$ exist. Recall that the existence of the left (respectively, right) derivative of a function y at a point implies that the function itself is left (respectively, right) continuous at that point.

Definition 1 *A function $f : [a,b] \to X$ is called absolutely continuous, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that the implication*

$$\sum_{k=1}^{n}(b_k - a_k) < \delta \implies \sum_{k=1}^{n} \|f(b_k) - f(a_k)\| < \varepsilon$$

holds, for every sequence of intervals $]a_k, b_k[\subset [a,b]$ such that $]a_k, b_k[\cap]a_j, b_j[= \emptyset$ for $k \neq j$. We denote by $AC([a,b];X)$ the space of all absolutely continuous functions $f : [a,b] \to X$.

Here are some properties of the absolutely continuous functions :

Let f be a function from the interval $[a,b]$ to X

1- If the function f is absolutely continuous on $[a,b]$, then it is continuous.

2- If f is absolutely continuous, then f is almost everywhere differentiable.

3- A function f is an indefinite integral if and only if it is absolutely continuous.

4- Every absolutely continuous function is the indefinite integral of its derivative.

5- If f satisfies the Lipschitz condition, then it is absolutely continuous.

The space of piecewise continuous functions :

Now we introduce some Banach spaces which are very useful for the study of impulsive differential equations.

We define the following space of functions:

$\mathcal{PC}\left([0,T];X\right) = \{y, y : [0,T] \to X$ *such that* $y(t)$ *is continuous at* $t \neq t_k$, $y\left(0^+\right), y\left(T^-\right)$, $y\left(t_k^-\right), y\left(t_k^+\right)$ *exist, for every* $k \in \sigma_1^m\}$, where σ_p^q is a subset of $\mathbb{N}$ given by

$$\sigma_p^q = \{p, p+1, ..., q\}, \quad p < q, \ p, q \in \mathbb{N}.$$

Evidently, $\mathcal{PC}\left([0,T];X\right)$ is a Banach space with respect to the norm

$$\|y\|_{\mathcal{PC}} = \sup_{t\in[0,T]} \|y(t)\|.$$

In particular, if $\{t_k\}_{k\in\sigma_1^m} = \varnothing$, then the space $\mathcal{PC}\left([0,T];\{t_k\}_{k\in\sigma_1^m}, X\right)$ coincides with $C\left([0,T], X\right)$.

On the other hand, we define the subspaces $\mathcal{PLC}$ (respectively, $\mathcal{PRC}$)$= \{y, y \in \mathcal{PC}$ *such that* $y(t)$ *is left (*respectively, *right) continuous at* $t = t_k$, *for every* $k \in \sigma_1^m\}$.

If $\phi \in \mathcal{PC}$, then one can define a function

$$\phi^{left} \in \mathcal{PLC}\left([0,T];\{t_k\}_{k\in\sigma_1^m}, X\right)$$

respectively,

$$\phi^{right} \in \mathcal{PRC}\left([0,T];\{t_k\}_{k\in\sigma_1^m}, X\right)$$

such that

$$\phi^{left}(t) = \phi(t) = \phi^{right}(t)$$

everywhere, except possibly at points $t = t_k$, that is,

$$\phi^{left}(t) = \begin{cases} \phi(t) & \text{if } t \neq t_k \\ \phi(t_k^-), k \in \sigma_1^m & \text{otherwise.} \end{cases}$$

respectively,

$$\phi^{right}(t) = \begin{cases} \phi(t) & \text{if } t \neq t_k \\ \phi(t_k^+), k \in \sigma_1^m & \text{otherwise.} \end{cases}$$

We shall call the function ϕ^{left} *(*respectively, ϕ^{right}*)* a left *(*respectively, *right)* extension of the function $\phi \in \mathcal{PC}$. Then the function $\phi^{left} \in \mathcal{PLC}\left([0,T), X\right)$ *(*respectively, $\phi^{right} \in \mathcal{PRC}\left([0,T), X\right)$*)* can be written as

$$\phi^{left}(t) = \begin{cases} \phi_{[0]}^{left}(t) & \textit{if} \quad t \in [t_0, t_1], \\ \phi_{[1]}^{left}(t) & \textit{if} \quad t \in (t_1, t_2], \\ \ldots\ldots\ldots & \ldots. \quad \ldots\ldots\ldots \\ \phi_{[m]}^{left}(t) & \textit{if} \quad t \in (t_m, T], \end{cases}$$

respectively

$$\phi^{right}(t) = \begin{cases} \phi_{[0]}^{right}(t) & \textit{if} \quad t \in [t_0, t_1), \\ \phi_{[1]}^{right}(t) & \textit{if} \quad t \in [t_1, t_2), \\ \ldots\ldots. & \ldots\ldots \quad \ldots.. \\ \phi_{[m]}^{right}(t) & \textit{if} \quad t \in [t_m, T]. \end{cases}$$

For $y \in \mathcal{PC}\left([0,T]; \{t_k\}_{k=1}^{k=m}; X\right)$, we consider the functions

$$y_{[k]} := y\left|_{[t_k, t_{k+1}]}\right.$$

where $y_{[k]}(t) = y(t)$, if $t \in (t_k, t_{k+1}]$ and $y_{[k]}(t_k) = y_{[k]}(t_k^+)$.Thus, $\mathcal{PC}$ can be identified with the Banach space $\prod\limits_{k=0}^{m} \mathcal{C}\left([t_k, t_{k+1}]; X\right)$ and hence, $\mathcal{PC}\left([0,T]; \{t_k\}_{k \in \sigma_1^m}; X\right)$ is also a Banach space with respect to the norm

$$\sum_{k=0}^{m} \left\| y\left|_{[t_k, t_{k+1}]}\right. \right\|_{C([t_k, t_{k+1}]; X)},$$

Furthermore, we define the space

$$\mathcal{PC}^1 \doteqdot \left\{ y \in \mathcal{PC} : y_{[k]} \in C^1\left([t_k, t_{k+1}]; X\right), \text{ for each } k \in \sigma_0^m \right\}$$

and for $y \in \mathcal{PC}^1$, we define $y^{'}$ as a function

$$y^{'}(t) = \begin{cases} y^{'}_{[0]}(t) \;\; if \;\; t \in [t_0, t_1]\,, \\ y^{'}_{[1]}(t) \;\; if \;\; t \in [t_1, t_2]\,, \\ \qquad\qquad \cdots\cdots\cdots\cdots \\ y^{'}_{[m]}(t) \;\; if \;\; t \in [t_m, T]\,. \end{cases}$$

We also need the following spaces,

$$\begin{aligned} \mathcal{PAC} &\doteqdot \left\{ y \in \mathcal{PC} : \; y_{[k]} \in AC\left([t_k, t_{k+1}]\,; X\right), \text{ for each } k \in \sigma_0^m \right\}, \\ \mathcal{PAC}^1 &\doteqdot \left\{ y \in \mathcal{PC} : \; y_{[k]} \in AC^1\left([t_k, t_{k+1}]\,; X\right), \text{ for each } k \in \sigma_0^m \right\}, \end{aligned}$$

Next, we define some classes of piecewise continuous functions. Let $a, b \in \mathbb{R}$, with $a < b$ and let X be a Banach space. Define

$$\begin{aligned} \mathcal{PC}\left([a,b]\,; X\right) = \; & \{ y : [a,b] \to X \; | y\left(t^+\right) = y\left(t\right), \; \forall t \in [a,b)\,, \; y\left(t^-\right) \text{ exists in } X \\ & \text{for all } t \in (a,b] \text{ and } y\left(t^-\right) = y\left(t\right), \text{ for all but at most a finite number} \\ & \text{of points } t \in [a,b) \}\,, \end{aligned}$$

$$\begin{aligned} \mathcal{PC}\left([a,b)\,; X\right) = \; & \{ y : [a,b] \to X \; | y\left(t^+\right) = y\left(t\right), \; \forall t \in [a,b)\,, \; y\left(t^-\right) \text{ exists in } X \\ & \text{for all } t \in (a,b) \text{ and } y\left(t^-\right) = y\left(t\right), \text{ for all but at most a finite number} \\ & \text{of points } t \in (a,b) \}\,. \end{aligned}$$

We state the following definition :

Definition 2 *A function $f : [t_0, T] \to X$ is called piecewise absolutely continuous function of class $AC^{(n)}\left([0,T] \setminus \{t_k\}_{k \in \sigma_1^m}\,; X\right)$ if $f^{(j)}\left|_{(t_0,t_1]}\right. \in C^n\left([(t_0,t_1]\,, X\right)$, $f^{(j)}\left|_{[t_k,t_{k+1}]}\right. \in C^n\left([t_k, t_{k+1}], X\right)$, for $k \in \sigma_1^{m-1}$ and $f^{(j)}\left|_{[t_m,T)}\right. \in C^n\left([[t_m, T)\,, X\right), \; j \leq n$.*

The following lemmas are easy to demonstrate :

Lemma 3 [50] *If $y \in \mathcal{PC}\left([0,T]\,; X\right) \cap C^1\left([0,T] \setminus \{t_k\}_{k \in \sigma_1^m}\,; X\right)$, then*

$$y(t) = y(0) + \int_0^t y^{'}(s)ds + \sum_{0<t_k<t} \left[y(t_k^+) - y(t_k) \right],$$

for all $t \in [0, T]$.

Lemma 4 [50] *If* $y \in \mathcal{PC}^1([0,T];X) \cap C^2\left([0,T] \setminus \{t_k\}_{k\in\sigma_1^m};X\right)$, *then*

$$y^{'}(t) = y^{'}(0) + \int_0^t y^{''}(s)ds + \sum_{0<t_k<t} \left[y^{'}(t_k^+) - y^{'}(t_k) \right],$$

for all $t \in [0, T]$, $t \notin \{t_k\}_{k\in\sigma_1^m}$ *and*

$$\begin{aligned} y(t) &= y(0) + ty^{'}(0) + \int_0^t (t-s)\, y^{''}(s)ds \\ &\quad + \sum_{0<t_k<t} \left\{ \left[y(t_k^+) - y(t_k) \right] + (t - t_k) \left[y^{'}(t_k^+) - y^{'}(t_k) \right] \right\}, \end{aligned}$$

for all $t \in [0, T]$.

1.2 The model formulation

In this section we present some examples that motivate the study of impulsive evolution equations.

Example 5 *(A Bouncing ball* [9, 48, 93]*) In this example, we consider a ball that is jumping on a flat horizontal surface (seeFigure 1). The loss of energy, caused by the friction of surface, is characterized by constant* μ.

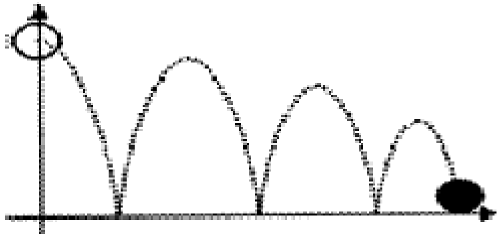

Figure 1 A Bouncing ball

This process is simulated by a differential equation of second order

$$m\frac{d^2z}{dt^2} = F,$$

where m *is the mass of the ball,* $F = -mg$, *is the force (*$g \approx 9.81m\left/s^2\right.$ *is the acceleration of the Earth's gravitation). Each time when the ball touches the ground the surface vertical component of the velacity vector changes its sign.*

We consider a ball of mass m *subject to the action of gravity. We let it fall from an altitude* $z_0 > 0$ *with a zero initial velocity. The altitude* $z(t)$ *of the ball follows the differential equation issued from the classical mechanics* $mz^{''}(t) = -mg$, *when* $z(t) = 0$, *the ball touches the ground and bounces loosing a fraction of its energy:*

$$z^{''}(t) = -cz(t), \;\; with\; c \leq 1.$$

Let us look what happens for the case of impulsive setting with the same bouncing ball. At time t_0 *we let the ball fall from an altitude* $z_0 > 0$. *The variable of the system is* $x = (x_1, x_2)$, *where* x_1 *is the altitude of the ball and* x_2 *its velocity. The initial condition of the impulsive system is* $(t_0, (z_0, 0))$. *As long as the altitude of the ball is positive one has*

$$\begin{aligned} x_1^{'}(t) &= x_2(t) \; and \; x_2^{'}(t) = -g \Longrightarrow \\ x_2(t) &= -g(t-t_0) \; and \; x_1(t) = \frac{-g}{2}(t-t_0)^2 + z_0. \end{aligned}$$

The impact of the ball on the ground (and thus the impulsive system transition) occurs at a *time t_1 such that $x_1(t_1) = 0$, and so*

$$t_1 = t_0 + \sqrt{\frac{2z_0}{g}},\ x_2(t_1^-) = -\sqrt{2gz_0}.$$

At this moment, the ball bounces:

$$x_2(t_1^+) = -cx_2(t_1^-) = c\sqrt{2gz_0}.$$

Let t_k be the moment when occurs the k^{th} bouncing. Until the next impact of the ball on the ground, one has

$$x_2(t) = -g(t - t_k) + x_2(t_k^+) \text{ and } x_1(t) = \frac{-g}{2}(t - t_k)^2 + x_2(t_k^+)(t - t_k).$$

At time t_{k+1}, the ball touches the ground

$$x_1(t_{k+1}) = 0 \Longrightarrow t_{k+1} = t_k + \frac{2}{g}x_2(t_k^+),$$

and boumces

$$x_2(t_{k+1}^+) = -cx_2(t_{k+1}^-) = cx_2(t_k^+) = ... = c^k x_2(t_1^+) = c^{k+1}\sqrt{2gz_0}.$$

Thus, the impulsive system admits an infinite impulsion and the necessary time to attain the rest position is

$$T = \sum_{k=0}^{k=\infty}(t_{k+1} - t_k) = \sqrt{\frac{2z_0}{g}} + \sum_{k=1}^{k=\infty}\frac{2}{g}x_2(t_k^+) = \sqrt{\frac{2z_0}{g}} + \sum_{k=1}^{k=\infty}c^k 2\sqrt{\frac{2z_0}{g}},$$

if $c < 1$. Therefore,

$$\sum_{k=0}^{k=\infty}(t_{k+1} - t_k) = \sqrt{\frac{2z_0}{g}}(1 + \frac{2c}{1-c}) = T < \infty.$$

Example 6 [09, 13, 100] ***A body attached by a spring to a fixed point***

A body M attached by a spring to a fixed point A and excited by a force $F = h\sin(pt+\alpha)$, vibrates along a horizontal line and collides with a rigid wall B as shown in Fig. 2

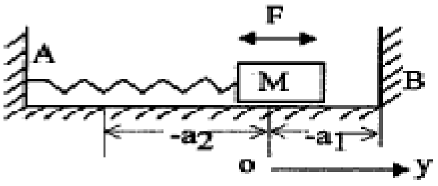

Figure 2. A body M attached by a spring to a fixed point.

The system can be described by the impulsive differential equations as follows

$$my^{''} + cy^{'} + ky = h\sin(pt+\alpha), \quad y \in [-a_2, a_1],$$

$$y^{'+} = \begin{cases} -\mu y^{'}, & y = a_1, \quad y^{'} \in (0, b_1], \quad \mu \in (0,1], \\ y^{'}, & y = a_1, \qquad\quad y^{'} \in [-b_2, 0), \end{cases}$$

where $y^{'+}$ is the velocity of the body after the impact is applied, $y^{'} = 0$ for $y = -a_2$, and all the constants are positive.

A multi-body system vibrating with impact is given by

$$Ny^{''} + Cy^{'} + Ky = H\sin(pt+\alpha), \quad g_i\left(t, y, y^{'}\right) \neq 0,$$

$$y^{'+} = By^{'}, \quad g_i\left(t, y, y^{'}\right) = 0, \quad i = 1, 2, ...,$$

where $y \in \mathbb{R}^n$; N, C, K and $B \in \mathbb{R}^n \times \mathbb{R}^n$; N, K are positive definite matrices and C is nonnegative definite matrix.

The results in this example are applicable to economic problems.

Example 7 [41] ***The Impulsive Solow equation.***

The seminal differential equation of Solow (1956) becomes an impulsive differential equation, when shocks to capital intensity are modelled with jumps. This statement results from an analy-

sis of unit roots in four German macroeconomic time series.

IDE modelling of the Solow equation

- *Model the jumps of German capital* $K(t)$ *and the capital intensity* $r(t) = \dfrac{K(t)}{L(t)}$ *with real-valued piecewise continuous functions.*
- *Let* $t_1, t_2, ..., t_k... > 0$ *be the moments, when the stock of capital* $K(t)$ *is subject to shock effects changing from the positions* $K(t_k^-)$ *into the position* $K(t_k^+)$ *and* $r(t)$ *is intrinsic to the system itself. An adequate mathematical model of the growth of capital in this case will be the impulsive differential equation of the form:*

$$\begin{cases} \dot{K}(t) = sF(K(t), L_0 e^{nt}), & t > t_0, \quad t \neq t_k \\ K(t_0 + 0) = K_0, & \\ \Delta K(t) = J_k\left(K(t_k)\right) & t = t_k \quad k = 1, 2, 3, ..., \end{cases} \tag{1.1}$$

where functions J_k *characterize the magnitude of the impulse effect at times* t_k; $K(t_0 - 0)$ *and* $K(t_0 + 0)$ *are respectively the capital level before and after the impulsive effect,* K_0 *is the initial capital.*

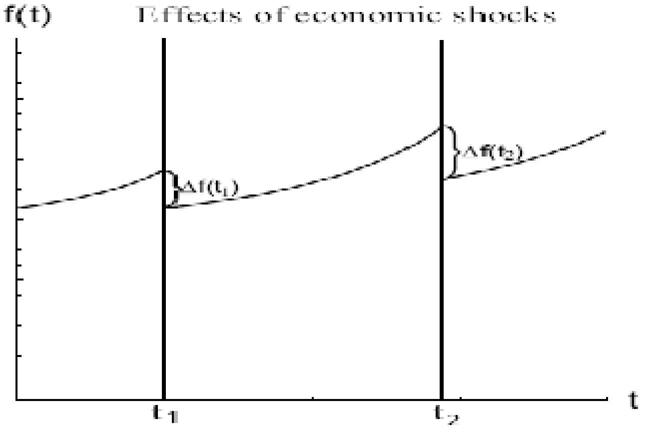

Figure 3. Appearance of economic shocks in macroeconomic variables

- *It can be used in the economic studies of business cycles in situation when the total stock of capital* $K(t)$ *is subject to shock effects.*
- *By means of the models of type (1.1) it is possible to investigate one of the most important*

problems of economics - the problem of the optimal control of the business cycles , (see R. M. May, [83]*).*

- *In the model (1.1) the moments of impulse effect is caused by an interior effect. But the moments of impulse effect can be caused by an exterior effect.*
- *The solution of the impulsive Solow equation is the following*

$$\begin{aligned} K(t) &= L_0 e^{nt}\left[\left(r\left(t_0\right)^{1-\alpha}-\frac{s}{n}\right)e^{-n(1-\alpha)t}+\frac{s}{n}\right]^{\frac{1}{1-\alpha}} \\ &\quad+\sum_{t_0\leq t_k<t} J_k\left(K(t_k)\right). \end{aligned}$$

The impulsive differential equations can be successfully used to the mathematical simulation of biotechnological processes as seen in the following

Example 8 [12] *Consider the equation of Verhulst*

$$\frac{dN}{dt}=\frac{\mu N}{k}\left(K-N\right),$$

where $N = N(t)$ *denotes the biomass of a given population at the moment* $t \geq 0$*,* K *is the capacity of the environment and* μ *is the difference between the birth-rate and death-rate.*
The case when external disturbances act upon the population is often met. We shall consider the cases when the external disturbances take place at fixed moments of time and are expressed as adding to or taking off certain quantities of biomass. The impulsive analogue of the equation of Verhulst in this case has the form

$$\begin{cases} \dfrac{dN}{dt}=\dfrac{\mu N}{k}\left(K-N\right) \quad t\neq t_k \\ \Delta N\left(t_k\right)=N\left(t_k^{+}\right)-N\left(t_k^{-}\right)=-I_k \quad k=1,2,... \end{cases}$$

where $0 < t_1 < t_2 < t_3 < ...$ *are the moments of external effect,* I_k*,* $k = 1, 2, ...$ *are the amounts of biomass added to* $(I_k < 0)$ *or taken off* $(I_k > 0)$ *at the moments* $t_1, t_2, t_3, ...$
Impulsive differential equations arise naturally in various fields such as population dynamics and optimal control. It seems that the first treatment of impulsive systems goes back to the monograph by Krylov and Bogolyubov [64]*.*

Example 9 [15]*(Population Dynamics) The impulsive boundary value problem*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = u(x,t)\left(a - bu^2(x,t)\right), & \text{on} \quad \Omega \times (0,T), \quad t \neq t_k, \\ u(\sigma,t) = 0, & \text{in} \quad \partial\Omega \times (0,T), \\ u(x,t_k^+) = (1+\alpha_k)u(x,t_k), & \text{on} \quad \Omega \times \{t_k\}, \quad t = t_k, \\ u(x,0^+) = u^0(x), & \text{on} \quad \overline{\Omega} \times \{0\}, \ k = 1,2,.... \end{cases}$$

describes a single species population in bounded environment. The function $y(x,t)$ represents the population density at the point $x \in \overline{\Omega}$ and time $t \geq 0$. Condition $u(x,t_k^+) = (1+\alpha_k)u(x,t_k)$, describes instantaneous changes in the population density due to phenomena as: harvesting, disasters, immigration, etc.

Example 10 [72] *In this example, we assume that the host population is in a stationary demographic state, whose total size is constant N. Let $N(a)$, $0 \leq a \leq r_m$ (r_m denotes the highest age attained by the individuals in the host population) be the age density of the total number of individuals, and $N(a)$ satisfies*

$$N(a) = \mu^* N e^{-\int_0^a \mu(s)ds},$$

$\mu_1(a)$ is the instantaneous death rate at age a of the host population, μ^* is the crude death rate, we assume that $\mu(a)$ is nonnegative, locally integrable on $[0,r_m)$, and satisfies

$$\int_0^{r_m} \mu(a)da = \infty,$$

μ^* satisfies

$$\mu^* \int_0^{r_m} f(a)da = 1,$$

where $f(a) = e^{-\int_0^a \mu(s)ds}$ is the survival function. We can get the relation $N(a) = \mu^* N f(a)$. The host population is divided into two groups: susceptible $S(a,t)$ (who are healthy but can be infected), infected $I(a,t)$ (which includes latent individuals, since individuals in incubation period can also infect susceptible population), $S(a,t)$, $I(a,t)$ is the age-densities of respectively *the susceptible and infected population at time* t*.* $N(a)$ *also satisfies*

$$N(a) = S(a,t) + I(a,t).$$

Let $M(t)$ *denote the number of susceptible vectors (mosquito population) at time* t*,* $P(t)$ *the number of infected vectors at time* t*.* b*,* μ_2 *is the birth and death rate of vectors, respectively. Since blood transfusion, or using contaminated needles and syringes, susceptibles* $S(a,t)$ *can be infected, and become infected individuals at a transmission* $\beta_1(a)$*. Susceptibles* $S(a,t)$ *are infected by infected vectors, and go into infected class at a transmission rate* β_2*. The number of new vectors by infected hosts depend on the transmission rate* $\gamma(a)$*. The infected population can recover, and go into susceptible population at a transmission rate* α*. In order to control the size of mosquito, we apply the pulse spraying strategy of insecticides, We spray insecticides upon mosquito at time* $n\tau$ *every* τ *months,* τ *is the period of spraying,* $n\tau$ *is the time at which we apply the* n^{th} *(*$n \in \mathbb{N}^*$*) pulse, and* $n\tau^-$ *is the time just before applying the* n^{th} *pulse. Every pulse can reduce a function* p *of mosquito population. We obtain the following system of equations that describes the dynamics of the model :*

$$\begin{cases}
\dfrac{\partial S}{\partial t} + \dfrac{\partial S}{\partial a} = -\left(\mu_1(a) + \int_0^{r_m} \beta_1(a) I(a,t)da + \beta_2 p(t)\right) S(a,t) + \alpha I, \\
\qquad\quad \text{for } 0 < a < r_m,\ t \neq n\tau,\ n \in \mathbb{N}^*, \\
S(a,n\tau) = S(a,n\tau^-),\ \text{for } t = n\tau,\ n \in \mathbb{N}^*, \\
\dfrac{\partial I}{\partial t} + \dfrac{\partial I}{\partial a} = -\left(\int_0^{r_m} \beta_1(a) I(a,t)da + \beta_2 p(t)\right) S(a,t) - \left(\mu_1(a) + \alpha\right) I, \\
\qquad\quad \text{for } 0 < a < r_m,\ t \neq n\tau,\ n \in \mathbb{N}^*, \\
I(a,n\tau) = I(a,n\tau^-),\ \text{for } 0 \leq a < r_m, \\
\dfrac{dM}{dt} = b - M \int_0^{r_m} \gamma(a) I(a,t)da - \mu_2 M,\ \text{for } t \neq n\tau,\ n \in \mathbb{N}^*, \\
M(n\tau) = (1-p) M(n\tau^-),\ \text{for } t \neq n\tau,\ n \in \mathbb{N}^*, \\
\dfrac{dP}{dt} = M \int_0^{r_m} \gamma(a) I(a,t)da - \mu_2 P,\ \text{for } t \neq n\tau,\ n \in \mathbb{N}^*, \\
P(n\tau) = (1-p) P(n\tau^-),\ \text{for } t \neq n\tau,
\end{cases}$$

with boundary conditions

$$S(0,t) = \mu^* N, \quad I(0,t) = 0,$$

and initial conditions

$$S(0) = S_0(a) \geq 0, \quad I(a,0) = I_0(a) \geq 0, \quad M(0) = M_0 \geq 0, \quad P(0) = P_0 \geq 0,$$

where $S_0(a),\ I_0(a) \in L\,(0, r_m)\,.$

It is clear that this is an impulsive differential system with infinite impulses.

Chapter 2

Existence and Uniqueness Results for Impulsive Evolution Equations

The problem of existence and uniqueness of the solution of impulsive evolution equations is similar to that of the corresponding ordinary evolution equations. The linear impulsive evolution equations in a Banach space have, for the first time, been considered by D. D. Bainov [12 − 15]. The existence of solutions, *classical and mild,* are established by Hernandez [54, 55], J. H. Liu, [78], Y. V. Rogovchenko [94], W. Zhang, R. P. Agarwal, E. Akin-Bohner [116], Chen Fangqi, chen Yushu [43], Benchohra [20, 21, 24] and Lakshmikanthan, [67] for linear and nonlinear cases.

In this chapter, we present some basic properties of the impulsive problem in a Banach space. For more details one may refer to Hernandez [54, 55] and J. H. Liu [78] . We construct a new impulsive evolution operator corresponding to the impulsive evolution system and introduce a suitable definition of a $\mathcal{PC}$-mild solution. The impulsive evolution operator can be used to reduce the existence of $\mathcal{PC}$-mild solution for nonhomogeneous linear impulsive system to the existence of fixed points for some operator equation.

2.1 First order impulsive Cauchy problem

We define impulsive differential equations at fixed moments (fixed impulses) as follows:

$$\begin{cases} y^{'}(t) = Ay(t) + f(t, y), & t \in (0, T) \Big\backslash \{t_k\}_{k \in \sigma_1^m} , \\ y(0) = y^0, & \\ y(t_k^+) - y(t_k^-) = \Delta y(t_k) = I_k y(t_k), & k \in \sigma_1^m, \end{cases} \tag{2.1}$$

where the final time T is a positive number, y^0 is an initial condition in a Banach space X, endowed with a norm $\|.\|$, $y : [0,T] \to X$ is a vector function, and finally, $\{t_k\}_{k\in\sigma_1^m}$ is an increasing sequence of numbers in the open interval $(0,T)$, and $\Delta y\left(t_k\right)$ denotes the jump of $y\left(t\right)$ at $t = t_k$

$$\Delta y\left(t_k\right) = y\left(t_k^+\right) - y\left(t_k^-\right),$$

where $y\left(t_k^+\right)$ and $y\left(t_k^-\right)$ represent the right and left limits of $y\left(t\right)$ at $t = t_k$, respectively. On the other hand, the operators A, $I_k : H \to H$ are given linear bounded or unbounded operators. The function $f : [0,T] \times X \longrightarrow X$ is continuous on every closed interval $[t_k, t_{k+1}]$, and it is non-linear in general.

The corresponding homogeneous system plays an important role in controllability studies,

$$\begin{cases} \varphi^{'}(t) = A\varphi(t), & t \in (0,T) \Big\backslash \{t_k\}_{k\in\sigma_1^m}, \\ \varphi(0) = \varphi^0, \\ \Delta\varphi\mid_{t=t_k} = I_k(\varphi(t_k)), & k \in \sigma_1^m, \end{cases} \tag{2.2}$$

and the following linear homogeneous impulsive differential system

$$\begin{cases} \tilde{\varphi}^{'} = -A^*(t)\tilde{\varphi}, & t \in (0,T) \Big\backslash \{t_k\}_{k\in\sigma_1^m}, \\ \tilde{\varphi}(0) = \tilde{\varphi}^0, \\ \Delta\tilde{\varphi}(t_k) = -(I + I_k^*)^{-1} I_k^* \tilde{\varphi}(t_k), & k \in \sigma_1^m, \end{cases}$$

is called the adjoint system to the impulsive system (2.2).

In the next section, we give some abstract results, some basic properties as well as the notion of solutions for the impulsive evolution equations.

Notion of Solution for the Impulsive Evolution Equation :

We first give the definition of a classical solution for impulsive evolution equations.

Definition 11 (Classical solution) *By a classical solution of an impulsive evolution equation we mean a piecewise absolutely continuous mapping with discontinuities of first kind at the points $t = t_k$ which, for almost all t, satisfies the system (2.1) and for $t = t_k$, satisfies the jump condition. In other words, a classical solution of the impulsive equation (2.1) is a function $y \in \mathcal{PC}([0,T]\,; X) \cap C^1((0,T) \Big\backslash \{t_k\}_{k\in\sigma_1^m}, X)$, $y(t) \in D(A)$, for $t \in (0,T) \backslash \{t_k\}_1^m$, such that y*

satisfies (2.1) in $[0,T)$.

Note that the classical solutions for evolution equations without impulsive conditions are defined in an obvious way, see Pazy [90].

To be able to apply the method in [90], we need the following lemmas.

Lemma 12 *[90] Consider the evolution problem*

$$\begin{cases} y^{'}(t) = Ay(t) + f(t,y), & t \in (0,T), \\ y(0) = y^0. \end{cases}$$

If $y^0 \in D(A)$, *and* $f \in C^1((0,T)\times X; X)$, *then it has a unique classical solution which satisfies*

$$y(t) = S(t)y^0 + \int_0^t S(t-s)f(s,y(s))ds, \quad t \in [0,T),$$

where $S(t)$ *is the semigroup generated by* A.

Lemma 13 *[78] Let assumptions* ***(H1)-(H2)*** *be satisfied, and assume that* $y^0 \in D(A)$ *and that* $f \in C^1((0,T)\times X; X)$. *Then, for the unique classical solution* $y(.,y^0)$ *on* $[0,t_1)$ *of system (2.1) without impulses (guaranteed by Lemma* ***12****), one can define* $y(t_1)$ *in such a way that* y *is left continuous at* t_1 *and* $y(t_1) \in D(A)$.

Proof. Consider the following evolution problem without impulses in $(0,T)$,

$$\begin{cases} w^{'}(t) = Aw(t) + f(t,w(t)), \; 0 < t < T, \\ w(0) = y^0, \end{cases}$$

From Lemma **12**, there is a classical solution given by

$$w(t) = S(t)y^0 + \int_0^t S(t-s)f(s,w(s))ds, \quad t \in [0,T),$$

with $w(t) \in D(A)$, for $t \in [0,T)$. Next, applying Lemma **12** one has, for $t \in [0,t_1) \subset [0,T)$

$$y(t) = S(t)y^0 + \int_0^t S(t-s)f(s,y(s))ds, \quad t \in [0,t_1).$$

Next, we define

$$y(t_1) = S(t_1)y^0 + \int_0^{t_1} S(t_1 - s)f(s, y(s))ds,$$

so that $y(.)$ is left continuous at t_1. Then apply Lemma **12** in $[0, t_1]$ to get

$$y(t) = w(t),\ [0, t_1].$$

Thus, we have, $y(t_1) = w(t_1) \in D(A)$ which completes the proof.■

Before proving the main theorem, we need the following Lemma.

Lemma 14 *[78] Assume that* $y^0 \in D(A),\ q_k \in D(A), k \in \sigma_1^m$ *and that* $f \in C^1((0,T) \times X; X)$. *Then the impulsive system*

$$\begin{cases} y^{'}(t) = Ay(t) + f(t, y(t)), & t \in (0,T) \Big\backslash \{t_k\}_{k \in \sigma_1^m}, \\ y(0) = y^0, & \\ \Delta y(t_k) = q_k, & k \in \sigma_1^m, \end{cases} \tag{2.3}$$

has a unique classical solution y *which, for* $t \in [0, T)$, *satisfies*

$$y(t) = S(t)y^0 + \int_0^t S(t-s)f(s, y(s))ds + \sum_{0 < t_k < t} S(t - t_k)q_k. \tag{2.4}$$

Proof. First consider the interval $J_1 = [0, t_1)$ and apply Lemma **12** to the equation

$$y^{'}(t) = Au(t) + f(t, y(t)), \quad 0 < t < t_1, \quad y(0) = y^0.$$

We obtain a unique classical solution y_1 satisfying

$$y_1(t) = S(t)y^0 + \int_0^t S(t-s)f(s, y_1(s))ds, t \in [0, t_1),$$

Next, define

$$y_1(t_1) = S(t_1)y^0 + \int_0^{t_1} S(t_1 - s)f(s, y_1(s))ds,$$

Applying Lemma **12**, we see that $y_1(.)$ is left continuous at t_1, and $y_1(t_1) \in D(A)$. On the other hand in $J_2 = [t_1, t_2)$, consider the equation

$$y^{'}(t) = Au(t) + f(t, y(t)), \quad t_1 < t < t_2, \quad y(t_1) = y_1(t_1) + q_1$$

Since $y(t_1) = y_1(t_1) + q_1 \in D(A)$, we can once again use Lemma **12** to get a unique classical solution y_2 satisfying

$$y_2(t) = S(t - t_1)\left[y_1(t_1) + q_1\right] + \textstyle\int_{t_1}^{t} S(t - s) f(s, y_2(s)) ds, \quad t \in [t_1, t_2),$$

so that

$$y_2(t_2) = S(t_2 - t_1)\left[y_1(t_1) + q_1\right] + \int_{t_1}^{t_2} S(t - s) f(s, y_2(s)) ds.$$

Therefore, $y_2(.)$ is left continuous at t_2 and $y_2(t_2) \in D(A)$. It is easily seen that this procedure can be repeated in $J_k = [t_{k-1}, t_k),\ k \in \sigma_3^{m+1}$ to get a classical solution

$$y_k(t) = S(t - t_{k-1})\left[y_{k-1}(t_{k-1}) + q_{k-1}\right] + \textstyle\int_{t_{k-1}}^{t} S(t - s) f(s, y_k(s)) ds, \quad t \in \left[t_{k-1}, t_k\right),$$

with $y_k(.)$ left continuous at t_k and $y_k(t_k) \in D(A)$, $k \in \sigma_1^m$.

Now, define

$$y(t) = \begin{cases} y_1(t), & 0 < t < t_1, \\ y_k(t), & t_{k-1} < t < t_k,\ k \in \sigma_2^m, \\ y_{m+1}(t), & t_m < t < T. \end{cases}$$

It is clear that $y(.)$ is the unique impulsive classical solution of (2.3)**.**

Next, we use induction to show that (2.4) is satisfied in $[0, T)$. In fact, (2.4) is satisfied in $[0, t_1]$. If (2.4) is satisfied in $(t_{k-1}, t_k]$, then for $t \in (t_{k-1}, t_k]$

$$\begin{aligned} y(t) &= y_{k+1}(t) = S(t - t_k)\left[y_k(t_k) + q_k\right] + \int_{t_k}^{t} S(t - s) f(s, y(s)) ds \\ &= S(t - t_k)\Bigg[S(t_k) y^0 + \int_{t_k}^{t} S(t_k - s) f(s, y(s)) ds \\ &\quad + \sum_{0 < t_i < t_k} S(t_k - t_i) q_i + q_k\Bigg] + \int_{t_k}^{t} S(t - s) f(s, y_{k+1}(s)) ds \end{aligned}$$

$$\begin{aligned} y(t) &= S(t-t_k)T(t_k)y^0 + \int_0^{t_k} S(t-s)f(s,y(s))ds + \sum_{0<t_i<t_k} S(t-t_i)q_i \\ &\quad + S(t-t_k)q_k + \int_{t_k}^{t} S(t-s)f(s,y(s))ds \\ &= S(t)y^0 + \int_0^t S(t-s)f(s,y(s))ds + \sum_{0<t_i<t} S(t-t_i)q_i. \end{aligned}$$

Thus (2.4) is also true on $(t_k, t_{k+1}]$. Therefore (2.4) is true on $[0,T)$.■

In what follows, we study the existence and uniqueness of mild solutions using the fixed point argument. First we start with the definition.

Definition 15 (Mild solution) *A function* $y(.) \in \mathcal{PC}([0,T];X) \cap C^1((0,T)\Big\backslash\{t_k\}_{k\in\sigma_1^m}, X)$, *is a mild solution for the problem (2.1) if it satisfies the impulsive condition and*

$$y(t) = S(t)y^0 + \int_{t_0}^{t} S(t-s)f(s,y(s))ds + \sum_{t_0<t_k<t} S(t-t_k)I_k y(t_k),\ \forall t \in [0,T). \tag{2.5}$$

We assume the following hypotheses:

(H1) $f : [0,T]\times X \to X$ and $I_k : X \to X$, $k = 1,..m$, are continuous and there exist constants $L(f) > 0$, $L(I_k) > 0$, $k \in \sigma_1^m$, such that

$$\left\|f(t,x) - f(t,x^{'})\right\| \le L(f)\left\|x - x^{'}\right\|,\ t\in[0,T],\quad x,x^{'}\in X$$

$$\left\|I_k(x) - I_k(x^{'})\right\| \le L(I_k)\left\|x - x^{'}\right\|,\ \ x,x^{'}\in X.$$

(H2) Let $S(.)$ be the strongly continuous semigroup generated by the unbounded

operator A. Let $\mathcal{L}(X)$ be the Banach space of all linear and bounded operators on X. We suppose that

$$M\left[L(f)T + \sum_{k=1}^{m} L(I_k)\right] < 1,$$

where

$$M = \sup_{t\in[0,T]} \|S(t)\|_{\mathcal{L}(X)}.$$

Under these assumptions, we can establish the existence and uniqueness of mild solutions.

Theorem 16 *[78] Let assumptions* ***(H1)-(H2)*** *be satisfied. Then for every* $y^0 \in D(A)$, *the problem (2.1) has a unique mild solution.*

Proof. Let $y^0 \in X$ be fixed. Define the operator F on $\mathcal{PC}([0,T]\,;X)$ by

$$(Fx)(t) = S(t)y^0 + \int_0^t S(t-s)f(s,x(s))ds + \sum_{0<t_k<t} S(t-t_k)I_k x(t_k),$$

Then, it is clear that $F : \mathcal{PC}([0,T]\,;X) \to \mathcal{PC}([0,T]\,;X)$. On the other hand, we have from assumption **(H1)**,

$$\begin{aligned}
& \|(Fx)(t)-(Fz)(t)\| \\
\leq\ & \int_0^t \|S(t-s)\|_{\mathcal{L}(X)} \|f(s,x(s))-f(s,z(s))\|\, ds \\
& + \sum_{0<t_k<t} \|S(t-t_k)\|_{\mathcal{L}(X)} \|I_k x(t_k) - I_k z(t_k)\| \\
\leq\ & MLT \|x-z\|_{\mathcal{PC}} + \sum_{0<t_k<t} Mh_k \|x(t_k)-z(t_k)\| \\
\leq\ & MLT \|x-z\|_{\mathcal{PC}} + M\|x-z\|_{\mathcal{PC}} \sum_{0<t_k<t} h_k \\
\leq\ & M\left[LT + \sum_{k=1}^{k=m} h_k\right] \|x-z\|_{\mathcal{PC}}\,,\ x,\ z \in \mathcal{PC}([0,T]\,;X).
\end{aligned}$$

Now from assumption **(H2)**, we see that F is a contraction operator on $\mathcal{PC}([0,T]\,;X)$. We conclude by the fixed point theorem that there is a unique mild solution $y \in \mathcal{PC}([0,T]\,;X)$ such that

$$y = Fy.$$

This completes the proof.■

Next, we study the existence of mild solutions for the initial value problem (2.1)

We set the following assumptions:

(A1) The function $a : [0,T] \to [0,T]$ is continuous and $a(t) \leq t$, for every $t \in [0,T]$;

(A2) The function $f : [0,T] \times X^2 \to X$, satisfies the Caratheodory condition *i.e.*

$$\begin{aligned}(a)\ f(t,.) \quad &: \quad X \to X \text{ is continuous for almost all } t \in [0,T]\,; \\ (b)\ f(.,x) \quad &: \quad [0,T] \longmapsto X \text{ is integrable for each } x \in X;\end{aligned}$$

and there exists a continuous function $g : [0,T] \to [0,\infty)$ and a nonincreasing function

$$W : [0,\infty) \to [0,\infty)\,,$$

such that

$$\|f(t,x,y)\| \leq g(t)W(\|x\| + \|y\|),$$

for all $t \in [0,T]$ and x, $y \in X$.

