الجمهورية الجزائرية الديمقراطية الشعبية

وزارة التعليم العالي والبحث العلمي

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Course handout

For the use of students of (1st Year of Natural and Life Sciences degree LMD**)**

Mathematics Statistics

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Foreword

This educational material has been specifically designed for first-year university students in Natural and Life Sciences (SNV1). Its objective is to provide a solid foundation in mathematical analysis as well as probability and statistics, two essential tools for understanding biological, environmental, and physiological phenomena.

In scientific studies, mathematics is not merely a theoretical subject; it is a universal language used to model population growth, analyze laboratory experiments, and interpret experimental data. With this in mind, this document was created to bridge theory with biological applications.

The first part of the material focuses on mathematical analysis, covering functions, limits, differentiability, integrals, series, and multivariable functions. Each concept is introduced progressively, with illustrative examples related to the biological field whenever possible.

The second part deals with probability and statistics, which are indispensable for reading scientific results, analyzing experiments, and understanding quantitative studies. Guided exercises are included to help students practice and gradually build their autonomy.

The main goal of this document is to help students approach mathematics not as a challenge, but as a valuable tool in the service of natural sciences.

Contents

Part I

Mathematical analysis

Chapter 1

Function of a real variable

An integral component of the analysis work created in the first year of SNV1 is the study of functions with real values of a real variable, that is, functions defined in $\mathbb{R}$ with values in $\mathbb{R}$. Our objective is to thoroughly develop the aspects specific to these concepts and to make explicit the conclusions that may be drawn from them, considering that many of the notions studied here (such as continuity and differentiability) have already been introduced in the final year of high school. In this chapter, we begin by defining the general notions of real functions, followed by the local concepts of limit and continuity, with particular emphasis on the use of comparison relations.

1 General

1.1 Functions

Definition 1.1

A function f is a mathematical object that associates a real number $f(x)$ with each real number x in a set D_f. The set D_f is called **the domain (of definition) of the function**, x the **variable** and the real number $f(x)$ the **value or image** of f in x.

Example 1

On a rod of length ℓ heated at one of its ends, the temperature at distance x from this end is a real number that we denote $T(x)$. This defines a function T on $[0;\ell]$.

Example 2

To reassure ourselves after this warning, here are some examples of functions given by formulas:$f(x) = x + 1$, $g(x) = x^2$ $h(x) = \frac{1}{x}$, $i(x) = x + \frac{2}{x-1}$.

1.1.1 Ways to Represent a Function

Representation	Description	Example
Formula (*Analytical*)	Given by an equation	$f(x) = 2x + 3$
Table (*Numrical*)	Inputs/Outputs listed	$x : 1 \to 2 \to 3 \to 4;$ $f(x) : 3 \to 5 \to 7 \to 9$
Graphical	Visual on Cartesian plane	Plot of $y = x^2$

1.1.2 Types of Functions

Type	Definition	Example
Constant	Output always the same	$f(x) = 5$
Linear	Degree 1 polynomial	$f(x) = ax + b$
Quadratic	Degree 2	$f(x) = ax^2 + bx + c$
Polynomial	Sum of powers	$f(x) = x^3 - 4x + 1$
Rational	Ratio of polynomials	$f(x) = \frac{1}{x-1}$
Exponential	Variable in exponent	$f(x) = 2^x$
Logarithmic	Inverse of exponential	$f(x) = \ln(x)$
Trigonometric	Based on angles	$f(x) = \sin(x)$

1.2 Definition domain

Definition 1.2

The domain D_f of a function f is made up of real numbers x for which, on the one hand, the value $f(x)$ has meaning and, on the other hand, is of interest in the situation under consideration. The precise determination of D_f therefore depends on the context in which the function f appears, and it is therefore essential to always specify the domain we have chosen; we will then say that we are studying f on D_f.

In simpler terms

$$D_f = \{x \in R / \ \ f(x) \ is \ defined\}$$

Example

For the functions in Example 2,

- $D_f = \mathbb{R}$ (we can always add a real x and 1),
- $D_g = \mathbb{R}$ (we can always multiply a real x with itself, which amounts to calculating x^2),
- $D_h = \mathbb{R} \setminus \{0\}$ (we can only divide by a real x if it's not 0),
- $D_i = \mathbb{R} \setminus \{1\}$ (we can only divide by $x-1$ when $x \neq 1$).

How to Find the Domain : Let $P(x)$ and $Q(x)$ be two functions.

1st*cas* Function of type $f(x) = P(x)/Q(x)$: f is defined for any $Q(x) \neq 0$.

2^{e}*cas* Function of type $f(x) = \sqrt{Q(x)}$: f is defined for all $Q(x) \geq 0$.

3$^{\tau h}$*cas* Function of type $f(x) = P(x)/\sqrt{Q(x)}$: f is defined for all $Q(x) > 0$.

4 th*cas* Function of type $f(x) = \ln(P(x))$: f is defined for all $P(x) > 0$.

Example

Determine the domain of definition of the following functions.

$$f(x) = \frac{x^2 - 2x + 3}{3x^2 - 7x + 2}, \qquad g(x) = \frac{x^3 + 5x}{\sqrt{8 - 3x}}, \qquad h(x) = \ln(x+1)$$

1. For $f(x)$:

$$D_f = \left\{ x \in \mathbb{R} \,\middle|\, 3x^2 - 7x + 2 \neq 0 \right\}.$$

Solve $3x^2 - 7x + 2 = 0$:

$$\Delta = (-7)^2 - 4 \cdot 3 \cdot 2 = 49 - 24 = 25 \Rightarrow \sqrt{\Delta} = 5$$

$$\begin{cases} x_1 = \dfrac{7-5}{6} = \dfrac{1}{3}, \\ x_2 = \dfrac{7+5}{6} = 2 \end{cases} \quad \Rightarrow \quad D_f = \mathbb{R} \setminus \left\{ \frac{1}{3}, 2 \right\}.$$

2. For $g(x)$:

$$D_g = \left\{ x \in \mathbb{R} \,\middle|\, 8 - 3x > 0 \right\} = \left] -\infty, \frac{8}{3} \right[.$$

3. For $h(x)$: The natural logarithm $ln(u)$ is defined only for $u > 0$.

$$D_h = \left\{x \in \mathbb{R} \middle| x + 1 > 0\right\} =]-1, +\infty[.$$

1.3 Graphical representations

Definition 1.3: Graphical Representation of a Function

The graphical representation of a function is the set of all points $(x, f(x))$ in a coordinate plane that show how the input values (x) relate to the output values $f(x)$.

Definition 1.4: Formal Definition

If f is a function defined on a domain $D \subseteq R$, then the graph of f is the set:

$$G_f = \{(x, f(x))/\ x \in D\}$$

This means for every x in the domain, we plot the point whose coordinates are $(x, f(x))$.

Purpose A graph helps to:

- Visualize how the function behaves (increases, decreases, constant, etc.).
- Identify important features: intercepts, maxima, minima, asymptotes, and limits.
- Interpret real-world relationships (for example, how one biological or physical variable depends on another).

Steps to Draw a Graph

- Determine the domain of the function.
- Calculate several values of $f(x)$ for the chosen x.
- Plot the points $(x, f(x))$ in a coordinate system.
- Connect the points smoothly if the function is continuous.

Example

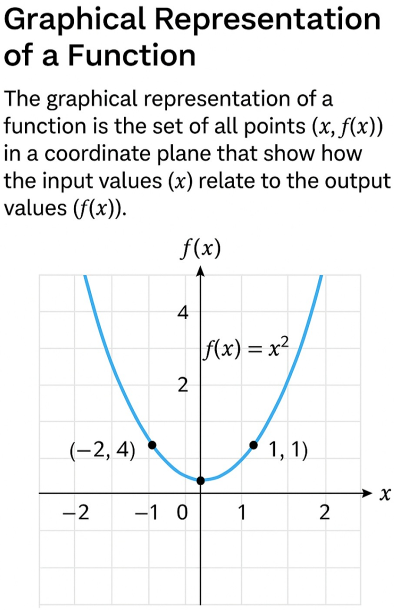
A biologist studies the growth of bacteria over time, modeled by:

$$f(t) = 100e^{0.3t}$$

where t is time (hours) and $f(t)$ is the population size.

Domain: t ≥ 0

Graph: Exponential curve increasing over time.



It shows that as time increases, the population grows rapidly.

1.4 Operations on functions

Given two functions $f(x)$ and $g(x)$, we can create new functions applying operations such as addition, subtraction, multiplication, division, and composition.

1- Addition of Functions

$$\forall x \in D, (f + g)(x) = f(x) + g(x).$$

Example

Let $f(x) = x^2$ and $g(x) = 3x$

$$(f + g)(x) = x^2 + 3x$$

Graphically, the graph of $(f + g)$ is obtained by adding the y-values of both functions at each x.

2- Subtraction of Functions

$$\forall x \in D, (f-g)(x) = f(x) - g(x).$$

Example

$$(f-g)(x) = x^2 - 3x.$$

3- Multiplication of Functions

$$\forall x \in D, (f \times g)(x) = f(x) \times g(x).$$

Example

$$(f \times g)(x) = x^2\,(3x) = 3x^3.$$

4-Division of Functions

$$\forall x \in D, \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, where \;\; g(x) \neq 0.$$

Example

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2}{3x} = \frac{x}{3}, \quad x \neq 0.$$

5- Composition of Functions (Most Important!)

$$\forall x \in D, \; (fog)(x) = f(g(x))$$

Example

$$f(x) = x^2, \quad g(x) = 3x+1$$

Then:

$$(fog)(x) = f(g(x)) = f(3x+1) = (3x+1)^2$$

Order matters!

$$\forall x \in D, \; (fog)(x) \neq (gof)(x)$$

Let's check:

$$(gof)\,(x) = g(x^2) = 3x^2 + 1.$$

1.5 Major, minor and bounded functions

These concepts are used to compare the size or growth of functions over a given interval or as x approaches some value (often ∞).

1.5.1 Major (Dominant) Function

Definition 1.5

A function $f(x)$ is said to be **majorized (or dominated)** by another function $g(x)$ if

$$f(x) \leq g(x) \quad \textit{for all } x \textit{ in a given interval}$$

In this case, we say $g(x)$ is a major function of $f(x)$.

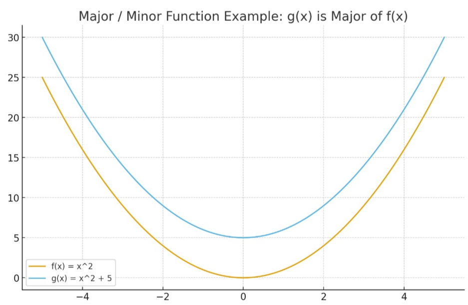
Example

$$f(x) = x^2, \quad g(x) = x^2 + 5$$

Then:

$$f(x) \leq g(x), \qquad \forall x$$

So, $g(x)$ is a major function of $f(x)$.



Therefore, $g(x)$ is a major (dominant) function of $f(x)$.

1.5.2 Minor Function

Definition 1.6

Similarly, a function $f(x)$ is a minor of $g(x)$ if:

$$f(x) \geq g(x) \text{ for all } x$$

This simply reverses the comparison.

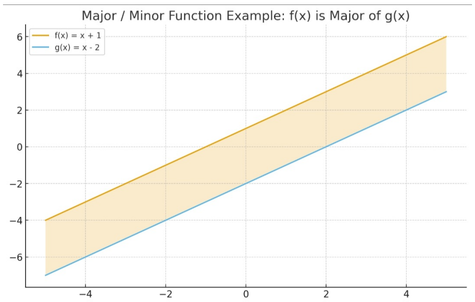
Example

$$f(x) = x + 1, \quad g(x) = x - 2$$

Then:

$$f(x) \geq g(x), \quad \forall x$$

So, $g(x)$ is a minor function of $f(x)$.



Therefore, $g(x)$ is a minor function of $f(x)$.

1.5.3 Bounded Functions

Definition 1.7

A function $f(x)$ is called bounded if it does not grow beyond certain limits.

More formally:

- **Bounded above** if, $\exists M \in \mathbb{R}, \forall x \in D,\ f(x) \leq M$.
- **Bounded below** if , $\exists m \in \mathbb{R}, \forall x \in D,\ f(x) \geq m$.
- **Bounded (both sides)** if, $\exists m, M \in \mathbb{R}, \forall x \in D,\ m \leq f(x) \leq M$.

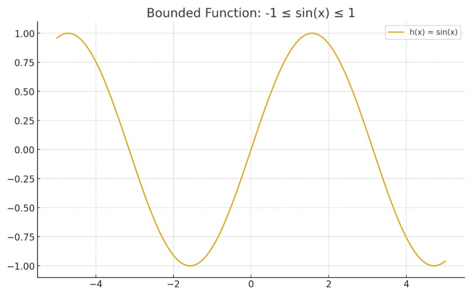
Example

$$f(x) = sin(x)$$

We know:

$$-1 \leq \sin(x) \leq 1$$

So, $\sin(x)$ is a bounded function.



The function $sin(x)$ oscillates between -1 and 1, so it is bounded above and below. This is a classic example of a bounded function.

Example

$$\begin{aligned} f(x) &= x^2 \\ f(x) &\geq 0 \quad \text{(Bounded below)} \end{aligned}$$

But it can grow to infinity, so it's not bounded above.

1.6 Parity and periodicity of Functions

1.6.1 Parity of Functions

Parity refers to whether a function is **even**, **odd**, or **neither.**

1- **Even Function:** A function $f(x)$ is even if:

$$\forall -x \in D \;\; and \;\; \forall x \in D \;\; f(-x) = f(x).$$

Graphical Property: Symmetric with respect to the y-axis.

Example

$Function$	$Check$	$Conclusion$
$f(x) = x^2$	$f(-x) = (-x)^2 = x^2$	$Even$
$f(x) = cos(x)$	$cos(-x) = cos(x)$	$Even$

2- Odd Function: A function $f(x)$ is **odd** if:

$$\forall -x \in D \quad and \;\; \forall x \in D, \quad f(-x) = -f(x).$$

Graphical Property: Symmetric with respect to the **origin**.

Example

$Function$	$Check$	$Conclusion$
$f(x) = x^3$	$f(-x) = (-x)^3 = -x^3$	Odd
$f(x) = \sin(x)$	$\sin(-x) = -\sin(x)$	Odd

Neither Even nor Odd

Some functions do not satisfy either of these conditions.

Example

$f(x) = x + 1$

1.6.2 Periodicity of Functions

Definition 1.8

A function is periodic if it repeats its values at regular intervals.

$$f(x+T) = f(x) \quad for \quad all \quad x$$

Here, T is called the period.

Common Periodic Functions

Function	*Rule*	*Fundamental Period*
$\sin(x)$	$sin(x+2\pi) = \sin(x)$	2π
$cos(x)$	$cos(x+2\pi) = \cos(x)$	2π
$tan(x)$	$tan(x+\pi) = \tan(x)$	π

Example

- The function defined on $\mathbb{R}$ by
$$f(x) = x^{2n}, \qquad (n \in \mathbb{N})$$
is even.
- The function defined on $\mathbb{R}$ by
$$f(x) = x^{2n+1}, \qquad (n \in \mathbb{N})$$
is odd.
- The function $\cos : \mathbb{R} \to \mathbb{R}$ is even. The function $\sin : \mathbb{R} \to \mathbb{R}$ is odd.
- The sine and cosine functions are 2π-periodic. The tangent function is π-periodic.

1.7 Monotone Functions

A monotone function is a function that preserves the order of values it either never increases or never decreases over its domain.

In other words, as x increases, the function $f(x)$ either always goes up, always goes down, or stays constant.

1.7.1 Types of Monotone Functions

1- Monotonically Increasing Function

Definition 1.9

A function $f(x)$ is monotonically increasing on an interval I if:

$$\forall x_1, x_2 \in I, \quad x_1 \leq x_2 \qquad then \qquad f(x_1) \leq f(x_2)$$

That means:

when x increases, $f(x)$ does not decrease.

Example

Let $f(x)$ be a function defined by

$$f(x) = 2x + 1$$

This is increasing because its slope is positive.

Graph: A straight line going upward from left to right.

2. Strictly Increasing Function

Definition 1.10

A function $f(x)$ is strictly increasing if:

$$\forall x_1, x_2 \in I, \quad x_1 < x_2 \qquad then \qquad f(x_1) < f(x_2)$$

So it always rises it never stays flat.

Example

Let $f(x)$ be a function defined by

$$f(x) = e^x$$

3- Monotonically Decreasing Function

Definition 1.11

A function $f(x)$ is monotonically decreasing if:

$$\forall x_1, x_2 \in I, \quad x_1 \leq x_2 \qquad then \qquad f(x_1) \geq f(x_2)$$

That means:

when x increases, $f(x)$ does not increase.

Example

Let $f(x)$ be a function defined by

$$f(x) = -3x + 4$$

Graph: A line going downward from left to right.

4- Strictly Decreasing Function

Definition 1.12

A function $f(x)$ is strictly decreasing if:

$$\forall x_1, x_2 \in I, \quad x_1 < x_2 \quad \textit{then} \quad f(x_1) > f(x_2)$$

Example

The function $f(x) = 2x + 3$ is increasing on R; the function $f(x) = \frac{1}{x}$ is decreasing on $]0, +\infty[$; the function $f(x) = x^2$ is decreasing on $]-\infty, 0]$ and increasing on $[0; +\infty[$ Growth and decrease are easily readable on the graphical representation of the function: when the graph "rises" above a certain interval, this means that an increase in the variable causes an increase in the value of the function, and. Conversely, a curve that "**descends**" indicates that the function is decreasing over this interval.

Example Biological

In biology, monotone functions can describe growth or decay processes:

- **Increasing Function Example:** Population growth:

$$f(t) = 100e^{0.2t}$$

As time t increases, the population size $f(t)$ increases it's strictly increasing.

- **Decreasing Function Example:** Radioactive decay of a biological tracer:

$$f(t) = 200e^{-0.3t}$$

As time increases, the amount of tracer decreases strictly decreasing.

2 Limit of a function

2.1 Limit at a point

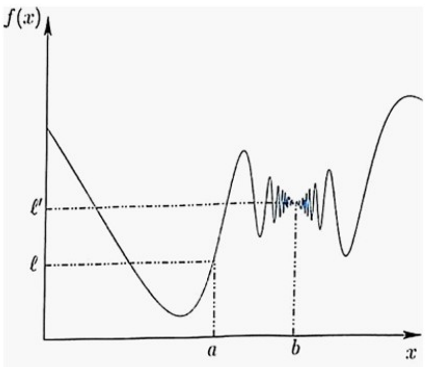
Definition 1.13

When a and ℓ are fixed real numbers, a function f is said to have limit ℓ when x tends towards a if, as x approaches x_0 (remaining in D_f), the value $f(x)$ of f at x approaches ℓ

We'll note

$$\lim_{x \to x_0} f(x) = \ell$$

and we'll also say that f tends to ℓ when x tends to a. An example of a graphical representation of a function with a limit in two real numbers a and b is given in figure 1.2



Definition 1.14

Let $\ell \in R$. We say that ℓ is the limit of f at x_0 if:

$$\forall \varepsilon > 0, \exists \delta > 0,\ \forall x \in I : |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon$$

Remark

1. To say that $f(x)$ tends to ℓ when x tends to x_0 or that ℓ is the limit of the function f when x tends to x_0, means that the values $f(x)$, taken for all x close to x_0, eventually accumulate "around" ℓ.

2. The inequality $|x - x_0| < \delta$ equals $x \in]x_0 - \delta; x_0 + \delta[$, and the inequality $|f(x) - \ell| < \varepsilon$ equals $f(x) \in]\ell - \varepsilon; \ell + \varepsilon[$

3. Some strict inequalities $<$ can be replaced by broad inequalities $\leq$ in the definition

$$\forall \varepsilon > 0, \exists \delta > 0,\ \forall x \in I : |x - x_0| \leq \delta \Rightarrow |f(x) - \ell| \leq \varepsilon$$

4. The order of quantifiers is important, we can't swap the $\forall \varepsilon$ with the $\exists \delta$ that usually depends on ε. To mark this dependence we can write: $\forall \varepsilon > 0, \exists \delta > 0$.

5. The definition of the limit does not allow us to calculate the limit but only to verify that a number is the limit.

Example

Let's show that if $f(x) = 2x - 1$ then $lim_{x \to \frac{1}{2}} f(x) = 0$. We write the definition of the limit with $x_0 = \frac{1}{2}$ and $\ell = 0$.

$$\begin{aligned} \forall \varepsilon &> 0, \exists \delta > 0,\ \forall x \in I : \left|x - \frac{1}{2}\right| < \delta \Rightarrow |f(x) - 0| < \varepsilon \\ |2x - 1| &< \varepsilon \Rightarrow 2\left|x - \frac{1}{2}\right| < \varepsilon \Rightarrow \left|x - \frac{1}{2}\right| < \frac{\varepsilon}{2} \end{aligned}$$

It then suffices to take $\delta = \frac{\varepsilon}{2}$ for condition (3) to be fulfilled.

Example

Let $f(x) = \frac{x^2-1}{x+1}$ verify that $lim_{x \to -1} f(x) = -2$ Indeed:

$$\forall \varepsilon > 0, \exists \delta > 0,\ \forall x \in I : |x + 1| \leq \delta \Rightarrow |f(x) - \ell| = \left|\frac{x^2 - 1}{x + 1} + 2\right| = |x + 1| < \varepsilon$$

just take $\delta = \varepsilon$

2.2 Operations on limits

2.2.1 Proposition

Let f and g be two functions such that $\lim_{x \to x_0} f(x) = \ell$ and $\lim_{x \to x_0} g(x) = l'$ (with ℓ and ℓ' real numbers , or $+\infty$ or -∞), then

1. $\lim_{x \to x_0} (\lambda f(x)) = \lambda \ell$ for all $\lambda \in R$.

2. $\lim_{x \to x_0} (f(x) + g(x)) = \ell + \ell'$.

3. $\lim_{x \to x_0} (f(x) - g(x)) = \ell - \ell'$.

4. If $\ell \neq 0$, then $\lim_{x\to x_0}(f(x)g(x)) = \ell\ell'$.

5. If $\ell' \neq 0$,then $\lim_{x\to x_0}\frac{1}{g(x)} = \frac{1}{\ell'}$.

6. $\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{\ell}{\ell'}$ (Note that the quotient only makes sense if $g(x)$ doesn't cancel out as x approaches a),

7. If $\lim_{x\to x_0} f(x) = \pm\infty$, then $\lim_{x\to x_0}\frac{1}{f(x)} = 0$.

2.2.2 Indeterminate shapes

There are situations where nothing can be said about the limits. For example, if $\lim\limits_{x\to x_0} f(x) = +\infty$, $\lim\limits_{x\to x_0} g(x) = -\infty$. then a priori nothing can be said about the limit of $f+g$. We then say that $+\infty-\infty$ is an indeterminate form. Other indeterminate forms are:

$$\infty-\infty;\quad 0\times\infty,\ \frac{0}{0},\ \frac{\infty}{\infty}, 1^{\infty}, \infty^{0}$$

Here is a very important proposition that means we can go to the limit in a wide inequality.

2.2.3 Proposition

If $f(x) \leq g(x),\ \forall x$ and $\lim_{x\to x_0} f(x) = \ell$ *and* $\lim_{x\to x_0} g(x) = \ell'$ then $\ell \leq \ell'$

If $f(x) \geq g(x),\ \forall x$ and $\lim_{x\to x_0} g(x) = +\infty$ *and* $\lim_{x\to x_0} f(x) = +\infty$

2.2.4 Proposition

Let f and g be two functions such that $\lim_{x\to a} f(x) = \ell$ with $\ell \neq 0$, from $\lim_{x\to a} g(x) = 0$ and $g(x) > 0$ for any x near but not equal to a. Then

1. If $\ell > 0$, $\lim_{x\to a}\frac{f(x)}{g(x)} = +\infty$;

2. If $\ell < 0$, $\lim_{x\to a}\frac{f(x)}{g(x)} = -\infty$;

If, on the other hand, $g(x) < 0$ for any x close to a, we have :

1. If $\ell > 0$, $\lim_{x\to a}\frac{f(x)}{g(x)} = -\infty$;

2. If $\ell < 0$, $\lim_{x\to a}\frac{f(x)}{g(x)} = +\infty$;

We have the limits of the usual functions :

1. $\lim_{x\to+\infty} x^\alpha = +\infty$ if $\alpha > 0$ and $\lim_{x\to+\infty} x^\alpha = +\infty$ if $\alpha < 0$.
2. $\lim_{x\to+\infty} e^x = +\infty$ and $\lim_{x\to-\infty} e^x = 0$.
3. $\lim_{x\to+\infty} \ln(x) = +\infty$ and $\lim_{x\to 0, x>0} \ln(x) = -\infty$.
4. $\lim_{x\to+\infty} \frac{e^x}{x^\alpha} = +\infty$ and $\lim_{x\to-\infty} |x|^\alpha e^x = 0$.
5. $\lim_{x\to+\infty} \frac{\ln(x)}{x^\alpha} = 0$ and $\lim_{x\to 0, x>0} x^\alpha \ln(x) = 0$.

3 Continuous functions

A continuous function is a function whose graph can be drawn without lifting your pencil from the paper.

In mathematical terms, it means there are no breaks, jumps, or holes in the function's graph.

3.1 Continuity at a point

In this paragraph, we consider I to be a non-empty interval of R.

Definition 1.15

Let f be a function defined on I, and let $x_0 \in I$.

We say that f is continuous in x_0 if

$$\lim_{x\to x_0} f(x) = f(x_0)$$

We say that f is continuous on I if f is continuous at any point on I. The set of continuous functions on I.

A function is continuous over an interval if its graph can be drawn "**without lifting a pencil**", i.e. if its curve admits no jumps.

A function that is not continuous at a point is said to be discontinuous at that point.

3.2 Continuity on the left, continuity on the right

Definition 1.16: Continuity on the right

A function f is said to be continuous on the right at a point x_0 if:

$$\lim_{x\to x_0^+} f(x) = f(x_0)$$

Definition 1.17: Continuity on the left

A function f is said to be left-continuous at a point x_0 if:

$$\lim_{x \to x_0^-} f(x) = f(x_0)$$

Remark

Obviously, the continuity of f to the right and left in x_0 entails the continuity of f in x_0

3.3 Different forms of discontinuity

A function that is not continuous at a point is said to be discontinuous at that point.

Form 1: f is discontinuous through all:

The function $f : R \to R$ defined by:

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in Q \\ 1 & \text{if} \quad x \notin Q \end{cases}$$

is discontinuous at any point $x_0 \in R$.

Form 2 : $f(x) \to_{x \to x_0} +\infty$ or $f(x) \to_{x \to x_0} +\infty$

The function $f : R \to R$ defined by:

$$f(x) = \begin{cases} \frac{1}{x} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

is discontinuous in $x_0 = 0$ as $\lim_{x \to 0^+} f(x) = +\infty$ and $\lim_{x \to 0^-} f(x) = -\infty$

Form 3 : $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$ but they are different from $f(x_0)$

The function f :$R \to R$ defined by:

$$f(x) = \begin{cases} 1 & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is discontinuous in $x_0 = 0$ as $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = 1 \neq f(0) = 0$

Form 4: $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$ but they are different from $f(x_0)$

The function f :$R \to R$ defined by:

$$f(x) = \begin{cases} 1 & \text{si} \quad x < 0 \\ 2 & \text{si} \quad x > 0 \\ 0 & \text{si} \quad x = 0 \end{cases}$$

We have $\lim_{x \to x_0^-} f(x) = 1 \neq \lim_{x \to x_0^+} f(x) = 2$. 2: The jump from f to point $x_0 = 0$ est égal à 1.

3.4 Operations on continuous functions

Theorem 1.1

Let $f, g : I \to R$ be two functions continuous at a point $x_0 \in I$, and $\lambda \in R$. Then the functions $f + g$, λf, $f.g$ and $|f|$ are continuous at x_0. If $g(x) = 0$ then the functions $\frac{1}{g}$and $\frac{f}{g}$ are also continuous in x_0

Proof

The proof is a direct consequence of the properties of the limits of functions. Suppose, for example, that f and g are continuous at the point $x_0 \in I$, So

$$\lim_{x \to x_0} f(x) = f(x_0) \text{ and } \lim_{x \to x_0} g(x) = g(x_0)$$

as

$$\lim_{x \to x_0} (f+g)(x) = f(x_0) + g(x_0) = (f+g)(x), \text{ so } f+g \text{ is continuous at } x_0$$

The same proof for λf, $f \times g$, $\frac{f}{g}$ and $|f|$.

■

Example

The previous theorem allows us to verify that other common functions are continuous:

- Power functions $x \to x^n$ on $\mathbb{R}$ (as a product $x \times x \times x$.....),
- Polynomials on $\mathbb{R}$ (sum and product of power functions and constant functions),
- Rational fractions $x \to \frac{P(x)}{Q(x)}$ on any interval where the polynomial $Q(x)$ does not vanish.

Composition preserves continuity (but care must be taken at which points the assumptions apply).

Theorem 1.2

Let $f : I \to R$ and $g : J \to R$ be two functions such that $f(I) \subset J$ .If f is continuous at a point $x_0 \in I$ and if g is continuous at $f(x_0)$, then gof is continuous at x_0.

3.5 Continuous functions on a closed interval

Definition 1.18

Let $f : I \subset R \to R$ be a function. Then

- f is said to be major over I if the set $\{f(x), x \in I\}$ is major.

- f is said to be minor on I , if the set $\{f(x), x \in I\}$ is minor.

- f is bounded on I , if f is both major and minor on I.

- If f is major on I and the set $\{f(x), x \in I\}$ is non-empty, by defnition

$$supf = sup\{f(x), x \in I.\}$$

- If f is minor on I and the set $\{f(x), x \in I\}$ is non-empty, by definition

$$\inf f = \inf\{f(x), x \in I\}$$

- If there exists $x_0 \in I$ such that $sup\{f(x), x \in I\} = f(x_0)$, then $\sup f$ is called the maximum value of f and x_0 is called the maximum or maximum solution of f on I.

Similarly if there exists $x_1 \in I$ such that $\inf\{f(x), x \in I\} = f(x_1)$, then $\inf f$ is called the minimum value of f and x_1 is called the minimum or minimum solution of f on I.

Theorem 1.3

Let $I = [a, b]$ be a fermed and bounded interval and $f : [a, b] \to R$ continuous. Then f is bounded on $[a, b]$.

Theorem 1.4

Any continuous function on a closed and bounded interval $[a, b]$, reaches its bounds at least once, in other words, there exist $x_1, x_2 \in [a; b]$ such that :

$$x_1 = \sup_{x\in[a,b]} f(x) = \max_{x\in[a,b]} f(x), \qquad x_2 = \inf_{x\in[a,b]} f(x) = \min_{x\in[a,b]} f(x)$$

Theorem 1.5

Let $f : [a, b] \to R$ be a continuous function on a segment. If $f(b).f(a) < 0$ then there exists $c \in]a, b[$such that $f(c) = 0$.

The previous theorem can be used to frame the solutions of an equation.

Example

Consider, for example, the equation

$$f(x) = x^5 - 3x + 1 = 0$$

We have $f(0) = 1$ and $f(1) = -1$. The continuity of f allows us to conclude that one or more solutions belonging to $]0, 1[$ are eixtent. Substitution values would reduce this interval.

Exercise: Let $P(x) = x^5 - 3x - 2$ and $f(x) = x2^x - 1$ be two functions defined on R. Show that the equation $P(x) = 0$ has a root in $[1, 2]$, the equation $f(x) = 0$ has at least one root in $[0, 1]$.

Theorem 1.6

Let $f : [a; b] \to R$ be a continuous function. Then for any real y between $f(a)$ and $f(b)$, there exists $c \in]a; b[$ such that $f(c) = y$.

Remark

If I is not closed, the interval $f(I)$ is not necessarily of the same nature as I. For example, the image of the open interval $]-1, 1[$ by $x \to x^2$ is the semi-open interval $[0, 1[$ and the image by $x \to \sin x$ of the open interval $]-\pi, \pi[$ is the closed interval $[-1, 1]$.

4 Reciprocal function

Theorem 1.7

Let $f : [a, b] \to R$ be a continuous, strictly increasing function. Then f is a bijective application of $[a, b]$ on $[f(a), f(b)]$. The reciprocal application $f^{-1} : [f(a), f(b)] \to [a, b]$ is continuous and strictly increasing.

Theorem 1.8

Let $f : [a, b] \to R$ be a continuous, strictly decreasing function. Then f is a bijective application from $[a, b]$ onto $[f(a), f(b)]$. The reciprocal application $f - ^1 : [f(a), f(b)] \to [a, b]$ is continuous and strictly decreasing.

4.1 Graph of f^{-1}

The graph of f is the set of points $(x, f(x))$ of R^2 for x in $[a, b]$. The graph of g is the set of points $(y; (f^{-1}(y))$ for y in $[f(a), f(b)]$ or, what amounts to the same thing, the set of points$(f(x), x)$ for x in $[a, b]$. The points $(x, f(x))$ and $(f(x), x)$ are symmetrical with respect to the first bisector (if we have taken an orthonormal basis). Both graphs are therefore symmetrical with respect to the first bisector.

4.2 Extensions

Results analogous to the previous theorems can be obtained when starting from a strictly monotone continuous function on an interval that is not necessarily closed and not necessarily bounded.

> ***Example***
>
> For example Let $f : [a, b[\to R$ be a continuous and strictly increasing function. It has a limit, finite or not, when x tends to b by lower values. Assume that $lim_{x \to b^-} f(x) = +\infty$. Then f is a bijective application from $[a, b[$ to $[f(a), +\infty[$ and f^{-1} is continuous and strictly increasing.

5 Derivative functions

Having defined the notion of continuity, in this section we'll study the notion of derivation and establish some fundamental results concerning the study of real functions.

5.1 Derivative at a point, derivative function

Let I be an open interval of R and $f : I \to R$ a function. Let $x_0 \in I$.

Definition 1.19

f is derivable at x_0 if the quantity $\frac{f(x)-f(x_0)}{x-x_0}$ admits a finite limit at x_0. The limit is then called the derivative number of f at x_0 and is denoted $f'(x_0)$. Thus

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Definition 1.20

f is derivable on I if f is derivable at any point $x_0 \in I$. The function $x \to f^{'}(x)$ is the derivative function of f, it is denoted $f^{'}$ or $\frac{df}{dx}$.

Example

1) If f is constant, then $f'(x) = 0, \forall x \in R$..

