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Asymptotic profiles for some problems of wave equations in the Fourier space and other spaces

A Doctoral Thesis presented By:

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Contents

Abstract

In this thesis, we consider the study of some hyperbolic problems (equations and system of equations) with the presence of a viscoelastic term and under some assumptions on initial data and boundary conditions, conditions on damping and source terms. The focuss of the study is on the existence and asymptotic behavior of solutions .

Key Words: Hyperbolic equation, viscoelastic wave equations, system of wave equations, degenerately damped systems, strong nonlinear source, decay rate .
subject classification 2000 : 35L05,35L20,35L70,35L80 58J45, 35B40 , 58G16.

RESUME

Dans cette these, on considere l'études théorique de quelques problémes de type hyperbolique (équations et systemes d'équations) à terme viscoélastique et sous quelques hypothéses sur les conditions initiale et au bord, des conditions sur les termes de dissipation, termes sources. Nous avons étudié l'existence et le comportement asymptotique de la décroissance de l'énergie des solutions.

Mots Clés: Equation hyperbolique, viscoélastique, équation d'ondes, systéme d'équation d'ondes , dissipation nonlinéaire, source nonlinéaire, énergie initiale , déccroissance de l'énergie.

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INTRODUCTION

Nonlinear evolution equations, i.e., partial differential equations with time t as one of the independent variables, arise not only in many fields of mathematics, but also in other branches of science such as physics, mechanics and material science. For example, NavierStokes and Euler equations of fluid mechanics, nonlinear reaction-diffusion equations of heat transfers and biological sciences, nonlinear Klein- Gorden equations and nonlinear Schrodinger equations of quantum mechanics and Cahn-Hilliard equations of material science, to name just a few, are special examples of nonlinear evolution equations. Complexity of nonlinear evolution equations and challenges in their theoretical study have attracted a lot of interest from many mathematicians and scientists in nonlinear sciences.

From the physical point of view, these types of problems arise usually in viscoelasticity. These types of problems have been considered first by Dafermos [15], in 1970, where the general decay was discussed; this type has attracted a great deal of attention in the last two decades, and many results have been appeared on the existence and long time behavior of solutions. See, in this direction, [[8], [6], [12], [11], [22]] and references therein. As these materials have wide applications in the natural sciences, their dynamics are interesting and of great importance. Hence, questions related to the behavior of the solutions for the partial differential equations (PDE) system attracted considerable attention in recent years. To motivate our present work, let us recall some previous results regarding the viscoelastic wave equation, for example, Cavalcanti et al. [[12], [13]] studied the problem of the form

$$u'' - \Delta u + \int_0^t g(t-s)\Delta u(s,x)ds + \alpha(x).u' = 0, x \in \mathbb{R}^n, t \in \mathbb{R}_*^+$$

in $\Omega \times (0,\infty)$ subject to initial condition and boundary conditions of Dirichlet type. The authors showed an exponential decay result under some restrictions on $\alpha(x)$ and $g(t)$. To be specific, they assumed that $a : \Omega \longrightarrow \mathbb{R}$ is a non negative and bounded function, and the kernel g in the memory term decays exponentially. In [19] the authors have studied the following system:

$$u'' - \Delta u - \alpha(t)\Delta u' \int_0^t g(t-s)\Delta u(s,x)ds = |u|^{p-2}u, in\ \Omega \times (0,\infty),$$

where they have applied the Faedo-Galerkin method combined with the fixed point theorem. They showed the existence and uniqueness of a local in time solution and under some restrictions on the initial data, the solution continues to exist globally in time. On the other hand, if the interior source dominates the boundary damping, they proved that the solution is unbounded and grows as an exponential function. In addition, in the absence of the strong damping, they also proved the solution ceases to exist and blows up in finite time. Given an initial value problem or initial boundary value problem for a nonlinear evolution equation, if a priori estimates, which is often called the compactness estimates in literature, can be obtained, then we can use a systematic approach that is called the compactness method to deal with the issue of local existence or global existence. there are three major steps for this method:

1. Use the Faedo-Galerkin method, i.e, choose certain basis functions in an appropriate Sobolev space, and solve the approximate problems in any finite dimensional space spanned by finite basis functions. This often turns out to be an initial value problem for nonlinear ordinary differential equations. By the well-known local existence theorem for ordinary differential equations, local existence of solution to the approximate problem follows.

2. Obtain the compactness estimates for the solution of the approximate problem. It also turns out that the solution to the approximate problem globally exists.

3. Further use of the obtained compactness estimates allows one to choose a subsequence of solutions of the approximate problem obtained in the second step, converging to a solu-

tion of the original problem; uniqueness of the solution for the original problem has to be proved separately. However the compactness estimates obtained in the second step are still very useful for this purpose. For a given nonlinear hyperbolic equation, once it is known that a solution exists for all time $t > 0$, a natural and interesting question is to ask about the asymptotic behavior of the solution as $t \longrightarrow \infty$. Let us first look at some fundamental integral inequalities introduced by A. Haraux[53], V. Komornik[34], P. Martinez[42] and A.Guesmia[20] to estimate the decay rate of the energy of some dissipative problems.

The main aim of this thesis is to study some hyperbolic systems with the presence of different mechanisms of damping and under assumptions on initial data and boundary conditions. Our main goal is to investigate the existence of the solutions and their behavior when the time evolves. In fact, we prove that under some assumptions on the parameters in the systems and on the size of the initial data, the solutions can be proved to be either global in time or may blow up in finite time (i.e some norms of the solution will be unbounded in finite time). If the solutions are global in time, then the natural question is about their convergence to the steady state and the rate of convergence. The study of the asymptotic behavior of solutions of nonlinear evolution equations, particularly those governing gas dynamics, quantum theory and thermoelasticity, has been an important area for the interaction between the partial differential equations and physics.

The main results of this work

The thesis divided in to five chapters beginning by a general introduction.

Chapter 1

This chapter summarizes some concepts, definitions and results which are mostly relevant to the undergraduate curriculum and are thus assumed as basically known, or have specific roots in rather distant areas and have rather auxiliary character with respect to the purpose of this study. In the next four chapters, we develop our main results for nonlinear evolution problems of hyperbolic type

Chapter 2

In this chapter, we consider a viscoelastic wave plate equation with memory fading in $\mathbb{R}^N$

$$\begin{cases} u'' + \Delta^2 u + \alpha(t) \int_0^t g(t-s)\Delta u(s,x)ds - \Delta u' = 0, x \in \mathbb{R}^n, t \in \mathbb{R}_*^+ \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^n), u'(0,x) = u_1(x) \in L^2(\mathbb{R}^n), x \in \mathbb{R}^n. \end{cases}$$

In the second section we establish a general decay rate properties of solutions
the system 3.1. We exploit a density function to introduce weighted spaces for solutions and using an appropriate perturbed energy method.
This is the subject of a paper accepted in Facta Universitatis, Series Mathematics and Informatics, Vol 31 (4) (2016), 559–568.

Chapter 3

This chapter is devoted to the study of the decay rate estimate of solutions to damped wave equation with memory term in Fourier spaces of the following problem:

$$\begin{cases} u'' - \Delta u - \Delta u' + \alpha(t) \int_0^t g(t-s)\Delta u(s,x)ds = 0, x \in \mathbb{R}^n, t \in \mathbb{R}_*^+ \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^n), u'(0,x) = u_1(x) \in L^2(\mathbb{R}^n), x \in \mathbb{R}^n. \end{cases}$$

The main idea of the proof is to construct an appropriate Lyapunov function of the system obtained after taking the Fourier transform.
This is the subject of an international publication in Global Journal of pure and Applied Mathematics. Vo 11, No 5 (2015),pp 3027-3038.

Chater 4

This chapter considers the study of the following system:

$$\begin{cases} \left(|u'|^{q-2}u'\right)' - \phi(x)\left(\Delta_x u - \int_0^t g_1(t-s)\Delta_x u(s)ds\right) + \alpha v = 0 \\ \left(|v'|^{q-2}v'\right)' - \phi(x)\left(\Delta_x v - \int_0^t g_2(t-s)\Delta_x v(s)ds\right) + \alpha u = 0 \end{cases}$$

We establish explicit and very general decay rate properties of solutions. We exploit the density to introduce weighted spaces for solutions and using an appropriate Lyapunov function. This result is accepted in Demonstratio Mathematica, vol 50, Issue No. 4, 2017,pp 1-15.

Chapter 5

The first section is devoted to prove the existence and uniqueness of solutions for a coupled system of a viscoelastic wave equations in $\mathbb{R}^n$ given by system 5.1. We prove the existence of a unique weak solution of the restricted problem on B_R; the main ingredient used here is the Galerkin approximation method introduced in [37].
In the second section, we establish a general decay rate property of solutions
of system 5.1.
We exploit a density function to introduce weighted spaces for solutions and using an appropriate perturbed energy method.

Chapter 1

Functional analysis preliminaries

The aim of this chapter is to recall the essential notions and results used throughout this work. First, we recall some definitions and results on Sobolev spaces and the spaces $L^p(0, T, X)$ and give the statement of some important theorems in the analysis of problems to be studied and eventually some notations used throughout this study.

1.1 Function Annalysis

1.1.1 Normed spaces, Banach spaces and their properties

Defintion 1.1.1 *The linear space V is endowed by a binary operation*
$(v_1, v_1) \longrightarrow v_1 + v_1 : V \times V \longrightarrow V$ *which makes it a commutative group and furthermore it is equipped with a multiplication* $(a, x) \longrightarrow ax : \mathbb{R} \times V \longrightarrow V$ *satisfying*
$(a_1 + a_2)v = a_1 v + a_2 v, a(v_1 + v_2) = av_1 + av_2, (a_1 a_2)v = a_1(a_2 v)$*, and* $1.v = v$.

Defintion 1.1.2 *Let V be linear space. A non-negative, degree-1 homogeneous, subadditive functional* $\|.\|_V : V \longrightarrow \mathbb{R}$ *is called a norm if it vanishes only at 0, often, we will write briefly* $\|.\|$ *instead of* $\|.\|_V$ *if the following properties are satisfing respectively* $\|v\| \geq 0, \|av\| = |a|\|v\|, \|u + v\| \leq \|u\| + \|v\|$ *for any* $v \in V$ *and* $a \in \mathbb{R}$ *and* $\|v\| = 0 \longrightarrow v = 0$.
A linear space equipped with a norm is called a normed linear space. If the last (i.e.$\|v\|_v = 0 \longrightarrow$

$v = 0$)is missing, we call such a functional a seminorm.

Defintion 1.1.3 *A Banach space is a complete normed linear space X. Its dual space X' is the linear space of all continuous linear functional $f : X \longrightarrow \mathbb{R}$*

Proposition 1.1.1 *V' equipped with the norm $\|.\|_{V'}$ defined by*
$\|u\|_{V'} = sup|u(x)| : \|x\| \leq 1,$
is also a Banach space.
If V is a Banach space such that, for any $v \in V, V \longrightarrow \mathbb{R} : u \longrightarrow \|u+v\|^2 - \|u-v\|^2$.is linear, then V is called a Hilbert space. In this case, we define the inner product (also called scalar product) by
$(u, v) = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2.$

Defintion 1.1.4 *Since u is linear we see that*

$$u : V \longrightarrow V'',$$

is a linear isometry of V onto a closed subspace of V'' , we denote this by

$$V \longrightarrow V''.$$

Let V be a Banach space and $u \in V'$. Denote by

$$\phi_u : V \longrightarrow \mathbb{R}$$

$$x \longmapsto \phi_u(V),$$

when u covers V', we obtain a family of applications to $V \in \mathbb{R}$.

Defintion 1.1.5 *The weak topology on V, denoted by $\sigma(V, V')$, is the weakest topology on V for which every $(\phi_u)_{u \in V'}$ is continuous. We will define the third topology on V', the weak star topology, denoted by $\sigma(V', V)$. For all $x \in V$, denote by*

$$\begin{array}{rcl} \phi_x : V' & \longrightarrow & \mathbb{R} \\ u & \longmapsto & \phi_x(u) = < u, x >_{V',V} \end{array}$$

when x cover V, we obtain a family $(\phi_x)_{x \in V}$, of applications to V' in $\mathbb{R}$.

Theorem 1.1.1 *Let V be Banach space. Then, V is reflexive, if and only if,*

$$B_V = \{x \in V : \|x\| \leq 1\},$$

is compact with the weak topology $\sigma(V, V')$.

Corollary 1.1.1 *Every weakly* y^* *convergent sequence in* V' *must be bounded if V is a Banach space. In particular, every weakly convergent sequence in a reflexive Banach V must be bounded.*

Defintion 1.1.6 *Let V be a Banach space and let* $(u_n)_{n \in N}$ *be a sequence in* V. *Then* u_n *converges strongly to u in V if and only if*

$$\lim_{t \longrightarrow \infty} \|u_n - u\|_V = 0$$

and this is denoted by $u_n \longrightarrow u$, *or*

$$\lim_{t \longrightarrow \infty} u_n = u$$

.

1.2 Functional spaces

1.2.1 The $L^p(\Omega)$ spaces

Defintion 1.2.1 *Let* $1 \leq p \leq \infty$ *; and let* Ω *be an open domain in* $\mathbb{R}^n$ *;* $n \in \mathbb{N}$. *Define the standard Lebesgue space* $L^p(\Omega)$*; by:*

$$L^p(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}, f \quad \text{is} \quad \text{measurable} \quad \text{and} \int_\Omega |f|^p dx < \infty\}.$$

Notation 1.2.1 *For* $p \in \mathbb{R}$ *and* $1 \leq p \leq \infty$, *denote by:*

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}},$$

if $p = \infty$, *we have*

$$L^\infty(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}, f \text{ measurable} \quad \text{and } \exists C \in \mathbb{R}_+, |f(x)| \leq C \quad _{a.e}\}.$$

Theorem 1.2.1 *It is well known that $L^p(\Omega)$ equipped with the norm $\|.\|_p$ is a Banach space for all $1 \leq p \leq \infty$.*

Remark 1.2.1 *In particular, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$< f, g >_{L^2(\Omega)} = \int_\Omega f(x).g(x)dx,$$

is a Hilbert space.

Theorem 1.2.2 *For $1 \leq p \leq \infty$, $L^p(\Omega)$ is a reflexive space.*

1.2.2 Sobolev spaces

Modern theory of differential equations is based on spaces of functions whose derivatives exist in a generalized sense and enjoy a suitable integrability.

Proposition 1.2.1 *Let Ω be an open domain in $\mathbb{R}^N$, then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$if there exists a function $f \in L^p(\Omega)$ such that*

$$< T, \phi >= \int_\Omega f(x)\phi(x)dx, \forall\phi \in D(\Omega),$$

where $1 \leq p \leq \infty$, and it's well-known that f is unique.

Defintion 1.2.2 *Let $m \in \mathbb{N}$ and $p \in [0, \infty]$. The $W_{m,p}(\Omega)$ is the space of all $f \in L^p(\Omega)$, defined as*

$$\begin{aligned} W_{m,p}(\Omega) &= \{f \in L^p(\Omega) \quad where, \partial^\alpha f \in L^p(\Omega) \forall \alpha \in \mathbb{N}^m \quad such \quad that, \\ |\alpha| &= \sum_{j=1}^{n} \alpha_j < m \quad with, \partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}...\partial_n^{\alpha_n}\}. \end{aligned}$$

Theorem 1.2.3 *$W_{m,p}(\Omega)$ is a Banach space with its usual norm*

$$\|f\|_{W_{m,p}(\Omega)} = \sum_{\alpha<m} \|\partial^\alpha f\|_{L^p(\Omega)}, 1 \leq p < \infty \forall f \in W^{m,p}(\Omega).$$

Defintion 1.2.3 *Denote by $W_0^{m,p}(\Omega)$ the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$.*

1.2.3 Space $H^m(\Omega)$

Defintion 1.2.4 *When $p = 2$, we write $W^{m,2}(\Omega) = H^m(\Omega)$ and $W_0^{m,2}(\Omega) = H_0^m(\Omega)$ endowed with the norm*

$$\|f\|_{H^m(\Omega)} = \left(\sum_{\alpha < m} (\|\partial^\alpha f\|)^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

which renders $H^m(\Omega)$ a real Hilbert space with their usual scalar product

$$< u, v >_{H^m(\Omega)} = \sum_{\alpha < m} \int_\Omega \partial^\alpha u \partial^\alpha v dx.$$

Theorem 1.2.4 *1)$H^m(\Omega)$ endowed with inner product $< ., . >_{H^m(\Omega)}$ is a Hilbert space.*
2) If $m < m'$, $H^m(\Omega) \longrightarrow H^{m'}(\Omega)$, with continuous imbedding .

Lemma 1.2.1 *Since $D(\Omega)$ is dense in $H_0^m(\Omega)$, we identify a dual $H^{-m}(\Omega)$ of $H_0^m(\Omega)$ in a weak subspace on Ω and we have*

$$D(\Omega) \longrightarrow H_0^m(\Omega) \longrightarrow L^2(\Omega) \longrightarrow H_0^{-m}(\Omega) \longrightarrow D'(\Omega)$$

1.3 Integral Inequalities used in the multiplied method

Lemma 1.3.1 *For any two functions $g, v \in C^1(\mathbb{R})$ and $\theta \in [0,1]$, we have*

$$\begin{aligned} v'(t) \int_0^t g(t-s)v(s)ds &= -\frac{1}{2}\frac{d}{dt}\int_0^t g(t-s)|v(t)-v(s)|^2 ds + \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s)ds\right)|v(t)|^2 \\ &+\frac{1}{2}\int_0^t g'^2 ds - \frac{1}{2}g(t)|v(t)|^2. \end{aligned} \tag{1.1}$$

and

$$\left| \int_0^t g(t-s)(v(t)-v(s))ds \right|^2 \leq \left(\int_0^t |g(s)|^{2(1-\theta)} ds \right) \int_0^t |g(t-s)|^{2\theta} |v(t)-v(s)|^2 ds. \tag{1.2}$$

Lemma 1.3.2 *Let ρ satisfy (**A2**), then for any $u \in D(A^{1/2})$, we have*

$$\|u\|_{L^q_\rho(\mathbb{R}^n)} \le \|\rho\|_{L^s(\mathbb{R}^n)} \|A^{1/2}u\|_{L^2(\mathbb{R}^n)}, \tag{1.3}$$

with,

$$s = \frac{2n}{2n - qn + 2q}, 2 \le q \le \frac{2n}{n-2}.$$

We define the function spaces of our problem and their norm as follows:

$$\mathcal{H}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla_x f \in (L^2(\mathbb{R}^n))^n \right\}$$

and the space $L^2_\rho(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(f, h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx. \tag{1.4}$$

Lemma 1.3.3 *Let ρ satisfy (1.4), then for any $u \in \mathcal{H}(\mathbb{R}^n)$,for $1 < p < \infty$, if f is a measurable function on $\mathbb{R}^n$,*

$$\|u\|_{L^p_\rho(\mathbb{R}^n)} \le \|\rho\|_{L^s(\mathbb{R}^n)} \|\nabla_x u\|_{L^2(\mathbb{R}^n)}, \tag{1.5}$$

with $s = \frac{2n}{2n-pn+2p}, 2 \le p \le \frac{2n}{n-2}$.

1.4 Some algebraic and integral inequalities

We give here some important integral inequalities. These inequalities play an important role in applied mathematics and are also very useful in the next chapters.

Theorem 1.4.1 *Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \le p < \infty$,then $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} \quad \le \|f\|_{L^p(\Omega)}.\|g\|_{L^{p'}(\Omega)}$$

when $p = p' = 2$ one finds the Cauchy-Schwarz inequality.

Assume $f \in L^p(\Omega) \cap L^q(\Omega)$ then $f \in L^r(\Omega)$ for $r \in [p, q]$ and

$$\|f\|_{L^r(\Omega)} \le \|f\|^\alpha_{L^p(\Omega)} \|f\|^{1-\alpha}_{L^q(\Omega)},$$

with

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q} \quad for\, some \quad 0 \le \alpha \le 1.$$

Theorem 1.4.2 *Let a and b be strictly positive realities p and q such as,* $\dfrac{1}{p}+\dfrac{1}{q}=1$ *and* $1<p<\infty$*, we have :*

$$ab \leq \frac{a^p}{p}+\frac{b^q}{q}$$

Proof 1.4.1 *The function f defined by:*

$$f(x)=\frac{x^p}{p}-x$$

reached its minimum point $x=1$ *indeed :*

$$y' = x^{p-1} \quad et \quad y'' = (p-1)x^{p-2} > 0$$

from where

$$f(ab^{1-q}) \geq f(1)$$

which gives

$$\frac{\left(ab^{1-q}\right)^p}{p} - ab^{1-q} \geq \frac{1}{p} - 1 = -\frac{1}{q}$$

so that

$$\frac{a^p}{p}b^{(1-q)p} - ab^{1-q} + \frac{1}{q} \quad \geq 0$$

By dividing the two members by $b^{(1-q)p}$ *we obtain :*

$$\frac{a^p}{p} - ab^{(1-q)-p+pq} + \frac{b^q}{q} \geq 0$$

which yieldes

$$\frac{a^p}{p} - ab + \frac{b^q}{q} \geq 0$$

so that

$$ab \leq \frac{a^p}{p}+\frac{b^q}{q}.$$

Remark 1.4.1 *A simple case of Young's inequality is the inequality for* $p=q=2$ *:*

$$ab \leq \frac{a^2}{2}+\frac{b^2}{2}$$

which also gives Young's inequality for all $\delta>0$ *:*

$$ab \leq \delta a^2 + \frac{1}{4\delta}b^2$$

Theorem 1.4.3 *(Young)*
Let $f \in L^1(\mathbb{R}^n)$ *and* $g \in L^p(\mathbb{R}^n$ *with* $1 \leq p \leq \infty$, $1 \leq q \leq \infty$.
Then for a.e.$x \in \mathbb{R}^N$ *the function is integrable on* $\mathbb{R}^n$ *and we define:*

$$(f * g) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

In addition

$$(f * g) \in L^p(\mathbb{R}^n)$$

and

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p$$

The following is an extension of Theorem 1.4.3

Theorem 1.4.4 *(Young)*
Assume $f \in L^1(\mathbb{R}^n)$ *and* $g \in L^p(\mathbb{R}^n$ *with* $1 \leq p \leq \infty$, $1 \leq p \leq \infty$.
and $x \in \mathbb{R}^N$ *the function is integrable on* $\mathbb{R}^n$ *and* $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$ *.Therefore*

$$(f * g) \in L^r(\mathbb{R}^n)$$

and

$$\|f * g\|_r \leq \|f\|_p \|g\|_p$$

Remark 1.4.2 *Young's inequality can sometimes be written in the form :*

$$ab \leq \delta a^p + C(\delta) b^q, \qquad C(\delta) = \delta^{-\frac{1}{p-1}}$$

1.4.1 Holder's inequalities

Theorem 1.4.5 *{ Assume that* $f \in L^p(\Omega)$ *and* $g \in L^{p'}(\Omega)$ *with* $1 \leq p < \infty$*,Then* $fg \in L^1(\Omega)$ *and:*

$$\|fg\|_{L^1(\Omega)} \quad \leq \|f\|_{L^p(\Omega)} . \|g\|_{L^{p'}(\Omega)},$$

when $p = p' = 2$ *,we get the inequality of Cauchy-Schwartz inequality*

Corollary 1.4.1 *(Holder's inequality general form)*
Let $f_1, f_2, ...f_k$ *be k functions such that,* $f_i \in L^{p_i}(\Omega)$, $1 \leq i \leq k$, *and*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_k} \leq 1.$$

Then, the product $f_1, f_2, ...f_k \in L^p(\Omega)$ *and* $\|f_1 f_2 ... f_k\|_p \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_k\|_{p_k}$.