Theorem 17 [55] *Let $y^0 \in X$ and let the following assumptions hold.*

(1) *For each $k \in \sigma_1^m$, the operator I_k is completely continuous and bounded in X, with*

$$N_k = \sup\left\{\|I_k(x)\| : x \in X\right\};$$

(2) *For every $t \in [0,T]$ and $r > 0$ the region $\{S(t)f(s,x_1) : s \in [0,t]\,,\ \|x_1\| \leq r,\}$ is relatively compact in X.*

If

$$2M \int_0^T g(s)ds < \int_c^\infty \frac{ds}{W(s)},$$

where

$$c = 2(M\,\|u_0\| + \textstyle\sum_{k=1}^n MN_k), \quad \textit{and} \quad M = \sup_{t\in[0,T]} \|S(t)\|\,,$$

then the problem (2.1) has a unique mild solution.

Remark 1

(1) *If y is a solution of (2.1) then $y \in \mathcal{PLC}^1([0,T]\,;X)$.*

(2) *Since the solution y of (2.1) has jumps, then it is not continuous, but $y \in BV((0,T);X)$,*

($BV((0,T);X)$ being the space of bounded variation functions) and

$$y(t) = S(t)y^0 + \int_0^t S(t-s)f(s,y(s),)ds + \sum_{0<t_k<t} S(t-t_k)I_k(y(t_k)), \quad \forall t \in [0,T] \ .$$

In fact, it has two parts: the first part is a continuous function $y_c \in W^{1,1}(0;T;X)$, with

$$y_c(t) = S(t)y^0,$$

while the second one is the jump function defined by

$$y_d(t) = \sum_{0<t_k<t} S(t-t_k)I_k(y(t_k)), \quad \forall t \in [0,T] \, .$$

***(3)** The problem of existence and uniqueness of the solutions of impulsive differential equations is similar to that of the corresponding ordinary differential equations. We can represent the solution $y(t)$ of equation (2.1) with initial condition $y(0) = y^0$, as follows,*

$$y(t) = \begin{cases} S(t)y^0 + \int_{t_0}^t S(t-s)f(s,y(s))ds + \sum\limits_{t_0<t_k<t} S(t-t_k)I_k y(t_k), & t \in [t_0,T]^+ \, , \\ S(t)y^0 + \int_{t_0}^t S(t-s)f(s,y(s))ds - \sum\limits_{t_0<t_k<t} S(t-t_k)I_k y(t_k), & t \in [t_0,T]^- \end{cases}$$

where $[t_0,T]^+$ and $[t_0,T]^-$ are maximal intervals on which the solution can be continued to the right or to the left of the point $t = t_0$, respectively.
***(4)** As a result, the solution of (2.2) is given by*

$$\varphi(t) = S(t-t_k)\varphi(t_k), \quad \textit{for} \quad t \in [t_k, t_{k+1}) \, , \ k = 0,1,2...$$

***(5)** The relationship between nonhomogeneous equation (2.1) and the corresponding homogeneous equation is the following: if $\varphi(t)$ is a classical solution of (2.2) without an initial condition and $y(t)$ is a classical solution of (2.1) without an initial condition, then the function $\varphi(t)+y(t)$ is again a classical solution of (2.1) without initial condition. Conversely, if $y_1(t)$ and $y_2(t)$ are two solutions of (2.1) without an initial condition, then the difference $y_1(t)-y_2(t)$ is a solution of (2.2) without initial condition.*

(6) *If $I_k = 0$ for $k \in \sigma_1^m$, then the equation (2.2) reduces to the ordinary evolution equation*

$$\begin{cases} \varphi'(t) = A\varphi(t), & t \in (0,T) \Big\backslash \{t_k\}_{k\in\sigma_1^m}\ , \\ \varphi(0) = \varphi^0, \end{cases}$$

and the solution (2.2) reduces to

$$\varphi(t) = S(t)\varphi^0.$$

Regarding the solution of (2.2), we have the following result.

Theorem 18 *[80] If I_k maps $D(A)$ to $D(A)$, $k = 1, 2...$ and $\varphi^0 \in D(A)$, then problem (2.2) has a unique solution $\varphi(t)$ given by*

$$\varphi(t) = \begin{cases} S(t)\varphi^0 & 0 \le t \le t_1 \\ S(t)\varphi^0 + \sum\limits_{i=1}^{k} S(t-t_i)I_i(\varphi(t_i)), & t_k < t \le t_{k+1},\ k \in \sigma_0^m. \end{cases} \tag{2.6}$$

Proof: It follows from A.Pazy, [89] that the function $\varphi(t)$ defined by (2.2) satisfies $(2.2)_1$ for $0 < t \le t_1$ and $\varphi(t_1^-) = S(t_1)\varphi^0 = \varphi(t_1)$ and such a solution $\varphi(t)$ is unique. From the given assumption, we have $\Delta\varphi(t_1) = I_1(\varphi(t_1)) \in D(A)$. If $\varphi(t)$, defined by (2.2), satisfies equation $(2.2)_1$ for $t_i < t < t_{i+1}$, $i \in \sigma_1^k$ and

$$\begin{aligned} \varphi(t_{i+1}^-) = \varphi(t_{i+1}) = S(t_{i+1})\varphi^0 + \sum_{j=1}^{i} S(t_i - t_j)I_j(\varphi(t_j)), &\quad i \in \sigma_1^k, \\ \Delta\varphi(t_i) = I_i(\varphi(t_i)) \in D(A), &\quad i \in \sigma_1^k, \end{aligned}$$

then, for $t_{k+1} < t < t_{k+2}$, we have

$$\begin{aligned} \varphi'(t) &= AS(t)\varphi^0 + \sum_{i=1}^{k+1} AS(t-t_i)I_i(\varphi(t_i)) = A\varphi(t), \\ \varphi(t_{k+2}^-) &= \varphi(t_{k+2}) = S(t_{k+2})\varphi^0 + \sum_{i=1}^{k+1} S(t_{i+1} - t_i)I_i(\varphi(t_i)), \end{aligned}$$

and

$$\begin{array}{rcl}\Delta\varphi(t_{k+1}) & = & \varphi(t_{k+1}^{+})-\varphi(t_{k+1}^{-}) \\ & = & S(t_{k+1})\varphi^{0}+\sum\limits_{j=1}^{k+1}S(t_{k+1}-t_{j})I_{j}(\varphi(t_{j})) \\ & & -S(t_{k+1})\varphi^{0}+\sum\limits_{j=1}^{k}S(t_{k+1}-t_{j})I_{j}(\varphi(t_{j})) \\ & = & I_{k+1}(\varphi(t_{k+1}))\in D(A).\end{array}$$

Thus, the theorem is proved by induction.■

Example 19 *Let* $\lambda, \alpha_k \in \mathbb{R}$, $k \in \sigma_1^m$ *and* $\alpha_k \neq -1$ *for every* $k \in \sigma_1^m$*. Then, the solution* $\varphi \in \mathcal{PLC}\left([0,T]\,;\{t_k\}_{k\in\sigma_1^m}, X\right)$ *of*

$$\begin{cases} \varphi^{'}(t)+\lambda\varphi(t)=0 & (0,T),\ t\neq t_k,\quad k\in\sigma_1^m, \\ \varphi(t_k^{+})=(1+\alpha_k)\varphi(t_k) & \\ \varphi(0)=\varphi^0. & \end{cases}$$

is given explicitly by

$$\varphi(t)=\prod_{0<t_k<t}(1+\alpha_k)\varphi^0.$$

The next result is a consequence of Theorem **2.4.**

Impulsive Partial Differential Equations

Consider the partial differential problem :

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t}+a(x)\dfrac{\partial y(x,t)}{\partial x}=f(x,y), \quad t>0,\ t\neq t_k,\ k\in\sigma_1^m \quad x\in\mathbb{R}, \\ y(x,0)=\phi(x), \\ \Delta y(x,t_k)=I_k(y(x,t_k)), \qquad\qquad k\in\sigma_1^m \end{cases} \tag{2.7}$$

where $I_k \in C(Y,Y)$, and

$$Y=C_0(\mathbb{R})\equiv\left\{\phi\in C(\mathbb{R}):\lim_{|x|\to+\infty}\phi\left(x\right)=0\right\},$$

with norm

$$\|\phi\| = \sup_{x\in I} |\phi(x)| \, .$$

We define the operator $\mathbf{A}$ on Y as follows :

$$\mathbf{A}(x) = a(x)\frac{d\phi(x)}{dx}, \tag{2.8}$$

$$\begin{aligned} D(\mathbf{A}) \quad = \quad & \Big\{ \phi \in Y : \phi^{'}(x) \text{ exists and is continuous at } x, \lim_{|x|\to+\infty} a(x)\phi(x) = 0, \\ & \lim_{x\to x_0} a(x)\phi(x) \text{ exists when } a(x_0) = 0 \Big\} \, . \end{aligned}$$

We set the following assumptions :

(i) $a(x)$ is positive and continuous on $\mathbb{R}$,

(ii) $\int_0^\infty (\dfrac{1}{a(x)})dx = +\infty$ and $\lim\limits_{|x|\to+\infty} a(x) = 0,$

(iii) $f \in C^1(\mathbb{R}\times\mathbb{R})$, there exist a real positive number M such that

$$\left|\frac{\partial f(x,u)}{\partial x}\right| \le M \text{ and } \left|\frac{\partial f(x,u)}{\partial u}\right| \le M, \text{ for every } (x,u) \in \mathbb{R}\times\mathbb{R}, \ \lim_{x\to\pm\infty} f(x,u) = 0, \text{ for each } u \in \mathbb{R};$$

(iv) $I_k \in \mathcal{L}(X)$ and $I_k(D(-\mathbf{A})) \subset D(-\mathbf{A})$.

Next, we are going to discuss the existence of solutions for the first order impulsive evolution problem (2.7).

Theorem 20 *[80] Suppose (i)-(iv) are satisfied. Then problem (2.7) has a unique mild solution in* $[0,\infty)$.

Proof: First, problem (2.7) can be written in an abstract form,

$$\begin{cases} y^{'}(t) + \mathbf{A}y(t) = g(y(t)), \ \ t \neq t_k \, , \\ y(0) = y^0, \\ \Delta y \,|_{t=t_k} \ = I_k(y(t_k)), \, k \in \sigma_1^m, \end{cases} \tag{2.9}$$

where $y(t) = y(t,.), g(y(t)) = f(.,y(t,.))$, $y^0 = \phi(x)$, and $\mathbf{A}$ is defined by (2.8). Define

$$\Phi(t)=\begin{cases} S(t), & [0,t_1] \\ S(t-t_k)(I_k+I)S(t_k-t_{k-1})...(I_1+I)S(t_1), & (t_k,t_{k+1}]\,,\ k=1,2..., \end{cases}$$

where $S(t)$ is the semigroup generated by $-\mathbf{A}$.

Taking $y_1(t)=y^0$, then $g(y_1(t))\in D(-\mathbf{A})$ and $\Phi(t)g(y_1(t))$ is continuous in t except at t_k, it has discontinuities of first kind at t_k and it is integrable in t. Hence, we can define

$$u_2(t)=\Phi(t)y^0+\int_0^t\Gamma(t,\tau)g(y_1(\tau))d\tau,$$

where

$$\Gamma(t,\tau)=\begin{cases} S(t-t_k)(I_k+I)S(t_k-t_{k-1})... & \\ \qquad ...(I_1+I)S(t_1-\tau) & \\ S(t-t_k)(I_k+I)S(t_k-t_{k-1})... & \text{if } t\in(t_k,t_{k+1}]\,,\tau\in[0,t_1] \\ \qquad ...(I_l+I)S(t_l-\tau), & \text{if } k\geq l \text{ and } t\in(t_k,t_{k+1}]\,,\ \tau\in(t_{l-1},t_l]\,, \\ \qquad S(t-\tau), & \text{if } t\geq\tau \text{ and } t,\tau\in(t_k,t_{k+1}] \\ & \quad \text{or } t,\tau\in[0,t_1]\,,\ k=1,2...,\ l=1,2.... \end{cases}$$

It is easy to check that $y_2(t)\in D(-\mathbf{A})$, if $t\neq t_k$, and satisfies the following problem:

$$\begin{cases} w^{'}(t)+\mathbf{A}w(t)=g(y_1(t)),\ t\neq t_k\,,\ k=1,2..., \\ w(0)=y^0, \\ \Delta w\,|_{t=t_k}\,=I_k(w(t_k)),\ k=1,2.... \end{cases}$$

In a similar way, we can define

$$y_n(t)=\Phi(t)y^0+\int_0^t\Gamma(t,\tau)g(y_{n-1}(\tau))d\tau, \tag{2.10}$$

for $n = 2, ...,$ so that $y_n(t)$ satisfies

$$\begin{cases} w^{'}(t) + \mathbf{A}w(t) = g(y_{n-1}(t)),\ 0 < t < \infty,\ \ t \neq t_k, \\ w(0) = y^0, \\ \Delta w\mid_{t=t_k} = I_k(w(t_k)),\ k = 1, 2... \end{cases} \tag{2.11}$$

For a given $T > 0$, let $k_0 = \max\{k \mid t_k \in [0, T]\}$ and $N = \max\{\|I_k\| \mid \|I_k\| \geq 1, \text{ and } 1 \leq k \leq k_0\}$, we get

$$\|y_{n+1}(t) - y_n(t)\|_Y \leq NM \int_0^t \|y_n(\tau) - y_{n-1}(\tau)\|_Y \, d\tau,$$

for $n = 2, 3, ...$ Therefore,

$$\|y_{n+1}(t) - y_n(t)\|_Y \leq (NM)^{n-1} \frac{t^{n-1}}{n!} \max_{0 \leq \tau \leq S} \|y_2(\tau) - y_1(\tau)\|, \tag{2.12}$$

for $n = 2, 3, ...$It follows from (2.12) that there exists a real valued function $y(t)$ on $[0, T]$ such that $y_n(t)$ converges uniformly to $y(t)$ on $[0, T]$. Since $y_n(t)$ satisfies (2.11), then by virtue of (2.10) and assumption on f, we see that $y(t)$ satisfies

$$y(t) = \Phi(t)y^0 + \int_0^t \Gamma(t, \tau) g(y(\tau)) d\tau.$$

The argument above shows that (2.9) has a mild solution on $[0, T]$, for each $T > 0$. If the problem (2.9) has another mild solution $\tilde{y}(t)$ on $[0, T]$, then

$$\|y(t) - \tilde{y}(t)\|_Y \leq M \int_0^t \|\Gamma(t, \tau)\| \, \|y(t) - \tilde{y}(t)\|_Y \, d\tau.$$

By Gronwall's inequality, $y(t) = \tilde{y}(t)$ on $[0, T]$. So, we see that problem (2.9) has a unique global mild solution on $[0, \infty)$. The theorem is proved.■

Representation of Solutions of Some Impulsive Linear Systems

This section deals with the solution representation of linear impulsive evolution equations. We consider the following homogeneous linear impulsive system with time-varying generating operators

$$\begin{cases} y^{'}(t) = A(t)y(t) + f(t), \quad t \in (0,T) \Big\backslash \{t_k\}_{k \in \sigma_1^m}, \\ y(0) = y^0, \\ \Delta y(t_k) = I_k y(t_k) + \alpha_k, \quad k \in \sigma_1^m, \end{cases} \tag{2.13}$$

on some infinite dimensional Banach space X, where $0 = t_0 < t_1 < t_2 < .... < t_k ... < t_m < T$, and $\{A(t),\ t \in [0,T]\}$ is a family of unbounded operators on X satisfying the following assumptions :

For $t \in [0,T]$ one has

(H1) The domain $D(A(t)) = D$ is independent of t and is dense in X.

(H2) $A(t)$ is a closed operator in X.

(H3) For $t \geq 0$, the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists, for all λ with $\operatorname{Re}\lambda \leq 0$, and there exists a real constant M independent of λ and t such that

$$\|R(\lambda, A(t))\| \leq M\,(1 + |\lambda|)^{-1} \quad \text{for} \quad \operatorname{Re}\lambda \leq 0.$$

(H4) The resolvent of the unbounded operator $A(t)$ is compact.

(H5) There exist constants $L > 0$ and $0 < \alpha \leq 1$ such that

$$\left\|(A(t) - A(s))\,A^{-1}(r)\right\| \leq L\,|t - s|^{\alpha}, \quad \text{for} \quad t, s, r \in [0,T].$$

Let us begin with the following lemma.

Lemma 21 *(See Ahmed* [2] *p. 159) Under assumptions* ***(H1)-(H5),*** *the Cauchy problem*

$$x^{'}(t) = A(t)x(t), \quad t \in (0,T] \quad \textit{with} \quad x(0) = x^0, \tag{2.14}$$

has a unique evolution system $\{U(t,s) \mid 0 \leq s \leq t \leq T\}$ *in* X *satisfying the following properties*

(1) $U(t,s) \in \mathcal{L}(X)$, *for* $0 \leq s \leq t \leq T$.

(2) $U(t,r)U(r,s) = U(t,s)$, *for* $0 \leq s \leq t \leq T$.

(3) $U(.,.)x \in C\left(\Delta_T, X\right)$, *for* $x \in X$, $\Delta_T = \{(t,s) : s \in [0,T], t \in [s,T]\}$.

(4) For $0 \leq s < t \leq T$, $U(t,s) : X \to D$ *and* $t \to U(t,s)$ *is strongly differentiable in* X. *The*

derivative $\dfrac{\partial}{\partial t}U(t,s) \in \mathcal{L}(X)$ *and it is strongly continuous on* $0 \leq s < t \leq T$. *Moreover,*

$$\frac{\partial}{\partial t}U(t,s) = -A(t)U(t,s), \quad \textit{for} \quad 0 \leq s < t \leq T,$$

$$\left\| \frac{\partial}{\partial t}U(t,s) \right\|_{\mathcal{L}(X)} = \|A(t)U(t,s)\|_{\mathcal{L}(X)} \leq \frac{C}{t-s},$$

$$\left\| A(t)U(t,s)A^{-1}(s) \right\|_{\mathcal{L}(X)} \leq C, \quad \textit{for} \quad 0 \leq s < t \leq T.$$

(5) For every $z \in D$ *and* $t \in (0,T]$, $U(t,s)z$ *is differentiable with respect to* s *on* $0 \leq s \leq t \leq T$ *and*

$$\frac{\partial}{\partial s}U(t,s)z = U(t,s)A(s)z.$$

Furthermore, for each $x^0 \in X$, *the Cauchy problem (2.14) has a unique classical solution* $x \in C^1([0,T];X)$ *given by*

$$x(t) = U(t,0)x^0, \quad t \in [0,T].$$

In order to construct an impulsive evolution operator and investigate its properties, we shall consider the following impulsive Cauchy problem

$$\begin{cases} y^{'}(t) = A(t)y(t), & t \in (0,T) \Big\backslash \{t_k\}_{k \in \sigma_1^m}, \\ y(0) = y^0, \\ \Delta y(t_k) = I_k y(t_k), & k \in \sigma_1^m, \end{cases} \tag{2.15}$$

where $m \in \mathbb{N}^*$, is a fixed positive integer. For every $y^0 \in X$, D is an invariant subspace of I_k. Using Lemma 1 and 2 in ([12] p. 19-20.) step by step, one can check that the impulsive Cauchy problem (2.15) has a unique classical solution $y \in \mathcal{PC}([0,T];X) \cap C^1((0,T) \Big\backslash \{t_k\}_{k \in \sigma_1^m}, X)$, represented by $y(t) = G(t,0)y^0$ where $G(.,.) : \Delta_T \to X$ is given by

$$G(t,s) = \begin{cases} U_k(t,s), \quad \text{for } s \le t \in [t_{k-1}, t_k], \\ U_{k+1}(t, t_k^+)\,(I + I_k)\,U_k(t_k, s), \\ \qquad\qquad \text{for } t_{k-1} < s \le t_k < t \le t_{k+1},\ k \in \sigma_1^m, \\ U_k(t, t_k)\,(I + I_k)^{-1}\,U_{k+1}(t_k^+, s), \\ \qquad\qquad \text{for } \ t_{k-1} < s \le t_k < t \le t_{k+1},\ k \in \sigma_1^m, \\ U_{k+1}(t, t_k) \left[\displaystyle\prod_{j=k}^{i+1} (I + I_j)\,U_j(t_j, t_{j-1}^+) \right] (I + I_i)\,U_i(t_i, s), \\ \qquad\qquad \text{for } t_{i-1} < s \le t_i < t_k < t \le t_{k+1},\ k \in \sigma_1^m, \\ U_i(t, t_i) \left[\displaystyle\prod_{j=i}^{k-1} (I + I_j)^{-1}\,U_{j+1}(t_j^+, t_{j+1}) \right] (I + I_k)^{-1}\,U_{k+1}(t_k^+, s), \\ \qquad\qquad \text{for } t_{i-1} < t \le t_i < t_k < s \le t_{k+1},\ k \in \sigma_1^m. \end{cases}$$

The operator $G(t,s)$, $(t,s) \in \Delta_T$ is called the impulsive evolution operator associated with $\{I_k, t_k\}_{k \in \sigma_1^m}$.

Proposition 22 *[105] The impulsive evolution operator $G(t,s)$, $(t,s) \in \Delta_T$ has the following properties:*

(a) $G(t,s) \in \mathcal{L}(X)$ for $0 \le s \le t \le T$.

(b) $G(t,r)G(r,s) = G(t,s)$ for $0 \le r \le s \le t \le T$, and $G(s,s) = I$.

(c) If $\{U(t,s)\}\, 0 \le s < t \le T$ is a compact operator, then $G(t,s)$ is also compact for $0 \le s < t \le T$.

Proof : (a) is due to property (1) of Lemma **21** and assumption $I_k \in \mathcal{L}(X)$ for $k \in \sigma_1^m$,

(b) is evident,

(c) Since $\{U(t,s)\}$, $0 \le s < t \le T$ are compact operators, one can deduce that $G(t,s)$ is also compact for $0 \le s < t \le T$.■

Now we are able to introduce the $\mathcal{PC}-$mild solution of impulsive Cauchy problem (2.13) as follows :

Definition 23 *For every $y^0 \in X$, $f \in L^1([0,T]\,; X)$, the function $y \in \mathcal{PC}((0,T) \Big\backslash \{t_k\}_{k \in \sigma_1^m}, X)$*

given by

$$x(t) = G(t,0)x_0 + \textstyle\int_0^t G(t,s)f(s)ds + \sum\limits_{0<t_k<t} G(t,t_k^+)\alpha_k, \quad \textit{for} \quad t \in [0,T] .$$

is said to be a mild solution of the impulsive Cauchy problem (2.13).

On the other hand, we have the following existence theorem :

Theorem 24 *[49] If* $\{U(t,s)\}$, $t,s \in \Delta_T$ *is an evolution operator in a Banach space* X *with infinitesimal generator* $A(t)$ *and* $I_k \in \mathcal{L}(X)$, *for* $k \in \sigma_1^m$, *then the problem*

$$\begin{cases} y^{'}(t) = Ay(t) + f(t),\, t \in (0,T) \;\; t \neq t_k \,,\, k \in \sigma_1^m, \\ y(0) = y_0, \\ \Delta y(t_k) = I_k(y(t_k)),\, k \in \sigma_1^m, \end{cases} \tag{2.16}$$

has a unique solution $y(t)$ *given by*

$$y(t) = U(t,t_{k-1}^+)y(t_{k-1}^+) + \textstyle\int_{t_{k-1}}^t U(t,s)f(s)ds, \quad t \in (t_k, t_{k+1}] \,, \quad k \in \sigma_1^m, \tag{2.17}$$

such that

$$\begin{aligned} y(t_k^+) \;\; &= \;\; \prod_{j=k}^{1} (I + I_j)\, U(t_j, t_{j-1}) y_0 + \sum_{i=1}^{k} \prod_{j=k}^{i} (I + I_j)\, U(t_j, t_{j-1}) \\ &\quad \times \int_{t_{i-1}}^{t_i} U(t_i, s) f(s) ds, \end{aligned} \tag{2.18}$$

for each $k \in \sigma_1^m$.

Proof : We have

$$\begin{aligned} &y^{'}(t) = Ay(t) + f(t), t \in \; [t_0, t_1] \,, \\ &y(0) = y^0, \end{aligned}$$

and the ordinary variation of parameters leads to

$$y(t) = U(t,t_0)y(t_0^+) + \textstyle\int_{t_0}^t U(t,s)f(s)ds, \quad t \in (t_0, t_1]$$

implies that

$$y(t_1) = U(t_1,t_0)y(t_0^+) + \int_{t_0}^{t_1} U(t_1,s)f(s)ds.$$

Since $\Delta y(t_1) = I_1\left(y(t_1)\right)$, then

$$y(t_1) = (I+I_1)\,U(t_1,t_0)y(t_0) + (I+I_1)\,U(t_1,t_0)\int_{t_0}^{t_1} U(t_0,s)f(s)ds. \tag{2.19}$$

Moreover, for $t \in (t_1,t_2]$,

$$y(t) = U(t,t_1)y(t_1^+) + \textstyle\int_{t_1}^{t} U(t,s)f(s)ds, \quad t \in (t_1,t_2],$$

where $y(t_1^+)$ is given by (2.18). This implies that Theorem **24** holds for $k=1$. Now, suppose that (2.16) and (2.17) are true when $k=m$, namely, as $t \in (t_m,t_{m+1}]$, one has

$$y(t) = U(t,t_m)y(t_m^+) + \int_{t_m}^{t} U(t,s)f(s)ds, \tag{2.20}$$

where

$$\begin{aligned} y(t_m^+) &= \prod_{j=m}^{1}(I+I_j)\,U(t_j,t_{j-1})y_0 + \sum_{i=1}^{m}\prod_{j=m}^{i}(I+I_j)\times U(t_j,t_{j-1}) \\ &\quad \times \int_{t_{i-1}}^{t_i} U(t_{i-1},s)f(s)ds. \end{aligned} \tag{2.21}$$

then, (2.20) leads to

$$y(t) = U(t_{m+1},t_m)y(t_m^+) + \int_{t_m}^{t_{m+1}} U(t,s)f(s)ds.$$

It follows from the impulsive system (2.15) and (2.21) that

$$\begin{aligned} & y(t_{m+1}^+) = (I+I_{m+1})\,y(t_{m+1}) \\ = \ & (I+I_{m+1})\left[U(t_{m+1},t_m)y(t_m^+) + \int_{t_m}^{t_{m+1}} U(t_{m+1},s)f(s)ds\right] \end{aligned}$$

$$\begin{aligned}
& y(t_{m+1}^{+}) \\
= \; & (I+I_{m+1})\,U(t_{m+1},t_m)y(t_m^{+}) + (I+I_{m+1})\,U(t_{m+1},t_m)\int_{t_m}^{t_{m+1}} U(t_m,s)f(s)ds \\
= \; & (I+I_{m+1})\,U(t_{m+1},t_m)\prod_{j=m}^{1}(I+I_j)\,U(t_j,t_{j-1})y_0 \\
& +(I+I_{m+1})\,U(t_{m+1},t_m)\sum_{i=1}^{m}\prod_{j=m}^{i}(I+I_j)\times U(t_j,t_{j-1})\int_{t_{i-1}}^{t_i} U(t_{i-1},s)f(s)ds \\
& +(I+I_{m+1})\,U(t_{m+1},t_m)\int_{t_m}^{t_{m+1}} U(t_m,s)f(s)ds \\
= \; & \prod_{j=m+1}^{1}(I+I_j)\,U(t_j,t_{j-1})y_0+\sum_{i=1}^{m+1}\prod_{j=m+1}^{i}(I+I_j)\times U(t_j,t_{j-1})\int_{t_{i-1}}^{t_i} U(t_{i-1},s)f(s)ds.
\end{aligned}$$

Accordingly, as $t \in (t_{m+1}, t_{m+2}]$, we have

$$y(t) = U(t,t_{m+1})y(t_{m+1}^{+}) + \int_{t_{m+1}}^{t} U(t,s)f(s)ds,$$

which means that (2.16) and (2.17) hold when $k = m+1$. Then the Theorem is true for any $t \in (t_k, t_{k+1}]$, and $k \in \sigma_1^m$. This, completes the proof.■

The next theorem is of theoretical importance as it relates the solutions of (2.13) with the impulsive evolution operator $G(t,0)$.

Theorem 25 *[7] Let $x(t) = x(t,0,x_0)$ be a solution of the impulsive problem (2.13). Then $x(t)$ has the representation*

$$x(t) = G(t,0)x_0 + \int_0^t G(t,s)f(s)ds + \sum_{0<t_k<t} G(t,t_k^{+})\alpha_k$$

where

$$x(0) = x_0, \quad \textit{and} \quad t \geq 0.$$

Proof. Let $x(t,0,x_0)$ be the solution of (2.13) and let $\phi_t(s) = G(t,s)x(s)$. Clearly,

$$\phi(t) - \phi(0) = \int_0^t \phi'(s)\,ds + \sum_{0<t_k<t} \Delta\phi(t_k).$$

Since we have

$$\begin{aligned}\Delta\phi\left(t_k\right) &= G(t,t_k^+)x\left(t_k^+\right)-G(t,t_k^-)x\left(t_k^-\right)\\ &= G(t,t_k^+)\left[x\left(t_k^+\right)-x\left(t_k^-\right)\right]+G(t,t_k^+)x\left(t_k^-\right)-G(t,t_k^-)x\left(t_k^-\right)\\ &= G(t,t_k^+)\Delta x\left(t_k\right)+\Delta G(t,t_k)x\left(t_k^-\right),\end{aligned}$$

we obtain that

$$\begin{aligned}&\sum_{0<t_k<t}\Delta\phi\left(t_k\right)\\ =\;&\sum_{0<t_k<t}\Delta G(t,t_k)x\left(t_k^-\right)+G(t,t_k^+)\Delta x\left(t_k\right)\\ =\;&\sum_{0<t_k<t}\Delta G(t,t_k)x\left(t_k^-\right)+\sum_{0<t_k<t}G(t,t_k^+)I_kx\left(t_k^-\right)+\sum_{0<t_k<t}G(t,t_k^+)\alpha_k\\ =\;&\sum_{0<t_k<t}\Delta G(t,t_k)x\left(t_k^-\right)+\sum_{0<t_k<t}\left[\Delta G(t,t_k)+G(t,t_k)\right]I_kx\left(t_k^-\right)\\ &+\sum_{0<t_k<t}G(t,t_k^+)\alpha_k\\ =\;&\sum_{0<t_k<t}\Delta G(t,t_k)\left[I+I_k\right]x\left(t_k^-\right)+\sum_{0<t_k<t}G(t,t_k)I_kx\left(t_k^-\right)\\ &+\sum_{0<t_k<t}G(t,t_k^+)\alpha_k\\ =\;&\left[\Delta G(t,t_k)\left(I+I_k\right)+G(t,t_k)I_k\right]x\left(t_k^-\right)+\sum_{0<t_k<t}G(t,t_k^+)\alpha_k.\end{aligned}$$

On the other hand, by differentiating the relation $\phi\left(s\right)=G(t,s)x(s)$ we obtain

$$\begin{aligned}\phi^{'}\left(s\right) &= \left[\frac{\partial}{\partial s}G(t,s)\right]x(s)+G(t,s)x^{'}(s)\\ &= AG(t,s)x(s)+G(t,s)\left[-Ax(s)\right]+G(t,s)Bu(s).\end{aligned}$$

Integrating both sides from 0 to t we get

$$\begin{aligned} & G(t,t)x(t)-G(t,0)x(0) \\ = & \int_0^t G(t,s)f(s)ds+\sum_{0<t_k<t}\left[\Delta G(t,t_k)\left(I+I_k\right)+G(t,t_k)I_k\right]x\left(t_k^-\right) \\ & +\sum_{0<t_k<t}G(t,t_k^+)\alpha_k. \end{aligned}$$

Thus, we obtain

$$x(t)=G(t,0)x(0)+\int_0^t G(t,s)f(s)ds+\sum_{0<t_k<t}G(t,t_k^+)\alpha_k.$$

This completes the proof.■

Suppose that X is a Hilbert space. Consider the following Cauchy problem

$$\begin{cases} \varphi'(t)+A(t)\varphi(t)=0, & t\in(0,T)\setminus\{t_k\}_{k\in\sigma_1^m}, \\ \varphi(0)=\varphi^0, & \\ \Delta\varphi(t_k)=I_k\varphi(t_k), & k\in\sigma_1^m. \end{cases} \tag{2.22}$$

Consider the corresponding adjoint impulsive problem :

$$\begin{cases} -\tilde{\varphi}'(t)+A^*(t)\tilde{\varphi}(t)=0, & t\in(0,T)\setminus\{t_k\}_{k\in\sigma_1^m}, \\ \tilde{\varphi}(T)=\varphi^0, & \\ \Delta\tilde{\varphi}(t_k)=-I_k^*\left(I+I_k^*\right)^{-1}\tilde{\varphi}(t_k^-) & k\in\sigma_1^m, \end{cases} \tag{2.23}$$

where $A^*(t)$, I_k^* are the adjoint operators of $A(t)$ and I_k, respectively.

Remark 2 *It is easy to verify also that the adjoint problem of (2.22) is (2.23), i.e. they are mutually adjoint of each other.*

Similarly to the discussion on the adjoint impulsive Cauchy problem of homogenous linear impulsive system with time-varying generating operator, the mild solution of the impulsive problem (2.23) can be given as

$$\tilde{x}(\theta) = \Psi(T,\theta)\tilde{x}_0, \quad \theta < T,$$

where

$$\Psi(T,\theta) = \begin{cases} U^*(T,\theta), \qquad \text{for } \theta \in (t_{k-1}, T]\,, \\ U^*(t_{k-1},\theta)\,(I + I_k^*)\,U^*(T,t_{k-1}), \qquad \text{for } t_{k-2} < s \leq t_{k-1} < T, \\ U^*(t_i,\theta)\,(I+I_i^*)\left[\prod\limits_{\theta < t_J < T} (I+I_j)\,U(t_j,t_{j-1})\right]^* U^*(T,t_{k-1}), \\ \text{for } \; t_{i-1} < \theta \leq t_i < t_k < ... < T. \end{cases}$$

The next theorem relates the solution of (2.23) with the impulsive evolution operator $\Psi(T,\theta)$.

Theorem 26 *[7] Let* $\tilde{x}(\theta) = \tilde{x}(T,\theta,\tilde{x}_0)$ *be a solution of the impulsive problem (2.25). Then* $\tilde{x}(s)$ *has the representation*

$$\tilde{x}(\theta) = \Psi(T,\theta)\tilde{x}_0 + \int_\theta^T \Psi(s,\theta)Bu(s)ds + \sum_{\theta < t_k < T} \Psi(t_k^-,\theta)D_k v_k$$

with

$$\tilde{x}(T) = \tilde{x}_0.$$

Proof. The proof is similar to that of Theorem **25**. Let $\tilde{x}(T,\theta,\tilde{x}_0)$ be the solution of (2.23) and let $\psi_\theta\,(s) = \Psi(s,\theta)\tilde{x}(s)$. Clearly,

$$\psi\,(\theta) - \psi\,(T) = \int_\theta^{T'} \psi'\,(s)\,ds + \sum_{\theta < t_k < T} \Delta^- \psi\,(t_k)$$

Recall that

$$\begin{aligned}
\Delta^{-}\psi\left(t_k\right) &= \Psi(t_k^{-},\theta)\tilde{x}\left(t_k^{-}\right)-\Psi(t_k^{+},\theta)\tilde{x}\left(t_k^{+}\right)\\
&= \Psi(t_k^{-},\theta)\left[\tilde{x}\left(t_k^{-}\right)-\tilde{x}\left(t_k^{+}\right)\right]+\Psi(t_k^{-},\theta)\tilde{x}\left(t_k^{+}\right)\\
&\quad -\Psi(t_k^{+},\theta)\tilde{x}\left(t_k^{+}\right)\\
&= \Psi(t_k^{-},\theta)\Delta^{-}\tilde{x}\left(t_k\right)+\Delta^{-}\Psi(t_k,\theta)\tilde{x}\left(t_k^{+}\right),
\end{aligned}$$

Let $J_k \doteqdot I_k^{*}\left(I+I_k^{*}\right)^{-1}$, we obtain that

$$\begin{aligned}
\sum_{\theta<t_k<T}\Delta^{-}\psi\left(t_k\right) &= \sum_{\theta<\theta_k<T}\Delta^{-}\Psi(t_k,\theta)\tilde{x}\left(t_k^{+}\right)+\Psi(t_k^{-},\theta)\Delta^{-}\tilde{x}\left(t_k\right)\\
&= \sum_{\theta<t_k<T}\Delta^{-}\Psi(t_k,\theta)\tilde{x}\left(t_k^{+}\right)+\Psi(t_k^{-},\theta)\left[J_k\right]\tilde{x}\left(t_k^{+}\right)\\
&\quad +\Psi(t_k^{-},\theta)D_k^{*}w_k\\
&= \sum_{\theta<t_k<T}\Delta^{-}\Psi(t_k,\theta)\tilde{x}\left(\theta_k^{+}\right)\\
&\quad +\left[\Delta^{-}\Psi(t_k,\theta)+\Psi(t_k,\theta)\right]\times\left[J_k\right]\tilde{x}\left(t_k^{+}\right)\\
&\quad +\Psi(t_k^{-},\theta)D_k^{*}w_k\\
&= \sum_{\theta<\theta_k<T}\Delta^{-}\Psi(t_k,\theta)\left[I+J_k\right]\tilde{x}\left(t_k^{+}\right)\\
&\quad +\sum_{\theta<\theta_k<T}\Psi(\theta_k,\theta)\left[J_k\right]\tilde{x}\left(\theta_k^{+}\right)+\sum_{\theta<\theta_k<\rho}\Psi(\theta_k^{-},\theta)D_k^{*}w_k\\
&= \sum_{\theta<\theta_k<\rho}\left\{\Delta^{-}\Psi(\theta_k,\theta)\left[I+J_k\right]+\Psi(\theta_k,\theta)\left[J_k\right]\right\}\tilde{x}\left(\theta_k^{+}\right)\\
&\quad +\sum_{\theta<t_k<T}\Psi(\theta_k^{-},\theta)D_k^{*}w_k.
\end{aligned}$$

On the other hand, by differentiating the function $\psi\left(s\right)=\Psi(s,\theta)\tilde{x}(s)$ we obtain

$$\begin{aligned}
\psi^{'}\left(s\right) &= \frac{d}{ds}\left[\Psi(s,\theta)\tilde{x}(s)\right]=\left[\frac{\partial}{\partial s}\Psi(s,\theta)\right]\tilde{x}(s)+\Psi(s,\theta)\tilde{x}^{'}(s)\\
&= -A^{*}G(t,s)x(s)+G(t,s)\left[A^{*}x(s)\right]+\Psi(s,\theta)B^{*}u^{*}(s).
\end{aligned}$$

Hence

$$\begin{aligned} & \Psi(\theta,\theta)\tilde{x}(\theta)-\Psi(T,\theta)\tilde{x}(T) \\ = & \int_T^\theta \Psi(s,\theta)B^*\xi(s)ds \\ & + \sum_{\theta<t_k<T} \left[\Delta^-\Psi(t_k,\theta)\left(I+J_k\right)+\Psi(t_k,\theta)J_k\right]\tilde{x}\left(t_k^+\right) \\ & + \sum_{\theta<t_k<T} \Psi(t_k^-,\theta)D_k^* w_k. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \tilde{x}(\theta) &= \Psi(T,\theta)\tilde{x}(T)-\int_\theta^T \Psi(s,\theta)B^*\xi(s)ds \\ & + \sum_{\theta<\theta_k<T} \left[\Delta^-\Psi(\theta_k,\theta)\left(I+J_k\right)+\Psi(\theta_k,\theta)J_k\right]\tilde{x}\left(t_k^+\right) \\ & + \sum_{\theta<\theta_k<T} \Psi(\theta_k^-,\theta)D_k^* w_k. \end{aligned}$$

This completes the proof.■

Proposition 27 *[7] The impulsive evolution operators $G(t,s)$ related to (2.22) can be represented explicitly as follows*

$$G(t,t)-G(t,0)=\int_0^t G(t,s)A\left(s\right)ds-\sum_{0<t_k<t}\left[I_k\left(I+I_k\right)^{-1}\right]G(t,t_k^-)$$

Proof. Let $G(t,s)$ be an impulsive evolution operator of (2.22). Then,

$$\begin{cases} \dfrac{\partial}{\partial s}G(t,s)=G(t,s)A\left(s\right), \ \ t\in\left[t_k,t_{k+1}\right), \qquad k\in\sigma_0^m, \\ \Delta_2^+G(t,t_k)=G(t,t_k^+)-G(t,t_k^-)=-G(t,t_k^-)\left[I_k\left(I+I_k\right)^{-1}\right] \quad k\in\sigma_0^{m-1}. \end{cases}$$

It is not difficult to see that the integration of $\dfrac{\partial}{\partial s}G(t,s)$ with respect to s over an interval $[0,t]$ leads to

$$G(t,t)-G(t,0)=\int_0^t G(t,s)A\left(s\right)ds-\sum_{0<t_k<t}\left[I_k\left(I+I_k\right)^{-1}\right]G(t,t_k^-).$$

This completes the proof.■

Proposition 28 *[7] The impulsive evolution operators* $\Psi(s,\theta)$ *of (2.23) can be explicitly represented as follows*

$$\Psi(T,\theta) = \Psi(\theta,\theta) + \int_{\theta}^{T} -\Psi(s,\theta)A^{*}(s)\,ds + \sum_{\theta<t_k<T} \left[J_k\left(I+J_k\right)^{-1}\right]\Psi(t_k,\theta)$$

Proof. Let $\Psi(s,\theta)$ be the impulsive evolution operators of (2.23). Then,

$$\begin{cases} \dfrac{\partial}{\partial s}\Psi(s,\theta) = -\Psi(s,\theta)A^{*}(s)\,, \ \ t\in(t_k,t_{k+1}]\,, \qquad k\in\sigma_0^m, \\ \Delta_1^{-}\Psi(t_k,\theta) = \left[-J_k\left(I+J_k\right)^{-1}\right]\Psi(t_k,\theta), \quad k\in\sigma_0^{m-1}. \end{cases}$$

It is not difficult to see that the integration of $\dfrac{\partial}{\partial s}\Psi(s,\theta)$ with respect to s over an interval $[\theta,T]$ leads to

$$\Psi(\theta,\theta) - \Psi(T,\theta) = \int_{T}^{\theta} -\Psi(s,\theta)A^{*}(s)\,ds + \sum_{\theta<t_k<T} \left[-J_k\left(I+J_k\right)^{-1}\right]\Psi(t_k,\theta)$$

which gives

$$\Psi(T,\theta) = \Psi(\theta,\theta) + \int_{\theta}^{T} -\Psi(s,\theta)A^{*}(s)\,ds + \sum_{\theta<t_k<T} \left[J_k\left(I+J_k\right)^{-1}\right]\Psi(t_k,\theta).$$

This completes the proof.■

A direct consequence of the Proposition **2.27** is the following.