2) The function f defined by $f(x) = \sqrt{x}$ is derivable in $x_0 = 1$. Indeed.:

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}$$

3) The function defined by $f(x) = x^2$ is derivable at any point $x_0 \in R$. Indeed:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{(x + x_0)(x - x_0)}{x - x_0} = \lim_{x \to x_0} x + x_0 \to 2x_0$$

We've even shown that the derivative number of f at x_0 is $2x_0$, in other words:

$$f'(x) = 2x$$

5.2 Right derivative, left derivative

Definition 1.21

If the quantity $\frac{f(x)-f(x_0)}{x-x_0}$ admits a limit to the right of x_0, f is said to be derivable to the right of x_0 and is denoted $f'_d(x_0)$ or $f'_{x_0^+}$, If the quantity $\frac{f(x)-f(x_0)}{x-x_0}$ admits a limit to the left of x_0, we say that f is derivable to the left of x_0 and note $f'_g(x_0)$ or $f'_{x_0^-}$ In other words

$$f'_d(x_0) = \lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}, \qquad f'_g(x_0) = \lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}$$

Example

The function $x \to |x|$ defined on R has at the point $x_0 = 0$, a right derivative equal to $+1$ and a left derivative equal to 1. Indeed,

If $x > 0$, then $f(x) = x$, so $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = 1$.

If $x < 0$, then $f(x) = -x$, so $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = -1$.

The function $f(x) = |x|$ is therefore not derivable at the 0 point.

Remark

The definition of the derivative can be extended by considering infinite limits.If $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = +\infty$ $(ou$ -$\infty)$,we'll say that f has a derivative equal to $+\infty$ (ou -∞). On the graph of f, this leads to the existence of a vertical tangent at $(x_0, f(x_0))$.

Theorem 1.9: (Link to continuity)

Let f be a function derivable at a point x_0, then f is continuous at that point.

5.3 Calculating derivatives

Theorem 1.10

Say $f, g : I \to R$ are two functions derivable on I. Then for all $x \in I$, we have :

- $(f+g)'(x) = f'(x) + g'(x)$.
- $(\lambda f)'(x) = \lambda f'(x)$ or λ *est un réel fixé.*
- $(f \times g)'(x) = f'(x)g(x) + f(x)g'(x)$.

- $\left(\frac{1}{f}\right)'(x) = \frac{f'(x)}{f^2(x)}$, $(If\ f(x) \neq 0)$.
- $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{f^2(x)}$, $(If\ g(x) \neq 0)$..

5.4 Derivatives of common functions

The table on the left is a summary of the main formulas you need to know, where x is a variable. The table on the right is that of compositions (see next paragraph), u represents a function $x \to u(x)$

$Function$	$Derivative$	$Function$	$Derivative$
x^n	$nx^{n-1}, n \in Z$	u^n	$nu'u^{n-1}, n \in Z$
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\frac{1}{u}$	-$\frac{u'}{u^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\sqrt{u}$	$\frac{u'}{2\sqrt{u}}$
x^α	$\alpha x^{\alpha-1}; \alpha \in R$	u^α	$\alpha u'u^{\alpha-1}; \alpha \in R$
e^x	e^x	e^u	e^u
$ln(x)$	$\frac{1}{x}$	$ln(u)$	$\frac{u'}{u}$
$cos(x)$	$-sin(x)$	$cos(u)$	$-u'\sin(u)$
$sin(x)$	$cos(x)$	$sin(u)$	$u'\cos(u)$
$tan(x)$	$1+tan^2(x) = \frac{1}{\cos^2(x)}$	$tan(u)$	$\frac{u'}{\cos^2(u)}$

5.5 Derivative of a compound function

Theorem 1.11

If f is derivable in x and g is derivable in $f(x)$, then gof is derivable in x and we have

$$(gof)'(x) = f'(x)g'(f(x))$$

Example

Let's calculate the derivative of $ln(1+x^2)$. We have $g(x) = ln(x)$ with $g\prime(x) = \frac{1}{x}$ and $f(x) = 1+x^2$ with $f\prime(x) = 2x$. Then

$$(ln\left(1+x^2\right))' = (gof)'(x) = f'(x)g'(f(x)) = 2xg'(1+x2) = \frac{2x}{1+x^2}$$

5.6 $n^{\text{ième}}$ derivatives of a function

Let $f : I \to R$ be a differentiable function and let $f\prime$ be its derivative. If the function $f' : I \to R$ is also derivable we note $f'' = (f')'$ the second derivative of f. More generally we note: $f^{(0)} = f; f^{(1)} = f'; f^{(2)} = f''$ and $f^{(n+1)} = \left(f^{(n)}\right)'$,If the derivative of order n $f^{(n)}$ exists we say that f is n times derivable.

6 Rolle's theorem

Definition 1.22

Rolle's Theorem is a fundamental result in differential calculus that describes the behavior of a function on a closed interval when it starts and ends at the same value.

6.1 Statement of the Theorem

Let f be a real-valued function defined on a closed interval $[a, b]$.

Then Rolle's Theoremstates:

If the function f satisfies the following three conditions:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)
3. $f(a) = f(b)$

Then there exists at least one point $c \in (a, b)$ such that:

$$f'(c) = 0$$

6.2 Geometric Interpretation

On the graph of $f(x)$:

- If the curve starts and ends at the same height (i.e.,$f(a) = f(b)$),

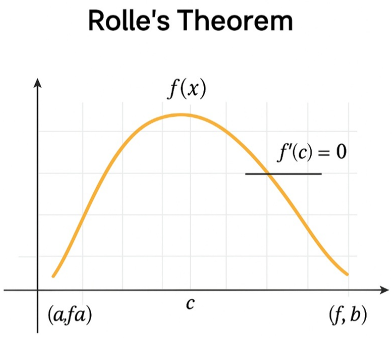
- Then there must be at least one point between a and b

- Where the tangent is horizontal, meaning $f'(c) = 0$.

Example

Let:

$$f(x) = x^2 - 4x + 3$$



On the interval $[1, 3]$.

1. $f(1) = 1^2 - 4(1) + 3 = 0$

$f(3) = 3^2 - 4(3) + 3 = 0$

$$f(1) = f(3)$$

2. $f(x)$ is a polynomial $\rightarrow$ continuous and differentiable everywhere.

By Rolle's Theorem, there exists $c \in (1, 3)$ such that:

$$f'(c) = 0$$

Compute derivative:

$$f'(x) = 2x - 4$$

$$f'(c) = 0 \Rightarrow 2c - 4 = 0 \Rightarrow c = 2$$

So, at $x = 2$, the tangent is horizontal.

Graphical Interpretation

* The parabola $f(x) = x^2 - 4x + 3$ touches the x-axis at $x = 1$ and $x = 3$.

* The tangent line at $x = 2$ is horizontal because $f'(2) = 0$.

6.3 Biological Example

In biology, Rolle's Theorem can describe points where a process momentarily stops changing.

Example

If a species' population $f(t)$ starts and ends at the same level during a time period $[a, b]$,

> then at some point (c), the rate of change of population (growth rate) is zero meaning the population was momentarily stable.
>
> $$f'(c) = 0 \Rightarrow \text{ no growth or decline at that moment.}$$

7 Theorem of finite increments

The Theorem of Finite Increments also known as the Mean Value Theorem (MVT) is one of the most important results in differential calculus. It generalizes Rolle's Theorem and establishes a connection between the average rate of change of a function and its instantaneous rate of change (derivative).

7.1 Statement of the Theorem

Let f be a real-valued function defined on the closed interval $[a, b]$.

Then there exists at least one point $c \in (a, b)$ such that:

$$f(b) - f(a) = (b - a)f'(c) \quad \text{or even} \quad \frac{f(b) - f(a)}{b - a} = f'(c)$$

7.2 Conditions

For the theorem to apply, f must satisfy:

1. f is continuous on the closed interval $[a, b]$;
2. f is differentiable on the open interval (a, b).

7.3 Interpretation

The theorem says that somewhere between a and b,

the instantaneous slope of the function (its derivative)

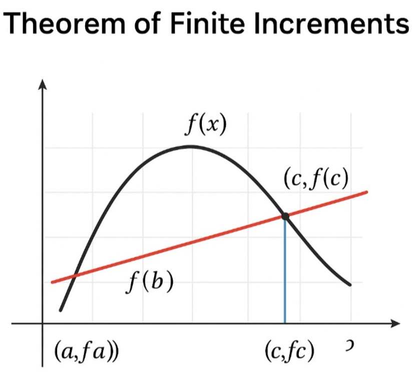
equals the average slope between the endpoints.

Visually:

* The secant line connecting $(a, f(a))$ and $(b, f(b))$ has slope

$$\frac{f(b) - f(a)}{b - a}.$$

* There is at least one point (c) where the tangent line to the curve is parallel to this secant line.



Example

Let:

$$f(x) = x^2 \qquad \text{on} \quad [1, 3]$$

1. $f(1) = 1^2 = 1$, $f(3) = 9$.
2. Average slope:

$$\frac{f(b) - f(a)}{b - a}. = \frac{9 - 1}{2} = 4.$$

3. Derivative:

$$f'(x) = 2x$$

Set

$$f\prime(c) = 4 \Rightarrow 2c = 4 \Rightarrow c = 2$$

Therefore, at $x = 2$, the tangent line has slope 4,

7.4 Geometric Meaning

The theorem guarantees that for any smooth curve between two points, there is at least one point where the tangent is parallel to the line joining those two points.

7.5 Biological Example

In biology, the theorem can model rates of change in time-dependent processes.

Example

Let $f(t)$ be the population of bacteria at time t

If the population increases from 500 to 900 in 4 hours:

$$\frac{f(b) - f(a)}{b - a}. = \frac{900 - 500}{4} = 100.$$

Then, by the theorem,

there exists a moment $t = c$ in $(0, 4)$ where

$$f'(c) = 100$$

meaning the instantaneous growth rate of the bacteria was 100 per hour at that moment.

8 Hopital rule

Definition 1.23

L'Hôpital's Rule is a powerful tool in calculus used to evaluate limits that produce indeterminate forms such as:

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}$$

8.1 Statement of the Rule

Let$f(x)$ and $g(x)$ be two functions differentiable on an open interval containing a point a, except possibly at a itself.

If:

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a} f(x) = \pm\infty$$

and $g(x) \neq 0$ near a, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (finite or infinite).

8.2 Conditions

1. f and g are differentiable on an open interval containing a;
2. $g'(x) \neq 0$ near a;
3. The limit produces an indeterminate form

$$\frac{0}{0} \; or \; \frac{\infty}{\infty}$$

Example

Let's calculate $\lim_{x \to 0} \frac{\ln(1+x)}{x} = \frac{0}{0}$ (indeterminate form). We have

$$f(x) = \ln(1 + x), f'(x) = \frac{1}{1 + x},$$

$$g(x) = x \Longrightarrow g'(x) = 1 \Longrightarrow g'(0) = 1 \neq 0$$

Apply L'hopital's Rule:

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = \frac{f'(0)}{g'(0)} = \frac{1}{1} = 1.$$

Example

Calculate the limit in 1 of $\frac{\ln(x^2+x-1)}{\ln(x)}$.

Check that:

$$\begin{aligned} f(x) &= \ln(x^2+x-1), \quad f(1)=0, \\ f'(x) &= \frac{2x+1}{x^2+x-1} \end{aligned}$$

$$g(x) = \ln(x), \quad g(1) = 0, \quad g'(x) = \frac{1}{x}$$

Take $I =]0,1]$, $x_0 = 1$, then g' does not cancel on $I - \{x_0\}$.
$\lim_{x\to 1} \frac{f'(x)}{g'(x)} = \lim_{x\to 1} \frac{2x^2+x}{x^2+x-1} = 3$, so $\lim_{x\to 1} \frac{f(x)}{g(x)} = 3$.

8.3 Biological Example

Let $f(t)$ represent the growth of bacteria over time, and $g(t)$ represent the available nutrient concentration.

If both $f(t)$ and $g(t)$ tend to infinity over time,

Then L'Hôpital's Rule helps find the limiting ratio:

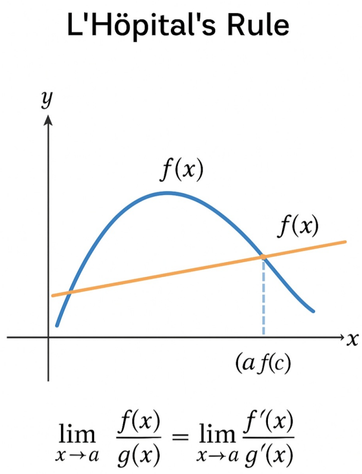
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

→ which shows how growth rate compares to nutrient consumption rate.

8.4 Geometric Meaning

The rule compares the slopes of $f(x)$ and $g(x)$ near the point where their ratio is indeterminate.

It effectively finds how fast each function grows or approaches zero.



9 Exercises

Exercise 1: Determine the domain of definition of the following functions:

$$f(x) = 2x^2 + x + 1, \quad g(x) = \frac{2x+1}{x^2+x-2}, \quad h(x) = \ln\left(\frac{x+1}{1-x}\right)$$

Exercise 2: Calculate the following limits

$$\lim_{x\to 1} \frac{x^2-1}{x+1}, \quad \lim_{x\to 2} x+2-\sqrt{x^2+4}, \quad \lim_{x\to 0} \frac{\sin 5x}{3x}, \quad \lim_{x\to 2} \frac{x^3-8}{x^2-4}$$

Exercise 3: Study the continuity of the following functions at the indicated points:

1. $f(x) = \begin{cases} \frac{x^3-8}{x^2-4} & x \neq 2 \\ 3 & x = 2 \end{cases}$
2. $f(x) = |4x-5|$ sur $\mathbb{R}$

Exercise 4: Determine the real numbers a and b so that the function:

$$f(x) = \begin{cases} 2a\ln(e+x)+b & \text{si} \quad x<0 \\ 2 & si \quad x=0 \\ a\cos x - b(x+1)^3 & \text{si } x>2 \end{cases} \qquad \text{Either continue on } \mathbb{R}$$

Exercise 5

Let f be the function defined on $\mathbb{R}$ by :

$$f(x) = 4x^3 - 3x - \frac{1}{2}$$

.

1) Calculate $f(-1)$, $f(\frac{-1}{2})$, $f(0)$ and $f(1)$

2) Justify that the equation $f(x) = 0$ admits 3 distict solutions included between -1 and 1 .

Exercise 6

The function $f(x) = x^2 + \sqrt{x^2}$ is it derivable at the point $x_0 = 0$?

Exercise 7:

Calculate the derivations of the following functions:

$$f(x) = e^{3x}\cos 4x, \quad g(x) = x^{2x}, \quad h(x) = \ln\left(\frac{x+1}{4-x}\right), \quad k(x) = \sqrt{x^4+2x} + 2\ln(x)$$

Exercise 8:

By using the rule of Hospital , calculate the following limits :

$$1. \lim_{x\to 0}\frac{\sin 2x}{\sin 3x}, \quad 2. \lim_{x\to\frac{\pi}{4}}\frac{\sin x - \cos x}{1-\tan x}, \quad 3. \lim_{x\to 0^+}(\cos x)^{\frac{1}{x^2}}$$

10 The answers

Exercise 1 : Determine the Domain of Definition

The domain of definition (D) is the set of all real numbers for which the function is defined.

1. $f(x) = 2x^2 + x + 1$

This is a polynomial function. Polynomials are defined for all real numbers.

$$D_f = \mathbb{R}$$

2. $g(x) = \frac{2x+1}{x^2+x-2}$

This is a rational function. It's defined everywhere except where the denominator is zero.

We must solve $x^2 + x - 2 = 0$.

The roots can be found by factoring: $(x+2)(x-1) = 0$.

The values that make the denominator zero are $x = -2$ and $x = 1$.

$$D_g = \mathbb{R}\setminus\{-2, 1\}$$

3. $h(x) = \ln\left(\frac{x+1}{1-x}\right)$

This function involves a logarithm (ln). The argument of the logarithm must be strictly positive.

We require $\frac{x+1}{1-x} > 0$. We analyze the signs of the numerator and denominator:

$Interval$	$x+1$	$1-x$	$\frac{x+1}{1-x}$
$x < -1$	$-$	$+$	$-$
$-1 < x < 1$	$+$	$+$	$+$
$x > 1$	$+$	$-$	$-$

The expression is positive when $-1 < x < 1$

$$D_h =]-1, 1[$$

Exercise 2:

Calculate the Following Limits

1.$\lim_{x\to 1} \frac{x^2-1}{x+1}$

This limit is not indeterminate at x=1 since the denominator is not zero. We can substitute the value directly.

$$\lim_{x\to 1} \frac{x^2-1}{x+1} = \frac{1^2-1}{2} = 0$$

2. $\lim_{x\to 1}(x + 2 - \sqrt{x^2+4})$

This limit is also not indeterminate at $x = 2$. We can substitute the value directly.

$$\lim_{x\to 2}(x + 2 - \sqrt{x^2+4}) = 4 - \sqrt{8}$$

3.$\lim_{x\to 0} \frac{\sin 5x}{3x}$

This limit is in the indeterminate form $\frac{0}{0}$. We use the notable trigonometric limit $\lim_{x\to 0} \frac{\sin u}{u} = 1$

$$\lim_{x\to 0} \frac{\sin 5x}{3x} = \lim_{x\to 0} \frac{\sin 5x}{5x}.\frac{5}{3} = 1.\frac{5}{3} = \frac{5}{3}$$

4.$\lim_{x\to 2} \frac{x^3-8}{x^2-4}$.

his limit is in the indeterminate form $\frac{0}{0}$ We factor the numerator (difference of cubes) and the denominator (difference of squares).

$$\begin{aligned} x^3 - 8 &= (x-2)(x^2+2x+4) \\ x^2 - 4 &= (x-2)(x+2) \end{aligned}$$

So,

$$\begin{aligned} \lim_{x\to 2} \frac{x^3-8}{x^2-4} &= \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} \\ &= \lim_{x\to 2} \frac{x^2+2x+4}{x+2} = \frac{4+4+4}{4} = 3 \end{aligned}$$

Exercise 3: Study the Continuity

A function $f(x)$ is continuous at a point c if $\lim_{x\to c} f(x) = f(c)$

1. $f(x) = \begin{cases} \frac{x^3-8}{x^2-4} & x \neq 2 \\ 3 & x = 2 \end{cases}$ at $x = 2$

We check the three conditions for continuity:

- $f(2)$ is defined: $f(2) = 3$.
- $\lim_{x\to 2} f(x)$ exists: From Exercise 2, we found $\lim_{x\to 2} \frac{x^3-8}{x^2-4} = 3$.
- $\lim_{x\to 2} f(x) = f(2) = 3$

Since the limit equals the function value, $f(x)$ is continuous at $x = 2$.

2. $f(x) = |4x-5|$ sur $\mathbb{R}$

The absolute value function, $| u |$, is continuous for all values of u. Since the argument $u = 4x - 5$ is a polynomial (and therefore continuous everywhere), the composition of a continuous function with a continuous function is also continuous.

Therefore, $f(x) = |4x - 5|$ is continuous on $\mathbb{R}$

(The "corner" point at $x = 5/4$ affects differentiability, not continuity.)

Exercise 4: Determine a and b for Continuity on $\mathbb{R}$

The function $f(x) = \begin{cases} 2a\ln(e+x) + b & \text{si } x < 0 \\ 2 & si \quad x = 0 \\ a\cos x - b(x+1)^3 & \text{si } x > 0 \end{cases}$ must be continuous on $\mathbb{R}$.

The functions on the subintervals ($x < 0$ and $x > 0$) are compositions of elementary continuous functions (logarithm, polynomial, cosine), so they are continuous everywhere except possibly where their definitions join: at $x = 0$.

For $f(x)$ to be continuous at $x = 0$, we must have:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

1. **Calculate** $f(0)$:

$$f(0) = 2$$

2. **Calculate the Left Limit** $\lim_{x\to 0^-} f(x)$:

$$\lim_{x\to 0^-} (2a\ln(e+x)+b) = 2a\ln(e+0)+b = 2a+b$$

3. **Calculate the Right Limit** $\lim_{x\to 0^+} f(x)$:

$$\lim_{x\to 0^+} a\cos x - b(x+1)^3 = a\cos(0) - b(1)^3 = a-b$$

4. **Set up the System of Equations:**

Equating the limits to $f(0) = 2$:

$$\begin{cases} 2a+b=2 & from\ left\ limit \\ a-b=2 & from\ right\ limit \end{cases}$$

5. Solve the System:

Add the two equations:

$$(2a+b)+(a-b) = 2+2$$

$$3a = 4 \Rightarrow a = \frac{4}{3}$$

Substitute $a = \frac{4}{3}$ into the second equation $a - b = 2$

$$\frac{4}{3} - b = 2 \Rightarrow b = -\frac{2}{3}$$

The function $f(x)$ is continuous on $\mathbb{R}$ when $a = \frac{4}{3}$ and $b = -\frac{2}{3}$.

Exercise 5

Let f be the function defined on $\mathbb{R}$ by :

$$f(x) = 4x^3 - 3x - \frac{1}{2}$$

.

1) Calculate $f(-1)$, $f(\frac{-1}{2})$, $f(0)$ and $f(1)$

- $f(-1) = 4(-1)^3 - 3(-1) - \frac{1}{2} = -\frac{3}{2}$.
- $f(\frac{-1}{2}) = 4(\frac{-1}{2})^3 - 3(\frac{-1}{2}) - \frac{1}{2} = \frac{1}{2}$.
- $f(0) = 4(0)^3 - 3(0) - \frac{1}{2} = -\frac{1}{2}$.
- $f(1) = 4(1)^3 - 3(1) - \frac{1}{2} = \frac{1}{2}$.

2) Justify that the equation $f(x) = 0$ admits 3 distinct solutions included between -1 and 1

The function $f(x)$ is a **polynomial function**, so it is continuous and differentiable on R, and thus on any interval $[-1, 1]$. We use the **Intermediate Value Theorem (IVT)** on the intervals where the function changes sign.

1. Interval $\left[-1, \frac{-1}{2}\right]$;

- $f(-1) = -\frac{3}{2} < 0$
- $f(\frac{-1}{2}) = \frac{1}{2} > 0$
- Since $f(-1)$ and $f(\frac{-1}{2})$ have opposite signs, and $f(x)$ is continuous on $\left[-1, \frac{-1}{2}\right]$, the IVT guarantees at least **one solution** c_1 in the open interval $\left[-1, \frac{-1}{2}\right]$ such that $f(c_1) = 0$

2. Interval $\left[-\frac{1}{2}, 0\right]$;

- $f(\frac{-1}{2}) = \frac{1}{2} > 0$
- $f(0) = -\frac{1}{2} < 0$
- Since $f(-\frac{1}{2})$ and $f(0)$ have opposite signs, and $f(x)$ is continuous on $\left[-\frac{1}{2}, 0\right]$, the IVT guarantees at least **one solution** c_2 in the open interval $\left[-\frac{1}{2}, 0\right]$ such that $f(c_2) = 0$

3. Interval $[0, 1]$;

- $f(0) = -\frac{1}{2} < 0$
- $f(1) = \frac{1}{2} > 0$
- Since $f(0)$ and $f(1)$ have opposite signs, and $f(x)$ is continuous on $[0, 1]$, the IVT guarantees at least **one solution** c_3 in the open interval $[0, 1]$ such that $f(c_3) = 0$

Since the intervals $\left[-1, \frac{-1}{2}\right]$, $\left[-\frac{1}{2}, 0\right]$, and $[0, 1]$ are disjoint, the three solutions c_1, c_2 , and c_3 must be **distinct.** Also, all these solutions are clearly included between -1 and 1.

A cubic equation can have at most three distinct real roots, and we've found three, so we have justified the existence of **3 distinct solutions** between -1 and 1.

Exercise 6

The function $f(x) = x^2 + \sqrt{x^2}$ We can simplify this function first.

Since $\sqrt{x^2} = |x|$, the function is $f(x) = x^2 + |x|$

A function $f(x)$ is derivable at a point x_0 if the limit of the difference quotient exists:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Here, $x_0 = 0$, and $f(0) = 0^2 + |0| = 0$.

We need to check the left and right derivatives at $x_0 = 0$.

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h\to 0} \frac{(0 + h)^2 + |0 + h| - 0}{h} = \lim_{h\to 0} \frac{h^2 + |h|}{h}$$

1. **Right Derivative** $(h \to 0^+)$:

For $h > 0$, $|h| = h$

$$f'_+(0) = \lim_{h\to 0^+} \frac{h^2 + h}{h} = \lim_{h\to 0^+} \frac{h(h + 1)}{h} = \lim_{h\to 0^+} (h + 1) = 1$$

2. **Left Derivative** $(h \to 0^-)$:

For $h < 0$, $|h| = -h$

$$f'_-(0) = \lim_{h\to 0^-} \frac{h^2 + (-h)}{h} = \lim_{h\to 0^-} \frac{h(h - 1)}{h} = \lim_{h\to 0^-} (h - 1) = -1$$

Since the left and right derivatives are not equal $f'_{-}(0) \neq f'_{+}(0)$, the limit does not exist.

Conclusion: The function $f(x) = x^2 + \sqrt{x^2}$ is not derivable at the point $x_0 = 0$.

Exercise 7: Calculate the derivations of the following functions:

We will use the differentiation rules: $(uv)' = u'v + uv'$ (product rule),$\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}$ (quotient rule), $(u^v)' = u^v(\ln(u).v' + \frac{v}{u}.u')$, $(\ln u)' = \frac{u'}{u}$, and chain rule.

1. $f(x) = e^{3x}\cos 4x$

This is a product of two functions, $u(x) = e^{3x}$ and $v(x) = \cos 4x$

- $u'(x) = \frac{d}{dx}(e^{3x}) = 3e^{3x}$
- $v'(x) = \frac{d}{dx}(\cos 4x) = -4\sin(4x)$

Using the product rule, $f'(x) = u'v + uv'$

$$\begin{aligned} f'(x) &= (3e^{3x})\cos 4x + e^{3x}(-4\sin(4x)) \\ &= e^{3x}(3\cos 4x - 4\sin(4x)) \end{aligned}$$

2. $g(x) = x^{2x}$

This is a function of the form $u(x)^{v(x)}$, which requires logarithmic differentiation

Let $y = x^{2x}$

$$\ln(y) = \ln(x^{2x}) = 2x\ln(x)$$

Differentiate both sides with respect to x

$$\frac{y'}{y} = \frac{d}{dx}(2x\ln x)$$

Usingtheproductrulefortherightside:

$$\frac{y'}{y} = 2\left(\ln(x) + \frac{x}{x}\right) = 2(\ln x + 1),$$

Now, solve for y' :

$$y' = y.2(\ln x + 1) = x^{2x}.2(\ln x + 1)$$

$$g'(x) = 2x^{2x}(\ln x + 1)$$

3. $h(x) = \ln\left(\frac{x+1}{4-x}\right)$

Using the logarithm property $ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

$$\begin{aligned} h(x) &= \ln(x+1) - \ln(4-x) \\ h'(x) &= \frac{1}{x+1} + \frac{1}{4-x} \\ &= \frac{(4-x)+(x+1)}{(4-x)(x+1)} \\ &= \frac{5}{(4-x)(x+1)} \end{aligned}$$

4. $k(x) = \sqrt{x^4+2x} + 2\ln(x)$

Differentiate term by term.

- $\frac{d}{dx}\left(\sqrt{x^4+2x}\right) = \frac{u'}{2\sqrt{u}} = \frac{4x^3+2}{2\sqrt{x^4+2x}} = \frac{2x^3+1}{\sqrt{x^4+2x}}$
- $\frac{d}{dx}(2\ln x)$:

$$\frac{d}{dx}(2\ln x) = 2.\frac{1}{x} = \frac{2}{x},$$

  Combining the terms:

$$k'(x) = \frac{2x^3+1}{\sqrt{x^4+2x}} + \frac{2}{x}$$

Exercise 8:

L'Hôpital's Rule (L'Hospital's Rule) applies to limits of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$. it states that $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$, provided the latter limit exists

1. $\lim_{x\to 0}\frac{\sin 2x}{\sin 3x}$

- **Check Indeterminate Form**: As $x \to 0$, $\sin 2x \to \sin 0 = 0$ and $\sin 3x \to \sin 0 = 0$

  The form is $\frac{0}{0}$

- **Apply L'Hôpital's Rule:**

$$\lim_{x\to 0}\frac{(\sin 2x)'}{(\sin 3x)'} = \lim_{x\to 0}\frac{2\cos 2x}{3\cos 3x}$$

- **Evaluate Limit:**

$$\frac{2\cos 2\,(0)}{3\cos 3\,(0)} = \frac{2}{3}$$

2. $\lim_{x\to\frac{\pi}{4}}\frac{\sin x - \cos x}{1-\tan x}$

- **Check Indeterminate Form: As** $x \to \frac{\pi}{4}$:

  - Numerator: $sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 0.$
  - Denominator: $1 - tan\left(\frac{\pi}{4}\right) = 1 - 1 = 0$
  - The form is $\frac{0}{0}$

- **Apply L'Hôpital's Rule:**

$$\lim_{x\to\frac{\pi}{4}} \frac{(\sin x - \cos x)'}{(1 - \tan x)'} = \lim_{x\to\frac{\pi}{4}} \frac{\cos x - (-\sin x)}{0 - \sec^2 x} = \lim_{x\to\frac{\pi}{4}} \frac{\cos x + \sin x}{-\sec^2 x}$$

- **Evaluate Limit:** Recall

$$secx = \frac{1}{\cos x}, \text{ so } \sec^2 x = \frac{1}{\cos^2 x}$$

$$\frac{\cos(\frac{\pi}{4}) + \sin(\frac{\pi}{4})}{-\sec^2(\frac{\pi}{4})} = \frac{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}{-\left(\frac{1}{\cos(\frac{\pi}{4})}\right)^2} = -\frac{\sqrt{2}}{2}$$

3.$\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}}$

- **Check Indeterminate Form: As** $x \to 0^+$, $\cos x \to \cos 0 = 1$ and $\frac{1}{x^2} \to \infty$ The form is 1^∞.

- Convert to $\frac{\infty}{\infty}$ or $\frac{0}{0}$:

Let $l = \lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}}$
Take the natural logarithm of the expression:

$$\ln((\cos x)^{\frac{1}{x^2}}) = \frac{1}{x^2} \ln(\cos x)$$

Now we find the limit of the logarithm:

$$\lim_{x\to 0^+} \frac{1}{x^2} \ln(\cos x)$$

As $x \to 0^+$, $\ln(\cos x) \to \ln(\cos 0) = \ln(1) = 0$, and $x^2 \to 0$. The form is $\frac{0}{0}$.

- **Apply L'Hôpital's Rule (First time):**

$$\lim_{x\to 0^+} \frac{(\ln(\cos x))'}{(x^2)'} = \lim_{x\to 0^+} \frac{\frac{-\sin x}{\cos x}}{2x} = \lim_{x\to 0^+} \frac{-\tan x}{2x}$$

As $x \to 0^+$, $-\tan x \to 0$, and $2x \to 0$. The form is $\frac{0}{0}$.

- **Apply L'Hôpital's Rule (Second time):**

$$\lim_{x\to 0^+} \frac{(-\tan x)'}{(2x)'} = \lim_{x\to 0^+} \frac{-\sec^2 x}{2}$$

- **Evaluate Limit:**

$$\frac{-\sec^2 0}{2} = \frac{-\left(\frac{1}{\cos 0}\right)^2}{2} = -\frac{1}{2}$$

- **Final Result: Since**

$$\lim_{x\to 0^+} \ln(f(x)) = -\frac{1}{2},$$

The original limit L is:

$$L = e^{\lim_{x\to 0^+} \ln(f(x))} = e^{-1/2} = \frac{1}{\sqrt{e}}$$

Chapter 2

The primitive function and integrals

In calculus, primitive functions and integrals are fundamental concepts used to compute areas, accumulated quantities, and reverse the process of differentiation.

- A primitive function (also called an antiderivative) is the inverse process of differentiation.
- An integral represents either:

- An indefinite integral $\rightarrow$ general antiderivative (no limits).
- A definite integral $\rightarrow$ numerical value representing area or accumulated quantity over an interval.

1 The primitive function

Definition 2.1

A primitive of a continuous function f on an interval I is a function F derivable on I such that $F\prime(x) = f(x)$ for all x in I. As soon as there is a primitive F of a function f, there are in fact infinitely many: if C is a constant, then $F + C$ is still a primitive of f. All primitives of a function f differ from each other by a constant: if F and G are two primitives of f, then there exists a constant C such that $F = G + C$.

In this section, we'll look at a few techniques for calculating primitives.

1. If $F(x)$ and $G(x)$ are primitives of $f(x)$ and $g(x)$, and if α and β are real numbers, then $\alpha F(x) + \beta G(x)$ is a primitive of $\alpha f(x) + \beta g(x)$.

2. If $F(x)$ is a primitive of $f(x)$ and r is a non-zero real number, then $\frac{1}{r}F(rx)$ is a primitive of $f(rx)$.

3. A primitive of $f(x) = exp(x)$ is $F(x) = exp(x)$.

4. A primitive of $f(x) = \frac{1}{x}$ is $F(x) = ln(|x|)$.

5. If $\alpha \neq -1$, a primitive of $f(x) = x^\alpha$ is $F(x) = \frac{1}{\alpha+1}x^{\alpha+1}$.

Example

Say f and F are two functions defined on $]-1,+\infty[$ as follows.

$$f(x) = -\frac{2x^2+4x}{(x+1)^2}, \qquad F(x) = \frac{x-1}{x+1} - 2x$$

Show that F is a primitive of f on $]-1,+\infty[$.

Solution

F is derivable on $]-1,+\infty[$, so

$$F'(x) = \frac{1(x+1)-(x-1)}{(x+1)^2} - 2 = \frac{x+1-x+1-2(x+1)^2}{(x+1)^2} = -\frac{2x^2+4x}{(x+1)^2}$$

So $F'(x) = f(x)$ for all $x \in\ ]-1,+\infty[$.

Example

Let f be the function defined on $\mathbb{R}$

$$f(x) = 2x + \cos(x)$$

1) Give the primitive of f ∩∩on $\mathbb{R}$

2) Determine the primitive F on $\mathbb{R}$ that verifies $F(\pi) = -1$.