Lemma 1.4.1 *(Minkowski inequality)*
For $1 \leq p \leq \infty$, *we have*

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p$$

.

Lemma 1.4.2 *(Cauchy-Schwarz inequality)*
Every inner product satisfies the Cauchy-Schwarz inequality

$$< x_1, x_2 > \leq \|x_1\| \|x_2\|.$$

The equality sign holds if and only if x_1 *and* x_2 *are dependent.*
We will give here some integral inequalities. These inequalities play an important role in applied mathematics and are also very useful in the next chapters.

Lemma 1.4.3 *let* $1 \leq p \leq r \leq q$, $\frac{1}{r} = \frac{\alpha}{p} + \frac{1}{q}$ *and* $1 \leq \alpha \leq 1$. *Then*

$$\|u\|_r \leq \|u\|_p^{\alpha} \|u\|_q^{1-\alpha}$$

.

Lemma 1.4.4 *For any* $v \in C^1\left(0, T, H^1(\mathbb{R}^n)\right)$ *we have*

$$\begin{aligned}
& -\int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s) A v(s) v'(t) ds dx \\
= \; & \frac{1}{2}\frac{d}{dt}\alpha(t)\left(g \circ A^{1/2} v\right)(t) \\
- \; & \frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(s)\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dx ds\right] \\
& -\frac{1}{2}\alpha(t)\left(g'^{1/2}v\right)(t) + \frac{1}{2}\alpha(t)g(t)\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dx ds \\
& -\frac{1}{2}\alpha'(t)\left(g \circ A^{1/2}v\right)(t) + \frac{1}{2}\alpha'(t)\int_0^t g(s) ds \int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dx ds.
\end{aligned}$$

Proof 1.4.2

$$\begin{aligned}
&\int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s) A v(s) v'(t) ds dx \\
= \; & \alpha(t) \int_0^t g(t-s) \int_{\mathbb{R}^n} A^{1/2} v'^{1/2} v(s) dx ds \\
= \; & \alpha(t) \int_0^t g(t-s) \int_{\mathbb{R}^n} A^{1/2} v'(t) \left[A^{1/2} v(s) - A^{1/2} v(t) \right] dx ds \\
& + \alpha(t) \int_0^t g(t-s) \int_{\mathbb{R}^n} A^{1/2} v'^{1/2} v(t) dx ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s) A v(s) v'(t) ds dx \\
= \; & -\frac{1}{2} \alpha(t) \int_0^t g(t-s) \frac{d}{dt} \int_{\mathbb{R}^n} \left| A^{1/2} v(s) - A^{1/2} v(t) \right|^2 dx ds \\
& + \alpha(t) \int_0^t g(s) \left(\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^n} \left| A^{1/2} v(t) \right|^2 dx \right) ds
\end{aligned}$$

which implies,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s) A v(s) v'(t) ds dx \\
= \; & -\frac{1}{2} \frac{d}{dt} \left[\alpha(t) \int_0^t g(t-s) \int_{\mathbb{R}^n} \left| A^{1/2} v(s) - A^{1/2} v(t) \right|^2 dx ds \right] \\
& + \frac{1}{2} \frac{d}{dt} \left[\alpha(t) \int_0^t g(s) \int_{\mathbb{R}^n} \left| A^{1/2} v(t) \right|^2 dx ds \right] \\
& + \frac{1}{2} \alpha(t) \int_0^t g'(t-s) \int_{\mathbb{R}^n} \left| A^{1/2} v(s) - A^{1/2} v(t) \right|^2 dx ds \\
& - \frac{1}{2} \alpha(t) g(t) \int_{\mathbb{R}^n} \left| A^{1/2} v(t) \right|^2 dx ds. \\
& + \frac{1}{2} \alpha'(t) \int_0^t g(t-s) \int_{\mathbb{R}^n} \left| A^{1/2} v(s) - A^{1/2} v(t) \right|^2 dx ds \\
& - \frac{1}{2} \alpha'(t) \int_0^s g(s) ds \int_{\mathbb{R}^n} \left| A^{1/2} v(t) \right|^2 dx ds.
\end{aligned}$$

This completes the proof.

Theorem 1.4.6 *(Gronwell lemma in integral form)*

Let $T > 0$, and let ψ be a function such that,$\psi \in L^1[0,T], \phi \geq 0$, almost everywhere and ψ be a *function such that* $\phi \in L^1[0,T], \phi \geq 0$*, almost everywhere and* $\phi\psi \in L^1[0,T], C_1, C_2 \geq 0$*. Suppose that*

$$\phi(t) \leq C_1 + C_2 \int_0^t \phi(t)\psi(t)ds, for \quad a.e \quad t \in]0,T[,$$

then

$$\phi(t) \leq C_1 \exp\left(C_2 \int_0^t \phi(t)\psi(t)ds\right), for \quad a.e \quad t \in]0,T[.$$

Chapter 2

Energy decay of solution to plate equation with memory in $\mathbb{R}^n$

2.1 Intrduction

This chapter aims at investigating the energy decay of solution to plate equation with memory in $\mathbb{R}^n$ *to the following equation:*

$$\begin{cases} u'' + \Delta^2 u + \alpha(t)\int_0^t g(t-s)\Delta u(s,x)ds - \Delta u' = 0, x \in \mathbb{R}^n, t \in \mathbb{R}_*^+ \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^n), u'(0,x) = u_1(x) \in L^2(\mathbb{R}^n), x \in \mathbb{R}^n. \end{cases} \tag{2.1}$$

where $n \geq 2$*. It is well known that the presence of a viscoelastic term with and without the weighted function* α *doest not preclude the question of existence, but its effects are only on the stability of the existing solution. For the existence, we refer to the works [24], [26], [25], [31], [40], [44], [16]. This type of problems is usually encountered in viscoelasticity in various areas of mathematical physics. It was first considered by Dafermos in [15], where the general decay was discussed. The problems related to (2.1) attracted a great deal of attention in the last decades and numerous results appeared on the existence and long time behavior of solutions but their results is by now rather developed, especially in any space dimension when it comes to nonlinear problems.*

For the literature, in $\mathbb{R}^n$ *we quote essentially the results of [10], [28], [29], [30], [32], [33], [45]. In [29],the authors showed that, for compactly supported initial data and for an exponentially decaying* *relaxation function, the decay of the energy of solution of a linear Cauchy problem related of (2.1) is polynomial. The finite-speed propagation is used to compensate for the lack of Poincares inequality. In [28], the author looked into a linear Cauchy viscoelastic equation with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate for the lack of Poincare's inequality. The same problem treated in [28], was considered in [30], where they consider a Cauchy problem for a viscoelastic wave equation. Under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. Conditions used, on the relaxation function g and its derivative* g' *are different from the usual ones.*

Ikehata in [24] considered, in the one-dimensional half space, the mixed problem of the equation

$$v_{tt} - v_{xx} + v_t = 0 \tag{2.2}$$

with a weighted initial data and presented a new decay estimate of solutions which can be also derived for the Cauchy problem in $\mathbb{R}^n$. *Let us mention that a pioneer question on the long time asymptotic of strongly damped wave equations in [25], where the authors studied the Cauchy problem for abstract dissipative equations in Hilbert spaces generalizing wave equations with strong damping terms in* $\mathbb{R}^n$ *or exterior domains*

$$u_{tt}(t) + Au(t) + Au'(t) = 0, \qquad t \in (0, \infty). \tag{2.3}$$

$$u(0) = u_0, \qquad u_t(0) = u_1, \tag{2.4}$$

where $A : D(A) \subset H \to H$ *is a nonnegative self-adjoint operator in* $(H, \|.\|)$ *with a dense domain* $D(A)$. *Using the energy method in the Fourier space and its generalization based on the spectral theorem for self-adjoint operators. Their main result was a combination of solutions of diffusion and wave equations.*

2.2 Statement and Preliminairies

We omit the space variable x *of* $u(x,t), u'(x,t)$ *and, for simplicity reason, denote* $u(x,t) = u$ *and* $u'(x,t) = u'$, *when there is no confusion. The constants* c *used throughout this paper are* *positive generic constants which may be different in various settings, here* $u' = du(t)/dt$ *and* $u'' = d^2u(t)/dt^2$.

The following notation will be used throughout this chapter

$$(g \circ \Psi) = \int_0^t g(t-\tau)\,|\Psi(t) - \Psi(\tau)|^2\,d\tau, \text{ for any } \Psi \in L^\infty(0,T;L^2(\mathbb{R}^n)). \tag{2.5}$$

In order to investigate the decay structure based on the weak- memory and the damping terms, we also consider the following assumptions:

$g, \alpha : \mathbb{R}+ \longrightarrow \mathbb{R}^+$ *are non-increasing differentiable functions of class* C^1 *satisfying:*

$$|\xi|^2 - \alpha(t)\int_0^t g(t)dt \geq k > 0, \quad g(0) = g_0 > 0, \tag{2.6}$$

$$\infty > \int_0^\infty g(t)dt, \quad \alpha(t) > 0. \tag{2.7}$$

Here ξ *is the variable associated with the Fourier transform.*

In addition, there exists a non-increasing differentiable function $\beta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *satisfying*

$$\beta(t) > 0, \quad g'(t) + \beta(t)g(t) \leq 0, \quad \forall t \geq 0, \quad \lim_{t\to\infty} = \frac{\alpha'(t)}{\beta(t)\alpha(t)} = 0. \tag{2.8}$$

We give some notations to be used below. Let F *denote the Fourier transform in* $L^2(\mathbb{R}^n)$ *defined as follows:*

$$F[f](\xi) = \widehat{f}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \exp(-ix.\xi)f(x)dx, \tag{2.9}$$

where $i = \sqrt{-1}$, $\quad x.\xi = \sum_{i=1}^n x_i\xi_i$ *and denote its inverse transform by* F^{-1}. *The operator* $-\Delta$ *is defined by*

$$-\Delta v(x) = F^{-1}\left(|\xi|^2 F(v)(\xi)\right)(x), v \in H^2(\mathbb{R}^n), x \in \mathbb{R}^n.$$

For $1 \leq p \leq \infty$, *we denote by* $L^p(\mathbb{R}^n)$ *the usual Lebesgue space on* $\mathbb{R}^n$ *with the norm* $\|.\|_{L^p}$. *For a nonnegative integer* m, $H^m(\mathbb{R}^n)$ *denotes the Sobolev space of* $L^2(\mathbb{R}^n)$ *functions on* $\mathbb{R}^n$, *equipped with the norm* $\|.\|_{H^m}$. *By direct calculations, we have the following technical Lemma which will play an important role in the sequel.*

Lemma 2.2.1 *([16], Lemma 2.1) For any two functions* $g \in C^1(\mathbb{R}^n)$, *and* $v \in W^{1,2}(0,T)$, *it holds that*

$$
\begin{aligned}
Re\left\{\alpha(t)\int_0^t g(t-s)v(s)ds\overline{v}'(t)\right\} &= -\frac{1}{2}\alpha(t)g(t)|v(t)|^2 + \frac{1}{2}\alpha(t)(g'\circ v)(t) \\
&- \frac{1}{2}\frac{d}{dt}\alpha(t)(g\circ v)(t) + \frac{1}{2}\frac{d}{dt}\alpha(t)\int_0^t g(s)ds|v(t)|^2 \\
&+ \frac{1}{2}\alpha'(t)(g\circ v)(t) - \frac{1}{2}\alpha'(t)\int_0^t g(s)ds|v|^2
\end{aligned}
\tag{2.10}
$$

and

$$
|\int_0^t g(t-s)(v(t)-v(s))ds|^2 \leq \int_0^t |g(s)|ds \int_0^t |g|(t-s)|v(t)-v(s)|^2 ds
$$

We can now state and prove the asymptotic behavior of the solution of the system (2.1). Throughout this chapter, let us set $\widehat{u}(t,\xi) = F(u(t,.))(\xi)$.

2.3 Main result

We show that our solution decays time asymptotically to zero and the rate of decay for the solution is similar to both α *and* g.

Theorem 2.3.1 *Assume u is the solution of (2.1), then the next general exponential estimate satisfies in the Fourier space*

$$
E(t) \leq W\exp\left(-\omega\int_0^t \alpha(s)\beta(s)ds\right), \qquad \forall t \geq 0, \tag{2.11}
$$

for some positive constants W, *and* ω.

We take the Fourier transform of both sides of (2.1). Then one has the reduced equation for $\xi \in \mathbb{R}^n, t \in \mathbb{R}_*^+$:

$$
\begin{cases}
\widehat{u}(t,\xi)'' + |\xi|^4\widehat{u}(t,\xi) - |\xi|^2\alpha(t)\int_0^t g(t-s)\widehat{u}(s,\xi)ds + |\xi|^2\widehat{u}'(t,\xi) = 0, \\
\widehat{u}(0,\xi) = \widehat{u}_0(\xi) \in H^1(\mathbb{R}^n), \widehat{u}'(0,\xi) = \widehat{u}_1(\xi) \in L^2(\mathbb{R}^n).
\end{cases}
\tag{2.12}
$$

We apply the multiplier technique in Fourier space in order to obtain useful estimates and prepare some functionals associated with the nature of our problem to introduce an appropriate Lyapunov functions.

First, to derive the equality for the physical energy, we multiply both sides of (2.12) by $\overline{\widehat{u}'}$. We denote by

$$E_1(t) = \frac{1}{2}|\widehat{u}'|^2 + \frac{1}{2}|\xi|^2(|\xi|^2 - \alpha(t)\int_0^t g(s)ds)|\widehat{u}|^2 + \frac{1}{2}|\xi|^2\alpha(t)(g \circ \widehat{u})(t)$$

and

$$\begin{aligned} e_1(t) &= \frac{1}{2}|\xi|^2\left(\alpha(t)g(t)|\widehat{u}|^2 - \alpha(t)(g' \circ \widehat{u})(t) + 2|\widehat{u}'|^2\right) \\ &+ \frac{1}{2}|\xi|^2\left(\alpha'(t)(g \circ \widehat{u})(t) - \alpha'(t)\int_0^t g(s)ds|\widehat{u}|^2\right). \end{aligned}$$

Then, taking the real part of the resulting identities and by Lemma 2.2.1, we obtain

$$\frac{d}{dt}E_1(t) + e_1(t) = 0. \tag{2.13}$$

Second, the existence of the memory term forces us to make the first modification of the energy by multiplying (2.12) by $\left(-\frac{d}{dt}\left(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right)\right)$ and taking the real part, we have that

$$\begin{aligned} &0 = -Re\left\{\widehat{u}''\frac{d}{dt}\left(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right)\right\} \\ &-Re\left\{|\xi|^4\widehat{u}\frac{d}{dt}\left(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right)\right\} \\ &+\frac{1}{2}|\xi|^2\frac{d}{dt}\left(\left|\alpha(t)\int_0^t g(t-s)\widehat{u}(s)ds\right|^2\right) - Re\left\{|\xi|^2\widehat{u}'\frac{d}{dt}\left(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right)\right\} \end{aligned} \tag{2.14}$$

Since

$$\begin{aligned} \frac{d}{dt}\left(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right) &= \alpha'(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds + \alpha(t)\frac{d}{dt}\left(\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right) \\ &= \alpha'(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds + \alpha(t)g_0\overline{\widehat{u}} + \alpha(t)\int_0^t g'(t-s)\overline{\widehat{u}}(s). \end{aligned} \tag{2.15}$$

The first term in Eq.(2.14) takes the form

$$\begin{aligned}
&- Re\left\{\widehat{u}''\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\} \\
&= -Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}' \\
&+ Re\left\{\widehat{u}'\frac{d^2}{dt^2}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\} \\
&= -Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}' + \alpha(t)g_0|\widehat{u}'^2 \\
&+ Re\left\{\widehat{u}'\Big(\alpha(t)\frac{d}{dt}\Big(\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\Big) + \alpha'(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}.
\end{aligned}$$

Denote by

$$E_2(t) = \frac{1}{2}\left(|\xi|^2\Big|\alpha(t)\int_0^t g(t-s)\widehat{u}(s)ds\Big|^2 - Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right),$$

and

$$\begin{aligned}
e_2(t) &= \alpha(t)g_0|\widehat{u}'|^2 - Re\left\{|\xi|^2\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\} \\
&+ Re\left\{\alpha'(t)\widehat{u}'\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right\} \\
R_2(t) &= -Re\left\{|\xi|^4\widehat{u}\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\} \\
&+ Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}.
\end{aligned}$$

Then,

$$\frac{d}{dt}E_2(t) + e_2(t) + R_2(t) = 0. \tag{2.16}$$

Next, to make the second modification of the energy which corresponds to the strong damping, we multiply (2.12) by $\overline{\widehat{u}}$ *and taking the real part, we have*

$$\begin{aligned}
0 &= (Re\{\widehat{u}'\overline{\widehat{u}}\})' - |\widehat{u}'|^2 + |\xi|^4|\widehat{u}|^2 \\
&- |\xi|^2 Re\left\{\alpha(t)\int_0^t g(t-s)\widehat{u}(s)\overline{\widehat{u}}(t)ds\right\} + \frac{1}{2}|\xi|^2(|\widehat{u}|^2)',
\end{aligned}$$

using results in Lemma 2.2.1, we get

$$\begin{aligned} 0 &= (Re\{\widehat{u}'\overline{\widehat{u}}\})' - |\widehat{u}'|^2 + |\xi|^4|\widehat{u}|^2 + \frac{1}{2}|\xi|^2(|\widehat{u}|^2)' \\ &- |\xi|^2\left(\alpha(t)\int_0^t g(s)ds|\widehat{u}|^2 - Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\overline{\widehat{u}}(s)ds\right\}\right). \end{aligned}$$

Denote by

$$E_3(t) = Re\{\widehat{u}'\overline{\widehat{u}}\} + \frac{1}{2}|\xi|^2|\widehat{u}|^2,$$

and

$$\begin{aligned} e_3(t) &= |\xi|^2\Big(|\xi|^2 - \alpha(t)\int_0^t g(s)ds\Big)|\widehat{u}|^2. \\ R_3(t) &= |\widehat{u}'|^2 - Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\overline{\widehat{u}}(s)ds\right\} \end{aligned}$$

Then,

$$\frac{d}{dt}E_3(t) + e_3(t) + R_3(t) = 0. \tag{2.17}$$

Let us define for some constants ε_1, and $\varepsilon_2 > 0$ to be chosen later.

$$\begin{aligned} E_4(t) &= E_1(t) + \varepsilon_1\alpha(t)E_2(t) + \varepsilon_2\alpha(t)E_3(t) \\ &= \frac{1}{2}\left\{|\widehat{u}'|^2 + |\xi|^2\Big(1-\alpha(t)\int_0^t g(s)ds\Big)|\widehat{u}|^2 + |\xi|^2\alpha(t)(g\circ\widehat{u})(t)\right\} \\ &+ \frac{\varepsilon_1\alpha(t)}{2}\left(|\xi|^2\left|\alpha(t)\int_0^t g(t-s)\widehat{u}(s)ds\right|^2 - Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right) \\ &+ \varepsilon_2\alpha(t)\left(Re\left\{\widehat{u}'\overline{\widehat{u}}\right\} + \frac{1}{2}|\xi|^2|\widehat{u}|^2\right). \end{aligned}$$

and

$$\begin{aligned} e_4(t) &= e_1(t) + \varepsilon_1\alpha(t)e_2(t) + \varepsilon_2\alpha(t)e_3(t) \\ &= \frac{|\xi|^2}{2}\alpha(t)\left(g(t)|\widehat{u}|^2 - (g' \circ \widehat{u})(t) + 2\alpha^{-1}(t)|\widehat{u}'|^2\right) \\ &+ \frac{|\xi|^2}{2}\alpha'(t)\left((g \circ \widehat{u})(t) - \widehat{u}' \int_0^t g(s)ds|\widehat{u}|^2\right) \\ &+ \varepsilon_1\alpha(t)\left(\alpha(t)g_0|\widehat{u}'|^2 - Re\left\{|\xi|^2\widehat{u}'\frac{d}{dt}\left(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right)\right\}\right) \\ &+ \varepsilon_1\alpha(t)\left(Re\left\{\alpha'(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right\}\right) \\ &+ \varepsilon_2|\xi|^2\alpha(t)\left(|\xi|^2 - \alpha(t)\int_0^t g(s)ds\right)|\widehat{u}|^2. \end{aligned}$$

and

$$\begin{aligned} R_4(t) &= \varepsilon_1\alpha(t)R_2(t) + \varepsilon_2\alpha(t)R_3(t) \\ &= \varepsilon_1\alpha(t)\left(-Re\left\{|\xi|^4\widehat{u}\frac{d}{dt}\left(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right)\right\}\right) \\ &+ \varepsilon_1\alpha(t)\left(Re\left\{\widehat{u}'\frac{d}{dt}\left(\alpha(t)\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\right)\right\}\right) \\ &+ \varepsilon_2\alpha(t)\left(-|\widehat{u}'|^2 - Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s) - \widehat{u}(t))\overline{\widehat{u}}(s)ds\right\}\right). \end{aligned}$$

At this point, we introduce as in [33], the Lyapunov functions as

$$L_1(t) = \left\{|\widehat{u}'|^2 + k|\xi|^2|\widehat{u}|^2 + |\xi|^2\alpha(t)(g \circ \widehat{u})(t)\right\} \tag{2.18}$$

and

$$L_2(t) = \alpha(t)g(t)|\widehat{u}|^2 + \alpha(t)\beta(t)(g \circ \widehat{u})(t). \tag{2.19}$$