Corollary 29 *[7] Let* $\left(I+\left[I_k\left(I+I_k\right)^{-1}\right]\right)^{-1} = I_k$, $k\in\sigma_1^m$. *For each* $s\in[0,T]$ *we have*

$$\begin{cases} \dfrac{\partial}{\partial t}G(t,s) = G(t,s)A(t)\,, \ \ t\in[t_k,t_{k+1})\,, \qquad k\in\sigma_0^m, \\ \Delta_1^{+}G(t_k,s) = G(t_k^{+},s) - G(t_k^{-},s) = I_kG(t_k^{-},s) \ k\in\sigma_1^m. \end{cases}$$

Proof. By Proposition **2.27** we have

$$G(t,t) - G(t,\sigma) = \int_{\sigma}^{t} G(t,s)A(s)\,ds - \sum_{\sigma<t_k<t}\left[I_k\left(I+I_k\right)^{-1}\right]G(t,t_k^{-})$$

It is easy to verify that

$$\frac{\partial}{\partial t}G(t,\sigma) = G(t,\sigma)A\left(t\right), \quad \forall \sigma \in \left[0,T\right].$$

We obtain for each $\sigma \in \left[0,T\right]$, $t+h \in \left[\sigma,T\right]$, that

$$\begin{aligned} & G(t+h,t+h) - G(t+h,\sigma) \\ = & \int_{\sigma}^{t+h} G(t+h,s)A\left(s\right)ds - \sum_{\sigma<t_k<t+h} \left[I_k\left(I+I_k\right)^{-1}\right]G(t,t_k^-). \end{aligned}$$

Leting $h \downarrow 0^+$ we obtain

$$\begin{aligned} & G(t^+,t^+) - G(t^+,t) \\ = & \int_{t}^{t^+} G(t^+,s)A\left(s\right)ds - \sum_{t<t_k<t^+} \left[I_k\left(I+I_k\right)^{-1}\right]G(t,t_k^-). \end{aligned}$$

This implies that

$$G(t_k^+,t_k) = \left(I + \left[I_k\left(I+I_k\right)^{-1}\right]\right)^{-1} G(t_k,t_k) = I_k.$$

It follows immediately from $\left(I + \left[I_k\left(I+I_k\right)^{-1}\right]\right)^{-1} = I_k$, and the semigroup property of $G(t,s)$ that

$$G(t_k^+,s) = G(t_k^+,t_k)G(t_k,s) = I_kG(t_k,s), \quad \forall s \in \left[0,T\right].$$

Using the definition of $\Delta_1^+G(t_k,s)$ we have

$$\Delta_1^+G(t_k,s) = G(t_k^+,s) - G(t_k^-,s) = I_kG(t_k,s) \ k \in \sigma_1^m.$$

This completes the proof.■

Corollary 30 *[7] Let* $\left(I + \left[J_k\left(I+J_k\right)^{-1}\right]\right)^{-1} = J_k$, $k \in \sigma_1^m$. *For each* $s \in \left[0,T\right]$ *we have*

$$\begin{cases} \dfrac{\partial}{\partial\theta}\Psi(s,\theta) = \Psi(s,\theta)A^*\left(s\right), \ t \in \left(t_k,t_{k+1}\right], \qquad k \in \sigma_0^m, \\ \Delta_2^-\Psi(\theta,t_k) = J_k\Psi(\theta,t_k) \ \ k \in \sigma_0^{m-1}. \end{cases}$$

In the following we will study the existence of weak solutions of first order impulsive evolution equations and we will apply these results to study exact controllability by HUM method in chapter 3.

For impulsive evolution equations with an unbounded linear operator A of the form

$$\begin{cases} y^{'}(t) + A(t,y) = f(t,y), & t \in (t_0, T) \Big\backslash \{t_k\}_{k\in\sigma_1^m}\ , \\ y(0) = y^0 \in H, & \\ \Delta y(t_k) = I_k y(t_k) & k \in \sigma_1^m \end{cases} \tag{2.24}$$

where the operators $A : [t_0, T] \times V \to V^*$, $f : [t_0, T] \times H \to V^*$ and for each $k \in \sigma_1^m$, $I_k : H \to H$, H and V being Hilbert spaces such that V is a dense subspace of H having a structure of a reflexive Banach space, with continuous embedding $V \hookrightarrow H \hookrightarrow V^*$ and $V \hookrightarrow H$ is compact, V^* being the topological dual space of V, have been considered in several papers, for instance, Ahmed [3.4] , Liu [78] , Rogovchenko [93] and Pongchalee [91] . The question of existence and regularity of solutions have been discussed there. However, these questions are still open when the operator A is nonlinear.

2.2 Second order impulsive Cauchy problem

In this section we discuss the existence of mild and classical solutions for some abstract second order impulsive equations.

We consider the following impulsive second order initial value problem of the form

$$\begin{cases} y^{''}(t) = Ay(t) + f(t, y(t), y^{'}(t)), \quad t \in (0,T) \setminus \{t_k\}_{k\in\sigma_1^m}, \\ y(0) = y^0, \quad y^{'}(0) = y^1, \\ \Delta y^{'}(t_k) = I_k^1(y(t_k)), \quad k \in \sigma_1^m, \\ \Delta y^{'}(t_k) = I_k^2(y^{'}(t_k)), \quad k \in \sigma_1^m, \end{cases} \tag{2.25}$$

where the final time T is a positive number, $\left\{y^0, y^1\right\}$ is an initial condition in a Banach space X, $y\left(t\right) : [0,T] \to H$ is a vector function, and finally, $\{t_k\}_{k\in\sigma_1^m}$ is an increasing sequence of numbers in the open interval $\left(t_0, T\right)$, and $\Delta y\left(t_k\right)$, $\Delta y^{'}(t_k)$ denote the jump of $y\left(t\right)$ and $y^{'}\left(t\right)$ at $t = t_k$ respectively, i.e,

$$\Delta y\left(t_k\right) = y\left(t_k^+\right) - y\left(t_k^-\right), \quad \Delta y^{'}\left(t_k\right) = y^{'}\left(t_k^+\right) - y^{'}\left(t_k^-\right),$$

where $y\left(t_k^+\right), y^{'}\left(t_k^+\right)$ and $y\left(t_k^-\right), y^{'}\left(t_k^-\right)$ represent the right and left limits of $y\left(t\right), y^{'}\left(t\right)$ at $t = t_k$ respectively. We assume that A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators, $\{C(t)\}$, $t \in \mathbb{R}$, with dense domain

$$D(A) = \{x \in X,\ C(t)x \text{ is twice continuously differentiable } \}$$

and $I_k^i : X \to X, i = 1, 2$, are given linear bounded operators.

We denote by E the set

$$E = \{x \in X,\ C(t)x \text{ is once continuously differentiable } \} \tag{2.26}$$

and $S = \{S(t)\}, t \in \mathbb{R}$, the associated sine function that is

$$S(t)x = \int_0^t C(\theta)x d\theta, \quad x \in X, \quad t \in \mathbb{R}.$$

In this section we discuss the existence of mild and classical solutions for the impulsive initial value problem (2.25) by comparison with the second order abstract Cauchy problem. We begin with the following definitions :

Definition 31 *A function* $(y,z) \in \mathcal{PLC}_b^1((0,T);X)$ *is a mild solution of the impulsive problem (2.25) if the impulsive conditions in (2.25) is satisfied and*

$$\begin{aligned} y(t) &= C(t)y^0 + S(t)y^1 + \int_0^t S(t-s)f(s,y(s),y^{'}(s))ds \\ &+ \sum_{0<t_k<t} C(t-t_k)I_k^1(y(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(y^{'}(t_k)) \end{aligned}$$

for every $t \in (0,T)$.

Definition 32 *A function* $(y,z) \in \mathcal{PLC}_b^1((0,T);X)$ *is a classical solution of the impulsive problem (2.25) if* $y \in C^2((0,T)\Big\backslash\{t_k\}_{k\in\sigma_1^m}, X)$ *and (2.25) are satisfied.*

In order to establish the existence of mild solutions, we introduce the following technical assumptions :

(H1) $f:[0,T]\times X \to X$ and $I_k^i : X \to X, i=1.2,\ k\in\sigma_1^m$, is a continuous function and there exist positive constants $L_1(f)$ and $L_2(f)>0$ such that

$$\|f(t,x_1,\tilde{x}_1) - f(t,x_2,\tilde{x}_2)\| \le L_1(f)\|x_1-\tilde{x}_1\| + L_2(f)\|x_2-\tilde{x}_2\|, \quad t\in[0,T],$$

for every $x_1, \tilde{x}_1$, x_2 and $\tilde{x}_2 \in X$.

(H2) The functions $I_k^i : X \to X, i=1.2,\ k\in\sigma_1^m$, are continuous and there exist positive constants $L(I_k^i)>0, i=1.2,\ k\in\sigma_1^m$, such that

$$\left\|I_k^i(x) - I_k^i(\tilde{x})\right\| \le L(I_k^i)\|x-\tilde{x}\|, \quad \forall x,\tilde{x}\in X.$$

(H3) There exist functions $B:[0,T]\to\mathcal{L}(X)$, $F_k : X\to X$, $k\in\sigma_1^m$ such that:

(i) $B(0)=0$, B is strongly continuous and $\dfrac{d}{dt}C(t)I_k^1x = B(t)F_kx$, $\forall\ x\in X$, $\forall k\in\sigma_1^m$.

(ii) for each $k \in \sigma_1^m$, there exists a positive constant $L(F_k)$ such that

$$\|F_k(x) - F_k(\tilde{x})\| \leq L(F_k)\,\|x - \tilde{x}\|\,, \quad \forall x, \tilde{x} \in X.$$

We have the following result :

Theorem 33 *[54] Let $y^0 \in E$, $y^1 \in X$ and let assumptions **(H1)-(H3)** hold. If*

$$\max\{\Lambda_1, \Lambda_2\} < 1,$$

where

$$\Lambda_1 = L_1(f)(\|S\|_{\mathcal{PLC}_b} + \|C\|_{\mathcal{PLC}_b})T + \textstyle\sum_{k=1}^m(\|C\|_{\mathcal{PLC}_b}\,L(I_k^1) + \|B\|_{\mathcal{PLC}_b}\,L(F_k)),$$

and

$$\Lambda_2 = (\|S\|_{\mathcal{PLC}_b} + \|C\|_{\mathcal{PLC}_b})(L_2(f)T + \textstyle\sum_{k=1}^m L(I_k^2),$$

then there exists a unique mild solution of the impulsive problem (2.25).

Proof: For $(y, z) \in \mathcal{PLC}_b^1((0,T); X)$ we define $\Phi(y, z) = (\Phi_1(y, z), \Phi_2(y, z))$, where

$$\begin{array}{rcl} \Phi_1(y,z)(t) & = & C(t)y^0 + S(t)y^1 + \int_0^t S(t-s)f(s, y(s), y'(s))ds \\ & & + \sum_{0<t_k<t} C(t-t_k)I_k^1(y(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(y'(t_k)), \end{array}$$

and

$$\begin{array}{rcl} \Phi_2(y,z)(t) & = & AS(t)x_0 + C(t)y_0 + \int_0^t C(t-s)f(s, y(s), y'(s))ds \\ & & + \sum_{0<t_k<t} B(t-t_k)F_k(y(t_k)) + \sum_{0<t_k<t} C(t-t_k)I_k^2(y'(t_k)), \end{array}$$

Clearly $\Phi(y, z) \in \mathcal{PLC}_b^1((0,T); X)$. In order to prove that Φ is a contraction mapping on $\mathcal{PLC}_b^1((0,T); X)$, we take (y, z), (v, w) in $\mathcal{PLC}_b^1((0,T); X)$. From assumption (**H1**) we see that for $t \in (0,T)$

$$\begin{aligned}
& \left\| \Phi_1(y,z)(t) - \Phi_1(v,w)(t) \right\| \qquad (2.27) \\
\leq \; & \int_0^t \|S\|_{\mathcal{PLC}_b} \left(L_1(f) \|y - v\|_{\mathcal{PLC}_b((0,\theta);X)} + L_2(f) \left\| y^{'} - v^{'} \right\|_{\mathcal{PLC}_b((0,\theta);X)} \right) d\theta \\
& + \sum_{0<t_k<t} \|C\|_{\mathcal{PLC}_b} L(I_k^1) \|y(t_k) - v(t_k)\| + \sum_{0<t_k<t} \|S\|_{\mathcal{PLC}_b} L(I_k^2) \left\| y^{'}(t_k) - v^{'}(t_k) \right\|,
\end{aligned}$$

thus

$$\begin{aligned}
& \left\| \Phi_1(y,z)(t) - \Phi_1(v,w) \right\|_{\mathcal{PLC}_b} \qquad (2.28) \\
\leq \; & (\|S\|_{\mathcal{PLC}_b} (L_1(f)(T) + \sum_{k=1}^{m} \|C\|_{\mathcal{PLC}_b} L(I_k^1) \|y - v\|_{PC_b} \\
& + \|S\|_{\mathcal{PLC}_b} (L^2(f)(T) + \sum_{k=1}^{m} L(I_k^2) \|z - w\|_{\mathcal{PLC}_b}.
\end{aligned}$$

Likewise we have

$$\begin{aligned}
& \left\| \Phi_2(y,z)(t) - \Phi_2(v,w) \right\|_{\mathcal{PLC}_b} \\
\leq \; & (\|C\|_{PC_b} (L^1(f)(T) + \|B\|_{\mathcal{PLC}_b} \sum_{k=1}^{m} L(F_k) \|y - v\|_{\mathcal{PLC}_b} \\
& + \|C\|_{\mathcal{PLC}_b} (L_2(f)(T) + \sum_{k=1}^{m} L(I_k^2) \|z - w\|_{\mathcal{PLC}_b}.
\end{aligned}$$

Inequalities (2.27)-(2.28) and the assumption $\max\{\Lambda_1, \Lambda_2\} < 1$, imply that Φ is a contraction mapping. We deduce from the fixed point Theorem that there exists a unique mild solution y of (2.25). The proof is complete.■

Next we establish the existence of a classical solution for the impulsive initial value problem (2.25) under the assumption that f is continuously differentiable. For this purpose, we need the following lemmas.

Lemma 34 *[54] If $g \in C((0,b);X) \cap C^1((0,b);X)$, $b > 0$, then*

$$h(t) = \int_0^t C(t-s)g(s)ds \in E, \quad \forall t \in [0,b].$$

where E is given in (2.26)

Lemma 35 *[54] Let $y^0 \in D(A)$, $y^1 \in E$ and let the assumptions of Theorem **33** hold. If f is continuously differentiable and there exists $L_3(f) > 0$ such that*

$$\max\left\{\left\|\frac{\partial f}{\partial x_j}(t,x_1,x_2) - \frac{\partial f}{\partial x_j}(t,x_1,\tilde{x}_2)\right\|,\ j=1,2,3\right\} \leq L_3(f)\left\|x_2-\tilde{x}_2\right\|$$

for every $\tilde{x}_1, \tilde{x}_2 \in X$, then there exits a unique classical solution $y(.)$ of

$$\begin{cases} y^{''}(t) = Ay(t) + f(t,y(t),y^{'}(t)),\ (0,t_{k+1}), \\ y(t_k) = y^0, \quad y^{'}(t_k) = y^1. \end{cases} \tag{2.29}$$

Moreover, $y(t_{k+1}^-) \in D(A)$ and $y^{'}(t_{k+1}^-) \in E$.

Proof. Let $w : (0,T) \to X$ be the unique mild solution of

$$\begin{cases} y^{''}(t) = Ay(t) + f(t,y(t),y^{'}(t)),\ (0,T), \\ y(t_k) = y^0,\ y^{'}(t_k) = y^1. \end{cases}$$

and $y : (0,t_{k+1}) \to X$ the mild solution of the initial value problem (2.29). From the proof of Proposition 3.3 in [103] we infer that w and y are classical solutions, and thus, $y(t_{k+1}) = w(t_{k+1}) \in D(A)$, since $y = w$ on $(0,t_{k+1})$.

On the other hand,

$$y^{'}(t_{k+1}^-) = AS(t_{k+1})y^0 + C(t_{k+1})y^1 + \int_0^{t_{k+1}} C(t_{k+1}-s)f(s,y(s),y^{'}(s))ds = w^{'}(t_{k+1}). \tag{2.30}$$

Clearly, $AS(t_{k+1})y^0 + C(t_{k+1})y^1 \in E$ and from Lemma **36**, the integral term in (2.30) is in E, therefore $w^{'}(t_{k+1}) \in E$. The proof is complete.■

Theorem 36 (Classical solution) *[54] Let $y^0 \in D(A)$, $y^1 \in E$ and assume that assumptions in Theorem **33** and Lemma **35** hold. If $(y,z) : (0,T) \to X$ is the mild solution of (2.25) and $I_k^1(y(t_k)) \in D(A)$, $I_k^2(y(t_k)) \in E$, for each $k \in \sigma_1^m$, then (y,z) is a classical solution.*

Proof: Let $w : (0,T) \to X$ be the unique classical solution of

$$\begin{cases} w^{''}(t) = Aw(t) + f(t, w(t), w^{'}(t)), \quad 0 < t < t_1, \\ \quad w(0) = y^0, \quad w^{'}(0) = y^1. \end{cases}$$

and $(y_1, z_1) : (0, t_1] \to X \times X$ the function defined by $(y_1, z_1)(t) = (w(t^-), w^{'}(t^-))$. From Lemma **37** we know that $(y_1(t_1), z_1(t_1)) \in D(A) \times E$, thus there exists a unique classical solution $w(.)$ of the abstract Cauchy problem

$$\begin{cases} w^{''}(t) = Aw(t) + f(t, w(t), w^{'}(t)), \quad 0 < t < t_2, \\ \quad w(t_1) = y_1(t_1) + I_1^1(w(t_1)), \\ \quad w^{'}(t_1) = z_1(t_1) + I_1^2(w^{'}(t_1)). \end{cases}$$

Similarly to the previous case, let $(y_2, z_2) : (0, t_2] \to X \times X$ be the function defined by

$$(y_2, z_2)(t) = (w(t^-), w^{'}(t^-)).$$

If w is the classical solution of the second order Cauchy problem

$$\begin{cases} w^{''}(t) = Aw(t) + f(t, w(t), w^{'}(t)), \qquad 0 < t < t_k, \\ w(t_{k-1}) = y_{k-1}(t_1) + I_{k-1}^1(w(t_{k-1})), \\ w^{'}(t_{k-1}) = z_{k-1}(t_{k-1}) + I_{k-1}^2(w^{'}(t_{k-1})). \end{cases}$$

W shall denote by (y_k, z_k) the function $(y_k, z_k) : (0, t_k] \to X \times X$ such that

$$(y_k, z_k)(t) = (w(t^-), w^{'}(t^-)).$$

Let $(\tilde{y}(t), \tilde{z}(t)) : (0,T) \to X \times X$ be the function defined by

$$(\tilde{y}(t), \tilde{z}(t)) = \begin{cases} (y_1(t), z_1(t)) & \text{if} \quad 0 < t < t_1, \\ (y_k(t), z_k(t)) & \text{if} \quad t_{k-1} < t < t_k, \\ (y_m(t), z_m(t)) & \text{if} \quad t_m < t < T. \end{cases}$$

It is easy to see that $(\tilde{y}(t), \tilde{z}(t))$ is the unique classical solution of the impulsive problem

$$\begin{cases} y^{''}(t) = Ay(t) + f(t, y(t), y^{'}(t)), & (0,T)\,, \quad t \neq t_k \\ y(0) = y^0, \quad y^{'}(0) = y^1 & \\ \Delta y(t_k) = I_k^1(y(t_k)), \quad \Delta y^{'}(t_k) = I_k^2(y^{'}(t_k)), & k \in \sigma_1^m. \end{cases}$$

Next, we show that $(\tilde{y}(t), \tilde{z}(t)) = (y, z)$. In order to reduce the proof, which is only a tedious calculation, we introduce the group of linear operators

$$U(t) = \begin{pmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{pmatrix}_{t \in \mathbb{R}}$$

on $D(A) \times E$. The function $(\tilde{y}, \tilde{z})$ is a solution of the first order impulsive Cauchy problem

$$\begin{cases} W^{'}(t) = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} W(t) + F(t, W(t)), \quad t \neq t_k\,, \\ W(0) = \begin{pmatrix} y^0 \\ y^1 \end{pmatrix}, \\ \Delta W(t_k) = I_k \begin{pmatrix} y(t_k) \\ y^{'}(t_k) \end{pmatrix}, \qquad k \in \sigma_1^m, \end{cases}$$

where I_k and F are defined by

$$F(t) = \begin{pmatrix} 0 \\ f(t, x(t), x^{'}(t)) \end{pmatrix}, \quad I_k(u(t_k)) = \begin{pmatrix} I_k^1(y(t_k)) \\ I_k^2(y^{'}(t_k)) \end{pmatrix}.$$

We have

$$W(t) = U(t) \begin{pmatrix} y^0 \\ y^1 \end{pmatrix} + \int_0^t U(t-s)F(s, W(s))ds + \sum_{0<t_k<t} U(t-t_k) I_k \begin{pmatrix} y(t_k) \\ y^{'}(t_k) \end{pmatrix},$$

hence

$$\begin{aligned} \tilde{y}(t) \;\; = \;\; & C(t)y^0 + S(t)y^1 + \textstyle\int_0^t S(t-s) f(s, \tilde{y}(s), \tilde{z}(s))ds \\ & + \sum_{0<t_k<t} C(t-t_k) I_k^1(y(t_k)) + \sum_{0<t_k<t} S(t-t_k) I_k^2(y^{'}(t_k)) \end{aligned} \tag{2.31}$$

and

$$\begin{aligned}\tilde{z}(t) \quad = \quad & AS(t)y^0 + C(t)y^1 + \textstyle\int_0^t C(t-s)f(s, \tilde{y}(s), \tilde{z}(s))ds \\ & + \sum_{0<t_k<t} AS(t-t_k)I_k^1(y(t_k)) + \sum_{0<t_k<t} C(t-t_k)I_k^2(y^{'}(t_k)).\end{aligned}$$

Finally, let $\theta(t) = \max\{\|y(t) - \tilde{y}(t)\|, \ \|z(t) - \tilde{z}(t)\|\}$. From assumption **(H1)** and (2.30)-(2.31) we obtain that

$$\theta(t) \leq C_1 \int_0^t \theta(s)ds,$$

where C_1 is a constant independent of $t \in (0, T)$. The Gronwall's inequality implies that $y = \tilde{y}$. The proof is complete.■

Chapter 3

Exact Controllability of Some Impulsive Evolution Equation of First Order

3.1 Null Controllability

The problem of exact controllability of linear systems represented by infinite conservative systems has been extensively studied by several authors A. Haraux [53], R.Triggiani [104], Z.H. Guan, T.H. Qian, and X.Yu [49], see also the references [2, 3, 4, 47]. In the sequel, we shall be concerned with the problem of null controllability of some first order evolution equation subject to impulsive conditions and so we shall derive a necessary and sufficient condition under which null controllability occurs. Actually, we shall establish an equivalence between the null-controllability and some "observability" inequality in a somehow more general framework than that proposed by A Haraux [53]. Regarding the literature on the impulsive differential equations we refer the reader to the works of D.D. Bainov and P.S. Simeonov [11, 13] and the references [1, 22, 33, 46 ,73, 75, 76]. We are going to study the following problem

$$\begin{cases} y^{'}(t) + Ay(t) = Bu(t), \quad t \in (0,T) \setminus \{t_k\}_{k \in \sigma_1^m}, \\ y(0) = y^0, \\ \Delta y(t_k) = I_k y(t_k) + D_k v_k, \;\; k \in \sigma_1^m, \qquad\qquad 3.1_k \end{cases} \tag{3.1}$$

where the final time T is a positive number, y^0 is an initial condition in a Hilbert space H endowed with an inner product $\langle .,. \rangle_H$, $y(t) : [0,T] \to H$ is a vector function, σ_1^m is a subset of $\mathbb{N}$ given by $\sigma_1^m = \{1, 2, ..., m\}$, and finally, $\{t_k\}_{k \in \sigma_1^m}$ is an increasing sequence of numbers in the open interval $(0,T)$, and $\Delta y(t_k)$ denotes the jump of $y(t)$ at $t = t_k$,i.e,

$$\Delta y(t_k) = y\left(t_k^+\right) - y\left(t_k^-\right)$$

where $y\left(t_k^+\right)$ and $y\left(t_k^-\right)$ represent the right and left limits of $y(t)$ at $t = t_k$ respectively. On the other hand, the operators A, B, I_k, $D_k : H \to H$ are given linear operators.

Moreover, we set the following assumptions:

(H1) $A^* = -A$,

(H2) $I_k^* = -I_k$, for every $k \in \sigma_1^m$, and for each $k \in \sigma_1^m$, the operator $\mathcal{I}_k = I_k + I$ is invertible,

(H3) $B^* = B \geq 0$ and there is $d_0 > 0$ such that

$$(Bu, u)_H \leq d_0 \|u\|_H^2, \;\; \text{for all } u \in H,$$

(H4) $D_k^* = D_k \geq 0$, for every $k \in \sigma_1^m$, and for each $k \in \sigma_1^m$ there is $d_k > 0$ such that

$$(D_k u, u)_H \leq d_k \|u\|_H^2, \;\; \text{for all } u \in H,$$

In the sequel we shall designate by h the function

$$h(t) = \left(u(t), \{v_k\}_{k \in \sigma_1^m}\right),$$

where $u(t) \in L^2\left((0,T) \setminus \{t_k\}_{k\in\sigma_1^m}; H\right)$ and

$$\{v_k\}_{k\in\sigma_1^m} \in l^2(\sigma_1^m; H) \doteqdot \left\{\{v_k\}_{k\in\sigma_1^m}, v_k \in H\right\}.$$

We point out that the space $\mathcal{K}_m = L^2\left((0,T) \setminus \{t_k\}_{k\in\sigma_1^m}; H\right) \times l^2(\sigma_1^m; H)$ is a Hilbert space with respect to the inner product

$$\left(h, \widetilde{h}\right)_{\mathcal{K}_m} = \int_0^T (u(t), \widetilde{u}(t))_H\, dt + \sum_{k=1}^m (v_k, \widetilde{v}_k)_H,$$

defined for all $h = (u(t), \{v_k\}_{k=1}^m)$ and $\widetilde{h} = (\widetilde{u}(t), \{\widetilde{v}_k\}_{k=1}^m) \in \mathcal{K}_m$.

We shall denote by $\mathcal{B}$ the control operator given by

$$\mathcal{B} = \left(B, \{D_k\}_{k\in\sigma_1^m}\right) \in \mathcal{L}\left(L^2\left((0,T) \setminus \{t_k\}_{k\in\sigma_1^m}; H\right) \times l^2(\sigma_1^m; H)\right),$$

so that

$$\mathcal{B}h(t) = \left(Bu(t), \{D_k v_k\}_{k\in\sigma_1^m}\right).$$

We have for every $h = (u(t), \{v_k\}_{k=1}^m) \in \mathcal{K}_m$

$$\begin{aligned} (\mathcal{B}h, h)_{\mathcal{K}_m} &= \int_0^T (Bu(t), u(t))_H\, dt + \sum_{k=1}^m (D_k v_k, v_k)_H \\ &= \int_0^T (u(t), Bu(t))_H\, dt + \sum_{k=1}^m (v_k, D_k v_k)_H \\ &= (h, \mathcal{B}h)_{\mathcal{K}_m}, \end{aligned}$$

which shows that $\mathcal{B}^* = \mathcal{B}$, that is, $\mathcal{B}$ is self-adjoint. On the other hand, we have

$$\begin{aligned} (\mathcal{B}h, h)_{\mathcal{K}_m} &= \int_0^T (Bu(t), u(t))_H\, dt + \sum_{k=1}^m (D_k v_k, v_k)_H \\ &\le d_0 \int_0^T \|u(t)\|_H^2\, dt + \sum_{k=1}^m d_k \|v_k\|_H^2 \\ &\le \delta \|h\|_{\mathcal{K}_m}^2, \end{aligned}$$

where $\delta = \max\{d_0, d_1, ..., d_m\}$. Thus, the operator $\mathcal{B}$ is bounded in $\mathcal{K}_m$.

Next, we consider the homogeneous system associated with (3.1) :

$$\begin{cases} \varphi^{'}(t) + A\varphi(t) = 0, \quad t \in (0,T) \setminus \{t_k\}_{k \in \sigma_1^m}, \\ \varphi(0) = \varphi^0, \\ \Delta\varphi(t_k) = I_k \varphi(t_k), \ k \in \sigma_1^m. \qquad 3.2_k \end{cases} \tag{3.2}$$

We point out that on each interval $[t_k, t_{k+1})$, for $k = 0, ..., m$, the solution φ is left continuous at each time t_k.

Consider the corresponding homogeneous backward problem :

$$\begin{cases} -\tilde{\varphi}^{'}(t) + \mathbf{A}\tilde{\varphi}(t) = 0, \quad t \in (0,T) \setminus \{t_k\}_{k \in \sigma_1^m}, \\ \tilde{\varphi}(T) = \varphi^0, \\ \Delta\tilde{\varphi}(t_{m-(k-1)}) = -\tilde{I}_{m-(k-1)}\tilde{\varphi}(t^+_{m-(k-1)}), \ k \in \sigma_1^m. \qquad 3.3_k \end{cases} \tag{3.3}$$

where

$$\mathbf{A} = A^* = -A, \quad \tilde{I}_{m-(k-1)} = I^*_{m-(k-1)} = -I_{m-(k-1)}, \ k \in \sigma_1^m.$$

We observe that the problem (3.3) on the interval $[t_m, T]$ is equivalent to the classical backward problem

$$\begin{cases} -\tilde{\varphi}^{'}(t) + \mathbf{A}\tilde{\varphi}(t) = 0, \ t \in [t_m, T], \\ \tilde{\varphi}(T) = \varphi^0. \end{cases}$$

We introduce the following space :

$\mathcal{PC}([0,T];X) = \{y, y : [0,T] \to X$ *such that* $y(t)$ *is continuous at* $t \neq t_k$, $y(0^+), y(T^-)$, $y(t_k^-), y(t_k^+)$ *exist, for every* $k \in \sigma_1^m\}$.

Evidently, $\mathcal{PC}([0,T];H)$ is a Banach space with respect to the norm

$$\|y\|_{\mathcal{PC}} = \sup_{t \in (0,T)} \|y(t)\|.$$

On the other hand, we define the subspaces $\mathcal{PLC}$, (respectively, $\mathcal{PRC}$)$= \{y, y \in \mathcal{PC}$ *such that* $y(t)$ *is left (*respectively, *right) continuous at* $t = t_k$*, for every* $k \in \sigma_1^m\}$.

Remark 3 *1) The space* $\mathcal{PLC}$*, (respectively,* $\mathcal{PRC}$*) can be identified to a subspace of* $\mathcal{K}_m$*. That is, to each* $y \in \mathcal{PLC}$*, (respectively,* $\tilde{y} \in \mathcal{PRC}$*) is assigned the function* h *(respectively,* $\widetilde{\mathrm{h}}$*) defined* *by*

$$\mathrm{h}\,(t)=\Big(y\,(t)\,,\{y\,(t_k)\}_{k\in\sigma_1^m}\Big)\,,$$

and

$$\widetilde{\mathrm{h}}\,(t)=\Big(\tilde{y}\,(t)\,,\{\tilde{y}\,(t_k)\}_{k\in\sigma_1^m}\Big)\,.$$

The identity $y\longmapsto \mathrm{h}\,(t)$ *(respectively,* $\tilde{y}\longmapsto\widetilde{\mathrm{h}}$*) is a linear injection.*

2) Let $\widetilde{y}\in\mathcal{PRC}$*, the function* y *can be written as :*

$$\widetilde{y}(t)=\begin{cases}\widetilde{y}_{[0]}(t) & \text{if} \quad t\in[t_0,t_1)\\ \widetilde{y}_{[k]}(t) & \text{if} \quad t\in\big[t_k,t_{k+1)}\big)\\ \widetilde{y}_{[m]}(t) & \text{if} \quad t\in[t_m,T]\,.\end{cases}$$

Next, let $\tau_k=t_k-t_{k-1}$*, we define the operator* $\mathcal{T}:D(\mathcal{T})=\mathcal{PRC}\subset\mathcal{K}_m\to\mathcal{K}_m$ *by*

$$(\mathcal{T}\widetilde{y})(t)=\begin{cases}\widetilde{y}_{[0]}((T-t)\dfrac{\tau_1}{\tau_{m+1}}+t_0) & \text{if} \quad t\in[t_m,T]\,,\\ \widetilde{y}_{[k]}((t_{m-(k-1)}-t)\dfrac{\tau_{k+1}}{\tau_{m-(k-1)}}+t_k) & \text{if} \quad t\in\big[t_{m-k},t_{m-(k-1)}\big)\,,\quad k\in\sigma_1^{m-1},\\ \widetilde{y}_{[m]}((t_1-t)\dfrac{\tau_{m+1}}{\tau_1}+t_m) & \text{if} \quad t\in(0,t_1]\,.\end{cases}\tag{3.4}$$

We note that the range of $\mathcal{T}$ *is* $\mathcal{PLC}$*. The function* $(\mathcal{T}\widetilde{y})(t)$ *can be written as :*

$$(\mathcal{T}\widetilde{y})(t)=\begin{cases}y_{[0]}(t) & \text{if} \quad t\in[t_0,t_1]\,,\\ y_{[k]}(t) & \text{if} \quad t\in(t_k,t_{k+1}]\,,\quad k\in\sigma_1^{m-1},\\ y_{[m]}(t) & \text{if} \quad t\in(t_m,T]\,.\end{cases}$$

Let $X\,(t)$ be the resolvent solution of the operator system

$$\begin{cases}X'(t)+AX(t)=0,\ 0=t_0<t<t_{m+1}=T,\ t\neq t_k,\ k=1,2,...,m,\\ X(0)=I,\\ X(t_k+0)-X(t_k-0)=I_kX(t_k),\ k=1,2,...,m,\end{cases}$$

where $I:H\to H$ is the identity operator. We shall suppose that the operator $\mathcal{I}_k=I_k+I$ has a bounded inverse.

Definition 37 *A function* $y \in \mathcal{PC}([0,T];H)$ *is a mild solution of the impulsive problem (3.1) if the impulsive conditions are satisfied and*

$$\begin{aligned} y(t) \quad = \quad & G(t,0^{+})y^{0} + \textstyle\int_{0}^{t} G(t,s)Bu(s)\,ds \\ & + \textstyle\sum_{0<t_k\leq t} G(t,t_k)(D_k v_k) \text{ for every } t \in (0,T), \end{aligned}$$

where the evolution operator $G(t,s)$ *is given by*

$$G(t,s) = X(t)X^{-1}(s).$$

It is not hard to check that the operator $G(t,t_k)$ *satisfies the operator system*

$$\begin{cases} G^{'}(t,t_k) + AG(t,t_k) = 0, \;\; t \in [t_k, t_{k+1}), \;\; k \in \sigma_0^m, \\ G(t_k,t_k) = I, \\ G(t_{k+1}^{+},t_k) - G(t_{k+1}^{-},t_k) = I_{k+1}G(t_{k+1}^{-},t_k). \end{cases}$$

It is well known that (3.1) has a unique solution y with

$$y \in \mathcal{PLC}([0,T];H) \cap C^{1}\left([0,T] \setminus \{t_k\}_{k\in\sigma_1^m}; H\right).$$

Now, we define the concept of mild solution for the backward impulsive system (3.3) associated to the system (3.2).

Definition 38 *We say that* $\tilde{\varphi} \in \mathcal{PRC}([0,T];H)$ *is a mild solution for the backward impulsive system (3.3) if* $\mathcal{T}\tilde{\varphi}$ *is a mild solution for the homogeneous impulsive system (3.2).*

Let us introduce the notion of the null controllability of the initial state as follows:

Definition 39 *We say that the initial state* $y^0 \in H$ *is null controllable at time* T*, if there is a control function* $\mathrm{h} \in \mathcal{K}_m$ *for which the solution* y *of system (3.1) satisfies* $y(T) = 0$.

First we begin by the following lemma.

Lemma 40 *Assume that* $\xi(t), \zeta(t) \in L^{1}([0,T];H)$ *and* $\{\xi_k\}_{k=1}^{m}, \{\zeta_k\}_{k=1}^{m} \in l^{1}(\sigma_1^m, H)$*. Then,* *for every vector functions*

$$\gamma(t) \in \mathcal{PLC}([0,T];H) \cap C^1\left([0,T] \setminus \{t_k\}_{k\in\sigma_1^m};H\right)$$

and

$$\eta(t) \in \mathcal{PRC}([0,T];H) \cap C^1\left([0,T] \setminus \{t_k\}_{k\in\sigma_1^m};H\right)$$

satisfying the problem

$$\begin{cases} \dfrac{d}{dt}\langle \gamma(t), \eta(t)\rangle = \langle \xi(t), \zeta(t)\rangle, \quad t \neq t_k, \ \textit{for } k \in \sigma_1^m, \\ \Delta\langle \gamma(t_k), \eta(t_k)\rangle = \langle \Delta\gamma(t_k), \eta(t_k)\rangle + \langle \gamma(t_k), \Delta\eta(t_k)\rangle = \langle \xi_k, \zeta_k\rangle, \ k \in \sigma_1^m, \end{cases}$$

we have the following identity

$$\begin{aligned} \langle \gamma(t), \eta(t)\rangle\big|_0^T &= \langle \gamma(T), \eta(T)\rangle - \langle \gamma(0), \eta(0)\rangle \\ &= \int_0^T \langle \xi(t), \zeta(t)\rangle dt + \sum_{k=1}^m \langle \xi_k, \zeta_k\rangle. \end{aligned} \tag{3.5}$$

Proof. Let us define the functions Φ and Ψ as follows

$$\Phi(t) = \begin{cases} \langle \xi(t), \zeta(t)\rangle, \quad t \neq t_k, \ \text{for } k \in \sigma_1^m \\ \langle \xi_k, \zeta_k\rangle, \ t = t_k, \quad k \in \sigma_1^m, \end{cases}$$

and $\Psi(t) = \langle \gamma(t), \eta(t)\rangle$, then, Φ and Ψ satisfy the problem

$$\begin{aligned} \Psi'(t) &= \Phi(t), \quad t \neq t_k, \ \text{for } k \in \sigma_1^m \\ \Delta\Psi(t_k) &= \Phi(t_k), \ k \in \sigma_1^m, \end{aligned}$$

and we have

$$\Psi\left(t_1^-\right) - \Psi(0) = \int_0^{t_1} \Phi(s)\, ds$$

$$\Psi\left(t_2^-\right) - \Psi\left(t_1^+\right) = \int_{t_1}^{t_2} \Phi(s)\, ds$$

....

$$\Psi\left(t_k^-\right)-\Psi\left(t_{k-1}^+\right)=\int_{t_{k-1}}^{t_k}\Phi(s)\,ds, \qquad k=3,...m-1$$

$$\Psi(T)-\Psi\left(t_m^+\right)=\int_{t_m}^{T}\Phi(s)\,ds.$$

Adding together we get

$$\begin{aligned}\int_0^T\Phi(t)\,dt &= \int_0^{t_1}\Phi(t)\,dt+\sum_{k=1}^{m-1}\int_{t_k}^{t_{k+1}}\Phi(t)\,dt+\int_{t_m}^{T}\Phi(t)\,dt\\ &= \Psi\left(t_1^-\right)-\Psi\left(0^+\right)+\sum_{k=1}^{m-1}\left[\Psi\left(t_{k+1}^-\right)-\Psi\left(t_k^+\right)\right]+\Psi(T)-\Psi\left(t_m^+\right)\\ &= -\sum_{k=1}^{m}\Phi(t_k)+\Psi(T)-\Psi(0).\end{aligned}$$

So that

$$\Psi(T)-\Psi(0)=\int_0^T\Phi(t)\,dt+\sum_{k=1}^{m}\Phi(t_k),$$

which shows that (3.5) is satisfied.■

We also need the following Lemmas.