Solution

1) The primitives of f on $\mathbb{R}$

$$F(x) = x^2 + \sin(x) + c$$

2) $F(\pi) = -1$ means that $\pi^2 + \sin(\pi) + c = -1 \Longrightarrow c = -1 - \pi$

$$F(x) = x^2 + \sin(x) - (1+\pi)$$

2 Integrals

2.1 Indefinite integrals

Definition 2.2

The set of all primitives of the function $I \to \mathbb{R}$ is called the indefinite integral of f and is denoted $\int f(x)dx$ i.e.

$$\int f(x)dx = F(x) + C$$

Example

Calculate the following integrals:

$$\int x - \frac{1}{x^2}dx, \quad \int -2x^5 + x^3 - x + 3dx, \int \frac{1}{\sqrt{5x}}dx, \int \frac{9}{x^4}dx$$

Solution

1) $\int x - \frac{1}{x^2}dx = \frac{x^2}{2} + \frac{1}{x} + C.$
2) $\int -2x^5 + x^3 - x + 3dx = -\frac{2}{6}x^6 + \frac{x^4}{4} - \frac{x^2}{2} + 3x + C.$
3) $\int \frac{1}{\sqrt{5x}}dx = \frac{2}{5}\int \frac{5}{2\sqrt{5x}}dx = \frac{2}{5}\sqrt{5x} + C.$
4) $\int \frac{9}{x^4}dx = -\frac{9}{3x^3} + C.$

2.2 Definite integrals

Definition 2.3

The integral between two real numbers $a \leq b$ of a continuous function f on the interval $[a, b]$, denoted $\int_a^b f(x)\ dx$, is the airc located between the vertical lines at absciss a and b, the graphical representation of f and the abscissa axis, counting this area positively in the parts where f is positive and negatively dam the parts where f is negative (see Figure2.1)

2.3 Property of integrals

An integral is defined over an interval I if the function is continuous over I. Here are some properties that we apply to integrals for calculations:

1. $\int_a^a f(x)dx = 0.$

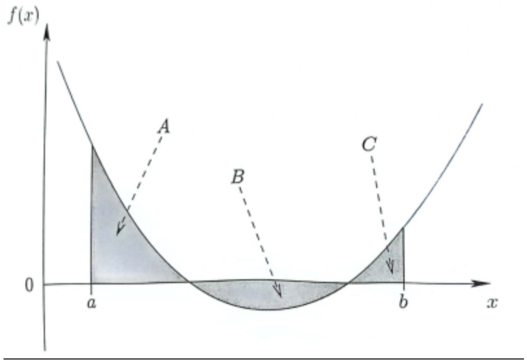


Figure 2.1: The integral of f between a and b is $\int_a^b f(x)\,dx = A - B + C$.

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

3. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$. (Chasles relationship)

4. Let k be a real number, $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.

5. $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

6. $\int_a^b (f(x)-g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx$.

Example

Let g be a function defined on the interval $[1, +\infty[$ by:

$$g(x) = \frac{1}{x(x^2-1)}$$

1) Determine the real numbers a, b and c such that, for all $x > 1$:

$$g(x) = \frac{\alpha}{x} + \frac{b}{x+1} + \frac{c}{x-1}$$

2) Calculate the following integral:

$$\int_2^3 g(x)dx$$

Solution

1) Real numbers a, b and c :

$$\begin{aligned} g(x) &= \frac{\alpha}{x} + \frac{b}{x+1} + \frac{c}{x-1} \\ &= \frac{a(x+1)(x-1) + bx(x-1) + cx(x+1)}{x(x^2-1)} \\ &= \frac{(a+b+c)x^2 + (c-b)x - a}{x(x^2-1)} \end{aligned}$$

corresponding to the function $g(x)$ we obtain: $\begin{cases} -a = 1 \\ c - b = 0 \\ a + b + c = 0 \end{cases}$

After the calculations, we obtain $\begin{cases} a = -1 \\ b = \frac{1}{2} \\ c = \frac{1}{2} \end{cases}$ So $g(x) = -\frac{1}{x} + \frac{1}{2(x+1)} + \frac{1}{2(x-1)}$

2) Calculate the integral:

$$\begin{aligned} \int_2^3 g(x)dx &= \int_2^3 -\frac{1}{x} + \frac{1}{2(x+1)} + \frac{1}{2(x-1)}dx \\ &= -\int_2^3 \frac{1}{x}dx + \frac{1}{2}\int_2^3 \frac{1}{(x+1)}dx + \frac{1}{2}\int_2^3 \frac{1}{(x-1)}dx \\ &= [-\ln(x)]_2^3 + \frac{1}{2}[\ln(x+1)]_2^3 + \frac{1}{2}[\ln(x-1)]_2^3 \\ &= -\frac{3}{2}\ln(3) + \frac{3}{2}\ln(2) + \frac{1}{2}\ln(4) \end{aligned}$$

2.4 Effective Calculation of integrals

2.4.1 Integration by parts

Theorem 2.1

So let f, g be two functions continuously derivable on $[a, b]$, Then,

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b g(x)f'(x)dx$$

Recall that $[f(x)g(x)]_a^b = f(b)g(b) - f(a)g(a)$

Example

calculate$\int_0^1 xe^x dx$

Let $f(x) = x, \;\; g'(x) = e^x$

$$\begin{aligned} f(x) &= x & \Longrightarrow f'(x) = 1dx \\ g\prime(x) &= e^x dx & \Longrightarrow g(x) = e^x \end{aligned}$$

So

$$\int_0^1 xe^x dx = [xe^x]_0^1 - \int_0^1 e^x dx = [xe^x]_0^1 - [e^x]_0^1 = 1$$

Example

We are looking for a primitive of the function $ln(x)$ on its interval of definition $]0, +\infty[$, $I = \int_1^t \ln(x) dx$

Let $f(x) = \ln(x), \;\; g'(x) = 1$

$$\begin{aligned} f(x) &= \ln(x) & \Longrightarrow f'(x) = \frac{1}{x} dx \\ g\prime(x) &= 1dx & \Longrightarrow g(x) = x \end{aligned}$$

So

$$\int_1^t \ln(x) dx = [x \ln(x)]_1^t - \int_1^t 1dx = [x \ln(x)]_1^t - [x]_1^t = t \ln t - t + 1$$

2.4.2 Change of variable

Theorem 2.2

Let φ be a function continuously derivable on an interval I, with va their in an interval J. If a and b are two points of I and f a continuous function on J,Then

$$\int_{\varphi(a)}^{\varphi(b)} f(x) dx = \int_a^b (fo\varphi(t))\, \varphi'(t) d(t)$$

Example

Calculate the primitive of the following function.

$$\int 4\cos^3 x - 3\cos x dx$$

Let $t = \sin x \implies dt = \cos x dx \implies dx = \frac{dt}{\cos x}$

$$\begin{aligned}\int 4\cos^3 x - 3\cos x dx &= \int \cos x(4\cos^2 x - 3)dx \\ &= \int \cos x(4\cos^2 x - 3)\frac{dt}{\cos x} \\ &= \int (4\cos^2 x - 3)dt\end{aligned}$$

as $\cos^2 x + \sin^2 x = 1 \implies \cos^2 x = 1 - \sin^2 x$

$$\begin{aligned}\int (4\cos^2 x - 3)dt &= \int 4\left(1 - \sin^2 x\right) - 3dt \\ &= \int 4(1 - t^2) - 3dt \\ &= \int 1 - 4t^2 dt = \int dt - 4\int t^2 dt \\ &= t - \frac{4}{3}t^3 + C \\ \int 4\cos^3 x - 3\cos x dx &= \sin x - \frac{4}{3}\sin^3 x + C\end{aligned}$$

3 Integral approximation method

Integral approximation methods, often called numerical integration or quadrature, are techniques used to estimate the value of a definite integral $\int_a^b f(x)dx$ when finding the exact analytical solution is impossible or too complicated.

These methods work by replacing the function $f(x)$ with a simpler function (like a constant or a line) over small subintervals and summing the areas of the resulting geometric shapes

3.1 Riemann Sums

Riemann sums are the foundational concept for all integral approximations. They estimate the area under the curve by dividing the interval $[a, b]$ into n rectangles and summing their areas. The difference between types lies in how the height of each rectangle is determined:

- **Left Endpoint Rule**: The height of each rectangle is taken as the function value at the **left endpoint** of the subinterval.

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

- **Right Endpoint Rule**: The height is taken as the function value at the **right**

endpoint of the subinterval.

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

- **Midpoint Rule**: The height is taken as the function value at the **midpoint** of the subinterval. This is generally more accurate than the Left or Right Endpoint rules.

$$M_n = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x$$

Where $\bar{x}_i = \frac{x_{i-1}+x_i}{2}$. In all cases, the width of the subintervals is $\Delta x = \frac{b-a}{n}$.

Example

We will approximate the area under the curve for the integral $\int_1^3 \frac{1}{x}dx$ using four ($n = 4$) subintervals and two different rules: the Left Riemann Sum and the Right Riemann Sum. **1. Initial Setup and Parameters**

- Function: $f(x) = \frac{1}{x}$
- Interval: $[a, b] = [1, 3]$
- Number of Subintervals: $n = 4$
- Width (Δx):

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = 0.5$$

Our evaluation points (x_i) start at $a = 1$ and increase by $0.5 : x_0 = 1.0$, $x_1 = 1.5$, $x_2 = 2.0$, $x_3 = 2.5$, and $x_4 = 3.0$ **2. Left Riemann Sum** (L_4) The Left Riemann Sum uses the left endpoint of each subinterval to determine the height. We use x_0, x_1, x_2, x_3.

$$L_4 = \Delta x \left[f(x_0) + f(x_1) + f(x_2) + f(x_3)\right]$$

Left Endpoint (x_i)	$f(x_i) = 1/x_i$
$x_0 = 1.0$	1.0000
$x_1 = 1.5$	0.6667
$x_2 = 2.0$	0.5000
$x_3 = 2.5$	0.4000
sum	2.5667

Calculation:

$$L_4 = 0.5 \times 2.5667 \approx 1.2834$$

3. Right Riemann Sum (R_4) The Right Riemann Sum uses the right endpoint of each subinterval to determine the height. We use x_0, x_1, x_2, x_3

$$R_4 = \Delta x\left[f(x_1) + f\left(x_2\right) + f\left(x_3\right) + f\left(x_4\right)\right]$$

Left Endpoint (x_i)	$f(x_i) = 1/x_i$
$x_1 = 1.5$	0.6667
$x_2 = 2.0$	0.5000
$x_3 = 2.5$	0.4000
$x_4 = 3.0$	0.3333
sum	1.9000

Calculation:

$$R_4 = 0.5 \times 1.9 \approx 0.95$$

Summary

- Left Riemann Sum (L_4) : 1.2834
- Right Riemann Sum (R_4) : 0.95

(The exact value of the integral is $ln(3) \approx 1.0986$. Notice that the true value lies between the Left Sum, which overestimates the area for this decreasing function, and the Right Sum, which underestimates it.)

3.2 Trapezoidal Rule

The Trapezoidal Rule improves upon Riemann sums by approximating the area under the curve using trapezoids instead of rectangles. It connects the endpoints of the subinterval $[x_{i-1}, x_i]$ with a straight line, which often fits the curve better than a horizontal line.

The approximation T_n is the average of the Left and Right Riemann sums:

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + ...2f(x_{n-1}) + f(x_n)\right]$$

Example

Let's use the Trapezoidal Rule to approximate the definite integral $\int_1^3 \frac{1}{x} dx$ We will use $n = 4$ subintervals for this approximation.

1. Determine Parameters and Width

First, we define our parameters and calculate the width of the subintervals (Δx).

- Function: $f(x) = \frac{1}{x}$.
- Interval:$[a, b] = [1, 3]$
- Number of Subintervals: $n = 4$
- Width:

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = 0.5$$

2. Calculate Function Values

Since $\Delta x = 0.5$, our five evaluation points are $x_0 = 1.0$, $x_1 = 1.5$, $x_2 = 2.0$, $x_3 = 2.5$, and $x_4 = 3.0$

Point (x_i)	$f(x_i) = 1/x_i$ (Approx.)	Trapezoid coeff.	$f(x_i) \times Coeff.$
$x_0 = 1.0$	1.0000	1	1.0000
$x_1 = 1.5$	0.6667	2	1.3334
$x_2 = 2.0$	0.5000	2	1.0000
$x_3 = 2.5$	0.4000	2	0.8000
$x_4 = 3.0$	0.3333	1	0.3333
			sum $\approx$ 4.4667

3. Apply the Trapezoidal Rule Formula

Now we apply the formula using the calculated sum:

$$\begin{aligned} T_n &= \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + .2f(x_3) + f(x4)] \\ &= \frac{0.5}{2} [4.4667] = 0.25 \times 4.4667 \approx 1.1167 \end{aligned}$$

Conclusion

The Trapezoidal Rule gives an approximation of 1.1167 for the area under the curve. (For comparison, the exact value of $\int_1^3 \frac{1}{x} dx = [\ln(x)]_1^3 = \ln(3) - \ln(1) \approx 1.0986$. Our approximation is very close!)

3.3 Simpson's Rule

Simpson's Rule (also known as the parabolic rule) offers an even better approximation by fitting **parabolic segments** (second-degree polynomials) to the function over every pair of subintervals.

Because it uses pairs of intervals, the number of subintervals, n, must be **even.**

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4)........ + 4f(x_{n-1}) + f(x_n)\right]$$

Where:

- $\Delta x = \frac{b-a}{n}$ is the width of each subinterval.
- n must be an even integer.
- The coefficients follow a repeating pattern: $1, 4, 2, 4, 2, ..., 4, 1$. The first and last terms are multiplied by 1, and the interior terms alternate between 4 (for odd-indexed points) and 2 (for even-indexed points)

Example

We will approximate the value of the integral $\int_0^4 x^2 dx$ using $n = 4$ subintervals.

The exact value of this integral is $\int_0^4 x^2 dx = \left[\frac{x^3}{3}\right]_0^4 = \frac{4^3}{3} \approx 21.3333$.

1. **Determine Parameters and Width**

First, we define our parameters and calculate the width of the subintervals (Δx).

- Function: $f(x) = x^2$
- Interval: $[a, b] = [0, 4]$
- Number of Subintervals: $n = 4$ (must be even)
- Width:

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$$

2. Calculate Function Values

Since $\Delta x = 1$, our five evaluation points are $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and

$x_4 = 4$

Point (x_i)	$f(x_i) = x_i^2$	Simpson's Coefficient	$f(x_i) \times Coeff.$
$x_0 = 0$	$f(0) = 0$	1	$1 \times 0 = 0$
$x_1 = 1$	$f(1) = 1$	4	$4 \times 1 = 4$
$x_2 = 2$	$f(2) = 4$	2	$2 \times 4 = 8$
$x_3 = 3$	$f(3) = 9$	4	$4 \times 9 = 36$
$x_4 = 4$	$f(4) = 16$	1	$1 \times 16 = 16$

3. **Apply Simpson's Rule Formula**

Now we apply the formula using the calculated sum:

$$\begin{aligned} S_n &= \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)] \\ &= \frac{1}{3}[0 + 4 + 8 + 36 + 16] \\ &\approx 21.3333 \end{aligned}$$

Conclusion

In this specific example, the approximation using Simpson's Rule with $n = 4$ is **exactly equal** to the true value of the integral ($\frac{64}{3}$). This highlights a key property of Simpson's Rule: it provides an exact value for any integral of a quadratic or cubic polynomial.

4 Exercises

Exercise 1

A biologist observes that the rate of population growth of a bacteria culture (in thousands) after t hours is given by:

$$f(t) = 2t + 1$$

where $f(t)$ is measured in thousands of bacteria per hour.

At time $t = 0$, the population is 5000 bacteria

Exercise 2

Use the Composite Simpson's $\frac{1}{3}$ Rule with $n = 4$ subintervals to approximate the definite integral:

$$I = \int_1^3 \frac{1}{x} dx$$

(**Note:** The exact value of this integral is $ln(3) \approx 1.098612$ to compare our result.)

Exercise 3

Use the Composite Trapezoidal Rule with $n = 4$ subintervals to approximate the definite integral:

$$I = \int_0^2 x^2 dx$$

(**Note:** The exact value of this integral is $\frac{8}{3} \approx 2.66667$ for comparison.)

Exercise 4

Use the Left Riemann Sum with $n = 4$ subintervals to approximate the definite integral:

$$I = \int_1^3 \frac{1}{x} dx$$

(**Note**: The exact value of this integral is $ln(3) \approx 1.0986$ for comparison.)

Exercise 5

Use the Right Riemann Sum with $n = 4$ subintervals to approximate the definite integral:

$$I = \int_0^4 (4x - x^2) dx$$

(**Note**: The exact value of this integral is $\frac{32}{3} \approx 10.6667$ for comparison.)

5 The answers

Exercise 1

1. Finding the Primitive Function (Population P(t))

The population $P(t)$ is the primitive function (antiderivative) of the rate of growth $f(t)$.

The rate of growth is $f(t) = 2t + 1$ (thousands of bacteria per hour).

The population $P(t)$ is the indefinite integral of $f(t)$:

$$P(t) = \int f(t)dt = \int (2t + 1)dt$$

Using the power rule for integration:

$$\begin{aligned} P(t) &= 2.\frac{t^2}{2} + t + c \\ &= t^2 + t + C \end{aligned}$$

Here, $P(t)$ is measured in thousands of bacteria, and C is the constant of integration.

2. Determining the Constant of Integration C

You are given the initial condition: At time $t = 0$, the population is 5000 bacteria.

Since the population function $P(t)$ is in thousands, we use $P(0) = 5$.

Substitute $t = 0$ and $P(t) = 5$ into the primitive function:

$$\begin{aligned} P(0) &= (0^2) + 0 + C \\ 5 &= 0 + 0 + C \\ C &= 5 \end{aligned}$$

3. The Population Function

Substituting $C = 5$ back into the primitive function gives the specific equation for the bacteria population over time:

$$P(t) = t^2 + t + 5$$

This function gives the population in thousands of bacteria after t hours. For example, after 3 hours, the population would be $P(3) = 3^2 + 3 + 5 = 17$ or $17,000$ bacteria.

4. **Interpretation**

* $f(t) = 2t + 1 \rightarrow$ growth rate (how fast population increases).

* $P(t) = t^2 + t + 5 \rightarrow$ primitive function giving total population.

* The definite integral of $f(t)$ over an interval gives the total increase in population during that time.

Exercise 2

Step 1: Determine the Interval Width (Δx)

The integration is over the interval $[a, b] = [1, 3]$ with n=4 subintervals.

The width of each subinterval, Δx (or h), is calculated as:

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = 0.5$$

Step 2: Define the Subinterval Points $(\mathbf{x}_i)$

We start at $x_0 = a = 1$ and add $\Delta x = 0.5$ for each subsequent point until we reach $x_4 = b = 3$

Index (i)	x_i
x_0	1.0
x_1	1.5
x_2	2
x_3	2.5
x_4	3

Step 3: Calculate the Function Values $f(\mathbf{x}_i)$

The function is $f(x) = 1/x$. We calculate the value of the function at each point x_i

i	x_i	$f(x_i) = x_i^2$
0	1.0	$f(1.0) = 1$
1	1.5	$f(1.5) = 0.66667$
2	2	$f(2) = 0.5$
3	2.5	$f(2.5) = 0.4$
4	3	$f(3) = 0.33333$

Step 4: Apply the Composite Simpson's Rule Formula

The Composite Simpson's $\frac{1}{3}$ Rule is:

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + ...4f(x_{n-1}) + f(x_n)\right]$$

For $n = 4$, the formula is:

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

Substitute $\Delta x = 0.5$ and the function values:

$$\begin{aligned} S_4 &= \frac{0.5}{3}\left[(1.000000) + 4(0.666667) + 2(0.500000) + 4(0.400000) + (0.333333)\right] \\ &= \frac{1}{6}\left[1.000000 + 2.666668 + 1.000000 + 1.600000 + 0.333333\right] \approx 1.1 \end{aligned}$$

Conclusion

The Simpson's Rule approximation for the integral $\int_1^3 1/x dx$ with $n = 4$ is 1.1

Comparison to Exact Value:

- Simpson's Rule Approximation: 1.1
- Exact Value (ln(3)): 1.098612
- Error $\mid 1.100000 - 1.098612 \mid = 0.001388$

This shows that Simpson's Rule provides a very accurate estimate, even with a small number of subintervals ($n = 4$).

Exercise 3

Step 1: State the Formula and Determine the Interval Width (Δx)

The Composite Trapezoidal Rule formula is:

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + ...2f(x_{n-1}) + f(x_n)\right]$$

The integration is over the interval $[a, b] = [0, 2]$ with $n = 4$ subintervals.

The width of each subinterval, Δx (or h), is calculated as:

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$$

Step 2: Define the Subinterval Points ($\mathbf{x}_i$)

We start at $x_0 = a = 0$ and add $\Delta x = 0.5$ for each subsequent point until we reach $x_4 = b = 2$

Index (i)	x_i
x_0	0
x_1	0.5
x_2	1
x_3	1.5
x_4	2

Step 3: Calculate the Function Values $f(x_i)$

The function is $f(x) = x^2$. We calculate the value of the function at each point x_i

i	x_i	$f(x_i) = x_i^2$
0	0	$f(0) = 0$
1	0.5	$f(0.5) = 0.25$
2	1	$f(1) = 1$
3	1.5	$f(1.5) = 2.25$
4	2	$f(2) = 4$

Step 4: Apply the Trapezoidal Rule

Substitute $\Delta x = 0.5$ and the function values into the formula for $n = 4$:

$$\begin{aligned} T_n &= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right] \\ &= \frac{0.5}{2}[(0.0) + 2(0.25) + 2(1.0) + 2(2.25) + (4.0)] \\ &= 0.25[0.0 + 0.5 + 2.0 + 4.5 + 4.0] = 2.75 \end{aligned}$$

Conclusion

The Trapezoidal Rule approximation for the integral $\int_0^2 x^2 dx$ with $n = 4$ is 2.75.

Comparison to Exact Value:

- Trapezoidal Rule Approximation: 2.75

- Exact Value (2.66667

Since $f(x) = x^2$ is concave up on $[0, 2]$, the trapezoids lie slightly above the curve, resulting in an overestimate (which is consistent with $2.75 > 2.66667$).

Exercise 4

Step 1: State the Formula and Determine the Interval Width (Δx)

The Left Riemann Sum formula for n equal subintervals is:

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x = \Delta x\,[f(x_0) + f(x_1) + ... + f(x_{n-1})]$$

The integration is over the interval $[a, b] = [1, 3]$ with $n = 4$ subintervals.

The width of each subinterval, Δx, is:

$$\Delta x = \frac{b-a}{n} = 0.5$$

Step 2: Define the Subinterval Points ($\mathbf{x}_i$)

We start at $x_0 = a = 1$ and add $\Delta x = 0.5$ for each subsequent point. For a Left Riemann Sum with $n = 4$, we need to evaluate the function up to $x_{n-1} = x_3$

Index (i)	Interval	x_i (Left Endpoint)
x_0	$[1, 1.5]$	1
x_1	$[1.5, 2]$	1.5
x_2	$[2, 2.5]$	2
x_3	$[2.5, 3]$	2.5
x_4	Not used for height	3

Step 3: Calculate the Function Values $f(x_i)$

The function is $f(x) = 1/x$. These values represent the **height** of each rectangle.

i	x_i	$f(x_i)$
0	1	$f(1) = 1$
1	1.5	$f(1.5) \approx 0.6667$
2	2	$f(2) = 0.5$
3	2.5	$f(2.5) = 0.4$

Step 4: Calculate the Left Riemann Sum $(\mathbf{L}_4)$

Substitute $\Delta x = 0.5$ and the heights into the formula:

$$\begin{aligned} L_4 &= \Delta x \left[f(x_0) + f(x_1) + f(x_2) + f(x_3) \right] \\ &= 0.5[1.0 + 0.66667 + 0.5 + 0.4] \\ &= 0.5[2.56667] \approx 1.283335 \end{aligned}$$

Conclusion

The Left Riemann Sum approximation for the integral $\int_1^3 \frac{1}{x} dx$ with $n = 4$ is approximately 1.2833.

Context: Since the function $f(x) = 1/x$ is monotonically decreasing on the interval $[1, 3]$, the Left Riemann Sum uses the highest value in each interval as the height. Therefore, this approximation (1.2833) is an overestimate compared to the exact value (≈ 1.0986). $n = 4$

Exercise 5

Step 1: State the Formula and Determine the Interval Width (Δx)

The Right Riemann Sum formula for n equal subintervals is:

$$R_n = \int_{i=1}^{n} f(x_i).\Delta x = \Delta x \left[f(x_1) + f(x_2) .... + f(x_n) \right]$$

The integration is over the interval $[a, b] = [0, 4]$ with $n = 4$ subintervals.

The width of each subinterval, Δx, is:

$$\Delta x = \frac{b - a}{n} = \frac{4 - 0}{4} = 1$$

Step 2: Define the Subinterval Points (x_i)

We start at $x_0 = a = 0$ and add $\Delta x = 1$ for each subsequent point. For a Right Riemann Sum with $n = 4$, we evaluate the function at the right endpoints, which are x_1 through x

Index (i)	Interval	x_i (Right Endpoint)
x_0	Not used for height	0
x_1	$[0, 1]$	1
x_2	$[1, 2]$	2
x_3	$[2, 3]$	3
x_4	$[3, 4]$	4

Step 3: Calculate the Function Values $f(x_i)$

The function is $f(x) = 4x - x^2$. These values represent the **height** of each rectangle.

i	x_i	$f(x_i)$
1	1	$f(1) = 3$
2	2	$f(2) = 4$
3	3	$f(3) = 3$
4	4	$f(4) = 0$

Step 4: Calculate the Right Riemann Sum $(\mathbf{R}_4)$

Substitute $\Delta x = 1$ and the heights into the formula:

$$\begin{aligned} R_4 &= \Delta x \left[f(x_1) + f(x_2) + f(x_3) + f(x_4) \right] \\ &= 1 \left[3 + 4 + 3 + 0 \right] = 10 \end{aligned}$$

Conclusion

The Right Riemann Sum approximation for the integral $\int_0^4 (4x - x^2) dx$ with $n = 4$ is exactly 10.

Context: The function $f(x) = 4x - x^2$ is an upward-opening parabola that peaks at $x = 2$. Because the function is increasing on the left half of the interval $[0, 2]$ and decreasing on the right half $[2, 4]$, the Right Riemann Sum in this case includes both over- and under-estimations, and the result is quite close to the true value (≈ 10.6667).

Chapter 3

Numerical Series

1 Introduction

The theory of series aims to impart meaning, if possible, to the sum of an infinite number of numbers. Suppose we have a cake and a perfect knife. that allows us to cut it without ever losing a crumb! If we start by eating half of the cake, then half of what remains, and so on indefinitely, no one doubts that we will eventually eat the whole cake. And if we note the fraction of cake eaten the first time $\left(u_1 = \frac{1}{2}\right)$, u_2 the fraction of cake eaten the second time $\left(u_1 = \frac{1}{2}, u_2 = \frac{1}{4}\right)$, and in general the fraction of cake eaten the n^{th} time.To say that we have eaten everything means that if we add up $u_1, u_2, ...u_n, ...$, we will find 1, which we will. note as:

$$u_1 + u_2 + ... + u_n + ... = \sum_{n=1}^{+\infty} u_n = 1$$

2 Fundamental definitions

2.1 Definitions of a series

Let $(u_n)_{n\geq 1}$ define a numerical sequence and, for each $n \in \mathbb{N}$, let S be the sum of the first $(n+1)$ terms of this sequence.

Definition 3.1: Sequence

A sequence $(u_n)_{n\geq 1}$ is an ordered list of real numbers

$$(u_n) = u_1, u_2, u_3......$$

Definition 3.2: Series

The series associated with (u_n) is the formal sum:

$$S = \sum_{n=1}^{\infty} u_n = u_1 + u_2 + \dots.$$

Definition 3.3: Partial sums

The partial sum S_N is defined as:

$$S_N = \sum_{n=1}^{N} u_n$$

If the sequence of partial sums S_N has a finite limit:

$$\lim_{N\to\infty} S_N = S$$

Definition 3.4: Convergence of a series

The series $\sum u_n$ converges if the sequence of partial sums (S_N) converges to a finite limit S : If the limit does not exist or is infinite, the series diverges.

2.2 Series with Positive Terms

A positive-term series satisfies $u_{n\geq 0}$ for all n.

Property

If $u_{n\geq 0}$, then S_n is monotonic increasing (since each new term adds a nonnegative quantity).

Therefore:

- If S_n is bounded above, the series converges.
- If S_n is unbounded, the series diverges to infinity.

3 Illustrative examples

1. Geometric series: $\sum_{n=0}^{\infty} q^n$

- Convergent if $|q| < 1$, with sum $\frac{1}{1-q}$.
- Divergent if $|q| \geq 1$

2. Harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n}$

Divergent (slowly)

3. p-series:

- Convergent if $p > 1$
- Divergent if $p \leq 1$.

4. Positive biological model example: Suppose bacterial population growth rate each day is $r_n = 1/n^2$ (relative units). Then r_n converges, meaning the total growth effect is finite.

4 Elementary properties

4.1 A necessary condition for convergence

Statement. If $\sum_{n=0}^{\infty} u_n$ converges, then $u_n \to 0$

Proof

$u_n = S_n - S_{n-1}$ If $S_n \to S$, then $u_n = S_n - S_{n-1} = S - S = 0$

■

Remark

$u_n \to 0$ is necessary but not sufficient (e.g. harmonic series: $u_n = \frac{1}{n} \to 0$ but series diverges).

4.2 Monotonicity for positive terms

If $u_n \geq 0$ then $S_N = \sum_{n=1}^{N} u_n$ is nondecreasing in (N). Hence a positive-term series converges **iff** its partial sums are bounded above.

Proof

For every $N \geq 1$ Trivial: $S_N - S_{N-1} = \sum_{n=1}^{N} u_n - \sum_{n=1}^{N-1} u_n = u_{N+1} \geq 0$. If bounded, monotone increasing sequence converges.

■

4.3 Boundedness test:

Theorem 3.1: Statement

Let $\sum_{n=1}^{\infty} u_n$ be an infinite series where all terms are non-negative, i.e., $u_n \geq 0$ for all n. Let $S_n = \sum_{n=1}^{N} u_n$ be the sequence of partial sums. The series $\sum_{n=1}^{\infty} u_n$ converges if and only if the sequence of partial sums $\{S_N\}$ is bounded above.

Proof

1. Monotonicity. *For every N, we have*

$$S_{N+1} - S_N = u_{N+1} \geq 0,$$

so (S_N) is non-decreasing.

2. Bounded above $\Rightarrow$ convergent. *Since (S_N) is non-decreasing and bounded above, the completeness of $\mathbb{R}$ implies it has a least upper bound (supremum). Let*

$$S := \sup\{S_N : N \geq 1\}.$$

We show that $S_N \to S$. Fix $\varepsilon > 0$. By definition of the supremum, there exists N_0 such that

$$S - \varepsilon < S_{N_0} \leq S.$$

Because (S_N) is non-decreasing, for every $N \geq N_0$ we have

$$S - \varepsilon < S_{N_0} \leq S_N \leq S,$$

so $|S_N - S| \leq S - S_{N_0} < \varepsilon$. Thus $S_N \to S$, i.e. $\sum_{n=1}^{\infty} u_n$ converges and its sum equals S.

3. Converse (easy). *If $\sum u_n$ converges to L, then (S_N) converges to L, hence it is bounded above. Therefore, for series with nonnegative terms, boundedness above of the partial sums is* equivalent *to convergence.*

■

4.4 Comparison test

Statement. Let $0 \leq u_n \leq b_n$ for all large n

- If $\sum b_n$ converges then $\sum u_n$ converges.
- If $\sum u_n$ diverges then $\sum b_n$ diverges.

Proof

For convergence: partial sums satisfy $\sum_{k=1}^{N} u_n \leq \sum_{k=1}^{N} b_n \leq M$, so $(\sum a_k)$ is monotone bounded $\rightarrow$ converges.

The divergence part is contrapositive.

■

4.5 Limit comparison test

Statement. If $u_n > 0$, $b_n > 0$ and $L = \lim_{n\to\infty} \frac{u_n}{b_n}$ exists in $]0, \infty[$, then $\sum u_n$ and $\sum b_n$ either both converge or both diverge.

Sketch of proof.

For large n, u_n is between $\frac{L}{2} b_n$ and $\frac{3L}{2} b_n$. Then apply comparison test.

5 Comparison theorem with its integral

Theorem 3.2

A series whose general term is of the form $u_n = f(x)$; where f is a functioncontinuous, positive, and decreasing towards 0; it is of the same nature as the sequence $(\int_1^n f(x)dx)_n$

$$\sum u_n \textit{ converge} \Longleftrightarrow \lim_{n\to+\infty} \int_1^n f(x)dx \textit{ exists}$$

Proof

Suppose that $u_n = f(n)$; where f is a continuous, positive, decreasing function towards 0 from $n = 1$. On each segment $[n, n+1]$ we have $f(n+1) \leq f(x) \leq f(n)$. from where

$$\begin{aligned} \int_1^{n+1} f(n+1)dx &\leq \int_n^{n+1} f(x)dx \leq \int_n^{n+1} f(n)dx \\ f(n+1) &\leq \int_n^{n+1} f(x)dx \leq f(n) \end{aligned}$$

as

$$\int_1^{n+1} f(x)dx = \int_1^2 f(x)dx + \int_2^3 f(x)dx + ... + \int_n^{n+1} f(x)dx$$

we will have

$$f(2) + f(3) + ... + f(n+1) \leq \int_1^{n+1} f(x)dx \leq f(1) + f(2) + f(2) + ... + f(n)$$

Or $f(n) = u_n$, $\forall n \geq 1, i.e;$

$$S_{n+1} - u_1 = u_2 + u_2 + ... + u_{n+1} \leq \int_1^{n+1} f(x)dx \leq u_1 + u_2 + ... + u_n = S_n$$

- *If* $\lim_{n\to\infty} \left(\int_1^n f(x)dx\right)$ *exists, we have*

$$S_{n+1} \leq \int_1^{n+1} f(x)dx + u_1 \Longrightarrow \lim_{n\to+\infty} S_{n+1} \leq u_1 + \lim_{n\to+\infty} \left(\int_1^n f(x)dx\right)$$

  $(S_n)_{n\in N}$ *converges so the series* $\sum u_n$ *converges.*

- *If* $\lim_{n\to\infty} \left(\int_1^n f(x)dx\right) = +\infty$ *(because* f *is positive), we have* $\int_1^{n+1} f(x)dx \leq S_n$ *which implies that* $\lim_{n\to\infty} S_n \geq \lim_{n\to\infty} \left(\int_1^n f(x)dx\right) = +\infty$ *and that* $(S_n)_{n\in N}$ *diverge.*

Conversely, if the series converges, $\lim_{n\to\infty} \left(\int_1^n f(x)dx\right)$ *exists, and if the series diverges, the limit* $\lim_{n\to\infty} \left(\int_1^n f(x)dx\right) = +\infty$.