It is easy to verify that there exist positive constants $c_1(g_0)$, and $c_2(g_0)$ such that

$$c_1L_1(t) \le E_1(t) \le c_2L_1(t), \forall t > 0. \tag{2.20}$$

Thanks to Holder's and Young's inequalities, one gets for some constant $c_3 > 0$

$$|\varepsilon_1E_2(t) + \varepsilon_2E_3(t)| \le c_3L_1(t),$$

which means that $L_1(t) \sim E(t)$. Using again (2.8), Holder's and Young's inequalities and assumptions on g, we obtain

$$\begin{aligned}
|R_4(t)| &= \varepsilon_1 R_2(t) + \varepsilon_2 R_3(t) \\
&\leq \varepsilon_1 Re\left\{|\xi|^4 \widehat{u}\alpha(t)\frac{d}{dt}\Big(\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\} \\
&+ \varepsilon_1 Re\left\{\widehat{u}'\alpha(t)\frac{d}{dt}\Big(\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\Big)\right\} \\
&+ \varepsilon_2\left(|\widehat{u}'|^2 + Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\overline{\widehat{u}}(s)ds\right\}\right) \\
&\leq \varepsilon_1|\widehat{u}'|^2 + c_4\varepsilon_1|\xi|^4|\widehat{u}|^2 + c_5\varepsilon_1|\xi|^2 L_2(t) \\
&+ \varepsilon_2\left[|\widehat{u}'|^2 + c_6|\xi|^2\left(\lambda|\widehat{u}|^2 + c_\lambda\alpha(t)(g\circ\widehat{u})(t)\right)\right] \\
&\leq (\varepsilon_1+\varepsilon_2)|\widehat{u}'|^2 + (c_4\varepsilon_1|\xi|^2 + \varepsilon_2 c_6\lambda)|\xi|^2|\widehat{u}|^2 + (c_5\varepsilon_1 + c_\lambda\varepsilon_2)|\xi|^2 L_2(t).
\end{aligned}$$

Since $L_2(t) \leq c_3\, e_1(t)$, one can easily check that there exist positive constants $\varepsilon_1, \varepsilon_2, \lambda, c_4, c_5, c_6$ such that

$$|R_4(t)| \leq ce_4(t), c > 0. \tag{2.21}$$

By (2.13), (2.16) and (2.17), we get

$$\frac{d}{dt}E_4(t) = \frac{d}{dt}E_1(t) + \varepsilon_1\alpha(t)\frac{d}{dt}E_2(t) + \varepsilon_2\alpha(t)\frac{d}{dt}E_3(t) + \varepsilon_1\alpha'(t)E_2(t) + \varepsilon_2\alpha'(t)E_3(t).$$

We use $\lim\limits_{t\to\infty}\frac{\alpha'(t)}{\alpha(t)} = 0$ by (2.6)-(2.8) to choose $t_1 > t_0$ and since $e_4(t) \geq cE_4(t)$, then (2.21) gives for some positive constant N

$$\frac{d}{dt}E_4(t) \leq -N\alpha(t)E_4(t) + c\alpha(t)(g\circ\widehat{u})(t). \tag{2.22}$$

Multiplying (2.22) by $\beta(t)$ and using (2.8), (2.19), we obtain

$$\begin{aligned}
\beta(t)\frac{d}{dt}E_4(t) &\leq -N\beta(t)\alpha(t)E_4(t) + c\beta(t)\alpha(t)(g\circ\widehat{u})(t) \\
&\leq -N\beta(t)\alpha(t)E_4(t) - c\alpha(t)(g'\circ\widehat{u})(t) \\
&\leq -N\beta(t)\alpha(t)E_4(t) - c|\xi|^2\alpha'(t)\int_0^t g(s)ds|\widehat{u}|^2 - 2c\frac{d}{dt}E_4(t), \quad \forall t > t_1
\end{aligned} \tag{2.23}$$

Since $\beta'(t) \leq 0$*, we set* $L(s) = (\beta(s) + 2c)E_4(s)$ *which is equivalent to* $E_4(t)$*, then*

$$\begin{aligned}
\frac{d}{dt}L(t) &\leq -N\beta(t)\alpha(t)E_4(t) - c|\xi|^2\alpha'(t)\int_0^t g(s)ds|\widehat{u}|^2 \\
&\leq -\beta(t)\alpha(t)\left[N - \frac{2\alpha'(t)}{k\beta(t)\alpha(t)}\int_0^t g(s)ds\right]E_4(t), \quad \forall t > t_1. \qquad (2.24)
\end{aligned}$$

By (2.8), we can choose $t_2 > t_1$ *such that*

$$\begin{aligned}
\frac{d}{dt}L(t) &\leq -c\beta(t)\alpha(t)E_4(t) \\
&\leq -c\beta(t)\alpha(t)L(t), \quad \forall t > t_2. \qquad (2.25)
\end{aligned}$$

Integrating (2.25) over $[t_2, t]$ *using the equivalence between Lyapunov function and the energy function, it yields*

$$E_4(t) \leq W\exp(-\omega\int_0^t \alpha(s)\beta(s)ds), W, \omega > 0.$$

Chapter 3

Decay rate estimate of solution to damped wave equation with memory term in Fourier spaces

3.1 Preliminaries and position of the problem

Let us consider the weak-viscoelastic case in the following problem:

$$\begin{cases} u'' - \Delta u - \Delta u' + \alpha(t) \int_0^t g(t-s)\Delta u(s,x)ds = 0, x \in \mathbb{R}^n, t \in \mathbb{R}_*^+, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^n), u'(0,x) = u_1(x) \in L^2(\mathbb{R}^n), x \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $n \geq 2$. It is well known that the presence of a viscoelastic term with and without the weighted function α does not preclude the question of existence, but its effects are on the stability of the existing solution. For the existence, we refer the reader to works in [24], [26], [25], [31], [40], [44], [16] and [57].

The energy of u at time t is given by

$$E(t) = \frac{1}{2}\|u'\|_2^2 + \frac{1}{2}\left(1 - \alpha(t)\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}\alpha(t)(g \circ \nabla u), \tag{3.2}$$

and the following energy functional law holds.

$$
\begin{aligned}
E'(t) &= \frac{1}{2}\alpha(t)(g' \circ \nabla u) - \frac{1}{2}\alpha(t)g(t)\|\nabla u\|_2^2 - \|\nabla u'^{2}_{2} \\
&+ \frac{1}{2}\alpha'(t)(g \circ \nabla u) - \frac{1}{2}\alpha'(t)\int_0^t g(s)ds\|\nabla u\|_2^2.
\end{aligned} \tag{3.3}
$$

This type of problems is usually encountered in viscoelasticity in various areas of mathematical physics. It was first considered by Dafermos in [15], where the general decay was discussed. The problems related to (3.1) attracted a great deal of attention in the last decades and numerous results appeared on the existence and long time behavior of solutions but their results is by now rather developed, especially in any space dimension when it comes to nonlinear problems.

For the literature, in $\mathbb{R}^n$ we quote essentially the results of [10], [28], [29], [30], [32], [45]. In [29], the authors showed that, for compactly supported initial data and for an exponentially decaying relaxation function, the decay of the energy of solution of a linear Cauchy problem related of (3.1) is polynomial. The finite-speed propagation is used to compensate for the lack of Poincaré's inequality. In [28], the author looked into a linear Cauchy viscoelastic equation with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate for the lack of Poincaré's inequality. The same problem treated in [28], was considered in [30], where they consider a Cauchy problem for a viscoelastic wave equation. Under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. Conditions used, on the relaxation function g and its derivative g' are different from the usual ones.

Ikehata in [24] considered, in the one-dimensional half space, the mixed problem of the equation

$$
v_{tt} - v_{xx} + v_t = 0 \tag{3.4}
$$

with a weighted initial data and presented a new decay estimates of solutions which also can be derived for the Cauchy problem in $\mathbb{R}^n$. Let us mention that a pioneer question on the long time asymptotic of strongly damped wave equations in [25].The authors, studied the Cauchy problem for abstract dissipative equations in Hilbert spaces generalizing wave equations with strong damping terms in $\mathbb{R}^n$ or exterior domains

$$
u_{tt}(t) + Au(t) + Au'(t) = 0, \qquad t \in (0, \infty). \tag{3.5}
$$

$$u(0) = u_0, \qquad u_t(0) = u_1, \tag{3.6}$$

where $A : D(A) \subset H \to H$ is a nonnegative self-adjoint operator in $(H, \|.\|)$ with a dense domain $D(A)$. Using the energy method in the Fourier space and its generalization based on the spectral theorem for self-adjoint operators, their main result was a combination of solutions of diffusion and wave equations.

Recently, in [26], Ryo Ikehata considered the Cauchy problem in $\mathbb{R}^n$ for strongly damped wave equations (3.5) with $A = -\Delta$. He derived asymptotic profiles of its solutions with weighted $L^{1,1}(\mathbb{R}^n)$ data by using a method introduced in [24] and developed in [25]. The same author, extends his results in [23] when the initial data belongs to a weighted $L^{1,2}(\mathbb{R}^n)$ space.

3.2 Statement

In order to investigate the decay structure based on the memory and the weighted function, we also consider the following assumptions:

$g, \alpha : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are non-increasing differentiable functions of class C^1 satisfying:

$$1 - \alpha(t) \int_0^t g(s)ds \geq k > 0, \quad g(0) = g_0 > 0 \tag{3.7}$$

$$\infty > \int_0^\infty g(t)dt, \quad \alpha(t) > 0, \tag{3.8}$$

In addition, there exists a non-increasing differentiable function $\beta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

$$\beta(t) > 0, \quad g'(t) + \beta(t)g(t) \leq 0, \quad \forall t \geq 0, \quad \lim_{t\to\infty} = \frac{\alpha'(t)}{\beta(t)\alpha(t)} = 0. \tag{3.9}$$

We give the definition of weak solutions for the problem (3.1).

Defintion 3.2.1 *A weak solution of (3.1) is u such that*

- $u \in C([0,T]; H^1(\mathbb{R}^n)), \quad u'^1([0,T]; L^2(\mathbb{R}^n)),$
- *For all $v \in C_0^\infty([0,T] \times \mathbb{R}^n)$, u satisfies the generalized formulae:*

$$\begin{aligned} 0 &= \int_0^T (u'', v)ds + \int_0^T \int_{\mathbb{R}^n} \nabla u \nabla v dxds + \int_0^T \int_{\mathbb{R}^n} \nabla u' \nabla v dxds \\ &- \int_0^T \int_{\mathbb{R}^n} \alpha(t) \int_0^s g(s-\tau)\nabla u(\tau)d\tau \nabla v(s) dxds, \end{aligned} \tag{3.10}$$

- *u satisfies the initial conditions* $u_0(x) \in H^1(\mathbb{R}^n), \quad u_1(x) \in L^2(\mathbb{R}^n).$

We can now state and prove the asymptotic behavior of the solution of (3.1). Throughout this chapter, let us set $\widehat{u}(t,\xi) = F(u(t,.))(\xi)$.

3.3 Decay rate results

We show that our solution decays time asymptotically to zero and the rate of decay for the solution is fast and similar to both α *and* g.

Theorem 3.3.1 *Assume u is the solution of (3.1), then the next general exponential estimate satisfies in the Fourier space*

$$E(t) \leq W \exp\left(-\omega \int_0^t \alpha(s)\beta(s)ds\right), \qquad \forall t \geq 0, \tag{3.11}$$

for some positive constants W, ω.

We take the Fourier transform of both sides of (3.1). Then one has the reduced equation for $\xi \in \mathbb{R}^n, t \in \mathbb{R}_*^+$:

$$\begin{cases} \widehat{u}(t,\xi)'' + |\xi|^2\widehat{u}(t,\xi) + |\xi|^2\widehat{u}(t,\xi)' - |\xi|^2\alpha(t)\int_0^t g(t-s)\widehat{u}(s,\xi)ds + |\xi|^2\widehat{u}'(t,\xi) = 0, \\ \widehat{u}(0,\xi) = \widehat{u}_0(\xi) \in H^1(\mathbb{R}^n), \widehat{u}'(0,\xi) = \widehat{u}_1(\xi) \in L^2(\mathbb{R}^n). \end{cases} \tag{3.12}$$

We apply the multiplier techniques in Fourier space in order to obtain useful estimates and construct some functionals associated with the nature of our problem to introduce an appropriate Lyapunov functions.

First, to derive the equality for the physical energy, we multiply both sides of (3.12) by $\overline{\widehat{u}'}$. *Then, taking the real part of the resulting identities, we obtain*

$$E_1(t) = \frac{1}{2}|\widehat{u}'|^2 + \frac{1}{2}|\xi|^2(1-\alpha(t)\int_0^t g(s)ds)|\widehat{u}|^2 + \frac{1}{2}|\xi|^2\alpha(t)(g\circ\widehat{u})(t),$$

and

$$\begin{aligned} e_1(t) &= \frac{1}{2}|\xi|^2\left(\alpha(t)g(t)|\widehat{u}|^2-\alpha(t)(g'\circ\widehat{u})(t)+2|\widehat{u}'|^2\right) \\ &+ \frac{1}{2}|\xi|^2\left(\alpha'(t)(g\circ\widehat{u})(t)-\alpha'(t)\int\limits_0^t g(s)ds|\widehat{u}|^2\right). \end{aligned} \tag{3.13}$$

Then,

$$\frac{d}{dt}E_1(t)+e_1(t)=0. \tag{3.14}$$

Second, the existence of the memory term forces us to make the first modification of the energy by multiplying (3.12) by $\Big(-\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\Big)$ *and taking the real part, we have that*

$$\begin{aligned} &0=-Re\left\{\widehat{u}''(t,\xi)\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\} \\ &-Re\left\{|\xi|^2\widehat{u}\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\} \\ &+\frac{1}{2}|\xi|^2\frac{d}{dt}\left(\left|\alpha(t)\int_0^t g(t-s)\widehat{u}(s)ds\right|^2\right)-Re\left\{|\xi|^2\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}, \end{aligned} \tag{3.15}$$

since

$$\begin{aligned} \frac{d}{dt}\left(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right) &= \alpha'(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds+\alpha(t)\frac{d}{dt}\left(\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right) \\ &= \alpha'(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds+\alpha(t)g_0\overline{\widehat{u}}+\alpha(t)\int_0^t g'(t-s)\overline{\widehat{u}}(s). \end{aligned} \tag{3.16}$$

The first term in Eq.(3.15) takes the form

$$
\begin{aligned}
&- \quad Re\left\{\widehat{u}''(t,\xi)\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\\
&= \quad -Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}'\\
&+ \quad Re\left\{\widehat{u}'\frac{d^2}{dt^2}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\\
&= \quad -Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}' + \alpha(t)g_0|\widehat{u}'^2\\
&+ \quad Re\left\{\widehat{u}'\Big(\alpha(t)\frac{d}{dt}\Big(\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\Big) + \alpha'(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}.
\end{aligned}
$$

Denote by

$$
E_2(t) = \frac{1}{2}\left(|\xi|^2\Big|\alpha(t)\int_0^t g(t-s)\widehat{u}(s)ds\Big|^2 - Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right),
$$

and

$$
\begin{aligned}
e_2(t) \quad &= \quad \alpha(t)g_0|\widehat{u}'|^2 - Re\left\{|\xi|^2\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\\
&+ \quad Re\left\{\alpha'(t)\widehat{u}'\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right\}\\
R_2(t) \quad &= \quad -Re\left\{|\xi|^2\widehat{u}\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\\
&+ \quad Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}.
\end{aligned}
$$

Then,

$$
\frac{d}{dt}E_2(t) + e_2(t) + R_2(t) = 0. \tag{3.17}
$$

Next, to make the second modification of the energy which corresponds to the strong damping, we multiply (3.12) by $\overline{\widehat{u}}$ and taking the real part,we have

$$
\begin{aligned}
0 \quad &= \quad (Re\{\widehat{u}'\overline{\widehat{u}}\})' - |\widehat{u}'|^2 + |\xi|^2|\widehat{u}|^2\\
&- \quad |\xi|^2 Re\left\{\alpha(t)\int_0^t g(t-s)\widehat{u}(s)\overline{\widehat{u}}(t)ds\right\} + \frac{1}{2}|\xi|^2(|\widehat{u}|^2)',
\end{aligned}
$$

using results in Lemma 2.2.1, we get

$$\begin{aligned} 0 &= (Re\{\widehat{u}'\overline{\widehat{u}}\})' - |\widehat{u}'|^2 + |\xi|^2|\widehat{u}|^2 + \frac{1}{2}|\xi|^2(|\widehat{u}|^2)' \\ &- |\xi|^2\left(\alpha(t)\int_0^t g(s)ds|\widehat{u}|^2 + Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\overline{\widehat{u}}(s)ds\right\}\right). \end{aligned}$$

Denote

$$E_3(t) = Re\{\widehat{u}'\overline{\widehat{u}}\} + \frac{1}{2}|\xi|^2|\widehat{u}|^2,$$

and

$$\begin{aligned} e_3(t) &= |\xi|^2\Big(1-\alpha(t)\int_0^t g(s)ds\Big)|\widehat{u}|^2. \\ R_3(t) &= -|\widehat{u}'|^2 - Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\overline{\widehat{u}}(s)ds\right\} \end{aligned}$$

Then,

$$\frac{d}{dt}E_3(t) + e_3(t) + R_3(t) = 0. \tag{3.18}$$

Let us define for some constants ε_1 and $\varepsilon_2 > 0$ to be chosen later

$$\begin{aligned} E_4(t) &= E_1(t) + \varepsilon_1\alpha(t)E_2(t) + \varepsilon_2\alpha(t)E_3(t) \\ &= \frac{1}{2}\left\{|\widehat{u}'|^2 + |\xi|^2\Big(1-\alpha(t)\int_0^t g(s)ds\Big)|\widehat{u}|^2 + |\xi|^2\alpha(t)(g\circ\widehat{u})(t)\right\} \\ &+ \frac{\varepsilon_1\alpha(t)}{2}\left(|\xi|^2\left|\alpha(t)\int_0^t g(t-s)\widehat{u}(s)ds\right|^2 - Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right) \\ &+ \varepsilon_2\alpha(t)\left(Re\left\{\widehat{u}'\overline{\widehat{u}}\right\} + \frac{1}{2}|\xi|^2|\widehat{u}|^2\right). \end{aligned}$$

and

$$\begin{aligned}
e_4(t) &= e_1(t) + \varepsilon_1\alpha(t)e_2(t) + \varepsilon_2\alpha(t)e_3(t) \\
&= \frac{|\xi|^2}{2}\alpha(t)\left(g(t)|\widehat{u}|^2 - (g'\circ\widehat{u})(t) + 2\alpha^{-1}(t)|\widehat{u}'^2\right) \\
&+ \frac{|\xi|^2}{2}\alpha'(t)\left((g\circ\widehat{u})(t) - \widehat{u}'\int_0^t g(s)ds|\widehat{u}|^2\right) \\
&+ \varepsilon_1\alpha(t)\left(\alpha(t)g_0|\widehat{u}'^2 - Re\left\{|\xi|^2\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right) \\
&+ \varepsilon_1\alpha(t)\left(Re\left\{\alpha'(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right\}\right) \\
&+ \varepsilon_2|\xi|^2\alpha(t)\left(1-\alpha(t)\int_0^t g(s)ds\right)|\widehat{u}|^2.
\end{aligned}$$

and

$$\begin{aligned}
R_4(t) &= \varepsilon_1\alpha(t)R_2(t) + \varepsilon_2\alpha(t)R_3(t) \\
&= \varepsilon_1\alpha(t)\left(-Re\left\{|\xi|^2\widehat{u}\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right) \\
&+ \varepsilon_1\alpha(t)\left(Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right) \\
&+ \varepsilon_2\alpha(t)\left(-|\widehat{u}'|^2 - Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\overline{\widehat{u}}(s)ds\right\}\right).
\end{aligned}$$

At this point, we introduce the Lyapunov functions as

$$L_1(t) = \left\{|\widehat{u}'|^2 + k|\xi|^2|\widehat{u}|^2 + |\xi|^2\alpha(t)(g\circ\widehat{u})(t)\right\}, \tag{3.19}$$

and

$$L_2(t) = \alpha(t)g(t)|\widehat{u}|^2 + \alpha(t)\beta(t)(g\circ\widehat{u})(t). \tag{3.20}$$

It is easy to verify that there exist positive constants $c_1(g_0)$ and $c_2(g_0)$ such that

$$c_1L_1(t) \le E_1(t) \le c_2L_1(t), \forall t > 0. \tag{3.21}$$

Thanks to Holder's and Young's inequalities, one gets for some constant c_3

$$|\varepsilon_1E_2(t) + \varepsilon_2E_3(t)| \le c_3L_1(t),$$

which means that $L_1(t) \sim E(t)$. Using again (3.9), Holder's and Young's inequalities and assumptions on g we obtain

$$\begin{aligned}|R_4(t)| &= \varepsilon_1\alpha(t)R_2(t)+\varepsilon_2\alpha(t)R_3(t)\\ &\leq \varepsilon_1\alpha(t)Re\left\{|\xi|^2\widehat{u}\alpha(t)\frac{d}{dt}\Big(\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\\ &+ \varepsilon_1\alpha(t)Re\left\{\widehat{u}'\alpha(t)\frac{d}{dt}\Big(\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\\ &+ \varepsilon_2\alpha(t)\left(|\widehat{u}'^2+Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\overline{\widehat{u}}(s)ds\right\}\right)\\ &\leq \varepsilon_1\alpha(t)|\widehat{u}'|^2+c_4\varepsilon_1\alpha(t)|\xi|^2|\widehat{u}|^2+c_5\varepsilon_1|\xi|^2L_2(t)\\ &+ \varepsilon_2\alpha(t)\left[|\widehat{u}'|^2+c_6|\xi|^2\left(\lambda|\widehat{u}|^2+c_\lambda\alpha(t)(g\circ\widehat{u})(t)\right)\right]\\ &\leq (\varepsilon_1+\varepsilon_2)\alpha(t)|\widehat{u}'|^2+(c_4\varepsilon_1+\varepsilon_2c_6\lambda)\alpha(t)|\xi|^2|\widehat{u}|^2+(c_5\varepsilon_1+c_\lambda\varepsilon_2)|\xi|^2L_2(t).\end{aligned}$$

Since $L_2(t)\leq c_3\, e_1(t)$, one can easily check that there exist positive constants $\varepsilon_1,\varepsilon_2,\lambda,c_4,c_5,c_6$ such that

$$|R_4(t)|\leq ce_4(t), c>0. \tag{3.22}$$

By (3.14), (3.17) and (3.18), we get

$$\frac{d}{dt}E_4(t) = \frac{d}{dt}E_1(t)+\varepsilon_1\alpha(t)\frac{d}{dt}E_2(t)+\varepsilon_2\alpha(t)\frac{d}{dt}E_3(t)+\varepsilon_1\alpha'(t)E_2(t)+\varepsilon_2\alpha'(t)E_3(t).$$

We use $\lim\limits_{t\to\infty}\frac{\alpha'(t)}{\alpha(t)}=0$ by (5.11)-(3.9) to choose $t_1>t_0$ and since $e_4(t)\geq cE_4(t)$, then (3.22) gives for some positive constant N

$$\frac{d}{dt}E_4(t)\leq -N\alpha(t)E_4(t)+c\alpha(t)(g\circ\widehat{u})(t). \tag{3.23}$$

Multiplying (3.23) by $\beta(t)$ and using (3.9), (3.20), we obtain

$$\begin{aligned}\beta(t)\frac{d}{dt}E_4(t) &\leq -N\beta(t)\alpha(t)E_4(t)+c\beta(t)\alpha(t)(g\circ\widehat{u})(t)\\ &\leq -N\beta(t)\alpha(t)E_4(t)-c\alpha(t)(g'\circ\widehat{u})(t)\\ &\leq -N\beta(t)\alpha(t)E_4(t)-c|\xi|^2\alpha'(t)\int_0^t g(s)ds|\widehat{u}|^2-2c\frac{d}{dt}E_4(t), \quad \forall t>t_1.\end{aligned} \tag{3.24}$$

Since $\beta'(t) \leq 0$, we set $L(s) = (\beta(s) + 2c)E_4(s)$ which is equivalent to $E_4(t)$, then

$$\begin{aligned}\frac{d}{dt}L(t) &\leq -N\beta(t)\alpha(t)E_4(t) - c|\xi|^2\alpha'(t)\int_0^t g(s)ds|\widehat{u}|^2 \\ &\leq -\beta(t)\alpha(t)\left[N - \frac{2\alpha'(t)}{k\beta(t)\alpha(t)}\int_0^t g(s)ds\right]E_4(t), \quad \forall t > t_1. \end{aligned} \tag{3.25}$$

By (3.9), we can choose $t_2 > t_1$ such that

$$\begin{aligned}\frac{d}{dt}L(t) &\leq -c\beta(t)\alpha(t)E_4(t) \\ &\leq -c\beta(t)\alpha(t)L(t), \quad \forall t > t_2. \end{aligned} \tag{3.26}$$

Integrating (3.26) over $[t_2, t]$ using equivalence between Lyapunov function and the energy function, it yields that

$$E(t) \leq W\exp(-\omega\int_0^t \alpha(s)\beta(s)ds), W, \omega > 0.$$

Chapter 4

General decay of solution for a coupled system of viscoelstic wave equations with density in $\mathbb{R}^n$

4.1 Introduction

In this chapter, we give the general decay of solution for a coupled system of viscoelstic wave equations with density in $\mathbb{R}^n$. We consider the following system

$$\begin{cases} \left(|u'^{q-2}u'\right)' - \phi(x)\left(\Delta_x u - \int_0^t g_1(t-s)\Delta_x u(s)ds\right) + \alpha v = 0, \\ \left(|v'^{q-2}v'\right)' - \phi(x)\left(\Delta_x v - \int_0^t g_2(t-s)\Delta_x v(s)ds\right) + \alpha u = 0, \end{cases} \tag{4.1}$$

where $x \in \mathbb{R}^n, \alpha \neq 0, t > 0, q, n \geq 2$ and the scalar functions $g_i(s), i = 1, 2$ (so-called relaxation kernel) are assumed to satisfy (A1)given below.