Lemma 41 [100] *If $\mathcal{B}\in\mathcal{L}(\mathcal{K}_m)$ is self-adjoint and nonnegative, then*

$$\|\mathcal{B}\mathrm{h}\|\leq\|\mathcal{B}\|^{1/2}(\mathcal{B}\mathrm{h},\mathrm{h})_{\mathcal{K}_m}^{1/2}, \quad \mathrm{h}\in\mathcal{K}_m.$$

Lemma 42 *If $\tau_{k+1}=\tau_{m-(k-1)}$, $k\in\sigma_0^{m-1}$, then for the mild solution $\widetilde{\varphi}$ of (3.3), the identity holds :*

$$\int_0^T|B\widetilde{\varphi}|_H^2\,dt+\sum_{k=1}^{m}\left|D_k\widetilde{\varphi}\left(t_k^+\right)\right|_H^2=\int_0^T|B\varphi|_H^2\,dt+\sum_{k=1}^{m}\left|D_k\varphi\left(t_{m-(k-1)}\right)\right|_H^2. \tag{3.6}$$

Proof. For each $k\in\sigma_0^m$, using the change of variable $t\to(t_{m-(k-1)}-t)\dfrac{\tau_{k+1}}{\tau_{m-(k-1)}}+t_k$ we have

$$
\begin{aligned}
& \int_{t_{m-k}}^{t_{m-(k-1)}} (B\varphi_{[m-k]}(t), B\varphi_{[m-k]}(t))dt \\
= \; & \int_{t_{m-k}}^{t_{m-(k-1)}} (B\widetilde{\varphi}_{[k]}((t_{m-(k-1)}-t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}}+t_k), B\widetilde{\varphi}_{[k]}((t_{m-(k-1)}-t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}}+t_k))dt \\
= \; & \frac{-\tau_{m-(k-1)}}{\tau_{k+1}} \int_{t_{m-k}}^{t_{m-(k-1)}} (B\widetilde{\varphi}_{[k]}((t_{m-(k-1)}-t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}}+t_k), B\widetilde{\varphi}_{[k]}((t_{m-(k-1)}-t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}}+t_k)) \\
& \times \frac{-\tau_{k+1}}{\tau_{m-(k-1)}}dt \\
= \; & \frac{-\tau_{m-(k-1)}}{\tau_{k+1}} \int_{t_{k+1}}^{t_k} (B\widetilde{\varphi}_{[k]}(s), B\widetilde{\varphi}_{[k]}(s))ds \\
= \; & \int_{t_k}^{t_{k+1}} (B\widetilde{\varphi}_{[k]}(s), B\widetilde{\varphi}_{[k]}(s))ds.
\end{aligned}
$$

Summing up with respect to k, we get

$$\sum_{k=0}^{m} \int_{t_{m-k}}^{t_{m-(k-1)}} (B\varphi_{[m-k]}((t)), B\varphi_{[m-k]}(t))dt = \sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} (B\widetilde{\varphi}_{[k]}(t), B\widetilde{\varphi}_{[k]}(t))dt.$$

Thus, we obtain that

$$\int_0^T |B\widetilde{\varphi}|_H^2 \, dt = \int_0^T |B\varphi|_H^2 \, dt.$$

On the other hand, from the definition of the function $\widetilde{\varphi}$ we get

$$\varphi\left(t_{m-k}\right) = \widetilde{\varphi}\left(t_{k+1}\right), \quad k \in \sigma_0^{m-1}.$$

Also, we have

$$\varphi\left(t_{m-(k-1)}\right) = \widetilde{\varphi}\left(t_k\right), \quad k \in \sigma_1^m,$$

and

$$\widetilde{\varphi}\left(t_{m-k}\right) = \varphi\left(t_{k+1}\right), \quad k \in \sigma_0^{m-1}.$$

This implies that

$$\begin{aligned}\sum_{k=1}^{m}\left|D_k\widetilde{\varphi}\left(t_k\right)\right|_H^2 &= \sum_{k=0}^{m-1}\left\langle D_{m-k}\widetilde{\varphi}\left(t_{m-k}\right), D_{m-k}\widetilde{\varphi}\left(t_{m-k}\right)\right\rangle_H \\ &= \sum_{k=0}^{m-1}\left\langle D_{m-k}\varphi\left(t_{k+1}\right), D_{m-k}\varphi\left(t_{k+1}\right)\right\rangle_H \\ &= \sum_{l=1}^{m}\left\langle D_l\varphi\left(t_{m-(l-1)}\right), D_l\varphi\left(t_{m-(l-1)}\right)\right\rangle_H \\ &= \sum_{k=1}^{m}\left\langle D_k\varphi\left(t_{m-(k-1)}\right), D_k\varphi\left(t_{m-(k-1)}\right)\right\rangle_H \\ &= \sum_{k=1}^{m}\left|D_k\varphi\left(t_{m-(k-1)}\right)\right|_H^2\end{aligned}$$

which gives (3.6).■

Corollary 43 *If $\tau_{k+1} = \tau_{m-(k-1)}$, for $k \in \sigma_0^{m-1}$, and B, D_k are nonnegative in H, then the following holds:*

$$\begin{aligned}&\int_0^T \langle B\widetilde{\varphi}(t), \widetilde{\varphi}(t)\rangle dt + \sum_{k=1}^{k=m}\langle D_k\widetilde{\varphi}(t_k), \widetilde{\varphi}(t_k)\rangle \\ = &\int_0^T \langle B\varphi(t), \varphi(t)\rangle dt + \sum_{k=1}^{k=m}\langle D_k\varphi(t_{m-(k-1)}), \varphi(t_{m-(k-1)})\rangle.\end{aligned}$$

Proof. This follows immediately from Lemma **42** if we replace B by $B^{\frac{1}{2}}$, and D_k by $D_k^{\frac{1}{2}}$.■

Now, we state and establish the following Theorem.

Theorem 44 *Let $y^0 \in H$ be a given initial state for the system (3.1), then y^0 is null controllable at time T if and only if there is a positive constant C such that*

$$\left|\langle y^0, \tilde{\varphi}^0\rangle_H\right| \le C\left\{\int_0^T |B\varphi|_H^2\, dt + \sum_{k=1}^{m}\left|D_k\varphi\left(t_{m-(k-1)}\right)\right|_H^2\right\}^{1/2}, \ \forall\tilde{\varphi}^0 \in H, \tag{3.7}$$

where $\varphi \in \mathcal{PLC}\left([0,T];H\right)$ is the unique mild solution to (3.2) *with $\varphi\left(T\right) = \tilde{\varphi}^0$.*

Proof. It suffices to prove this Theorem in the special case $\tau_{k+1} = \tau_{m-(k-1)}$, for $k \in \sigma_0^{m-1}$, because the norm $\|||\| \doteqdot \left\{ \sum_{k=0}^{m} \frac{\tau_{m-(k-1)}}{\tau_{k+1}} \int_{t_k}^{t_{k+1}} ||_H^2 \, dt \right\}^{1/2}$ is equivalent to the usual norm in $L^2\left([0,T];H\right)$.

We shall proceed in several steps.

Step 1: Let y and $\widetilde{\varphi}$ be strong solutions to (3.1) and (3.3), respectively. Then, for $t \neq t_k$, $k \in \sigma_1^m$, we have

$$\begin{aligned} \frac{d}{dt}\langle y\left(t\right), \widetilde{\varphi}\left(t\right)\rangle &= \langle y\left(t\right), \widetilde{\varphi}^{'}\left(t\right)\rangle + \langle y^{'}\left(t\right), \widetilde{\varphi}\left(t\right)\rangle \\ &= \langle y\left(t\right), -A\widetilde{\varphi}\left(t\right)\rangle + \langle -Ay\left(t\right) + Bu\left(t\right), \widetilde{\varphi}\left(t\right)\rangle \\ &= \langle y\left(t\right), -A\widetilde{\varphi}\left(t\right)\rangle + \langle -Ay\left(t\right), \widetilde{\varphi}\left(t\right)\rangle + \langle Bu\left(t\right), \widetilde{\varphi}\left(t\right)\rangle \\ &= \langle Bu\left(t\right), \widetilde{\varphi}\left(t\right)\rangle. \end{aligned} \tag{3.8}$$

Multiplying equation (3.3_k) in (3.3) from the left by $y\left(t_{m-(k-1)}\right)$ the solution of (3.1), and multiplying equation (3.1_k) in (3.1) from the right by $\widetilde{\varphi}\left(t_k\right)$ the solution of (3.3), and finally adding memberwise we get

$$\begin{aligned} \Delta\langle y(t), \widetilde{\varphi}(t)\rangle_{|t=t_k} &= \langle y(t_k), \Delta\widetilde{\varphi}(t_k)\rangle + \langle \Delta y(t_k), \widetilde{\varphi}(t_k)\rangle \\ &= \langle y(t_k), I_k\widetilde{\varphi}(t_k)\rangle + \langle I_k y(t_k) + D_k v_k, \widetilde{\varphi}(t_k)\rangle \\ &= \langle y(t_k), I_k\widetilde{\varphi}(t_k)\rangle + \langle I_k y(t_k), \widetilde{\varphi}(t_k)\rangle + \langle D_k v_k, \widetilde{\varphi}(t_k)\rangle \\ &= \langle D_k v_k, \widetilde{\varphi}(t_k)\rangle. \end{aligned} \tag{3.9}$$

Setting $\gamma(t) = y(t)$, $\eta(t) = \widetilde{\varphi}(t)$, $\xi(t) = Bu(t)$, $\zeta(t) = \widetilde{\varphi}(t)$, $\xi_k = D_k v_k$, $\zeta_k = \widetilde{\varphi}(t_k)$, then equations (3.5), (3.8) and (3.9) give

$$\langle y(T), \widetilde{\varphi}(T)\rangle - \langle y(0), \widetilde{\varphi}(0)\rangle = \int_0^T \langle Bu(t), \widetilde{\varphi}(t)\rangle dt + \sum_{k=1}^{k=m} \langle D_k v_k, \widetilde{\varphi}(t_k)\rangle. \tag{3.10}$$

Since $\mathcal{B}$ is bounded, self-adjoint and $\mathcal{B} \geq 0$, then by density the latter identity is still valid for mild solutions y of (3.1). Identity (3.10) can be written as follows

$$\langle y(T), \widetilde{\varphi}(T)\rangle - \langle y(0), \widetilde{\varphi}(0)\rangle = \int_0^T \langle u(t), B\widetilde{\varphi}(t)\rangle dt + \sum_{k=1}^{k=m} \langle v_k, D_k\widetilde{\varphi}(t_k)\rangle \tag{3.11}$$

Next, if there is a certain $h(t) \in \mathcal{K}_m$ such that the mild solution of (3.1) with $y(0) = y^0$ satisfies $y(T) = 0$, then

$$-\langle y(0), \widetilde{\varphi}(0)\rangle = \int_0^T \langle u(t), B\widetilde{\varphi}(t)\rangle dt + \sum_{k=1}^{k=m} \langle v_k, D_k\widetilde{\varphi}(t_k)\rangle$$

and so by Cauchy-Schwarz Inequality we obtain

$$\begin{aligned} |\langle y(0), \widetilde{\varphi}(0)\rangle_H| \ &\le \ \left\{ \int_0^T \|u(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 \right\}^{1/2} \\ &\quad \times \left\{ \int_0^T \|B\widetilde{\varphi}(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|D_k\widetilde{\varphi}((t_k)\|_H^2 \right\}^{1/2} . \end{aligned} \tag{3.12}$$

Using Lemma **42**, and equation (3.12) we have

$$\begin{aligned} |\langle y(0), \widetilde{\varphi}(0)\rangle_H| \ &\le \ \left\{ \int_0^T \|u(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 \right\}^{1/2} \\ &\quad \times \left\{ \int_0^T \|B\varphi(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \left\|D_k\varphi(t_{m-(k-1)})\right\|_H^2 \right\}^{1/2} . \end{aligned}$$

Setting

$$C = \|h(t)\|_{\mathcal{K}_m} = \left\{ \int_0^T \|u(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 \right\}^{1/2}$$

we find that

$$|(\langle y(0), \widetilde{\varphi}(0)\rangle_H| \le C \left\{ \int_0^T \|B\varphi(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \left\|D_k\varphi(t_{m-(k-1)})\right\|_H^2 \right\}^{1/2} .$$

This shows the necessary condition of the Theorem.

Step 2: To prove the sufficiency we need the following result when $\mathcal{B} \ge \alpha > 0$.

Claim 1 Assume that there is $\alpha > 0$ such that

$$\left\{ \int_0^T \|Bu(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|D_k v_k\|_H^2 \right\} \ge \alpha \left\{ \int_0^T \|u(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 \right\}$$

then, for every $y^0 \in H$ there is $\varphi^0 \in H$ such that the mild solution of (3.1) with

$$h(t) = (\widetilde{\varphi}(t), \widetilde{\varphi}(t_1), .., \widetilde{\varphi}(t_k).., \widetilde{\varphi}(t_m)) \in \mathcal{K}_m \text{ and } y(0) = y^0$$

satisfies $y(T) = 0$.

To prove this Claim, we consider for every $z \in H$ the solution φ of (3.2) satisfying $\varphi(T) = z$ and the unique mild solution y to the problem

$$\begin{cases} y^{'}(t) + Ay(t) = B\widetilde{\varphi}(t), t \in (0,T) \Big\backslash \{t_k\}_{k\in\sigma_1^m} , \\ \Delta y(t_k) = I_k y(t_k) + D_k\widetilde{\varphi}(t_k), \\ y(T) = 0. \end{cases}$$

Next, we introduce a bounded linear operator $\Lambda : H \to H$ defined by

$$\Lambda z = -y(0).$$

According to formula (3.11) and the Corollary **43** we have

$$\begin{aligned} |\langle \Lambda z, z\rangle| &= |-\langle y(0), \widetilde{\varphi}(0)\rangle| = \left| \int_0^T \langle B\widetilde{\varphi}(t), \widetilde{\varphi}(t)\rangle dt + \sum_{k=1}^{k=m} \langle D_k\widetilde{\varphi}(t_k), \widetilde{\varphi}(t_k)\rangle \right| \\ &= \left| \int_0^T \langle B\varphi(t), \varphi(t)\rangle dt + \sum_{k=1}^{k=m} \langle D_k\varphi(t_{m-(k-1)}), \varphi(t_{m-(k-1)})\rangle \right| \\ &\le \varsigma \left\{ \int_0^T \|\varphi(t)\|^2 \, dt + \sum_{k=1}^{k=m} \|\varphi(t_k)\|^2 \right\}, \end{aligned}$$

where

$$\varsigma = \sup_{k\in\sigma_0^m} \{d_k\} < \infty.$$

We have

$$\int_0^T \|\varphi(t)\|^2 \, dt = \int_0^{t_1} \|\varphi(t)\|^2 \, dt + \int_{t_1}^{t_2} \|\varphi(t)\|^2 \, dt + ... + \int_{t_m}^{T} \|\varphi(t)\|^2 \, dt.$$

Since there is no impulse in the interval $[t_k, t_{k+1})$ we have

$$\|\varphi(t)\| = \left\|\varphi(t_k^+)\right\|, \text{ for every } t \in [t_k, t_{k+1}), k \in \sigma_0^m.$$

$$\left\|\varphi(t_{k+1}^-)\right\| = \left\|\varphi(t_{k-1}^+)\right\|, \quad k \in \sigma_0^m. \tag{3.13}$$

Therefore, there are $\tau_{k+1} = t_{k+1} - t_k > 0$, $k \in \sigma_0^m$ such that

$$\int_{t_k}^{t_{k+1}} \|\varphi(t)\|^2 dt \leq \rho_k \left\|\varphi(t_k^+)\right\|^2 = \tau_{k+1} \left\|I_k\varphi(t_k^-) + \varphi(t_k^-)\right\|^2, \quad k \in \sigma_1^m. \tag{3.14}$$

On the other hand, the continuity of I_k implies that

$$\left\|\varphi(t_k^+)\right\|^2 = \left\|(I_k + I)\varphi(t_k^-)\right\|^2 \leq (1 + L(I_k))^2 \left\|\varphi(t_k^-)\right\|^2, \quad k \in \sigma_1^m. \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\int_{t_k}^{t_{k+1}} \|\varphi(t)\|^2 dt \leq \tau_{k+1}(1 + L(I_k))^2 \left\|\varphi(t_k^-)\right\|^2, \quad k \in \sigma_1^m. \tag{3.16}$$

Since m is finite, and due to (3.13),(3.16), then there is a constant $0 < \mu < \infty$ such that $\langle \Lambda z, z\rangle \leq \mu \|z\|^2$, and thus, Λ is bounded.

Now, as $\mathcal{B}$ is nonnegative in $\mathcal{K}_m$, we have

$$\|\mathcal{B}\xi(t)\| \geq \alpha \left\{(\xi(t), \xi(t))_{\mathcal{K}_m}\right\}^{1/2} \quad ,$$

for all $\xi \in \mathcal{K}_m$; thus, by virtue of Lemma **41**, we have

$$\begin{aligned} &\left\{\int_0^T (Bu(t), u(t))_H dt + \sum_{k=1}^{k=m} (D_k v_k, v_k)_H\right\} \\ \geq \quad &\alpha \|\mathcal{B}\| \left\{\int_0^T \|u(t)\|_H^2 dt + \sum_{k=1}^{k=m} \|v_k\|_H^2\right\}. \end{aligned} \tag{3.17}$$

It follows from (3.11) and (3.17) and Corollary **43** that

$$\begin{aligned}
\langle \Lambda z, z\rangle &= -\langle y(0), \widetilde{\varphi}(0)\rangle = \int_0^T \langle B\varphi(t), \varphi(t)\rangle dt + \sum_{k=1}^{k=m} \langle D_k \varphi(t_{m-(k-1)}), \varphi(t_{m-(k-1)})\rangle \\
&\geq \alpha \left\| \mathcal{B} \right\| \left\{ \int_0^T \left\| \varphi(t) \right\|^2 dt + \sum_{k=1}^{k=m} \left\| \varphi(t_k) \right\|^2 \right\} \\
&\geq \alpha \left\| \mathcal{B} \right\| \int_0^{t_1} \left\| \varphi(t) \right\|^2 dt = \left\| \mathcal{B} \right\| \alpha t_1 \left\| z \right\|^2 = \theta \left\| z \right\|^2,
\end{aligned}$$

because there is no impulse before time t_1. Therefore, Λ is coercive on H.

To show that there is a bijection from H onto H, it suffices to prove that $\Lambda + I$ is a bijection from H onto H. Clearly, $\Lambda + I$ is injective since

$$\langle \Lambda z + z, z\rangle = \langle \Lambda z, z\rangle + \langle z, z\rangle \geq (\theta + 1) \left\| z \right\|^2.$$

On the other hand, let $y^0 \in H$, as the form $a(f, g) + \langle f, g\rangle = \langle \Lambda f, g\rangle + \langle f, g\rangle$ is symmetric and coercive, then, by virtue of Lax-Milgram Theorem, there is an element $f \in H$ such that

$$a(f, g) + \langle f, g\rangle = \langle y^0, g\rangle, \text{ for all } g \in H.$$

This implies that $\Lambda(H) = H$. Thus, for every $y^0 \in H$, there is a unique $z \in H$ such that $\Lambda(z) = -y^0$, which completes the proof of Claim **1**.

Step 3: Assume that $B, D_k \geq 0$, then $\mathcal{B} \geq 0$,

$$\widetilde{B}^2 = B, \widetilde{D}_k^2 = D_k.$$

We define for $\varepsilon > 0$,

$$\beta^\varepsilon \doteqdot \widetilde{B}^2 + \varepsilon I,$$

$$\delta_k^\varepsilon \doteqdot \widetilde{D}_k^2 + \varepsilon I,$$

and

$$\mathcal{B}^\varepsilon \doteqdot (\beta^\varepsilon; \delta_1^\varepsilon, .., \delta_m^\varepsilon) = (\widetilde{B}^2 + \varepsilon I; \widetilde{D}_1^2 + \varepsilon I, .., \widetilde{D}_m^2 + \varepsilon I).$$

According to Claim **1**, there is $\tilde{\varphi}^{0,\varepsilon} \in H$ such that the mild solution y_ε of (3.1) with $y_\varepsilon(0) = y^0$ satisfies $y_\varepsilon(T) = 0$; where $\mathcal{B}(h)$ has been replaced by

$$\mathcal{B}^\varepsilon(\widetilde{\varphi}(t), \widetilde{\varphi}(t_1), .., \widetilde{\varphi}(t_k).., \widetilde{\varphi}(t_m)) \in \mathcal{K}_m.$$

We obtain from (3.11) and Corollary **43**

$$-\langle y(0), \widetilde{\varphi}_\varepsilon(0)\rangle = \int_0^T \langle \beta_\varepsilon^\varepsilon \widetilde{\varphi}(t), \widetilde{\varphi}_\varepsilon(t)\rangle dt + \sum_{k=1}^{k=m} \langle \delta_k^\varepsilon \widetilde{\varphi}_\varepsilon(t_k), \widetilde{\varphi}_\varepsilon(t_k)\rangle, \tag{3.18}$$

and (3.7) gives

$$-\langle y(0), \widetilde{\varphi}_\varepsilon(0)\rangle \leq C \left\{ \int_0^T \langle \widetilde{B}^2 \varphi_\varepsilon(t), \varphi_\varepsilon(t)\rangle dt + \sum_{k=1}^{k=m} \langle \widetilde{D}_k^2 \varphi_\varepsilon(t_{m-(k-1)}), \varphi_\varepsilon(t_{m-(k-1)})\rangle \right\}^{1/2}. \tag{3.19}$$

Whence,

$$-\langle y(0), \widetilde{\varphi}_\varepsilon(0)\rangle \leq C \left\{ \int_0^T \langle \beta^\varepsilon \varphi_\varepsilon(t), \varphi_\varepsilon(t)\rangle dt + \sum_{k=1}^{k=m} \langle \delta_k^\varepsilon \varphi_\varepsilon(t_{m-(k-1)}), \varphi_\varepsilon(t_{m-(k-1)})\rangle \right\}^{1/2}. \tag{3.20}$$

It follows at once from (3.18) (3.19) and (3.20) that

$$\begin{aligned}
&\varepsilon \left\{ \textstyle\int_0^T \|\varphi_\varepsilon(t)\|^2 dt + \sum\limits_{k=1}^{k=m} \|\varphi_\varepsilon(t_k)\|^2 \right\} \\
&+ \textstyle\int_0^T \langle \widetilde{B}\varphi_\varepsilon(t), \widetilde{B}\varphi_\varepsilon(t)\rangle dt + \sum\limits_{k=1}^{k=m} \langle \widetilde{D}_k \varphi_\varepsilon(t_{m-(k-1)}), \widetilde{D}_k \varphi_\varepsilon(t_{m-(k-1)})\rangle \\
&= \textstyle\int_0^T (\beta^\varepsilon \varphi_\varepsilon(t), \varphi_\varepsilon(t))dt + \sum\limits_{k=1}^{k=m} \langle \delta_k^\varepsilon \varphi_\varepsilon(t_{m-(k-1)}), \varphi_\varepsilon(t_{m-(k-1)})) \leq C^2.
\end{aligned} \tag{3.21}$$

Step 4: According to the estimate (3.20) the family

$$\begin{aligned}
b_\varepsilon &= \mathcal{B}^\varepsilon(\widetilde{\varphi}_\varepsilon(t); \widetilde{\varphi}_\varepsilon(t_1)..., \widetilde{\varphi}_\varepsilon(t_m) \\
&= (\widetilde{B}_\varepsilon^2 \widetilde{\varphi}(t); \widetilde{D}_1^2 \widetilde{\varphi}_\varepsilon(t_1)..., \widetilde{D}_m \widetilde{\varphi}_\varepsilon(t_m)) + \varepsilon(\widetilde{\varphi}_\varepsilon(t); \widetilde{\varphi}_\varepsilon(t_1)..., \widetilde{\varphi}_\varepsilon(t_m))
\end{aligned}$$

is contained in a bounded subset $\mathcal{K}_m$.

Thus, both of the families

$$\sqrt{\varepsilon}(\widetilde{\varphi}_\varepsilon(t);\widetilde{\varphi}_\varepsilon(t_1)...,\widetilde{\varphi}_\varepsilon(t_m)) \text{ and } (B\widetilde{\varphi}_\varepsilon(t);D_1\widetilde{\varphi}_\varepsilon(t_1)...,D_m\widetilde{\varphi}_\varepsilon(t_m))$$

are bounded in $\mathcal{K}_m$. Therefore, we may extract a subfamily, say

$$(B\widetilde{\varphi}_\varepsilon(t);D_1\widetilde{\varphi}_\varepsilon(t_1)...,D_m\widetilde{\varphi}_\varepsilon(t_m)) \rightharpoonup h, \text{ weakly in } \mathcal{K}_m.$$

Then clearly

$$(\widetilde{B}^2\widetilde{\varphi}_\varepsilon(t);\widetilde{D}_1^2\widetilde{\varphi}_\varepsilon(t_1)...,\widetilde{D}_m^2\widetilde{\varphi}_\varepsilon(t_m))+\varepsilon(\widetilde{\varphi}_\varepsilon(t);\widetilde{\varphi}_\varepsilon(t_1)...,\widetilde{\varphi}_\varepsilon(t_m)) \rightharpoonup \mathcal{B}h, \text{ weakly in } \mathcal{K}_m.$$

Step 5: Taking the limit as $\varepsilon \to 0$, we see that the solution y of (1) with initial condition $y(0)=y^0$, h being as in step 4 satisfies $y(T)=0$. This completes the proof of Theorem **44**.■

As an immediate application of the previous Theorem we give the following example.

Example. One dimensional impulsive Schrödinger equation :

We consider the problem

$$\begin{cases} \dfrac{\partial y\,(t,x)}{\partial t}+i\dfrac{\partial^2 y}{\partial x^2}(t,x)=\chi_{\omega_0}u\,(t,x)\,, \quad t\in(0,T)\setminus\{t_k\}_{k\in\sigma_1^m}\,,\ x\in\Omega=(0,2\pi), \\ y(t,0)=y(t,2\pi)=0, \\ y\,(0,x)=y^0, \\ \Delta y\,(t_k,x)=i\alpha_k y\,(t_k,x)+\chi_{\omega_k}v_k(x),\ \ k\in\sigma_1^m, \end{cases} \tag{3.22}$$

where

$$t_{k+1}-t_k>2\pi, \quad \omega_k=(a_1^k,a_2^k)\subset\Omega,\ k\in\sigma_0^m, \quad \{\alpha_k\}_{k\in\sigma_1^m}\subset\mathbb{R}^+.$$

Let

$$H=L^2(\Omega,\mathbb{C}), Aw(x)=i\frac{\partial^2 w}{\partial x^2}(x), \quad D(A)=\left\{w\in H,\frac{\partial^2 w}{\partial x^2}\in H, w(0)=w(\pi)=0\right\},$$

and $I_k w(x)=i\alpha_k w(x)$ and the control operator is given by $B=\chi_{\omega_0}$, $D_k=\chi_{\omega_k}$,then the system (3.22) becomes an abstract formulation of (3.1). As a consequence of Theorem **44**, the initial state $y^0\in L^2(\Omega,\mathbb{C})=H$ of the solution of (3.22) is null-controllable at $t=T$, if and only if, there is $C > 0$ such that

$$\begin{aligned}&\left|\int_\Omega y^0(x)\widetilde{\varphi}^0(x)dx\right| \qquad (3.23)\\ \leq\ & C\left\{\int_0^T\int_{\omega_0}|\varphi|^2(t,x)dxdt+\sum_{k=1}^m\int_{\omega_k}|\varphi|^2(t_{m-(k-1)},x)\right\}^{\frac{1}{2}},\ \forall\widetilde{\varphi}^0\in L^2(\Omega,\mathbb{C}),\end{aligned}$$

where $\widetilde{\varphi}^0(x)=\varphi(T,x)$ and φ is the mild solution of

$$\begin{cases}\dfrac{\partial\varphi(t,x)}{\partial t}+i\dfrac{\partial^2\varphi(t,x)}{\partial x^2}=0, \quad t\in(0,T)\setminus\{t_k\}_{k\in\sigma_1^m},\ x\in\Omega,\\ \varphi(t,0)=\varphi(t,2\pi)=0,\\ \varphi(0,x)=\varphi^0(x),\ x\in\Omega,\\ \Delta\varphi(t_k,x)=i\alpha_k\varphi(t_k,x),\ x\in\Omega,\ k\in\sigma_1^m.\end{cases}$$

Here φ is given by

$$\varphi(t)=\begin{cases}\varphi_{[0]}(t) & \text{if} \quad t\in[t_0,t_1)\\ \varphi_{[k]}(t) & \text{if} \quad t\in\left[t_k,t_{k+1})\right)\\ \varphi_{[m]}(t) & \text{if} \quad t\in[t_m,T],\end{cases}$$

where $\varphi_{[k]}(t)$ is a solution of the classical Schrödinger equation

$$\begin{cases}\dfrac{\partial\varphi_{[k]}(t,x)}{\partial t}+i\dfrac{\partial^2\varphi_{[k]}}{\partial x^2}(t,x)=\chi_{\omega_0}u(t,x), \quad t\in(t_0,t_1),\ x\in\Omega=(0,2\pi),\\ \varphi_{[k]}(t,0)=\varphi_{[k]}(t,2\pi)=0,\\ \varphi_{[0]}(t_0,x)=\varphi^0(x),\ x\in\Omega,\end{cases}$$

and

$$\begin{cases}\dfrac{\partial\varphi_{[k]}(t,x)}{\partial t}+i\dfrac{\partial^2\varphi_{[k]}}{\partial x^2}(t,x)=\chi_{\omega_0}u(t,x), \quad t\in(t_k,t_{k+1}),\ x\in\Omega=(0,2\pi),\\ \varphi_{[k]}(t,0)=\varphi_{[k]}(t,2\pi)=0,\\ \varphi_{[k]}(t_k,x)=(1+i\alpha_k)\varphi_{[k-1]}(t_k,x),\ x\in\Omega,\ k\in\sigma_1^m.\end{cases}$$

Then a standard application of a variant to Ingham's Inequality [53] shows that

$$\int_{t_k}^{t_{k+1}}\int_{w_0}\left|\varphi_{[k]}\right|(t,x)dtdx\geq c(\tau_k,w_0)\int_\Omega\left|\varphi_{[k]}\right|(t_k^+,x)dx,$$

for some positive constants $c(\tau_k, w_0) > 0$. Summing up we get

$$\begin{aligned}\sum_{k=0}^{m}\int_{t_k}^{t_{k+1}}\int_{w_0}\left|\varphi_{[k]}\right|(t,x)dtdx &= \int_0^T\int_{\omega_0}\left|\varphi\right|^2(t,x)dxdt \\ &\geq c_1\sum_{k=0}^{m}\int_\Omega\left|\varphi_{[k]}\right|(t_k^+,x)dx \geq c_1\sum_{k=1}^{m}\int_\Omega\left|\varphi_{[k]}\right|(t_k^+,x)dx,\end{aligned}$$

where $c_1 = \min\limits_{k\in\sigma_0^m} c(\tau_k, w_0) > 0$.

On the other hand, there is a positive constant $c_2 > 0$ such that

$$\sum_{k=1}^{m}\int_{\omega_k}\left|\varphi\right|^2(t_{m-(k-1)},x) \geq c_2\sum_{k=1}^{m}\int_\Omega\left|\varphi_{[k]}\right|^2(t_k^+,x)dx.$$

It follows that

$$\begin{aligned}\int_0^T\int_{\omega_0}\left|\varphi\right|^2(t,x)dxdt + \sum_{k=1}^{m}\int_{\omega_k}\left|\varphi\right|^2(t_{m-(k-1)},x) &\geq (c_1+c_2)\sum_{k=1}^{m}\int_\Omega\left|\varphi_{[k]}\right|^2(t_k^+,x)dx \\ &\geq (c_1+c_2)\int_\Omega\left|\varphi_{[m]}\right|^2(t_m^+,x)dx \\ &= (c_1+c_2)\int_\Omega\left|\varphi\right|^2(T,x)dx.\end{aligned}$$

Now, since

$$\widetilde{\varphi}^0(x) = \widetilde{\varphi}(0,x) = \varphi(T,x),$$

then,

$$\int_0^T\int_{\omega_0}\left|\varphi\right|^2(t,x)dxdt + \sum_{k=1}^{m}\int_{\omega_k}\left|\varphi\right|^2(t_{m-(k-1)},x) \geq m(c_1+c_2)\int_\Omega\left|\widetilde{\varphi}^0\right|^2(x)dx,$$

from which we get

$$\int_\Omega\left|\widetilde{\varphi}^0\right|^2(x)dx \leq \frac{1}{m(c_1+c_2)}\left(\int_0^T\int_{\omega_0}\left|\varphi\right|^2(t,x)dxdt + \sum_{k=1}^{m}\int_{\omega_k}\left|\varphi\right|^2(t_{m-(k-1)},x)\right).$$

We conclude by Cauchy-Schwarz inequality that

$$\begin{aligned}
\left|\int_\Omega y^0(x)\widetilde{\varphi}^0(x)dx\right| &\leq \left\{\int_\Omega \left|y^0\right|^2(x)dx\int_\Omega \left|\widetilde{\varphi}^0\right|^2(x)dx\right\}^{1/2} \\
&\leq \left\{\frac{\int_\Omega \left|y^0\right|^2(x)dx}{m(c_1+c_2)}\right\}^{1/2}\left(\int_0^T\int_{\omega_0}|\varphi|^2(t,x)dxdt\right. \\
&\quad \left.+\sum_{k=1}^m\int_{\omega_k}|\varphi|^2(t_{m-(k-1)},x)dx\right)^{1/2},
\end{aligned}$$

which establishes the necessary and sufficient condition of null controllability stated in Theorem **44.■**

We conclude by a special case when our initial state is a eigensolution of the following linear operator $\Gamma : H \to H$ defined by

$$\Gamma(\psi) = \int_0^T X^{-1}(s)B^2X(s)\psi ds + \sum_{k=1}^{k=m} X^{-1}(t_k)D_k^2X(t_k)\psi.$$

We have the following result of null-controllability.

Proposition 45 *Let $\lambda > 0$ be an eigenvalue of Γ with eigenvector $\psi \in H$. Then, the solution y to the problem*

$$\begin{cases} y'(t) + Ay(t) = -\dfrac{1}{\lambda}B^2(X(t)\psi), & t \in (0,T)\Big\backslash\{t_k\}_{k\in\sigma_1^m}, \\ \Delta y(t_k) = I_k y(t_k) - \dfrac{1}{\lambda}D_k^2(X(t_k)\psi), & k \in \sigma_1^m \\ y(0) = \psi, \end{cases} \tag{3.24}$$

satisfies

$$y(T) = 0.$$

Proof. Write system (3.24) into the form

$$\begin{cases} y'(t) + Ay(t) = -\dfrac{1}{\lambda}B^2(X(t)\psi), & t \in (0,T)\Big\backslash\{t_k\}_{k\in\sigma_1^m}, \\ y(t_k^+) = \mathcal{I}_k y(t_k) - \dfrac{1}{\lambda}D_k^2(X(t_k)\psi), & k \in \sigma_1^m \\ y(0) = \psi. \end{cases}$$

Therefore, this impulsive problem has a solution which can be represented explicitly as follows

$$y(t) = X(t)\psi + \int_0^t G(t,s)\left[-\frac{1}{\lambda}B^2(X(s)\psi\right] ds + \sum_{0<t_k\leq t} G(t,t_k)\left[-\frac{1}{\lambda}D_k^2 X(t_k)\psi\right],$$

where the evolution operator $G(t,s)$ is given by

$$G(t,s) = X(t)X^{-1}(s).$$

On the other hand, the system (3.24) yields

$$\begin{aligned} y(T) &= X(T)\psi + \int_0^T G(T,s)\left\{-\frac{1}{\lambda}B^2(X(s)\psi\right\} ds \\ &\quad + \sum_{0<t_k\leq T} G(T,t_k)\left\{-\frac{1}{\lambda}D_k^2 X(t_k)\psi\right\} \\ &= X(T)\left[\psi + \int_0^T X^{-1}(T)G(T,s)\left\{-\frac{1}{\lambda}B^2(X(s)\psi\right\} ds\right. \\ &\quad \left. -\frac{1}{\lambda}\sum_{0<t_k\leq T} X^{-1}(T)G(T,t_k)\left\{D_k^2 X(t_k)\psi\right\}\right] \\ &= X(T)\left[\psi + \int_0^T X^{-1}(s)\left\{-\frac{1}{\lambda}B^2(X(s)\psi\right\} ds\right. \\ &\quad \left. -\frac{1}{\lambda}\sum_{0<t_k\leq T} X^{-1}(t_k)\left\{D_k^2 X(t_k)\psi\right\}\right] \\ &= X(T))\left[\psi - \frac{1}{\lambda}\Gamma(\psi)\right] = 0. \end{aligned}$$

This shows that the initial state ψ is null-controllable at time T with control

$$h(t) = \left(u(t), \{v_k\}_{k\in\sigma_1^m}\right) = \left(-\frac{1}{\lambda}X(t)\psi, \left\{-\frac{1}{\lambda}X(t_k)\psi\right\}_{k\in\sigma_1^m}\right).$$

which completes the proof of the Proposition.■

3.2 Description of the HUM method

In this section, we shall apply Hilbert's Uniqueness Method (**HUM**) to derive an exact controllability result.

Consider the impulsive equation

$$\begin{cases} y^{'}(t) + Ay(t) = B^2 u(t), \quad t \in (0,T) \setminus \{t_k\}_{k\in\sigma_1^m}, \\ y(0) = y^0, \\ \Delta y(t_k) = I_k y(t_k) + D_k^2 v_k, \ \ k \in \sigma_1^m, \qquad\qquad 3.25_k \end{cases} \tag{3.25}$$

where $A : V \to V^*$, $B^2 : H \to V^*$ are bounded operators and for each $k \in \sigma_1^m$, $I_k : H \to H$, H is a suitable Hilbert space such that V is a dense subspace of H having the structure of a reflexive Banach space, with continuous embedding $V \hookrightarrow H \hookrightarrow V^*$ and $V \hookrightarrow H$ is compact, V^* being the topological dual space of V.

Definition 46 *Let* $T > 0$ *be given, we say that the system (3.25) is exactly controllable at time* T, *if for any* $y^0 \in H$, *there exists a control function* $\mathrm{h} = (u(t), \{v_k\}_{k=1}^m) \in \mathcal{K}_m(H)$ *such that*

$$y(T) = 0.$$

Moreover, we set the following assumptions:

(H1) $A^* = -A$,

(H2) $I_k^* = I_k$, for every $k \in \sigma_1^m$, and for each $k \in \sigma_1^m$, the operators

$$\mathcal{I}_k = (I_k + I), \quad \left[I - (I + I_k)^{-1} I_k\right]$$

are invertible, with

$$(I + I_k^*)^{-1} = I - (I + I_k^*)^{-1} I_k^*.$$

(H3) $B^* = B \geq 0$, and there is $d_0 > 0$ such that

$$(Bu, u)_H \leq d_0 \|u\|_H^2, \ \text{ for all } u \in H,$$

(H4) For every $k \in \sigma_1^m$, $D_k^* = D_k \geq 0$, and

$$D_k^2 \left(\left[I - (I + I_k)^{-1} I_k \right]^{-1} \right)^* = \left(\left[I - (I + I_k)^{-1} I_k \right]^{-1} \right)^* D_k^2, \quad \forall k \in \sigma_1^m,$$

and for each $k \in \sigma_1^m$, there is $d_k > 0$, such that

$$(D_k u, u)_H \leq d_k \|u\|_H^2, \text{ for all } u \in H,$$

Theorem 47 *Let $y^0 \in H$ be a given initial state for the system (3.25), then y^0 is null controllable at time T if and only if there is a positive constant C such that*

$$\left| \langle y^0, \varphi^0 \rangle_H \right| \leq C \left\{ \int_0^T |B\varphi|_H^2 \, dt + \sum_{k=1}^m \left| D_k \varphi \left(t_{m-(k-1)} \right) \right|_H^2 \right\}^{1/2}, \ \forall \varphi^0 \in H, \tag{3.26}$$

where φ is the mild solution of the impulsive homogeous system (3.27) below.