■

Remark

Condition $u_n = f(n)$; where f is a continuous, positive function decreasing towards 0; does not need to be true from $n = 1$; it only needs to be true from a certain rank. The positive condition can be replaced by a function of **constant sign**

5.1 The Riemann Series (p-series)

The *p-series* or *Riemann series* is defined by

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

where $p > 0$.

Convergence.

- The series converges if $p > 1$.
- The series diverges if $p \leq 1$.

Proof

Let $f(x) = \frac{1}{n^p}$*, continuous, positive, and decreasing on* $[1, \infty[$.
By the Integral Test:

$$\sum_{n=1}^{\infty} f(x) \text{ converges} \iff \int_1^{\infty} f(x)dx \text{ converges}$$

Compute:

$$\int_1^{\infty} f(x)dx = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p \leq 1 \end{cases}$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if and only if* $p > 1$.

■

Example

What error do we make when we approximate the sum $S = \sum_1^{+\infty} \frac{1}{n^2}$ by the partial sum of order n of the series?

We know that $R = s - S_n = \sum_{k=n+1}^{+\infty} \frac{1}{k^2}$, which is between $\int_{n+1}^{+\infty} \frac{1}{x^2}dx = \frac{1}{n+1}$ and $\int_n^{+\infty} \frac{1}{x^2}dx = \frac{1}{n}$.

To obtain 3 exact numbers, we take the partial sum of order $n = 1000$, in fact $S_{1000} = 1.64393$ and $S = \frac{\pi^2}{6} = 1.6449$.

6 Theorems for comparing two series with positive terms

Theorem 3.3

Let $\sum u_n$ and $\sum v_n$ be two series verifying $0 \leq u_n \leq v_n$ from a certain rank n_0.

- If the series $\sum v_n$ converges, then series $\sum u_n$ also converges $\sum_{k=n_0}^{+\infty} u_k \leq \sum_{k=n_0}^{+\infty} v_k$ converges as well.
- If series $\sum u_n$ diverges, then $\sum v_n$ diverges.

Example

1) Let the series be $\sum_{n\geq 1} \frac{|\cos n|}{n^2}$; its general term, $u_n = \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$ or $\sum_{n\geq 1} \frac{1}{n^2}$, is a convergent Riemann series, so $\sum_{n\geq 1} \frac{|\cos n|}{n^2}$ also converges.

2) The series $\sum_{n\geq 1} \frac{3+\sin(\ln n)}{n^2}$ is a series with positive terms such that $\forall n \geq 1$, $\frac{3+\sin(\ln n)}{n^2} \geq \frac{3}{n}$, As $\sum_{n\geq 1} \frac{3}{n}$ is (the harmonic series) is divergent, then $\sum_{n\geq 1} \frac{3+\sin(\ln n)}{n^2}$ also diverges.

Theorem 3.4

Given two series with positive terms, whose general terms are u_n and v_n, equivalent at infinity ($u_n \sim v_n$ in the neighborhood of $+\infty$), then $\sum u_n$ and $\sum v_n$ are of the same nature:

$$\sum u_n \text{ converge} \iff \sum v_n \text{ converge}$$
$$\sum u_n \text{ diverge} \iff \sum v_n \text{ diverge.}$$

Example

Consider the series $\sum_{n\geq 1} \left(\frac{2}{3}\right)^n \left(1+\frac{1}{n}\right)^n$. We have $\left(1+\frac{1}{n}\right)^n \sim e$ when $n \in V(+\infty)$, hence $\left(\frac{2}{3}\right)^n \left(1+\frac{1}{n}\right)^n \sim \left(\frac{2}{3}\right)^n e$. Since $\sum_{n\geq 0} \left(\frac{2}{3}\right)^n$ is a convergent geometric series, then the series $\sum_{n\geq 1} \left(\frac{2}{3}\right)^n \left(1+\frac{1}{n}\right)^n$ is also convergent.

6.1 Riemann's Rule

Let $\sum_{n\geq 1} u_n$ be a series with positive terms. Assume that a $\exists \alpha \in \mathbb{R}$ and $I \in \mathbb{R}^+ \cup \{+\infty\}$ are such that $\lim_{n\to+\infty} n^\alpha u_n = l$.

- If $l \neq +\infty$ and $\alpha > 1$, then $\sum u_n$ a converges.

- If $l \neq 0$ and $\alpha \leq 1$, then $\sum u_n$ diverges.

Proof

- *If $l \neq +\infty$ and $\alpha > 1$: $\lim_{n\to+\infty} n^\alpha u_n = l$ means that $u_n \sim \frac{1}{n^\alpha}$ or $\sum_{n\geq 1} \frac{1}{n^\alpha}$ is a convergent Riemann series (because $\alpha > 1$) therefore $\sum u_n$ converges*

- *If $l = +\infty$ and $\alpha \leq 1$:*

$\lim_{n\to+\infty} n^\alpha u_n = l \implies \exists$integer such that $n \geq N \implies n^\alpha u_n \geq 1$ i.e $u_n \geq \frac{1}{n^\alpha}$,as $\sum_{n\geq 1} \frac{l}{n^\alpha}$ diverges $(\alpha \leq 1)$ then $\sum u_n$ diverges.

If $l \in \mathbb{R}^*$ and $\alpha \leq 1$: we have $u_n \sim \frac{1}{n^\alpha}$ and as $\sum_{n\geq 1} \frac{l}{n^\alpha}$ diverges $(\alpha \leq 1)$ then $\sum u_n$ diverges.

■

Example

Let's study the nature of the series: $\sum_{n\geq 2} \frac{1}{(\ln n)^2}$

$\lim_{n\to+\infty} n\frac{1}{(\ln n)^2} = +\infty$, so $\sum_{n\geq 2} \frac{1}{(\ln n)^2}$ diverge.

6.2 D'Alembert's Rule

Let $\sum u_n$ be a series with positive terms.

Let us denote

$$l = \lim_{n\to+\infty} \frac{u_{n+1}}{u_n} \text{ such as } I \in \mathbb{R}^+ \cup \{+\infty\}$$

- If $l < 1$ the series $\sum u_n$ converge.

- If $l > 1$ the series $\sum u_n$ diverge.

- If $l = 1$ we can't conclude anything

Example

- Let the series $\sum_{n\geq 0} \frac{1}{n!}$ have a general term $u_n = \frac{1}{n!}$, so

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

from which the series converges according to the D'Alembert criterion.

- For the series $\sum_{n\geq 0} \frac{1}{n}$, we have $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$, so we cannot conclude anything by applying D'Alembert's criterion, although we have already demonstrated its divergence.
- The series $\sum_{n\geq 0} \frac{1}{n^2}$ is a convergent Riemann series and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$

6.3 Cauchy Rule

Let the positive series with general term u_n. Note that

$$l = \lim_{n\to\to+\infty} \sqrt[n]{u_n} \textit{ such as } I \in \mathbb{R}^+ \cup \{+\infty\}$$

- If $l < 1$, the series $\sum u_n$ converges.
- If $l > 1$, the series $\sum u_n$ diverges.
- If $l = 1$: in case of doubt, we cannot conclude anything.

Example

Let $\sum_{n\geq 1} \left(\frac{n+1}{n}\right)^{-n^2}$. We have $\sqrt[n]{u_n} = \left(1 + \frac{1}{n}\right)^{-n} \to \frac{1}{e} < 1$, so the series converges.

Corollary 3.1

If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$ then $\lim_{n\to\infty} \sqrt[n]{u_n} = l$
and if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = +\infty$ then $\lim_{n\to\infty} \sqrt[n]{u_n} = +\infty$
So if we encounter the doubtful case when applying D'Alembert's rule, we should not apply Cauchy's rule because we will also be unable to conclude anything.

6.4 Raabe-Duhamel Rule

Let $u_n > 0, \forall n \geq n_0$ and denote by $l = lim_{n\to+\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right)$ where $l \in \mathbb{R}$.

Then

- If $l > 1 \Longrightarrow$ the series $\sum u_n$ converges
- If $l < 1 \Longrightarrow$ the series $\sum u_n$ diverges

Example

Let us consider the series $\sum_{n\geq 0} \frac{1}{n^2}$ once again. The limit $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, D'Alembert's criterion, does not allow us to conclude its nature, so let us apply the Raabe-Duhamel rule:

$$lim_{n\to+\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = lim_{n\to+\infty} n\left(\frac{1+2n}{n^2}\right) = 2 > 1$$

which proves the convergence of the series.

6.5 Gauss's Rule

Let $u_n > 0, \forall n \geq n_0$ and assume that $\exists\,(\alpha, \beta) \in \mathbb{R}\times]1, +\infty[$ such that $\frac{u_{n+1}}{u_n} = 1 - \frac{\alpha}{n} + o\left(\frac{1}{n^\beta}\right)$.

Then: $\exists k \in \mathbb{R}_+^*$ such that $u_n \sim \frac{k}{n^\alpha}$ when n tends to infinity, i.e.,

- If $\alpha > 1 \Longrightarrow$ the series $\sum u_n$ converges
- If $\alpha < 1 \Longrightarrow$ the series $\sum u_n$ diverges

Example

Let $\sum u_n$ be with $u_n = \sqrt{n!}\sin 1 \sin \frac{1}{\sqrt{2}} .... \sin \frac{1}{\sqrt{n}}, \forall n \geq 1$.

$$\frac{u_{n+1}}{u_n} = 1 - \frac{1}{6n} + o\left(\frac{1}{n^2}\right)$$

hence $\alpha = 1/6$. This proves that the series diverges.

7 Series with arbitrary terms

7.1 Absolutely convergent series

Definition 3.5

The series $\sum u_n$ is said to be absolutely convergent when $\sum |u_n|$ converges.

Example

The series $\sum_{n\geq 2} \frac{(-1)^n e^{in}}{n^2+n}$ converges absolutely because the modulus of the general term is equal to $\frac{1}{n^2+n}$ which can be bounded above by $\frac{1}{n^2}$.

The series $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n(n+1)}}$ is not absolutely convergent, because $\left|\frac{(-1)^n}{\sqrt{n(n+1)}}\right| = \frac{1}{\sqrt{n(n+1)}} \geq \frac{1}{n+1}$ and $\sum \frac{1}{n+1}$ diverges.

Lemma 3.1

Any absolutely convergent series is convergent.

Definition 3.6

The series $\sum u_n$ is said to be semi-convergent when $\sum u_n$ converges but $\sum |u_n|$ diverges

D'Alembert and Cauchy rules for series with arbitrary terms

Rule 1: Let $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = l$

- If $l < 1 \implies$ the series $\sum u_n$ converges absolutely
- If $l > 1 \implies$ the series $\sum u_n$ does not converge absolutely
- If $l = 1$: in case of doubt.

Rule 2: Let $\lim_{n\to\infty} \sqrt[n]{|u_n|} = l$

- If $l < 1 \implies$ the series $\sum u_n$ converges absolutely
- If $l > 1 \implies$ the series $\sum u_n$ does not converge absolutely
- If $l = 1$: in case of doubt.

7.2 Abel's Criterion

Theorem 3.5

Let the series $\sum u_n$ be such that , $u_n = a_n b_n$ with

- The sequence is $(a_n)_{n\in\mathbb{N}}$ positive, decreasing towards 0.
- $\exists M$ constant such that $\forall n \in \mathbb{N}$; $|\sum_{k=0}^{n} b_k| \leq M$

Then the series $\sum u_n$ converges.

Fundamental examples

Example

Series of the form $\sum_{n\geq 1} \frac{\cos(nx)}{n^\alpha}$ or $\sum_{n\geq 1} \frac{\sin(nx)}{n^\alpha}$ where $x \in \mathbb{R} \setminus \{2m\pi\,;\, m \in \mathbb{Z}\}$ and $\alpha > 0$.

Let

$$A = 1 + e^{ix} + e^{i2x} + \cdots + e^{inx} = \frac{1 - e^{i(n+1)x}}{1 - e^{ix}} = e^{in\frac{x}{2}} \frac{\sin\left(\frac{x(n+1)}{2}\right)}{\sin\frac{x}{2}}.$$

Which gives

$$1 + \cos x + \cos 2x + \cdots + \cos nx = \Re(A) = \cos\left(\frac{nx}{2}\right) \frac{\sin\left(\frac{x(n+1)}{2}\right)}{\sin\frac{x}{2}},$$

and

$$1 + \sin x + \sin 2x + \cdots + \sin nx = \Im(A) = \sin\left(\frac{nx}{2}\right) \frac{\sin\left(\frac{x(n+1)}{2}\right)}{\sin\frac{x}{2}}.$$

Hence

$$|1 + \cos x + \cos 2x + \cdots + \cos nx| \leq \frac{1}{\left|\sin\frac{x}{2}\right|},$$

and

$$|1 + \sin x + \sin 2x + \cdots + \sin nx| \leq \frac{1}{\left|\sin\frac{x}{2}\right|}.$$

These upper bounds are independent of n, so we can apply Abel's criterion to demonstrate the convergence of the series $\sum_{n\geq 1} \frac{\cos(nx)}{n^\alpha}$ and $\sum_{n\geq 1} \frac{\sin(nx)}{n^\alpha}$, knowing that the sequence $\left(\frac{1}{n^\alpha}\right)_{n\geq 1}$ is positive and decreasing towards 0 for all $\alpha > 0$ and $x \neq 2m\pi$. Moreover, for $\alpha > 1$, the series converges absolutely.

$$\sum_{n\geq 1} \frac{\cos(nx)}{n^\alpha} \quad \text{converges for } \alpha > 0 \text{ and } x \not\equiv 0 \pmod{2\pi}.$$

$$\sum_{n\geq 1} \frac{\sin(nx)}{n^\alpha} \quad \text{converges for } \alpha > 0,\ \forall x \in \mathbb{R}.$$

Example: Alternating Series

Definition 3.7

An alternating series $\sum u_n$ is a series whose general term is alternately positive and then negative, i.e.,

$$u_n = (-1)^n a_n \text{ such as } a_n \geq 0\ ,\ \forall n \geq 0$$

Corollary 3.2: Leibnitz criterion

Any alternating series $\sum(-1)^n a_n$ whose absolute value of the general term decreases towards 0 is convergent.

Indeed: For any integer n, $\left|\sum_{k=0}^{n}(-1)^k\right| \leq 2$, if the sequence $(a_n)_n$ also decreases towards 0, then $\sum(-1)^n a_n$ converges according to Abel's criterion.

Special case: The series $\sum_{n\geq 1} \frac{(-1)^{n-1}}{n^\alpha}$ is a convergent alternating series for $\alpha > 0$. Moreover, the series is semi-convergent for $0 < \alpha \leq 1$ and absolutely convergent for $\alpha > 1$.

Proposition 3.1

The remainder of order n of a convergent alternating series is bounded above by:

$$|R_n| = \left|\sum_{k=n+1}^{+\infty} u_k\right| \leq |u_{n+1}|$$

Example

Let us find an integer n such that the partial sum S_n approaches the sum of the series $\sum_{n\geq 1} \frac{(-1)^n}{n^3}$ to within 10^{-6}.

This series is an absolutely convergent alternating series, therefore

$$|R_n| \leq \frac{1}{(n+1)^3} < 10^{-6} \Longrightarrow n \geq 100.$$

7.3 Using asymptotic expansion:

This is a widely used technique for series with arbitrary terms for which the preceding criteria do not apply. In many situations, conclusions about the nature of a series are drawn.One can draw conclusions by reducing to a simpler series. For series with positive terms, for example, it is sufficient to reduce to an equivalent; this is no longer the case with series with arbitrary terms. Furthermore, an equivalent corresponds to a first-order approximation, which does not necessarily allow a conclusion. In these cases, it is sufficient to provide an asymptotic expansion of the general term.

Example

Given the series $\sum_{n\geq 2} \frac{(-1)^n}{n+(-1)^n}$, the sequence $(|u_n|)_{n\geq 2}$ is not decreasing, so we cannot apply Abel's criterion. But we have $u_n = \frac{(-1)^n}{n}\frac{1}{1+\frac{(-1)^n}{n}}$ with $\lim_{n\to+\infty} \frac{(-1)^n}{n} = 0$, in which case we can use the limited expansion of $\frac{1}{1+u}$ near 0:

$$u_n = \frac{(-1)^n}{n}\frac{1}{1+\frac{(-1)^n}{n}} = \frac{(-1)^n}{n}\left[1 - \frac{(-1)^n}{n} + \frac{1}{n^2} + O\left(\frac{1}{n^2}\right)\right]$$

Hence

$$u_n = \frac{(-1)^n}{n} - \frac{1}{n^2} + \frac{(-1)^n}{n^3} + O\left(\frac{1}{n^3}\right)$$

we have:

- $\sum \frac{(-1)^n}{n}$ is a convergent alternating series.
- $\sum \frac{1}{n^2}$ is a convergent Riemann series.
- $\sum \frac{(-1)^n}{n^2}$ is an absolutely convergent alternating series.

Landau notation: We say that a function f is an $O(x^k)$ near 0, if there exist $c_1 > 0$ and $c_2 > 0$ such that, near 0, we have $c_1\left|x^k\right| \leq |f(x)| \leq c_2\left|x^k\right|$.

Hence $O(\frac{1}{n^3})$ is the general term of an absolutely convergent series, then the series $\sum_{n\geq 2} \frac{(-1)^n}{n+(-1)^n}$ converges.

Remark

Using the asymptotic expansion, we must expand to a sufficiently high order to obtain an **absolutely convergent** remainder.

Chapter 4

Multivariable functions

A multivariable function (or multivariate function) is a mathematical function whose input consists of multiple independent variables and/or whose output consists of multiple dependent variables. It extends the familiar concept of a single-variable function, $y = f(x)$, to higher dimensions.

1 General definitions

Definition 4.1

A *multivariable function* (or *function of several variables*) is a rule that assigns one output value to each ordered set of input values coming from two or more variables. It can be written as:

$$f(x, y) \quad \text{or} \quad f(x, y, z) \quad \text{or more generally} \quad f(x_1, x_2, \dots, x_n).$$

- **Domain:** A subset of $\mathbb{R}^n$ — all possible inputs.
- **Codomain:** Usually $\mathbb{R}$ (real number output).

Example in Calculus

Multivariable functions are the central object of study in multivariable (or multivariate) calculus.

Function Type	Notation	Description	Visualized as
Two-Variable Input , Single Output	$z = f(x, y)$ $= x^2 + y^2$	Takes two inputs (x and y) and returns one output (z).	A surface in 3D space.
Three-Variable Input, Single Output	$w = f(x, y, z)$ $= x + 2y - z$	Takes three inputs (x,y,z) and returns one output (w).	Hard to graph, but its level sets (w=k) are surfaces in 3D space.
Single-Variable Input, Vector Output	$r(t) = \langle \cos(t), \sin(t) \rangle$	Takes one input (t) and returns a vector output (2 components).	A curve in 2D space (a parametric curve).
Multivariable Input, Vector Output	$F(x, y) = \langle -y, x \rangle$	Takes two inputs (x and y) and returns a vector output.	A vector field in 2D space.

1.1 $n-$dimensional space

The space $\mathbb{R}^n$ is the set of points described by n coordinates $(x_1, x_2, ...x_n)$, each of these coordinates being a real number.

- For one variable: $y = f(x) \to$ a curve in $2D$ space
- For two variables: $z = f(x, y) \to$ a surface in $3D$ space
- For three variables: $w = f(x, y, z) \to$ a hypersurface (harder to visualize)

1.2 Level Curves and Level Surfaces

Level curves and level surfaces are tools used to visualize multivariable functions by showing where the function takes constant values

1.2.1 Level Curves (for functions of two variables)

Definition 4.2

If $f(x, y)$ is a function of two variables, then a level curve is defined by:

$$f(x, y) = c \quad \text{(where } c \text{ is a constant)}$$

Each value of c gives one curve in the $xy-$plane.

Example

Let

$$f(x,y) = x^2 + y^2$$

Level curves are defined by:

$$x^2 + y^2 = c$$

- If $c = 1 : x^2 + y^2 = 1 \to$ Circle of radius 1
- If $c = 4 : x^2 + y^2 = 4 \to$ Circle in the $xy-$plane.

So level curves are concentric circles.

These curves help visualize how the function behaves without drawing a $3D$ plot.

1.2.2 Level Surface (for functions of three variables)

Definition 4.3

If $f(x,y,z)$ is a function of three variables, then a level surface is defined by:

$$f(x,y,z) = c$$

Each value of (c) gives one surface in $3D$ space.

Example

Let:

$$f(x,y,z) = x^2 + y^2 + z^2$$

Level surfaces are :

$$x^2 + y^2 + z^2 = c$$

- If $c = 1$: Sphere of radius 1
- If $c = 4$: Sphere of radius 2

So the level surfaces are spheres centered at the origin.

1.3 Functions of several variables

Definition 4.4

A function of n variables is a function f that associates with n real numbers $n = 3$ $(espace) \to \mathbb{R}^3$, lying in its definition set D_f, a real number

$$f(x_1, x_2, x_3).$$

Definition 4.5

In other words: a function of n variables is a function f that associates with a point $x = (x_1, x_2, ...x_n)$ of $\mathbb{R}^n$ located in its definition set, the real number

$$f(x) = f(x_1, x_2, ...x_n)$$

Example

The function $(x, y, x) \to \log(x^2 + y^2 + z^2)$ *is uyne function of three variables whose natural domain of definition.*

Example

The internal energy of a Van der Waals gas

$$U(P, V) = -\frac{a}{V} + \frac{3}{2}\left(P\,(V - b) + a\frac{V - b}{V^2}\right)$$

which is a function of the two variables (P, V)

1.4 Graphical representation

Definition 4.6

The graphical representation of a function of n variables is the subset of $\mathbb{R}^{n+1}$ formed by all points $(x_1, x_2, \ldots, x_n, x_{n+1})$ such that

$$x_{n+1} = f(x_1, x_2, \ldots, x_n),$$

where $(x_1, x_2, \ldots, x_n)$ ranges over the domain D_f.

Example

Figure 1.4 shows the surface graph of the function of two variables giving the internal energy of one mole of helium when Van der Waals' law is applied

$$U(P,V) = -\frac{a}{V} + \frac{3}{2}\left(P\,(V-b) + a\frac{V-b}{V^2}\right)$$

(a = $3.45 kPa.dm^6$ and b = $0.0237 dm^3$ for one mole of helium). This surface already gives a number of ideas about the properties of the U function. We'll have the opportunity to see more of them later.

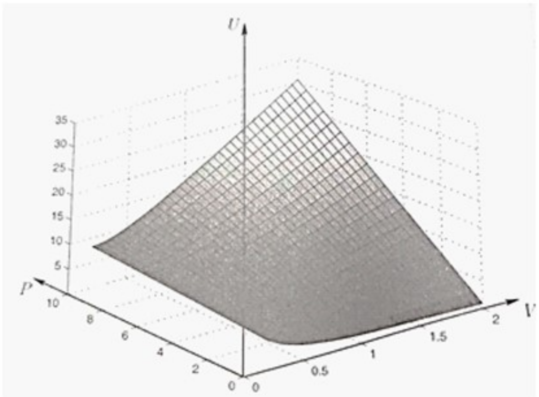


Figure 1.4 Surface graph of the internal energy function of a Van der Waals gas.

2 Partial functions

A partial function of $f(x_1, x_2, ...x_n)$ is a function of the type

$$t \rightarrow f(p_1, ..., p_{i-1}, t, p_{i+1}, ..., p_n),$$

where the real numbers $p_1, p_2, ...p_{i-1}, p_{i+1}, p_{i+2}, ...p_n$ have fixed values.

Example

consider the function of two variables $f(x,y) = -yx^2 - y$, whose definition set is the plane.

Fixing $x = 2$, we obtain the partial function

$$y \rightarrow f(2,y) = -5y$$

If we fix $x = 0$, the partial function becomes $y \to f(0, y) = -y$, If $x = -2$, we again obtain $y \to f(-2, y) = -5y$.

Now let's set a value for y and let x vary. Let's start with $y = 2$. The partial function is then

$$x \to f(x, 2) = 2x^2 + 2$$

For $y = 0$, this is the null function, while for $y = -2$ we find $x \to f(x, -2) = -2x^2 - 2$

3 Partial derivatives

The $i - th$ partial derivative of f in $p = (p_1, ...p_n)$ is the derivative in $t = p_i$ of the $i - th$ partial function

$$t \to f\left(p_1, ...p_{i-1}, t, p_{i+1}, p_{i+2}, ...p_n\right)$$

When this derivative exists, i.e. if this partial function is derivable.

Its value is denoted

$$\frac{\partial f}{\partial x_i}(p) \qquad \text{ou} \qquad \frac{\partial f}{\partial x_i}(p_1, ...p_n)$$

Example

Let the function $f(x, y) = -yx^2 - y$, To calculate the partial derivatives of f

$$\begin{aligned} \frac{\partial}{\partial x} f(x, y) &= -2xy \\ \frac{\partial}{\partial y} f(x, y) &= -\left(x^2 + 1\right) \end{aligned}$$

4 Higher-order derivatives

Let f be a function with two variables x and y Consider the two partial derivatives of f define two functions $(x, y) \to \frac{\partial f}{\partial x}(x, y)$ and $(x, y) \to \frac{\partial f}{\partial x}(x, y)$, which we can re-derive to obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \ et \ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

W*hich we call partial derivatives of order 2 and are noted respectively:*

$$\left(\frac{\partial^2 f}{\partial x^2}\right), \left(\frac{\partial^2 f}{\partial x \partial y}\right), \left(\frac{\partial^2 f}{\partial y \partial x}\right) \ et \ \left(\frac{\partial^2 f}{\partial^2 y}\right)$$

Example

Calculate the 1st and 2nd order partial derivatives of the function

$$f(x,y) = x^2 + xy + e^x$$

1) Partial derivatives of order 1:

$$\begin{aligned} \frac{\partial}{\partial x} f(x,y) &= 2x + y + e^x \\ \frac{\partial}{\partial y} f(x,y) &= x \end{aligned}$$

2) Partial derivatives of order 2:

$$\begin{aligned} \frac{\partial^2}{\partial x^2} f(x,y) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x + y + e^x) = 2 + e^x \\ \left(\frac{\partial^2 f}{\partial x \partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x) = 1 \\ \left(\frac{\partial^2 f}{\partial y \partial x}\right) &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x + y + e^x) = 1 \\ \frac{\partial^2}{\partial y^2} f(x,y) &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(x) = 0 \end{aligned}$$

5 Differential

Let $z = f(x;y)$, then

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x,y) = \Delta f$$

Where Δ is the straight line of equation $y = -x$, is called the increment of f.

If f has continuous first partial derivatives in a domain D, then

$$\begin{aligned} \Delta z &= \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \\ &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y = \Delta f \end{aligned}$$

Where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ when $\Delta x \to 0$ and $\Delta y \to 0$.

In this case we say that f is continuously differentiable in D:

The expression

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df$$

is called the total differential of z or f.

Example

Let $U = x^2 e^{\frac{y}{x}}$. Calculate dU.

We have

$$\begin{aligned} \frac{\partial U}{\partial x} &= 2xe^{\frac{y}{x}} + x^2\left(-\frac{y}{x^2}\right)e^{\frac{y}{x}} \\ \frac{\partial U}{\partial y} &= x^2\frac{e^{\frac{y}{x}}}{x} \end{aligned}$$

hence

$$\begin{aligned} dU &= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy \\ &= (2x - y)\, e^{\frac{y}{x}} dx + xe^{\frac{y}{x}} dy \end{aligned}$$

6 Higher-order derivatives

Let f be a function with two variables x and y.

Consider the two partial derivatives of f define two functions $(x, y) \to \frac{\partial f}{\partial x}(x, y)$ and $(x, y) \to \frac{\partial f}{\partial x}(x, y)$, which we can re-derive to obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \ et\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

which we call partial derivatives of order 2 and are noted respectively:

$$\left(\frac{\partial^2 f}{\partial x^2}\right), \left(\frac{\partial^2 f}{\partial x \partial y}\right), \left(\frac{\partial^2 f}{\partial y \partial x}\right) \ et\ \left(\frac{\partial^2 f}{\partial^2 y}\right)$$

Example

Calculate the 1st and 2nd order partial derivatives of the function

$$f(x, y) = x^2 + xy + e^x$$

1) Partial derivatives of order 1:

$$\begin{aligned} \frac{\partial}{\partial x} f(x,y) &= 2x + y + e^x \\ \frac{\partial}{\partial y} f(x,y) &= x \end{aligned}$$

2) Partial derivatives of order 2:

$$\begin{aligned} \frac{\partial^2}{\partial x^2} f(x,y) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x + y + e^x) = 2 + e^x \\ \left(\frac{\partial^2 f}{\partial x \partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x) = 1 \\ \left(\frac{\partial^2 f}{\partial y \partial x}\right) &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x + y + e^x) = 1 \\ \frac{\partial^2}{\partial y^2} f(x,y) &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(x) = 0 \end{aligned}$$

7 Differentiation of compound functions

7.1 Case of two independent variables:

Let $z = f(x,y)$ where $x = g(r,s)$ and $y = h(r,s)$, so z is a function of r and s. Then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Higher-order derivatives are obtained by successive application of these rules.

Example

If $z = f(x,y)$ where $x = uv$ and $y = \frac{u}{v}$, we find

$$\begin{aligned} \frac{\partial z}{\partial u} &= f'_x(x,y)v + f'_y(x,y)\frac{1}{v} \\ \frac{\partial z}{\partial v} &= f'_x(x,y)u - f'_y(x,y)\frac{u}{v^2} \end{aligned}$$

7.2 Case of an independent variable

Let $z = f(x, y)$ where $x = \varphi(t)$ *and* $y = \Psi(t)$, from which $z = f(\varphi(t); \Psi(t))$ consequently

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example

If $z = e^{3x+2y}$ where $x = \cos t$ et $y = t^2$ then we have

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\\ &= -3\sin t e^{3x+2y} + 4te^{3x+2y}\\ &= (-3\sin t + 4t)\, e^{3x+2y}\\ &= (-3\sin t + 4t)\, e^{3\cos t+2t^2}\end{aligned}$$

Example

Show that $(3x^2y - 2y^2)\, dx + (x^3 - 4xy + 6y^2)\, dy$ can be written as the total differential of a function $\Phi(x, y)$ and find this function.

Solution We have

$$\frac{\partial \Phi}{\partial x} = 3x^2y - 2y^2 \Longrightarrow \Phi(x, y) = \int 3x^2y - 2y^2 dx = x^3y - 2y^2x + c(y)$$

hence

$$\frac{\partial \Phi}{\partial y} = x^3 - 4xy + 6y^2 = x^3 - 4yx + c'(y) \Longrightarrow c'(y) = 6y^2 \Longrightarrow c(y) = 2y^3 + c,$$

so

$$\Phi(x, y) = x^3y - 2y^2x + 2y^3 + c$$

8 Double integrals

Definition 4.7

The double integral of a function of two variables over the domain D is called the integral of the function and is given by: $\int\int_D f(x, y)dxdy$

8.1 Calculating double integrals

8.1.1 Fubini formulas

Theorem 4.1

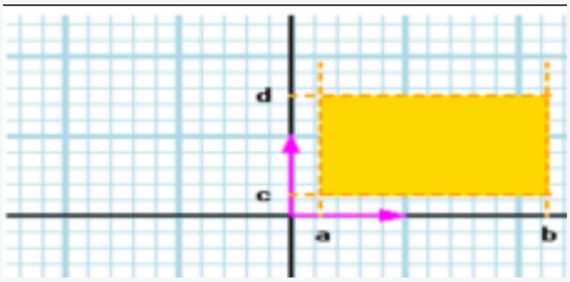
Let f be a continuous function on a rectangle $D = [a,b] \times [c,d]$. We have

$$\int\int_D f(x,y)dxdy = \int_a^b \left[\int_c^d f(x,y)dy\right] dx = \int_c^d \left[\int_a^b f(x,y)dx\right] dy$$

So we calculate a double integral over a rectangle by calculating two simple integrals:

- First, by integrating with respect to x between a and b (leaving y constant). The result is a function of y
- By integrating this expression of y between c and d.

Alternatively, you can do the same by integrating first in y and then in x.