Problem (4.1) is equipped with the following initial data.

$$u\,(0,x) = u_0(x) \in \mathcal{H}(\mathbb{R}^n), \quad u'\,(0,x) = u_1(x) \in L^q_\rho(\mathbb{R}^n), \tag{4.2}$$

$$v\,(0,x) = v_0(x) \in \mathcal{H}(\mathbb{R}^n), \quad v'\,(0,x) = v_1(x) \in L^q_\rho(\mathbb{R}^n), \tag{4.3}$$

where the weighted space $\mathcal{H}$ *is given in Definition 4.2.1 below and a density function* $(\phi(x))^{-1} = \rho(x)$ *satisfies*

$$\rho : \mathbb{R}^n \to \mathbb{R}^*_+, \quad \rho(x) \in C^{0,\tilde{\gamma}}(\mathbb{R}^n) \tag{4.4}$$

with $\tilde{\gamma} \in (0,1)$ *and* $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, *where* $s = \frac{2n}{2n-qn+2q}$.

There are many results about the existence by the standard Galerkin method, (see [36], [28], [43], [45], [52], [57]). It is well known that, for any initial data $u_0, v_0 \in \mathcal{H}(\mathbb{R}^n)$ *and* $u_1, v_1 \in L^q_\rho(\mathbb{R}^n)$, *the problem (4.1)-(4.3) has a unique weak solution, under hypotheses* $(A1) - (A2)$*given below. Problem (4.1) is usually encountered in viscoelasticity in various areas of mathematical physics. It was first considered by Dafermos in [15], where the general decay was discussed. The problems related to (4.1) attracted a great deal of attention in the last decades and numerous results appeared on the existence and long time behavior of solutions but their results are by now rather developed, especially in any space dimension.*

The work with weighted spaces was studied by many authors (see in this direction [10], [31], [48] and [55]). For the decay rate of solution for equations in $\mathbb{R}^n$, *we quote essentially the results of [2], [28], [29], [30], [45]. In [29], the authors showed that, for compactly supported initial data and for an exponentially decaying relaxation function, the decay of the energy of solution of a linear Cauchy problem* $(4.1), (4.3)$ *in one equation with* $\alpha = 0, q = 2, \rho(x) = 1$ *is polynomial. The finite-speed propagation is used to compensate for the lack of Poincare's inequality. In the case* $\alpha = 0, q = 2$, *in [28]. The author looked into a linear Cauchy viscoelastic problem with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate for the lack of Poincare's inequality. The same problem treated in [28], was considered in [30], where they considered a Cauchy problem for a viscoelastic wave equation. Under suitable conditions on the initial data and the relaxation function, they proved a polynomial decay result of solutions. Conditions used on the relaxation function g and its derivative* g' *are different from the usual ones.*

The problem (4.1)-(4.3) in the case $\alpha = 0, x \in \mathbb{R}^n$ *with relaxation function g is a positive nonincreasing function was considered as one equation for Kirchhoff type in [57], where the author established a general decay rate result for relaxation functions satisfying assumption* $(A1) - (A2)$ *given below. The main purpose of the present paper is to extend this result for a coupled system of* *linear equations.*

4.2 Preliminaries and assumptions

First, we recall and make use of the following assumption on the functions $g_i, i = 1, 2$ *as:*

(A1) We assume that the functions $g_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *are of class* C^1 *satisfying:*

$$1 - \overline{g_i} = l_i > 0, \quad g_i(0) = g_{0i} > 0, \tag{4.5}$$

where $\overline{g_i} = \int_0^\infty g_i(t)dt$.

(A2) There exists a positive function $H \in C^1(\mathbb{R}^+)$ *such that*

$$g_i'(t) + H(g_i(t)) \leq 0, t \geq 0, \quad H(0) = 0 \tag{4.6}$$

and H is linear or strictly increasing and strictly convex C^2 *function on* $(0, r]$, $1 > r$.

Remark 4.2.1 *i- We can deduce that there exists* $t_1 > 0$ *large enough such that for* $i = 1, 2$:

<u>*1)* $\forall t \geq t_1$:</u> *we have* $\lim_{s \to +\infty} g_i(s) = 0$, *which implies that* $\lim_{s \to +\infty} \Big(- g_i'(s) \Big)$ *cannot be positive, so* $\lim_{s \to +\infty} \Big(- g_i'(s) \Big) = 0$. *Then* $g_i(t_1) > 0$ *and*

$$\max\{g_1(s), g_2(s), -g_1'(s), -g_2'(s)\} < \min\{r, H(r), H_0(r)\}, \tag{4.7}$$

where $H_0(t) = H(D(t))$ *provided that D is a positive* C^1 *function, with* $D(0) = 0$, *for which* H_0 *is strictly increasing and strictly convex* C^2 *function on* $(0, r]$ *and*

$$\int_0^{+\infty} g_i(s) H_0(-g_i'(s)) ds < +\infty.$$

<u>*2)* $\forall t \in [0, t_1]$:</u> *As* g_i *are nonincreasing,* $g_i(0) > 0$ *and* $g_i(t_1) > 0$ *then* $g_i(t) > 0$ *and*

$$g_i(0) \geq g_i(t) \geq g_i(t_1) > 0.$$

Therefore, since H is a positive continuous function, then

$$a' \leq H(g_1(t)) \leq b'$$

$$c' \leq H(g_2(t)) \leq d'$$

for some positive constants a', b', c' *and* d'. *Consequently,*

$$g_i'(t) \leq -H(g_i(t)) \leq -kg_i(t), \quad k > 0$$

which gives

$$g_i'(t) + kg_i(t) \leq 0, k > 0. \tag{4.8}$$

ii- Let H_0^* *be the convex conjugate of* H_0 *in the sense of Young (see [4], pages 61-64), then*

$$H_0^*(s) = s(H_0'^{-1}(s) - H_0[(H_0'^{-1}(s)], \quad s \in (0, H_0'(r)),$$

and satisfies the following Young's inequality

$$AB \leq H_0^*(A) + H_0(B), \quad A \in (0, H_0'(r)), B \in (0, r]. \tag{4.9}$$

Defintion 4.2.1 ([28], [48]) *We define the function spaces of our problem and their norm as follows:*

$$\mathcal{H}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla_x f \in (L^2(\mathbb{R}^n))^n \right\}, \tag{4.10}$$

and the space $L^2_\rho(\mathbb{R}^n)$ *to be the closure of* $C_0^\infty(\mathbb{R}^n)$ *functions with respect to the inner product*

$$(f, h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx.$$

For $1 < p < \infty$, *if* f *is a measurable function on* $\mathbb{R}^n$, *we define*

$$\|f\|_{L^p_\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \rho |f|^q dx \right)^{1/p}. \tag{4.11}$$

The space $L^2_\rho(\mathbb{R}^n)$ *is a separable Hilbert space.*

The following technical Lemma will play an important role in the sequel.

Lemma 4.2.1 *(Lemma 1.1 of [11]) For any two functions* $h, w \in C^1(\mathbb{R})$ *and* $\theta \in [0, 1]$ *we have*

$$\begin{aligned} w'(t) \int_0^t h(t-s) w(s) ds &= -\frac{1}{2} \frac{d}{dt} \int_0^t h(t-s) |w(t) - w(s)|^2 ds + \frac{1}{2} \frac{d}{dt} \left(\int_0^t h(s) ds \right) |w(t)|^2 \\ &\quad + \frac{1}{2} \int_0^t h'^2 ds - \frac{1}{2} h(t) |w(t)|^2, \end{aligned} \tag{4.12}$$

and

$$\left|\int_0^t h(t-s)(w(t)-w(s))ds\right|^2$$
$$\leq \left(\int_0^t |h(s)|^{2(1-\theta)}ds\right)\left(\int_0^t |h(t-s)|^{2\theta}|w(t)-w(s)|^2 ds\right). \qquad (4.13)$$

The energy of (u,v) at time t is defined by

$$\begin{aligned} E(t) &= \frac{(q-1)}{q}\left[\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \|v'\|^q_{L^q_\rho(\mathbb{R}^n)}\right] + \alpha\int_{\mathbb{R}^n}\rho uv dx \\ &+ \frac{1}{2}\Big(1-\int_0^t g_1(s)ds\Big)\|\nabla_x u\|_2^2 + \frac{1}{2}\Big(1-\int_0^t g_2(s)ds\Big)\|\nabla_x v\|_2^2. \\ &+ \frac{1}{2}(g_1\circ\nabla_x u) + \frac{1}{2}(g_2\circ\nabla_x v) \end{aligned} \qquad (4.14)$$

For α small enough we use Lemma **??** *to deduce, for $c>0$, that:*

$$\begin{aligned} E(t) &\geq (1-c|\alpha|\|\rho\|^{-1}_{L^{n/2}})\frac{(q-1)}{q}\left[\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \|v'\|^q_{L^q_\rho(\mathbb{R}^n)}\right] \\ &+ \frac{1}{2}\Big(1-\int_0^t g_1(s)ds\Big)\|\nabla_x u\|_2^2 + \frac{1}{2}\Big(1-\int_0^t g_2(s)ds\Big)\|\nabla_x v\|_2^2 \\ &+ \frac{1}{2}(g_1\circ\nabla_x u) + \frac{1}{2}(g_2\circ\nabla_x v) \end{aligned} \qquad (4.15)$$

and the following energy functional law holds:

$$E'(t) \leq \frac{1}{2}(g_1'\circ\nabla_x u)(t) + \frac{1}{2}(g_2'\circ\nabla_x v)(t), \quad \textit{for all} \quad t\geq 0. \qquad (4.16)$$

which means that, our energy is uniformly bounded and decreasing along the trajectories.

The following notation will be used throughout this paper

$$(g_i\circ\nabla_x\psi)(t) = \int_0^t g_i(t-\tau)\|\nabla_x\psi(t)-\nabla_x\psi(\tau)\|_2^2 d\tau, i=1,2, \qquad (4.17)$$

for $\psi(t)\in\mathcal{H}(\mathbb{R}^n), t\geq 0$.

We are now ready to state and prove our main results.

4.3 Decay of solution to system of nonlinear wave equations with degenerate damping

The next Lemma can be easily shown (see [31], Lemma 2.1).

Lemma 4.3.1 *Let ρ satisfy (4.4), then for any $u \in \mathcal{H}(\mathbb{R}^n)$*

$$\|u\|_{L^p_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^s(\mathbb{R}^n)} \|\nabla_x u\|_{L^2(\mathbb{R}^n)}, \tag{4.18}$$

with $s = \frac{2n}{2n-pn+2p}, 2 \leq p \leq \frac{2n}{n-2}$

Our main result reads as follows.

Theorem 4.3.1 *Let $(u_0, v_0) \in (\mathcal{H}(\mathbb{R}^n))^2, (u_1, v_1) \in (L^q_\rho(\mathbb{R}^n))^2$ and suppose that $(A1) - (A2)$ hold. Then there exist positive constants a, b, c, d such that the energy of solution of problem (4.1)-(4.3) satisfies,*

$$E(t) \leq d H_1^{-1}(bt + c), \quad \textit{for all} \quad t \geq 0,$$

where

$$H_1(t) = \int_t^1 \frac{1}{s H_0'(as)} ds. \tag{4.19}$$

To prove Theorem 4.3.1, let us define

$$L(t) = \xi_1 E(t) + \psi_1(t) + \xi_2 \psi_2(t), \tag{4.20}$$

for $\xi_1, \xi_2 > 1$. In order to obtain useful estimates, we construct some functionals associated with the nature of our problem introduced in Lyapunov function L as

$$\psi_1(t) = \int_{\mathbb{R}^n} \rho(x) \left[u|u'^{q-2}u' + v|v'^{q-2}v' \right] dx, \tag{4.21}$$

and the existence of the memory terms forces us to introduce the next functional

$$\begin{aligned} \psi_2(t) &= -\int_{\mathbb{R}^n} \rho(x)|u'^{q-2}u' \int_0^t g_1(t-s)(u(t)-u(s)) ds dx \\ &- \int_{\mathbb{R}^n} \rho(x)|v'^{q-2}v' \int_0^t g_2(t-s)(v(t)-v(s)) ds dx. \end{aligned} \tag{4.22}$$

Lemma 4.3.2 *Under the assumptions (A1) and (A2), the functional ψ_1 satisfies*

$$\begin{aligned} \psi_1'(t) &\leq (1 - c|\alpha| \|\rho\|^{-1}_{L^{n/2}}) \left[\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \|v'\|^q_{L^q_\rho(\mathbb{R}^n)} \right] \\ &+ (\sigma - l) \left[\|\nabla_x u\|_2^2 + \|\nabla_x v\|_2^2 \right] + \frac{(1-l)}{4\sigma} \left[(g_1 \circ \nabla_x u) + (g_2 \circ \nabla_x v) \right], \end{aligned}$$

where $l = \min\{l_1, l_2\}$, along the solution of (4.1)-(4.3) .

From (4.21), integrating over $\mathbb{R}^n$*, we have*

$$\begin{aligned}
\psi_1'(t) &= \int_{\mathbb{R}^n} \rho(x)|u'|^q dx + \int_{\mathbb{R}^n} \rho(x)u \left(|u'|^{q-2}u'\right)' dx \\
&+ \int_{\mathbb{R}^n} \rho(x)|v'|^q dx + \int_{\mathbb{R}^n} \rho(x)v \left(|v'|^{q-2}v'\right)' dx \\
&= \int_{\mathbb{R}^n} \left(\rho(x)|u'|^q + u\Delta_x u - u\int_0^t g_1(t-s)\Delta_x u(s,x)ds - \alpha\rho(x)uv\right) dx \\
&+ \int_{\mathbb{R}^n} \left(\rho(x)|v'|^q + v\Delta_x v - v\int_0^t g_2(t-s)\Delta_x v(s,x)ds - \alpha\rho(x)uv\right) dx \\
&= \int_{\mathbb{R}^n} \left(\rho(x)|u'|^q - \nabla_x u\nabla_x u + \nabla_x u\int_0^t g_1(t-s)\nabla_x u(s,x)ds - \alpha\rho(x)uv\right) dx \\
&+ \int_{\mathbb{R}^n} \left(\rho(x)|v'|^q - \nabla_x v\nabla_x v + \nabla_x v\int_0^t g_2(t-s)\nabla_x v(s,x)ds - \alpha\rho(x)uv\right) dx \\
&= \int_{\mathbb{R}^n} \left(\rho(x)|u'|^q - (\nabla_x u)^2 + \nabla_x u\int_0^t g_1(t-s)(\nabla_x u(s) - \nabla_x u(t))ds\right) dx \\
&+ \int_{\mathbb{R}^n} \left(\rho(x)|v'|^q - (\nabla_x v)^2 + \nabla_x v\int_0^t g_2(t-s)(\nabla_x v(s) - \nabla_x v(t))ds\right) dx \\
&+ \int_{\mathbb{R}^n} (\nabla_x u)^2 \int_0^t g_1(s)dsdx + \int_{\mathbb{R}^n} (\nabla_x v)^2 \int_0^t g_2(s)dsdx - 2\alpha\int_{\mathbb{R}^n} \rho(x)uvdx.
\end{aligned}$$

Using Young's inequality and Lemma 4.2.1 for $\theta = 1/2$*, we obtain*

for small enough positive constant σ

$$\begin{aligned}
&\nabla_x u\int_0^t g_1(t-s)(\nabla_x u(s) - \nabla_x u(t))dsdx \\
&\leq \sigma\|\nabla_x u\|_2^2 + \frac{1}{4\sigma}\int_{\mathbb{R}^n}\left(\int_0^t g_1(t-s)|\nabla_x u(s) - \nabla_x u(t)|ds\right)^2 dx \\
&\leq \sigma\|\nabla_x u\|_2^2 + \frac{1-l_1}{4\sigma}(g_1 \circ \nabla_x u)(t),
\end{aligned}$$

and

$$\begin{aligned}
&\nabla_x v\int_0^t g_2(t-s)(\nabla_x v(s) - \nabla_x v(t))dsdx \\
&\leq \sigma\|\nabla_x v\|_2^2 + \frac{1}{4\sigma}\int_{\mathbb{R}^n}\left(\int_0^t g_2(t-s)|\nabla_x v(s) - \nabla_x v(t)|ds\right)^2 dx \\
&\leq \sigma\|\nabla_x v\|_2^2 + \frac{1-l_2}{4\sigma}(g_2 \circ \nabla_x v)(t).
\end{aligned}$$

By (A1) and the fact that $\int_0^t g(s)ds < \int_0^\infty g(s)ds$,

$$\begin{aligned}\psi_1'(t) &\leq \|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \|v'\|^q_{L^q_\rho(\mathbb{R}^n)} + (\sigma - l_1)\|\nabla_x u\|_2^2 + (\sigma - l_2)\|\nabla_x v\|_2^2 - 2\alpha\int_{\mathbb{R}^n}\rho(x)uvdx \\ &+ \frac{1-l_1}{4\sigma}(g_1 \circ \nabla_x u) + \frac{1-l_2}{4\sigma}(g_2 \circ \nabla_x v).\end{aligned}$$

Using Holder's and Young's inequalities and Lemma 4.3.1 we get

$$\begin{aligned}\int_{\mathbb{R}^n}|\rho(x)uv|dx &= \int_{\mathbb{R}^n}|(\rho(x)^{1/2}u)(\rho(x)^{1/2}v)|dx \\ &\leq \left(\int_{\mathbb{R}^n}\rho(x)|u|^2dx\right)^{1/2}\left(\int_{\mathbb{R}^n}\rho(x)|v|^2dx\right)^{1/2} \\ &\leq \sigma\int_{\mathbb{R}^n}\rho(x)|u|^2dx + \frac{1}{4\sigma}\int_{\mathbb{R}^n}\rho(x)|v|^2dx \\ &\leq \|\rho\|_{L^{n/2}}\left[\sigma\int_{\mathbb{R}^n}|\nabla_x u|^2dx + \frac{1}{4\sigma}\int_{\mathbb{R}^n}|\nabla_x v|^2dx\right].\end{aligned}$$

Then, we obtain for $l = \min\{l_1, l_2\}, c > 0$,

$$\begin{aligned}\psi_1'(t) &\leq (1 - c|\alpha|\|\rho\|^{-1}_{L^{n/2}})\left[\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \|v'\|^q_{L^q_\rho(\mathbb{R}^n)}\right] \\ &+ (\sigma - l)\left(\|\nabla_x u\|_2^2 + \|\nabla_x v\|_2^2\right) + \frac{(1-l)}{4\sigma}\left((g_1 \circ \nabla_x u) + (g_2 \circ \nabla_x v)\right).\end{aligned}$$

Lemma 4.3.3 *Under the assumptions (A1) and (A2), the functional* ψ_2 *satisfies*

$$\begin{aligned}\psi_2'(t) &\leq \sigma\left(1 + \alpha\|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}\right)\left[\|\nabla_x u\|_2^2 + \|\nabla_x v\|_2^2\right] \\ &+ c_\sigma\left(1 + \alpha\|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}\right)\left[(g_1 \circ \nabla_x u) + (g_2 \circ \nabla_x v)\right] \\ &- c_\sigma\|\rho\|^q_{L^s(\mathbb{R}^n)}\left[(g_1' \circ \nabla_x u)^{q/2} + (g_2' \circ \nabla_x v)^{q/2}\right] \\ &+ \left(\sigma - \int_0^t g(s)ds\right)\left[\|u'\|^{q/2}_{L^q_\rho(\mathbb{R}^n)} + \|v'\|^{q/2}_{L^q_\rho(\mathbb{R}^n)}\right]\end{aligned}$$

where

$$\int_0^t g(s)ds \leq \min\left\{\int_0^t g_1(s)ds, \int_0^t g_2(s)ds\right\}. \tag{4.23}$$

along the solution of (4.1)-(4.3), for any $\sigma \in (0,1)$.