Proof. The proof is similar to that of Theorem **44**.■

Theorem 48 *Let $T > 0$ and let be given the initial condition y^0, then there exists a control* $\mathrm{h} = \left(u(t), \{v_k\}_{k=1}^m\right) \in \mathcal{K}_m(H)$ *such that system (3.25) is exactly controllable.*

Proof.

Step 1. We first begin with an initial condition $\phi^0 \in V_0$ and we consider the homogeneous system

$$\begin{cases} \varphi'(t) + A\varphi(t) = 0, \quad t \in (0,T) \setminus \{t_k\}_{k \in \sigma_1^m}, \\ \varphi(0) = \varphi^0, \\ \Delta\varphi(t_k) = I_k \varphi(t_k), \ k \in \sigma_1^m. \qquad 3.27_k \end{cases} \tag{3.27}$$

We recall that problem (3.27) admits a unique solution φ.

Step 2. We solve the backward problem

$$\begin{cases} \psi'(t) + A\psi(t) = B^2\varphi(t), \quad t \in (0,T) \setminus \{t_k\}_{k \in \sigma_1^m}, \\ \psi(T) = 0, \\ \Delta\psi(t_k) = I_k \psi(t_k) + D_k^2 \varphi(t_k), \ k \in \sigma_1^m. \qquad 3.28_k \end{cases} \tag{3.28}$$

System (3.28) is nonhomogeneous problem with backward character. This doesn't change the well-posedness character of the system which has a unique solution ψ.

We define then a linear operator Λ that associates to φ^0 the vector given by

$$\Lambda\varphi^0 = -\psi(0)$$

The operator Λ is well defined by virtue of the regularity of

$$\psi \in \mathcal{PLC}\left([0,T]\,;H\right) \cap C^1\left([0,T] \setminus \{t_k\}_{k\in\sigma_1^m}\,;H\right).$$

Step 3.

Now, we consider the initial data $\zeta^0 \in V_0$ and $\zeta = \zeta(t)$ the solution of the adjoint problem associated with (3.27).

$$\begin{cases} \zeta^{'}(t) + A\zeta(t) = 0, \quad t \in (0,T) \setminus \{t_k\}_{k\in\sigma_1^m}, \\ \zeta(0) = \zeta^0, \\ \Delta\zeta(t_k) = -I_k^*(I + I_k^*)^{-1}\zeta\left(t_k^-\right), \ k \in \sigma_1^m. \quad 3.29_k \end{cases} \tag{3.29}$$

Let ψ and ζ be strong solutions to (3.28) and (3.29), respectively. Then, for $t \neq t_k$, $k \in \sigma_1^m$, we have

$$\begin{aligned} \frac{d}{dt}\langle\psi(t), \zeta(t)\rangle &= \langle\psi(t), \zeta^{'}(t)\rangle + \langle\psi^{'}(t), \zeta(t)\rangle \\ &= \langle\psi(t), -A\zeta(t)\rangle + \langle -A\psi(t) + B^2u(t), \zeta(t)\rangle \end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}\langle \psi\left(t\right), \zeta\left(t\right)\rangle &= \langle \psi\left(t\right), -A\zeta\left(t\right)\rangle + \langle -A\psi\left(t\right), \zeta\left(t\right)\rangle + \langle B^2 u\left(t\right), \zeta\left(t\right)\rangle \\
&= \langle B^2 u\left(t\right), \zeta\left(t\right)\rangle.
\end{aligned}
$$

Multiplying equation $(3.29)_k$ in (3.29), from the left, by $\psi\left(t_k\right)$ the solution of (3.28), and multiplying equation $(3.28)_k$ in (3.28), from the right, by $\psi\left(t_k\right)$ the solution of (3.28), and finally adding memberwise we get

$$\Delta\zeta(t_k) = \zeta(t_k^+) - \zeta(t_k^-) = -I_k^*(I + I_k^*)^{-1}\zeta(t_k^-)$$

$$\zeta(t_k^+) = \left[-I_k^*(I + I_k^*)^{-1} + I\right]\zeta(t_k^-)$$

$$
\begin{aligned}
& \Delta\langle\psi(t_k), \zeta(t_k)\rangle_H \\
= \; & \langle\psi(t_k^+), \zeta(t_k^+)\rangle_H - \langle\psi(t_k^-), \zeta(t_k^-)\rangle_H \\
= \; & \langle\psi(t_k^+) - \psi(t_k^-) + \psi(t_k^-), \zeta(t_k^+)\rangle_H - \langle\psi(t_k^-), \zeta(t_k^-) - \zeta(t_k^+) + \zeta(t_k^+)\rangle_H \\
= \; & \langle\psi(t_k^+) - \psi(t_k^-) + \psi(t_k^-), \zeta(t_k^+)\rangle_H - \langle\psi(t_k^-), \zeta(t_k^-) - \zeta(t_k^+) + \zeta(t_k^+)\rangle_H \\
= \; & \langle\Delta\psi(t_k), \zeta(t_k^+)\rangle_H + \langle\psi(t_k^-), \zeta(t_k^+)\rangle_H - \langle\psi(t_k^-), -\Delta\zeta(t_k)\rangle_H - \langle\psi(t_k^-), \zeta(t_k^+)\rangle_H \\
= \; & \langle\Delta\psi(t_k), \zeta(t_k^+)\rangle_H + \langle\psi(t_k^-), \Delta\zeta(t_k)\rangle_H \\
= \; & \langle I_k\psi(t_k) + D_k^2 v_k, \zeta(t_k^+)\rangle_H + \langle\psi(t_k^-), \Delta\zeta(t_k)\rangle_H \\
= \; & \langle I_k\psi(t_k) + D_k^2 v_k, \left[-I_k^*(I + I_k^*)^{-1} + I\right]\zeta(t_k^-)\rangle_H + \langle\psi(t_k), -I_k^*(I + I_k^*)^{-1}\zeta(t_k^-)\rangle_H \qquad (3.30)
\end{aligned}
$$

Using the hypothesis $(I + I_k^*)^{-1} = \left[I - (I + I_k^*)^{-1} I_k^*\right]$, $I_k^* = I_k$, and equation (3.30) we infer that :

$$
\begin{aligned}
& \Delta\langle\psi(t_k), \zeta(t_k)\rangle_H \\
= \; & \langle I_k\psi(t_k) + D_k^2 v_k, \left[-I_k^*(I + I_k^*)^{-1} + I\right]\zeta(t_k^-)\rangle_H + \langle\psi(t_k), -I_k^*(I + I_k^*)^{-1}\zeta(t_k^-)\rangle_H
\end{aligned}
$$

$$\begin{aligned}
&\Delta\langle\psi(t_k),\zeta(t_k)\rangle_H \\
= \;& \langle I_k\psi(t_k) + D_k^2 v_k, \left[-I_k^*(I+I_k^*)^{-1} + I\right]\zeta(t_k^-)\rangle_H + \langle\psi(t_k), -I_k^*(\left[I - (I+I_k^*)^{-1}I_k^*\right]\zeta(t_k^-)\rangle_H \\
= \;& \langle I_k\psi(t_k) + D_k^2 v_k, \left[-I_k(I+I_k)^{-1} + I\right]\zeta(t_k^-)\rangle_H + \langle\psi(t_k), -I_k(\left[I - (I+I_k)^{-1}I_k\right]\zeta(t_k^-)\rangle_H \\
= \;& \langle\psi(t_k), I_k\left[I - I_k(I+I_k)^{-1}\right]\zeta(t_k^-)\rangle_H \\
& +\langle\psi(t_k), -I_k(\left[I - (I+I_k)^{-1}I_k\right]\zeta(t_k^-)\rangle_H \\
& +\langle D_k^2 v_k, \left[I - I_k(I+I_k)^{-1}\right]\zeta(t_k)\rangle_H \\
= \;& \langle\left[I - I_k(I+I_k)^{-1}\right]^* D_k^2 v_k, \zeta(t_k)\rangle_H.
\end{aligned}$$

Moreover, the fact that

$$\left[I - I_k(I+I_k)^{-1}\right]^* D_k^2 = D_k^2\left[I - I_k(I+I_k)^{-1}\right]^*$$

implies that

$$\Delta\langle\psi(t_k),\zeta(t_k)\rangle_H = \langle\left[I - I_k(I+I_k)^{-1}\right]^* D_k^2 v_k, \zeta(t_k)\rangle_H.$$

Setting

$$u(t) = \zeta(t)$$

$$v_k = \left(\left[I - I_k(I+I_k)^{-1}\right]^*\right)^{-1}\zeta(t_k)$$

$$\langle\psi(T),\zeta(T)\rangle - \langle\psi(0),\zeta(0)\rangle = \int_0^T\langle B^2u(t),\zeta(t)\rangle dt + \sum_{k=1}^{k=m}\langle D_k^2 v_k, \zeta(t_k)\rangle,$$

we get

$$\left\langle\Lambda\varphi^0,\zeta^0\right\rangle_H = \left\langle -\psi(0),\zeta^0\right\rangle_H. \tag{3.31}$$

So that, in particular, we have

$$\left\langle\Lambda\varphi^0,\varphi^0\right\rangle_H = \left\{\int_0^T |B\varphi|_H^2\,dt + \sum_{k=1}^m |D_k\varphi(t_k)|_H^2\right\}^{1/2}.$$

We introduce the semi-norm

$$\left\|\varphi^0\right\|_F = \left\{ \int_0^T |B\varphi|_H^2 \, dt + \sum_{k=1}^{m} |D_k\varphi\left(t_k\right)|_H^2 \right\}^{1/2}, \ \forall \varphi^0 \in V_0. \tag{3.32}$$

We point out that $\|.\|_F$ is a norm in V_0.

Under the uniqueness result, we define the Hilbert space F completion of V_0 with respect to the norm $\|.\|_F$. In view of (3.29) and (3.32) we have

$$\left\langle \Lambda\varphi^0, \zeta^0 \right\rangle_H = \left\langle \varphi^0, \zeta^0 \right\rangle_F, \quad \forall \varphi^0, \zeta^0 \in V_0, \tag{3.33}$$

where $(.,.)_F$ denotes the inner product associated with the norm $\|.\|_F$. Therefore,

$$\left| \left\langle \Lambda\varphi^0, \zeta^0 \right\rangle_H \right| = \left\langle \varphi^0, \zeta^0 \right\rangle_F \leq \left\|\varphi^0\right\|_F \left\|\zeta^0\right\|_F, \quad \forall \varphi^0, \zeta^0 \in V_0. \tag{3.34}$$

Inequality (3.34) allows us to extend, in a unique manner, Λ into a continuous linear operator of F into the dual space $F^{'}$

$$\Lambda : F \rightarrow F^{'}.$$

From (3.33) we infer

$$\left| \left\langle \Lambda\varphi^0, \zeta^0 \right\rangle_H \right| = \left\langle \varphi^0, \zeta^0 \right\rangle_F, \quad \forall \varphi^0, \zeta^0 \in F,$$

which implies that

$$\Lambda = \Lambda^\star,$$

where $\Lambda^\star$ denotes the adjoint operator of Λ. Thus, Λ is an isomorphism from F on $F^{'}$.

Next, the equation

$$\Lambda\varphi^0 = -y^0$$

has a unique solution $\varphi^0 \in F$ for any initial data y^0 such that

$$-y^0 \in F^{'}.$$

We choose the control function $h = (u(t), \{v_k\}_{k=1}^m)$ by

$$h = (u(t), \{v_k\}_{k=1}^m) = (\varphi(t), \{\varphi(t_k)\}_{k\in\sigma_1^m}),$$

where φ denotes the solution of (3.27) corresponding to the data φ^0 satisfying (3.27). Hence, according to the uniqueness of the solution of problem (3.25) we have

$$y(t;h) = \psi,$$

where ψ is the solution of (3.28) associated with φ. Finally, by definition of ψ, we see that $y = y(t;h)$ satisfies

$$y(T) = 0.\blacksquare$$

Remark 4 *In general, there is an infinity of controls* h$(t) \in \mathcal{K}_m(H)$ *for the given implsive evolution system (3.1).*

An important question to study is the exact controllability of impulsive evolution system (3.1) subject to some control on the point $\{t_k\}_{k\in\sigma_1^m}$ only.

Consider the following impulsive control system with control u :

$$\begin{cases} y'(t) + Ay(t) = Bu(t), & t \in (0,T) \Big\backslash \{t_k\}_{k\in\sigma_1^m} , \\ y(0) = y^0 & \\ \Delta y(t_k) = I_k y(t_k), & k \in \sigma_1^m \end{cases} \tag{3.35}$$

we have the following result

Proposition 49 *In addition to the hypotheses (**H1**)-(**H4**) above, assume that*

$$\mathcal{B}_c^* L \in \mathcal{L}\left(L^2\left((0,T)\setminus\{t_k\}_{k\in\sigma_1^m}; H\right)\right),$$

then the control operator of (3.35)

$$L : L^2\left((0,T)\setminus\{t_k\}_{k\in\sigma_1^m}; H\right) \to \mathcal{C}((0,T)\setminus\{t_k\}_{k\in\sigma_1^m}; H).$$

is continuous.

Proof: Let $u \in L^2\left((0,T)\setminus\{t_k\}_{k\in\sigma_1^m};H\right)$. Set

$$x(t)=\int_0^t G(0,s)\,Bu(s)ds.$$

In view of condition $\mathcal{B}_c^* L \in \mathcal{L}\left(L^2\left((0,T)\setminus\{t_k\}_{k\in\sigma_1^m};H\right)\right)$, we have

$$\begin{array}{rcl}
C_T\left\|u\right\|_{L^2\left((0,T)\setminus\{t_k\}_{k\in\sigma_1^m};H\right)} & \geq & \int_0^T ((\mathcal{B}_c^* Lu)(t),u(t))_H dt \\
& = & \int_0^T\left(\int_0^t G(t,s)\,Bu(s)ds, Bu(t)\right)_H dt \\
& = & \int_0^T\left(\int_0^t G(t,0)\,G(0,s)\,Bu(s)ds, Bu(t)\right)_H dt \\
& = & \int_0^T\left(\int_0^t G(0,s)\,Bu(s)ds, G(0,t)\,Bu(t)\right)_H dt \\
& = & \int_0^T\left(x(t),\dfrac{d}{dt}x(t)\right)_H \\
& = & \frac{1}{2}\alpha\left\|\int_0^T G(0,s)\,Bu(s)ds\right\|_H^2 \\
& = & \frac{1}{2}\alpha\left\|G(0,T)\int_0^T G(T,s)\,Bu(s)ds\right\|_H^2 \\
& \geq & c_T\left\|\int_0^T G(T,s)\,Bu(s)ds\right\| = c_T\left\|L_T u\right\|_H^2.
\end{array} \tag{3.36}$$

Then (3.36) says that

$$L_T : L^2\left((0,T)\setminus\{t_k\}_{k\in\sigma_1^m};H\right)\to H$$

is continuous, which implies that

$$L_T : L^2\left((0,T)\setminus\{t_k\}_{k\in\sigma_1^m};H\right)\to \mathcal{C}((0,T)\setminus\{t_k\}_{k\in\sigma_1^m};H)$$

is continuous.■

Chapter 4

Exact Controllability of a Second Order Impulsive Evolution Equation

4.1 Null Controllability

In this chapter, we generalize the foregoing results of null controllability for an impulsive evolution equation of second order.

Let us fix two real separable Hilbert spaces H, V with respective inner products $\langle . \rangle_H$, $\langle . \rangle_V$ and such that V is densely and continuously embedded into H. Identifying H with its dual $H^{'}$ we have the standard diagram:

$$V \hookrightarrow H = H^{'} \hookrightarrow V^{'}.$$

We shall represent a pair of functions by $\left(f^0, f^1\right)$ rather than the symbol $\langle f^0, f^1 \rangle_{H \times H}$ which will represent indifferently either the $H \times H$ inner product of $f^0 \in H$ and $f^1 \in H$ or the duality product $\langle f^0, f^1 \rangle_{V \times V'}$ when $f^0 \in V$ and $f^1 \in V^{'}$. We shall consider the following Hilbert space $\mathcal{H}$ defined by

$$\mathcal{H} \doteqdot H \times V$$

and its dual

$$\mathcal{H}^{'} = H \times V^{'}$$

and the space $\tilde{\mathcal{H}}$ given by

$$\tilde{\mathcal{H}} \doteq V \times H$$

and its dual

$$\tilde{\mathcal{H}}^{'} = V^{'} \times H.$$

For an invertible matrix $\mathcal{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ we define the norm of the space $\mathcal{H}$ in the usual way,

$$\left\| (x, \bar{x}) \right\|_{\tilde{\mathcal{H}}} = \left\| \mathcal{M} \left(x, \bar{x} \right) \right\|_{\mathcal{H}}.$$

On the other hand, the norm of the dual space $\tilde{\mathcal{H}}^{'}$ is the following

$$\left\| (x, \bar{x}) \right\|_{\tilde{\mathcal{H}}'} = \left\| \left(\mathcal{M}^{T} \right)^{-1} (x, \bar{x}) \right\|_{\mathcal{H}'}.$$

Here, $\mathcal{M}^{T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the transpose matrix of $\mathcal{M}$. For convenience we present the following diagram that relates the function spaces $\mathcal{H}, \mathcal{H}^{'}, \tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}^{'}$:

$$\begin{tikzcd}
\mathcal{H} & \mathcal{H}^{'} \arrow[l, "\tau"'] \arrow[d, "\mathcal{M}^{T}"] \\
\tilde{\mathcal{H}} \arrow[u, "\mathcal{M}"] & \tilde{\mathcal{H}}^{'} \arrow[l, "\tilde{\tau}"]
\end{tikzcd}$$

where τ and $\tilde{\tau}$ are the Riesz canonical isometries defined as follows

$$\left\langle (y, \bar{y}), (x, \bar{x}) \right\rangle_{\mathcal{H}' \times \mathcal{H}} = \left\langle \tau (y, \bar{y}), (x, \bar{x}) \right\rangle_{\mathcal{H}}, \quad \{(y, \bar{y}), (x, \bar{x})\} \in \mathcal{H}^{'} \times \mathcal{H}$$

and

$$\left\langle (y, \bar{y}), (x, \bar{x}) \right\rangle_{\tilde{\mathcal{H}}' \times \tilde{\mathcal{H}}} = \left\langle \tilde{\tau} (y, \bar{y}), (x, \bar{x}) \right\rangle_{\tilde{\mathcal{H}}}, \quad \{(y, \bar{y}), (x, \bar{x})\} \in \tilde{\mathcal{H}}^{'} \times \tilde{\mathcal{H}}.$$

We are going to study the following problem

$$\begin{aligned}
y^{''}(t)+Ay(t) &= Bu(t), \quad t\in(0,T)\setminus\{t_k\}_{k\in\sigma_1^m}, && (4.1)\\
y(0) &= y^0,\ y^{'}(0)=\bar{y}^0, &&\\
\Delta y(t_k) &= I_k y(t_k)+D_k v_k,\ \ k\in\sigma_1^m, && (4.1_k)\\
\Delta y^{'}(t_k) &= \bar{I}_k y^{'}(t_k)+\bar{D}_k w_k,\ \ k\in\sigma_1^m, && (4.1_k^{'})
\end{aligned}$$

where the final time T is a positive number, $\left(y^0,y^1\right)$ is an initial condition in the Hilbert space $\mathcal{H}^{'}$, $y(t):[0,T]\to H$ is a vector function, and finally, $\{t_k\}_{k\in\sigma_1^m}$ is an increasing sequence of numbers in the open interval $(0,T)$, and $\Delta y(t_k)$, $\Delta y^{'}(t_k)$ denote the jump of $y(t)$ and $y^{'}(t)$ at $t=t_k$ respectively, *i.e*,

$$\Delta y(t_k)=y\left(t_k^+\right)-y\left(t_k^-\right),\quad \Delta y^{'}(t_k)=y^{'}\left(t_k^+\right)-y^{'}\left(t_k^-\right),$$

where $y\left(t_k^+\right)$, $y^{'}\left(t_k^+\right)$ and $y\left(t_k^-\right)$, $y^{'}\left(t_k^-\right)$ represent the right and left limits of $y(t)$, $y^{'}(t)$ at $t=t_k$ respectively. We assume that A is a positive self-adjoint with dense domain $D(A)$ on a real Hilbert space H, and $I_k, D_k, \bar{D}_k, B: H\to H$ and $\bar{I}_k: V^{'}\to V^{'}$, are given linear bounded operators. On the other hand, the vector function $\left(u(t),\{(v_k,w_k)\}_{k\in\sigma_1^m}\right)$ are control. We also consider the space $V=D(A^{\frac{1}{2}})$ and its dual space $V^{'}$.

Moreover, we set the following assumptions:

(H1) $I_k=(I_k)^*$, and $\bar{I}_k=(\bar{I}_k)^*$, for every $k\in\sigma_1^m$, and for each $k\in\sigma_1^m$, the operator $\boldsymbol{I}_k=I_k+I$ and $\bar{\boldsymbol{I}}_k=\bar{I}_k+I$ is invertible in H respectively in $V^{'}$, it is clearly that for each $k\in\sigma_1^m$, the operator

$$\begin{aligned}
\mathcal{I}_k\ &:\ \mathcal{H}^{'}\to\mathcal{H}^{'}\\
&\mathcal{I}_k(y,\bar{y}) \doteqdot \begin{pmatrix} I_k & 0\\ 0 & \bar{I}_k\end{pmatrix}(y,\bar{y})^T=\left(I_k y,\bar{I}_k\bar{y}\right),\quad \forall(y,\bar{y})\in\mathcal{H}^{'}
\end{aligned}$$

is invertible and self adjoint in $\mathcal{H}^{'}$.

(H2) $B^* = B \geq 0$, $B : H \to H$ such that there is $b_0 > 0$ so that

$$\langle By, y\rangle_{H\times H} \leq b_0 \|y\|_H^2, \quad \forall y \in H.$$

(H3) $D_k^* = D_k \geq 0$, $\bar{D}_k^* = \bar{D}_k \geq 0$, for every $k \in \sigma_1^m$, and for each $k \in \sigma_1^m$, there is $d_k > 0$ and $\bar{d}_k > 0$ such that

$$\langle D_k y, y\rangle_{H\times H} \leq d_k \|y\|_H^2, \quad \forall y \in H,$$

for all $y \in H$, and

$$\left\langle \overline{D}_k y, y\right\rangle_{H\times H} \leq \bar{d}_k \|y\|_H^2, \quad \forall y \in H.$$

In the sequel we shall designate by h the function

$$h(t) = \left(u(t), \{(v_k, w_k)\}_{k\in\sigma_1^m}\right),$$

where $u(t) \in L^2\left((0,T)\setminus\{t_k\}_{k\in\sigma_1^m}; H\right)$ and$\{(v_k, w_k)\}_{k\in\sigma_1^m} \in l^2(\sigma_1^m; H\times H)$ with

$$l^2(\sigma_1^m; H) \doteqdot \left\{\{(v_k, w_k)\}_{k\in\sigma_1^m}, (v_k, w_k) \in H\times H\right\}.$$

We point out that the space $\mathcal{R}_m(H) = L^2\left((0,T)\setminus\{t_k\}_{k\in\sigma_1^m}; H\right) \times l^2(\sigma_1^m; H\times H)$ is a Hilbert space with respect to the inner product

$$\left(h, \widetilde{h}\right)_{\mathcal{R}_m(H)} = \int_0^T (u(t), \widetilde{u}(t))_H\, dt + \sum_{k=1}^m (v_k, \widetilde{v}_k)_H + \sum_{k=1}^m (w_k, \widetilde{w}_k)_H,$$

defined for all $h = \left(h(t), \{(v_k, w_k)\}_{k\in\sigma_1^m}\right)$ and $\widetilde{h} = \left(\widetilde{u}(t), \{\widetilde{v}_k, \widetilde{w}_k\}_{k\in\sigma_1^m}\right) \in \mathcal{R}_m(H)$.

We shall denote by $\mathcal{B}$ the control operator given by

$$\mathcal{B} = \left(B, \left\{D_k, \overline{D}_k\right\}_{k\in\sigma_1^m}\right) \in \mathcal{L}(\mathcal{R}_m(H)),$$

so that

$$\mathcal{B}h(t) = \left(Bu(t), \left\{D_k v_k, \overline{D}_k w_k\right\}_{k\in\sigma_1^m}\right).$$

We have for every $h=\left(u\left(t\right),\left\{\left(v_k,w_k\right)\right\}_{k\in\sigma_1^m}\right)\in\mathcal{K}_m(H)$

$$\begin{aligned}
\left(\mathcal{B}h,\right)_{\mathcal{K}_m(H)} &= \int_0^T\left\langle Bu\left(t\right),\widetilde{u}\left(t\right)\right\rangle_H dt+\sum_{k=1}^m\left\langle D_kv_k,\widetilde{v}_k\right\rangle_H+\sum_{k=1}^m\left\langle\overline{D}_kw_k,\widetilde{w}_k\right\rangle_H,\\
&= \int_0^T\left\langle u\left(t\right),B\widetilde{u}\left(t\right)\right\rangle_H dt+\sum_{k=1}^m\left\langle v_k,D_k\widetilde{v}_k\right\rangle_H+\sum_{k=1}^m\left\langle w_k,\overline{D}_k\widetilde{w}_k\right\rangle_H\\
&= \left(h,\mathcal{B}\right)_{\mathcal{R}_m(H)},
\end{aligned}$$

which shows that $\mathcal{B}^*=\mathcal{B}$, that is, $\mathcal{B}$ is self-adjoint. On the other hand, we have

$$\begin{aligned}
\left(\mathcal{B}h,h\right)_{\mathcal{R}_m(H)} &= \int_0^T\left\langle Bu\left(t\right),u\left(t\right)\right\rangle_H dt+\sum_{k=1}^m\left\langle D_kv_k,v_k\right\rangle_H+\sum_{k=1}^m\left\langle\overline{D}_kw_k,w_k\right\rangle_H\\
&\leq b_0\int_0^T\left\|u\left(t\right)\right\|_H^2dt+\sum_{k=1}^m d_k\left\|v_k\right\|_H^2+\sum_{k=1}^m\bar{d}_k\left\|w_k\right\|_H^2\\
&\leq \delta\left\|h\right\|_{\mathcal{K}_m(H)}^2,
\end{aligned}$$

where $\delta=\max\left\{b_0,d_1,...,d_m,\bar{d}_1,...,\bar{d}_m\right\}$. Thus, the operator is $\mathcal{B}$ bounded in $\mathcal{R}_m(H)$.

Next, we consider the homogeneous system associated with (4.1):

$$\left\{\begin{array}{ll}
\varphi^{''}\left(t\right)=-A\varphi\left(t\right),\quad t\in\left(0,T\right)\setminus\left\{t_k\right\}_{k\in\sigma_1^m}, & \\
\varphi\left(0\right)=\varphi^0,\varphi^{'}(0)=\bar{\varphi}^0, & \\
\Delta\varphi\left(t_k\right)=I_k\varphi\left(t_k\right),\ \ k\in\sigma_1^m, & 4.2_k\\
\Delta\varphi^{'}\left(t_k\right)=\bar{I}_k\varphi^{'}\left(t_k\right),\ \ k\in\sigma_1^m. & 4.2_k^{'}
\end{array}\right.\tag{4.2}$$

We point out that on each interval $[t_k,t_{k+1})$, for $k\in\sigma_0^m$, the solution $\left(\varphi,\varphi^{'}\right)\in\tilde{\mathcal{H}}$ is left continuous at each time t_k.

We consider the corresponding homogeneous backward problem

$$\left\{\begin{array}{ll}
\tilde{\varphi}^{''}(t)+\mathbf{A}\tilde{\varphi}(t)=0,,\quad t\in(0,T)\setminus\left\{t_k\right\}_{k\in\sigma_1^m}, & \\
\tilde{\varphi}(T)=\bar{\varphi}^0,\ \tilde{\varphi}^{'}(T)=-\varphi^0, & \\
\Delta\tilde{\varphi}(t_{m-(k-1)})=-J_{m-(k-1)}\tilde{\varphi}(t_{m-(k-1)}^+),\ k\in\sigma_1^m, & 4.3_k\\
\Delta\tilde{\varphi}^{'}(t_{m-(k-1)})=-\bar{J}_{m-(k-1)}\tilde{\varphi}^{'}(t_{m-(k-1)}^+),\ k\in\sigma_1^m, & 4.3_k^{'}
\end{array}\right.\tag{4.3}$$

where

$$\mathbf{A} = A^*, \quad J_{m-(k-1)} = \bar{I}_{m-(k-1)}, \quad \bar{J}_{m-(k-1)} = I_{m-(k-1)}, \quad \forall k \in \sigma_1^m.$$

We observe that the problem (4.4) on the interval $[t_m, T]$ is equivalent to the classical backward problem

$$\begin{cases} \tilde{\varphi}'' + (t) + A^*\tilde{\varphi}(t) = 0,\ t \in (t_m, T)\,, \\ \tilde{\varphi}(T) = \varphi^0,\ \tilde{\varphi}'(T) = -\varphi^1. \end{cases}$$

We introduce the following space: $\mathcal{PLC}^1\left([0,T]\,;\tilde{\mathcal{H}}\right)$, (respectively, $\mathcal{PRC}^1\left([0,T]\,;\tilde{\mathcal{H}}\right)$) by

$$\mathcal{PLC}^1\left([0,T]\,;\tilde{\mathcal{H}}\right) = \left\{\{x,\bar{x}\} \in \mathcal{PC}^1\left([0,T]\,;\tilde{\mathcal{H}}\right) : x \in \mathcal{PLC}\left([0,T]\,;V\right),\ \bar{x} \in \mathcal{PLC}\left([0,T]\,;H\right)\right\}$$

respectively,

$$\mathcal{PRC}^1\left([0,T]\,;\tilde{\mathcal{H}}\right) \doteqdot \left\{\{x,\bar{x}\} \in \mathcal{PC}^1\left([0,T]\,;\tilde{\mathcal{H}}\right) : x \in \mathcal{PRC}\left([0,T]\,;V\right),\ \bar{x} \in \mathcal{PRC}\left([0,T]\,;H\right)\right\}$$

Finally, $\mathcal{C}^2\left([0,T] \setminus \{t_k\}_{k\in\sigma_1^m}\,;\tilde{\mathcal{H}}\right)$ will be the space

$$\begin{aligned} &\mathcal{C}^2\left([0,T] \setminus \{t_k\}_{k\in\sigma_1^m}\,;\tilde{\mathcal{H}}\right) \\ \doteqdot\ &\Big\{\{x,\bar{x}\} : x \in \mathcal{C}^1\left([0,T] \setminus \{t_k\}_{k\in\sigma_1^m}\,;V\right),\ \bar{x} \in \mathcal{C}^1\left([0,T] \setminus \{t_k\}_{k\in\sigma_1^m}\,;H\right) \\ &\text{and } x'(t) = \bar{x}(t) \text{ for } t \neq t_k,\ k \in \sigma_1^m\Big\}. \end{aligned}$$

Remark 5 *1) The space* $\mathcal{PLC}^1\left([0,T]\,;\tilde{\mathcal{H}}\right)$*, (respectively,* $\mathcal{PRC}^1\left([0,T]\,;\tilde{\mathcal{H}}\right)$*) can be identified to a subspace of* $\mathcal{R}_m(H)$*. That is, to each* $(x,\bar{x}) \in \mathcal{PLC}^1\left([0,T]\,;\tilde{\mathcal{H}}\right)$*, (respectively,* $(x^*,\bar{x}^*) \in \mathcal{PRC}^1\left([0,T]\,;\tilde{\mathcal{H}}\right)$*) is assigned the function* h *(respectively,* $\widetilde{\mathrm{h}}$ *) defined by*

$$\mathrm{h}\,(t) = \left(x\,(t)\,, \{(x\,(t_k)\,, \bar{x}\,(t_k))\}_{k\in\sigma_1^m}\right),$$

resp.

$$\widetilde{\mathrm{h}}\,(t) = \left(x^*\,(t)\,, \{(x^*\,(t_k)\,, \bar{x}^*\,(t_k))\}_{k\in\sigma_1^m}\right).$$

The mapping $(x\,(t)\,, \bar{x}\,(t)) \longmapsto \mathrm{h}\,(t)$ *(respectively,* $(x^*\,(t)\,, \bar{x}^*\,(t)) \longmapsto \mathrm{h}^*\,(t)$ *) is a linear imbedding.*

2) Let $(x^*,\bar{x}^*) \in \mathcal{PRC}^1\left([0,T]\,;\tilde{\mathcal{H}}\right)$*, the function* $(x^*,\bar{x}^*)$ *can be written as :*

$$(x^*,\bar{x}^*)\,(t) = \begin{cases} (x^*_{[0]}(t),\bar{x}^*_{[0]}(t)) & \textit{if} \quad t\in[t_0,t_1) \\ (x^*_{[k]}(t),\bar{x}^*_{[k]}(t)) & \textit{if} \quad t\in[t_k,t_{k+1}), \quad k\in\sigma_1^{m-1}, \\ (x^*_{[m]}(t),\bar{x}^*_{[m]}(t)) & \textit{if} \quad t\in[t_m,T], \end{cases}$$

Next, let $\tau_k = t_k - t_{k-1}$*, we define the operator*

$$\mathcal{T} : D(\mathcal{T}) = \mathcal{PRC}^1\left([0,T]\,;\tilde{\mathcal{H}}\right) \subset \mathcal{R}_m(H) \to \mathcal{R}_m(H)$$

by

$$(\mathcal{T}\,(x^*,\bar{x}^*))(t) = \begin{cases} (x^*_{[0]}(T-t)\dfrac{\tau_1}{\tau_{m+1}}+t_0),\bar{x}^*_{[0]}(T-t)\dfrac{\tau_1}{\tau_{m+1}}+t_0)), \\ \textit{for } t\in[t_0,t_1], \\ (x^*_{[k]}(t_{m-(k-1)}-t)\dfrac{\tau_{k+1}}{\tau_{m-(k-1)}}+t_k),\bar{x}^*_{[k]}(t_{m-(k-1)}-t)\dfrac{\tau_{k+1}}{\tau_{m-(k-1)}}+t_k)) \\ \textit{for } t\in\left[t_{m-k},t_{m-(k-1)}\right),\ k\in\sigma_1^{m-1}, \\ (x^*_{[m]}((t_1-t)\,\dfrac{\tau_{m+1}}{\tau_1}+t_m),\bar{x}^*_{[m]}((t_1-t)\,\dfrac{\tau_{m+1}}{\tau_1}+t_m)) \\ \textit{for } t\in(0,t_1]. \end{cases} \tag{4.4}$$

We note that the range of $\mathcal{T}$ *is exactly* $\mathcal{PLC}^1\left([0,T]\,;\tilde{\mathcal{H}}\right)$*. The function* $(\mathcal{T}\,(x^*,\bar{x}^*))(t)$ *can be written as*

$$(\mathcal{T}\,(x^*,\bar{x}^*))(t) = \begin{cases} \left(x_{[0]}\,(t)\,,\bar{x}_{[0]}\,(t)\right)(t) & \textit{if} \quad t\in[t_0,t_1], \\ \left(x_{[k]}\,(t)\,,\bar{x}_{[k]}\,(t)\right)(t) & \textit{if} \quad t\in(t_k,t_{k+1}], \quad k\in\sigma_1^{m-1}, \\ \left(x_{[m}\,(t)\,,\bar{x}_{[m]}\,(t)\right)(t) & \textit{if} \quad t\in(t_m,T]. \end{cases}$$

Let $X\,(t)$ be the resolvent solution of the operator system

$$\begin{cases} X'(t)+\mathcal{A}X(t)=0,\ t\in(0,T)\setminus\{t_k\}_{k\in\sigma_1^m}, \\ X(0)=\mathcal{I}, \\ X(t_k^+)-X(t_k^-)=\mathcal{I}_kX(t_k),\ k\in\sigma_1^m, \end{cases}$$

where $\mathcal{A}$ is the operator matrix

$$\mathcal{A} \doteqdot \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \quad D(\mathcal{A}) \doteqdot \{(y, \bar{y}) \in D(A) \times H\} \doteqdot \tilde{\mathcal{H}},$$

$$\mathcal{I}_k \doteqdot \begin{pmatrix} I_k & 0 \\ 0 & \bar{I}_k \end{pmatrix}, \quad D(\mathcal{I}_k) \doteqdot \tilde{\mathcal{H}}$$

and $\mathcal{I} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ is the identity operator. We shall suppose that the operator $\mathcal{I}_k$ has a bounded inverse in $\mathcal{H}$.

It is easy to see that the impulsive system (4.2) can be transformed in $\tilde{\mathcal{H}}$ into a corresponding first-order linear impulsive system

$$\begin{cases} \binom{\varphi'(t)}{\bar{\varphi}'(t)} + \mathcal{A}\binom{\varphi(t)}{\bar{\varphi}(t)} = 0, \ t \in (0, T) \setminus \{t_k\}_{k \in \sigma_1^m}, \\ \binom{\varphi(0)}{\bar{\varphi}(0)} = \binom{\varphi^0}{\bar{\varphi}^0}, \\ \binom{\Delta\varphi(t_k)}{\Delta\bar{\varphi}(t_k)} = \mathcal{I}_k\binom{\varphi(t_k)}{\bar{\varphi}(t_k)} = \binom{I_k\varphi(t_k)}{\bar{I}_k\bar{\varphi}(t_k)}, \ k \in \sigma_1^m. \end{cases} \tag{4.5}$$

We now introduce a duality pairing $\mathcal{H}' \times \tilde{\mathcal{H}} \longmapsto \mathbb{R}$ in the following way,

$$\begin{aligned} \{(y, \bar{y}), (x, \bar{x})\} &= \langle (y, \bar{y}), \mathcal{M}(x, \bar{x}) \rangle_{\mathcal{H}' \times \mathcal{H}} \\ &= \left\langle \mathcal{M}^T (y, \bar{y}), (x, \bar{x}) \right\rangle_{\tilde{\mathcal{H}}' \times \tilde{\mathcal{H}}} \\ &= \langle (y, \bar{y}), (-\bar{x}, x) \rangle_{\mathcal{H}' \times \mathcal{H}} \\ &= \langle \bar{y}, x \rangle_{V' \times V} - \langle y, \bar{x} \rangle_{H \times H}, \end{aligned}$$

for all $(y, \bar{y}) \in \mathcal{H}'$ and $(x, \bar{x}) \in \tilde{\mathcal{H}}$.

The formal adjoint operator of $\mathcal{A}$ is the uniquely defined operator $\mathcal{A}^*$ given by

$$\langle \mathcal{A}(y, \bar{y}), (x, \bar{x}) \rangle_{\mathcal{H}' \times \mathcal{H}} = \langle (y, \bar{y}), \mathcal{A}^*(x, \bar{x}) \rangle_{\mathcal{H}' \times \mathcal{H}}, \quad \forall (y, \bar{y}), (x, \bar{x}) \in \mathcal{H}' \times \mathcal{H},$$

We will therefore consider another operator $\tilde{A}$ defined by

$$\langle \mathcal{A}(y, \bar{y}), (x, \bar{x}) \rangle_{\mathcal{H}' \times \mathcal{H}} = \langle (y, \bar{y}), \tilde{A}(x, \bar{x}) \rangle_{\mathcal{H}' \times \mathcal{H}}, \quad \forall (y, \bar{y}), (x, \bar{x}) \in \mathcal{H}' \times \tilde{\mathcal{H}}.$$

Clearly we have

$$\tilde{A} = \mathcal{M}^{-1}\mathcal{A}^{*}\mathcal{M}.$$

Since the matrix operator $\mathcal{A}$ is closed if and only if A is closed, then $\tilde{A}$ generates a strongly continuous semigroup $e^{t\tilde{A}} \in \mathcal{L}\left(\tilde{\mathcal{H}}\right)$.