Example

Calculate the following double integrals:

1. $\int\int_D x^2y - 3xy^2dxdy \quad D = \left\{(x,y) \in R^2, \quad 1 \leq x \leq 2, \quad -1 \leq y \leq 1\right\}$

2. $\int\int_D ye^{xy}dxdy \quad D = \left\{(x,y) \in R^2, \quad 1 \leq x \leq 2, \quad 0 \leq y \leq 2\right\}$

Solution

$$\begin{aligned} 1 - \int_{-1}^1 \left(\int_1^2 x^2y - 3xy^2dx\right) dy &= \int_{-1}^1 \left[\frac{1}{3}yx^3 - \frac{3}{2}y^2x^2\right]_1^2 dy \\ &= \int_{-1}^1 \frac{7}{3}y - \frac{9}{2}y^2dy = -\frac{18}{6} \end{aligned}$$

$$2 - \int_0^2 \left(\int_1^2 y e^{xy} dx \right) dy = \int_0^2 \left[e^{xy} \right]_1^2 dy = \int_0^2 e^{2y} - e^y dy$$
$$= \frac{1}{2} e^4 - e^2 + \frac{1}{2}$$

Corollary 4.1

A double integral of the form $\int \int_{[a,b]\times[c,d]} f(x)g(y)dxdy$ can be calculated by separating the variables:

$$\int \int_{[a,b]\times[c,d]} f(x)g(y)dxdy = \left(\int_a^b f(x)dx \right) \left(\int_c^d g(y)dy \right)$$

Example

Calculate the following double integrals:

$$1 - \int \int_{[0,1]\times\left[0,\frac{\pi}{2}\right]} x \cos y dxdy = \left(\int_0^1 xdx \right) \left(\int_0^{\frac{\pi}{2}} \cos ydy \right)$$
$$= \left[\frac{x^2}{2} \right]_0^1 [\sin y]_0^{\frac{\pi}{2}} = \frac{1}{2}$$

$$2 - \int \int_{[0,1]\times[1,2]} e^{x-y} dxdy = \left(\int_0^1 e^x dx \right) \left(\int_1^2 e^{-y} dy \right)$$
$$= [e^x]_0^1 \left[-e^{-y} \right]_1^2 = (e-1)\left(e^{-1} - e^{-2} \right)$$

8.2 Change of variable in a double integral

8.2.1 General case

Suppose we perform the change of variables $\begin{cases} x = \varphi(u,v) \\ y = \Psi(u,v) \end{cases}$

When (x, y) varies in domain D; (u, v) varies in a domain $D\prime$. Consider the determinant of the matrix $J = \begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} \\ \frac{\partial\Psi}{\partial u} & \frac{\partial\Psi}{\partial v} \end{vmatrix}$ which we call the Jacobian matrix of the coordinate change

We have:

$$\int \int_D f(x,y)dxdy = \int \int_D F(u,v).\left| J \right| dudv$$

We have:

$$\int \int_D f(x,y)dxdy = \int \int_D F(u,v).\left| J \right| dudv$$

Example

Calculate:

$$\int\int_D (x+y)^3(x-y)^2 dxdy \qquad \text{si } D \text{ est fermé par :}$$

$$y=-x+1; \qquad y=x-1; \quad y=3-x; \quad y=x+1$$

Solution

Change to polar coordinates

In this case : $u=\theta$, $v=r$: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

The Jacobian of the transformation of Cartesian coordinates x and y into polar coordinates and r is given by :

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

We therefore have:

$$\int\int_D f(x,y)dxdy = \int\int_{D_1} F(r,\theta).r.drd\theta$$

Example

Calculate $\int\int_D \frac{xdxdy}{1+x^2+y^2}$ or $D=\{(x,y)\in R^2,\ \ x^2+y^2\le 1,\ \ x\ge 0\}$

$$\Delta = \left\{(r,\theta)\in R_+\times R,\, 0\le r\le 1 \text{ and } -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\right\}$$

$$\begin{aligned}
\int\int_D \frac{xdxdy}{1+x^2+y^2} &= \int\int_\Delta \frac{r\cos\theta}{1+r^2} rdrd\theta \\
&= \left(\int_0^1 \frac{r^2}{1+r^2}dr\right)\left(\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta\right) \\
&= \left(\int_0^1 \frac{r^2+1-1}{1+r^2}dr\right)[\sin\theta]_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \\
&= 2\times\left(\int_0^1 1-\frac{1}{1+r^2}dr\right) \\
&= 2\left[r-\arctan r\right]_0^1 = 2\left(1-\frac{\pi}{4}\right) = 2-\frac{\pi}{2}
\end{aligned}$$

9 Triple integrals

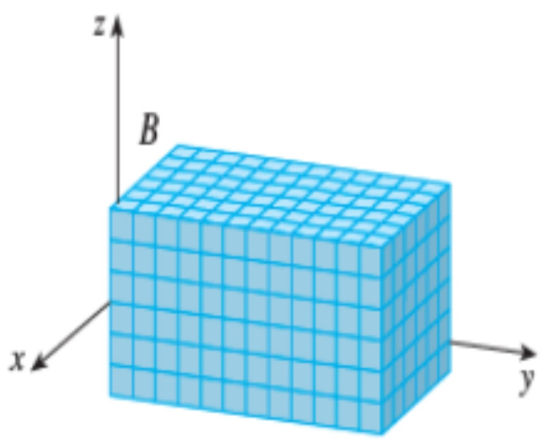
Triple integrals are powerful mathematical tools used to analyze three-dimensional biological systems. While double integrals apply to surfaces (like tissue layers or cell membranes), triple integrals extend integration to volumes, making them perfect for:

Biological Application	Example
Mass or density of tissues/organs	Computing mass of a tumor with non-uniform density
Diffusion of substances	Total drug concentration in a 3D organ
Cell population modeling	Number of cells distributed in an organoid (3D culture)
Oxygen / nutrient distribution	Total oxygen in lungs or blood plasma volume
Heat or chemical spread	Energy distribution in metabolic tissue

9.1 Mathematical Definition

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables. Let's first deal with the simplest case where f is defined on a rectangular box:

$$B = \{(x, y, z) |\, a \leq x \leq b,\ c \leq y \leq d,\ r \leq z \leq s\}$$



9.2 Theorem (Fubini's Theorem for Triple Integrals)

Theorem 4.2

If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\begin{aligned}
\int\int\int_B f(x,y,z)dV &= \int_a^b \int_c^d \int_r^s f(x,y,z)dzdydx \\
&= \int_a^b \int_r^s \int_c^d f(x,y,z)dydzdx \\
&= \int_c^d \int_a^b \int_r^s f(x,y,z)dxdzdy \\
&= \int_c^d \int_r^s \int_a^b f(x,y,z)dxdzdy \\
&= \int_r^s \int_a^b \int_c^d f(x,y,z)dydxdz \\
&= \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz
\end{aligned}$$

Example

Evaluate the triple integral $\int\int\int_B xyz^2 dV$, where B is the rectangular bo given by

$$B == \{(x,y,z)|\, 0 \le x \le 1, \ -1 \le y \le 2, \ 0 \le z \le 3\}$$

Solution We could use any of the six possible orders of integration. If we choose to integrate with respect to x, then y, and then z, we obtain

$$\begin{aligned}
\int\int\int_B xyz^2 dV &= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 dxdydz = \int_0^3 \int_{-1}^2 \left[\frac{x^2}{2} yz^2\right]_0^1 dydz \\
&= \int_0^3 \int_{-1}^2 \frac{yz^2}{2} dydz = \int_0^3 \left[\frac{y^2 z^2}{4}\right]_{-1}^2 dz = \int_0^3 \frac{3z^2}{4} dz \\
&= \left[\frac{z^3}{4}\right]_0^3 = \frac{27}{4}.
\end{aligned}$$

10 Calculation of Surfaces and Volumes in Calculus

10.1 Calculation of Surfaces (Areas)

In two-dimensional space, the primary tool for finding the area of complex regions is the definite integral.

10.1.1 Area Under a Curve (Single Integral)

The area A of the region bounded by a continuous, non-negative function $f(x)$, the $x - axis$, and the vertical lines $x = a$ and $x = b$ $(a < b)$ is given by the definite integral:

$$A = \int_a^b f(x)dx$$

Key Concepts

- **Non-Negativity:** The formula works directly when $f(x) \geq 0$ over the interval $[a, b]$. If the function dips below the $x - axis$ $(f(x) < 0)$, the integral calculates the signed area (negative value). To find the total physical area, you must integrate the **absolute value** of the function: $A = \int_a^b |f(x)|\, dx$

- **Riemann Sums:** The integral represents the limit of **Riemann sums**, where the area is approximated by summing the areas of an infinite number of infinitesimally thin rectangles under the curve.

Example

Find the area A under the curve $f(x) = x^3 + 1$ from $x = 0$ to $x = 2$.

Step 1: Set up the Integral Since $f(x) = x^3 + 1 > 0$ on the interval $[0, 2]$, we can apply the formula directly:

$$A = \int_0^2 (x^3 + 1)dx$$

Step 2: Find the Antiderivative Use the power rule for integration:

$$A = \left[\frac{x^4}{4} + x\right]_0^2$$

Step 3: Evaluate the Definite Integral Apply the Fundamental Theorem of Calculus (evaluate at the upper limit and subtract the evaluation at the lower limit):

$$\begin{aligned} A &= \frac{2^4}{4} + 2 - 0 \\ &= 6 \end{aligned}$$

The area under the curve is 6 square units.

10.1.2 Area of a Region in the Plane (Double Integral)

The Area of a Region in the Plane (A) is calculated using a double integral where the function being integrated is $f(x, y) = 1$. This method is particularly powerful for finding the area of complex or non-rectangular regions defined in the $xy - plane$.

The general formula is:

$$Area(R) = \int\int_R 1 dA$$

Where:

- R is the region in the $xy - plane$ whose area you want to find.
- dA represents the differential area element, which is $dxdy$ or $dydx$ (for Cartesian coordinates).

The double integral essentially sums up the area of infinitely small rectangles (with area dA) across the entire region R.

Example

Find the area A of the region R bounded by the parabola $y = x^2$ and the line $y = x+2$.

Step 1: Find Intersection Points

Set the equations equal to each other:

$$\begin{aligned} x^2 &= x + 2 \\ x^2 - x - 2 &= 0 \\ (x - 2)(x + 1) &= 0 \end{aligned}$$

The intersection points occur at $x = -1$ and $x = 2$.

Step 2: Set up the Double Integral

The region R is vertically simple, bounded below by $y = x^2$ and above by $y = x + 2$.

$$Area(R) = \int_{-1}^{2} \int_{x^2}^{x+2} 1dxdy$$

Step 3: Compute the Inner Integral (with respect to y)

$$\int_{x^2}^{x+2} 1dy = [y]_{x^2}^{x+2} = x + 2 - \left(x^2\right) = x + 2 - x^2.$$

Step 4: Compute the Outer Integral (with respect to x)

$$\begin{aligned} Area(R) &= \int_{-1}^{2} x + 2 - x^2 dx \\ &= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} \\ &= \frac{2^2}{2} + 4 - \frac{2^3}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) \\ &= \frac{9}{2} \end{aligned}$$

The area of the region is $\frac{9}{2}$ square units.

10.2 Calculation of Volumes

Volume calculations extend integration into three dimensions, allowing us to quantify the space occupied by biological solids.

10.3 Volume Under a Surface (Double Integral)

The Volume Under a Surface is calculated using a double integral over a two-dimensional region in the $xy - plane$. This method finds the volume of the three-dimensional region bounded above by the surface $z = f(x, y)$ and below by the region R.

Formula and Concept

The volume V under the surface $f(x, y)$ and above the region R in the $xy - plane$ is given by the double integral:

$$V = \int \int_R f(x, y)dA$$

Where:

- $f(x, y)$ is the height function (or "ceiling") of the solid at any point (x, y). The function must be non-negative, $f(x, y) \geq 0$, over R for the integral to represent the physical volume.

- R is the region of integration in the $xy - plane$ (the "floor" or domain).
- dA is the differential area element, which is typically $dxdy$ or $dydx$ in Cartesian coordinates.

Conceptually, the double integral sums the volumes of an infinite number of infinitesimally thin rectangular prisms (or "towers") of base area dA and height $f(x, y)$ across the entire region R.

Example

Find the volume V under the surface $f(x, y) = 4 - x - y$ and above the rectangular region R defined by $0 \leq x \leq 2$ and $0 \leq y \leq 1$.

Step 1: Set up the Iterated Integral

Since R is a rectangle, the integration limits are constants. We'll choose the order $dydx$:

$$V = \int_0^2 \int_0^1 (4 - x - y)dydx$$

Step 2: Compute the Inner Integral (with respect to y)

Treat x as a constant:

$$\begin{aligned} I_y &= \int_0^1 (4 - x - y)dy \\ &= \left[4y - xy - \frac{y^2}{2}\right]_{y=0}^{y=1} \\ &= 4 - x - \frac{1}{2} - 0 \\ &= \frac{7}{2} - x \end{aligned}$$

Step 3: Compute the Outer Integral (with respect to x)

Now, integrate the result $I_y = \frac{7}{2} - x$ from $x = 0$ to $x = 2$:

$$\begin{aligned} V &= \int_0^2 \frac{7}{2} - xdx \\ &= \left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^2 = \frac{7}{2}(2) - \frac{1}{2}(2)^2 - 0 = 5 \end{aligned}$$

The volume under the surface is 5 cubic units.

10.4 Volume of a 3D Region (Triple Integral)

The Volume of a $3D$ Region (V) is calculated using a triple integral where the function being integrated is $f(x, y, z) = 1$. This is the most general method for finding the volume of any defined region D in three-dimensional space.

$$V = \int\int\int_D 1.dV$$

The triple integral sums the volumes of an infinite number of infinitesimally small cubic volumes (dV) across the entire region D.

Setting up the Triple Integral

In Cartesian coordinates, the differential volume element is $dV = dxdydz$, $dydxdz$, or $dzdydx$ (and other permutations). The order of integration depends on how the boundaries of the region D are best described.

For a region D bounded above by the surface $z = g_2(x, y)$ and below by $z = g_1(x, y)$, and whose projection onto the $xy - plane$ is a region R, the volume is typically set up as:

$$V = \int\int_R \int_{g_1(x,y)}^{g_2(x,y)} 1dzdA$$

The Iterated Process

1. **Innermost Integral (with respect to z):** Integrates along the vertical dimension, from the lower surface $z = g_1(x, y)$ to the upper surface $z = g_2(x, y)$. This step converts the triple integral into a double integral over the base region R:

$$\int_{g_1(x,y)}^{g_2(x,y)} 1dz = [z]_{g_1(x,y)}^{g_2(x,y)} = g_2(x, y) - g_1(x, y)$$

   This result is the height of the region D at any point (x, y).

2. **Outer Double Integral (with respect to x and y):** The remaining double integral $\int\int_R (g_2(x, y) - g_1(x, y))\, dA$ calculates the volume by summing these heights over the entire base region R.

Example

Find the volume V of the solid region D bounded by the planes $z = 0$, $x = 0$, $y = 0$, and the plane $x + y + z = 1$ (in the first octant).

Step 1: Define the Boundaries

The solid D is a tetrahedron with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

- z **boundaries**: z goes from the bottom plane $z = 0$ to the top plane $z = 1-x-y$.
- **xy − plane region R boundaries:** This projection is a triangle bounded by $x = 0$, $y = 0$, and the line$x + y = 1$ (or $y = 1 - x$).

We set up the integral as $dzdydx$:

$$V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1dzdydx$$

Step 2: Compute the Innermost Integral (with respect to z)

$$I_z = \int_0^{1-x-y} 1dz = [z]_0^{1-x-y} = 1 - x - y$$

Step 3: Compute the Middle Integral (with respect to y)

$$\begin{aligned} I_y &= \int_0^{1-x} 1 - x - ydy \\ &= \left[y - xy - \frac{y^2}{2}\right]_0^{1-x} \\ &= \frac{1}{2} - x + \frac{x^2}{2} \end{aligned}$$

Step 4: Compute the Outermost Integral (with respect to x)

$$\begin{aligned} V &= \int_0^1 \frac{1}{2} - x + \frac{x^2}{2}dx \\ &= \left[\frac{1}{2}x - \frac{x^2}{2} + \frac{x^3}{6}\right]_0^1 \\ &= \frac{1}{2} - \frac{1}{2} + \frac{1}{6} - 0 \end{aligned}$$

The volume of the tetrahedron is $\frac{1}{6}$ cubic units.

11 Exercises

Exercise 1

1- *Calculate the 1st and 2nd order partial derivatives of the function*

$$f(x,y) = x^3y^2 + \sin(xy)$$

2- Calculate dU.

$$U = e^{3y}\sin x$$

Exercise 2

A limnologist is studying a species of toxic algae in a rectangular section of a lake defined by $R = [0,4] \times [0,6]$ (where x and y are measured in kilometers).

The population density of the algae, $\delta(x,y)$ (measured in millions of cells per square kilometer), is not uniform and is modeled by the function:

$$\delta(x,y) = 50 + 4xy$$

Calculate the total number of algae cells (in millions) in the entire region R.

Exercise 3

A microbiologist is growing a bacteria culture in a rectangular bioreactor defined by the volume V :

$$V = [0,4] \times [0,3] \times [0,1]$$

where x, y, z are measured in centimeters (cm).

The density of a specific functional protein within the culture, $\rho(x,y,z)$ (measured in milligrams per cubic centimeter, mg/cm^3), is modeled by the function:

$$\rho(x,y,z) = 6xz$$

Calculate the total mass (M, in milligrams) of this protein in the entire bioreactor volume V.

Exercise 4

A microbiologist models the growth of a small, developing bacterial colony in a cubic nutrient solution. The colony occupies the volume D defined by the coordinates:

$$0 \leq x \leq 2\ mm, 0 \leq y \leq 2\ mm, 0 \leq z \leq 2\ mm$$

The density of the bacteria at any point (x, y, z), measured in millions of cells per mm^3, is given by the function:

$$\rho(x, y, z) = xz^2$$

The task is to find the total number of bacteria (N) in the entire cube. Since density integrated over volume yields mass (or, in this case, total count), we use a triple integral:

$$N = \iiint_D \rho(x, y, z)\, dV$$

12 The answers

Exercise 1

I- **Partial Derivatives of** $f(x, y) = x^3y^2 + \sin(xy)$

A. First-Order Partial Derivatives

Derivative with respect to x $\left(\frac{\partial f}{\partial x} \text{ or } f_x\right)$:

Treat y as a constant.

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^3y^2) + \frac{\partial}{\partial x}(\sin(xy)) \\ &= 3x^2y^2 + y\cos(xy) \end{aligned}$$

Derivative with respect to y $\left(\frac{\partial f}{\partial y} \text{ or } f_y\right)$:

Treat x as a constant.

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^3y^2) + \frac{\partial}{\partial y}(\sin(xy)) \\ &= 2yx^3 + x\cos(xy) \end{aligned}$$

B. Second-Order Partial Derivatives

1. $\frac{\partial^2 f}{\partial x^2}$ **Differentiate f_x with respect to x.**

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\Big(3x^2y^2 + \cos(xy).(y)\Big)\\ &= 6xy^2 - y^2\sin(xy)\end{aligned}$$

2. $\frac{\partial^2 f}{\partial y^2}$ **Differentiate f_y with respect to y.**

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\Big(2yx^3 + x\cos(xy)\Big)\\ &= 2x^3 - x^2\sin(xy)\end{aligned}$$

3. **Mixed Partial Derivative** $\frac{\partial^2 f}{\partial x \partial y}$ **Differentiate $f_x$with respect to** y

$$\begin{aligned}\frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial}{\partial x}(2yx^3 + x\cos(xy))\\ &= 6x^2y + \cos(xy) - xy\sin(xy)\end{aligned}$$

II. **Calculate the Differential** dU

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$

Derivative with respect to x

$$\begin{aligned}\frac{\partial U}{\partial x} &= \frac{\partial}{\partial x}\Big(e^{3y}\sin x\Big)\\ &= e^{3y}\cos x\end{aligned}$$

Derivative with respect to y

$$\begin{aligned}\frac{\partial U}{\partial y} &= \frac{\partial}{\partial y}\Big(e^{3y}\sin x\Big)\\ &= 3e^{3y}\sin x\end{aligned}$$

Calculate the Total Differential (dU)

Substitute the partial derivatives back into the total differential formula:

$$dU = \Big(e^{3y}\cos x\Big)\,dx + \Big(3e^{3y}\sin x\Big)\,dy$$

We can factor out e^{3y} for a concise final form:

$$dU = e^{3y}\left(\cos x dx + 3\sin x dy\right)$$

Exercise 2

The total population (P) is found by integrating the density function over the region R:

$$P = \int\int_R \delta(x,y)dxdy = \int_0^6\int_0^4 50 + 4xydxdy$$

We will use iterated integration, integrating with respect to x first, then y.

Step 1: Integrate with respect to x (Inner Integral)

Treat y as a constant and integrate $50 + 4xy$ from $x = 0$ to $x = 4$:

$$\begin{aligned} I_x &= \int_0^4 (50 + 4xy)dx \\ &= 50\,[x]_0^4 + 2y\left[x^2\right]_0^4 \\ &= 200 + 32y \end{aligned}$$

Step 2: Integrate with respect to y (Outer Integral)

Now, integrate the result of the inner integral ($I_x = 200 + 32y$) from $y = 0$ to $y = 6$:

$$\begin{aligned} P &= \int_0^6 (200 + 32y)\,dy \\ &= 200\,(y)_0^6 + 32\,(y)_0^6 \\ &= 1776 \end{aligned}$$

Conclusion

The total estimated algae population in the rectangular section of the lake is $1,776$ million cells.

Exercise 3

The total mass (M) is found by integrating the density function over the three-dimensional volume V :

$$M = \int\int\int_V \rho(x,y,z)dV = \int_0^1\int_0^3\int_0^4 6xzdxdydz$$

We will perform three successive iterated integrations, starting with respect to x.

Step 1: Integrate with respect to x (Inner Integral)

Treat z as a constant and integrate $6xz$ from $x = 0$ to $x = 4$:

$$\begin{aligned} I_x &= \int_0^4 6xzdx \\ &= 6z\left(\frac{x^2}{2}\right)_0^4 \\ &= 48z \end{aligned}$$

Step 2: Integrate with respect to y (Middle Integral)

Now, substitute the result $I_x = 48z$ and integrate it with respect to y from $y = 0$ to $y = 3$. Treat z as a constant:

$$\begin{aligned} I_y &= \int_0^3 48zdy \\ &= 48z\,(y)_0^3 \\ &= 144z \end{aligned}$$

Step 3: Integrate with respect to z (Outer Integral)

Finally, integrate the result $I_y = 144z$ with respect to z from $z = 0$ to $z = 1$:

$$\begin{aligned} M &= \int_0^1 144zdz \\ &= 144\left(\frac{z^2}{2}\right)_0^1 = 72 \end{aligned}$$

Conclusion

The total mass of the specific protein in the bioreactor volume is 72 milligrams (mg).

Exercise 4

We solve the integral using three iterated steps.

Step 1: Integrate with respect to x (Innermost Integral)

Treat z^2 as a constant:

$$\begin{aligned} I_x &= \int_0^2 xz^2dx = z^2\left[\frac{x^2}{2}\right]_0^2 \\ &= z^2\left[\frac{2^2}{2} - 0\right] = 2z^2 \end{aligned}$$

Step 2: Integrate with respect to y (Middle Integral)

Now, substitute $I_x = 2z^2$ and integrate it with respect to y. Treat z^2 as a constant:

$$\begin{aligned} I_y &= \int_0^2 2z^2 dy = 2z^2 \left[y\right]_0^2 \\ &= 2z^2(2-0) = 4z^2 \end{aligned}$$

Step 3: Integrate with respect to z (Outermost Integral)

Finally, integrate the result $I_y = 4z^2$ with respect to z :

$$\begin{aligned} N &= \int_0^2 4z^2 dz = 4.\left(\frac{z^3}{3}\right)_0^2 \\ &= 4.\left(\frac{2^3}{3}\right) - 0 = \frac{32}{3} \end{aligned}$$

Conclusion

The total estimated number of bacteria in the colony is $\frac{32}{3}$ million cells, which is approximately 10.67 **million cells.**

Part II

Probability and Statistics

Chapter 5

One-dimensional descriptive statistics

1 Introduction

Statistics is a scientific method that involves gathering data on sets, then analysing, commenting on and criticising these data. Descriptive statistics can be defined as the statistical instrument that gives meaning and expression to the information gathered. It provides a concise, simplified picture of reality.

In biology, this is commonly used to describe:

* The weight of lab animals
* Length of plant leaves
* Number of bacteria colonies
* Hemoglobin levels in blood samples
* Enzyme activity in experiments

The goal is to summarize large datasets into meaningful indicators that reveal patterns, central tendencies, and variability.

2 Basic vocabulary

Population: The population is a set of similar items on which the statistical study is based. The number of elements within a population is called the size of the population.

Sample: A sample is a subset of the population having the same characteristics as it. Samples are used when the population sizes are too large so as it becomes impossible to include all possible observations. A sample should represent the population as a whole and not reflect any bias toward a specific attribute.

Statistical unit: Each element in the population is called a statistical unit or an individual.

Statistical variable: The character is a particular feature of the observations, in which the statistical study is interested.

Modalities of a character: The modalities of a character are the different situations taken by this character

3 Characteristics of the statistical variable

In a population, one or more characteristics can be studied. These are represented by statistical variables, which may be quantitative or qualitative.

3.1 Quantitative variables

A quantitative variable has a measurable or countable unit. For example, grades, the number of children in a family, company sales figures, etc.

There are 2 types of quantitative variable:

3.1.1 Discrete (discontinuous) quantitative variables

These are variables that take only isolated values.

> ***Example***
> The number of children in the class can only be 0, or 1, or 2, or 3, ...; it can never take a value strictly between 0 and 1, or 1 and 2, or 2 and 3, ...

3.1.2 Continuous quantitative variables

These are variables that can take any value within an interval.

> ***Example***
> Sales per SME can be 2900.1, 2900.12, 2500; the height of a group of individuals is 165.5 cm, 135 cm, 149.3 cm, etc...

3.2 Qualitative variables

These are variables that admit neither measurable nor countable units.

> ***Example***
> The study of the gender of a group of individuals (male or female); the study of eye colour: blue, brown, black, green or other; the study of the baccalaureate grade: very

good, good, fair, passable. Among qualitative statistical variables, we distinguish two types:

3.2.1 Ordinal qualitative variables

When its modalities can be classified in a certain natural order, as in the case of Mention du Bac.

3.2.2 Nominal qualitative variables

When its modalities cannot be classified in a natural order, as in the case of eye colour or gender.

4 Application to an Example

In the following examples, determine the population, the statistical unit, the statistical variable and the type of statistical variable studied.

Example 1

"Study of the size of a group of children".

- **Population:** Children,
- **Statistical unit:** One child,
- **Statistical variable:** Height,
- **Variable type:** Continuous quantitative.

Example 2

"Study of a group of families according to the number of children".

- **Population:** The set of families,
- **Statistical unit:** The family,
- **Statistical variable:** Number of children,
- **Variable type:** Discrete quantitative.

Example 3

"Study of the nationality of a company's workers".

- **Population:** Workers,

- **Statistical unit:** One worker,
- **Statistical variable:** Worker's nationality,
- **Variable type:** Nominal qualitative.

5 Statistical series

A statistical series is the sequence of values taken by a variable X over the units of observation.

Example

Let S be the series of all the ages of a group of children.S:

$$12 - 7 - 10 - 10 - 5 - 12 - 5 - 7 - 7 - 7 - 7 - 10 - 7 - 5 - 10$$

Let's write S in ascending order.

$$S \; : \; 5 - 5 - 5 - 7 - 7 - 7 - 7 - 7 - 7 - 10 - 10 - 10 - 10 - 12 - 12$$

$$S \; : \; 5_{(3)} - 7_{(6)} - 10_{(4)} - 12_{(2)}$$

$Max(x_i)$: The upper limit of S

$Min(x_i)$: The lower limit of S

The range of the S series is the difference between the largest and smallest values in the series.

$$R = Max(x_i) - min(x_i) = 12 - 5 = 7years.$$

6 Statistical tables

Observations gathered in a study can be presented in the form of tables called statistical tables, in which the values of the variable studied x_i and the number of times they are observed n_i (counts) are represented:

$x_i(variable)$	*Age*	x_1	x_2	x_3	$Total$
$n_i(absolute\ number)$	Number of children	n_1	n_2	n_3	$N = \sum n_i$

Example

S : "Group of workers by nationality"

$$S : Algerians(15) - Moroccans(3) - Tunisians(7)$$

x_i (*nationality*)	*Algerians*	*Moroccans*	*Tunisians*	*Total*
n_i(*workers*)	15	3	7	$N = \sum n_i = 25$

Example

In the case of a continuous variable, the observation is represented by a class or interval, as in the case of size.

x_i	$[6 - 8[$	$[8 - 10[$	$[10 - 12[$
n_i	3	7	4

The center of class $[a - b[$ is

$$C_i = \frac{a+b}{2}$$

Class length

$$l = b - a$$

7 Frequencies

- **Absolute number:** is n_i
- **Relative frequency: is** $f_i = \frac{n_i}{N}$ et $\sum f_i = 1$.
- **Cumulative Increasing Frequency** is $N_k^{\uparrow} = n_1 + n_2 + ... + n_k = \sum n_i$.
- **The decreasing cumulative number:** is $N_k^{\downarrow} = n - n_1 - n_2 - ... - n_k = n - \sum n_i$.
- **Increasing cumulative relative frequency:** is $F_i^{\uparrow} = \frac{N_i^{\uparrow}}{N}$.
- **Decreasing cumulative relative frequency:** is $F_i^{\downarrow} = \frac{N_i^{\downarrow}}{N}$

Example

Let the distribution

x_i	c_i	n_i	$N_i^{\uparrow}$	$N_i^{\downarrow}$	$F_i^{\uparrow}$	$F_i^{\downarrow}$
$[6-8[$	7	3	3	16	$3/16 = 0.1875$	$16/16 = 1$
$[8-10[$	9	7	10	13	$10/16 = 0.625$	$13/16 = 0.8125$
$[10-12[$	11	4	14	6	$14/16 = 0.875$	$6/16 = 0.375$
$[12-14[$	13	2	16	2	$16/16 = 1$	$2/16 = 0.125$
$\sum$	/	16	/	/	/	/

8 Graphical representation

Statistical tables can be accompanied by graphical representations to give a more comprehensive understanding of the data. Depending on your needs, you can represent :

- Numbers and cumulative numbers.

- Relative frequencies and cumulative relative frequencies.

Graphical representations differ according to the type of variable.

8.1 Discrete variable

8.1.1 Bar chart

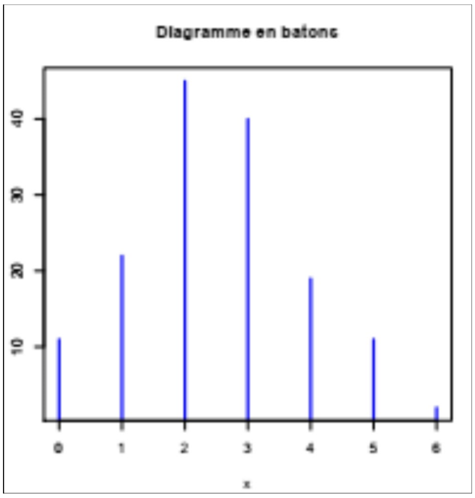
This is represented by plotting the values x_i taken by the statistical variable on the x-axis, and then tracing from each point x_i a bar whose length is proportional to n_i or f_i.

Example

Distribution of 150 frogs according to the number of trematode worms (parasites) they harbor.

Number of trimatodes per frog (x_i)	0	1	2	3	4	5	6
Number of matching frogs (n_i)	11	22	45	40	19	11	2
Relative frequency f_i	0.07	0.14	0.30	0.26	0.12	0.07	0.01

This produces the following bar chart:

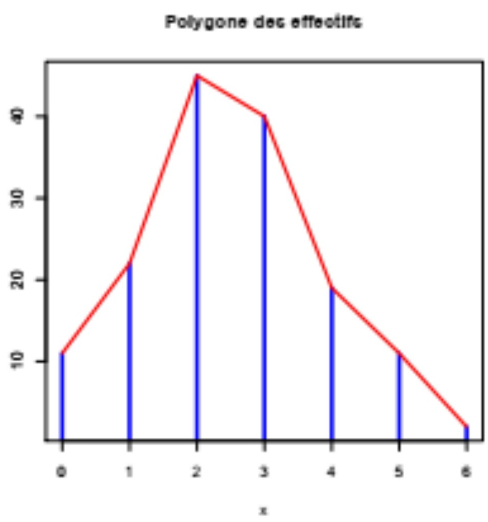


8.1.2 The frequency (or headcount) polygon

This representation is obtained by joining the bar vertices.

Example

In the case of (example 1), we obtain the following polygon:



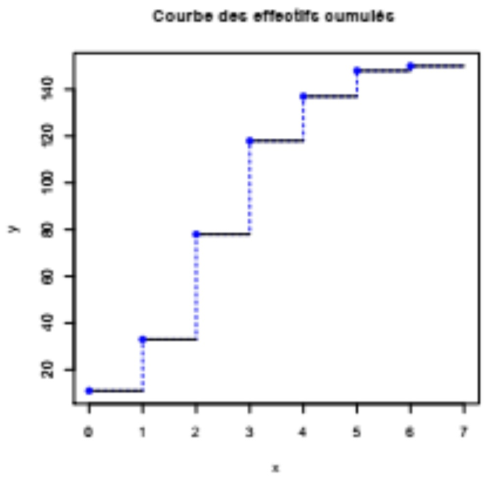
8.1.3 Cumulative headcount polygon (or cumulative headcount curve)

We define the function which associates with each value x , the sum of the numbers of all $x_i < x$ and which we call the number distribution function. The graphical representation of this function is called the cumulative headcount polygon.

Example

In the case of (example 1), we obtain the following Cumulative Diagram:

Number of trimatodes (x_i)	0	1	2	3	4	5	6
n_i	11	22	45	40	19	1	2
$N_i^{\uparrow}$	11	33	78	118	137	148	150



8.2 Continuous variable

8.2.1 Case of classes with equal extents

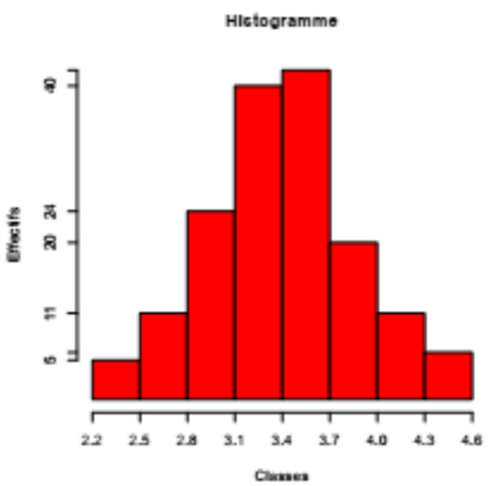
Histogram On the abscissa axis, we represent the boundaries of the different classes and we associate to each class a rectangle, in the base is a part of the abscissa axis included between the boundaries of this class and whose length is proportional to n_i or f_i.

Example

Newborn weights range from *2.240 kg to 4.490 kg.* See the following table:

$Class$	Class center	n_i	f_i
$[2.2, 2.5[$	2.350	5	0.031
$[2.5, 2.8[$	2.650	11	0.068
$[2.8, 3.1[$	2.950	24	0.148
$[3.1, 3.4[$	3.230	40	0.248
$[3.4, 3.7[$	3.550	42	0.259
$[3.7, 4.0[$	3.850	20	0.124
$[4.0, 4.3[$	4.150	13	0.080
$[4.3, 4.6[$	4.45	6	0.037
		160	1

The following histogram is obtained:



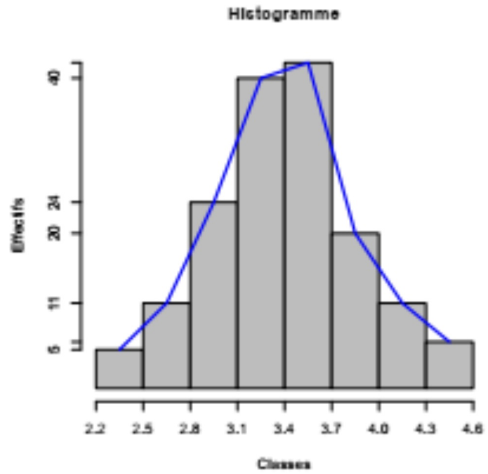
The frequency (or headcount) polygon This representation is obtained by joining the points (x_i, n_i) by line segments. It is completed by 2 extreme classes of the same amplitude.

1- The area of all rectangles is equal to 1 if we represent relative frequencies and n if we represent numbers.

2- The area between the number polygon and the $x-$axis is equal to the area of the histogram.

Example

See the previous example



8.2.2 Case of classes with different ranges

In biology, data are often grouped into **classes (intervals) of unequal** width. This happens because biological measurements are rarely uniform across scales.