Exploiting Eqs. (4.1) we get

$$
\begin{aligned}
\psi_2'(t) &= -\int_{\mathbb{R}^n} \rho(x) \left(|u'|^{q-2}u'\right)' \int_0^t g_1(t-s)(u(t)-u(s))dsdx \\
&- \int_{\mathbb{R}^n} \rho(x)|u'|^{q-2}u' \int_0^t g_1'(t-s)(u(t)-u(s))dsdx - \int_0^t g_1(s)ds\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} \\
&- \int_{\mathbb{R}^n} \rho(x) \left(|v'|^{q-2}v'\right)' \int_0^t g_2(t-s)(v(t)-v(s))dsdx \\
&- \int_{\mathbb{R}^n} \rho(x)|v'|^{q-2}v' \int_0^t g_2'(t-s)(v(t)-v(s))dsdx - \int_0^t g_2(s)ds\|v'\|^q_{L^q_\rho(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} \nabla_x u \int_0^t g_1(t-s)(\nabla_x u(t) - \nabla_x u(s))dsdx \\
&- \int_{\mathbb{R}^n} \left(\int_0^t g_1(t-s)\nabla_x u(s,x)ds\right)\left(\int_0^t g_1(t-s)(\nabla_x u(t) - \nabla_x u(s))ds\right)dx \\
&- \int_{\mathbb{R}^n} \rho(x)|u'|^{q-2}u' \int_0^t g_1'(t-s)(u(t)-u(s))dsdx \\
&- \int_0^t g_1(s)ds\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \alpha\int_{\mathbb{R}^n} \rho(x)v \int_0^t g_1(t-s)(u(t)-u(s))dsdx \\
&+ \int_{\mathbb{R}^n} \nabla_x v \int_0^t g_2(t-s)(\nabla_x v(t) - \nabla_x v(s))dsdx \\
&- \int_{\mathbb{R}^n} \left(\int_0^t g_2(t-s)\nabla_x v(s,x)ds\right)\left(\int_0^t g_2(t-s)(\nabla_x v(t) - \nabla_x v(s))ds\right)dx \\
&- \int_{\mathbb{R}^n} \rho(x)|v'|^{q-2}v' \int_0^t g_2'(t-s)(v(t)-v(s))dsdx \\
&- \int_0^t g_2(s)ds\|v'\|^q_{L^q_\rho(\mathbb{R}^n)} + \alpha\int_{\mathbb{R}^n} \rho(x)u \int_0^t g_2(t-s)(v(t)-v(s))dsdx,
\end{aligned}
$$

then

$$
\begin{aligned}
\psi_2'(t) &= \left(1-\int_0^t g_1(s)ds\right)\int_{\mathbb{R}^n}\nabla_x u\int_0^t g_1(t-s)(\nabla_x u(t)-\nabla_x u(s))dsdx \\
&+ \int_{\mathbb{R}^n}\left(\int_0^t g_1(t-s)(\nabla_x u(t)-\nabla_x u(s))ds\right)^2 dx \\
&- \int_{\mathbb{R}^n}\rho(x)|u'^{q-2}u'\int_0^t g_1'(t-s)(u(t)-u(s))dsdx \\
&- \int_0^t g_1(s)ds\|u'\|^q_{L^q_\rho(\mathbb{R}^n)}+c(g_1\circ\nabla_x u)(t) \\
&+ \left(1-\int_0^t g_2(s)ds\right)\int_{\mathbb{R}^n}\nabla_x v\int_0^t g_2(t-s)(\nabla_x v(t)-\nabla_x v(s))dsdx \\
&+ \int_{\mathbb{R}^n}\left(\int_0^t g_2(t-s)(\nabla_x v(t)-\nabla_x v(s))ds\right)^2 dx \\
&- \int_{\mathbb{R}^n}\rho(x)|v'^{q-2}v'\int_0^t g_2'(t-s)(v(t)-v(s))dsdx \\
&- \int_0^t g_2(s)ds\|v'\|^q_{L^q_\rho(\mathbb{R}^n)}+c(g_2\circ\nabla_x v)(t) \\
&+ \alpha\int_{\mathbb{R}^n}\rho(x)\left(v\int_0^t g_1(t-s)(u(t)-u(s))ds+u\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx.
\end{aligned}
$$

By Holder's and Young's inequalities and Lemma 4.3.1, we estimate the last term as

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\rho(x)v\int_0^t g_1(t-s)(u(t)-u(s))dsdx \\
&\le \left(\int_{\mathbb{R}^n}\rho(x)|v|^2dx\right)^{1/2}\times \\
&\left(\int_{\mathbb{R}^n}\rho(x)\left|\int_0^t g_1(t-s)(u(t)-u(s))ds\right|^2\right)^{1/2} \\
&\le \sigma\|v\|^2_{L^2_\rho(\mathbb{R}^n)}+c_\sigma\left\|\int_0^t g_1(t-s)(u(t)-u(s))ds\right\|^2_{L^2_\rho(\mathbb{R}^n)} \\
&\le \sigma\|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}\|\nabla_x v\|_2^2+c_\sigma\|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}(g_1\circ\nabla_x u)(t).
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \rho(x) u \int_0^t g_2(t-s)(v(t)-v(s))dsdx \\
&\leq \left(\int_{\mathbb{R}^n} \rho(x)|u|^2 dx\right)^{1/2} \times \\
&\left(\int_{\mathbb{R}^n} \rho(x)\Big| \int_0^t g_2(t-s)(v(t)-v(s))ds\Big|^2\right)^{1/2} \\
&\leq \sigma \|u\|^2_{L^2_\rho(\mathbb{R}^n)} + c_\sigma \Big\| \int_0^t g_2(t-s)(v(t)-v(s))ds\Big\|^2_{L^2_\rho(\mathbb{R}^n)} \\
&\leq \sigma \|\rho\|^2_{L^{n/2}(\mathbb{R}^n)} \|\nabla_x u\|_2^2 + c_\sigma \|\rho\|^2_{L^{n/2}(\mathbb{R}^n)} (g_2 \circ \nabla_x v)(t),
\end{aligned}
$$

and for the exponents $\frac{q}{q-1}, q$

$$
\begin{aligned}
&-\int_{\mathbb{R}^n} \rho(x)|u'^{q-2}u' \int_0^t g_1'(t-s)(u(t)-u(s))dsdx \\
&\leq \left(\int_{\mathbb{R}^n} \rho(x)|u'^{q} dx\right)^{(q-1)/q} \times \\
&\left(\int_{\mathbb{R}^n} \rho(x)\Big| \int_0^t g_1'(t-s)(u(t)-u(s))ds\Big|^q\right)^{1/q} \\
&\leq \sigma \|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + c_\sigma \Big\| \int_0^t -g_1'(t-s)(u(t)-u(s))ds\Big\|^q_{L^q_\rho(\mathbb{R}^n)} \\
&\leq \sigma \|u'\|^q_{L^q_\rho(\mathbb{R}^n)} - c_\sigma \|\rho\|^q_{L^s(\mathbb{R}^n)} (g_1' \circ \nabla_x u)^{q/2}(t),
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_{\mathbb{R}^n} \rho(x)|v'^{q-2}v' \int_0^t g_2'(t-s)(v(t)-v(s))dsdx \\
&\leq \left(\int_{\mathbb{R}^n} \rho(x)|v'^{q} dx\right)^{(q-1)/q} \times \\
&\left(\int_{\mathbb{R}^n} \rho(x)\Big| \int_0^t g_2'(t-s)(v(t)-v(s))ds\Big|^q\right)^{1/q} \\
&\leq \sigma \|v'\|^q_{L^q_\rho(\mathbb{R}^n)} + c_\sigma \Big\| \int_0^t -g_2'(t-s)(v(t)-v(s))ds\Big\|^q_{L^q_\rho(\mathbb{R}^n)} \\
&\leq \sigma \|v'\|^q_{L^q_\rho(\mathbb{R}^n)} - c_\sigma \|\rho\|^q_{L^s(\mathbb{R}^n)} (g_2' \circ \nabla_x v)^{q/2}(t).
\end{aligned}
$$

Using Young's and Poincare's inequalities and Lemma 4.2.1 for $\theta = 1/2$, we obtain

$$\begin{aligned}
\psi_2'(t) &\leq \sigma\left(1+\alpha\|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}\right)\left(\|\nabla_x u\|_2^2+\|\nabla_x v\|_2^2\right)\\
&+ c_\sigma\left(1+\alpha\|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}\right)\left((g_1\circ\nabla_x u)+(g_2\circ\nabla_x v)\right)\\
&- c_\sigma\|\rho\|^q_{L^s(\mathbb{R}^n)}\left((g_1'\circ\nabla_x u)^{q/2}+(g_2'\circ\nabla_x v)^{q/2}\right)\\
&+ \left(\sigma-\int_0^t g_1(s)ds\right)\|u'\|^{q/2}_{L^q_\rho(\mathbb{R}^n)}+\left(\sigma-\int_0^t g_2(s)ds\right)\|v'\|^{q/2}_{L^q_\rho(\mathbb{R}^n)}.
\end{aligned}$$

We need the next Lemma, which means that there is equivalence between the Lyapunov and energy functions, that is for $\xi_1, \xi_2 > 1$, we have

$$\beta_1 L(t) \leq E(t) \leq \beta_2 L(t), \tag{4.24}$$

which holds for two positive constants β_1 and β_2.

Lemma 4.3.4 *For $\xi_1, \xi_2 > 1$, we have*

$$L(t) \sim E(t). \tag{4.25}$$

By (4.20) we have

$$\begin{aligned}
|L(t)-\xi_1 E(t)| &\leq |\psi_1(t)|+\xi_2|\psi_2(t)|\\
&\leq \int_{\mathbb{R}^n}\left|\rho(x)u|u'^{q-2}u'\right|dx+\int_{\mathbb{R}^n}\left|\rho(x)v|v'^{q-2}v'\right|dx\\
&+ \xi_2\int_{\mathbb{R}^n}\left|\rho(x)|u'^{q-2}u'\int_0^t g_1(t-s)(u(t)-u(s))ds\right|dx\\
&+ \xi_2\int_{\mathbb{R}^n}\left|\rho(x)|v'^{q-2}v'\int_0^t g_2(t-s)(v(t)-v(s))ds\right|dx.
\end{aligned}$$

Thanks to Holder's and Young's inequalities with exponents $\frac{q}{q-1}, q$, since $q \geq 2$, we have by using Lemma 4.3.1

$$\begin{aligned}
\int_{\mathbb{R}^n}\left|\rho(x)u|u'^{q-2}u'\right|dx &\leq \left(\int_{\mathbb{R}^n}\rho(x)|u|^q dx\right)^{1/q}\left(\int_{\mathbb{R}^n}\rho(x)|u'^q dx\right)^{(q-1)/q}\\
&\leq \frac{1}{q}\left(\int_{\mathbb{R}^n}\rho(x)|u|^q dx\right)+\frac{q-1}{q}\left(\int_{\mathbb{R}^n}\rho(x)|u'^q dx\right)\\
&\leq c\|u'\|^q_{L^q_\rho(\mathbb{R}^n)}+c\|\rho\|^q_{L^s(\mathbb{R}^n)}\|\nabla_x u\|_2^q,
\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|\rho(x)v|v'^{q-2}v'\right| dx &\leq \left(\int_{\mathbb{R}^n} \rho(x)|v|^q dx\right)^{1/q} \left(\int_{\mathbb{R}^n} \rho(x)|v'^q dx\right)^{(q-1)/q} \\
&\leq \frac{1}{q}\left(\int_{\mathbb{R}^n} \rho(x)|v|^q dx\right) + \frac{q-1}{q}\left(\int_{\mathbb{R}^n} \rho(x)|v'^q dx\right) \\
&\leq c\|v'\|^q_{L^q_\rho(\mathbb{R}^n)} + c\|\rho\|^q_{L^s(\mathbb{R}^n)}\|\nabla_x v\|^q_2, \qquad (4.27)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|\left(\rho(x)^{\frac{q-1}{q}}|u'^{q-2}u'\right)\left(\rho(x)^{\frac{1}{q}}\int_0^t g_1(t-s)(u(t)-u(s))ds\right)\right| dx \\
&\leq \left(\int_{\mathbb{R}^n} \rho(x)|u'^q dx\right)^{(q-1)/q} \times \\
&\left(\int_{\mathbb{R}^n} \rho(x)\left|\int_0^t g_1(t-s)(u(t)-u(s))ds\right|^q dx\right)^{1/q} \\
&\leq \frac{q-1}{q}\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \frac{1}{q}\left\|\int_0^t g_1(t-s)(u(t)-u(s))ds\right\|^q_{L^q_\rho(\mathbb{R}^n)} \\
&\leq \frac{q-1}{q}\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \frac{1}{q}\|\rho\|^q_{L^s(\mathbb{R}^n)}(g_1 \circ \nabla_x u)^{q/2}(t),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|\left(\rho(x)^{\frac{q-1}{q}}|v'^{q-2}v'\right)\left(\rho(x)^{\frac{1}{q}}\int_0^t g_2(t-s)(v(t)-v(s))ds\right)\right| dx \\
&\leq \left(\int_{\mathbb{R}^n} \rho(x)|v'^q dx\right)^{(q-1)/q} \times \\
&\left(\int_{\mathbb{R}^n} \rho(x)\left|\int_0^t g_2(t-s)(v(t)-v(s))ds\right|^q dx\right)^{1/q} \\
&\leq \frac{q-1}{q}\|v'\|^q_{L^q_\rho(\mathbb{R}^n)} + \frac{1}{q}\left\|\int_0^t g_2(t-s)(v(t)-v(s))ds\right\|^q_{L^q_\rho(\mathbb{R}^n)} \\
&\leq \frac{q-1}{q}\|v'\|^q_{L^q_\rho(\mathbb{R}^n)} + \frac{1}{q}\|\rho\|^q_{L^s(\mathbb{R}^n)}(g_2 \circ \nabla_x v)^{q/2}(t).
\end{aligned}$$

Then, since $q \geq 2$, *we have*

$$\begin{aligned}
|L(t) - \xi_1 E(t)| &\leq c(E(t) + E^{q/2}(t)) \\
&\leq c(E(t) + E(t)E^{(q/2)-1}(t)) \\
&\leq c(E(t) + E(t)E^{(q/2)-1}(0)) \\
&\leq cE(t).
\end{aligned}$$

Therefore, we can choose ξ_1 so that

$$L(t) \sim E(t). \tag{4.28}$$

Proof of Theorem 4.3.1 *From (4.16), results of Lemma 4.3.2 and Lemma 4.3.3, we have*

$$\begin{aligned}
L'(t) &= \xi_1 E'(t) + \psi_1'(t) + \xi_2 \psi_2'(t) \\
&\leq (\frac{\xi_1}{2} - c_\sigma \xi_2 \|\rho\|^q_{L^s(\mathbb{R}^n)}) \left[(g_1' \circ \nabla_x u)^{q/2} + (g_2' \circ \nabla_x v)^{q/2} \right] \\
&+ M_0 \left[(g_2 \circ \nabla_x u) + (g_2 \circ \nabla_x v) \right] - M_1 \left[\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \|v'\|^q_{L^q_\rho(\mathbb{R}^n)} \right] \\
&- M_2 \left[\|\nabla_x u\|_2^2 + \|\nabla_x v\|_2^2 \right],
\end{aligned}$$

where

$$M_0 = (\frac{4\xi_2 c \left(1 + \alpha \|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}\right) + (1-l)}{4\sigma}),$$

$$M_1 = \left(\xi_2 \left(\int_0^{t_1} g(s)ds - \sigma \right) + c|\alpha| \|\rho\|^{-1}_{L^{n/2}(\mathbb{R}^n)} - 1 \right),$$

$$M_2 = \left(-\xi_2 \sigma \left(1 + \alpha \|\rho\|^2_{L^{n/2}(\mathbb{R}^n)} \right) + (l - \sigma) \right),$$

and t_1 was introduced in Remark 4.2.1.

We choose σ sufficiently small such that $\xi_1 > 2c_\sigma \|\rho\|^q_{L^s(\mathbb{R}^n)} \xi_2$. For σ fixed, we can choose ξ_1, ξ_2 large enough so that $M_1, M_2 > 0$, which yields

$$L'(t) \leq M_0 \left[(g_1 \circ \nabla_x u) + (g_2 \circ \nabla_x v) \right] - cE(t), \quad \textit{for all} \quad t \geq t_1. \tag{4.29}$$

Now we set $F(t) = L(t) + cE(t)$, which is equivalent to $E(t)$. Then by (4.29), we get for some positive constant c

$$\begin{aligned}
F'(t) &= L'(t) + cE'(t) \qquad (4.30)\\
&\leq -cE(t) + c \int_{\mathbb{R}^n} \int_{t_1}^t g_1(t-s) |\nabla_x u(t) - \nabla_x u(s)|^2 ds dx \\
&+ c \int_{\mathbb{R}^n} \int_{t_1}^t g_2(t-s) |\nabla_x v(t) - \nabla_x v(s)|^2 ds dx, \quad \textit{for all} \quad t \geq t_1.
\end{aligned}$$

By (4.8) and (4.16), we have for all $t \geq t_1$

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_0^{t_1} g_1(t-s)|\nabla_x u(t)-\nabla_x u(s)|^2 dsdx + \int_{\mathbb{R}^n}\int_0^{t_1} g_2(t-s)|\nabla_x v(t)-\nabla_x v(s)|^2 dsdx \\
&\leq -\frac{1}{k}\left(\int_{\mathbb{R}^n}\int_0^{t_1} g_1'(t-s)|\nabla_x u(t)-\nabla_x u(s)|^2 dsdx + \int_{\mathbb{R}^n}\int_0^{t_1} g_2'(t-s)|\nabla_x v(t)-\nabla_x v(s)|^2 dsdx\right) \\
\leq \quad & -cE'(t).
\end{aligned}$$

At this point, we define

$$\begin{aligned}
I(t) \quad = \quad & \int_{t_1}^{t} H_0(-g_1'(s))(g_1 \circ \nabla_x u)(t) ds \\
+ \quad & \int_{t_1}^{t} H_0(-g_2'(s))(g_2 \circ \nabla_x v)(t) ds.
\end{aligned} \tag{4.31}$$

Since $\int_0^{+\infty} H_0(-g_i'(s))g(s)ds < +\infty, i = 1,2$, from (4.16) we have

$$\begin{aligned}
& I(t) = \int_{t_1}^{t} H_0(-g_1'(s)) \int_{\mathbb{R}^n} g_1(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds \\
& + \int_{t_1}^{t} H_0(-g_2'(s)) \int_{\mathbb{R}^n} g_2(s)|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds \\
& \leq 2\int_{t_1}^{t} H_0(-g_1'(s))g_1(s) \int_{\mathbb{R}^n} |\nabla_x u(t)|^2 + |\nabla_x u(t-s)|^2 dxds \\
& +2\int_{t_1}^{t} H_0(-g_2'(s))g_2(s) \int_{\mathbb{R}^n} |\nabla_x v(t)|^2 + |\nabla_x v(t-s)|^2 dxds \\
& \leq cE(0)\left[\int_{t_1}^{t} H_0(-g_1'(s))g_1(s)ds + \int_{t_1}^{t} H_0(-g_2'(s))g_2(s)ds\right].
\end{aligned} \tag{4.32}$$

We have $I(t) < 1$ (see[45], Eq. (3.11)). Now, we define again a new functional $\lambda(t)$ related to $I(t)$ as

$$\begin{aligned}
\lambda(t) \quad = \quad & -\int_{t_1}^{t} H_0(-g_1'(s))g_1'(s) \int_{\mathbb{R}^n} g_1(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds \\
- \quad & \int_{t_1}^{t} H_0(-g_2'(s))g_2'(s) \int_{\mathbb{R}^n} g_2(s)|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds.
\end{aligned} \tag{4.33}$$

From (A1)-(A2) and Remark 4.2.1, we get

$$H_0(-g_i'(s))g_i(s) \leq H_0(H(g_i(s)))g_i(s) = D(g_i(s))g_i(s) \leq k_0,$$

for some positive constant k_0. Then, for all $t \geq t_1$

$$\begin{aligned}
\lambda(t) &\leq -k_0 \int_{t_1}^{t} g_1'(s) \int_{\mathbb{R}^n} |\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \\
&- k_0 \int_{t_1}^{t} g_2'(s) \int_{\mathbb{R}^n} |\nabla_x v(t) - \nabla_x v(t-s)|^2 dxds \\
&\leq -k_0 \int_{t_1}^{t} g_1'(s) \int_{\mathbb{R}^n} |\nabla_x u(t)|^2 + |\nabla_x u(t-s)|^2 dxds \\
&- k_0 \int_{t_1}^{t} g_2'(s) \int_{\mathbb{R}^n} |\nabla_x v(t)|^2 + |\nabla_x v(t-s)|^2 dxds \\
&\leq -cE(0) \left[\int_{t_1}^{t} g_1'(s) ds + \int_{t_1}^{t} g_2'(s) ds \right] \\
&\leq cE(0) \max \{g_1(t_1), g_2(t_1)\} \\
&< \min\{r, H(r), H_0(r)\}. \qquad (4.34)
\end{aligned}$$

Using the properties of H_0 (strictly convex in $(0,r], H_0(0) = 0$), then for $x \in (0,r], \theta \in [0,1]$,

$$H_0(\theta x) \leq \theta H_0(x).$$

Using Remark 4.2.1, (4.32), (4.34) and Jensen's inequality leads to

$$\begin{aligned}
\lambda(t) &= I^{-1}(t) \Big\{ \int_{t_1}^{t} I(t) H_0[H_0^{-1}(-g_1'(s))] H_0(-g_1'(s)) g_1'(s) \int_{\mathbb{R}^n} g_1(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \\
&+ \int_{t_1}^{t} I(t) H_0[H_0^{-1}(-g_2'(s))] H_0(-g_2'(s)) g_2'(s) \int_{\mathbb{R}^n} g_2(s) |\nabla_x v(t) - \nabla_x v(t-s)|^2 dxds \Big\} \\
&\geq I^{-1}(t) \Big\{ \int_{t_1}^{t} H_0[I(t) H_0^{-1}(-g_1'(s))] H_0(-g_1'(s)) g_1'(s) \int_{\mathbb{R}^n} g_1(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \\
&+ \int_{t_1}^{t} H_0[I(t) H_0^{-1}(-g_2'(s))] H_0(-g_2'(s)) g_2'(s) \int_{\mathbb{R}^n} g_2(s) |\nabla_x v(t) - \nabla_x v(t-s)|^2 dxds \Big\} \\
&\geq H_0 \Big(I^{-1}(t) \int_{t_1}^{t} I(t) H_0^{-1}(-g_1'(s)) H_0(-g_1'(s)) g_1'(s) \int_{\mathbb{R}^n} g_1(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \\
&+ I^{-1}(t) \int_{t_1}^{t} I(t) H_0^{-1}(-g_2'(s)) H_0(-g_2'(s)) g_2'(s) \int_{\mathbb{R}^n} g_2(s) |\nabla_x v(t) - \nabla_x v(t-s)|^2 dxds \Big) \\
&\geq H_0 \left(\int_{t_1}^{t} \int_{\mathbb{R}^n} g_1(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds + \int_{t_1}^{t} \int_{\mathbb{R}^n} g_2(s) |\nabla_x v(t) - \nabla_x v(t-s)|^2 dxds \right),
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_{t_1}^{t} \int_{\mathbb{R}^n} g_1(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds + \int_{t_1}^{t} \int_{\mathbb{R}^n} g_2(s) |\nabla_x v(t) - \nabla_x v(t-s)|^2 dxds \\
&\leq H_0^{-1}(\lambda(t)).
\end{aligned}$$