The linear homogeneous impulsive problem

$$\begin{cases} \binom{\tilde{\varphi}'(t)}{\tilde{\psi}'(t)} + \tilde{A}\binom{\tilde{\varphi}(t)}{\tilde{\psi}(t)} = 0, \ t \in (0,T) \setminus \{t_k\}_{k\in\sigma_1^m}, \\ \binom{\tilde{\varphi}(T)}{\tilde{\psi}(T)} = \binom{\varphi^0}{-\varphi^1}, \\ \binom{\Delta\tilde{\varphi}(t_k)}{\Delta\tilde{\psi}(t_k)} = \mathcal{I}_k\binom{\tilde{\varphi}(t_k)}{\tilde{\psi}(t_k)} = \binom{I_k\varphi(t_k)}{\bar{I}_k\bar{\varphi}(t_k)}, \ k \in \sigma_1^m. \end{cases} \tag{4.6}$$

is called the adjoint system to impulsive problem *(4.5).*

Definition 50 *A function* $(y, \bar{y}) \in \mathcal{PLC}^1\left([0,T]\,; \mathcal{H}'\right)$ *is a mild solution of the impulsive problem (4.1) if the impulsive conditions are satisfied and*

$$\begin{aligned} \binom{y}{\bar{y}}(t) \quad = \quad & G(t,0)\binom{y^0}{y^1} + \int_0^t G(t,s)\binom{0}{Bu(s)}Bu\,(s)\,ds \\ & \textstyle\sum_{0<t_k\leq t} G(t,t_k)\binom{D_k v_k}{\bar{D}_k w_k}, \ \ \textit{for every } t \in (0,T), \end{aligned}$$

where $G(t,s)$ *is evolution operator of (4.5) in* $\mathcal{H}'$*.*

It is well known that for each $\left(y^0, y^1\right) \in \mathcal{H}$, (4.1) has a unique solution $(y, \bar{y})$ with

$$(y, \bar{y}) \in \mathcal{PLC}^1\left([0,T]\,;\mathcal{H}\right) \cap \mathcal{C}^2\left([0,T] \setminus \{t_k\}_{k\in\sigma_1^m}\,; \mathcal{H}\right).$$

Now, we define the concept of mild solution for the backward impulsive system (4.3) associated with the system (4.2) and the operator given by (4.4).

Definition 51 *We say that* $(\tilde{\varphi}, \tilde{\psi}) \in \mathcal{PRC}\left([0,T]\,;\tilde{\mathcal{H}}\right)$ *is a* $\mathcal{T}-$*mild solution for the impulsive problem (4.6) if* $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathcal{T}\left((\tilde{\varphi}, \tilde{\psi})\right)$ *is a mild solution for the homogeneous impulsive problem (4.5).*

Definition 52 *We say that* $(\tilde{\varphi}, \tilde{\psi}) \in \mathcal{PRC}\left([0,T]\,;\tilde{\mathcal{H}}\right)$ *is a mild solution to the backward impulsive system (4.3) if* $(\tilde{\varphi}, \tilde{\psi})$ *is a* $\mathcal{T}-$*mild solution of the impulsive problem (4.6).*

Definition 53 *We say that the system (4.1) is exactly controllable at time* T*, if for every initial state* $\left(y^0, y^1\right) \in \mathcal{H}$*, there is a control function* $\mathrm{h} \in \mathcal{R}_m(H)$ *for which the solution* y *satisfies* $y\left(T\right) = y^{'}\left(T\right) = 0$.

Remark 6

1. *We observe that taking into account the linearity of (4.1) one can eventually steer the solution at time* T *to any desired state* $\left(y_d^0, y_d^1\right) \in \mathcal{H}$*, so that* $\left(y\left(T\right), y^{'}\left(T\right)\right) = \left(y_d^0, y_d^1\right)$

2. *Since the impulsive system (4.1) is linear and reversible in time null and exact controllability are equivalent notion. As we shall see, the situation is completely different in the case of the first order impulsive systems.*

3. *Null controllability is physically an interesting notion since the condition* $y\left(T\right) = y^{'}\left(T\right) = 0$ *is an equilibrium for system (4.1).*

Before studing impulsive null controllability of (4.1) we prove the following lemma :

Lemma 54 *Assume that*

$$\left(\xi\left(t\right), \zeta\left(t\right)\right) \in L^2\left(\left[0, T\right]; _{H \times H}\right), \quad \left(\left\{\xi_k, \bar{\xi}_k\right\}_k, \left\{\zeta_k, \bar{\zeta}_k\right\}_k\right) \in l^2(\sigma_1^m, \mathcal{H}^{'} \times \tilde{\mathcal{H}}).$$

Then, for every vector functions

$$\left(\gamma\left(t\right), \bar{\gamma}\left(t\right)\right) \in \mathcal{PLC}^1\left(\left[0, T\right]; \mathcal{H}^{'}\right) \cap \mathcal{C}^2\left(\left[0, T\right] \setminus \left\{t_k\right\}_{k \in \sigma_1^m}; \mathcal{H}^{'}\right)$$

and

$$\left(\eta\left(t\right), \bar{\eta}\left(t\right)\right) \in \mathcal{PRC}^1\left(\left[0, T\right]; \tilde{\mathcal{H}}\right) \cap \mathcal{C}^2\left(\left[0, T\right] \setminus \left\{t_k\right\}_{k \in \sigma_1^m}; \tilde{\mathcal{H}}\right)$$

satisfying the problem

$$\frac{d}{dt}\left[\left\langle \bar{\gamma}\left(t\right), \eta\left(t\right)\right\rangle_{V' \times V} - \langle \gamma(t), \bar{\eta}(t)\rangle_{H \times H}\right] = \left\langle \xi\left(t\right), \zeta\left(t\right)\right\rangle_{H \times H}, \quad t \neq t_k, \textit{ for } k \in \sigma_1^m$$

$$
\begin{aligned}
&\Delta\left[\langle\bar{\gamma}\left(t_k\right),\eta\left(t_k\right)\rangle_{V'\times V}-\langle\gamma(t_k),\bar{\eta}(t_k)\rangle_{H\times H}\right]\\
=\;&\left(\left[\langle\Delta\bar{\gamma}\left(t_k\right),\eta(t_k)\rangle_{V'\times V}\right]+\left[\langle\bar{\gamma}\left(t_k\right),\Delta\eta(t_k)\rangle_{V'\times V}\right]\right)-\left(\left[\langle\Delta\gamma\left(t_k\right),\bar{\eta}\left(t_k\right)\rangle_{H\times H}\right]\right.\\
&\left.+\left[\langle\gamma\left(t_k\right),\Delta\bar{\eta}\left(t_k\right)\rangle_{H\times H}\right]\right)\\
=\;&\langle\left(\xi_k,\bar{\xi}_k\right),\left(\zeta_k,\bar{\zeta}_k\right)\rangle_{\mathcal{H}'\times\tilde{\mathcal{H}}},\;k\in\sigma_1^m,
\end{aligned}
$$

we have the following identity

$$
\begin{aligned}
&\left[\langle\bar{\gamma}\left(t\right),\eta\left(t\right)\rangle_{V'\times V}-\langle\gamma(t),\bar{\eta}(t)\rangle_{H\times H}\right]\Big|_0^T \qquad (4.7)\\
=\;&\left[\langle\bar{\gamma}\left(T\right),\eta\left(T\right)\rangle_{V'\times V}-\langle\gamma(T),\bar{\eta}(T)\rangle_{H\times H}\right]-\left[\langle\bar{\gamma}\left(0\right),\eta\left(0\right)\rangle_{V'\times V}-\langle\gamma(0),\bar{\eta}(0)\rangle_{H\times H}\right]\\
=\;&\int_0^T\langle\xi\left(t\right),\zeta\left(t\right)\rangle_{H\times H}dt+\sum_{k=1}^{m}\langle\bar{\xi}_k,\zeta_k\rangle_{V'\times V}-\sum_{k=1}^{m}\langle\xi_k,\bar{\zeta}_k\rangle_{H\times H}.
\end{aligned}
$$

Proof. Let us define the functions Φ and Ψ as follows

$$
\Phi\left(t\right)=\begin{cases}\langle\xi\left(t\right),\zeta\left(t\right)\rangle_{H\times H}, & t\neq t_k,\text{ for }k\in\sigma_1^m\\ \langle\left\{\xi_k,\bar{\xi}_k\right\},\left\{\zeta_k,\bar{\zeta}_k\right\}\rangle_{\mathcal{H}'\times\tilde{\mathcal{H}}}, & t=t_k,\quad k\in\sigma_1^m,\end{cases}
$$

and $\Psi\left(t\right)=\left[\langle\bar{\gamma}\left(t\right),\eta\left(t\right)\rangle_{V'\times V}-\langle\gamma(t),\bar{\eta}(t)\rangle_{H\times H}\right]$, then, Φ and Ψ satisfy the problem

$$
\begin{cases}\Psi'\left(t\right)=\Phi\left(t\right), & t\neq t_k,\text{ for }k\in\sigma_1^m\\ \Delta\Psi\left(t_k\right)=\Phi\left(t_k\right), & k\in\sigma_1^m,\end{cases}
$$

and we have

$$
\Psi\left(t_1^-\right)-\Psi\left(0\right)=\int_0^{t_1}\Phi\left(s\right)ds
$$

$$
\Psi\left(t_2^-\right)-\Psi\left(t_1^+\right)=\int_{t_1}^{t_2}\Phi\left(s\right)ds
$$

....

$$
\Psi\left(t_k^-\right)-\Psi\left(t_{k-1}^+\right)=\int_{t_{k-1}}^{t_k}\Phi\left(s\right)ds,\quad k=3,...m-1
$$

$$
\Psi\left(T\right)-\Psi\left(t_m^+\right)=\int_{t_m}^{T}\Phi\left(s\right)ds.
$$

Adding together we get

$$\begin{aligned}\int_0^T \Phi(t)\,dt &= \int_0^{t_1} \Phi(t)\,dt + \sum_{k=1}^{m-1} \int_{t_k}^{t_{k+1}} \Phi(t)\,dt + \int_{t_m}^{T} \Phi(t)\,dt \\ &= \Psi\left(t_1^-\right) - \Psi\left(0^+\right) + \sum_{k=1}^{m-1} \left[\Psi\left(t_{k+1}^-\right) - \Psi\left(t_k^+\right)\right] + \Psi(T) - \Psi\left(t_m^+\right) \\ &= -\sum_{k=1}^{m} \Phi(t_k) + \Psi(T) - \Psi(0).\end{aligned}$$

So that

$$\Psi(T) - \Psi(0) = \int_0^T \Phi(t)\,dt + \sum_{k=1}^{m} \Phi(t_k),$$

which shows that (4.7) is satisfied.■

We also need the following Lemmas :

Lemma 55 [99] *If $\mathcal{B} \in \mathcal{L}(\mathcal{K}_m(H))$ is self-adjoint and nonnegative, then*

$$\|\mathcal{B}\mathrm{h}\| \leq \|\mathcal{B}\|^{1/2} (\mathcal{B}\mathrm{h}, \mathrm{h})_{\mathcal{K}_m(H)}^{1/2}, \quad \mathrm{h} \in \mathcal{R}_m(H).$$

Lemma 56 *If $\tau_{k+1} = \tau_{m-(k-1)}$, $k \in \sigma_0^{m-1}$, then for the mild solution $\left(\tilde{\varphi}, \tilde{\psi}\right)$ of (4.3), the identity holds :*

$$\begin{aligned}&\textstyle\int_0^T \|B\tilde{\varphi}\|_H^2\,dt + \sum_{k=1}^m \left\|D_k\tilde{\psi}(t_k)\right\|_H^2 + \sum_{k=1}^m \left\|\bar{D}_k\tilde{\varphi}(t_k)\right\|_H^2 \\ &= \textstyle\int_0^T \|B\varphi\|_H^2\,dt + \sum_{k=1}^m \left\|\bar{D}_k\varphi\left(t_{m-(k-1)}\right)\right\|_H^2 + \left\|D_k\psi\left(t_{m-(k-1)}\right)\right\|_H^2.\end{aligned} \tag{4.8}$$

Proof. For each $k \in \sigma_0^m$, using the change of variable $t \longmapsto (t_{m-(k-1)} - t)\dfrac{\tau_{k+1}}{\tau_{m-(k-1)}} + t_k$ we have

$$\begin{aligned}&\int_{t_{m-k}}^{t_{m-(k-1)}} \left\langle B\varphi_{[m-k]}(t), B\varphi_{[m-k]}(t)\right\rangle_H dt \\ =\ &\int_{t_{m-k}}^{t_{m-(k-1)}} \Big\langle B\widetilde{\varphi}_{[k]}((t_{m-(k-1)} - t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}} + t_k), \\ &B\widetilde{\varphi}_{[k]}((t_{m-(k-1)} - t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}} + t_k)\Big\rangle_H dt\end{aligned}$$

$$
\begin{aligned}
&\int_{t_{m-k}}^{t_{m-(k-1)}} \left\langle B\varphi_{[m-k]}(t), B\varphi_{[m-k]}(t)\right\rangle_H dt \\
=\ & \frac{-\tau_{m-(k-1)}}{\tau_{k+1}}\left[\int_{t_{m-k}}^{t_{m-(k-1)}} \left\langle (\widetilde{\varphi}_{[k]}((t_{m-(k-1)}-t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}}+t_k),\right.\right. \\
& \left.\left. B\widetilde{\varphi}_{[k]}((t_{m-(k-1)}-t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}}+t_k)\right\rangle_H\right] \times \frac{-\tau_{k+1}}{\tau_{m-(k-1)}} dt \\
=\ & \frac{-\tau_{m-(k-1)}}{\tau_{k+1}} \int_{t_{k+1}}^{t_k} \left\langle B\widetilde{\varphi}_{[k]}(s), B\widetilde{\varphi}_{[k]}(s)\right\rangle_H ds \\
=\ & \int_{t_k}^{t_{k+1}} (B\widetilde{\varphi}_{[k]}(s), B\widetilde{\varphi}_{[k]}(s)) ds.
\end{aligned}
$$

Summing up with respect to k, we get

$$
\sum_{k=0}^{m}\int_{t_{m-k}}^{t_{m-(k-1)}} \left\langle B\varphi_{[m-k]}(t), B\varphi_{[m-k]}(t)\right\rangle_H dt = \sum_{k=0}^{m}\int_{t_k}^{t_{k+1}} (B\widetilde{\varphi}_{[k]}(t), B\widetilde{\varphi}_{[k]}(t))dt.
$$

Thus, we obtain that

$$
\int_0^T |B\widetilde{\varphi}|_H^2\, dt = \int_0^T |B\varphi|_H^2\, dt.
$$

On the other hand, from the definition of the function $\left(\widetilde{\varphi}, \tilde{\psi}\right)$ we get

$$
\left(\varphi\left(t_{m-k}\right), \psi\left(t_{m-k}\right)\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\widetilde{\varphi}\left(t_{k+1}\right), \tilde{\psi}\left(t_{k+1}\right)\right), \quad k \in \sigma_0^{m-1}.
$$

Also, we have

$$
\left(\varphi\left(t_{m-(k-1)}\right), \psi\left(t_{m-(k-1)}\right)\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\widetilde{\varphi}\left(t_k\right), \tilde{\psi}\left(t_k\right)\right), \quad k \in \sigma_1^{m},
$$

and

$$
\left(\widetilde{\varphi}\left(t_{m-k}\right), \tilde{\psi}\left(t_{m-k}\right)\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\varphi\left(t_{k+1}\right), \psi\left(t_{k+1}\right)\right), \quad k \in \sigma_1^{m}.
$$

This implies that

$$\begin{aligned}
&\sum_{k=1}^{m}\left\|D_k\tilde{\psi}\left(t_k\right)\right\|_H^2+\left\|\bar{D}_k\tilde{\varphi}\left(t_k\right)\right\|_H^2\\
=\ &\sum_{k=0}^{m-1}\left\|\left(\bar{D}_{m-k}\tilde{\varphi}\left(t_{m-k}\right),D_k\tilde{\psi}\left(t_{m-k}\right)\right)\right\|_{H\times H}^2\\
=\ &\sum_{k=0}^{m-1}\left\langle\begin{pmatrix}\bar{D}_{m-k} & 0\\ 0 & D_{m-k}\end{pmatrix}\left(\tilde{\varphi}\left(t_{m-k}\right),\tilde{\psi}\left(t_{m-k}\right)\right),\right.\\
&\left.\begin{pmatrix}\bar{D}_{m-k} & 0\\ 0 & D_{m-k}\end{pmatrix}\left(\tilde{\varphi}\left(t_{m-k}\right),\tilde{\psi}\left(t_{m-k}\right)\right)\right\rangle_{H\times H}\\
=\ &\sum_{k=0}^{m-1}\left\langle\begin{pmatrix}\bar{D}_{m-k} & 0\\ 0 & D_{m-k}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\left(\varphi\left(t_{k+1}\right),\psi\left(t_{k+1}\right)\right),\right.\\
&\left.\begin{pmatrix}\bar{D}_{m-k} & 0\\ 0 & D_{m-k}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\left(\varphi\left(t_{k+1}\right),\psi\left(t_{k+1}\right)\right)\right\rangle_{H\times H}\\
=\ &\sum_{k=0}^{m-1}\left\|\begin{pmatrix}\bar{D}_{m-k} & 0\\ 0 & -D_{m-k}\end{pmatrix}\left(\varphi\left(t_{k+1}\right),\psi\left(t_{k+1}\right)\right)\right\|_{H\times H}^2\\
=\ &\sum_{l=1}^{m}\left\|\begin{pmatrix}\bar{D}_{l} & 0\\ 0 & -D_{l}\end{pmatrix}\left(\varphi\left(t_{m-(l-1)}\right),\psi\left(t_{m-(l-1)}\right)\right)\right\|_{H\times H}^2\\
=\ &\sum_{k=1}^{m}\left\|\begin{pmatrix}\bar{D}_{k} & 0\\ 0 & -D_{k}\end{pmatrix}\left(\varphi\left(t_{m-(k-1)}\right),\psi\left(t_{m-(k-1)}\right)\right)\right\|_{H\times H}^2\\
=\ &\sum_{k=1}^{m}\left\|\bar{D}_k\varphi\left(t_{m-(k-1)}\right)\right\|_H^2+\left\|-D_k\psi\left(t_{m-(k-1)}\right)\right\|_H^2\\
=\ &\sum_{k=1}^{m}\left\|\bar{D}_k\varphi\left(t_{m-(k-1)}\right)\right\|_H^2+\left\|D_k\psi\left(t_{m-(k-1)}\right)\right\|_H^2,
\end{aligned}$$

which gives (4.8).■

Corollary 57 *If* $\tau_{k+1}=\tau_{m-(k-1)}$, *for* $k\in\sigma_0^{m-1}$, *and* $B,D_k,\bar{D}_k$ *are nonnegative in* H, *then* *the following holds:*

$$\begin{aligned}&\int_0^T \langle B\widetilde{\varphi}(t),\widetilde{\varphi}(t)\rangle_H dt+\sum_{k=1}^{k=m}\langle D_k\tilde{\psi}(t_k),\tilde{\psi}(t_k)\rangle_H+\sum_{k=1}^{k=m}\langle \bar{D}_k\tilde{\varphi}(t_k),\tilde{\varphi}(t_k)\rangle_H\\ =\;&\int_0^T \langle B\varphi(t),\varphi(t)\rangle_H dt+\sum_{k=1}^{k=m}\langle \bar{D}_k\varphi(t_{m-(k-1)}),\varphi(t_{m-(k-1)})\rangle_H\\ &+\sum_{k=1}^{k=m}\langle D_k\psi(t_{m-(k-1)}),\psi(t_{m-(k-1)})\rangle_H.\end{aligned}$$

Proof. This follows immediately from Lemma **56** by replacing B by $B^{\frac{1}{2}}$, D_k by $D_k^{\frac{1}{2}}$, and $\bar{D}_k$ by $\bar{D}_k^{\frac{1}{2}}$.■

Now, we state and establish the following Theorem.

Theorem 58 *Let* $\left(y^0,y^1\right)\in\tilde{\mathcal{H}}$, *be a given initial state for the system (4.1), then* $\left(y^0,y^1\right)$ *is null controllable at time* T *if and only if there is a positive constant* C *such that*

$$\begin{aligned}\left|\langle\left(y^0,y^1\right),\left(\tilde{\varphi}^0,\tilde{\varphi}^1\right)\rangle_{\mathcal{H}'\times\tilde{\mathcal{H}}}\right| \;\leq\;& C\left\{\int_0^T\|B\varphi(t)\|_H^2\,dt+\sum_{k=1}^{k=m}\left\|\bar{D}_k\varphi((t_{m-(k-1)})\right\|_H^2\right.\\ &\left.+\sum_{k=1}^{k=m}\left\|D_k\psi((t_{m-(k-1)})\right\|_H^2\right\}^{1/2},\ \forall\left\{\varphi^0,\varphi^1\right\}\in\tilde{\mathcal{H}},\end{aligned}\tag{4.9}$$

where $(\varphi,\psi)\in\mathcal{PLC}\left([0,T];\tilde{\mathcal{H}}\right)$ *is the unique mild solution to* (4.2) *with*

$$\left(\tilde{\varphi}^0,\tilde{\varphi}^1\right)=(\tilde{\varphi}(0),\psi(0))=\left(\varphi\left(T\right),-\varphi^{'}\left(T\right)\right).$$

Proof. It suffices to prove this Theorem for the special case $\tau_{k+1}=\tau_{m-(k-1)}$, for $k\in\sigma_0^{m-1}$, because the norm $\||\|| \doteqdot \left\{\sum_{k=0}^m\frac{\tau_{m-(k-1)}}{\tau_{k+1}}\int_{t_k}^{t_{k+1}}\||\|_H^2\,dt\right\}^{1/2}$ is equivalent to the usual norm in $L^2\left([0,T];H\times H\right).$

We shall proceed in several steps.

Step 1: Let $(y,\bar{y})$ and $(\tilde{\varphi},\tilde{\psi})$ be strong solutions to (4.1) and (4.3), respectively.

Then, for $t\neq t_k$, $k\in\sigma_1^m$, we have

$$\begin{aligned}&\langle y^{''}(t),\tilde{\varphi}(t)\rangle_{V'\times V}-\langle y(t),\tilde{\varphi}^{''}(t)\rangle_H\\ =\;&\langle y^{''}(t),\widetilde{\varphi}\,(t)\rangle_H+\left[\langle\bar{y}(t),\tilde{\psi}(t)\rangle_{V'\times V}-\langle\bar{y}(t),\tilde{\psi}(t)\rangle_H\right]-\langle y(t),\tilde{\varphi}^{''}(t)\rangle_H\end{aligned}$$

$$\begin{aligned}
&\langle y^{''}(t), \tilde{\varphi}(t)\rangle_{V'\times V} - \langle y(t), \tilde{\varphi}^{''}(t)\rangle_H \qquad\qquad (4.10)\\
= & \left[\langle y^{''}(t), \widetilde{\varphi}\,(t)\rangle_{V'\times V} + \langle \bar{y}(t), \tilde{\psi}(t)\rangle_{V'\times V}\right] - \left[\langle \bar{y}(t), \tilde{\psi}(t)\rangle_{H\times H} + \langle y(t), \tilde{\varphi}^{''}(t)\rangle_H\right]\\
= & \frac{d}{dt}\left[\langle \bar{y}(t), \widetilde{\varphi}(t)\rangle_{V'\times V}\right] - \frac{d}{dt}\left[\langle y(t), \tilde{\psi}(t)\rangle_H\right]\\
= & \frac{d}{dt}\left[\langle \bar{y}(t), \widetilde{\varphi}(t)\rangle_{V'\times V} - \langle y(t), \tilde{\psi}(t)\rangle_H\right].
\end{aligned}$$

On the other hand, we have

$$\langle y^{''}(t), \tilde{\varphi}(t)\rangle_{V'\times V} = \langle -Ay(t) + Bu(t), \tilde{\varphi}(t)\rangle_{V'\times V}$$

and

$$\langle y(t), \tilde{\varphi}^{''}(t)\rangle_{H\times H} = \langle y(t), -A\tilde{\varphi}^{''}(t)\rangle_{H\times H}.$$

By subtracting these two identities we find

$$\begin{aligned}
\frac{d}{dt}\left[\langle \bar{y}(t), \widetilde{\varphi}(t)\rangle_{V'\times V} - \langle y(t), \tilde{\psi}(t)\rangle_H\right] &= \langle -Ay(t) + Bu(t), \widetilde{\varphi}(t)\rangle_{V'\times V} - \langle y(t), -A\widetilde{\varphi}(t)\rangle_H\\
&= \langle -Ay(t) + Bu(t), \widetilde{\varphi}(t)\rangle_H - \langle y(t), -A\widetilde{\varphi}(t)\rangle_H\\
&= \langle Bu(t), \widetilde{\varphi}(t)\rangle_H.
\end{aligned}$$

Multiplying equation $(4.3)_{m+1-k}$ in (4.3) from the left by $y^{'}(t_k)$ the solution of (4.1), and multiplying equation $(4.3^{'})_{m+1-k}$ in (4.3) by $y^{'}(t_k)$ the solution of (4.1), and multiplying equation $(4.1)_k$ in (4.1) from the right by $\widetilde{\varphi}^{'}(t_k)$ the solution of (4.3), and multiplying equation $(4.1^{'})_k$ in (4.1) by $\widetilde{\varphi}\,(t_k)$ finally; adding memberwise we get

$$\begin{aligned}
&\Delta\left[\langle \bar{y}(t_k), \widetilde{\varphi}(t_k)\rangle_{V'\times V} - \langle y(t_k), \tilde{\psi}(t_k)\rangle_H\right]\\
= \;&\Delta\left[\langle y^{'}(t_k), \widetilde{\varphi}(t_k)\rangle_{V'\times V}\right] - \Delta\left[\langle y(t_k), \tilde{\psi}(t_k)\rangle_H\right]
\end{aligned}$$

$$
\begin{aligned}
& \Delta\left[\langle\bar{y}(t_k),\widetilde{\varphi}(t_k)\rangle_{V'\times V}-\langle y(t_k),\tilde{\psi}(t_k)\rangle_H\right] \qquad (4.11)\\
= & \left\{\left[\langle\Delta y'(t_k),\widetilde{\varphi}(t_k)\rangle_{V'\times V}\right]+\left[\langle y'(t_k),\Delta\widetilde{\varphi}(t_k)\rangle_{V'\times V}\right]\right\}\\
& -\left\{\left[\langle\Delta y(t_k),\tilde{\psi}(t_k)\rangle_{H\times H}\right]+\left[\langle y(t_k),\Delta\tilde{\psi}(t_k)\rangle_H\right]\right\}\\
= & \left\{\langle\bar{I}_k y'(t_k)+\bar{D}_k w_k,\widetilde{\varphi}(t_k)\rangle_{V'\times V}+\langle y'(t_k),-\bar{I}_k\widetilde{\varphi}(t_k)\rangle_{V'\times V}\right\}\\
& -\left\{\langle I_k y(t_k)+D_k v_k,\tilde{\psi}(t_k)\rangle_H+\langle y(t_k),-I_k\tilde{\psi}(t_k)\rangle_H\right\}\\
& \langle\bar{D}_k w_k,\widetilde{\varphi}(t_k)\rangle_{V'\times V}-\langle D_k v_k,\tilde{\psi}(t_k)\rangle_H\\
& \langle\left(D_k v_k,\bar{D}_k w_k\right),\left(\widetilde{\varphi}(t_k),\tilde{\psi}(t_k)\right)\rangle_{\mathcal{H}'\times\tilde{\mathcal{H}}}.
\end{aligned}
$$

Setting

$$(\gamma\left(t\right),\bar{\gamma}\left(t\right))=(y(t),\bar{y}(t)),(\eta\left(t\right),\bar{\eta}\left(t\right))=(\widetilde{\varphi}(t),\tilde{\psi}(t)),(\xi\left(t\right),\zeta\left(t\right))=(Bu(t),\widetilde{\varphi}(t)),\ t\neq t_k,\ k\in\sigma_1^m,$$

$$(\gamma\left(t_k\right),\bar{\gamma}\left(t_k\right))=(y(t_k),y'(t_k)),(\eta\left(t_k\right),\bar{\eta}\left(t_k\right))=(\widetilde{\varphi}(t_k),\widetilde{\varphi}'(t_k)),$$

and

$$\left\{\xi_k,\bar{\xi}_k\right\}_{k=1}^m=\left\{D_k v_k,\bar{D}_k w_k\right\}_{k=1}^m,\left\{\zeta_k,\bar{\zeta}_k\right\}_{k=1}^m=(\widetilde{\varphi}(t_k),\tilde{\psi}(t_k)),$$

then equations (4.7), (4.10) and (4.11) give,

$$
\begin{aligned}
& \left[\langle\bar{y}\left(T\right),\widetilde{\varphi}\left(T\right)\rangle_{V'\times V}-\langle y(T),\tilde{\psi}(T)\rangle\right]-\left[\langle\bar{y}\left(0\right),\widetilde{\varphi}\left(0\right)\rangle_{V\times V'}-\langle y(0),\tilde{\psi}(0)\rangle_H\right] \quad (4.12)\\
= & \int_0^T\langle Bu(t),\widetilde{\varphi}(t)\rangle_H dt+\sum_{k=1}^{k=m}\langle\bar{D}_k w_k,\widetilde{\varphi}(t_k)\rangle_{V'\times V}-\sum_{k=1}^{k=m}\langle D_k v_k,\tilde{\psi}(t_k)\rangle_H\\
= & \int_0^T\langle Bu(t),\widetilde{\varphi}(t)\rangle_H dt+\langle\left(D_k v_k,\bar{D}_k w_k\right),\left(\widetilde{\varphi}(t_k),\tilde{\psi}(t_k)\right)\rangle_{\mathcal{H}'\times\tilde{\mathcal{H}}}\\
= & \int_0^T\langle Bu(t),\widetilde{\varphi}(t)\rangle_H dt+\sum_{k=1}^{k=m}\langle\bar{D}_k w_k,\widetilde{\varphi}(t_k)\rangle_{V'\times V}-\sum_{k=1}^{k=m}\langle D_k v_k,\tilde{\psi}(t_k)\rangle_H.
\end{aligned}
$$

By density. This identity is valid for mild solutions as well.

Next, if there is a certain $h(t) \in \mathcal{R}_m(\mathcal{H})$ such that the mild solution of (4.1) with

$$(y, \bar{y})(0) = \left(y^0, y^1\right) \quad \text{satisfies} \quad (y, \bar{y})(T) = (0, 0),$$

then

$$\begin{aligned} & -\left[\langle \bar{y}(0), \widetilde{\varphi}(0)\rangle_{V' \times V} - \langle y(0), \tilde{\psi}(0)\rangle_{H \times H}\right] \\ = & \int_0^T \langle u(t), B\widetilde{\varphi}(t)\rangle_{H \times H} dt + \sum_{k=1}^{k=m} \langle \bar{D}_k w_k, \widetilde{\varphi}(t_k)\rangle_{V' \times V} - \sum_{k=1}^{k=m} \langle D_k v_k, \tilde{\psi}(t_k)\rangle_{H \times H}, \end{aligned} \tag{4.13}$$

and so thanks to Cauchy-Schwarz Inequality we obtain

$$\begin{aligned} & \left|\left[\langle \bar{y}(0), \widetilde{\varphi}(0)\rangle_{V' \times V} - \langle y(0), \tilde{\psi}(0)\rangle_{H}\right]\right| \\ \leq & \left|\int_0^T \langle u(t), B\widetilde{\varphi}(t)\rangle_H dt + \sum_{k=1}^{k=m} \langle \bar{D}_k w_k, \widetilde{\varphi}(t_k)\rangle_{V' \times V} + \sum_{k=1}^{k=m} \langle D_k v_k, \tilde{\psi}(t_k)\rangle_H\right| \\ = & \left|\int_0^T \langle u(t), B\widetilde{\varphi}(t)\rangle_H dt + \sum_{k=1}^{k=m} \langle \bar{D}_k w_k, \widetilde{\varphi}(t_k)\rangle_H + \sum_{k=1}^{k=m} \langle D_k v_k, \tilde{\psi}(t_k)\rangle_H\right| \\ \leq & \left\{\int_0^T \|u(t)\|_H^2 dt + \sum_{k=1}^{k=m} \|w_k\|_H^2 + \sum_{k=1}^{k=m} \|v_k\|_H^2\right\}^{1/2} \\ & \times \left\{\int_0^T \|B\widetilde{\varphi}(t)\|_H^2 dt + \sum_{k=1}^{k=m} \left\|D_k \tilde{\psi}(t_k)\right\|_H^2 + \sum_{k=1}^{k=m} \left\|\bar{D}_k \widetilde{\varphi}(t_k)\right\|_H^2\right\}^{1/2}. \end{aligned} \tag{4.14}$$

Using Lemma **56**, and equation (4.14) we have

$$\begin{aligned} & \left|\left[\langle \bar{y}(0), \widetilde{\varphi}(0)\rangle_{V \times V'} - \langle y(0), \tilde{\psi}(0)\rangle_{H \times H}\right]\right| \\ \leq & \left\{\int_0^T \|u(t)\|_H^2 dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 + \sum_{k=1}^{k=m} \|w_k\|_H^2\right\}^{1/2} \\ & \times \left\{\int_0^T \|B\varphi(t)\|_H^2 dt + \sum_{k=1}^{k=m} \left\|\bar{D}_k \varphi((t_{m-(k-1)})\right\|_H^2 + \sum_{k=1}^{k=m} \left\|D_k \psi((t_{m-(k-1)})\right\|_H^2\right\}^{1/2}. \end{aligned}$$

Setting

$$C = \|h\|_{\mathcal{K}_m(H)} = \left\{ \int_0^T \|u(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 + \sum_{k=1}^{k=m} \|w_k\|_H^2 \right\}^{1/2},$$

we find that

$$\begin{aligned} & \left| \left[\langle \bar{y}\,(0)\,, \widetilde{\varphi}\,(0) \rangle_{V' \times V} - \langle y(0), \tilde{\psi}(0) \rangle_H \right] \right| \\ \leq \;\; & C \left\{ \int_0^T \|B\varphi(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \left\| \bar{D}_k \varphi((t_{m-(k-1)}) \right\|_H^2 + \sum_{k=1}^{k=m} \left\| D_k \psi((t_{m-(k-1)}) \right\|_H^2 \right\}^{1/2}. \end{aligned}$$

This shows that **i)** implies **ii)**.

Step 2. If $\mathcal{B} \geq \alpha > 0$ the proof of the second implication **ii)** $\Rightarrow$ **i)** in this special case is the subject of

Claim 2 Assume that there is $\alpha > 0$ such that

$$\begin{aligned} & \left\{ \int_0^T \|Bu(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|D_k v_k\|_H^2 + \sum_{k=1}^{k=m} \left\| \bar{D}_k w_k \right\|_H^2 \right\} \\ \geq \;\; & \alpha \left\{ \int_0^T \|u(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 + \sum_{k=1}^{k=m} \|w_k\|_H^2 \right\} \end{aligned}$$

then, for every $\left(y^0, y^1\right) \in \mathcal{H}$ there is $\left(\varphi^0, \varphi^1\right) \in \tilde{\mathcal{H}}$ such that the mild solution of (4.1) with

$$h(t) = \left((\tilde{\varphi}(t); \left\{ \left(\left(-\tilde{\psi}(t_k) \right), \tilde{\varphi}(t_k) \right) \right\}_{k \in \sigma_1^m} \right) \in \mathcal{R}_m\left(H\right)$$

satisfies

$$y(T) = y^{'}(T) = 0.$$

where $(\tilde{\varphi}, \tilde{\psi})$ is a solution of (4.3) with initial data $(\tilde{\varphi}(T), \tilde{\psi}(T)) = \left(\varphi^0, -\varphi^1\right)$.

Proof. For each $\left(\tilde{\varphi}^0, \tilde{\varphi}^1\right) \in \tilde{\mathcal{H}}$ there exists a unique $\left(\varphi^0, \varphi^1\right) \in \tilde{\mathcal{H}}$ such that

$$\left(\tilde{\varphi}(0), \tilde{\varphi}^{'}(0) \right) = \left(\tilde{\varphi}^0, \tilde{\varphi}^1 \right)$$

since $\mathcal{I}_k$ is invertible, and for every $(e, \mathring{e}) \in \tilde{\mathcal{H}}$ we consider the mild solution $(\tilde{\varphi}, \tilde{\psi})$ of (4.3) satisfying $\left(\tilde{\varphi}(0), \tilde{\varphi}^{'}(0)\right) = (e, \mathring{e})$ and the unique mild solution $(y, \bar{y})$ to the problem

$$\begin{cases} y^{''}(t) + Ay(t) = B\tilde{\varphi}(t), \quad t \in (0,T) \setminus \{t_k\}_{k \in \sigma_1^m}, \\ y(0) = y^0, y^{'}(0) = y^1, \\ \Delta y(t_k) = I_k^1 y(t_k) + D_k\left(-\tilde{\psi}(t_k)\right), \ \ k \in \sigma_1^m, \\ \Delta y^{'}(t_k) = I_k^2 y^{'}(t_k) + \bar{D}_k \tilde{\varphi}(t_k), \ \ k \in \sigma_1^m. \end{cases}$$

Next, we introduce a bounded linear operator $\Lambda : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}^{'}$ defined by

$$\Lambda(e, \mathring{e}) = (-y^{'}(0), Ay(0)).$$

According to formula (4.13) and Corollary **57** we get

$$\begin{aligned} & \left|\langle \Lambda(e, \mathring{e}), \{e, \mathring{e}\}\rangle_{\tilde{\mathcal{H}}' \times \tilde{\mathcal{H}}}\right| \\ = & \left|-\left[\langle z(0), \widetilde{\varphi}(0)\rangle_{V' \times V} - \langle y(0), \tilde{\psi}(0)\rangle_H\right]\right| \\ = & \left|\int_0^T \langle B\widetilde{\varphi}(t), \widetilde{\varphi}(t)\rangle_H dt + \sum_{k=1}^{k=m} \langle \bar{D}_k \widetilde{\varphi}(t_k), \widetilde{\varphi}(t_k)\rangle_{V' \times V} - \sum_{k=1}^{k=m} \langle D_k\left(-\tilde{\psi}(t_k)\right), \tilde{\psi}(t_k)\rangle_H\right| \\ \leq & \left|\int_0^T \langle B\varphi(t), \varphi(t)\rangle_H dt + \sum_{k=1}^{k=m} \langle \bar{D}_k \varphi(t_{m-(k-1)}), \varphi(t_{m-(k-1)})\rangle_H\right. \\ & \left. + \sum_{k=1}^{k=m} \langle D_k \psi(t_{m-(k-1)}), \psi(t_{m-(k-1)})\rangle_H\right| \\ \leq & \ \varsigma\left\{\int_0^T \|\varphi(t)\|^2 dt + \sum_{k=1}^{k=m} \|\varphi(t_k)\|_H^2 + \sum_{k=1}^{k=m} \|\psi(t_k)\|_H^2\right\}, \end{aligned}$$

where

$$\varsigma = \max\left\{b, \sup_{k \in \sigma_1^m}\{d_k\}, \sup_{k \in \sigma_1^m}\{\bar{d}_k\}\right\} < \infty.$$

We also have

$$\int_0^T \|(\varphi(t), \psi(t))\|_{\tilde{\mathcal{H}}}^2 dt = \int_0^{t_1} \|(\varphi(t), \psi(t))\|_{\tilde{\mathcal{H}}}^2 dt + \int_{t_1}^{t_2} \|(\varphi(t), \psi(t))\|_{\tilde{\mathcal{H}}}^2 dt + ... + \int_{t_m}^T \|(\varphi(t), \psi(t))\|_{\tilde{\mathcal{H}}}^2 dt.$$

Since there is no impulse in the interval $[t_k, t_{k+1})$ we have

$$\left\|\left(\varphi(t_k^+), \psi(t_k^+)\right)\right\|_{\tilde{\mathcal{H}}}^2 = \left\{ \left\|\varphi(t_k^+)\right\|_V^2 + \left\|\varphi^{'}(t_k^+)\right\|_H^2 \right\}, \text{ for every } t \in [t_k, t_{k+1}), \ k \in \sigma_0^m.$$

and

$$\left\{ \left\|\varphi(t_{k+1}^-)\right\|_V^2 + \left\|\varphi^{'}(t_{k+1}^-)\right\|_H^2 \right\} = \left\{ \left\|\varphi(t_k^+)\right\|_V^2 + \left\|\varphi^{'}(t_k^+)\right\|_H^2 \right\}, \quad k \in \sigma_0^m. \tag{4.15}$$

Therefore, there are $\tau_{k+1} = t_{k+1} - t_k > 0$, $k \in \sigma_0^m$, such that

$$\begin{aligned} & \int_{t_k}^{t_{k+1}} \left\|(\varphi(t), \psi(t))\right\|_{\tilde{\mathcal{H}}}^2 dt \\ \leq \ & \rho_k \left\{ \left\|\varphi(t_k^+)\right\|_V^2 + \left\|\varphi^{'}(t_k^+)\right\|_H^2 \right\} \\ = \ & \tau_{k+1} \left\{ \left\|I_k^1 \varphi(t_k^-) + \varphi(t_k^-)\right\|_V^2 + \left\|I_k^2 \varphi^{'}(t_k^-) + \varphi^{'}(t_k^-)\right\|_H^2 \right\}, \ k \in \sigma_1^m. \end{aligned} \tag{4.16}$$

On the other hand, the continuity of $I_k, \bar{I}_k$ implies that

$$\left\|\varphi(t_k^+)\right\|_V^2 = \left\|(I_k + I)\varphi(t_k^-)\right\|_V^2 \leq (1 + L(I_k))^2 \left\|\varphi(t_k^-)\right\|_V^2, \quad k \in \sigma_1^m. \tag{4.17}$$

$$\left\|\varphi^{'}(t_k^+)\right\|_H^2 = \left\|(\bar{I}_k + I)\varphi^{'}(t_k^-)\right\|_H^2 \leq (1 + L(\bar{I}_k))^2 \left\|\varphi^{'}(t_k^-)\right\|_H^2, \quad k \in \sigma_1^m, \tag{4.18}$$

where

$$L(I_k), \ L(\bar{I}_k) > 0, \quad \forall k \in \sigma_1^m.$$

It follows from (4.16)-(4.18) that

$$\textstyle\int_{t_k}^{t_{k+1}} \left\|(\varphi(t), \psi(t))\right\|_{\tilde{\mathcal{H}}}^2 dt \leq r_k \left\{ \left\|\varphi(t_k^-)\right\|_V^2 + \left\|\varphi^{'}(t_k^-)\right\|_H^2 \right\}, \quad k \in \sigma_1^m, \tag{4.19}$$

where

$$r_k = \tau_k \max\left\{ (1 + L(I_k^1)), (1 + L(I_k^2)) \right\}.$$

Since m is finite, and due to (4.15), (4.19), then there is a constant $0 < \mu < \infty$ such that

$$\left\langle \Lambda\left(e, \mathring{e}\right), \left(e, \mathring{e}\right)\right\rangle_{\tilde{\mathcal{H}}' \times \tilde{\mathcal{H}}} \leq \mu \left\|(e, \mathring{e})\right\|_{\tilde{\mathcal{H}}}^2,$$

and thus, Λ is bounded.