Common biological examples

* **Age groups:** 0–1 year, 1–5 years, 5–10 years

* **Body size / mass:** narrow intervals for small organisms, wider for large ones

* **Cell size:** many small cells, few very large ones

* **Concentration levels**: fine resolution at low doses, coarse at high doses

* **Time intervals**: in growth or survival studies (short early phases, longer later ones)

Why unequal class widths are a problem:

If we only look at **frequencies**, wider classes naturally contain more observations even if the biological phenomenon isn't more intense there.

This can **bias interpretation**, for example:

* Overestimating survival in long age intervals

* Misrepresenting growth or abundance patterns

* Distorting population structure

To compare classes fairly, we use frequency density:

$$Frequency\ density = \frac{Number\ of\ observations\ in\ the\ class}{Class\ width}$$

If relative frequencies are used:

$$Density = \frac{Relative\ frequency}{Class\ width}$$

Example

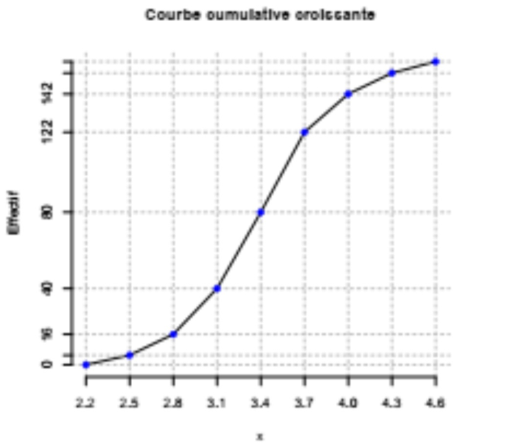
Study of cell diameters (μm) :

Diameter class	Width	Number of cells	Density
[2, 4[	2	40	20
[4, 8[	4	60	15
[8, 16[	8	40	5

Even though the second class has more cells, the highest density is in the smallest size class $\longrightarrow$ small cells dominate the population.

Example

In the case of (example 2), we obtain the following cumulative curve:



The polygon of decreasing cumulative numbers The polygon of decreasing cumulative numbers is obtained by joining, by straight segments, the points whose abscissa is the lower limit of the classes and whose ordinate is the decreasing cumulative number (or cumulative relative frequency) corresponding to the class in question. The first point is $(a_k, 0)$

9 Central tendency and dispersion parameters

The description of a statistical series does not provide all the information about the population under consideration, which is why it will be studied through parameters such as the mean, median, mode, variance and shape coefficients.

9.1 The arithmetic mean $\bar{x}$

Is equal to the sum of the observed values divided by the number of observations.

9.1.1 Discrete variable

This is the sum of the values divided by their number.

$$\bar{X} = \sum_{i=1}^{N} \frac{n_i x_i}{N}$$

9.1.2 Continuous variable

$$\bar{X} = \sum_{i=1}^{N} \frac{n_i c_i}{N}$$

Where c_i class center

Example

Calculate average student grade

x_i	5	7	8	9	$\sum$
n_i	3	4	2	1	10
$n_i x_i$	15	28	16	9	68

$\bar{X} = \sum_{i=1}^{N} \frac{n_i x_i}{N} = \frac{68}{10} = 6.8$

Interpretation: The average of his students is 6.8 pts

Example

Calculate the average salary for this table.

x_i	$6 - 8$	$8 - 10$	$10 - 12$	$12 - 14$	$\sum$
n_i	8	10	20	8	48
c_i	7	9	11	13	/
$n_i c_i$	56	90	220	104	470

$\bar{X} = \sum_{i=1}^{N} \frac{n_i c_i}{N} = \frac{470}{48} = 9.79$ da

Interpretation: The average wage is 9.79 da

9.2 Mode (Mo) or dominant value

This is the most represented value of any variable in a given population.

9.2.1 Discrete variable

This is the value with the largest number of individuals (n_i).

$$Mo = max(n_i)$$

9.2.2 Continuous variable

$$Mo = L_{mo} + \frac{d_1}{d_1 + d_2} * l$$

Where

$\boldsymbol{d}_1 = \boldsymbol{n}_{mo} - \boldsymbol{n}_{mo-1}$ and $\boldsymbol{d}_2 = \boldsymbol{n}_{mo} - \boldsymbol{n}_{mo+1}$

L_{mo} : Lower limit of the modal class.

$\boldsymbol{d}_1$: The difference between the modal class size and the previous size.

$\boldsymbol{d}_2$:The difference between the size of the modal class and the next size.

l : The length of the modal class.

Example

Find the mode of this statistical series.

x_i	3	7	10	12
n_i	100	95	300	20

$Max(n_i) = 300$ so the mode is 10.

Example

Calculate mode

x_i	$0-4$	$4-8$	$8-12$	$12-16$	$16-20$
n_i	40	250	300	100	40

$Max(n_i) = 300$ So the modal class is $[8 - 12[$

$$\begin{aligned} Mo &= L_{mo} + \frac{d_1}{d_1 + d_2} * l \\ &= 8 + \left(\frac{50}{50 + 200}\right) * 4 \\ &= 8.8 \end{aligned}$$

Where

$L_{mo} = 8$

$\boldsymbol{d}_1 = \boldsymbol{n}_{mo} - \boldsymbol{n}_{mo-1} = 300 - 250 = 50$

$\boldsymbol{d}_2 = \boldsymbol{n}_{mo} - \boldsymbol{n}_{mo+1} = 300 - 100 = 200$

$l = 12 - 8 = 4$

9.3 The median Me

This is the value that divides the series into two parts such that 50% of observations have values below Me and 50% of observations have values above Me.

9.3.1 Discrete variable

$$Me = X_{\left(\frac{N}{2} + \frac{1}{2}\right)}$$

9.3.2 Continuous variable

The median class: this is the class whose cumulative ascending number of observations is greater than or equal to $\frac{N}{2}$.

$$Me = L_{me} + \frac{\frac{N}{2} - N^{\uparrow}_{me-1}}{n_{me}} * l$$

L_{me} : Lower medial class limit.

$N^{\uparrow}_{me-1}$: The cumulative increasing number of the previous class.

n_{me} : Absolute number of the medial class.

l : The length of the medial class.

Example

Find the median of this statistical series :

x_i	3	7	10	12
n_i	100	95	300	20
$N_i^{\uparrow}$	100	195	495	515

$Me = X_{\left(\frac{N}{2}+\frac{1}{2}\right)} = X_{\left(\frac{515}{2}+\frac{1}{2}\right)} = X_{258}$
$258 \in 495 \quad \Rightarrow Me = 10.$

Example

Calculating the median

x_i	$0-4$	$4-8$	$8-12$	$12-14$	$\sum$
n_i	40	250	300	100	690
$N_i^{\uparrow}$	40	290	590	690	/

$N/2 = 690/2 = 345$ so the medial class is $[8-12[$

$$\begin{aligned} Me &= L_{me} + \frac{\frac{N}{2} - N_{me-1}^{\uparrow}}{n_{me}} * l \\ &= 8 + \frac{345-290}{300} * 4 = 8.7333 \end{aligned}$$

9.4 The variance

$$V(x) = \frac{1}{N}\sum_{i=1}^{N} n_i(x_i - \bar{x})^2 = \frac{1}{N}\sum_{i=1}^{N} n_i x_i^2 - (\bar{x})^2$$

9.5 Standard deviation

$$\delta(x) = \sqrt{V(x)}$$

10 Shape characteristic

10.0.1 Moments

$$\begin{aligned} \alpha_k &= \frac{1}{N}\sum_{i=1}^{N} n_i x_i^k \\ \mu_k &= \frac{1}{N}\sum_{i=1}^{N} n_i \left(x_i^k - \bar{x}\right)^2 \end{aligned}$$

10.0.2 Fisher coefficient :

$$\gamma_1 = \frac{\mu_3}{\delta^3}$$

$\gamma_1 < 0$ this is the negative asymmetry (on the left),

$\gamma_1 = 0$ the distribution is symmetrical,

$\gamma_1 > 0$ we have positive asymmetry (on the right).

11 Interquartile range

Quartiles are the values that divide the ordered statistical series into 4 parts of equal numbers.

- The first quartile is the number Q_1 such that 25% of the values are less than or equal to Q_1.

- The third quartile is the number Q_3 such that 75% of values are less than or equal to Q_3.

- The second quartile Q_2 is the median.

- The first quartile Q_1 is the median of the first half of the statistical series.

- The third quartile Q_3 is the median of the second half of the statistical series.

- The method for calculating quartiles is therefore identical to that for calculating the median.

The interquartile range is the number IQR such that $IQR = Q_3 - Q_1$. It gives the range of the central half of the observations.

1) Given the results obtained by a student in the statistics module

The ordered series is: $10, 9, 12, 10, 13, 14, 18, 13, 15$

$$\underbrace{9, 10, 10, 12}, 13, \underbrace{13, 14, 15, 15}$$

The 1^{st} half of the series contains $4 (= 2 \times 2 = 2 \times k)$ values, so the median of this part is

$$Q_1 = \frac{x_k + x_{k+1}}{2} = \frac{x_2 + x_3}{2} = 10 = Q_1.$$

The 2^{th}half of the series also contains 4 values, so therefore the median of this half is

$$Q_3 = \frac{x_{4+1+k} + x_{4+1+k+1}}{2} = \frac{x_7 + x_8}{2} = \frac{14 + 15}{2} = 14.5$$

2) We keep the same series and add the value 11, so the ordered series becomes :

$$\underbrace{9, 10, 10, 11, 12}, \underbrace{13, 13, 14, 15, 15}.$$

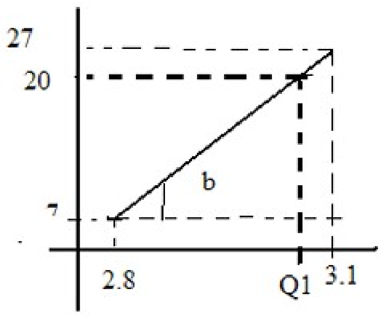
The 1^{st} half of the series contains $5(= 2 \times 2 + 1 = 2 \times k + 1)$ values, so the median of this part is

$$Q_1 = x_{k+1} = x_3 = 10.$$

The 2^{th} half of the series also contains 5 values, so the median of this half is

$$Q_3 = x_{5+k+1} = \frac{x_7 + x_8}{2} = 14$$

Returning to Example 8.1 , $\frac{N}{4} = 20$, then according to the table Q_1 is $[2.8, 3.1[$. So, using linear interpolation, we obtain



$\tan b = \frac{27-7}{3.1-2.8} = \frac{20-7}{Q_1-2.8} \Longleftrightarrow Q_1 = 2.8 + \frac{27-7}{3.1-2.8}(3.1 - 2.8) = 2.995kg$

12 Exercises

Exercise1

In each case, determine the population, the statistical unit , the studied character, and its type (qualitative/quantitative – discrete/continuous).

1. A survey on blood type has been conducted among students of a medical school.
2. The study of the heart rate (beats per minute) of patients in a hospital ward.
3. A survey conducted among nurses dealt with the type of diet they follow (vegetarian / keto / balanced / fast-food).
4. The study of the number of plants in each classroom of a biology department.
5. The analysis of the daily water consumption (liters) of laboratory animals (rats).

Exercise 2

A microbiologist is studying different bacterial strains found in water samples from a lake. After analyzing 120 samples, the results are classified as follows:

Bacteria Type	E. coli	Salmonella	Staphylococcus	Pseudomonas	Lactobacillus
Number (n)	30	18	25	22	25

1. Identify the population, the studied character, its type, and its modalities.
2. Compute the relative frequencies for each bacteria type.
3. Determine the most common bacteria type (mode).
4. Draw a pie chart to visualize proportions.

Exercise 3

A biologist is analyzing the weight (grams) of laboratory mice. The grouped data for 70 mice is given below:

Weight (g)	[18,20[	[20,22[	[22,24[	[24,26[	[26,28[	[28,30[
Number	5	12	25	10	13	5

1. Identify the population, character, type, and modalities.
2. Calculate relative frequencies.
3. Determine the modal class.
4. Calculate mean and variance.
5. Draw a histogram.

Exercise 4

A plant biologist counted the number of seeds in 100 tomato fruits. The results are:

Seeds (X)	2	3	4	5	6	7
Frequency	6	14	28	30	12	10

1. Identify population, character, type, modalities.
2. Compute relative frequencies.
3. Determine the mode.
4. Calculate mean, quartiles, and variance.
5. Draw suitable graphical representations (bar chart, pie chart, etc.).

13 The answers

Exercise 1

For each item give: population, statistical unit, studied character, type.

1. Blood type survey among medical students

* **Population**: All students in the medical school (or the sampled students if a sample).

* **Statistical unit**: One student.

* **Studied character**: Blood type (A, B, AB, O, possibly Rh factor).

* **Type:** Qualitative (categorical), Nominal.

2. Heart rate of patients in a ward

* **Population**: Patients in the ward (or the sampled patients).

* **Statistical unit**: One patient.

* **Studied character**: Heart rate (beats per minute).

* **Type:** Quantitative continuous (or discrete if measured as integer bpm, but usually treated continuous).

3. Diet type among nurses

* **Population**: Nurses in the surveyed group (or hospital).

* **Statistical unit:** One nurse.

* **Studied character**: Diet type (vegetarian, keto, balanced, fast-food, etc.).

* **Type:** Qualitative Nominal.

4. Number of plants per classroom

* **Population:** Classrooms in the biology department.

* **Statistical unit:** One classroom.

* **Studied character:** Number of plants.

* **Type:** Quantitative discrete.

5. Daily water consumption (liters) of laboratory rats

* **Population:** The lab rats (in the experiment).

* **Statistical unit:** One rat.

* **Studied character**: Water consumption (liters/day).

* **Type:** Quantitative continuous.

Exercise 2

Bacteria Type	E. coli	Salmonella	Staphylococcus	Pseudomonas	Lactobacillus
Number (n)	30	18	25	22	25

1. **Identifying the Population, Studied Character, Type, and Modalities.**

- **Population**: Water samples from the lake (or the set of lake samples).
- **Studied character**: Type of bacteria found in a sample.
- **Type:** Qualitative Nominal.
- **Modalities**; E. coli, Salmonella, Staphylococcus, Pseudomonas, Lactobacillus.

2. Relative frequencies (as proportions and percentages):

* E. coli: 30 / 120 = 0.250 = 25%

* Salmonella: 18 / 120 = 0.150 = 15%

* Staphylococcus: 25 / 120 = 0.2083 = 20.83%

* Pseudomonas: 22 / 120 = 0.1833 = 18.33%

* Lactobacillus: 25 / 120 = 0.2083 = 20.83%

3. Determine the Most Common Bacteria Type (Mode)

The mode is the modality (bacteria type) with the highest frequency.

The highest number of samples is 30, which corresponds to the E. coli strain.

$$Mode = E.\ coli$$

4. Pie chart: use the percentages above. Labels and slices:

Exercise 3

1) **Population , statistical unit , character , type and modalities:**

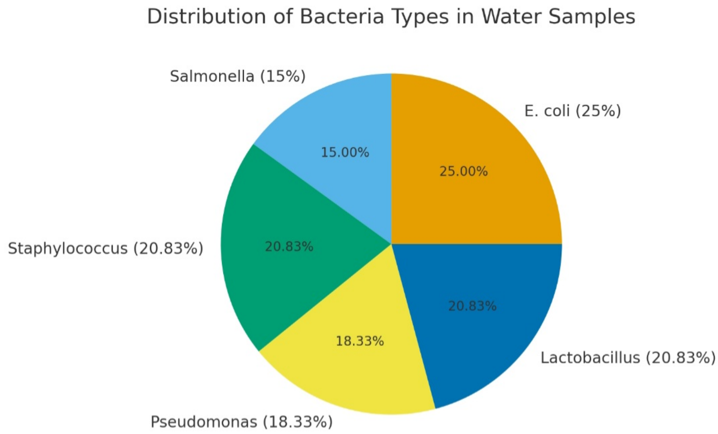
* **Population:** The 70 laboratory mice under study..

* **Statistical unit:** One mouse.

* **Studied character:** Weight (grams).

* **Type:** Quantitative continuous (grouped).

* **Modalities:** the class intervals ([18,20[,,[20,22[,,...,[28,30[).



2) **Calculation of Relative Frequencies**

The relative frequency (f_i) is the proportion of the total sample size ($N = 70$) that falls into each class.

$Class$	c_i	n_i	$f_i = \frac{n}{N}$	$Percent$	$n_i c_i$	$n_i c_i^2$	$N_i^{\uparrow}$
$[18, 20[$	19	5	0.0714	7.14%	132	1805	5
$[20, 22[$	21	12	0.1714	17.14%	252	5292	17
$[22, 24[$	23	25	0.3571	35.71%	575	13225	42
$[24, 26[$	25	10	0.1428	14.29%	250	6250	52
$[26, 28[$	27	13	0.1857	18.57%	351	9477	65
$[28, 30[$	29	5	0.0714	7.14%	145	4205	70
$Total$		70	1	100%	1705	46504	

3) **Determination of the Modal Class**

The modal class is the class interval with the highest frequency.

The highest frequency is 25, which corresponds to the class $[22, 24[$

$$[22, 24[$$

4) **Calculation of Mean and Variance**

These calculations use the midpoint (x_i) of each class to approximate the group's total weight.

a) Mean

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} n_i c_i = \frac{1705}{70} = 23.8285$$

b) Variance and Standard Deviation

- **Variance:**

$$\begin{aligned} Var(x) &= \frac{1}{N}\sum_{i=1}^{N} n_i c_i^2 - \bar{x}^2 \\ &= \frac{46504}{70} - (23.8285)^2 = 96.5454 \end{aligned}$$

- **Standard Deviation:**

$$s = \sqrt{Var(x)} = \sqrt{96.5454} = 9.8257$$

5) **Histogram:** Since all the class widths are equal (width =2g), the height of each bar corresponds directly to the frequency

Exercise 4

1.**Identification of Statistical Components**

* Population: the 100 tomato fruits observed.

* Character (variable): number of seeds per fruit (X).

* Type: quantitative, discrete (counts of seeds).

* Modalities: the possible seed counts: $2, 3, 4, 5, 6, 7$.

2. **Calculation of Relative Frequencies**

The table below shows the frequencies, relative frequencies (f_i), and cumulative frequencies ($N_i^{\uparrow}$) used for subsequent calculations.

x_i	n_i	$f_i = \frac{n}{N}$	$Percent$	$N_i^{\uparrow}$
2	6	0.06	6%	6
3	14	0.14	14%	20
4	28	0.28	28%	48
5	30	0.30	30%	78
6	12	0.12	12%	90
7	10	0.10	10%	100
$Total$	100	1	100%	

3. **Determination of the Mode**

The mode is the modality with the highest frequency.

The highest frequency is 30, which corresponds to $X = 5$ seeds.

$$Mode = 5$$

4. **Calculation of Mean, Quartiles, and Variance**

a) **Mean**

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} n_i x_i = \frac{458}{100} = 4.58$$

b) Quartiles

Quartiles are calculated based on the cumulative frequency ($N = 100$) :

- Q_1 : (25th position): $100 \times 0.25 = 25$. The $N_i^{\uparrow}$ value that first reaches or exceeds 25 is 48, corresponding to $X = 4$.

$$Q_1 = 4$$

- Q_2 :(50th position - Median): $100 \times 0.50 = 50$. The $N_i^{\uparrow}$ value that first reaches or exceeds 50 is 78, corresponding to $X = 5$.

$$Q_2 = 5$$

- Q_3 :(75th position): $100 \times 0.75 = 75$. The $N_i^{\uparrow}$ value that first reaches or exceeds 75 is 78, corresponding to $X = 5$

$$Q_3 = 5$$

Interquartile range:

$$IQR = Q_3 - Q_1 = 5 - 4 = 1.$$

c) Variance and Standard Deviation

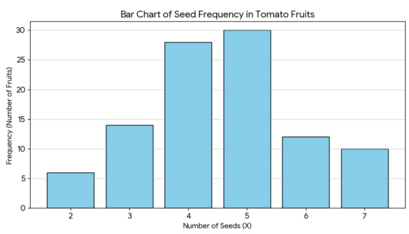
- **Variance:**

$$\begin{aligned} Var(x) &= \frac{1}{N}\sum_{i=1}^{N} n_i x_i^2 - \bar{x}^2 \\ &= \frac{2270}{100} - (4.58)^2 = 1.7336 \end{aligned}$$

- **Standard Deviation:**

$$\sigma(x) = \sqrt{Var(x)} = \sqrt{1.7336} = 1.3166$$

5. Draw suitable graphical representations



Chapter 6

Combinatorial Analysis and Probability

1 Reminders and complements of combinatorial analysis

- Combinatorial analysis is a branch of mathematics that studies how to count objects.
- Why count?

Example

- How do I classify 2 students?*3?4? ...20?*
- In a store, I can choose between three phones of the same size and 2 pouches.

How many choices do I have?

- The password for your Gmail account (BIO????) has four digits between 0 and 9.

How many different passwords are there?

- We want to park 3 cars in a 5-space parking lot. How many ways are there to park the cars?

Understanding how to count quickly becomes important:

Definition 6.1

A permutation without repetition of n distinct (different) elements is an ordered sequence of these n elements.

Example

The number 2537 is a permutation of the number 3752.

The number 7523 is a permutation of the number 3752.

The number 7533 is not a permutation of the number 3752.

Property: The number of permutations of n distinct elements is: $n!$

Example

1- How many 3-digit numbers can be formed using $1, 3$ and 5?

2- Anagrams can be found in most statements about permutations. Take the letters $R; O; M$ and E, for example. How many ways can they be arranged?

Solution

There are 4 possibilities for the first letter. For the second, there are only 3. Then 2, then one letter. The cardinal of the set of possibilities is $4 \times 3 \times 2 \times 1 = 24$.

There are 24 possibilities, i.e. 4!

So, to return to our question, there are $n!$ ways of arranging n distinct elements.

R	O	M	E	M	R	O	E
R	O	E	M	M	R	E	O
R	M	O	E	M	O	R	E
R	M	E	O	M	O	E	R
R	E	O	M	M	E	R	O
R	E	M	E	M	E	O	R
O	R	M	E	E	R	O	M
O	R	E	M	E	R	M	O
O	M	R	E	E	O	R	M
O	M	E	R	E	O	M	R
O	E	R	M	E	M	R	O
O	E	M	R	E	M	O	R

1.1.2 Permutations with repetitions

Definition 6.2

Consider a set of n objects divided into p groups of identical elements, the groups comprising $n_1, n_2...., and\ n_p$ objects respectively (so we have $n_1 + n_2 + ... + np = n$). Then the number of permutations of this set is :

$$P_n = \frac{n!}{\pi\,(number\ of\ repetitions)!}$$

Example

Find the number of words that can be formed by permuting different letters.

a- From the word *TABLEAU.*

b- From the word *VIVA.*

c- From the word *ANAGRAM*

Solution

The solution involves understanding the number of permutations with repetitions:

a- An anagram of the word $TABLEAU$ is a permutation of the 7 letters of this word. So, a priori, there are 7 of them! But if, within these anagrams, we "permute" the two letters A, we end up with the same word.

In other words, within the 7! anagrams, the words in which the two letters A are swapped are counted twice.

To avoid counting these anagrams twice, divide 7! by the number of possible permutations of the two letters A, i.e. $2! = 2$

The number of different anagrams of the word $TABLEAU$ is therefore equal to: $P_7 = \frac{7!}{2!}$

b- The number one anagram of the word $VIVA$.

$$P_4 = \frac{4!}{2!}$$

c- The number of anagrams of the word $ANAGRAMME$.

$$P_9 = \frac{9!}{3!2!}$$

Considering two identical letter groups: A (3 times) and M (2 times) d The number of anagrams for the word $CELLULE$.

$$P_7 = \frac{7!}{3!2!}$$

Considering two identical letter groups: L (3 times) and E (2 times)

Remark

In the rest of this course, n and p are two natural numbers.

1.2 Arrangements

Given a set E of n objects, an arrangement of p of these objects is an ordered sequence of p objects taken from these n objects.

Attention

The order of the objects is taken into account = order is important.

1.2.1 Arrangements without repetition

Definition 6.3

In a non-repeating arrangement, the p objects in the list are all **distinct (different).** This corresponds to a random draw with **order (order is important)**.
The number of arrangements without repetition of p objects among n (with $0 \leq p \leq n$)) is given by

$$A_n^P = \frac{n!}{(n-p)!}$$

Remark

The permutation without repetition of n elements = an arrangement of $p = n$ objects taken from n objects $(A_n^n = n!)$.

Example

- How many different six-digit numbers can be formed with the digits $0, 1, 2, 3, 4, 5, 6, 7$.

$$\begin{cases} \text{The order is important and without repetition} \implies \text{Arrangement without repetition} \\ A_8^6 = \dfrac{8!}{(8-6)!} \end{cases}$$

- The number of words of 5 different letters formed with 26 letters of the alphabet.

$$\begin{cases} \text{The order is important and without repetition } \Longrightarrow \text{Arrangement without repetition} \\ A_{26}^{5} = \dfrac{26!}{(26-5)!} \end{cases}$$

- How many different ways can a president and vice president be elected from 10 people?

$$\begin{cases} \text{The order is important and without repetition } \Longrightarrow \text{Arrangement without repetition} \\ A_{10}^{2} = \dfrac{10!}{(10-2)!} \end{cases}$$

- An urn contains 6 balls numbered from 1 to 6. In how many different ways can 3 balls be removed from the urn?

$$\begin{cases} \text{The order is important and without repetition } \Longrightarrow \text{Arrangement without repetition} \\ A_{6}^{3} = \dfrac{6!}{(6-3)!} \end{cases}$$

1.2.2 Arrangements with repetition = List

In the case of an arrangement with repetition, the p objects in the list are not necessarily all distinct.

This corresponds to a draw with discount and order. i.e., the order is important, but an element can be reused (p times).

Property: The number of repeated arrangements of p items among n is n^p .

1- Mathematically, A_n^P is the number of injections from a set E containing p elements to a set F containing n elements.

2- n^P is the number of functions from a set E containing p elements to a set F containing n elements.

Example

- What is the number of possible four-digit bank card codes?

$$\begin{cases} \text{The order is important and with repetition } \Longrightarrow \text{Arrangement with repetition} \\ A_{10}^{4} = 10^{4} \end{cases}$$

- How many 8-symbol passwords can be created with 66 characters?

$$\begin{cases} \text{The order is important and with repetition } \Longrightarrow \text{Arrangement with repetition} \\ A_{66}^{8} = 66^{8} \end{cases}$$

- A DNA sequence is made up of a sequence of 4 nucleotides [A (Adenine), C (Cytosine), G (Guanine), and T (Thymine)]. There are different possible arrangements of two nucleotides or dinucleotides with $p = 2$ and $n = 4$:

$$\begin{cases} \text{The order is important and with repetition } \Longrightarrow \text{Arrangement with repetition} \\ A_{4}^{2} = 4^{2} \end{cases}$$

- By rolling a standard die 4 times in a row, how many different sequences can be obtained?

$$\begin{cases} \text{The order is important and with repetition } \Longrightarrow \text{Arrangement with repetition} \\ A_{6}^{4} = 6^{4} \end{cases}$$

1.3 Number of combinations

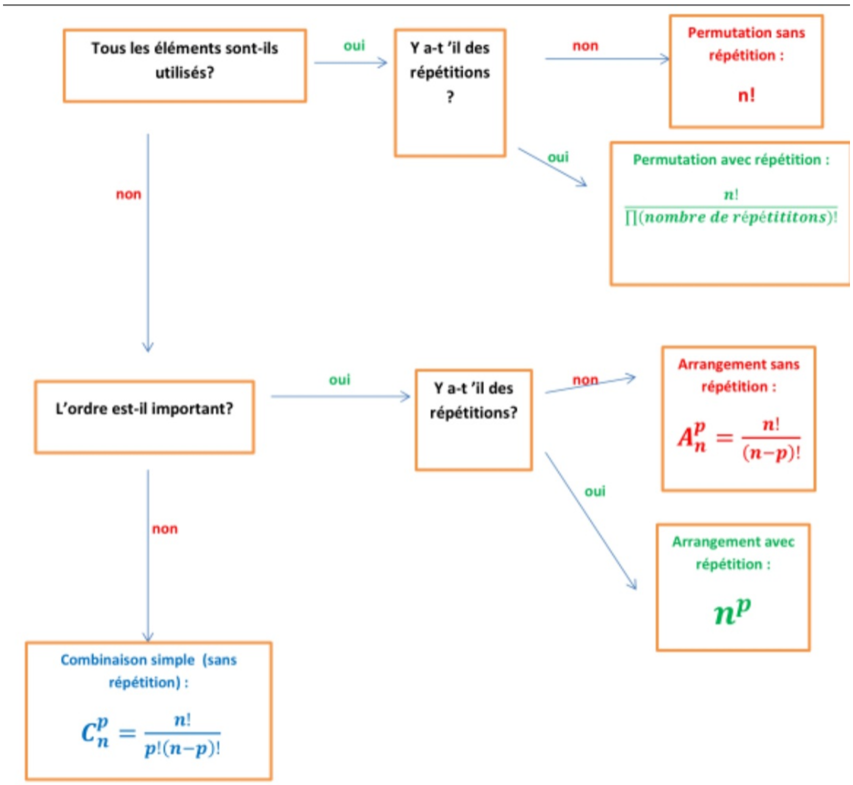
- Unlike the arrangement, the order of the p elements taken from n is of no importance. The notion of combination is essential in statistics and probability.
- A combination of p objects taken from E is a subset of p of these n objects.
- Order is irrelevant.
- A combination without repetition corresponds to a draw without discount and without order.

- The order of the objects is not taken into account = order is not important.

The number of combinations without repetition is given by:

$$C_n^p = \frac{A_n^p}{p!} = \frac{n!}{p!\,(n-p)!}$$

$$\begin{cases} C_n^p : \; notation\ francophone \\ \begin{pmatrix} n \\ p \end{pmatrix} : \; \text{Anglo-Saxon notation} \end{cases}$$



Which $n! = n(n-1)(n-2)\ldots 1$?

1.3.1 Properties of combinations

1. $\forall n \geq 1$, $C_n^0 = C_n^n = 1$ et $C_n^1 = C_n^{n-1} = n$.

2. $\forall n \geq 0$, $\forall 0 \leq p \leq n, C_n^P = C_n^{n-P}$.

3. $\forall n \geq 0$, $\forall 1 \leq p \leq n-1, C_n^P = C_{n-1}^{P-1} + C_{n-1}^P$.

Example A- How many 3-person teams can be formed from 7 people?

$$\begin{cases} \textit{Order is not important and without repetition} \Rightarrow \textit{Combination without repetition} \\ C_7^3 = \frac{7!}{3!(7-3)!} \end{cases}$$

b- How many different ways can a delegation of 3 men and 2 women be chosen from a group of 7 men and 5 women?

$$\begin{cases} \textit{Order is not important and without repetition} \Rightarrow \textit{Combination without repetition} \\ C_7^3 \times C_5^2 = \frac{7!}{3!(7-3)!} \times \frac{5!}{2!(5-2)!} \end{cases}$$

d- Random drawing of 5 cards from a deck of 32

$$\begin{cases} \textit{Order is not important and without repetition} \Rightarrow \textit{Combination without repetition} \\ C_{32}^5 \end{cases}$$

e- A student has to answer 7 questions out of 10 questions in an exam in how many ways can he choose them?

$$\begin{cases} \textit{Order is not important and without repetition} \Rightarrow \textit{Combination without repetition} \\ C_{10}^7 = \frac{10!}{7!(10-7)!} = 120 \end{cases}$$

1.4 Pascal's triangle

Property 3 allows us to construct the following Pascal triangle

	0	1	2	3	4	....
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
.						
.						
.						

1.5 Newton's binomial theorem

Let be a non-zero integer n and two real numbers a and b. Then we have

$$(a+b)^n = \sum_{k=0}^{n} C_n^k \, a^k b^{n-k}$$

Remark

1.
$$2^n = \sum_{k=0}^{n} C_n^k$$

2.
$$0 = \sum_{k=0}^{n} (-1)^k C_n^k$$

2 Probability

2.1 Probability space

2.1.1 Random experiment

A random experiment is any experiment whose outcome is governed by chance when repeated under the same conditions.

> ***Example***
>
> - Tossing a coin: there are 2 possible outcomes: heads or tails.
>
> - Throwing a 6-sided die: 6 possible outcomes $1, 2, 3, 4, 5, 6$.

2.1.2 Event space

This is the set of all possible outcomes of a random experiment, denoted Ω.

> ***Example***
>
> - Throwing a coin: $\Omega = \{P, F\}$.
>
> - Throwing a 6-sided die: $\Omega = \{1, 2, 3, 4, 5, 6\}$.

2.1.3 Elementary and compound events

-An elementary event is a single-element subset of Ω.

- A compound event is a set of elementary events.

> ***Example***
>
> When rolling a die, "having the number 2" is an elementary event, "having an even number" is a compound event.

2.2 Relation and operations on events

2.2.1 Inclusion

Let A and B be two events. A is said to be included in B, when the realization of A leads to the realization of B.

We write : $A \subset B$

2.2.2 Complement of an event

This event is realized when A is not. It is denoted C_{Ω}^{A} or$\overline{A}$.

The space of events Ω is said to be certain. Its complementary is noted Φ called an impossible event.

2.2.3 Union

Realized when at least one of the events A or B is realized It is noted $A \cup B$.

2.2.4 Intersection

Realized when events A and B are realized at the same time. It is noted $A \cap B$.

2.2.5 Difference

Realized when events A are realized without B. It is noted $A - B$.

2.2.6 Incompatible events

A and B are said to be incompatible or disjoint if their intersection is the impossible event $(A \cap B = \Phi)$.

2.3 General Definition of Probability

2.3.1 Introduction

Probability is a mathematical concept used to measure uncertainty. It provides a systematic way to quantify how likely an event is to occur in random or stochastic experiments.

In biology, probability is widely applied to describe uncertainties such as:

* The chance of inheriting a genetic trait.

* The probability of survival of organisms under certain conditions.
* The spread of a disease in a population.
* The likelihood of detecting a specific protein in a sample.

2.3.2 Definition of Probability

Let Ω be a finite sample space (the set of all possible outcomes).