Then

$$F'(t) \quad \leq \quad -cE(t) + cH_0^{-1}(\lambda(t)), \quad \textit{for all} \quad t \geq t_1.$$

Now, we will be following the steps in ([45]) and using the fact that $E'(t) \leq 0, 0 < H_0', 0 < H_0''$ *on* $(0, r]$ *to define the functional*

$$F_1(t) = H_0'\left(a\frac{E(t)}{E(0)}\right)F(t) + cE(t), \quad a < r, 0 < c,$$

where $F_1(t) \sim E(t)$ *and*

$$\begin{aligned} F_1'(t) &= a\frac{E'(t)}{E(0)}H_0''\left(a\frac{E(t)}{E(0)}\right)F(t) + H_0'\left(a\frac{E(t)}{E(0)}\right)F'(t) + cE'(t) \\ &\leq -cE(t)H_0'\left(a\frac{E(t)}{E(0)}\right) + cH_0'\left(a\frac{E(t)}{E(0)}\right)H_0^{-1}(\lambda(t)) + cE'(t). \end{aligned}$$

Let H_0^* *be given in Remark 4.2.1 and using Young's inequality (4.9) with* $A = H_0'\left(a\frac{E(t)}{E(0)}\right), B = H_0^{-1}(\lambda(t))$*, we get*

$$\begin{aligned} F_1'(t) &\leq -cE(t)H_0'\left(a\frac{E(t)}{E(0)}\right) + cH_0^*\left(H_0'\left(a\frac{E(t)}{E(0)}\right)\right) + c\lambda(t) + cE'(t) \\ &\leq -cE(t)H_0'\left(a\frac{E(t)}{E(0)}\right) + ca\frac{E(t)}{E(0)}H_0'\left(a\frac{E(t)}{E(0)}\right) - c'E'(t) + cE'(t). \end{aligned}$$

Choosing a, c, c'*, such that for all* $t \geq t_1$ *we have*

$$\begin{aligned} F_1'(t) &\leq -k\frac{E(t)}{E(0)}H_0'\left(a\frac{E(t)}{E(0)}\right) \\ &= -kH_2\left(\frac{E(t)}{E(0)}\right), \end{aligned}$$

where $H_2(t) = tH_0'(\alpha_0 t)$*. Using the strict convexity of* H_0 *on* $(0, r]$*, we find that* H_2', H_2 *are strictly positives on* $(0, 1]$*, and then*

$$R(t) = \tau\frac{k_1F_1(t)}{E(0)} \sim E(t), \quad \tau \in (0, 1) \tag{4.35}$$

and

$$R'(t) \leq -\tau k_0 H_2(R(t)), \quad k_0 \in (0, +\infty), t \geq t_1.$$

Then, a simple integration and a suitable choice of τ *yield,*

$$R(t) \leq H_1^{-1}(bt + c), \quad b, c \in (0, +\infty), t \geq t_1,$$

where $H_1(t) = \int_t^1 H_2^{-1}(s)ds$. *From (4.35), for a positive constant* α_3, *we have*

$$E(t) \leq dH_1^{-1}(bt + c).$$

The fact that H_1 *is a strictly decreasing function on* $(0, 1]$ *and due to the properties of* H_2, *we have*

$$\lim_{t \to 0} H_1(t) = +\infty.$$

Therefore

$$E(t) \leq dH_1^{-1}(bt + c), \quad \textit{for all} \quad t \geq 0.$$

This completes the proof of Theorem 4.3.1.

Chapter 5

Existence and decay of solution to the coupled system of viscoelastic wave equations with strong damping in $\mathbb{R}^n$

5.1 Introduction and previous results

The viscoelastic materials have intermediate properties (between the elastic materials and viscous fluids). These two types of materials are usually studied in basic texts in the field of continuum mechanics. At each material point of an elastic material, the current level of stresses depends only on the current level of strains. On the other hand, for incompressible viscous fluids, the level of stresses at a given point is a function of the current value of the velocity gradient at this point. These materials have memory: the stresses depend not only on the current values of strains and/or velocity gradient but also on the entire time history of motion. In this direction, let us consider the following problem:

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + \alpha u_2 + \phi(x)A\left(u_1 + \int_0^t g_1(s)u_1(t-s,x)ds + u_1'\right) = 0, \\ \left(|u_2'|^{l-2}u_2'\right)' + \alpha u_1 + \phi(x)A\left(u_2 + \int_0^t g_2(s)u_2(t-s,x)ds + u_2'\right) = 0, \\ (u_1(0,x), u_2(0,x)) = (u_{10}(x), u_{20}(x)) \in (D(\mathbb{R})^2, \\ (u_1'(0,x), u_2'(0,x)) = (u_{11}(x), u_{21}(x)) \in (L^l_\rho(\mathbb{R})^2, \end{cases} \tag{5.1}$$

where $x \in \mathbb{R}^n, t \in \mathbb{R}^+_*$ *where the space* $D(\mathbb{R}^n)$ *defined below in (5.14) and* $l \geq 2, n > 2$,
$\phi(x) > 0, \forall x \in \mathbb{R}^n, (\phi(x))^{-1} = \rho(x)$ *defined below in (**A2**).*

This type of problems is usually encountered in viscoelasticity in various areas of mathematical physics. It was first considered by Dafermos in [15], where the general decay was discussed. The problems related to (5.1) attract a great deal of attention in the last decades and numerous results appeared on the existence and long time behavior of solutions but their results is by now rather developed, especially in any space dimension when it comes to nonlinear problems.

The term $\int_0^t g_i(t)Au_i(t-s)ds$ *corresponds to the memory term and the scalar functions* $g_i(t)$ *(so-called relaxation kernel) is assumed to satisfy (5.11)-(5.13)below and* A *is a linear, selfadjoint operator in* $L^2(\mathbb{R}^n)$.

The energy of (u_1, u_2) *at time t is defined by*

$$\begin{aligned} E(t) &= \frac{(l-1)}{l}\sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \frac{1}{2}\sum_{i=1}^{2}\left(1-\int_0^t g_i(s)ds\right)\|A^{1/2}u_i\|_2^2 \\ &+ \frac{1}{2}\sum_{i=1}^{2}(g_i \circ A^{1/2}u_i) + \alpha\int_{\mathbb{R}^n}\rho u_1 u_2 dx. \end{aligned} \tag{5.2}$$

For α *small enough we use Lemma 5.2.1 below to deduce that:*

$$E(t) \geq \frac{1}{2}(1-c|\alpha|\|\rho\|^{-1}_{L^{n/2}})\Big[\frac{2(l-1)}{l}\sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho} + \sum_{i=1}^{2}\left(1-\int_0^t g_i(s)ds\right)\|A^{1/2}u_i\|_2^2$$
$$+\sum_{i=1}^{2}(g_i \circ A^{1/2}u_i)\Big], \tag{5.3}$$

and the following energy functional law holds, which means that our energy is uniformly bounded and decreasing along the trajectories.

$$E'(t) \leq \frac{1}{2}\sum_{i=1}^{2}(g_i' \circ A^{1/2}u_i)(t) - \sum_{i=1}^{2}\|A^{1/2}u_i'\|_2^2, \forall t \geq 0. \tag{5.4}$$

The following notation will be used throughout this chapter

$$(g \circ \Psi)(t) = \int_0^t g(t-\tau)\,\|\Psi(t)-\Psi(\tau)\|_2^2\,d\tau, \text{ for any } \Psi \in L^\infty(0,T;L^2(\mathbb{R}^n)) \tag{5.5}$$

In the present chapter we consider the solutions in an appropriate space weighted by the density function $\rho(x)$ *in order to compensate for the lack of Poincare's inequality which plays a decisive role in the proof.*

To motivate our work, we present a similar models. We start with some results related to viscoelastic plate equations with strong damping in [39]:

$$u_{tt} + \Delta^2 u - \Delta_p u - \int_0^t g(t-s)\Delta u(s,x)ds - \Delta u_t + f(u) = 0, x \in \Omega \times \mathbb{R}^+,$$

supplemented with the following conditions:

$$u(t,x) = \Delta u = 0, \text{ on } \partial\Omega \times \mathbb{R}^+, \quad u(0,x) = u_0, u_t(0,t) = u_1, \text{ on } \Omega, \tag{5.6}$$

in this chapter, Liu and al[38] extend the exponential rate result obtained in [3] to the general case, and show that the rate of decay for the solution is similar to that of the memory term under the following assumption on the function g is

$$g'(t) \leq -\xi(t)g(t), \quad \text{where } \xi(t) \text{ satisfies } \xi'(t) \leq 0, \int_0^t \xi(t)dt = \infty.$$

The reference [27] is concerned with a class of plate equations with memory in a history space setting and perturbations of $p-$*Laplacian type*

$$u_{tt} + \alpha\Delta^2 u - \Delta_p u - \int_{-\infty}^t g(t-s)\Delta^2 u(s,x)ds - \Delta u_t + f(u) = h, x \in \Omega \times \mathbb{R}^+, \tag{5.7}$$

and results on the well-posedness and asymptotic stability of the problem were proved.

In many existing works on this field, the following condition on the kernel

$$g'(t) \geq -\lambda g'(t), \quad t \geq 0, p \geq 0, \tag{5.8}$$

is crucial in the proof of the stability.

For a viscoelastic system with oscillating kernels, we mention the work by Rivera and al[46]. The authors proved that if the kernel satisfies $g(0) > 0$ *and decays exponentially to zero, that is for* $p = 1$ *in (5.8), then the solution also decays exponentially to zero. On the other hand, if the kernel decays polynomially, i.e.* $(p > 1)$ *in the inequality (5.8), then the solution also decays polynomially with the same rate of decay.*
Recently problem related to (5.1) in a bounded domain $\Omega \subset \mathbb{R}^n$, $(n \geq 1)$ *with a smooth boundary* $\partial\Omega$

and g is a positive nonincreasing function was considered as equation in [45], where they established an explicit and very general decay rate result for relaxation functions satisfying:

$$g'(t) \leq -H(g(t)), t \geq 0, H(0) = 0, \tag{5.9}$$

for a positive function $H \in C^1(\mathbb{R}^+)$ *and* H *is linear or strictly increasing and strictly convex* C^2 *function on* $(0, r], 1 > r$.

For the literature, in $\mathbb{R}^n$*, we quote essentially the results of [5], [2], [10], [28]-[32], [45]-[49] and the references therein. In [29], the authors showed for one equation that, for compactly supported initial data and for an exponentially decaying relaxation function, the decay of the energy of solution of a linear Cauchy problem (5.1) without strong damping in the case* $l = 2, \rho(x) = 1$ *is polynomial. The finite-speed propagation is used to compensate for the lack of Poincare's inequality. In the case* $l = 2$*, in [28],the author looked into a linear Cauchy viscoelastic equation with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate for the lack of Poincare's inequality in the absence of strong damping. The same problem treated in [28], was considered in [30], where under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. Conditions used, on the relaxation function g and its derivative* g' *are different from the usual ones.*

Coupled systems in $\mathbb{R}^n$*, we mention, for instance, the work of [Takashi Narazaki, 2009. Global solutions to the Cauchy problem for the weakly coupled system of damped wave equations. Discrete And Continuous Dynamical Systems, 592-601], where the following weakly coupled system of a damped wave equations was considered:*

$$\begin{cases} u'' - \Delta u + u' = f(v), \quad t > 0, x \in \mathbb{R}^n, \\ v'' - \Delta v + v' = f(u), \quad t > 0, x \in \mathbb{R}^n, \\ (u(0,x), v(0,x)) = (\phi_0(x), \psi_0(x)), \quad x \in \mathbb{R}^n, \\ (u'(0,x), v'(0,x)) = (\phi_1(x), \psi_1(x)), \quad x \in \mathbb{R}^n. \end{cases} \tag{5.10}$$

The authors have shown the sufficient condition for the Cauchy problem (5.10) to admit global solutions when $n = 1, 2, 3$ *provided that the initial data are sufficiently small in an associate space. Moreover, they have also shown the asymptotic behavior of the above solutions, to generalize the existence result in [39] to the case* $n = 1, 2, 3$ *and improve time decay estimates when* $n = 3$.

5.2 Function spaces and statements

In this section we introduce some notation and construct some material needed for our work. We omit the space variable x of $u(x,t), u'(x,t)$ *and for simplicity reason denote* $u(x,t) = u$ *and* $u'(x,t) = u'$*, when no confusion arises. The constants c used throughout this chapter are positive generic constants which may be different in various occurrences also the functions considered are all real valued. Here* $u' = du(t)/dt$ *and* $u'' = d^2u(t)/dt^2$, $A = -\Delta$. *We denote by* B_R *the open ball of* $\mathbb{R}^n$ *with center* 0 *and radius R.*

First we recall and make use the following assumptions on the functions ρ *and g for* $i = 1,2$ *as:*

*(**A1**) We assume that the function* $g_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$*(for* $i = 1,2$*) is of class* C^1 *satisfying:*

$$1 - \int_0^\infty g_i(t)dt \geq k_i > 0, g_i(0) = g_{i0} > 0, \tag{5.11}$$

and there exist nonincreasing continuous functions ξ_1,ξ_2: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *satisfying*

$$\xi'(t) \leq 0, \quad \forall t > 0, \quad \int_0^\infty \xi(t) = \infty, \quad \xi(t) = \min\{\xi_1(t), \xi_2(t)\}, \tag{5.12}$$

where

$$g_i'(t) + \xi(t)g_i(t) \leq 0. \tag{5.13}$$

*(**A2**) The function* $\rho : \mathbb{R}^n \to \mathbb{R}^*_+, \rho(x) \in C^{0,\gamma}(\mathbb{R}^n)$ *with* $\gamma \in (0,1)$ *and* $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*, where* $s = \frac{2n}{2n-qn+2q}$.

Defintion 5.2.1 ([28], [48]) *We define the function spaces of our problem and its norm as follows:*

$$D(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-2)}(\mathbb{R}^n) : A^{1/2} f \in (L^2(\mathbb{R}^n)^n \right\}, \tag{5.14}$$

and the spaces $L^2_\rho(\mathbb{R}^n)$ *to be the closure of* $C_0^\infty(\mathbb{R}^n)$ *functions with respect to the inner product:*

$$(f,h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx. \tag{5.15}$$

For $1 < l < \infty$*, if f is a measurable function on* $\mathbb{R}^n$*, we define*

$$\|f\|_{L^l_\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \rho |f|^l dx \right)^{1/l}. \tag{5.16}$$

The space $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space.

So we are able to construct the necessary evolution triplet for the space setting of our problem, which is:

$$D(\mathbb{R}^n) \subset L^2_\rho(\mathbb{R}^n) \subset D^{-1}(\mathbb{R}^n), \tag{5.17}$$

where all the embedding are compact and dense.

The following technical lemma will play an important role in the sequel.

Lemma 5.2.1 *[10] Let ρ satisfies (**A2**), then for any $u \in D(A^{1/2})$*

$$\|u\|_{L^q_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^s(\mathbb{R}^n)} \|A^{1/2}u\|_{L^2(\mathbb{R}^n)}, \tag{5.18}$$

with,

$$s = \frac{2n}{2n - qn + 2q}, 2 \leq q \leq \frac{2n}{n-2}.$$

Lemma 5.2.2 *If $q = 2$, then Lemma 5.2.1. yields*

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^{n/2}(\mathbb{R}^n} \|A^{1/2}u\|_{L^2(\mathbb{R}^n)}, \tag{5.19}$$

where we can assume $\|\rho\|_{L^{n/2}(\mathbb{R}^n)} = c > 0$ to get

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \leq c\|A^{1/2}u\|_{L^2(\mathbb{R}^n)}. \tag{5.20}$$

To study the properties of the operator ϕA, we consider as in [32], the equation

$$\phi(x)Au(x) = \eta(x), \quad x \in \mathbb{R}^n, \tag{5.21}$$

without boundary conditions. Since for every u, v in $C_0^\infty(\mathbb{R}^n)$

$$(\phi Au, v)_{L^2_\rho} = \int_{\mathbb{R}^n} A^{1/2}uA^{1/2}v dx, \tag{5.22}$$

and $L^2_\rho(\mathbb{R}^n)$ are defined with respect to the inner product (5.15), we may consider equation (5.21) an operator equation:

$$A_0 u = \eta, \quad A_0 : D(A_0) \subseteq L^2_\rho(\mathbb{R}^n) \to L^2_\rho(\mathbb{R}^n), \quad \eta \in L^2_\rho(\mathbb{R}^n). \tag{5.23}$$

Relation(5.22) implies that the operators ϕA*, with domain of definition* $D(A_0) = C_0^\infty(\mathbb{R}^n)$ *are symmetric. Let us note that the operator* ϕA *is not symmetric in the standard Lebesgue space* $L^2(\mathbb{R}^n)$*, because of the appearance of* $\phi(x)$ *(see [[51], pages 185-187]). From (5.20) and (5.22), we have*

$$\|u\|_{L^2_\rho} \leq c(A_0 u, u)_{L^2_\rho}, \quad \text{for all } u \in D(A_0), \tag{5.24}$$

From (5.22) and (5.24) we conclude that A_0 *is symmetric, strongly monotone operator on* $L^2_\rho(\mathbb{R}^n)$*. The energy scalar product is given by:*

$$(u,v)_E = \int_{\mathbb{R}^n} A^{1/2} u A^{1/2} v dx, \tag{5.25}$$

and the energy space is the completion of $D(A_0)$ *with respect to* $(u,v)_E$*. It is obvious that the energy space* X_E *is the homogeneous Sobolev space* $D(\mathbb{R}^n)$*. The energy extension* A_E*, namely*

$$\phi A : D(\mathbb{R}^n) \to \mathcal{D}^{-1}(\mathbb{R}^n), \tag{5.26}$$

is defined to be the duality mapping of $D(\mathbb{R}^n)$*. For every* $\eta \in D^{-1}(\mathbb{R}^n)$ *the equation (5.21), has a unique solution. Define* $D(A_1)$ *to be the set of all solutions of the equations (5.21) for arbitrary* $\eta \in L^2_\rho(\mathbb{R}^n)$*. The operator extension* A_1 *of* A_0*, [see [56], Theorem 19.C] is the restriction of the energy extension* A_E *to the set* $D(A_1)$*. The operator* A_1 *is self-adjoint and therefore graph-closed. Its domain is a Hilbert space with respect to the graph scalar product*

$$(u,v)_{D(A_1)} = (u,v)_{L^2_\rho} + (A_1 u, A_1 v)_{L^2_\rho}, \quad \text{for all } u, v \in D(A_1).$$

The norm induced by the scalar product $(u,v)_{D(A_1)}$ *is*

$$\|u\|_{D(A_1)} = \left\{ \int_{\mathbb{R}^n} \rho |u|^2 dx + \int_{\mathbb{R}^n} \phi |Au|^2 dx \right\}^{\frac{1}{2}}.$$

which is equivalent to the norm

$$\|A_1 u\|_{L^2_\rho} = \left\{ \int_{\mathbb{R}^n} \phi |Au|^2 dx \right\}^{\frac{1}{2}}.$$

So we have established the evolution quartet

$$D(A_1) \subset D(\mathbb{R}^n) \subset L^2_\rho(\mathbb{R}^n) \subset D^{-1}(\mathbb{R}^n), \tag{5.27}$$

where all the embedding are dense and compact. A consequence of the compactness of the embedding in (5.27) is that the eigenvalue problem

$$Au = \mu u, x \in \mathbb{R}^n, \tag{5.28}$$

has a complete system of eigensolutions $\{w_n, \mu_n\}$ with the following properties:

$$\begin{cases} Aw_j = \mu w_j, & j = 1,2\cdots, \quad w_j \in \mathcal{D}(\mathbb{R}^n), \\ \\ 0 < \mu_1 \leq \mu_2 \leq \cdots, \quad \mu_j \to \infty, & as \quad j \to \infty. \end{cases} \tag{5.29}$$

It can be shown, as in [10], that every solution of (5.28) is such that

$$u(x) \longrightarrow 0, \quad as \quad |x| \longrightarrow \infty, \tag{5.30}$$

uniformly with respect to x.

We may define the fractional powers in the following way. For every $s > 0$, $(\phi A)^s$ is an unbounded selfadjoint operator in $L^2_\rho(\mathbb{R}^n)$ with its domain $D((\phi A)^s)$ to be a dense subset in $L^2_\rho(\mathbb{R}^n)$. The operator $(\phi A)^s$ is strictly positive and injective. Also $D((\phi A)^s)$ endowed with the scalar product

$$(u,v)_{D((\phi A)^s)} = (u,v)_{L^2_\rho} + ((\phi A)^s u, (\phi A)^s v)_{L^2_\rho},$$

becomes a Hilbert space and we have the following identifications

$$If \quad s = 0: \quad D((\phi A)^0) = L^2_\rho(\mathbb{R}^n),$$

and

$$If \quad s = \frac{1}{2}: \quad D((\phi A)^{1/2}) = D(\mathbb{R}^n) \text{ and } \quad D^{-1}((\phi A)^{1/2}) = D^{-1}(\mathbb{R}^n).$$

Furthermore, as a consequence of (5.27) the injection $D((\phi A)^{s_1}) \subset D((\phi A)^{s_2})$ is compact and dense, for every $s_1 > s_2$.

Finally, we give the definition of weak solutions for the problem (5.1).