Now, as $\mathcal{B}$ is nonnegative in $\mathcal{R}_m(H)$, we have

$$\|\mathcal{B}(t)\| \geq \alpha \left\{ (\xi(t), \xi(t))_{\mathcal{R}_m(H)} \right\}^{1/2},$$

for all $\xi \in \mathcal{R}_m(H)$; thus, by virtue of Lemma **55**, we have

$$\begin{aligned} & \left\{ \int_0^T (Bu(t), u(t))_H dt + \sum_{k=1}^{k=m} \left\langle \bar{D}_k w_k, w_k \right\rangle_{V' \times V} + \sum_{k=1}^{k=m} \langle D_k v_k, v_k \rangle_{H \times H} \right\} \\ \geq \quad & \alpha \|\mathcal{B}\| \left\{ \int_0^T \|u(t)\|_H^2 dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 + \sum_{k=1}^{k=m} \|w_k\|_H^2 \right\}. \end{aligned} \tag{4.20}$$

It follows from (4.13), (4.20) and Corollary **57** that

$$\begin{aligned} & \langle \Lambda \{e, \mathring{e}\}, \{e, \mathring{e}\} \rangle_{\mathcal{H}'} = -\left[\langle \bar{y}(0), \widetilde{\varphi}(0) \rangle_H - \langle y(0), \tilde{\psi}(0) \rangle_{V' \times V} \right] \\ = \quad & \int_0^T \langle B\widetilde{\varphi}(t), \widetilde{\varphi}(t) \rangle_H dt + \sum_{k=1}^{k=m} \langle \bar{D}_k \widetilde{\varphi}(t_k), \widetilde{\varphi}(t_k) \rangle_{V' \times V} - \sum_{k=1}^{k=m} \langle D_k \left(-\tilde{\psi}(t_k) \right), \tilde{\psi}(t_k) \rangle_{H \times H} \\ \geq \quad & \left\{ \int_0^T \langle B\varphi(t), \varphi(t) \rangle_H dt + \sum_{k=1}^{k=m} \langle D_k \psi(t_{m-(k-1)}), \psi(t_{m-(k-1)}) \rangle_H \right. \\ & \left. + \sum_{k=1}^{k=m} \langle \bar{D}_k \varphi(t_{m-(k-1)}), \varphi(t_{m-(k-1)}) \rangle_H \right\} \\ \geq \quad & \alpha \|\mathcal{B}\| \left\{ \int_0^T \|\varphi(t)\|_H^2 dt + \sum_{k=1}^{k=m} \left\|\varphi(t_{m-(k-1)}))\right\|_H^2 + \sum_{k=1}^{k=m} \left\|\varphi^{'}(t_{m-(k-1)}))\right\|_H^2 \right\} \\ \geq \quad & \alpha \|\mathcal{B}\| \int_{t_m}^T \|\varphi(t)\|^2 dt = \|\mathcal{B}\| \alpha \tau_{m+1} \left\{ \|\varphi(T)\|_H^2 + \left\|\varphi^{'}(T)\right\|_H^2 \right\} = \theta \|\{e, \mathring{e}\}\|_{\tilde{\mathcal{H}}}^2, \end{aligned}$$

because there is no impulse over the time interval $(t_m, T]$.

Therefore, $\Lambda : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}^{'}$ is coercive. To show that there is a bijection from $\tilde{\mathcal{H}}$ onto $\tilde{\mathcal{H}}^{'}$, it suffices to prove that $\Lambda + \mathcal{I}$ is a bijection from $\tilde{\mathcal{H}}$ onto $\tilde{\mathcal{H}}^{'}$. Clearly, $\Lambda + \mathcal{I}$ is one to one since

$$\langle \Lambda (e, \mathring{e}) + (e, \mathring{e}), (e, \mathring{e}) \rangle_{\tilde{\mathcal{H}}' \times \tilde{\mathcal{H}}} = \langle \Lambda (e, \mathring{e}), (e, \mathring{e}) \rangle_{\tilde{\mathcal{H}}' \times \tilde{\mathcal{H}}} + \langle (e, \mathring{e}), (e, \mathring{e}) \rangle_{\tilde{\mathcal{H}}} \geq (\theta + 1) \|(e, \mathring{e})\|_{\tilde{\mathcal{H}}}^2.$$

On the other hand, let $\left(y^0, y^1\right) \in \mathcal{H}^{'}$, as the form

$$a(\left(f, \mathring{f},\right)(g, \mathring{g})) + \langle\left(f, \mathring{f}\right), (g, \mathring{g})\rangle_{\tilde{\mathcal{H}}} = \langle\Lambda\left(f, \mathring{f}\right), (g, \mathring{g})\rangle_{\tilde{\mathcal{H}}' \times \tilde{\mathcal{H}}} + \langle\left(f, \mathring{f}\right), (g, \mathring{g})\rangle_{\tilde{\mathcal{H}}}$$

is symmetric and coercive, then, by virtue of Lax-Milgram Theorem, there is an element $\left(f, \mathring{f}\right) \in \tilde{\mathcal{H}}$ such that

$$a(\left(f, \mathring{f}\right), (g, \mathring{g})) + \langle\left(f, \mathring{f}\right), (g, \mathring{g})\rangle_{\tilde{\mathcal{H}}} = \langle\left(y^0, y^1\right), (g, \mathring{g})\rangle_{\tilde{\mathcal{H}}' \times \tilde{\mathcal{H}}}, \text{ for all } (g, \mathring{g}) \in \tilde{\mathcal{H}}.$$

This implies that $\Lambda(\tilde{\mathcal{H}}) = \tilde{\mathcal{H}}^{'}$. Thus, for every $\left(y^0, y^1\right) \in \mathcal{H}$, there is a unique $(e, \mathring{e}) \in \tilde{\mathcal{H}}$ such that $\Lambda((e, \mathring{e})) = \left\{-y^{'}(0), Ay(0)\right\}$, which completes the proof of Claim **2**.

Step 3: Assume that : $B, D_k, \bar{D}_k \geq 0$, then $\mathcal{B} \geq 0$,

$$\check{B}^2 = B, \quad \check{D}_k^2 = D_k, \quad \overline{\check{D}}^2 = \bar{D}_k.$$

We define for $\varepsilon > 0$,

$$\beta^\varepsilon \doteqdot \check{B}^2 + \varepsilon I,$$

$$\delta_k^\varepsilon \doteqdot \check{D}_k^2 + \varepsilon I,$$

$$\bar{\delta}_k^\varepsilon \doteqdot \overline{\check{D}}^2 + \varepsilon I,$$

and

$$\mathcal{B}^\varepsilon \doteqdot (\beta^\varepsilon; \delta_1^\varepsilon, .., \delta_m^\varepsilon; \bar{\delta}_1^\varepsilon, .., \bar{\delta}_m^\varepsilon) = \left(\check{B}^2 + \varepsilon I; \left\{\left(\check{D}_k^2 + \varepsilon I, \overline{\check{D}}^2 + \varepsilon I\right)\right\}_{k \in \sigma_1^m}\right).$$

According to Claim **2**, there is $\left(\tilde{\varphi}^{0,\varepsilon}, \tilde{\varphi}^{1,\varepsilon}\right) \in \tilde{\mathcal{H}}$ such that the mild solution $(y_\varepsilon, \bar{y}_\varepsilon)$ of (4.1) with $(y_\varepsilon(0), \bar{y}_\varepsilon(0)) = \left(y^0, y^1\right)$ satisfies $y_\varepsilon(T) = y_\varepsilon^{'}(T) = 0$; where $\mathcal{B}(h)$ has been replaced by

$$\mathcal{B}^\varepsilon\left(\tilde{\varphi}(t); \left\{\left(-\tilde{\psi}(t_k), \tilde{\varphi}(t_k)\right)\right\}_{k \in \sigma_1^m}\right) \in \mathcal{R}_m(H).$$

We obtain from (4.13) and Corollary **57**

$$\begin{aligned}
&-\left[\langle \bar{y}_\varepsilon\left(0\right), \widetilde{\varphi}_\varepsilon\left(0\right)\rangle_{V'\times V} - \langle y_\varepsilon(0), \tilde{\psi}_\varepsilon(0)\rangle_{H\times H}\right] \\
= \; & \int_0^T \langle \beta^\varepsilon \tilde{\varphi}_\varepsilon(t), \tilde{\varphi}_\varepsilon(t)\rangle_H dt + \sum_{k=1}^{k=m} \langle \bar{\delta}_k^\varepsilon \tilde{\varphi}_\varepsilon(t_k), \tilde{\varphi}_\varepsilon(t_k)\rangle_{V'\times V} \\
& - \sum_{k=1}^{k=m} \langle \delta_k^\varepsilon \left(-\tilde{\psi}_\varepsilon(t_k)\right), \tilde{\psi}_\varepsilon(t_k)\rangle_{H\times H},
\end{aligned} \tag{4.21}$$

and (4.9) gives

$$\begin{aligned}
&-\left[\langle \bar{y}\left(0\right), \widetilde{\varphi}_\varepsilon\left(0\right)\rangle_{V'\times V} - \langle y(0), \tilde{\psi}_\varepsilon(0)\rangle_{H\times H}\right] \\
\leq \; & C\left\{\int_0^T \left\|B\varphi_\varepsilon(t)\right\|_H^2 dt + \sum_{k=1}^{k=m} \left\|\overline{\check{D}}_k \varphi_\varepsilon(t_{m-(k-1)})\right\|_H^2 + \sum_{k=1}^{k=m} \left\|\check{D}_k \psi_\varepsilon(t_{m-(k-1)})\right\|_H^2\right\}^{1/2}.
\end{aligned} \tag{4.22}$$

Whence,

$$\begin{aligned}
&\left[-\langle \bar{y}\left(0\right), \widetilde{\varphi}_\varepsilon\left(0\right)\rangle_{V'\times V} - \langle y(0), \tilde{\psi}_\varepsilon(0)\rangle_{H\times H}\right] \\
\leq \; & C\left\{\int_0^T \langle \beta^\varepsilon \tilde{\varphi}_\varepsilon(t), \tilde{\varphi}_\varepsilon(t)\rangle_H dt + \sum_{k=1}^{k=m} \langle \bar{\delta}_k^\varepsilon \tilde{\varphi}_\varepsilon(t_k), \tilde{\varphi}_\varepsilon(t_k)\rangle_{V'\times V} \right. \\
& \left. - \sum_{k=1}^{k=m} \langle\langle \delta_k^\varepsilon \left(-\tilde{\psi}_\varepsilon(t_k)\right), \tilde{\psi}_\varepsilon(t_k)\rangle\rangle_{H\times H}\right\}^{1/2}.
\end{aligned} \tag{4.23}$$

It follows at once from (4.21) (4.22) and (4.23) that

$$\begin{aligned}
&\varepsilon\left\{\int_0^T \left\|\tilde{\varphi}_\varepsilon(t)\right\|_H^2 dt + \sum_{k=1}^{k=m} \left\|\tilde{\varphi}_\varepsilon(t_k)\right\|_H^2 + \sum_{k=1}^{k=m} \left\|\tilde{\psi}_\varepsilon(t_k)\right\|_H^2\right\} \\
&+ \int_0^T \langle \check{B}^2 \tilde{\varphi}_\varepsilon(t), \tilde{\varphi}_\varepsilon(t)\rangle_H dt + \textstyle\sum_{k=1}^{k=m} \langle \overline{\check{D}}_k^2 \tilde{\varphi}_\varepsilon(t_k), \tilde{\varphi}_\varepsilon(t_k)\rangle_{V'\times V} + \sum_{k=1}^{k=m} \langle \check{D}_k \tilde{\psi}_\varepsilon(t_k), \tilde{\psi}_\varepsilon(t_k)\rangle_H \\
&= \int_0^T \langle \beta^\varepsilon \tilde{\varphi}_\varepsilon(t), \tilde{\varphi}_\varepsilon(t)\rangle_H dt + \textstyle\sum_{k=1}^{k=m} \langle \bar{\delta}_k^\varepsilon \tilde{\varphi}_\varepsilon(t_{m-(k-1)}), \tilde{\varphi}_\varepsilon(t_{m-(k-1)})\rangle_H \\
&+ \textstyle\sum_{k=1}^{k=m} \langle \delta_k^\varepsilon \psi_\varepsilon(t_{m-(k-1)}), \psi_\varepsilon(t_{m-(k-1)}))\rangle_H \\
&\leq C^2.
\end{aligned} \tag{4.24}$$

Step 4: According to the estimate (4.23) the family

$$\begin{aligned} b_\varepsilon &= \mathcal{B}^\varepsilon \left((\tilde{\varphi}_\varepsilon(t); \left\{ \left(-\tilde{\psi}_\varepsilon(t_k), \tilde{\varphi}_\varepsilon(t_k) \right) \right\}_{k\in\sigma_1^m} \right) \\ &= \left((\check{B}^2 \tilde{\varphi}_\varepsilon(t); \left\{ \left(\check{D}_k^2 \left(-\tilde{\psi}_\varepsilon(t_k) \right), \overline{\check{D}}_k^2 \tilde{\varphi}_\varepsilon(t_k) \right) \right\}_{k\in\sigma_1^m} \right) \\ &\quad + \varepsilon \left((\tilde{\varphi}_\varepsilon(t); \left\{ \left(-\tilde{\psi}_\varepsilon(t_k), \tilde{\varphi}_\varepsilon(t_k) \right) \right\}_{k\in\sigma_1^m} \right) \end{aligned}$$

is contained in a bounded subset $\mathcal{R}_m(H)$.

Thus, both of the families

$$\begin{aligned} &\sqrt{\varepsilon} \left((\tilde{\varphi}_\varepsilon(t); \left\{ \left(-\tilde{\psi}_\varepsilon(t_k), \tilde{\varphi}_\varepsilon(t_k) \right) \right\}_{k\in\sigma_1^m} \right) \\ \text{and } &\left((B\tilde{\varphi}_\varepsilon(t); \left\{ \left(D_k \left(-\tilde{\psi}_\varepsilon(t_k) \right), \bar{D}_k \tilde{\varphi}_\varepsilon(t_k) \right) \right\}_{k\in\sigma_1^m} \right) \end{aligned}$$

are bounded in $\mathcal{R}_m(H)$. Therefore, we may extract a subfamily, say

$$\left((B\tilde{\varphi}_\varepsilon(t); \left\{ \left(D_k \left(-\tilde{\psi}_\varepsilon(t_k) \right), \bar{D}_k \tilde{\varphi}_\varepsilon(t_k) \right) \right\}_{k\in\sigma_1^m} \right) \rightharpoonup h, \text{ weakly in } \mathcal{R}_m(H).$$

Then clearly

$$\left(\check{B}^2 (\tilde{\varphi}_\varepsilon(t); \left\{ \left(\check{D}_k^2 \left(-\tilde{\psi}_\varepsilon(t_k) \right), \overline{\check{D}}_k^2 \tilde{\varphi}_\varepsilon(t_k) \right) \right\}_k \right) + \varepsilon\sqrt{\varepsilon} \left((\tilde{\varphi}_\varepsilon(t); \left\{ \left(-\tilde{\psi}_\varepsilon(t_k), \tilde{\varphi}_\varepsilon(t_k) \right) \right\}_k \right)$$

$\rightharpoonup \mathcal{B}h$, weakly in $\mathcal{R}_m(H)$.

Step 5: Taking the limit as $\varepsilon \to 0$, we see that the solution y of (4.1) with initial condition $\left(y(0), y^{'}(0) \right) = \left(y^0, y^1 \right)$, h being as in step 4, satisfies $y(T) = y^{'}(T) = 0$. This completes the proof of Theorem **58**.$\blacksquare$

As an immediate application of the previous Theorem we give the following example.

Example 59 *One dimensional impulsive wave equation :*

We consider the problem

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \dfrac{\partial^2 y}{\partial x^2}\,(t,x) = \chi_\omega u\,(t,x)\,, \qquad t \in (0,T) \setminus \{t_k\}_{k\in\sigma_1^m}\,,\ x \in \Omega, \\ \qquad\qquad\qquad\qquad y(t,0) = y(t,2\pi) = 0 \\ y(x,t_k^+) - y(x,t_k^-) = \alpha_k y\,(t_k,x) + p_k \chi_{\omega_k^a} v_k(x), \quad k \in \sigma_1^m, \\ y_t(x,t_k^+) - y_t(x,t_k^-) = \beta_k y\,(t_k,x) + q_k \chi_{\omega_k^b} w_k(x), \quad k \in \sigma_1^m, \\ y(x,0^+) = y^0(x), \quad y_t(x,0^+) = y^1(x), \quad x \in \Omega. \end{cases} \tag{4.25}$$

where $t_{k+1} - t_k > 2\pi$, $\Omega = (0,2\pi)$ *and*

$$\omega,\ \omega_k^a = (a_1^k, a_2^k),\ \omega_k^b = (b_1^k, b_2^k) \subset (0,2\pi),\ k \in \sigma_1^m, \quad \{\alpha_k, \beta_k\}_{k\in\sigma_1^m}\,, \ \{p_k, q_k\}_{k\in\sigma_1^m} \subset \mathbb{R}^+ \times \mathbb{R}^+.$$

Let $V \times H = H_0^1(\Omega) \times L^2(\Omega)$,

$$Aw(x) = -\frac{\partial^2 w}{\partial x^2}(x), \quad D(A) = H_0^1(\Omega),$$

and

$$I_k w(x) = \alpha_k w(x), \quad \bar{I}_k w(x) = \beta_k w(x)$$

and the control operators are given by

$$B = \chi_\omega, \quad D_k = p_k \chi_{\omega_k^a}, \quad \bar{D}_k = q_k \chi_{\omega_k^b}$$

then the system (4.25) becomes an abstract formulation of (4.1).
*As a consequence of Theorem **58**, the initial state* $\left(y^0, y^1\right) \in H_0^1(\Omega) \times L^2(\Omega) = \tilde{\mathcal{H}}$ *of the solution of (4.25) is null-controllable at* $t = T$, *if and only if, there exists a* $C > 0$ *such that*

$$\begin{aligned} &\left| \left\langle y^0(x), \tilde{\varphi}^1(x) \right\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}) dx - \int_\Omega y^1(x) \tilde{\varphi}^0(x) dx \right| \\ \leq \;\; & C \left\{ \int_0^T \int_\omega |\varphi|^2\,(t,x) dx dt + \sum_{k=1}^m \int_{\omega_k^b} p_k^2 \left|\varphi^{'}\right|^2 (t_{m-(k-1)}, x) dx + \sum_{k=1}^m \int_{\omega_k^b} q_k^2 \,|\varphi|^2\,(t_{m-(k-1)}, x) dx \right\}^{\frac{1}{2}} \end{aligned}$$

where where $(\tilde{\varphi}^0, \tilde{\varphi}^1) = (\tilde{\varphi}(0), \psi(0)) = \left(\varphi(T), -\varphi'(T)\right)$ *and* $\left(\varphi, \varphi'\right)$ *is the mild solution of*

$$\begin{cases} \dfrac{\partial^2 \varphi}{\partial t^2} - \dfrac{\partial^2 \varphi}{\partial x^2}(t,x) = 0, \quad t \in (0,T) \setminus \{t_k\}_{k \in \sigma_1^m}, \ x \in \Omega, \\ \varphi(t,0) = \varphi(t,2\pi) = 0 \\ \varphi(x,t_k^+) - \varphi(x,t_k^-) = \alpha_k \varphi(t_k,x), \ k \in \sigma_1^m, \\ \varphi_t(x,t_k^+) - \varphi_t(x,t_k^-) = \beta_k \varphi_t(t_k,x), \ k \in \sigma_1^m, \\ \varphi(x,0^+) = \varphi^0(x), \quad \varphi_t(x,0^+) = \varphi^1(x), \qquad x \in \Omega, \end{cases}$$

in $H_0^1(\Omega) \times L^2(\Omega)$.

Here $\left(\varphi, \varphi'\right)$ *is given by*

$$\left(\varphi(t), \varphi'(t)\right) = \begin{cases} (\varphi_{[0]}(t), \psi_{[0]}(t)) & \text{if} \quad t \in [t_0, t_1) \\ (\varphi_{[k]}(t), \psi_{[k]}(t)) & \text{if} \quad t \in \left[t_k, t_{k+1)}\right) \\ (\varphi_{[m}(t), \psi_{[m]}(t)) & \text{if} \quad t \in [t_m, T], \end{cases}$$

where $(\varphi_{[k]}(t), \psi_{[k]}(t))$ *is a solution of the classical wave equation*

$$\begin{cases} \dfrac{\partial^2 \varphi_{[0]}}{\partial t^2} - \dfrac{\partial^2 \varphi_{[0]}}{\partial x^2}(t,x) = 0, & t \in (t_0, t_1), \ x \in \Omega, \\ \varphi_{[0]}(t,0) = \varphi_{[0]}(t,2\pi) = 0 \\ \varphi_{[0]}(x,0^+) = \varphi^0(x), \quad \left(\varphi_{[0]}\right)_t(x,0^+) = \varphi^1(x), & x \in \Omega, \end{cases}$$

and

$$\begin{cases} \dfrac{\partial^2 \varphi_{[k]}}{\partial t^2} - \dfrac{\partial^2 \varphi_{[k]}}{\partial x^2}(t,x) = 0, \quad t \in (t_k, t_{k+1}), \ x \in \Omega, \\ \varphi_{[k]}(t,0) = \varphi_{[k]}(t,2\pi) = 0 \\ \varphi_{[k]}((x,t_k^+) = (1+\alpha_k)\varphi(t_k,x), \ k \in \sigma_1^m, \\ \varphi_t(x,t_k^+) - \varphi_t(x,t_k^-) = (1+\beta_k)\varphi_t(t_k,x), \ k \in \sigma_1^m, \\ \varphi(x,0^+) = \varphi^0(x), \quad \varphi_t(x,0^+) = \varphi^1(x), \quad x \in \Omega, \end{cases}$$

Then a standard application of a variant to Ingham's Inequality [53] shows that

$$\int_{t_k}^{t_{k+1}} \int_\omega \left|\varphi_{[k]}\right|(t,x)dtdx \geq c(\tau_k, \omega)\left[\int_\Omega \left|\varphi_{[k]}\right|(t_k^+, x)dx + \int_\Omega \left|\varphi'_{[k]}\right|(t_k^+, x)dx\right],$$

for some positive constants $c(\tau_k, \omega) > 0$. *Summing up we get*

$$\begin{aligned}
\sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} \int_{\omega} \left|\varphi_{[k]}\right| (t,x)dtdx &= \int_0^T \int_{\omega} |\varphi|^2 (t,x)dxdt \\
&\geq c_1 \sum_{k=0}^{m} \left[\int_{\Omega} \left|\varphi_{[k]}\right| (t_k^+, x)dx + \int_{\Omega} \left|\varphi_{[k]}^{'}\right| (t_k^+, x)dx \right] \\
&\geq c_1 \sum_{k=1}^{m} \left[\int_{\Omega} \left|\varphi_{[k]}\right| (t_k^+, x)dx + \int_{\Omega} \left|\varphi_{[k]}^{'}\right| (t_k^+, x)dx \right],
\end{aligned}$$

where $c_1 = \min\limits_{k \in \sigma_0^m} c(\tau_k, \omega) > 0$.

On the other hand, there is a positive constant $c_2 > 0$ such that

$$\begin{aligned}
&\sum_{k=1}^{m} \left[\int_{\omega_k^a} |\varphi| (t_{m-(k-1)}, x)dx + \int_{\omega_k^b} \left|\varphi^{'}\right| (t_{m-(k-1)}, x)dx \right] \\
\geq\ & c_2 \sum_{k=1}^{m} \left[\int_{\Omega} \left|\varphi_{[k]}\right| (t_k^+, x)dx + \int_{\Omega} \left|\varphi_{[k]}^{'}\right| (t_k^+, x)dx \right].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_0^T \int_{\omega} |\varphi|^2 (t,x)dxdt + \sum_{k=1}^{m} \int_{\omega_k^a} \left|\varphi^{'}\right|^2 (t_{m-(k-1)}, x) + \sum_{k=1}^{m} \int_{\omega_k^b} |\varphi|^2 (t_{m-(k-1)}, x) \\
\geq\ & (c_1 + c_2) \sum_{k=1}^{m} \left[\int_{\Omega} \left|\varphi_{[k]}\right|^2 (t_k^+, x)dx + \int_{\Omega} \left|\varphi_{[k]}^{'}\right|^2 (t_k^+, x)dx \right] \\
\geq\ & (c_1 + c_2) \left[\int_{\Omega} \left|\varphi_{[m]}\right|^2 (t_k^+, x)dx + \int_{\Omega} \left|\varphi_{[m]}^{'}\right|^2 (t_k^+, x)dx \right] \\
=\ & (c_1 + c_2) \left[\int_{\Omega} |\varphi|^2 (T, x)dx + \int_{\Omega} \left|\varphi^{'}\right|^2 (T, x)dx \right].
\end{aligned}$$

Now, since

$$\left(\tilde{\varphi}^0, \tilde{\varphi}^1\right) = (\tilde{\varphi}(0), \psi(0)) = \left(\varphi\left(T\right), -\varphi^{'}\left(T\right)\right)$$

then,

$$\begin{aligned}
&\int_0^T \int_{\omega} |\varphi|^2 (t,x)dxdt + \sum_{k=1}^{m} \int_{\omega_k^a} \left|\varphi^{'}\right|^2 (t_{m-(k-1)}, x) + \sum_{k=1}^{m} \int_{\omega_k^b} |\varphi|^2 (t_{m-(k-1)}, x) \\
\geq\ & m(c_1 + c_2) \left[\int_{\Omega} \left|\widetilde{\varphi}^0\right|^2 (x)dx + \int_{\Omega} \left|\widetilde{\varphi}^1\right|^2 (x)dx \right],
\end{aligned}$$

from which we get

$$\begin{aligned}
&\left[\int_\Omega \left|\widetilde{\varphi}^0\right|^2(x)dx + \int_\Omega \left|\widetilde{\varphi}^1\right|^2(x)dx\right] \\
\leq\ & \frac{1}{m(c_1+c_2)}\left(\int_0^T\int_\omega |\varphi|^2(t,x)dxdt + \sum_{k=1}^m \int_{\omega_k^a}\left|\varphi'\right|^2(t_{m-(k-1)},x) + \sum_{k=1}^m\int_{\omega_k^b}|\varphi|^2(t_{m-(k-1)},x)\right).
\end{aligned}$$

We conclude by Cauchy-Schwarz inequality that

$$\begin{aligned}
&\left|\left\langle y^0(x),\tilde{\varphi}^1(x)\right\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)})dx - \int_\Omega y^1(x)\tilde{\varphi}^0(x)dx\right| \\
\leq\ & \left\{\left[\int_\Omega \left|y^0\right|^2(x)dx + \int_\Omega \left|y^1\right|^2(x)dx\right]\left[\int_\Omega \left|\widetilde{\varphi}^0\right|^2(x)dx + \int_\Omega \left|\widetilde{\varphi}^1\right|^2(x)dx\right]\right\}^{1/2} \\
\leq\ & \left\{\frac{\left[\int_\Omega \left|\widetilde{\varphi}^0\right|^2(x)dx + \int_\Omega \left|\widetilde{\varphi}^1\right|^2(x)dx\right]}{m(c_1+c_2)}\right\}^{1/2}\left(\int_0^T\int_\omega |\varphi|^2(t,x)dxdt\right. \\
&\left. + \sum_{k=1}^m \int_{\omega_k^a}\left|\varphi'\right|^2(t_{m-(k-1)},x) + \sum_{k=1}^m\int_{\omega_k^b}|\varphi|^2(t_{m-(k-1)},x)\right)^{1/2},
\end{aligned}$$

which establishes the necessary and sufficient condition of null controllability stated in Theorem **58.**■

Conclusions

In this thesis we has studied the controllability of a class of impulsive systems in infinite dimensional Hilbert spaces. After introducing some results about impulsive evolution systems in chapter 1, existence results have been obtained for impulsive evolution systems in Chapter 2. These results have been applied to the controllability problem. Null and exact controllability questions of evolution systems with fixed impulses have been solved in Chapter 3 and 4 by the well-known Hilbert Uniqueness Method (**HUM**).

A necessary and sufficient condition for the controllability of such systems has been established. Finally, a concrete example in Partial Differential Equations has been given to illustrate the results.

The impulsive controllability results obtained before by other authors used fixed point theorems, hawever the proofs presented in chapter 3 and 4 do not depend on this method.

We intend in future works to study the exact controllability of impulsive systems with infinite impulses with respect to some suitable topologies.

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Electronic Journal of Qualitative Theory of Differential Equations
2007, No. **19**, 1-21; **http://www.math.u-szeged.hu/ejqtde/**

Null Controllability of Some Impulsive Evolution Equation in a Hilbert Space

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Abstract

We shall establish a necessary and sufficient condition under which we have the null controllability of some first order impulsive evolution equation in a Hilbert space.

MSC(2000) : 34A37, 93B05, 93C15.
Keywords: *Null-controllability, impulsive conditions, mild solutions, evolution equation.*

1 Introduction

The problem of exact controllability of linear systems represented by infinite conservative systems has been extensively studied by several authors A. Haraux [8], R.Triggiani [16], Z.H. Guan, T.H. Qian, and X.Yu [7], see also the references [1, 2, 6, 10,15]. In the sequel, we shall be concerned with the problem of null controllability of some first order evolution equation subject to impulsive conditions and so we shall derive a necessary and sufficient condition under which null controllability occurs. Actually, we shall establish an equivalence between the null-controllability and some "observability" inequality in somehow more general framework than that proposed by A Haraux [8]. Regarding the literature on the impulsive differential equations we refer the reader to the works of D.D. Bainov and P.S. Simeonov [3, 4] and the references [5, 9,11, 12, 13]. We are going to study the following problem

$$\begin{aligned} y^{'}(t)+Ay(t) &= Bu(t), \quad t\in(0,T)\setminus\{t_k\}_{k\in\sigma_1^m}, && (1)\\ y(0) &= y^0, \\ \Delta y(t_k) &= I_k y(t_k)+D_k v_k, \ \ k\in\sigma_1^m, && (1_k) \end{aligned}$$

where the final time T is a positive number, y^0 is an initial condition in a Hilbert space H endowed with an inner product $\langle .,.\rangle_H$, $y(t):[0,T]\to H$ is a vector function, σ_1^m is a subset of $\mathbb{N}$ given by $\sigma_1^m=\{1,2,...,m\}$, and finally, $\{t_k\}_{k\in\sigma_1^m}$ is an increasing sequence of numbers in the open interval $(0,T)$, and $\Delta y(t_k)$ denotes the jump of $y(t)$ at $t=t_k$, *i.e.*,

$$\Delta y(t_k)=y\left(t_k^+\right)-y\left(t_k^-\right)$$

where $y\left(t_k^+\right)$ and $y\left(t_k^-\right)$ represent the right and left limits of $y(t)$ at $t=t_k$ respectively. On the other hand, the operators A, B, I_k, $D_k : H\to H$ are given linear bounded operators. Moreover, we set the following assumptions:

(H1) $A^*=-A$,

(H2) $I_k^*=-I_k$, for every $k\in\sigma_1^m$, and for each $k\in\sigma_1^m$, the operator $\mathcal{I}_k=I_k+I$ is invertible,

(H3) $B^*=B\geq 0$ and there is $d_0>0$ such that

$$(Bu,u)_H\leq d_0\|u\|_H^2, \ \text{ for all } u\in H,$$

(H4) $D_k^*=D_k\geq 0$, for every $k\in\sigma_1^m$, and for each $k\in\sigma_1^m$ there is $d_k>0$ such that

$$(D_k u,u)_H\leq d_k\|u\|_H^2, \ \text{ for all } u\in H.$$

In the sequel we shall designate by h the function

$$h(t)=\left(u(t),\{v_k\}_{k\in\sigma_1^m}\right),$$

where $u(t)\in L^2\left((0,T)\setminus\{t_k\}_{k\in\sigma_1^m};H\right)$ and

$$\{v_k\}_{k\in\sigma_1^m}\in l^2\left(\sigma_1^m;H\right)\doteqdot\left\{\{v_k\}_{k\in\sigma_1^m}, v_k\in H\right\}.$$

We point out that the space $\mathcal{K}_m = L^2\left((0,T) \setminus \{t_k\}_{k\in\sigma_1^m}; H\right) \times l^2(\sigma_1^m; H)$ is a Hilbert space with respect to the inner product

$$\left(h, \widetilde{h}\right)_{\mathcal{K}_m} = \int_0^T (u(t), \widetilde{u}(t))_H \, dt + \sum_{k=1}^m (v_k, \widetilde{v}_k)_H ,$$

defined for all $h = (u(t), \{v_k\}_{k=1}^m)$ and $\widetilde{h} = (\widetilde{u}(t), \{\widetilde{v}_k\}_{k=1}^m) \in \mathcal{K}_m$.
We shall denote by $\mathcal{B}$ the control operator given by

$$\mathcal{B} = \left(B, \{D_k\}_{k\in\sigma_1^m}\right) \in \mathcal{L}\left(L^2\left((0,T) \setminus \{t_k\}_{k\in\sigma_1^m}; H\right) \times l^2(\sigma_1^m; H)\right),$$

so that

$$\mathcal{B}h(t) = \left(Bu(t), \{D_k v_k\}_{k\in\sigma_1^m}\right).$$

We have for every $h = (u(t), \{v_k\}_{k=1}^m) \in \mathcal{K}_m$

$$\begin{aligned}
(\mathcal{B}h, h)_{\mathcal{K}_m} &= \int_0^T (Bu(t), u(t))_H \, dt + \sum_{k=1}^m (D_k v_k, v_k)_H \\
&= \int_0^T (u(t), Bu(t))_H \, dt + \sum_{k=1}^m (v_k, D_k v_k)_H \\
&= (h, \mathcal{B}h)_{\mathcal{K}_m} ,
\end{aligned}$$

which shows that $\mathcal{B}^* = \mathcal{B}$, that is, $\mathcal{B}$ is self-adjoint. On the other hand, we have

$$\begin{aligned}
(\mathcal{B}h, h)_{\mathcal{K}_m} &= \int_0^T (Bu(t), u(t))_H \, dt + \sum_{k=1}^m (D_k v_k, v_k)_H \\
&\leq d_0 \int_0^T \|u(t)\|_H^2 \, dt + \sum_{k=1}^m d_k \|v_k\|_H^2 \\
&\leq \delta \|h\|_{\mathcal{K}_m}^2 ,
\end{aligned}$$

where $\delta = \max\{d_0, d_1, ..., d_m\}$. Thus, the operator is $\mathcal{B}$ bounded in $\mathcal{K}_m$.

Next, we consider the *homogeneous system* associated with (1) :

$$\begin{aligned}
\varphi'(t) + A\varphi(t) &= 0, \quad t \in (0,T) \setminus \{t_k\}_{k\in\sigma_1^m}, &(2)\\
\varphi(0) &= \varphi^0, \\
\Delta\varphi(t_k) &= I_k \varphi(t_k), \; k \in \sigma_1^m. &(2_k)
\end{aligned}$$

We point out that on each interval $[t_k, t_{k+1})$, for $k = 0, ..., m$, the solution φ is left continuous at each time t_k.
Consider the corresponding homogeneous backward problem :

$$\begin{aligned} -\tilde{\varphi}'(t) + \mathbf{A}\tilde{\varphi}(t) &= 0,, \quad t \in (0,T) \setminus \{t_k\}_{k \in \sigma_1^m}, && (3) \\ \tilde{\varphi}(T) &= \varphi^0, \\ \Delta\tilde{\varphi}(t_{m-(k-1)}) &= -\tilde{I}_{m-(k-1)}\tilde{\varphi}(t^+_{m-(k-1)}),\ k \in \sigma_1^m, && (3_k) \end{aligned}$$

where

$$\mathbf{A} = A^* = -A, \quad \tilde{I}_{m-(k-1)} = I^*_{m-(k-1)} = -I_{m-(k-1)},\ k \in \sigma_1^m.$$

We observe that the problem (3) on the interval $[t_m, T]$ is equivalent to the classical backward problem

$$\begin{aligned} &-\tilde{\varphi}'(t) + \mathbf{A}\tilde{\varphi}(t) = 0,\ t \in [t_m, T], \\ &\tilde{\varphi}(T) = \varphi^0. \end{aligned}$$

We introduce the following space : $\mathcal{PC}([0,T]; H) = \{y, y : [0,T] \to H$ *such that* $y(t)$ *is continuous at* $t \neq t_k$, *and has discontinuities of first kind at* $t = t_k$, *for every* $k \in \sigma_1^m\}$.
Evidently, $\mathcal{PC}([0,T]; H)$ is a Banach space with respect to the norm

$$\|y\|_{\mathcal{PC}} = \sup_{t\in(0,T)} \|y(t)\|.$$

On the other hand, we define the subspaces $\mathcal{PLC}$, (respectively, $\mathcal{PRC}$)$=$ $\{y, y \in \mathcal{PC}$ *such that* $y(t)$ *is left (*respectively, *right) continuous at* $t = t_k$, *for every* $k \in \sigma_1^m\}$.

Remark 1 *1) The space* $\mathcal{PLC}$*, (respectively,* $\mathcal{PRC}$*) can be identified to a subspace of* $\mathcal{K}_m$*. That is, to each* $y \in \mathcal{PLC}$*, (respectively,* $\tilde{y} \in \mathcal{PRC}$*) is assigned the function* h *(respectively,* $\widetilde{\mathrm{h}}$*) defined by*

$$\mathrm{h}(t) = \left(y(t), \{y(t_k)\}_{k\in\sigma_1^m}\right),$$

and

$$\widetilde{\mathrm{h}}(t) = \left(\tilde{y}(t), \{\tilde{y}(t_k)\}_{k\in\sigma_1^m}\right).$$

The mapping $y \mapsto \mathrm{h}(t)$ *(respectively,* $\tilde{y} \mapsto \widetilde{\mathrm{h}}$*) is a linear injection.*

2) Let $\widetilde{y} \in \mathcal{PRC}$, the function y can be written as :

$$\widetilde{y}(t) = \begin{cases} \widetilde{y}_{[0]}(t) & \text{if} \quad t \in [t_0, t_1) \\ \widetilde{y}_{[k]}(t) & \text{if} \quad t \in [t_k, t_{k+1}) \\ \widetilde{y}_{[m]}(t) & \text{if} \quad t \in [t_m, T]\,. \end{cases}$$

Next, let $\tau_k = t_k - t_{k-1}$, we define the operator $\mathcal{T} : D(\mathcal{T}) = \mathcal{PRC} \subset \mathcal{K}_m \to \mathcal{K}_m$ by

$$(\mathcal{T}\widetilde{y})(t) = \begin{cases} \widetilde{y}_{[0]}((T-t)\frac{\tau_1}{\tau_{m+1}} + t_0) & \text{if} \quad t \in [t_m, T]\,, \\ \widetilde{y}_{[k]}((t_{m-(k-1)} - t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}} + t_k) & \text{if} \quad t \in \left[t_{m-k}, t_{m-(k-1)}\right)\,, \quad k \in \sigma_1^{m-1}, \\ \widetilde{y}_{[m]}((t_1 - t)\frac{\tau_{m+1}}{\tau_1} + t_m) & \text{if} \quad t \in (0, t_1]\,. \end{cases} \tag{4}$$

We note that the range of $\mathcal{T}$ is exactly $\mathcal{PLC}$. The function $(\mathcal{T}\widetilde{y})(t)$ can be written as follows:

$$(\mathcal{T}\widetilde{y})(t) = \begin{cases} y_{[0]}(t) & \text{if} \quad t \in [t_0, t_1]\,, \\ y_{[k]}(t) & \text{if} \quad t \in (t_k, t_{k+1}]\,, \quad k \in \sigma_1^{m-1}, \\ y_{[m]}(t) & \text{if} \quad t \in (t_m, T]\,. \end{cases}$$

Let $X\,(t)$ be the resolvent solution of the operator system

$$\begin{aligned} &X'(t) + AX(t) = 0,\ 0 = t_0 < t < t_{m+1} = T,\ t \neq t_k,\ k = 1, 2, ..., m, \\ &X(0) = I, \\ &X(t_k + 0) - X(t_k - 0) = I_k X(t_k),\ k = 1, 2, ..., m, \end{aligned}$$

where $I : H \to H$ is the identity operator. We shall suppose that the operator $\mathcal{I}_k = I_k + I$ has a bounded inverse.