A probability measure is any map

$$P : P(\Omega) \to [0,1]$$

Satisfying:

1. **Normalization:** $P(\Omega) = 1$
2. **Additivity**: If $A \cap B = \phi$, then

$$P(A \cup B) = P(A) + P(B)$$

2.3.3 Fundamental Properties

Let $A, B \subseteq \Omega$. Then:

1. $P(\Phi) = 0$.
2. $0 \leq P(A) \leq 1$ for any event A.
3. $P(A^c) = 1 - P(A)$.
4. $P(A \backslash B) = P(A) - P(A \cap B)$
5. If $B \subseteq A$, then $P(A \backslash B) = P(A) - P(B)$.
6. If $B \subseteq A$, then $P(B) \leq P(A)$.
7. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
8. $P(A \Delta B) = P(A) + P(B) - 2P(A \cap B)$.

2.3.4 Independence in Probability

Definition 6.4

Two events A and B are **independent** if the occurrence of one does not affect the other:

$$P(A \cap B) = P(A) \times P(B).$$

If this condition does not hold, the events are dependent.

Remark

If A and B are independent, then so are A^c and B , A and B^c; A^c and B^c

2.4 Conditional Probability

From conditional probability:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0.$$

Events A and B are independent if

$$P(A/B) = P(A) \text{ and } P(B/A) = P(B)$$

2.5 Extension to Multiple Events

Events $A_1, A_2....A_n$ are mutually independent if for every subset:

$$P\left(A_{i1} \cap A_{i2} \cap ...A_{ik}\right) = P\left(A_{i1}\right).P\left(A_{i2}\right)......P\left(A_{ik}\right)$$

Example

- Event A: A randomly chosen individual carries a dominant allele : $P(A) = 0.6$.
- Event B: The individual has blue eyes : $P(B) = 0.2$. Suppose the probability of both is $P(A \cap B) = 0.12$.

Check independence:

$$P(A).P(B) = 0.6 \times 0.2 = 0.12 = P(A \cap B)$$

Thus, carrying the dominant allele and having blue eyes are independent events.

2.6 Equiprobability

Definition 6.5

Equiprobability occurs when all outcomes in a finite experiment are equally likely.

If $\Omega = \{\omega_1, \omega_{2,....}\omega_n\}$ with equal probabilities:

$$P(\omega_i) = \frac{1}{n}, \quad \forall i = 1, 2, ..., n$$

If event A has k favorable outcomes:

$$P(A) = \frac{k}{n}$$

Example

1. Tossing a fair coin to decide whether a cell receives an X or Y chromosome during meiosis:

$$P(X) = P(Y) = \frac{1}{2}$$

2. Choosing randomly among 12 fruit flies: 4 with red eyes, 4 with white eyes, 4 with brown eyes.

$$P(red\ eyes) = \frac{4}{12} = \frac{1}{3}$$

3. Selecting 1 out of 20 bacterial cultures, 5 of which show antibiotic resistance:

$$P(\text{resistant}) = \frac{5}{20} = 0.25$$

2.7 Bayes' Theorem

Definition 6.6

Let A and B be two events with non-zero probability, we have

$$P(A/B) = \frac{P(B/A) \times P(A)}{P(B)}.$$

Example

Consider the following:

- Event (A): A person has a genetic disease $\Rightarrow P(A) = 0.01$.
- Event (B): The medical test is positive.

Given that:

- Sensitivity: $P(B|A) = 0.95$,
- False positive rate: $P(B|A^c) = 0.05$.

Compute $P(A|B)$.

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|A^c)P(A^c) \\ &= (0.95)(0.01) + (0.05)(0.99) \\ &= 0.0095 + 0.0495 = 0.059 \end{aligned}$$

Thus,

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B)} = \frac{0.95 \times 0.01}{0.059} \approx 0.161$$

Interpretation: Even with a positive test result, the probability of actually having the disease is only about 16.1%, due to the low base rate.

2.8 Law of Total Probability

Definition 6.7

If $A_1, A_2, ..., A_k$ form a partition of Ω

$$P(B) = \sum_{i=1}^{k} P(B/A_i) \times P(A_i)$$

Example

Suppose a disease occurs differently across three age groups:

- (A_1): Children $\rightarrow$ 30% of population, $P(D|A_1) = 0.05$.
- (A_2): Adults $\rightarrow$ 50%, $P(D|A_2) = 0.10$.

- (A_3): Elderly $\rightarrow$ 20%, $P(D|A_3) = 0.25$.

Then, by the law of total probability:

$$\begin{aligned} P(D) &= \sum_{i=1}^{3} P(D|A_i) \times P(A_i) \\ &= (0.05)(0.3) + (0.10)(0.5) + (0.25)(0.2) \\ &= 0.015 + 0.05 + 0.05 = 0.115 \end{aligned}$$

Therefore, the overall probability of having the disease in the population is 11.5%.

2.9 Chain Rule of Probability

Theorem 6.1

Let $A_1, A_2, ..., A_k$ be a partition of Ω such that $P(A_i) \neq 0$ for all $i \in \{1, 2, ..., k\}$. We have for all $B \in P(\Omega)$ such that $P(B) \neq 0$

$$P(A_i/B) = \frac{P(B/A_i) \times P(A_i)}{\sum_{j=1}^{k} P(B/A_j) \times P(A_j)}, \qquad \forall i \in \{1, ..., k\}.$$

Example

Consider the following events:

- (A): An individual carries a certain gene mutation $\rightarrow P(A) = 0.20$.
- (B): If the mutation is present, the probability of developing the disease $\rightarrow P(B|A) = 0.30$.
- (C): If diseased, the probability of showing severe symptoms $\rightarrow P(C|A \cap B) = 0.40$.

Then, by the multiplication rule of probability:

$$P(A \cap B \cap C) = P(A) \times P(B|A) \times P(C|A \cap B) = 0.20 \times 0.30 \times 0.40 = 0.024$$

Thus, the probability that a randomly chosen individual carries the mutation, develops the disease, and shows severe symptoms is 2.4%.

3 Exercises

Exercise1

A survey is carried out in a biology department.

* 55% of students participate in field research (event F).

* 40% of students are involved in laboratory experiments (event L).

* 15% of students do both field research and laboratory experiments ($P(F \cap L) = 0.15$).

A student is chosen at random.

1. What is the probability that the student participates in field research or laboratory experiments (F $\cup$ L)?

2. What is the probability that the student does exactly one of the two activities?

3. What is the probability that the student does not participate in field research (F^c)?

4. If we randomly select a student who does field research (F), what is the probability that this student also does laboratory experiments $P(L/F)$?

5. Knowing that the student does not do field research (F^c), what is the probability that the student does laboratory experiments $P(L/F^c)$?

6. What is the probability that the student does neither field research nor laboratory experiments?

7. Among the students who do field research (F), there are 10 girls and 7 boys. What is the probability that a randomly chosen field-research student is a girl? And a boy?

Exercise 2

A botanist is studying a region with two dominant plant types: flowering plants and non-flowering plants.

From previous ecological surveys:

* 60% of the area is covered by flowering plants.

* 40% of the area is covered by non-flowering plants.

* In flowering plant areas, 70% of the plants are pollinated.

* In non-flowering plant areas, only 20% of the plants show spore reproduction activity.

If a randomly observed plant shows pollination or reproduction activity, what is the probability that it came from a flowering plant area?

Exercise 3

It is known that 20% of animals in a wildlife reserve carry a certain virus.

A diagnostic blood test is used to detect the virus:

* If an animal is infected, the test is positive with probability 0.92 (true positive rate).

* If an animal is not infected, the test is negative with probability 0.90 (true negative rate).

A random animal is tested.

1. If the test result is positive, what is the probability that the animal is not actually infected?

2. If the test result is negative, what is the probability that the animal is actually infected?

3. What is the overall diagnostic error rate of the test?

4 The answers

Exercise 1

Given: $P(F) = 0.55$; $P(L) = 0.40$ *and* $P(F \cap L) = 0.15$.

1. $P(F \cup L)$ (student does field or lab work):

$$P(F \cup L) = P(F) + P(L) - P(F \cap L) = 0.55 + 0.40 - 0.15 = 0.80,$$

2. Probability the student does exactly one of the two activities (either F or L but not both):

$$\begin{aligned} P(exactly\ one) &= P(F \backslash L) + P(L \backslash F) \\ &= P(F) + P(L) - 2P(F \cap L) \\ &= 0.55 + 0.40 - 2(0.15) = 0.65 \end{aligned}$$

3. $P(F^c)$ (the student does not do field research):

$$P(F^c) = 1 - P(F) = 1 - 0.55 = 0.45$$

4. $P(L/F)$ (given the student does field research, probability they also do lab work):

$$P(L/F) = \frac{P(L \cap F)}{P(F)} = \frac{0.15}{0.55} \approx 0.2727$$

5. $P(L/F^c)$ (given the student does not do field research, probability they do lab work):

First compute

$$P(L \cap F^c = P(L) - P(L \cap F) = 0.40 - 0.15 = 0.25.$$

Then

$$P\left(L/F^c\right) = \frac{P(L \cap F^c)}{P(F^c)} = \frac{0.25}{0.45} \approx 0.5556$$

6. Probability the student does neither activity

$$P\left((F \cup L)^c\right) = 1 - P\left(F \cup L\right) = 1 - 0.80 = 0.20$$

7. Among fieldwork students there are 10 girls and 7 boys (total 17).

* Probability a randomly chosen fieldwork student is a girl: $10/17 \approx 0.5882$.
* Probability a randomly chosen fieldwork student is a boy: $7/17 \approx 0.4118$.

Exercise 2:

Given:

$P(flowering\ area) = 0.60$, $P(non - flowering\ area) = 0.40$.

$P(fossils/flowering) = 0.70$, $P(fossils/non - flowering) = 0.20$.

We want $\mathbf{P(flowering/fossils)}$.

Use Bayes' theorem. First compute total probability of fossils:

$$\begin{aligned}
P(fossils) &= P(fossils/flowering)P(flowering) \\
&+ P(fossils/non - flowering)P(non - flowering) \\
&= 0.70 \times 0.60 + 0.20 \times 0.40 = 0.42 + 0.08 = 0.50.
\end{aligned}$$

Then

$$P(flowering/fossils) = \frac{0.70 \times 0.60}{0.50} = 0.84.$$

Exercise 3

Given:

Prevalence: $P(gold) = 0.20$.

Sensitivity: $P(+/gold) = 0.92$ (true positive).

Specificity $P(-/no\ gold) = 0.90$ (true negative) $\Rightarrow$ false positive rate $= 0.10$.

False negative rate $= 1 - 0.92 = 0.08$.

1. If test is positive, what is the probability the sample does not contain gold?

Compute $P(+)$:

$$\begin{aligned} P(+) &= P(+/gold)P(gold) + P(+/no\ gold)P(no\ gold) \\ &= 0.92 \times 0.20 + 0.10 \times 0.80 = 0.184 + 0.08 = 0.264 \end{aligned}$$

Then

$$P(no\ gold/+) = \frac{P(+/no\ gold)P(no\ gold)}{P\,(+)} = \frac{0.10 \times 0.80}{0.264} \approx 0.3030.$$

Answer: $\approx 30.30\%$ chance that the sample does not contain gold despite a positive test.

2. If test is negative, what is the probability the sample does contain gold? Compute $P(-)$:

$$\begin{aligned} P(-) &= P(-/gold)P(gold) + P(-/no\ gold)P(no\ gold) \\ &= 0.08 \times 0.20 + 0.90 \times 0.80 = 0.016 + 072 = 0.736 \end{aligned}$$

Then

$$P(\ gold/-) = \frac{P(-/\ gold)P(gold)}{P\,(-)} = \frac{0.08 \times 0.20}{0.736} \approx 0.02174.$$

Answer: $\approx 2.17\%$ chance the sample does contain gold given a negative test.

3. Overall diagnostic error rate of the test Error = probability of false positive or false negative:

$$\begin{aligned} Error &= P(false\ positive\}) + P(false\ negative\}) \\ &= P(no\ gold) \times 0.10 + P(gold) \times 0.08 \\ &= 0.80 \times 0.10 + 0.20 \times 0.08 = 0.08 + 0.016 = 0.096. \end{aligned}$$

Answer: 9.6% overall error rate.

Chapter 7

Random variables

1 Introduction

Random variables are crucial in biology because they allow researchers to model and analyze the inherent variability found in biological systems, whether studying genetics, ecology, or experimental results. They transform biological outcomes into numerical data that can be analyzed using probability and statistical methods.

Applications in Biological Analysis

- **Genetics and Genomics:** Random variables are used to model the inheritance of traits and the occurrence of mutations. The Binomial distribution (a discrete distribution) is often used to model the probability of inheriting a specific allele, where each offspring is a Bernoulli trial.

- **Ecology and Epidemiology**: In ecology, Poisson variables are often used to model the spatial distribution of rare species or the number of bacteria colonies on a petri dish. In epidemiology, they model the number of new infections (incidence) or deaths (mortality) over a fixed period

- **Experimental Design and Quality Control**: In laboratory experiments, random variables allow biologists to quantify experimental noise and determine if an observed effect is due to the treatment or merely random chance. The t-distribution and F-distribution (which are derived from the normal distribution) are used to perform hypothesis tests, such as comparing the mean yield of two different crop varieties.

2 Generalities

Definition 7.1

(Ω, F, P) being a probability space, we call random variable any application X defined by Ω in R such that

$$X : \Omega \longrightarrow \mathbb{R}$$
$$\omega \longmapsto X(\omega)$$

$X(\Omega)$ is called l-universe images.

2.1 Types of Random Variables

Random variables are categorized based on the nature of the values they can assume:

Type	Nature of Values	Origin	Example
Discrete	Finite or countably infinite number of values	Counting	Number of trials
Continuous	Any value within an interval	Measuring	Battery lifespan

Example

Toss a coin twice, the fundamental space consists of the following events:

$$\Omega = \{PP, PF, FP, FF\}$$

Let's look at the number of "heads" after the 2 tosses. Then

$$X(\Omega) = \{0, 1, 2\}$$

ω	PP	PF	FP	FF
$X(\omega)$	0	1	1	2

2.2 Key Characteristics

The behavior of a random variable is fully characterized by its probability distribution, often summarized using two key statistics:

1. **Expected Value** $E[X]$:This represents the long-run average value of the random variable if the experiment were repeated many times.

* Discrete: $E[X] = \sum_{i=1}^{n} x_i P(X = x_i)$

* Continuous: $E[X] = \int_{-\infty}^{+\infty} x f(x) dx$

2. **Variance ($Var[X]$):** This measures the expected squared deviation from the mean, indicating the spread or variability of the distribution.

$$Var(X) = E(X^2) - [E(X)]^2$$

3 Discrete random variables

Definition 7.2

A discrete random variable is random variable a whose image universe $X(\Omega)$ is finite and countable.

3.1 Probability law of a discrete random variable :

Definition 7.3

Given a probability space of fundamental space Ω and probability measure P, We call a random variable on this space, any application X from Ω into R such that:

$$P : X(\Omega) \to [0, 1]$$
$$X_i \to [X_i = x_i]$$

Where $X(\Omega) = \{x_1, x_2, ...x_n\}$

1. If X is discrete :$\sum_{i=1}^{n} P[X_i = x_i] = 1$.
2. $\forall x \in R : F_X(x) = \sum_{x_i \leq x} P[X_i]$.
3. The distribution function of X is a stepped function, it jumps to non-zero probability points.

Example

A balanced die is rolled, the random variable X : " represents the number observed on the top face of the die."

$$X(\Omega) = \{1, 2, 3, 4, 5, 6\}$$

The distribution of random variable X is :

X_i	X_1	X_2	X_3	X_4	X_5	X_6	$Total$
$P[x = x_i]$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\sum_{i=1}^{6} P[X = x_i] = \frac{6}{6} = 1$

3.2 Distribution function

Let X be an random variable The distribution function of the random variable X (discrete or continuous) is the function F_X defined by

$$F_X : R \to [0,1]$$
$$x \to P(X \leq x)$$

The practical importance of the distribution function is that it can be used to calculate the probability of any interval on R.

3.2.1 Properties of the distribution function

Let X be a random variable with distribution function F_X then :

1- $\forall t \in R : 0 \leq F_X \leq 1$.

2-F_X is increasing on R.

3- $\lim_{t\to-\infty} F_X(t) = 0$ and $\lim_{t\to+\infty} F_X(t) = 1$.

4- If $a \leq b$ $P[a \leq X \leq b] = F_X(b) - F_X(a)$.

Example

A coin is tossed 3 times, a random variable X defined by is introduced:

$$X(\omega) : \text{"The number of Piles.."}$$

1. *Find the probability law of X.*
2. *Determine the distribution function.*

Solution

1- X represents the number of stacks, so X can take the values :

$$X(\Omega) = \{0, 1, 2, 3\}$$

$X = 0$ corresponds to the case where there are no Piles P, i.e.:

$\{X = 0\} = \{FFF\} \quad \to \ P(0) = P(X = 0) = P(\{FFF\}) = \frac{1}{8}$.

$\{X = 1\} = \{PFF, FPF, FFP\} \to P(1) = P(X = 1) = P(\{PFF, FPF, FFP\}) = \frac{3}{8}$.

$\{X = 2\} = \{PPF, FPP, PFP\} \to P(2) = P(X = 2) = P(\{PPF, FPP, PFP\}) = \frac{3}{8}$.

$\{X=3\}=\{PPP\} \rightarrow \ P(3)=P(X=3)=P(\{PPP\})=\frac{1}{8}$.

Probability law of random variable X :

X_i	$X_1=0$	$X_2=1$	$X_3=2$	$X_4=3$	$Total$
$P[x=x_i]$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\sum_{i=1}^{6} P[X=x_i]=1$

2- Distribution function:

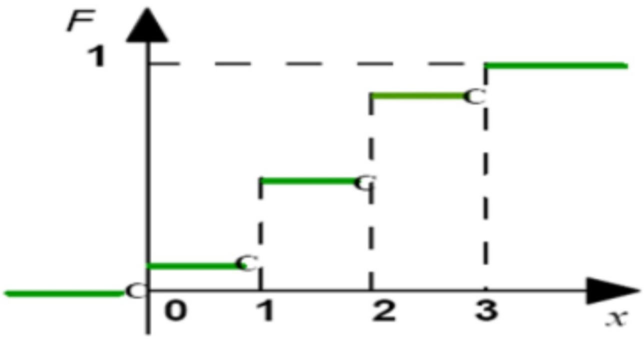
- If $x<0 \rightarrow F(x)=0$.

- If $0 \leq x<1 \rightarrow F(x)=\frac{1}{8}$.

- If $1 \leq x<2 \rightarrow F(x)=\frac{1}{8}+\frac{3}{8}=\frac{4}{8}$.

- If $2 \leq x<3 \rightarrow F(x)=\frac{4}{8}+\frac{3}{8}=\frac{7}{8}$.

- If $x \geq 3 \rightarrow F(x)=\frac{7}{8}+\frac{1}{8}=1$.



3.3 Mathematical expectation

Definition 7.4

Let X be a discrete random variable with possible values $x_1, x_2, \ldots$.and probability distribution $p(x)$. The mean value of a random variable is called the mathematical expectation of X, given by

$$E[X]=\sum_{i \geq 1} x_i p(x_i)=\sum_{i \geq 1} x_i P(X=x_i)$$

Provided that the above series is absolutely convergent, otherwise we'll say that X admits no mathematical expectation. In the same way, we can write

$$E\left[X^k\right]=\sum_{i \geq 1}(x_i)^k p(x_i)=\sum_{i \geq 1}(x_i)^k P(X=x_i)$$

which is called the $K-$order moment of the random variable X.

- If X has a finite number of values then $E[X]$ exists.
- We need the moment of order $2\,(E[X^2])$ to calculate the variance of the random variable X.
- If X and Y are two independent random variable $\Longrightarrow E(XY) = E(X)E(Y)$..

3.3.1 Properties of expectation

The properties of expectation apply equally to discrete and absolutely continuous random variable.

If X and Y are two definite random variables defined on the same universe Ω, admitting an expectation, then:

1. $E\,(X + Y) = E(X) + E(Y)$
2. $E(aX) = aE(X), \quad \forall a \in R.$
3. If $X \geq 0$ then $E(X) \geq 0$.
4. If X is a constant character such that: $\forall \varpi \in \Omega \;\; X(\varpi) = k$ then $E(X) = k$.

Example

We take example 01 again,

$$E\,[X] = \sum_{i\geq 1} (x_i)\,p(x_i) = \left(0.\frac{1}{8}\right) + \left(1.\frac{3}{8}\right)\left(2.\frac{3}{8}\right) + \left(3.\frac{1}{8}\right) = \frac{12}{8} = \frac{3}{2}.$$

The second-order moment of the a.v. X is given by

$$E\left[X^2\right] = \sum_{i\geq 1} (x_i)^2\,p(x_i) = \left(0^2.\frac{1}{8}\right) + \left(1^2.\frac{3}{8}\right)\left(2^2.\frac{3}{8}\right) + \left(3^2.\frac{1}{8}\right) = \frac{24}{8} = 3.$$

3.4 Mathematical variance

Definition 7.5

The variance $V(X)$ of an random variable is the mathematical expectation of the deviation from the mathematical expectation. It is a dispersion parameter that corresponds to the centered moment of order 2 of the random variable X.

If X is an random variable with expectation $E[X]$, we call the variance of X is the real

$$V(X) = E(X - E[X])^2 = E\left[X^2\right] - (E[X])^2$$

If X is a discrete random variable with probability distribution (x_i, p_i), defined over a finite number of elementary events, then the variance is equal to:

$$V(X) = \sum_{i=1}^{n}(x_i - E[X])^2 p_i = \left(\sum_{i=1}^{n} x_i^2 p_i\right) - (E[X])^2$$

The standard deviation of X is the quantity

$$\sigma = \sqrt{V(x)}$$

Remark

- Variance and standard deviation are never negative.
- The standard deviation represents the mean deviation (the average distance) between the variable and its mean. It measures the dispersion of a variable:

- The larger the standard deviation, the more the variable takes on values that may be far apart.

- The smaller the standard deviation the closer the variable is to its mean.

close to its mean.

- If $E[X] = 0$ we say that the random variable is centered.
- If $V(X) = 1$ we say the random variable is reduced.

Example

Let's go back to example 01,

$$V(X) = E\left[X^2\right] - (E\left[X\right])^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4} > 0$$

$$\sigma = \sqrt{V(X)} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{4}$$

4 Continuous random variables

Definition 7.6

A continuous random variable is a function X, going from a universe Ω into R

Definition 7.7

Let X be a continuous random variable. We call probability density of X, a positive and integrable application $f : R \to R^+$, verifying:

$$\begin{cases} f \geq, \quad \forall x \in R \\ \quad f \text{ is piecewise continuous} \\ \int_{-\infty}^{+\infty} f(x)dx = 1 \end{cases}$$

4.1 Probability law of a continuous random variable

The probability law of a continuous random variable is determined by the distribution function F defined for any real x by:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$$

The probability of an interval is obtained by integrating the density of X, or by using

$$\begin{aligned} P\left(X \in [x_1, x_2]\right) &= P\left(x_1 \leq X \leq x_2\right) \\ &= \int_{x_1}^{x_2} f(t)dt \\ &= F(x_2) - F(x_1) \end{aligned}$$

4.2 Mathematical expectation

It is defined by:

$$E(x) = \int_{-\infty}^{+\infty} xf(x)dx$$

4.3 Non-centered moments

The non-centered moment of order $r \in N^*$ of a continuous random variable X is the quantilé, when it exists:

$$m_r(X) = E(X^r) = \int_{-\infty}^{+\infty} x^r f(x)dx$$

4.4 Variance

It is defined by

$$V(X) = E\left[X - E(X)\right]^2 = E(X^2) - \left[E(X)\right]^2$$

When variance exists. For any real a:

$$V(X + a) = V(X) \text{ and } V(aX) = a^2V(X)$$

If X and Y are two independent random variables with variance, then:

$$V(X + Y) = V(X) + V(Y)$$

Example

Let the function

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

1- f *a density function?*

2. Determine the distribution function.

3. Calculate the expectation $E(X)$*.*

Solution

$$1) \begin{cases} f \text{ is positive } f \geq, \quad \forall x \in [0, 1] \\ f \quad is\ continuous\ on\ [0, 1] \\ \int_{-\infty}^{+\infty} f(x)dx \stackrel{?}{=} 1 \end{cases}$$

$$\begin{aligned}\int_{-\infty}^{+\infty} f(x)dx &= \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{+\infty} f(x)dx \\ &= \int_{0}^{1} 3x^2dx = 3\int_{0}^{1} x^2dx = 3\left[\frac{x^3}{3}\right]_0^1 = 1-0 = 1\end{aligned}$$

so f is a density function of the random variable X.

2- Distribution function :

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

- If $x < 0 \rightarrow F(x) = \int_{-\infty}^{x} 0dt = 0$.
- If $0 \leq x < 1 \rightarrow F(x) = \int_{-\infty}^{0} f(t)dt + \int_{0}^{x} f(t)dt = \int_{0}^{x} 3t^2dt = [t^3]_0^x = x^3$.
- If $x \geq 1 \rightarrow F(x) = \int_{-\infty}^{0} f(t)dt + \int_{0}^{1} f(t)dt + \int_{1}^{x} f(t)dt = \int_{1}^{x} 3t^2dt = 1$.

So

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^3 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 0 \end{cases}$$

3- Expectation $E(X)$

$E(x) = \int_{-\infty}^{+\infty} xf(x)dx = \int_{-\infty}^{0} xf(x)dx + \int_{0}^{1} xf(x)dx + \int_{1}^{+\infty} xf(x)dx$
$= 3\int_{0}^{1} xx^2dt = 3\left[\frac{x^4}{4}\right]_0^1 = \frac{3}{4}$

5 Exercises

Exercise 1

A geologist is observing the growth of mineral crystals in a lab experiment. The number of crystal facets that form on a particular mineral, denoted by F, is an integer between 6 and 12. The length of a specific crystal edge, denoted by L, varies between 0.5 mm and 2.0 mm. Let C be the total cumulative length of all edges on a given crystal.

1- Is F a discrete or continuous random variable? Justify your answer.

2- Is L a discrete or continuous random variable? Justify your answer.

3- Determine the possible values of C.

Exercise 2

A biologist is studying the number of eggs laid by a species of frog. She observes 50 frogs and records the number of eggs laid by each frog in a single day. The results are as follows:

Number of eggs (X)	0	1	2	3	4
Frequency	5	10	15	12	8

1. Define the probability mass function (pmf) of (X).
2. Calculate the probability that a frog lays at least 2 eggs.
3. Compute the expected number of eggs laid by a frog.
4. Calculate the variance of the number of eggs.

Exercise 3

Let (X) be the time (hours) it takes a bacterial culture in a lab to reach mid-log phase. The time is between 0 and 4 hours and its density is shaped like a symmetric triangle (short times and long times are less likely than intermediate times). The pdf is defined by

$$f(x) = \begin{cases} \frac{1}{4}x, & 0 \leq x \leq 2, \\ \frac{1}{4}(4-x), & 2 < x \leq 4 \\ 0 & otherwise \end{cases}$$

1. Is f a probability density?
2. Determine the distribution function.
3. Calculate: the expectation, the variance, and $P(1 \leq X \leq 3)$.

Exercise 4

A biologist is studying a new enzyme that breaks down a toxin. The reaction time (T, in minutes) required for the enzyme to break down 90% of the toxin in a solution is modeled by a continuous random variable with the following Probability Density Function (PDF):

$$f(t) = \begin{cases} \frac{1}{10}e^{-1/10t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

This function describes an Exponential Distribution, which is often used to model waiting times and lifespans.

1. Calculate the probability that the reaction takes between 5 and 15 minutes to complete. $P(5 \leq T \leq 15)$

2. Calculate the Expected Reaction Time ($E[T]$), which represents the average time for this enzyme.

3. Determine the Cumulative Distribution Function $(CDF), F(t)$, for $t \geq 0$.

Use the CDF to find the time t such that the probability of the reaction completing by that time is 0.8647 (i.e., find t such that $P(T \leq t) = 0.8647$).

6 The answers

Exercise 1

1- F is a discrete random variable. It can only take on specific, distinct integer values (6, 7, 8, 9, 10, 11, 12). You cannot have, for instance, 6.5 facets.

2- L is a continuous random variable. It can take on any value within a given range (0.5 mm to 2.0 mm), including decimals and fractions. For example, the length could be 0.75 mm, 1.234 mm, or any other value within that interval.

3- This question is a bit more complex in this context because the "total cumulative length of all edges" depends on both F (number of facets/edges) and L (length of each edge).

If we assume each of the F facets contributes an edge of length L, then $C = F * L$.

Since F ranges from 6 to 12 and L ranges from 0.5 to 2.0, the minimum possible value for C would be when F is at its minimum and L is at its minimum: $C_{min} = 6*0.5 = 3mm$.

The maximum possible value for C would be when F is at its maximum and L is at its maximum: $C_{max} = 12 * 2.0 = 24mm$.

Therefore, the possible values of C range from 3 mm to 24 mm. Since L is continuous, C will also be a continuous random variable within this range.

Exercise 2

This exercise deals with a Discrete Random Variable (X, the number of eggs laid) derived from observed frequencies. Since the total number of frogs observed is N=5+10+15+12+8=50, we can use the frequencies to define the empirical probabilities.

1. Define the Probability Mass Function (PMF)

The PMF, $P(X = x)$, is the probability that the random variable X takes on a specific value x. We calculate this by dividing the frequency of each outcome by the total number of observations ($N = 50$).

Number of eggs (x_i)	0	1	2	3	4	Total
Frequency	5	10	15	12	8	50
$P(X = x) = n_i/50$	0.10	0.20	0.30	0.25	0.16	2

The PMF is defined by the table in the last column. For example, $P(X = 2) = 0.30$.

2. Calculate $P(X \geq 2)$

The probability that a frog lays at least 2 eggs is the sum of the probabilities for laying $2, 3$, or 4 eggs (by the Additivity Axiom).

$$\begin{aligned} P(X &\geq 2) = P(X = 2) + P(X = 3) + P(X = 4) \\ &= 0.30 + 0.24 + 0.16 = 0.70 \end{aligned}$$

The probability that a frog lays at least 2 eggs is 0.70 (or 70%).

3. Compute the Expected Number of Eggs (E[X])

The expected value (or mean) is the weighted average of all possible outcomes, where the weights are the probabilities $P(X = x)$.

$$E[X] = \sum_{i=1}^{n} x_i \times P(X = x_i)$$

x	0	1	2	3	4	$Total$
$P(X = x)$	0.10	0.20	0.30	0.24	0.16	1
$x \times P(X = x)$	0	0.20	0.60	0.72	0.64	2.16
$x^2 \times P(X = x)$	0	0.20	1.20	2.16	2.56	6.12

$$E[X] = 2.16 \ eggs$$

4. Calculate the Variance (Var[X])

The variance measures the spread of the distribution around the mean $E[X] = 2.16$. We use the computational formula:

$$Var[X] = E[X^2] - (E[X])^2$$

First, calculate$E[X^2]$

$$\begin{aligned} E[X^2] &= \sum_{i=1}^{n} x_i^2 \times P(X = x_i) \\ &= 6.12 \end{aligned}$$

Now, calculate the variance:

$$\begin{aligned} Var[X] &= E[X^2] - (E[X])^2 \\ &= 6.12 - 2.16^2 = 1.4544 \end{aligned}$$

Exercise 3

1. Is f a probability density?

For a function to be a valid probability density function (PDF), two conditions must be met:

We must check two things: $\begin{cases} f(x) \geq 0 \text{ for all } x \\ \int_{-\infty}^{+\infty} f(x)dx = 1 \end{cases}$

Let's check these conditions for the given $f(x)$

$$f(x) = \begin{cases} \frac{1}{4}x, & 0 \leq x \leq 2, \\ \frac{1}{4}(4-x), & 2 < x \leq 4 \\ 0 & otherwise \end{cases}$$

Condition a: $f(x) \geq 0$

For $0 \leq x \leq 2$, x is non-negative, so $\frac{1}{4}x$ is non-negative.

For $2 < x \leq 4$, $4 - x$ is non-negative (since $x \leq 4$), so $\frac{1}{4}(4-x))$ is non-negative.

Otherwise, $f(x) = 0$, which is non-negative.

So, condition (a) is met.

Condition b: $\int_{-\infty}^{+\infty} f(x)dx = 1$

We need to integrate $f(x)$ over its defined intervals:

$$\begin{aligned}
\int_{-\infty}^{+\infty} f(x)dx &= \int_{-\infty}^{0} 0dx + \int_{0}^{2} \frac{1}{4}xdx + \int_{2}^{4} \frac{1}{4}(4-x)dx \\
&= \frac{1}{4}\int_{0}^{2} xdx + \frac{1}{4}\int_{2}^{4}(4-x)dx \\
&= \frac{1}{4}\left[\frac{x^2}{2}\right]_0^2 + \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_2^4 \\
&= \frac{1}{4}[2-0] + \frac{1}{4}\left[16 - \frac{16}{2} - 8 + \frac{4}{2}\right] \\
&= \frac{1}{2} + \frac{1}{2} = 1.
\end{aligned}$$

Since both conditions are met, yes, $f(x)$ is a probability density function.

2. Determine the distribution function.

The cumulative distribution function (CDF), $F(x)$, is defined as

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

We need to define $F(x)$ piecewise for different ranges of x.