Defintion 5.2.2 *A weak solution of (5.1) is (u_1, u_2) such that*

- $(u_1, u_2) \in (L^2[0,T; D(\mathbb{R}^n)])^2, \quad (u_1', u_2') \in (L^2[0,T; L^l_\rho(\mathbb{R}^n)])^2$ *and* $(u_1'', u_2'') \in (L^2[0,T; D^{-1}(\mathbb{R}^n)])^2$,

- *For all* $(v,w) \in (C_0^\infty([0,T] \times \mathbb{R}^n))^2$, (u_1,u_2) *satisfies the generalized formula:*

$$\begin{cases} \int_0^T \left(\left(|u_1'|^{l-2}u_1' \right)', v \right)_{L^l_\rho} ds + \alpha \int_0^T (u_2, v)_{L^2_\rho} ds + \int_0^T \int_{\mathbb{R}^n} A^{1/2}u_1 A^{1/2} v dx ds \\ + \int_0^T \int_{\mathbb{R}^n} A^{1/2}u_1' A^{1/2} v dx ds - \int_0^T \int_{\mathbb{R}^n} \int_0^s g_1(s-\tau) A^{1/2}u_1(\tau) d\tau A^{1/2} v(s) dx ds = 0, \\ \\ \int_0^T \left(\left(|u_2'|^{l-2}u_2' \right)', w \right)_{L^l_\rho} ds + \alpha \int_0^T (u_1, w)_{L^2_\rho} ds + \int_0^T \int_{\mathbb{R}^n} A^{1/2}u_2 A^{1/2} w dx ds \\ + \int_0^T \int_{\mathbb{R}^n} A^{1/2}u_2' A^{1/2} w dx ds - \int_0^T \int_{\mathbb{R}^n} \int_0^s g_2(s-\tau) A^{1/2}u_2(\tau) d\tau A^{1/2} w(s) dx ds = 0. \end{cases}$$

- (u_1,u_2) *satisfies the initial conditions*
  $(u_{10}(x), u_{20}(x)) \in (D(\mathbb{R}^n))^2, \quad (u_{11}(x), u_{21}(x)) \in (L^l_\rho(\mathbb{R}^n))^2.$

We are now ready to state and prove our existence results

5.3 Well-posedness results for the nonlinear case

This section is devoted to the proof of the existence and uniqueness of solutions to system (5.1) taking account the nonlinear case in the terms responsible on the relation between tow equations, that is replacing $\alpha u_1, \alpha u_2$ *by* $f_1(u_1,u_2), f_2(u_1,u_2)$ *introduced in the last section. First, we prove the existence of a unique weak solution of the restricted problem on* B_R*; the main ingredient used here is the Galerkin approximations introduced in [**?**].*

Lemma 5.3.1 *Assume that (**A1**), (**A2**), (5.67)-(5.71) are satisfied. Suppose that the constants* $T > 0,\ R > 0$ *and the initial conditions*

$$(u_{10}, u_{20}) \in (D(B_R))^2 =, (u_{11}, u_{21}) \in (L^l_\rho(B_R))^2,$$

are given. Then there exists a unique (weak) solution for the problem (5.1) , such that

$$u_i \in C[0,T; D(B_R)] \quad and \quad u_i' \in C[0,T; L^l_\rho(B_R)].$$

The existence is proved using the Galerkin method, which consists in constructing approximations of the solution. Then we obtain a priori estimates necessary to guarantee the convergence of these approximations. So, we take $\{w_i\}_{i=1}^\infty$ *be to the eigenfunctions of operator* A*. Then* $\{w_i\}_{i=1}^\infty$ *is* *orthogonal basis of* $D(B_R)$ *which is orthonormal in* $L^2_\rho(B_R)$.

Let

$$V_m = span\{w_1, w_2, \cdots, w_m\},$$

and the projection of the initial data on the finite dimensional subspace V_m *is given by:*

$$u^m_{10} = \sum_{j=0}^{m} a_j w_j, \quad u^m_{20} = \sum_{j=0}^{m} b_j w_j, \quad u^m_{11} = \sum_{j=0}^{m} c_j w_j, \quad u^m_{21} = \sum_{j=0}^{m} d_j w_j,$$

We consider the discrete problem in V_m *and look for the approximate solutions as*

$$u^m_1(x,t) := \sum_{j=0}^{m} h^m_j(t) w_j(x), \quad u^m_2(x,t) := \sum_{j=0}^{m} k^m_j(t) w_j(x),$$

Hence

$$\begin{cases} \int_{B_R} \left(\rho(x) \left(|u'^m_1|^{l-2} u'^m_1 \right)' w - \int_0^t g_1(t-s) A^{1/2} u^m_1(s,x) A^{1/2} w ds \right) dx \\ + \int_{B_R} \left(\rho(x) f_1(u^m_1, u^m_2) w + A^{1/2} u^m_1 A^{1/2} w + A^{1/2} u'^m_1 A^{1/2} w \right) dx = 0, \\ \\ \int_{B_R} \left(\rho(x) \left(|u'^m_2|^{l-2} u'^m_2 \right)' w - \int_0^t g_2(t-s) A^{1/2} u^m_2(s,x) A^{1/2} w ds \right) dx \\ + \int_{B_R} \left(\rho(x) f_2(u^m_1, u^m_2) w + A^{1/2} u^m_2 A^{1/2} w + A^{1/2} u'^m_2 A^{1/2} w \right) dx = 0, \\ \\ u^m_1(0) = u^m_{10}, u'^m_1(0) = u^m_{11}, u^m_2(0) = u^m_{20}, u'^m_2(0) = u^m_{21}. \end{cases} \tag{5.31}$$

Based on standard existence theory for differential equations, one can conclude as to the existence of a solution (u^m_1, u^m_2) *of (5.31) on a maximal time interval* $[0, t_m)$, *for each* $m \in \mathbb{N}$.

- *(A priori estimate 1): In (5.31), let* $w = (u^m_1)'$ *in the first equation and* $w = (u^m_2)'$ *in the second equation, add the resulting equations, and integrating by parts to obtain*

$$\frac{d}{dt} E^m(t) = \frac{1}{2} \sum_{i=1}^{2} (g'^{1/2}_i u^m_i)(t) - \frac{1}{2} \sum_{i=1}^{2} g_i(t) \|A^{1/2} u^m_i(t)\|_2^2 - \sum_{i=1}^{2} \|A^{1/2} u'^m_i\|_2^2. \tag{5.32}$$

*This means, using (**A1**), that, for some positive constant C independent of t and m, we have*

$$E^m(t) \leq E^m(0) \leq C. \tag{5.33}$$

• *(A priori estimate 2): In (5.31), let $w = Au_1'^m$ in the first equation and $w = Au_2'^m$ in the second equation, add the resulting equations, integrating by parts, and using (**A1**) to obtain*

$$
\begin{aligned}
&\frac{d}{dt}\sum_{i=1}^{2}\left(\frac{l-1}{l}\|A^{1/2}u_i'^m\|_{L_\rho^l}^l + \frac{1}{2}\left(1-\int_0^t g_i(s)ds\right)\|Au_i^m\|_2^2 + \frac{1}{2}(g_i\circ Au_i^m)\right)\\
= &\sum_{i=1}^{2}\left(\frac{1}{2}(g_i'\circ Au_i^m) - \frac{1}{2}g_i(t)\|Au_i^m\|_2^2 - \|Au_i'^m\|_2^2\right)\\
- &\sum_{i=1}^{2}\int_{B_R}\rho(x)f_i(u_1^m,u_2^m)Au_i'^m dx\\
\leq &-\sum_{i=1}^{2}\int_{B_R}\rho(x)f_i(u_1^m,u_2^m)Au_i'^m dx. \qquad (5.34)
\end{aligned}
$$

Then, integrating over $(0,t)$ yields

$$
\begin{aligned}
&\sum_{i=1}^{2}\left(\frac{l-1}{l}\|A^{1/2}u_i'^m\|_{L_\rho^l}^l + \frac{1}{2}\left(1-\int_0^t g_i(s)ds\right)\|Au_i^m\|_2^2 + \frac{1}{2}(g_i\circ Au_i^m)\right)\\
\leq &\sum_{i=1}^{2}\left(\|A^{1/2}u_{i1}^m\|_{L_\rho^l}^l + \|Au_{i0}^m\|_2^2 - \int_{B_R}\rho(x)f_i(u_1^m,u_2^m)Au_i^m dx\right)\\
+ &\sum_{i=1}^{2}\int_{B_R}\rho(x)\left(f_i(u_{10}^m,u_{20}^m)Au_{i0}^m\right)dx\\
+ &\int_0^t\int_{B_R}\rho(x)\left(\frac{\partial f_1}{\partial u_2}u_2'^m Au_1^m + \frac{\partial f_2}{\partial u_1}u_1'^m Au_2^m\right)dxds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_{B_R}\rho(x)f_i(u_1^m,u_2^m)Au_i^m \leq k\int_{B_R}\rho(x)\left(|u_1^m| + |u_2^m| + |u_1^m|^{\beta_{i1}} + |u_2^m|^{\beta_{i2}}\right)Au_i^m,\\
&\leq \delta\|Au_i^m\|_{L_\rho^2}^2 + \frac{c}{\delta}\int_{B_R}\rho(x)\left(|u_1^m|^2 + |u_2^m|^2 + |u_1^m|^{2\beta_{i1}} + |u_2^m|^{2\beta_{i2}}\right),\\
&\leq \delta\|Au_i^m\|_{L_\rho^2}^2 + \frac{c}{\delta}\left(\|u_1^m\|_{L_\rho^2}^2 + \|u_2^m\|_{L_\rho^2}^2 + \|u_1^m\|_{L_\rho^2}^{2\beta_{i1}} + \|u_2^m\|_{L_\rho^2}^{2\beta_{i2}}\right),\\
&\leq \delta\|Au_i^m\|_{L_\rho^2}^2 + \frac{c}{\delta}\left(\|A^{1/2}u_1^m\|_{L_\rho^2}^2 + \|A^{1/2}u_2^m\|_{L_\rho^2}^2 + \|A^{1/2}u_1^m\|_{L_\rho^2}^{2\beta_{i1}} + \|A^{1/2}u_2^m\|_{L_\rho^2}^{2\beta_{i2}}\right),\\
&\leq \delta\|Au_i^m\|_{L_\rho^2}^2 + \frac{c}{\delta}E^m(0)E^m(t),\\
&\leq \delta\|Au_i^m\|_{L_\rho^2}^2 + \frac{c}{\delta}. \qquad (5.35)
\end{aligned}
$$

since $1 \leq \beta_{ij}, i, j = 1, 2$.

Now, we estimate

$$I := \int_{B_R} \rho(x) \frac{\partial f_i}{\partial u_1} u_i'^m A u_i^m.$$

First, we observe that

$$\frac{\beta_{1j} - 1}{2\beta_{1j}} + \frac{1}{2\beta_{1j}} + \frac{1}{2} = 1,$$

so that

$$\begin{aligned} |I| &\leq d \int_{B_R} \rho(x) \left(1 + |u_1^m|^{\beta_{11}-1} + |u_2^m|^{\beta_{12}-1}\right) u_i'^m A u_i^m, \\ &\leq d \left(\|u_i'^m\|_{L^2_\rho} + \|u_i'^m\|_{L^{2\beta_{11}}_\rho} \|u_1^m\|^{\beta_{11}-1}_{L^{2\beta_{11}}_\rho} + \|u_i'^m\|_{L^{2\beta_{12}}_\rho} \|u_2^m\|^{\beta_{12}-1}_{L^{2\beta_{12}}_\rho} \right) \|A u_i^m\|_{L^2_\rho}. \end{aligned}$$

Hence

$$\begin{aligned} |I| &\leq c \left(1 + \|A^{1/2} u_1^m\|_2^{\beta_{11}-1} + \|A^{1/2} u_2^m\|_2^{\beta_{12}-1}\right) \|A^{1/2} u_i'^m\|_{L^2_\rho} \|A u_i^m\|_{L^2_\rho}, \\ &\leq c \left(\|A^{1/2} u_i'^m\|_{L^2_\rho} \cdot \|A u_i^m\|_{L^2_\rho} \right) \leq c \|A^{1/2} u_i'^m\|^2_{L^2_\rho} + c \|A u_i^m\|^2_{L^2_\rho}. \end{aligned} \tag{5.36}$$

Since the other terms in (5.35) can be similarly treated and the norms of the initial data are uniformly bounded, we combine to end up with

$$\sum_{i=1}^{2} \left(\|A^{1/2} u_i'^m\|^l_{L^l_\rho} + \|A u_i^m\|_2^2 \right) \leq c + c \sum_{i=1}^{2} \int_0^t \left(\|A^{1/2} u_i'^m\|^l_{L^l_\rho} + \|A u_i^m\|_2^2 \right) ds.$$

$$\sum_{i=1}^{2} \left(\|A_x^{1/2} u_i'^m\|^l_{L^l_\rho} + \|A u_i^m\|_2^2 \right) \leq C, \quad \forall t \in [0, T] \text{ and } m \in \mathbb{N}. \tag{5.37}$$

• *(A priori estimate 3): In (5.31), let* $w = \left(u_1^m\right)''$ *in the first equation and* $w = (u_2^m)''$ *in the second equation. Then, by exploiting the previous estimates and using similar arguments, we find*

$$\sum_{i=1}^{2} \|u_i''^m\|^l_{L^l_\rho} \leq C, \quad \forall t \in [0, T] \text{ and } m \in \mathbb{N}. \tag{5.38}$$

From (5.33), (5.37), and (5.38), we conclude that

$$u_i^m \text{ are uniformly bounded in } L^\infty(0, T; D(B_R)),$$

$$u_i^{m'} \text{ are uniformly bounded in } L^\infty(0, T; L^l_\rho(B_R)),$$

$$u_i^{m''} \text{ are uniformly bounded in } L^2(0, T; \mathcal{D}^{-1}(B_R)),$$

which implies that there exist subsequences of $\{u_i^m\}$, which we still denote in the same way, such that

$$\begin{aligned} u_i^m &\xrightarrow{*} \text{weak } u_i \text{ in } L^\infty(0,T;D(B_R)), \\ u_i^{m'} &\xrightarrow{*} \text{weak } u_i' \text{ in } L^\infty(0,T;L^l_\rho(B_R)), \\ u_i^{m''} &\xrightarrow{*} \text{weak } u_i'' \text{ in } L^2(0,T;D^{-1}(B_R)). \end{aligned} \tag{5.39}$$

According to lemma given in ([?]), we find, up to a subsequence, that

$$u_i^m \to u_i \text{ strongly in } L^2(0,T;L^l_\rho(B_R)). \tag{5.40}$$

Then,

$$u_i^m \to u_i \text{ almost everywhere in } (0,T)\times B_R, \tag{5.41}$$

and therefore, from (5.70), (5.71) below,

$$f_i(u_1^m,u_2^m) \to f_i(u_1,u_2) \text{ almost everywhere in } (0,T)\times B_R, \text{ for } i=1,2. \tag{5.42}$$

Also, as u_i^m are bounded in $L^\infty(0,T;L^2_\rho(B_R))$, then the use of (5.67)-(5.71) gives that $f_i(u_1^m,u_2^m)$ is bounded in $L^\infty(0,T;L^2_\rho(B_R))$. From (5.42), we can deduce that

$$f_i(u_1^m,u_2^m) \rightharpoonup f_i(u_1,u_2) \text{ in } L^2(0,T;L^2_\rho(B_R)), \text{ for } i=1,2.$$

Combining the results obtained above, we can go to the limit and conclude that (u_1,u_2) is a weak solution of system (5.1) restricted to B_R.

In the next result, we will extend our solutions to $\mathbb{R}^n$.

Theorem 5.3.1 *Assume that (**A1**), (**A2**), (5.67)-(5.71) are satisfied. Suppose that the initial conditions*

$$(u_{10},u_{11}) \in (C_0^\infty(B_R)^2, (u_{20},u_{21}) \in (C_0^\infty(B_R))^2,$$

are given. Then for the problem (5.1), there exists a unique (weak) solution such that

$$(u_1,u_2) \in (C[0,T;D(\mathbb{R}^n)])^2 \quad \text{and} \quad (u_1',u_2') \in (C[0,T;L^l_\rho(\mathbb{R}^n)])^2.$$

(a) ***Existence.*** *Let* $R_0 > 0$ *such that* $supp(u_{10}, u_{20}) \subset B_{R_0}$ *and* $supp(u_{11}, u_{21}) \subset B_{R_0}$. *Then, for* $R \geq R_0$, $R \in \mathbb{N}$, *we consider the approximating problem*

$$\begin{cases} \left(|u_1'^R|^{l-2}u_1'^R\right)' + f_1(u_1^R, u_2^R) + \phi(x)A\left(u_1^R + \int_0^t g_1(s)u_1^R(s-t,x)ds + u_1'^R\right) = 0, x \in B_R \times \mathbb{R}^+, \\ \\ \left(|u_2'^R|^{l-2}u_2'^R\right)' + f_2(u_1^R, u_2^R) + \phi(x)A\left(u_2^R + \int_0^t g_2(s)u_2^R(s-t,x)ds + u_2'^R\right) = 0, x \in B_R \times \mathbb{R}^+, \\ \\ (u_1^R(0,x), u_2^R(0,x)) = (u_1^0(x), u_2^0(x)) \in \left(C_0^\infty(B_R)\right)^2, \\ (u_1'^R(0,x), u_2'^R(0,x)) = (u_1^1(x), u_2^2(x)) \in \left(C_0^\infty(B_R)\right)^2. \end{cases} \tag{5.43}$$

By Lemma (5.3.1), problem (5.43) has a unique (weak) solution u_i^R *such that*

$$(u_1^R, u_2^R) \in (C[0,T; D(B_R)])^2 \quad \text{and} \quad ((u_1^R)', (u_2^R)'^l_\rho(B_R)])^2.$$

We extend the solution of the problem (5.43) as

$$(\widetilde{u}_1^R, \widetilde{u}_2^R) =: \begin{cases} (u_1^R, u_2^R), & if\ |x| \leq R, \\ 0, & otherwise. \end{cases} \tag{5.44}$$

The solution (u_1^R, u_2^R) *satisfies the estimates*

$$\begin{aligned} &\|\widetilde{u}_i^R\|_{L^\infty(0,T;D(\mathbb{R}^n))} \leq K, \qquad \|f(\widetilde{u}_i^R)\|_{L^\infty(0,T;\mathcal{D}(\mathbb{R}^n))} \leq K, \\ &\|(\widetilde{u}_i^R)'\|_{L^\infty(0,T;L^l_\rho(\mathbb{R}^n))} \leq K, \quad \|(\widetilde{u}_i^R)''\|_{L^\infty(0,T;D^{-1}(\mathbb{R}^n))} \leq K, \end{aligned} \tag{5.45}$$

where the constant K is independent of R. Estimates (5.45) applied imply that

$$\widetilde{u}_i^R \text{ is relatively compact in } C([0,T]; L^2_\rho(\mathbb{R}^n)). \tag{5.46}$$

Next using relations (5.45) and (5.46), the continuity of the embedding

$$C([0,T]; L^2_\rho(\mathbb{R}^n)) \subset L^2([0,T]; L^2_\rho(\mathbb{R}^n)),$$

and the continuity of f_i we can extract a subsequence of $\widetilde{u}_i^R$, denoted by $\widetilde{u}_i^{R_m}$, such that as $R_m \to \infty$ we get

$$\begin{aligned} \widetilde{u}_i^{R_m} &\overset{*}{\longrightarrow} \widetilde{u}_i \text{ in } L^\infty(0,T;D(B_R)), \\ (\widetilde{u}_i^{R_m})' &\overset{*}{\longrightarrow} u_i' \text{ in } L^\infty(0,T;L^l_\rho(B_R)), \\ (\widetilde{u}_i^{R_m})'' &\overset{*}{\longrightarrow} u_i'' \text{ in } L^\infty(0,T;D^{-1}(B_R)), \\ f(\widetilde{u}_i^{R_m}) &\overset{*}{\longrightarrow} f(\widetilde{u}_i) \text{ in } L^\infty(0,T;\mathcal{D}(B_R)). \end{aligned} \tag{5.47}$$

For fixed $R = R_m$, let L_m denote the operator restriction

$$L_m : [0,T] \times \mathbb{R}^n \to [0,T] \times B_R.$$

It is clear that the restricted subsequence $L_m \widetilde{u}_i^{R_m}$ satisfies the estimates obtained in Lemma 5.3.1. Therefore there exists a subsequence $\widetilde{u}_i^{R_{m_j}} = \widetilde{u}_i^j$ for which it can be shown by following the procedure of Lemma 5.3.1, that $L_m \widetilde{u}_i^j$ converges weakly to a (weak) solution $\widetilde{u}_i^m$. We have

$$\begin{cases} \int_0^T \left(L_m \left(|\widetilde{u}_1^{j'}|^{l-2} \widetilde{u}_1^{j'} \right)', v \right)_{L^l_\rho(B_R)} ds + \int_0^T \left(f_1(L_m \widetilde{u}_1^j, L_m \widetilde{u}_2^j), v \right)_{L^2_\rho(B_R)} ds \\ + \int_0^T \int_{B_R} A^{1/2} L_m \widetilde{u}_1^j A^{1/2} v dx ds - \int_0^T \int_0^t g_1(t-s) \int_{B_R} A^{1/2} \widetilde{u}_1^j A^{1/2} v dx ds \\ + \int_0^T \int_{B_R} A^{1/2} L_m \widetilde{u}_1'^j A^{1/2} v dx ds \\ = \int_0^T \left(\left(|\widetilde{u}_1'^j|^{l-2} \widetilde{u}_1^{j'} \right)', v \right)_{L^l_\rho(\mathbb{R}^n)} ds + \int_0^T \left(f_1(\widetilde{u}_1^j, \widetilde{u}_2^j), v \right)_{L^2_\rho(\mathbb{R}^n)} \\ + \int_0^T \int_{\mathbb{R}^n} A^{1/2} \widetilde{u}_1^j A^{1/2} v dx ds - \int_0^T \int_0^t g_1(t-s) \int_{\mathbb{R}^n} A^{1/2} \widetilde{u}_1^j A^{1/2} v dx ds, \\ \\ \int_0^T \left(L_m \left(|\widetilde{u}_2^{j'}|^{l-2} \widetilde{u}_2^{j'} \right)', v \right)_{L^l_\rho(B_R)} ds + \int_0^T \left(f_2(L_m \widetilde{u}_1^j, L_m \widetilde{u}_2^j), v \right)_{L^2_\rho(B_R)} ds \\ + \int_0^T \int_{B_R} A^{1/2} L_m \widetilde{u}_2^j A^{1/2} v dx ds - \int_0^T \int_0^t g_2(t-s) \int_{B_R} A^{1/2} \widetilde{u}_2^j A^{1/2} v dx ds \\ + \int_0^T \int_{B_R} A^{1/2} L_m \widetilde{u}_2'^j A^{1/2} v dx ds \\ = \int_0^T \left(\left(|\widetilde{u}_2'^j|^{l-2} \widetilde{u}_2^{j'} \right)', v \right)_{L^l_\rho(\mathbb{R}^n)} ds + \int_0^T \left(f_2(\widetilde{u}_1^j, \widetilde{u}_2^j), v \right)_{L^2_\rho(\mathbb{R}^n)} \\ + \int_0^T \int_{\mathbb{R}^n} A^{1/2} \widetilde{u}_2^j A^{1/2} v dx ds - \int_0^T \int_0^t g_2(t-s) \int_{\mathbb{R}^n} A^{1/2} \widetilde{u}_2^j A^{1/2} v dx ds, \end{cases} \tag{5.48}$$

for every $v \in C_0^\infty([0,T] \times B_R)$. Going to the limit in (5.48) as $j \longrightarrow \infty$, we obtain that $L_m \widetilde{u}_i = \widetilde{u}_i^m$. The equality (5.48) holds for any $v \in C_0^\infty([0,T] \times \mathbb{R}^n)$ since the radius R is arbitrarily chosen.