Definition 1 *A function $y \in \mathcal{PC}\left([0,T]\,; H\right)$is a mild solution to the impulsive problem (1) if the impulsive conditions are satisfied and*

$$\begin{aligned} y(t) \quad = & \quad G(t, 0^+)y^0 + \int_0^t G(t,s)Bu\,(s)\,ds \\ & + \sum\nolimits_{0<t_k\leq t} G(t, t_k)(D_k v_k),\ \textit{for every } t \in (0, T), \end{aligned}$$

where the evolution operator $G(t,s)$ is given by

$$G(t,s) = X(t)X^{-1}(s).$$

It is not hard to check that the operator $G(t,t_k)$ satisfies the operator system

$$\begin{array}{l} G^{'}(t,t_k)+AG(t,t_k)=0, \;\; t\in[t_k,t_{k+1}), \;\; k\in\sigma_0^m, \\ G(t_k,t_k)=I, \\ G(t_{k+1}^+,t_k)-G(t_{k+1}^-,t_k)=I_{k+1}G(t_{k+1}^-,t_k). \end{array}$$

It is well known that (1) has a unique solution y such that

$$y\in\mathcal{PLC}([0,T];H)\cap C^1\left([0,T]\setminus\{t_k\}_{k\in\sigma_1^m};H\right).$$

Now, we define the concept of mild solution for the backward impulsive system (3) associated with system (2).

Definition 2 *We say that* $\tilde{\varphi}\in\mathcal{PRC}([0,T];H)$ *is a mild solution for the backward impulsive system (3) if* $\mathcal{T}\tilde{\varphi}$ *is a mild solution for the homogeneous impulsive system (2).*

Let us introduce the notion of the null controllability of the initial state as follows:

Definition 3 *We say that the initial state* $y^0\in H$ *is null controllable at time* T*, if there is a control function* $\mathrm{h}\in\mathcal{K}_m$ *for which the solution* y *of system (1) satisfies* $y(T)=0$.

2 Main Results

First we begin by the following lemma.

Lemma 1 *Assume that* $\xi(t),\zeta(t)\in L^1([0,T];H)$ *and* $\{\xi_k\}_{k=1}^m,\{\zeta_k\}_{k=1}^m\in l^1(\sigma_1^m,H)$*. Then, for every vector functions*

$$\gamma(t)\in\mathcal{PLC}([0,T];H)\cap C^1\left([0,T]\setminus\{t_k\}_{k\in\sigma_1^m};H\right)$$

and

$$\eta(t)\in\mathcal{PRC}([0,T];H)\cap C^1\left([0,T]\setminus\{t_k\}_{k\in\sigma_1^m};H\right)$$

satisfying the problem

$$\begin{array}{rcl} \dfrac{d}{dt}\langle\gamma(t),\eta(t)\rangle &=& \langle\xi(t),\zeta(t)\rangle, \quad t\neq t_k, \text{ for } k\in\sigma_1^m, \\ \Delta\langle\gamma(t_k),\eta(t_k)\rangle &=& \langle\Delta\gamma(t_k),\eta(t_k)\rangle+\langle\gamma(t_k),\Delta\eta(t_k)\rangle=\langle\xi_k,\zeta_k\rangle, \;\; k\in\sigma_1^m, \end{array}$$

we have the following identity

$$\begin{aligned}\langle\gamma(t),\eta(t)\rangle|_0^T &= \langle\gamma(T),\eta(T)\rangle-\langle\gamma(0),\eta(0)\rangle \\ &= \int_0^T\langle\xi(t),\zeta(t)\rangle dt+\sum_{k=1}^m\langle\xi_k,\zeta_k\rangle.\end{aligned}\tag{5}$$

Proof. It is straightforward. □

We also need the following Lemmas.

Lemma 2 [14] *If $\mathcal{B}\in\mathcal{L}(\mathcal{K}_m)$ is self-adjoint and nonnegative, then*

$$\|\mathcal{B}\mathrm{h}\|\le\|\mathcal{B}\|^{1/2}(\mathcal{B}\mathrm{h},\mathrm{h})_{\mathcal{K}_m}^{1/2},\quad \mathrm{h}\in\mathcal{K}_m.$$

Lemma 3 *If $\tau_{k+1}=\tau_{m-(k-1)}$, $k\in\sigma_0^{m-1}$, then for the mild solution $\widetilde{\varphi}$ of (3), the identity holds :*

$$\int_0^T|B\widetilde{\varphi}|_H^2\,dt+\sum_{k=1}^m\left|D_k\widetilde{\varphi}\left(t_k^+\right)\right|_H^2=\int_0^T|B\varphi|_H^2\,dt+\sum_{k=1}^m\left|D_k\varphi\left(t_{m-(k-1)}\right)\right|_H^2.\tag{6}$$

Proof. For each $k\in\sigma_0^m$, using the change of variable $t\to(t_{m-(k-1)}-t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}}+t_k$ we have

$$\begin{aligned}&\int_{t_{m-k}}^{t_{m-(k-1)}}(B\varphi_{[m-k]}(t),B\varphi_{[m-k]}(t))dt\\ =&\int_{t_{m-k}}^{t_{m-(k-1)}}(B\widetilde{\varphi}_{[k]}((t_{m-(k-1)}-t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}}+t_k),B\widetilde{\varphi}_{[k]}((t_{m-(k-1)}-t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}}+t_k))dt\\ =&\frac{-\tau_{m-(k-1)}}{\tau_{k+1}}\int_{t_{k+1}}^{t_k}(B\widetilde{\varphi}_{[k]}(s),B\widetilde{\varphi}_{[k]}(s))ds\\ =&\int_{t_k}^{t_{k+1}}(B\widetilde{\varphi}_{[k]}(s),B\widetilde{\varphi}_{[k]}(s))ds.\end{aligned}$$

Summing up with respect to k, we get

$$\sum_{k=0}^m\int_{t_{m-k}}^{t_{m-(k-1)}}(B\varphi_{[m-k]}((t)),B\varphi_{[m-k]}(t))dt=\sum_{k=0}^m\int_{t_k}^{t_{k+1}}(B\widetilde{\varphi}_{[k]}(t),B\widetilde{\varphi}_{[k]}(t))dt.$$

Thus, we obtain

$$\int_0^T |B\widetilde{\varphi}|_H^2 \, dt = \int_0^T |B\varphi|_H^2 \, dt.$$

On the other hand, by virtue of the definition of the function $\widetilde{\varphi}$ we get

$$\varphi\left(t_{m-k}\right) = \widetilde{\varphi}\left(t_{k+1}\right), \quad k \in \sigma_0^{m-1}.$$

Also, we have

$$\varphi\left(t_{m-(k-1)}\right) = \widetilde{\varphi}\left(t_k\right), \quad k \in \sigma_1^m,$$

and

$$\widetilde{\varphi}\left(t_{m-k}\right) = \varphi\left(t_{k+1}\right), \quad k \in \sigma_0^{m-1}.$$

This implies that

$$\begin{aligned}
\sum_{k=1}^{m} \left|D_k \widetilde{\varphi}\left(t_k\right)\right|_H^2 &= \sum_{k=0}^{m-1} \left\langle D_{m-k} \widetilde{\varphi}\left(t_{m-k}\right), D_{m-k} \widetilde{\varphi}\left(t_{m-k}\right)\right\rangle_H \\
&= \sum_{k=0}^{m-1} \left\langle D_{m-k} \varphi\left(t_{k+1}\right), D_{m-k} \varphi\left(t_{k+1}\right)\right\rangle_H \\
&= \sum_{l=1}^{m} \left\langle D_l \varphi\left(t_{m-(l-1)}\right), D_l \varphi\left(t_{m-(l-1)}\right)\right\rangle_H \\
&= \sum_{k=1}^{m} \left\langle D_k \varphi\left(t_{m-(k-1)}\right), D_k \varphi\left(t_{m-(k-1)}\right)\right\rangle_H \\
&= \sum_{k=1}^{m} \left|D_k \varphi\left(t_{m-(k-1)}\right)\right|_H^2,
\end{aligned}$$

which gives (6). □

Corollary 1 *If $\tau_{k+1} = \tau_{m-(k-1)}$, for $k \in \sigma_0^{m-1}$, and B, D_k are nonnegative in H, then the following holds:*

$$\begin{aligned}
&\int_0^T \langle B\widetilde{\varphi}(t), \widetilde{\varphi}(t)\rangle dt + \sum_{k=1}^{k=m} \langle D_k \widetilde{\varphi}(t_k), \widetilde{\varphi}(t_k)\rangle \\
= &\int_0^T \langle B\varphi(t), \varphi(t)\rangle dt + \sum_{k=1}^{k=m} \langle D_k \varphi(t_{m-(k-1)}), \varphi(t_{m-(k-1)})\rangle.
\end{aligned}$$

Proof. This follows immediately from Lemma 3 if we substitute B by $B^{\frac{1}{2}}$, and D_k by $D_k^{\frac{1}{2}}$. $\square$

Now, we state and establish the following Theorem.

Theorem 1 *Let $y^0 \in H$ be a given initial state for the system (1), then y^0 is null controllable at time T if and only if there is a positive constant C such that*

$$\left|\langle y^0, \tilde{\varphi}^0\rangle_H\right| \leq C\left\{\int_0^T |B\varphi|_H^2\,dt + \sum_{k=1}^{m}\left|D_k\varphi\left(t_{m-(k-1)}\right)\right|_H^2\right\}^{1/2}, \ \forall\tilde{\varphi}^0 \in H, \tag{7}$$

where $\varphi \in \mathcal{PLC}\left([0,T];H\right)$ is the unique mild solution to (2) *with $\varphi(T) = \tilde{\varphi}^0$.*

Proof. It suffices to prove this Theorem for the special case $\tau_{k+1} = \tau_{m-(k-1)}$, for $k \in \sigma_0^{m-1}$, because the norm $|||.||| \doteqdot \left\{\sum_{k=0}^{m}\frac{\tau_{m-(k-1)}}{\tau_{k+1}}\int_{t_k}^{t_{k+1}}|.|_H^2\,dt\right\}^{1/2}$ is equivalent to the usual norm of $L^2\left([0,T];H\right)$.

We shall proceed in several steps.

Step 1: Let y and $\tilde{\varphi}$ be strong solutions to (1) and (3), respectively. Then, for $t \neq t_k$, $k \in \sigma_1^m$, we have

$$\begin{aligned}\frac{d}{dt}\langle y(t), \tilde{\varphi}(t)\rangle &= \langle y(t), \tilde{\varphi}'(t)\rangle + \langle y'(t), \tilde{\varphi}(t)\rangle \\ &= \langle y(t), -A\tilde{\varphi}(t)\rangle + \langle -Ay(t) + Bu(t), \tilde{\varphi}(t)\rangle \\ &= \langle y(t), -A\tilde{\varphi}(t)\rangle + \langle -Ay(t), \tilde{\varphi}(t)\rangle + \langle Bu(t), \tilde{\varphi}(t)\rangle \\ &= \langle Bu(t), \tilde{\varphi}(t)\rangle.\end{aligned} \tag{8}$$

Multiplying equation (3_k) in (3) from the left by $y\left(t_{m-(k-1)}\right)$ the solution of (1), and multiplying equation (1_k) in (1) from the right by $\tilde{\varphi}(t_k)$ the solution of (3), and finally adding memberwise we get

$$\begin{aligned}\Delta\langle y(t), \tilde{\varphi}(t)\rangle_{|t=t_k} &= \langle y(t_k), \Delta\tilde{\varphi}(t_k)\rangle + \langle \Delta y(t_k), \tilde{\varphi}(t_k)\rangle \\ &= \langle y(t_k), I_k\tilde{\varphi}(t_k)\rangle + \langle I_k y(t_k) + D_k v_k, \tilde{\varphi}(t_k)\rangle \\ &= \langle y(t_k), I_k\tilde{\varphi}(t_k)\rangle + \langle I_k y(t_k), \tilde{\varphi}(t_k)\rangle + \langle D_k v_k, \tilde{\varphi}(t_k)\rangle \\ &= \langle D_k v_k, \tilde{\varphi}(t_k)\rangle.\end{aligned} \tag{9}$$

Setting $\gamma(t) = y(t)$, $\eta(t) = \tilde{\varphi}(t)$, $\xi(t) = Bu(t)$, $\zeta(t) = \tilde{\varphi}(t)$, $\xi_k = D_k v_k$, $\zeta_k = \tilde{\varphi}(t_k)$, then equations (5), (8) and (9) give

$$\langle y(T), \widetilde{\varphi}(T)\rangle - \langle y(0), \widetilde{\varphi}(0)\rangle = \int_0^T \langle Bu(t), \widetilde{\varphi}(t)\rangle dt + \sum_{k=1}^{k=m} \langle D_k v_k, \widetilde{\varphi}(t_k)\rangle. \quad (10)$$

Since $\mathcal{B}$ is bounded, self-adjoint and $\mathcal{B} \geq 0$, then by density the latter identity is still valid for mild solutions y of (1). Identity (10) can be written as follows

$$\langle y(T), \widetilde{\varphi}(T)\rangle - \langle y(0), \widetilde{\varphi}(0)\rangle = \int_0^T \langle u(t), B\widetilde{\varphi}(t)\rangle dt + \sum_{k=1}^{k=m} \langle v_k, D_k\widetilde{\varphi}(t_k)\rangle. \quad (11)$$

Next, if there is a certain $h(t) \in \mathcal{K}_m$ such that the mild solution of (1) with $y(0) = y^0$ satisfies $y(T) = 0$, then

$$-\langle y(0), \widetilde{\varphi}(0)\rangle = \int_0^T \langle u(t), B\widetilde{\varphi}(t)\rangle dt + \sum_{k=1}^{k=m} \langle v_k, D_k\widetilde{\varphi}(t_k)\rangle,$$

and so by Cauchy-Schwarz Inequality we obtain

$$\begin{aligned} |\langle y(0), \widetilde{\varphi}(0)\rangle_H| \;\; &\leq \;\; \left\{ \int_0^T \|u(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 \right\}^{1/2} \qquad (12) \\ &\times \left\{ \int_0^T \|B\widetilde{\varphi}(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|D_k\widetilde{\varphi}((t_k)\|_H^2 \right\}^{1/2}. \end{aligned}$$

Using Lemma 3, and equation (12) we have

$$\begin{aligned} |\langle y(0), \widetilde{\varphi}(0)\rangle_H| \;\; &\leq \;\; \left\{ \int_0^T \|u(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 \right\}^{1/2} \\ &\times \left\{ \int_0^T \|B\varphi(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \left\|D_k\varphi(t_{m-(k-1)})\right\|_H^2 \right\}^{1/2}. \end{aligned}$$

Setting

$$C = \|h(t)\|_{\mathcal{K}_m} = \left\{ \int_0^T \|u(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 \right\}^{1/2}$$

we find that

$$|(\langle y(0),\widetilde{\varphi}(0)\rangle_H| \leq C\left\{\int_0^T \|B\varphi(t)\|_H^2\,dt + \sum_{k=1}^{k=m}\left\|D_k\varphi(t_{m-(k-1)})\right\|_H^2\right\}^{1/2}.$$

This shows the necessary condition of the Theorem.

Step 2: To prove the sufficiency we need the following result when $\mathcal{B} \geq \alpha > 0$.

Claim 1 *Assume that there is $\alpha > 0$ such that*

$$\left\{\int_0^T \|Bu(t)\|_H^2\,dt + \sum_{k=1}^{k=m}\|D_k v_k\|_H^2\right\} \geq \alpha\left\{\int_0^T \|u(t)\|_H^2\,dt + \sum_{k=1}^{k=m}\|v_k\|_H^2\right\}$$

then, for every $y^0 \in H$ there is $\varphi^0 \in H$ such that the mild solution of (1) with

$$\mathrm{h}(t) = (\widetilde{\varphi}(t), \widetilde{\varphi}(t_1), .., \widetilde{\varphi}(t_k).., \widetilde{\varphi}(t_m)) \in \mathcal{K}_m \;\; \textit{and}\;\; y(0) = y^0$$

satisfies $y(T) = 0$.

To prove this Claim, we consider for every $z \in H$ the solution φ of (2) satisfying $\varphi(T) = z$ and the unique mild solution y to the problem

$$\begin{aligned} y'(t) + Ay(t) &= B\widetilde{\varphi}(t), t \in (0,T)\Big\backslash\{t_k\}_{k\in\sigma_1^m}, \\ \Delta y(t_k) &= I_k y(t_k) + D_k\widetilde{\varphi}(t_k), \\ y(T) &= 0. \end{aligned}$$

Next, we introduce a bounded linear operator $\Lambda : H \to H$ defined by

$$\Lambda z = -y(0).$$

According to formula (11) and the Corollary 1 we have

$$\begin{aligned} |\langle\Lambda z, z\rangle| &= |-\langle y(0),\widetilde{\varphi}(0)\rangle| = \left|\int_0^T \langle B\widetilde{\varphi}(t),\widetilde{\varphi}(t)\rangle dt + \sum_{k=1}^{k=m}\langle D_k\widetilde{\varphi}(t_k),\widetilde{\varphi}(t_k)\rangle\right| \\ &= \left|\int_0^T \langle B\varphi(t),\varphi(t)\rangle dt + \sum_{k=1}^{k=m}\langle D_k\varphi(t_{m-(k-1)}),\varphi(t_{m-(k-1)})\rangle\right| \\ &\leq \varsigma\left\{\int_0^T \|\varphi(t)\|^2\,dt + \sum_{k=1}^{k=m}\|\varphi(t_k)\|^2\right\}, \end{aligned}$$

where

$$\varsigma = \sup_{k\in\sigma_0^m} \{d_k\} < \infty.$$

We have

$$\int_0^T \|\varphi(t)\|^2\, dt = \int_0^{t_1} \|\varphi(t)\|^2\, dt + \int_{t_1}^{t_2} \|\varphi(t)\|^2\, dt + ... + \int_{t_m}^{T} \|\varphi(t)\|^2\, dt.$$

Since there is no impulse in the interval $[t_k, t_{k+1})$ we have

$$\begin{aligned} \|\varphi(t)\| &= \left\|\varphi(t_k^+)\right\|, \text{ for every } t\in [t_k, t_{k+1})\,, k\in\sigma_0^m, \\ \left\|\varphi(t_{k+1}^-)\right\| &= \left\|\varphi(t_{k-1}^+)\right\|, \quad k\in\sigma_0^m. \end{aligned} \tag{13}$$

Therefore, there are $\tau_{k+1} = t_{k+1} - t_k > 0$, $k\in\sigma_0^m$ such that

$$\textstyle\int_{t_k}^{t_{k+1}} \|\varphi(t)\|^2\, dt \le \rho_k \left\|\varphi(t_k^+)\right\|^2 = \tau_{k+1}\left\|I_k\varphi(t_k^-) + \varphi(t_k^-)\right\|^2, \quad k\in\sigma_1^m. \tag{14}$$

On the other hand, the continuity of I_k implies that

$$\left\|\varphi(t_k^+)\right\|^2 = \left\|(I_k + I)\varphi(t_k^-)\right\|^2 \le (1 + L(I_k))^2 \left\|\varphi(t_k^-)\right\|^2, \quad k\in\sigma_1^m. \tag{15}$$

It follows from (14) and (15) that

$$\textstyle\int_{t_k}^{t_{k+1}} \|\varphi(t)\|^2\, dt \le \tau_{k+1}(1 + L(I_k))^2 \left\|\varphi(t_k^-)\right\|^2, \quad k\in\sigma_1^m. \tag{16}$$

Since m is finite, and due to (13),(16), then there is a constant $0 < \mu < \infty$ such that $\langle\Lambda z, z\rangle \le \mu \|z\|^2$, and thus, Λ is bounded.

Now, as $\mathcal{B}$ is nonnegative in $\mathcal{K}_m$, we have

$$\|\mathcal{B}\xi\,(t)\| \ge \alpha \left\{(\xi\,(t)\,, \xi\,(t))_{\mathcal{K}_m}\right\}^{1/2}$$

for all $\xi\in\mathcal{K}_m$; thus, by virtue of Lemma 2, we have

$$\begin{aligned} &\left\{\int_0^T (Bu(t), u(t))_H dt + \sum_{k=1}^{k=m} (D_k v_k, v_k)_H\right\} \\ \ge\ & \alpha\,\|\mathcal{B}\| \left\{\int_0^T \|u(t)\|_H^2\, dt + \sum_{k=1}^{k=m} \|v_k\|_H^2\right\}. \end{aligned} \tag{17}$$

It follows from (11), (17) and Corollary 1 that

$$\begin{aligned}\langle \Lambda z, z\rangle &= -\langle y(0), \widetilde{\varphi}(0)\rangle \\ &= \int_0^T \langle B\varphi(t), \varphi(t)\rangle dt + \sum_{k=1}^{k=m} \langle D_k \varphi(t_{m-(k-1)}), \varphi(t_{m-(k-1)})\rangle \\ &\geq \alpha \left\|\mathcal{B}\right\| \left\{ \int_0^T \left\|\varphi(t)\right\|^2 dt + \sum_{k=1}^{k=m} \left\|\varphi(t_k)\right\|^2 \right\} \\ &\geq \alpha \left\|\mathcal{B}\right\| \int_0^{t_1} \left\|\varphi(t)\right\|^2 dt = \left\|\mathcal{B}\right\| \alpha t_1 \left\|z\right\|^2 = \theta \left\|z\right\|^2,\end{aligned}$$

because there is no impulse before time t_1. Therefore, Λ is coercive on H. To show that there is a bijection from H onto H, it suffices to prove that $\Lambda + I$ is a bijection from H onto H. Clearly, $\Lambda + I$ is injective since

$$\langle \Lambda z + z, z\rangle = \langle \Lambda z, z\rangle + \langle z, z\rangle \geq (\theta + 1) \left\|z\right\|^2.$$

On the other hand, let $y^0 \in H$, as the form $a(f,g) + \langle f, g\rangle = \langle \Lambda f, g\rangle + \langle f, g\rangle$ is symmetric and coercive, then, by virtue of Lax-Milgram Theorem, there is an element $f \in H$ such that

$$a(f,g) + \langle f, g\rangle = \langle y^0, g\rangle, \text{ for all } g \in H.$$

This implies that $\Lambda(H) = H$. Thus, for every $y^0 \in H$, there is a unique $z \in H$ such that $\Lambda(z) = -y^0$, which completes the proof of Claim 1.

Step 3: Assume that $B, D_k \geq 0$, then $\mathcal{B} \geq 0$,

$$\widetilde{B}^2 = B, \widetilde{D}_k^2 = D_k.$$

We define for $\varepsilon > 0$,

$$\begin{aligned}\beta^\varepsilon &\doteqdot \widetilde{B}^2 + \varepsilon I, \\ \delta_k^\varepsilon &\doteqdot \widetilde{D}_k^2 + \varepsilon I,\end{aligned}$$

and

$$\mathcal{B}^\varepsilon \doteqdot (\beta^\varepsilon; \delta_1^\varepsilon, .., \delta_m^\varepsilon) = (\widetilde{B}^2 + \varepsilon I; \widetilde{D}_1^2 + \varepsilon I, .., \widetilde{D}_m^2 + \varepsilon I).$$

According to Claim 1, there is $\widetilde{\varphi}^{0,\varepsilon} \in H$ such that the mild solution y_ε of (1) with $y_\varepsilon(0) = y^0$ satisfies $y_\varepsilon(T) = 0$; where $\mathcal{B}(h)$ has been replaced by

$$\mathcal{B}^\varepsilon(\widetilde{\varphi}(t), \widetilde{\varphi}(t_1), .., \widetilde{\varphi}(t_k) .., \widetilde{\varphi}(t_m)) \in \mathcal{K}_m.$$

We obtain from (11) and Corollary 1

$$-\langle y(0),\widetilde{\varphi}_\varepsilon(0)\rangle = \int_0^T \langle \beta_\varepsilon^\varepsilon \widetilde{\varphi}(t), \widetilde{\varphi}_\varepsilon(t)\rangle dt + \sum_{k=1}^{k=m} \langle \delta_k^\varepsilon \widetilde{\varphi}_\varepsilon(t_k), \widetilde{\varphi}_\varepsilon(t_k)\rangle, \tag{18}$$

and (7) gives

$$-\langle y(0),\widetilde{\varphi}_\varepsilon(0)\rangle \leq C\left\{\int_0^T \langle \widetilde{B}^2 \varphi_\varepsilon(t), \varphi_\varepsilon(t)\rangle dt + \sum_{k=1}^{k=m} \langle \widetilde{D}_k^2 \varphi_\varepsilon(t_{m-(k-1)}), \varphi_\varepsilon(t_{m-(k-1)})\rangle\right\}^{1/2}. \tag{19}$$

Whence,

$$-\langle y(0),\widetilde{\varphi}_\varepsilon(0)\rangle \leq C\left\{\int_0^T \langle \beta^\varepsilon \varphi_\varepsilon(t), \varphi_\varepsilon(t)\rangle dt + \sum_{k=1}^{k=m} \langle \delta_k^\varepsilon \varphi_\varepsilon(t_{m-(k-1)}), \varphi_\varepsilon(t_{m-(k-1)})\rangle\right\}^{1/2}. \tag{20}$$

It follows at once from (18), (19) and (20) that

$$\begin{aligned}
&\varepsilon\left\{\int_0^T \left\|\varphi_\varepsilon(t)\right\|^2 dt + \sum_{k=1}^{k=m} \left\|\varphi_\varepsilon(t_k)\right\|^2\right\} \\
&\qquad + \int_0^T \langle \widetilde{B}\varphi_\varepsilon(t), \widetilde{B}\varphi_\varepsilon(t)\rangle dt + \sum_{k=1}^{k=m} \langle \widetilde{D}_k\varphi_\varepsilon(t_{m-(k-1)}), \widetilde{D}_k\varphi_\varepsilon(t_{m-(k-1)})\rangle \\
&= \int_0^T (\beta^\varepsilon \varphi_\varepsilon(t), \varphi_\varepsilon(t))dt + \sum_{k=1}^{k=m} \langle \delta_k^\varepsilon \varphi_\varepsilon(t_{m-(k-1)}), \varphi_\varepsilon(t_{m-(k-1)})) \leq C^2.
\end{aligned} \tag{21}$$

Step 4: According to the estimate (20) the family

$$\begin{aligned}
b_\varepsilon &= \mathcal{B}^\varepsilon(\widetilde{\varphi}_\varepsilon(t); \widetilde{\varphi}_\varepsilon(t_1)..., \widetilde{\varphi}_\varepsilon(t_m)) \\
&= (\widetilde{B}_\varepsilon^2 \widetilde{\varphi}(t); \widetilde{D}_1^2 \widetilde{\varphi}_\varepsilon(t_1)..., \widetilde{D}_m \widetilde{\varphi}_\varepsilon(t_m)) + \varepsilon(\widetilde{\varphi}_\varepsilon(t); \widetilde{\varphi}_\varepsilon(t_1)..., \widetilde{\varphi}_\varepsilon(t_m))
\end{aligned}$$

is contained in a bounded subset $\mathcal{K}_m$.
Thus, both of the families

$$\sqrt{\varepsilon}(\widetilde{\varphi}_\varepsilon(t); \widetilde{\varphi}_\varepsilon(t_1)..., \widetilde{\varphi}_\varepsilon(t_m)) \text{ and } (B\widetilde{\varphi}_\varepsilon(t); D_1\widetilde{\varphi}_\varepsilon(t_1)..., D_m\widetilde{\varphi}_\varepsilon(t_m))$$

are bounded in $\mathcal{K}_m$. Therefore, we may extract a subfamily, say

$$(B\widetilde{\varphi}_\varepsilon(t); D_1\widetilde{\varphi}_\varepsilon(t_1)..., D_m\widetilde{\varphi}_\varepsilon(t_m)) \rightharpoonup h, \text{ weakly in } \mathcal{K}_m.$$

Then clearly

$$(\widetilde{B}^2\widetilde{\varphi}_\varepsilon(t); \widetilde{D}_1^2\widetilde{\varphi}_\varepsilon(t_1)..., \widetilde{D}_m^2\widetilde{\varphi}_\varepsilon(t_m))+\varepsilon(\widetilde{\varphi}_\varepsilon(t);\widetilde{\varphi}_\varepsilon(t_1)...,\widetilde{\varphi}_\varepsilon(t_m)) \rightharpoonup \mathcal{B}h, \text{ weakly in } \mathcal{K}_m.$$

Step 5: Taking the limit as $\varepsilon \to 0$, we see that the solution y of (1) with initial condition $y(0) = y^0$, h being as in **step 4** satisfies $y(T) = 0$. This completes the proof of Theorem 1. □

As an immediate application of the foregoing Theorem we give the following example.

Example. One dimensional impulsive Schrödinger equation :
We consider the problem

$$\begin{aligned}
\frac{\partial y\,(t,x)}{\partial t}+i\frac{\partial^2 y}{\partial x^2}\,(t,x) &= \chi_{\omega_0}u\,(t,x)\,, \quad t\in(0,T)\setminus\{t_k\}_{k\in\sigma_1^m}\,, x\in\Omega=(0,2\pi),\\
y(t,0) &= y(t,2\pi)=0, \qquad\qquad (22)\\
y\,(0,x) &= y^0,\\
\Delta y\,(t_k,x) &= i\alpha_k y\,(t_k,x)+\chi_{\omega_k}v_k(x), \quad k\in\sigma_1^m,
\end{aligned}$$

where

$$t_{k+1}-t_k>2\pi, \quad \omega_k=(a_1^k,a_2^k)\subset\Omega, k\in\sigma_0^m, \quad \{\alpha_k\}_{k\in\sigma_1^m}\subset\mathbb{R}^+.$$

Let

$$H=L^2(\Omega,\mathbb{C}), Aw(x)=i\tfrac{\partial^2 w}{\partial x^2}(x), \quad D(A)=\left\{w\in H, \tfrac{\partial^2 w}{\partial x^2}\in H, w(0)=w(\pi)=0\right\},$$

and $I_k w(x) = i\alpha_k w(x)$ and the control operator is given by $B = \chi_{\omega_0}$, $D_k = \chi_{\omega_k}$, then the system (22) becomes an abstract formulation of (1). As a consequence of Theorem 1, the initial state $y^0 \in L^2(\Omega,\mathbb{C}) = H$ of the solution of (22) is null-controllable at $t = T$, if and only if, there is $C > 0$ such that

$$\begin{aligned}
&\left|\int_\Omega y^0(x)\widetilde{\varphi}^0(x)dx\right| \qquad\qquad (23)\\
\leq\ & C\left\{\int_0^T\int_{\omega_0}|\varphi|^2(t,x)dxdt+\sum_{k=1}^m\int_{\omega_k}|\varphi|^2(t_{m-(k-1)},x)\right\}^{\frac{1}{2}}, \ \forall\widetilde{\varphi}^0\in L^2(\Omega,\mathbb{C}),
\end{aligned}$$

where $\widetilde{\varphi}^0(x)=\varphi(T,x)$ and φ is the mild solution of

$$\begin{aligned}\frac{\partial\varphi(t,x)}{\partial t}+i\frac{\partial^2\varphi(t,x)}{\partial x^2} &= 0,\quad t\in(0,T)\setminus\{t_k\}_{k\in\sigma_1^m},\ x\in\Omega,\\ \varphi(t,0) &= \varphi(t,2\pi)=0,\\ \varphi(0,x) &= \varphi^0(x),\ x\in\Omega,\\ \Delta\varphi(t_k,x) &= i\alpha_k\varphi(t_k,x),\ x\in\Omega,\ k\in\sigma_1^m.\end{aligned}$$

Here φ is given by

$$\varphi(t)=\begin{cases}\varphi_{[0]}(t) & \text{, if}\quad t\in[t_0,t_1)\\ \varphi_{[k]}(t) & \text{, if}\quad t\in\big[t_k,t_{k+1)}\big)\\ \varphi_{[m]}(t) & \text{, if}\quad t\in[t_m,T],\end{cases}$$

where $\varphi_{[k]}(t)$ is a solution of the classical Schrödinger equation

$$\begin{aligned}\frac{\partial\varphi_{[k]}(t,x)}{\partial t}+i\frac{\partial^2\varphi_{[k]}}{\partial x^2}(t,x) &= \chi_{\omega_0}u(t,x),\quad t\in(t_0,t_1),\ x\in\Omega=(0,2\pi),\\ \varphi_{[k]}(t,0) &= \varphi_{[k]}(t,2\pi)=0,\\ \varphi_{[0]}(t_0,x) &= \varphi^0(x),\ x\in\Omega,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\varphi_{[k]}(t,x)}{\partial t}+i\frac{\partial^2\varphi_{[k]}}{\partial x^2}(t,x) &= \chi_{\omega_0}u(t,x),\quad t\in(t_k,t_{k+1}),\mathrm{x}\in\Omega=(0,2\pi),\\ \varphi_{[k]}(t,0) &= \varphi_{[k]}(t,2\pi)=0,\\ \varphi_{[k]}(t_k,x) &= (1+i\alpha_k)\varphi_{[k-1]}(t_k,x),x\in\Omega,k\in\sigma_1^m.\end{aligned}$$

Then a standard application of a variant of Ingham's Inequality [8] shows that

$$\int_{t_k}^{t_{k+1}}\int_{w_0}\left|\varphi_{[k]}\right|(t,x)dtdx\geq c(\tau_k,w_0)\int_\Omega\left|\varphi_{[k]}\right|(t_k^+,x)dx,$$

for some positive constants $c(\tau_k,w_0)>0$. Summing up we get

$$\begin{aligned}\sum_{k=0}^m\int_{t_k}^{t_{k+1}}\int_{w_0}\left|\varphi_{[k]}\right|(t,x)dtdx &= \int_0^T\int_{\omega_0}|\varphi|^2(t,x)dxdt\\ &\geq c_1\sum_{k=1}^m\int_\Omega\left|\varphi_{[k]}\right|(t_k^+,x)dx,\end{aligned}$$

where $c_1 = \min\limits_{k\in\sigma_0^m} c(\tau_k, w_0) > 0$.

On the other hand, there is a positive constant $c_2 > 0$ such that

$$\sum_{k=1}^{m} \int_{\omega_k} |\varphi|^2 (t_{m-(k-1)}, x) \geq c_2 \sum_{k=1}^{m} \int_{\Omega} \left|\varphi_{[k]}\right|^2 (t_k^+, x)dx.$$

It follows that

$$\begin{aligned} \int_0^T \int_{\omega_0} |\varphi|^2 (t,x)dxdt & \\ &+ \sum_{k=1}^{m} \int_{\omega_k} |\varphi|^2 (t_{m-(k-1)}, x) \\ &\geq (c_1+c_2) \sum_{k=1}^{m} \int_{\Omega} \left|\varphi_{[k]}\right|^2 (t_k^+, x)dx \\ &\geq (c_1+c_2) \int_{\Omega} \left|\varphi_{[m]}\right|^2 (t_m^+, x)dx \\ &= (c_1+c_2) \int_{\Omega} |\varphi|^2 (T, x)dx. \end{aligned}$$

Now, since $\widetilde{\varphi}^0(x) = \widetilde{\varphi}(0,x) = \varphi(T,x)$, then,

$$\int_0^T \int_{\omega_0} |\varphi|^2 (t,x)dxdt + \sum_{k=1}^{m} \int_{\omega_k} |\varphi|^2 (t_{m-(k-1)}, x) \geq m(c_1+c_2) \int_{\Omega} \left|\widetilde{\varphi}^0\right|^2 (x)dx,$$

from which we get

$$\int_{\Omega} \left|\widetilde{\varphi}^0\right|^2 (x)dx \leq \frac{1}{m(c_1+c_2)} \left(\int_0^T \int_{\omega_0} |\varphi|^2 (t,x)dxdt + \sum_{k=1}^{m} \int_{\omega_k} |\varphi|^2 (t_{m-(k-1)}, x) \right).$$

We conclude by Cauchy-Schwarz inequality that

$$\begin{aligned} \left| \int_{\Omega} y^0(x)\widetilde{\varphi}^0(x)dx \right| &\leq \left\{ \int_{\Omega} \left|y^0\right|^2 (x)dx \int_{\Omega} \left|\widetilde{\varphi}^0\right|^2 (x)dx \right\}^{1/2} \\ &\leq \left\{ \frac{\int_{\Omega} \left|y^0\right|^2 (x)dx}{m(c_1+c_2)} \right\}^{1/2} \left(\int_0^T \int_{\omega_0} |\varphi|^2 (t,x)dxdt \right. \\ &\quad \left. + \sum_{k=1}^{m} \int_{\omega_k} |\varphi|^2 (t_{m-(k-1)}, x)dx \right)^{1/2}, \end{aligned}$$

which establishes the necessary and sufficient condition of null controllability stated in Theorem 1.

We conclude our paper by a special case when our initial state is an eigensolution of the following linear operator $\Gamma : H \to H$ defined by

$$\Gamma(\psi) = \int_0^T X^{-1}(s)B^2X(s)\psi ds + \sum_{k=1}^{k=m} X^{-1}(t_k)D_k^2X(t_k)\psi.$$

We have the following result of null-controllability.

Proposition 1 *Let $\lambda > 0$ be an eigenvalue of Γ with eigenvector $\psi \in H$. Then, the solution y to the problem*

$$\left\{ \begin{array}{ll} y^{'}(t) + Ay(t) = -\frac{1}{\lambda}B^2(X(t)\psi), & t \in (0,T) \Big\backslash \{t_k\}_{k\in\sigma_1^m} , \\ \Delta y(t_k) = I_k y(t_k) - \frac{1}{\lambda}D_k^2(X(t_k)\psi), & k \in \sigma_1^m \\ y(0) = \psi, & \end{array} \right. \tag{24}$$

satisfies

$$y(T) = 0.$$

Proof.

Write system (24) into the form

$$\left\{ \begin{array}{ll} y^{'}(t) + Ay(t) = -\frac{1}{\lambda}B^2(X(t)\psi), & t \in (0,T) \Big\backslash \{t_k\}_{k\in\sigma_1^m} , \\ y(t_k^+) = \mathcal{I}_k y(t_k) - \frac{1}{\lambda}D_k^2(X(t_k)\psi), & k \in \sigma_1^m \\ y(0) = \psi. & \end{array} \right.$$

Therefore, this impulsive problem has a solution which can be represented explicitly as follows

$$y(t) = X(t)\psi + \int_0^t G(t,s)\left[-\frac{1}{\lambda}B^2(X(s)\psi\right] ds + \sum_{0<t_k\le t} G(t,t_k)\left[-\frac{1}{\lambda}D_k^2X(t_k)\psi\right],$$

where the evolution operator $G(t,s)$ is given by

$$G(t,s) = X(t)X^{-1}(s).$$

On the other hand, the system (24) yields

$$\begin{aligned}
y(T) &= X(T)\psi + \int_0^T G(T,s)\left\{-\frac{1}{\lambda}B^2(X(s)\psi\right\}ds \\
&\quad + \sum_{0<t_k\leq T} G(T,t_k)\left\{-\frac{1}{\lambda}D_k^2X(t_k)\psi\right\} \\
&= X(T)\left[\psi + \int_0^T X^{-1}(T)G(T,s)\left\{-\frac{1}{\lambda}B^2(X(s)\psi\right\}ds\right. \\
&\quad \left. -\frac{1}{\lambda}\sum_{0<t_k\leq T} X^{-1}(T)G(T,t_k)\left\{D_k^2X(t_k)\psi\right\}\right] \\
&= X(T)\left[\psi + \int_0^T X^{-1}(s)\left\{-\frac{1}{\lambda}B^2(X(s)\psi\right\}ds\right. \\
&\quad \left. -\frac{1}{\lambda}\sum_{0<t_k\leq T} X^{-1}(t_k)\left\{D_k^2X(t_k)\psi\right\}\right] \\
&= X(T))\left[\psi - \frac{1}{\lambda}\Gamma(\psi)\right] = 0.
\end{aligned}$$

This shows that the initial state ψ is null-controllable at time T with control

$$h(t) = \left(u(t), \{v_k\}_{k\in\sigma_1^m}\right) = \left(-\frac{1}{\lambda}X(t)\psi, \left\{-\frac{1}{\lambda}X(t_k)\psi\right\}_{k\in\sigma_1^m}\right),$$

which completes the proof of the Proposition. □

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(Received October 17, 2006)