Therefore $\widetilde{u}_i$ is the weak solution of the problem (5.43).

(b) ***Uniqueness.*** *Let us assume that $(u_{11}, u_{21}), (u_{12}, u_{22})$ are two strong solutions of (5.1). Then, $(z_1, z_2) = (u_{11} - u_{12}, u_{21} - u_{22})$ satisfies, for all $w \in D(\mathbb{R}^n)$*

$$\begin{cases} \int_{\mathbb{R}^n} \left(\rho(x) \left(|z_1'^{l-2} z_1'\right)'^{1/2} z_1 A^{1/2} w + \int_0^t g_1(s) A^{1/2} z_1(s-t,x) A^{1/2} w ds \right) dx \\ + \int_{\mathbb{R}^n} \rho(x) f_1(z_1, z_2) w dx + A^{1/2} z_1' A^{1/2} w = 0, \\ \int_{\mathbb{R}^n} \left(\rho(x) \left(|z_2'^{l-2} z_2'\right)'^{1/2} z_2 A^{1/2} w + \int_0^t g_2(s) A^{1/2} z_2(s-t,x) A^{1/2} w ds \right) dx \\ + \int_{\mathbb{R}^n} \rho(x) f_2(z_1, z_2) w dx + A^{1/2} z_2' A^{1/2} w = 0. \end{cases} \tag{5.49}$$

*Substituting $w = z_1'$ in the first equation and $w = z_2'$ in the second equation, adding the resulting equations, integrating by parts, and using (**A1**) yields*

$$\begin{aligned} & \frac{d}{dt} \sum_{i=1}^{2} \left(\frac{l-1}{l} \|z_i'\|_{L_\rho^l}^l + \frac{1}{2} \left(1 - \int_0^t g_i(s) ds \right) \|A^{1/2} z_i\|_2^2 + \frac{1}{2} (g_i \circ A^{1/2} z_i) \right) \\ \leq \; & \int_{\mathbb{R}^n} \left([f_1(u_{21}, u_{22}) + f_1(u_{11}, u_{12})] z_1' + [f_2(u_{21}, u_{22}) + f_2(u_{11}, u_{12})] z_2' \right) dx. \end{aligned}$$

Making use of (5.71) and following similar arguments used for obtaining (5.36), we find

$$\begin{aligned} & \int_{\mathbb{R}^n} \left([f_1(u_{21}, u_{22}) + f_1(u_{11}, u_{12})] z_1' + [f_2(u_{21}, u_{22}) + f_2(u_{11}, u_{12})] z_2' \right) dx \\ \leq \; & k \int_{\mathbb{R}^n} \left(1 + |u_{11}|^{\beta_{11}-1} + |u_{12}|^{\beta_{11}-1} + |u_{21}|^{\beta_{12}-1} + |u_{22}|^{\beta_{12}-1} \right) (|z_1| + |z_2|) z_1' dx \\ + \; & k \int_{B_R} \left(1 + |u_{11}|^{\beta_{21}-1} + |u_{12}|^{\beta_{21}-1} + |u_{21}|^{\beta_{22}-1} + |u_{22}|^{\beta_{22}-1} \right) (|z_1| + |z_2|) z_2' dx, \\ \leq \; & c \sum_{i=1}^{2} \left(\|z_i'\|_{L_\rho^l}^l + \|A^{1/2} z_i\|_2^2 . \right) \end{aligned} \tag{5.50}$$

Combining (5.49)and (5.50), integrating over $(0, t)$, we then deduce that

$$\sum_{i=1}^{2} \left(\|z_i'\|_{L_\rho^l}^l + \|z_i\|_2^2 \right) = 0, \tag{5.51}$$

which means that $(u_{11}, u_{21}) = (u_{12}, u_{22})$. This completes the proof.

We can now state and prove the asymptotic behavior of the solution of (5.1).

We point out that the theory of monotone operators and nonlinear semigroups method is applicable here to establish the existence of a unique local weak solution even with the presence of the nonlinear terms.

5.4 Decay rate for linear cases

We show that the solution decays asymptotically to zero in time. The rate of decay for the solution is similar to that of the memory terms. Allowing for some small perturbation in the associate energy, we introduce the next functional

$$\psi(t) = \sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) u_i |u_i'|^{l-2} u_i' dx. \tag{5.52}$$

The following Lemma will be useful in the proof of our next result.

Lemma 5.4.1 *Under the assumptions (**A1**)and (**A2**), the functional* ψ *satisfies,*

$$\psi'(t) \leq \sum_{i=1}^{2} \|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - (k - 1 - \delta + |\alpha| c) \sum_{i=1}^{2} \|A^{1/2} u_i\|_2^2 + c \sum_{i=1}^{2} (g_i \circ A^{1/2} u_i), \tag{5.53}$$

along the solution of (5.1) for positive constants c.

From (5.52), integrating by parts over $\mathbb{R}^n$*, we have*

$$\begin{aligned}
\psi'(t) &= \int_{\mathbb{R}^n} \rho(x) u_1'^l dx + \int_{\mathbb{R}^n} \rho(x) u_1 \left(|u_1'|^{l-2} u_1' \right)' dx \\
&+ \int_{\mathbb{R}^n} \rho(x) u_2'^l dx + \int_{\mathbb{R}^n} \rho(x) u_2 \left(|u_2'|^{l-2} u_2' \right)' dx, \\
&= \int_{\mathbb{R}^n} \left(\rho(x) u_1'^l - u_1 A u_1 - u_1 A u_1' - \alpha \rho(x) u_1 u_2 + u_1 \int_0^t g_1(t-s) A u_1(s,x) ds \right) dx \\
&+ \int_{\mathbb{R}^n} \left(\rho(x) u_2'^l - u_2 A^2 u_2 - u_2 A u_2' - \alpha \rho(x) u_1 u_2 + u_2 \int_0^t g_2(t-s) A u_2(s,x) ds \right) dx, \\
&= \sum_{i=1}^{2} \|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - \left(1 - \int_0^t g_i(s) ds \right) \sum_{i=1}^{2} \|A^{1/2} u_i\|_2^2 \\
&- \sum_{i=1}^{2} \|A^{1/2} u_i'\|_2^2 - 2\alpha \int_{\mathbb{R}^n} \rho(x) u_1 u_2 dx \\
&+ \sum_{i=1}^{2} \int_{\mathbb{R}^n} A^{1/2} u_i \int_0^t g_i(t-s) (A^{1/2} u_i(s) - A^{1/2} u_i(t)) ds dx.
\end{aligned}$$

Recalling that $\int_0^t g_i(s)ds \leq \int_0^\infty g_i(s)ds = 1 - k_i$, using Young's inequality, Lemma (5.18) and Lemma (5.2.1), we obtain

$$\begin{aligned}
\psi'(t) &\leq \sum_{i=1}^{2} \|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - \sum_{i=1}^{2} \|A^{1/2}u_i'\|_2^2 - (k_i - 1 + |\alpha|\|\rho\|^{-1}_{L^s(\mathbb{R}^n)}) \sum_{i=1}^{2} \|A^{1/2}u_i\|_2^2 \\
&+ \delta \sum_{i=1}^{2} \|A^{1/2}u_i\|_2^2 + \frac{1}{4\delta} \sum_{i=1}^{2} \int_{\mathbb{R}^n} \left(\int_0^t g_i(t-s)|A^{1/2}u_i(s) - A^{1/2}u_i(t)|ds \right)^2 dx, \\
&\leq \sum_{i=1}^{2} \|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - \sum_{i=1}^{2} \|A^{1/2}u_i'\|_2^2 - (k - 1 - \delta + |\alpha|c) \sum_{i=1}^{2} \|A^{1/2}u_i\|_2^2 \\
&+ \frac{(1-k)}{4\delta} \sum_{i=1}^{2} (g_i \circ A^{1/2}u_i).
\end{aligned}$$

for α small enough and $k = \min\{k_1, k_2\}$.

Our main result reads as follows

Theorem 5.4.1 *Let $(u_{10}, u_{11}), (u_{20}, u_{21}) \in D(\mathbb{R}^n) \times L^l_\rho(\mathbb{R}^n)$ and suppose that (**A1**), (**A2**) hold. Then there exist positive constants W, ω such that the energy of solution given by (5.1) satisfies,*

$$E(t) \leq WE(0) \exp\left(-\omega \int_0^t \xi(s)ds \right), \forall t \geq 0. \tag{5.54}$$

In order to prove this theorem, let us define

$$L(t) = N_1 E(t) + \varepsilon\psi(t), \quad \forall \varepsilon > 0. \tag{5.55}$$

for $N_1 > 1$. We need the next lemma, which means that there is an equivalence between the perturbed energy and energy functions.

Lemma 5.4.2 *For $N_1 > 1$, we have*

$$\beta_1 L(t) \leq E(t) \leq L(t)\beta_2, \quad \forall t \geq 0, \tag{5.56}$$

holding for two positive constants β_1 and β_2.

By (5.52) and (5.55), we have

$$\begin{aligned}
|L(t) - N_1 E(t)| &\leq \varepsilon|\psi_1(t)|, \\
&\leq \varepsilon \sum_{i=1}^{2} \int_{\mathbb{R}^n} \left| \rho(x) u_i |u_i'|^{l-2} u_i' \right| dx.
\end{aligned}$$

Thanks to Hölder's and Young's inequalities with respectivels exponents $\frac{l}{l-1}$, l*,and since* $\frac{2n}{n+2} \geq l \geq 2$*, we have using Lemma 5.2.1*

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|\rho(x) u_i |u_i'|^{l-2} u_i'\right| dx &\leq \left(\int_{\mathbb{R}^n} \rho(x)|u_i|^l dx\right)^{1/l} \left(\int_{\mathbb{R}^n} \rho(x)|u_i'|^l dx\right)^{(l-1)/l}, \\
&\leq \frac{1}{l}\left(\int_{\mathbb{R}^n} \rho(x)|u_i|^l dx\right) + \frac{l-1}{l}\left(\int_{\mathbb{R}^n} \rho(x)|u_i'|^l dx\right), \\
&\leq c\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + c\|\rho\|^l_{L^s(\mathbb{R}^n)} \|A^{1/2} u_i\|_2^l. \qquad (5.57)
\end{aligned}$$

Then, since $l \geq 2$*, we have by using (5.4)*

$$\begin{aligned}
|L(t) - N_1 E(t)| &\leq \varepsilon c \sum_{i=1}^{2} \left(\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \|A^{1/2} u_i\|_2^l \right), \\
&\leq \varepsilon c (E(t) + E^{l/2}(t)), \\
&\leq \varepsilon c E(t)(1 + E^{[(l/2)-1]}(t)), \\
&\leq \varepsilon c E(t)(1 + E^{[(l/2)-1]}(0)), \\
&\leq \varepsilon c E(t).
\end{aligned}$$

Consequently, (5.56) follows.

Proof of Theorem 5.4.1 *From (5.4)and results of Lemma 5.4.1, we have*

$$\begin{aligned}
L'(t) &= N_1 E'(t) + \varepsilon \psi'(t), \\
&\leq N_1 \Big(\frac{1}{2} \sum_{i=1}^{2} (g_i' \circ A^{1/2} u_i)(t) - \sum_{i=1}^{2} \|A^{1/2} u_i'\|_2^2 \Big) \\
&+ \varepsilon \sum_{i=1}^{2} \Big(\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - (k - 1 - \delta + |\alpha| c) \|A^{1/2} u_i\|_2^2 + c(g_i \circ A^{1/2} u_i) \Big).
\end{aligned}$$

At this point, we choose N_1 *large enough and* ε *sufficiently small such that*

$$L'(t) \leq M_0 \sum_{i=1}^{2} (g_i \circ A^{1/2} u_i) - \varepsilon E(t), \quad \forall t \geq 0. \qquad (5.58)$$

Multiplying (5.58) by $\xi(t)$ *gives*

$$\xi(t) L'(t) \leq -\varepsilon \xi(t) E(t) + M_0 \xi(t) \sum_{i=1}^{2} (g_i \circ A^{1/2} u_i). \qquad (5.59)$$

*The last term can be estimated using (**A1**), as follows*

$$
\begin{aligned}
\xi(t)\sum_{i=1}^{2}(g_i\circ A^{1/2}u_i) &\leq \sum_{i=1}^{2}\xi_i(t)\int_{\mathbb{R}^n}\int_0^t g_i(t-s)|u_i(t)-u_i(s)|^2dsdx,\\
&\leq \sum_{i=1}^{2}\int_{\mathbb{R}^n}\int_0^t \xi_i(t-s)g_i(t-s)|u_i(t)-u_i(s)|^2dsdx,\\
&\leq -\sum_{i=1}^{2}\int_{\mathbb{R}^n}\int_0^t g_i'(t-s)|u_i(t)-u_i(s)|^2dsdx,\\
&\leq -\sum_{i=1}^{2}(g_i'^{1/2}u_i)\leq -E'(t).
\end{aligned}
\tag{5.60}
$$

Thus, (5.58) becomes

$$
\xi(t)L'(t)+M_0E'(t) \leq -\varepsilon\xi(t)E(t) \quad \forall t\geq 0. \tag{5.61}
$$

Using the fact that ξ *is a nonincreasing continuous function as* ξ_1 *and* ξ_2 *are nonincreasing, and so* ξ *is differentiable, with* $\xi'(t)\leq 0$ *for a.e* t*, then*

$$
(\xi(t)L(t)+M_0E(t))' \leq \xi(t)L'(t)+M_0E'(t)\leq -\varepsilon\xi(t)E(t) \quad \forall t\geq 0. \tag{5.62}
$$

Using (5.56), we have

$$
F=\xi L+M_0E\sim E, \tag{5.63}
$$

Therefore, for some positive constant ω*, we get*

$$
F'(t)\leq -\omega\xi(t)F(t) \quad \forall t\geq 0. \tag{5.64}
$$

Integrating over $(0,t)$ *leads to,*

$$
F(t)\leq WF(0)\exp\left(-\omega\int_0^t\xi(s)ds\right), \forall t\geq 0. \tag{5.65}
$$

for some constant $\omega>0$ *, Recalling (5.63), estimate (5.65) yields the desired result (5.54). This completes the proof of*
Theorem (5.4.1).

5.5 Concluding comments

1- One can easily obtain the same result as in Theorem 5.4.1 in the nonlinear case

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + f_1(u_1,u_2) + \phi(x)A\left(u_1 + \int_0^t g_1(s)u_1(t-s,x)ds + u_1'\right) = 0, \\ \left(|u_2'|^{l-2}u_2'\right)' + f_2(u_1,u_2) + \phi(x)A\left(u_2 + \int_0^t g_2(s)u_2(t-s,x)ds + u_2'\right) = 0, \\ (u_1(0,x), u_2(0,x)) = (u_{10}(x), u_{20}(x)) \in (D(\mathbb{R}^n))^2, \\ (u_1'(0,x), u_2'(0,x)) = (u_{11}(x), u_{21}(x)) \in (L^l_\rho(\mathbb{R}^n))^2, \end{cases} \tag{5.66}$$

where the nonlinearity is given by the functions f_1, f_2 *satisfying the next assumptions:*

(***hyp1***) *The functions* $f_i : \mathbb{R}^2 \to \mathbb{R}$ *(for i=1,2) are of class* C^1 *and there exists a function F such that*

$$f_1(x,y) = \frac{\partial F}{\partial x}, \qquad f_2(x,y) = \frac{\partial F}{\partial y}, \tag{5.67}$$

$$F \geq 0,\ xf_1(x,y) + yf_2(x,y) - F(x,y) \geq 0. \tag{5.68}$$

and

$$\left|\frac{\partial f_i}{\partial x}(x,y)\right| + \left|\frac{\partial f_i}{\partial y}(x,y)\right| \leq d(1 + |x|^{\beta_{i1}-1} + |y|^{\beta_{i2}-1}) \quad \forall (x,y) \in \mathbb{R}, \tag{5.69}$$

for some constant $d > 0$ *and* $1 \leq \beta_{ij} \leq \frac{n}{n-2}$ *for* $i,j = 1,2$.

(***hyp2***) *There exists a positive constant k such that*

$$|f_i(x,y)| \leq k(|x| + |y| + |x|^{\beta_{i1}} + |y|^{\beta_{i2}}), \tag{5.70}$$

and

$$|f_i(x,y) - f_i(r,s)| \\ \leq k(1 + |x|^{\beta_{i1}-1} + |y|^{\beta_{i2}-1} + |r|^{\beta_{i1}-1} + |s|^{\beta_{i2}-1})(|x-r| + |y-s|), \tag{5.71}$$

for all $(x,y),(r,s) \in \mathbb{R}^2$ *and* $i = 1,2$. *Noting that we follow the same steps as a in the linear cases with the same perturbed function and some calculations related with to presence of* f_1 *and* f_2.

Our results also hold for other nonlinearities in $f_1(u_1,u_2)$ *and* $f_2(u_1,u_2)$. *We can show the same results for* $f_1(u_1,u_2) = |u_1|^{p-2}u_1|u_2|^p$ *and* $f_2(u_1,u_2) = |u_2|^{p-2}u_2|u_1|^p$.

2. Let us remark that it is similar to study the question of existence and decay of solution of the same problem with the presence of weak-viscoelasticity in the form

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + f_1(u_1,u_2) + \phi(x)A\left(u_1 + \alpha_1(t)\int_0^t g_1(s)u_1(t-s,x)ds + u_1'\right) = 0, \\ \left(|u_2'|^{l-2}u_2'\right)' + f_2(u_1,u_2) + \phi(x)A\left(u_2 + \alpha_2(t)\int_0^t g_2(s)u_2(t-s,x)ds + u_2'\right) = 0, \\ (u_1(0,x),u_2(0,x)) = (u_{10}(x),u_{20}(x)) \in (D(\mathbb{R}^n))^2, \\ (u_1'(0,x),u_2'(0,x)) = (u_{11}(x),u_{21}(x)) \in (L^l_\rho(\mathbb{R}^n))^2, \end{cases} \tag{5.72}$$

where we should need additional conditions on α as follows

$$1-\alpha_i(t)\int_0^t g_i(t)dt \geq k_i > 0, \int_0^\infty g_i(t)dt < +\infty, \alpha_i(t) > 0, \tag{5.73}$$

$$\lim_{t\to+\infty} \frac{-\alpha'(t)}{\alpha(t)\xi(t)} = 0 \tag{5.74}$$

where

$$\alpha(t) = \min\{\alpha_1(t),\alpha_2(t)\}, \quad \forall t \geq 0.$$

We give below an important technical lemma.

Lemma 5.5.1 *For any $v \in C^1\left(0,T,H^1(\mathbb{R}^n)\right)$, we have*

$$\begin{aligned} & -\int_{\mathbb{R}^n}\alpha(t)\int_0^t g(t-s)Av(s)v'(t)dsdx \\ = & \frac{1}{2}\frac{d}{dt}\alpha(t)\left(g\circ A^{1/2}v\right)(t) \\ - & \frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(s)\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dxds\right] \\ & -\frac{1}{2}\alpha(t)\left(g'^{1/2}v\right)(t) + \frac{1}{2}\alpha(t)g(t)\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dxds \\ & -\frac{1}{2}\alpha'(t)\left(g\circ A^{1/2}v\right)(t) + \frac{1}{2}\alpha'(t)\int_0^t g(s)ds\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dxds. \end{aligned}$$

It's not hard to see that

$$\begin{aligned} & \int_{\mathbb{R}^n}\alpha(t)\int_0^t g(t-s)Av(s)v'(t)dsdx \\ = & \alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} A^{1/2}v'^{1/2}v(s)dxds \\ = & \alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} A^{1/2}v'(t)\left[A^{1/2}v(s) - A^{1/2}v(t)\right]dxds \\ & +\alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} A^{1/2}v'^{1/2}v(t)dxds. \end{aligned}$$

Consequently

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s) Av(s) v'(t) ds dx \\
= \quad & -\frac{1}{2}\alpha(t) \int_0^t g(t-s) \frac{d}{dt} \int_{\mathbb{R}^n} \left| A^{1/2} v(s) - A^{1/2} v(t) \right|^2 dx ds \\
&+ \alpha(t) \int_0^t g(s) \left(\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^n} \left| A^{1/2} v(t) \right|^2 dx \right) ds
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s) Av(s) v'(t) ds dx \\
= \quad & -\frac{1}{2}\frac{d}{dt} \left[\alpha(t) \int_0^t g(t-s) \int_{\mathbb{R}^n} \left| A^{1/2} v(s) - A^{1/2} v(t) \right|^2 dx ds \right] \\
&+ \frac{1}{2}\frac{d}{dt} \left[\alpha(t) \int_0^t g(s) \int_{\mathbb{R}^n} \left| A^{1/2} v(t) \right|^2 dx ds \right] \\
&+ \frac{1}{2}\alpha(t) \int_0^t g'(t-s) \int_{\mathbb{R}^n} \left| A^{1/2} v(s) - A^{1/2} v(t) \right|^2 dx ds \\
&- \frac{1}{2}\alpha(t) g(t) \int_{\mathbb{R}^n} \left| A^{1/2} v(t) \right|^2 dx ds. \\
&+ \frac{1}{2}\alpha'(t) \int_0^t g(t-s) \int_{\mathbb{R}^n} \left| A^{1/2} v(s) - A^{1/2} v(t) \right|^2 dx ds \\
&- \frac{1}{2}\alpha'(t) \int_0^s g(s) ds \int_{\mathbb{R}^n} \left| A^{1/2} v(t) \right|^2 dx ds.
\end{aligned}
$$

This completes the proof.

Under these additional conditions on α*, the decay of energy associated with problem (5.72) is given in the next result*

Theorem 5.5.1 *Let* $(u_{i0}, u_{i1}) \in (D(\mathbb{R}^n) \times L^l_\rho(\mathbb{R}^n)), i = 1, 2$ *and suppose that (**A1**), (**A2**), (5.67)-(5.71) hold. Then there exist positive constants W and* ω *such that the energy of solution given by* (5.72) *satisfies,*

$$
E(t) \leq W E(t_0) \exp\left(-\omega \int_{t_0}^t \alpha(s) \xi(s) ds \right), \tag{5.75}
$$

where $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}, \quad \forall t \geq t_0 \geq 0.$

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