- For $x < 0$

$$F(x) = \int_{-\infty}^{x} 0dt = 0$$

- For $0 \leq x \leq 2$:

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} 0dt + \int_{0}^{x} f(t)dt \\
&= \int_{0}^{x} \frac{1}{4}tdt = \frac{1}{4}\left[\frac{t^2}{2}\right]_0^x = \frac{x^2}{8}
\end{aligned}$$

- For $2 \leq x \leq 4$

$$\begin{aligned}
F(t) &= \int_{-\infty}^{x} f(t)dt == \int_{-\infty}^{0} 0dt + \int_{0}^{2} f(t)dt + \int_{2}^{x} f(t)dt \\
&= \int_{0}^{2} \frac{1}{4}tdt + \int_{2}^{x} \frac{1}{4}(4-t)dt \\
&= \frac{1}{4}\left[\frac{t^2}{2}\right]_0^2 + \frac{1}{4}\left[4t - \frac{t^2}{2}\right]_2^x \\
&= x - \frac{x^2}{8} - 1
\end{aligned}$$

- For $x > 4$:

$$F(x) = \int_0^4 f(t)dt = 1 \text{ (as calculated inpart 1).}$$

Therefore, the distribution function $F(x)$ is:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x^2}{8} & \text{if } \ 0 \leq x \leq 2 \\ x - \frac{x^2}{8} - 1 & \text{if } 2 \leq x \leq 4 \\ 1, & \text{if } x > 4 \end{cases}$$

3. Calculate: the expectation, the variance, and $\mathbf{P(1 \leq X \leq 3)}$.

a. Expectation $(E[X])$:

$$\begin{aligned} E[X] &= \int_{-\infty}^{+\infty} xf(x)dx \\ &= \frac{1}{4}\int_0^2 x^2 dx + \frac{1}{4}\int_2^4 (4x - x^2)dx \\ &= \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2 + \frac{1}{4}\left[4\frac{x^2}{2} - \frac{x^3}{3}\right]_2^4 \\ &= \frac{2}{3} + \frac{4}{3} = 2. \end{aligned}$$

Since the distribution is symmetric about $x = 2$, the expectation (mean) is indeed 2 hours.

b. Variance $(Var[X])$

$$Var[X] = E\left[X^2\right] - (E\,[X])^2\,.$$

First, let's find $E\left[X^2\right]$

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{+\infty} x^2 f(x)dx \\ &= \frac{1}{4}\int_0^2 x^3 dx + \frac{1}{4}\int_2^4 (4x^2 - x^3)dx \\ &= \frac{1}{4}\left[\frac{x^4}{4}\right]_0^2 + \frac{1}{4}\left[4\frac{x^3}{3} - \frac{x^4}{4}\right]_2^4 \\ &= 1 + \frac{11}{3} = \frac{14}{3}. \end{aligned}$$

Now calculate the variance:

$$\begin{aligned} Var[X] &= E\left[X^2\right] - (E[X])^2 \\ &= \frac{14}{3} - 2^2 = \frac{2}{3}. \end{aligned}$$

The variance is 2/3 hours^2

c. $\mathbf{P(1 \leq X \leq 3)}$ =?

$$\mathbf{P(1 \leq X \leq 3)} = \int_1^3 f(x)dx$$

We need to split the integral because the function definition changes at $x = 2$:

$$\begin{aligned} \mathbf{P(1 \leq X \leq 3)} = & \int_1^2 f(x)dx + \int_2^3 f(x)dx \\ = & \frac{1}{4}\int_1^2 xdx + \frac{1}{4}\int_2^3 (4-x)dx \\ = & \frac{1}{4}\left[\frac{x^2}{2}\right]_1^2 + \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_2^3 \\ = & \frac{3}{8} + \frac{3}{8} = \frac{3}{4} \end{aligned}$$

The probability $\mathbf{P(1 \leq X \leq 3)}$ is 3/4 or 0.75.

Exercise 4

The random variable T follows an exponential distribution with rate parameter $\lambda = \frac{1}{10}$

1. Probability Calculation

The probability $P(5 \leq T \leq 15)$ is found by integrating the PDF over the interval $[5, 15]$.

$$P(5 \leq T \leq 15) = \int_5^{15} f(t)dt = \int_5^{15} \frac{1}{10}e^{-t/10}dt$$

Recall that the integral of $\lambda e^{-\lambda t}$ is $-e^{-\lambda t}$

$$P(5 \leq T \leq 15) = [-e^{-t/10}]_5^{15} = e^{-0.5} - e^{-1.5} \approx 0.3834$$

The probability that the reaction time is between 5 and 15 minutes is approximately 0.3834 (or 38.34%).

2. Expected Reaction Time (E[T])

For an exponential distribution with PDF $f(t) = \lambda e^{-\lambda t}$, the expected value is given simply by $1/\lambda$

Here, $\lambda = \frac{1}{10}$

$$E\left[T\right] = \int_{-\infty}^{+\infty} t.f(t)dt = \int_{-\infty}^{0} 0dt + \int_{0}^{+\infty} \frac{te^{-1/10t}}{10}dt = \frac{1}{\lambda} = \frac{1}{1/10} = 10\ minutes$$

The expected (average) time for the enzyme to complete the reaction is 10 minutes.

3. Cumulative Distribution Function (CDF)

The CDF, $F(t)$, gives the probability $P(T \leq t)$. It is found by integrating the PDF from $-\infty$ to t.

$$F(t) = \int_{-\infty}^{t} f(x)dx$$

- If $t < 0$

$$F(t) = \int_{-\infty}^{t} 0dx = 0.$$

- If $t \geq 0$

$$\begin{aligned} F(t) &= \int_{-\infty}^{t} f(x)dx = \int_{-\infty}^{0} 0dx + \int_{0}^{t} f(x)dx \\ &= \int_{0}^{t} \frac{1}{10} e^{-x/10} dx = -e^{-x*10}]_{0}^{t} = -e^{-t/10} - \left(-e^{-0/10}\right) \\ &= 1 - e^{-t/10} \quad \text{for } t \geq 0 \end{aligned}$$

Therefore, the distribution function $F(x)$ is

$$F(x) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-t/10} & \text{if } t \geq 0 \end{cases}$$

4. Finding the Time (t) from the CDF

We need to find t such that $P(T \leq t) = F(t) = 0.8647$.

$$1 - e^{-t/10} = 0.8647$$

Isolate the exponential term:

$$e^{-t/10} = 1 - 0.8647$$

Take the natural logarithm (ln) of both sides:

$$-\frac{t}{10} = -2$$

Solve for t :

$$t = 10 \times 2 = 20 \; minutes$$

The time at which there is an 86.47% chance of the reaction being complete is 20 minutes.

Chapter 8

Probability laws

1 Usual discrete laws

he **usual discrete laws** are a set of common probability distributions that are used to model the outcomes of experiments where the random variable can only take on a countable number of values (typically counts or integers).

The most frequently encountered discrete probability distributions are the Bernoulli, Binomial, and Poisson distributions.

1.1 Bernoulli's law

Definition 8.1

Bernoulli's Law describes the probability distribution of a discrete random variable X that can only take one of two values:

- 1 (Success): Occurs with probability p.
- 0 (Failure): Occurs with probability $q = 1 - p$.

This law models a single instance of a Bernoulli Trial a random experiment with exactly two mutually exclusive and exhaustive outcomes

Example

Head or Tail; Girl or Boy……

Example

Let be a Bernoulli trial and let X be a random variable taking the value 1 in the case of success and 0 in the case of failure.

$$\begin{aligned} X &: \Omega \to E = \{0,1\} \\ \omega &\to X(\omega) \end{aligned}$$

X follows a **Bernoulli distribution** with parameter p, denoted by : $X \rightsquigarrow B(1,p)$.

1.1.1 Probability Mass Function (PMF)

$P[X=1] = p$

$P[X=0] = 1-p = q$ or $p+q=1$

$P[X=k] = p^k(1-p)^{1-k}$

Parameters: The Bernoulli distribution is defined by a single parameter:

- $X \rightsquigarrow B(1,p)$
- **Expectation:** $E[X] = p$
- **Variance:** $V[X] = p \times q$ with $q = 1-p$

Proof

$E[X] = \sum_{i=1}^{k} X_i P[X=x_i] = 1 \times p + 0 \times (1-p) = p$

$V[X] = E[X^2] - (E[X])^2 = 1^2 \times p + 0^2 \times (1-p) - p^2 = p - p^2 = p(1-p) = p \times q.$

■

Example

A balanced coin is tossed. $\Omega = \{stack, face\}$.

Let X be a discrete random variable with:

$$\begin{aligned} X(\omega) &= 1 \text{ if } \omega = pile \\ X(\omega) &= 0 \text{ if } \omega = face \\ P[X=1] &= p = \frac{1}{2} \text{ so } X \rightsquigarrow B(1, p = \frac{1}{2}) \end{aligned}$$

Example

A disease M has a prevalence of 4%, an individual is chosen at random from the population. Let X be a discrete random variable:

$$X(\omega) = 1 \text{ if } \omega = illness$$
$$X(\omega) = 0 \text{ if } \omega = not\ ill$$
$$P[X = 1] = p = 0.04 \text{ therefore } X \rightsquigarrow B(1, p = 0.04)$$

1.2 Binomial law

The Binomial Law (or Binomial Distribution) is a discrete probability distribution used to model the number of successful outcomes in a fixed number of independent trials.

It is one of the most common distributions in statistics, frequently used when analyzing experiments with only two results: success or failure.

Conditions for a Binomial Experiment

For a random variable X to follow a Binomial Distribution, the underlying experiment must satisfy four specific criteria, often called the BINS conditions:

1- Binary: Each trial must have only two possible, mutually exclusive outcomes: Success or Failure.

2- Independent: The outcome of any single trial must not affect the outcome of any other trial.

3- Number (Fixed): The experiment must consist of a fixed number of trials (denoted by n).

4- Same Probability: The probability of success (p) must remain the same for every single trial.

Definition 8.2

Let be a Bernoulli scheme and let X be the random variable that associates with the number of successes:

$$X : \Omega \longrightarrow E = \{1, 2, \ldots, n\}$$
$$\omega \longmapsto X(\omega)$$

X follows a Binomial distribution with parameters n and p, noted as :$X \rightsquigarrow B(n, p)$.

1.2.1 Probability Mass Function (PMF)

Probability of obtaining k successes in n independent trials.

$$\begin{aligned} \forall k &\in \{0, 1, 2, ..., n\} \\ P[X = k] &= C_n^k \times p^k \times (1-p)^{n-k} \text{ avec } \quad C_n^k = \frac{n!}{k!(n-k)!} \end{aligned}$$

Where:

- $P(X = k)$ is the probability of exactly k successes.
- $C_n^k = \frac{n!}{k!(n-k)!}$ is the binomial coefficient, representing the number of ways to choose k successes from n trials.
- p^k is the probability of getting k successes.
- $(1-p)^{n-k}$ is the probability of getting n-k failures.

Property The sum of n independent Bernoulli variables with parameter p is a random variable following a binomial distribution with parameters n and p.

$$X_i \rightsquigarrow B(1, p), \ \forall i \in \{1, 2, ..., n\}, \ X_i \text{ independent}$$
$$S_n = \sum_{i=1}^{n} X_i$$

Remark

The Binomial Distribution is derived directly from Bernoulli's Law, modeling the number of successes in a sequence of n independent Bernoulli trials.

> ***Example***
>
> Let $S_n \rightsquigarrow B(n, p)$. Calculate $E[S_n]$ and $V[S_n]$
>
> $E[S_n] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = n \times p.$
>
> $V[S_n] = V[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} V[X_i] = n \times p \times (1-p).$
>
> To remember:
>
> $$X_i \rightsquigarrow B(n, p)$$

Key Statistics The mean and variance of a Binomial distribution are easily calculated directly from its parameters, n and p.

Expectation: $E[X] = np$.

Variance: $\mathbf{V}[X] = npq = np(1-p)$ with $q = 1-p$.

Example

A balanced coin is tossed 10 *times in a row. Let* X *be a random variable that associates the number of tails with these* 10 *coin tosses. What is the probability of having a total of* 8 *tails?*

$$X_i \rightsquigarrow B(n = 10, p = \frac{1}{2})$$

$P[X = k] = P[X = 8] = C_{10}^{8} \times \left(\frac{1}{2}\right)^8 \times (1 - \frac{1}{2})^{10-8} = 0.0439.$

Example

We want to model the number of boys in a family of 6 *children, each birth* $i(i \in \{1, 2, ...6\})$ *can be considered as a random variable* X*. What is the probability of having 4 boys?*

$$X_i \rightsquigarrow B(n = 6, p = \frac{1}{2})$$

$P[X = k] = P[X = 4] = C_6^4 \times \left(\frac{1}{2}\right)^4 \times (1 - \frac{1}{2})^{6-4} = 0.2344.$

Example

A student wants to take a MCQ-type competition consisting of 10 questions, for each question 4 answers are proposed of which only one is correct. What is the probability that he will pass? (to pass, he must answer at least 5 questions).

X represents the number of correct answers among 10 questions.

Here

$$X_i \rightsquigarrow B(n = 10, p = \frac{1}{4})$$

$P[X = k] = C_n^k \times \left(\frac{1}{4}\right)^k \times (1 - \frac{1}{4})^{n-k}, \;\; k = 0, 1, 2, ..., n$

$P[X \geq 5] = P[X = 5] + P[X = 6] + P[X = 7] + P[X = 8] + P[X = 9] + P[X = 10]$

$= 0.07812.$

1.3 Poisson distribution

The Poisson distribution is a discrete probability distribution that models the probability of a given number of events occurring in a fixed interval of time or space, provided these events occur with a known constant mean rate and independently of the time since the last event.

It is primarily used to count the occurrences of rare events.

Conditions for the Poisson Distribution

An experiment can be modeled by the Poisson distribution if it meets these conditions:

1. Events are Discrete: The variable X is a count (e.g., 0,1,2,3,. . .).

2. Independence: The occurrence of one event does not influence the occurrence of another event.

3. **Constant Rate (λ):** The average rate of events per unit of time or space is constant.

4. Simultaneity is Impossible: It is practically impossible for multiple events to occur at the exact same instant.

1.3.1 Parameters and Notation

The Poisson distribution is defined by a single parameter:

λ: The mean rate or average number of events that occur in the specified interval.

The notation is:

$$X \rightsquigarrow P(\lambda)$$

The random variable X can take any non-negative integer value (k=0,1,2,3,. . .).

1.3.2 Probability Mass Function (PMF)

The PMF gives the probability of observing exactly k events in the interval:

$$\forall k \in N : P\left[X = k\right] = \frac{e^{-\lambda}\lambda^{k}}{k!}$$

Where:

k is the actual number of occurrences (the outcome).

λ is the mean occurrence rate.

e is the base of the natural logarithm ($e \approx 2.71828$).

$k!$ is the factorial of k.

1.3.3 Key Statistics

A unique and important property of the Poisson distribution is that its mean and variance are equal to its parameter, λ.

$$E[X] = V[X] = \lambda$$

Proof

$$\begin{aligned} E[X] = \textstyle\sum_{k=0}^{\infty} k.P[X=k] = \sum_{k=0}^{\infty} k.\frac{e^{-\lambda}\lambda^k}{k!} &= e^{-\lambda} \textstyle\sum_{k=0}^{\infty} \frac{\lambda^k}{(k-1)!} \\ &= e^{-\lambda}\left[\lambda + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + ... + \frac{\lambda^k}{(k-1)!}\right] \\ &= \lambda e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + ... + \frac{\lambda^{k-1}}{(k-1)!}\right] = \lambda e^{-\lambda} e^{\lambda} = \lambda \end{aligned}$$

■

Remark

This law is often used in modeling queues: number of customers waiting at a supermarket checkout, number of drivers passing through a toll booth during a period of time.

Example

At a bank counter, we know that the average number of customers per hour is 12. It is assumed that the number of customers per hour is distributed according to a Poisson distribution. What is the probability that, in one hour, the teller will deal with more than 15 customers?

Let

$$X \rightsquigarrow P(\lambda) \ \ with\ \lambda = 12$$

$$P[X > 15] = 1 - P[X \leq 15] = 1 - \textstyle\sum_{k=0}^{15} \frac{e^{-12}12^{15}}{15!} = 0.1556$$

Example

A switchboard receives an average of 0.7 calls per minute. What is the probability that between 09h59 and 10 : 00 it receives:

1. *0 calls.*
2. *1 call.*

> *3. More than one call.*
>
> **Correction:**
>
> 1- $P\left[X=0\right]=\frac{e^{-0.7}(0.7)^{0}}{0!}=0.4965.$
>
> 2- $P\left[X=1\right]=\frac{e^{-0.7}(0.7)^{1}}{1!}=0.3476.$
>
> 3-$P\left[X\leq 1\right]=P\left[X=0\right]+P\left[X=1\right]=0.8441.$

1.4 Approximation of a binomial distribution by a Poisson distribution:

Given a binomial random variable $X\rightsquigarrow B(n,p)$, if n is large enough and p is small enough, then the binomial distribution $B(n,p)$ can be approximated by a Poisson distribution Poisson distribution with the same expectation $P(\lambda=np)$.

In practice, this approximation is satisfactory when:

$$\begin{cases} n\geq 50 \\ p\leq 0.1 \\ n\cdot\ p\leq 10 \end{cases}$$

Then:$X\rightsquigarrow B(n,p)\approx P\left(\lambda=np\right)$

> ***Example***
>
> *The probability that a person is allergic to drug M is $p=0.0002$, we consider a sample of 10000 people.*
>
> *Let X be the random variable whose value is the number of allergic people in the sample.*
>
> *1. Determine the distribution of X.*
>
> *2. Calculate the probability that 2 people are allergic.*
>
> *3. What is the probability of observing at least 3 patients?*
>
> **Correction:**
>
> 1- $X\rightsquigarrow B(n=10000,p=0.0002)$
>
> 2-$\begin{cases} n=10000\geq 50 \\ p=0.0002\leq 0.1 \\ n\cdot\ p=2<10 \end{cases}$
>
> then we can approximate the binomial distribution by the Poisson distribution
>
> $X\rightsquigarrow B(n=10000,p=0.0002)\approx P\left(\lambda=np=2\right)$

So

$$P[X=2] = \frac{e^{-2}(2)^2}{2!} = 0.2706$$

3- $P[X \geq 3] = 1 - P[X < 3] = 1 - P[X \leq 2]$

$$= 1 - (P[X=0] + P[X=1] + P[X=2])$$

$$= 1 - (0.1353 + 0.2706 + 0.2706) = 0.3235.$$

2 Continuous laws

Continuous probability laws (or distributions) model random variables that can take on any value within a specified range or interval. These variables typically arise from measurements (like weight, time, or temperature), rather than counting.

Unlike discrete laws, we cannot assign a probability to a single exact point; instead, we find the probability over a range by calculating the area under the Probability Density Function (PDF), f(x).

Here are the most common and important continuous probability laws:

2.1 Normal distribution

A random variable x is said to follow a Normal distribution with parameters μ and σ if

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad \forall x \in R$$

$f(x)$ depends on two parameters (μ, σ^2)

2.1.1 Expectation:

$$E(x) = \mu$$

2.1.2 The variance :

$$var(x) = \sigma^2$$

$$x \rightsquigarrow N(E(x), var(x)); \qquad x \rightsquigarrow N(\mu, \sigma^2)$$

The normal distribution curve has a bell shape.

2.1.3 The distribution function

$$P[a \leq x \leq b] = F(b) - F(a)$$

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

Since it is difficult to calculate the distribution function for this distribution, we use the change of variable to obtain a centered and reduced variable

$$Z = \frac{X - E(x)}{\sigma} = \frac{X - \mu}{\sigma} \rightsquigarrow N(0,1)$$

All values of the distribution function of the $N(0,1)$ distribution are tabulated.

If $X_1 \rightsquigarrow N(\mu_{x_1}, \sigma_{x_1})$ and $X_2 \rightsquigarrow N(\mu_{x_2}, \sigma_{x_2})$ knowing that $X_1 and\ X_2$ are independent, then the laws of the following variables are as follows:

$$X = aX_1 + bX_2 \rightsquigarrow N\left(a\mu_1 + b\mu_2; \sqrt{a^2\sigma_1^2 + b^2\sigma_2^2}\right)$$

$$X = aX + b \rightsquigarrow N\left(a\mu_x + b; \sqrt{a^2\sigma_x^2}\right)$$

Example

Let $X \rightsquigarrow N(-2,5)$

1-Give the probability distribution of $y = 3x - 2$

Let $X_1 \rightsquigarrow N(-1,6)$ and $X_2 \rightsquigarrow N(2,5)$, x_1 and x_2 be independent

2- Give the probability law of $Z = -2x_1 + 4x_2$

1- $X \rightsquigarrow N(-2,5)$, $y = 3x - 2 \rightsquigarrow N(3(-2) - 2; \sqrt{3^2 * 5})$, $y \rightsquigarrow N(-8,40)$

2- $X_1 \rightsquigarrow N(-1,6)\ et\ X_2 \rightsquigarrow N(2,5)$,

$Z = -2x_1 + 4x_2 \rightsquigarrow N(-2(-1) + 4 * 2; \sqrt{4 * 6 + 16 * 5})$

$$Z \rightsquigarrow N(10, \sqrt{104})$$

2.1.4 Probability calculation

Let $X \rightsquigarrow N(\mu_x, \sigma_x^2)$ and $Z = \frac{X-\mu}{\sigma} \rightsquigarrow N(0,1)$

Then the distribution function of Z is denoted by $F(Z)$ with $F(u) = P(Z < u)$ this function is tabulated

$P(X < a) = P\left(\frac{X-\mu}{\sigma} < \frac{a-\mu}{\sigma}\right) = P\left(Z < \frac{a-\mu}{\sigma}\right) = F\left(\frac{a-\mu}{\sigma}\right)$

$P(X \geq a) = 1 - P\left(\frac{X-\mu}{\sigma} < \frac{a-\mu}{\sigma}\right) = 1 - F(\frac{a-\mu}{\sigma})$

$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right) = F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$

$F(-Z) = 1 - F(Z).$

Example

1- Find : $F(1.84)$, $F(2.14)$, $F(0.8)$, $F(0.04)$, $F(0.89)$.

2- Determine the value of Z : $F(Z) = 0.9607$, $F(Z) = 0.7562$, $F(Z) = 0.8413$, $F(Z) = 0.8554$, $F(Z) = 0.9772$.

Solution :

From the normal distribution table we find :

$F(1.84) = 0.9671$, $F(Z) = 0.9607 \Rightarrow Z = 1.76$.

$F(2.14) = 0.9838, F(Z) = 0.7562 \Rightarrow Z = (0.69 + 0.7)/2 = 0.695.$

$F(0.8) = 0.7881, F(Z) = 0.8413 \Rightarrow Z = 1.$

$F(0.04) = 0.5160, F(Z) = 0.8554 \Rightarrow Z = 1.06.$

$F(0.89) = 0.8133, F(Z) = 0.9772 \Rightarrow Z = 2.$

Example

1- *Let* $X \rightsquigarrow N(\mu_x = 2, \sigma_x^2 = 25)$ *calculate* $P(x < 4)$ *and* $P(x > 3)$.

2- The random variable x follows a normal distribution with expectation 550 and standard deviation 100. What is the probability that x is less than 650 , more than 746 , less than 500?

Solution

1- Let $X \rightsquigarrow N(\mu_x = 2, \sigma_x^2 = 25)$ calculate

$P(x < 4) = P(\frac{x-2}{5} < \frac{4-2}{5}) = P(z < 0.4) = F(0.4) = 0.6554.$

$P(x > 3) = 1 - P(x < 3) = 1 - P(\frac{x-2}{5} < \frac{3-2}{5}) = 1 - P(z < 0.2) = 1 - F(0.2)$

$= 1 - 0.5793 = 0.4207$

2- Let $X \rightsquigarrow N(\mu_x = 550, \sigma_x^2 = 100^2)$ calculate

$P(x < 650) = P(\frac{x-550}{100} < \frac{650-550}{100}) = P(z < 1) = F(1) = 0.8413.$

$$P(x > 746) = 1 - P(x < 746) = 1 - P(\frac{x-550}{100} < \frac{746-550}{100}) = 1 - P(z < 1.96)$$

$$= 1 - F(1.96) = 1 - 0.975 = 0.025$$

$$P(550 < x < 600) = P\left(\frac{550-550}{100} < \frac{x-550}{100} < \frac{600-550}{100}\right) = P(0 < z < 0.5)$$
$$= F(0.5) - F(0) = 0.6915 - 0.5 = 0.1915$$

2.2 Chi-square law

The Chi-square (χ^2) distribution is a continuous probability distribution that is fundamental to inferential statistics, particularly for hypothesis testing. It primarily arises when analyzing variances and assessing the "goodness-of-fit" between observed data and expected data.

Definition 8.3

The Chi-square distribution describes the probability of the sum of squares of a number of independent standard normal random variables.

If $Z_1, Z_2, ..., Z_k$ are independent random variables, each following the Standard Normal Distribution (mean $\mu = 0$ standard deviation $\sigma = 1$), then the random variable X defined as:

$$X = Z_1^2 + Z_2^2 + ... + Z_k^2$$

follows a Chi-square distribution with k degrees of freedom v

$$X \rightsquigarrow \chi^2(v) \quad where\ v = k$$

2.2.1 Key Characteristics

* Parameter (v) : The shape of the Chi-square distribution is determined entirely by its degrees of freedom (df).

* Non-Negative: Since it is the sum of squared values, the χ^2 statistic can never be negative (it is defined on $[0, \infty[$).

* Skewed: The distribution is positively (right) skewed. As the degrees of freedom (v) increase, the distribution becomes more symmetrical and approaches the Normal distribution.

* Mean and Variance:
    * Mean $(E[X]) : v$
    * Variance $(Var[X]) : 2v$

2.2.2 Primary Applications

The Chi-square distribution is used in three main areas of statistical inference:

- Chi-square Test of Independence: Used to determine whether there is a statistically significant association between two categorical variables (e.g., Is the type of diet independent of blood pressure?).
- Chi-square Goodness-of-Fit Test: Used to test whether an observed frequency distribution significantly differs from a hypothesized theoretical distribution (e.g., Does the observed number of seeds with certain characteristics match the expected proportions from Mendelian genetics?).
- Inference on Variance: Used to construct confidence intervals for the population variance (σ^2) or to perform hypothesis tests about a single population variance.

2.3 Fisher-Snedecor law

The Fisher-Snedecor law, more commonly known as the F-distribution, is a continuous probability distribution that is fundamental to the statistical technique known as Analysis of Variance (ANOVA). It's primarily used to compare the variances of two or more populations.

Definition 8.4

The F-distribution describes the distribution of the ratio of two independent Chi-square random variables that have been divided by their respective degrees of freedom (v).
Let X_1 be a χ^2 random variable with v_1 degrees of freedom, and X_2 be a χ^2 random variable with v_2 degrees of freedom. If X_1 and X_2 are independent, then the random variable F :

$$F = \frac{X_1/v_1}{X_2/v_2}$$

follows an $F-$distribution.

The F-distribution is unique because it is defined by two separate degrees of freedom:

1. Numerator Degrees of Freedom (v_1): Corresponds to the degrees of freedom of the first Chi-square variable (the numerator).

2. Denominator Degrees of Freedom (v_2): Corresponds to the degrees of freedom of the second Chi-square variable (the denominator).

We write the notation as:

$$F \backsim F(v_1, v_2)$$

2.3.1 Key Characteristics

- **Non-Negative:** Similar to the Chi-square distribution, the F-distribution is defined only for values $F \geq 0$, since it represents the ratio of two squared (hence non-negative) quantities.
- **Asymmetric:** The distribution is typically positively (right) skewed. However, as the degrees of freedom increase, the skewness decreases and the distribution becomes more symmetric.
- **Critical Region:** In hypothesis testing, the F-distribution is predominantly used in right-tailed (one-sided) tests, as a significantly large F-statistic suggests the presence of a meaningful difference or effect.

2.3.2 Primary Applications

The F-distribution is the cornerstone of advanced statistical modeling:

1- . **Analysis of Variance (ANOVA)**

This is the most common use. ANOVA tests whether the means of two or more groups are equal. It does this by comparing two types of variability:

$$F = \frac{Variance\ Between\ Groups\ (Signal)}{Variance\ Within\ Groups\ (Noise)}$$

A large F-statistic suggests that the differences between the groups are much larger than the natural random differences within the groups, leading to the rejection of the null hypothesis that all means are equal.

2- **Comparing Two Population Variances**: The F-distribution can be used directly to test the null hypothesis that two population variances (σ_1^2 and σ_2^2) are equal. The test statistic is the ratio of the two sample variances:

$$F = \frac{s_1^2}{s_2^2}$$

3- Regression Analysis

It's also used to test the overall significance of a regression model (i.e., whether the entire set of predictor variables significantly explains the variation in the response variable).

3 Exercises

Exercise 1

A medical laboratory analyzes 3,000 blood samples per day to detect a certain type of bacterial contamination.

The probability that a blood sample is contaminated is 0.0015.

Let (X) be the random variable representing the number of contaminated samples per day.

1. Determine the distribution of (X). (Check the approximation conditions.)
2. Calculate the probability of getting exactly 3 contaminated samples in one day.
3. Calculate the probability of obtaining at least 2 contaminated samples.

Exercise 2

A plant biologist is testing the viability of a new hybrid seed. It is known that each seed has a 70% probability of germinating successfully, and the germination of any one seed is independent of the others. The biologist plants a random sample of five ($n = 5$) of these seeds.

Let X be the discrete random variable representing the total number of seeds that germinate successfully.

1. Identify the appropriate probability distribution for X and state its parameters.
2. Calculate the probability that exactly three seeds germinate successfully. $P(X = 3)$
3. Calculate the probability that at least four seeds germinate successfully. $P(X \geq 4)$
4. Calculate the expected number of seeds that will germinate.

4 The answers

Exercise 1

Given:

- Total number of blood samples per day, $n = 3000$
- Probability that a blood sample is contaminated, $p = 0.0015$
- X is the random variable representing the number of contaminated samples per day.

1. Determine the distribution of (X). (Check the approximation conditions.)

Initially, this scenario fits a **Binomial Distribution**, since:

- There's a fixed number of trials ($n = 3000$ samples).
- Each trial is independent (one sample's contamination doesn't affect another).
- There are two possible outcomes for each trial (contaminated or not contaminated).
- The probability of "success" (contamination) is constant (p=0.0015).

So,

$$X \rightsquigarrow B(n, p) = B(3000, 0.0015).$$

However, with n being large and p being small, the Binomial distribution can often be approximated by the Poisson Distribution. Let's check the conditions for this approximation:

- n is large ($n = 3000$, which is large).
- p is small ($p = 0.0015$, which is small, typically $p < 0.05$).
- $np < 10$ (or sometimes $np < 5$ is used as a stricter condition).

Let's calculate $\lambda = np$:

$$\lambda = np = 3000 * 0.0015 = 4.5$$

Since $\lambda = 4.5$, which is less than 10 (and even less than 5), the Poisson approximation is appropriate and provides a good estimate.

Therefore, the distribution of X can be approximated by a Poisson Distribution with parameter

$$X \rightsquigarrow P(\lambda = 4.5)$$

The probability mass function (PMF) for a Poisson distribution is:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

2. **Calculate the probability of getting exactly 3 contaminated samples in one day.**

Using the Poisson approximation with $\lambda = 4.5$ and $k = 3$.

$$P(X = 3) = \frac{e^{-4.5}(4.5)^3}{3!} = 0.1687$$

So, the probability of getting exactly 3 contaminated samples in one day is approximately 0.1687.

3. Calculate the probability of obtaining at least 2 contaminated samples.

"At least 2 contaminated samples" means $P(X \geq 2)$.

It's easier to calculate this using the complement rule: $P(X \geq 2) = 1 - P(x < 2)$.

And

$$P(X < 2) means P(X = 0) + P(X = 1).$$

First, calculate $P(X = 0)$:

$$P(X = 0) = \frac{e^{-0}(4.5)^0}{0!} \approx 0.0111119$$

Next, calculate $P(X = 1)$:

$$P(X = 1) = \frac{e^{-1}(4.5)^1}{1!} \approx 0.04999$$

Now, sum these probabilities:

$$P(X < 2) = P(X = 0) + P(X = 1) \approx 0.011109 + 0.04999 = 0.061099$$

Finally, calculate $P(X \geq 2)$:

$$P(X \geq 2) = 1 - P(X < 2) \approx 1 - 0.061099 = 0.938901$$

So, the probability of obtaining at least 2 contaminated samples is approximately 0.9389.

Exercise 2

1. Identify the Probability Distribution

The experiment consists of a fixed number of independent trials ($n = 5$), where each trial (seed) has only two possible outcomes (germinate or not germinate) with a constant probability of success ($p = 0.70$).

- Distribution: Binomial Distribution
- Parameters:

$$\begin{aligned} &\text{Number of trials, } n = 5\\ &\text{Probability of success, } p = 0.70\\ &\text{Probability of failure, } q = 1 - p = 0.30 \end{aligned}$$

The Binomial Probability Mass Function (PMF) is:

$$P(X = k) = C_n^k p^k q^{n-k} = C_5^k (0.70)^k (0.30)^{5-k}.$$

2. **Calculate P(X=3)**

We need to calculate the probability of exactly $k = 3$ successful germinations out of $n = 5$ trials.

$$\begin{aligned} P(X &= 3) = C_5^3 (0.70)^3 (0.30)^{5-3}.\\ &= 10 \times (0.343) \times (0.09)\\ &= 10 \times 0.03087 = 0.3087 \end{aligned}$$

The probability that exactly three seeds germinate is 0.3087 (*or* 30.87%).

3. Calculate $\mathbf{P(X \geq 4)}$

The probability of at least four germinations is the sum of the probabilities of exactly 4 or exactly 5 successful germinations:

$$P(X \geq 4) = P(X = 4) + P(X = 5)$$

Calculate $P(X = 4)$:

$$\begin{aligned} P(X &= 4) = C_5^4 (0.70)^4 (0.30)^{5-4} \\ &= 5 \times (0.2401) \times (0.30) = 0.36015 \end{aligned}$$

Calculate $P(X = 5)$:

$$\begin{aligned} P(X &= 5) = C_5^5 (0.70)^5 (0.30)^{5-5} \\ &= 1 \times (0.16807) \times (1) = 0.16807 \end{aligned}$$

Sum the probabilities:

$$P(X \geq 4) = 0.36015 + 0.16807 = 0.52822$$

The probability that at least four seeds germinate is approximately 0.5282 (*or* 52.82%).

4. **Calculate the Expected Number** $(E[X])$

For a Binomial distribution, the expected value is given by the simple formula:

$$E[X] = n \cdot p = 5 \times 0.70 = 3.5$$

The expected number of seeds to germinate successfully is 3.5. (Since the number of seeds must be discrete, this simply represents the long-run average).

References

1. A. Hamon. *Descriptive Statistics; Corrected Exercises*, PUR, 2008.
2. A. Oukacha. *Descriptive Statistics and Probability Calculus*, 2010.
3. A. Rebbouh. *Descriptive Statistics and Probability Calculus*, Houma, 2009.
4. T. Apostol. *Calculus, Vol. 1*, Addison-Wesley, 1957.
5. Bernard Grais. *Descriptive Statistics: With Course Reminders*, Dunod, 1998.
6. G. Calot. *Course in Descriptive Statistics*, Dunod, 1969.
7. R.M. Dudley. *Real Analysis and Probability*, Cambridge University Press, 2002.
8. D.S. Moore et al. *The Basic Practice of Statistics*, 5th Edition, Freeman, 2008.
9. W. Feller. *An Introduction to Probability Theory and its Applications*, 2nd Edition, Wiley, 1971.
10. P. Roger. *Probability, Statistics, and Stochastic Processes*, Pearson Education, 2004.
11. R. Courant and F. John. *Introduction to Calculus and Analysis*, 2nd Edition, Springer, 1965.
12. R.M. Walpole et al. *Probability and Statistics for Engineers and Scientists*, 9th Edition, Pearson, 2011.
13. Redjal, K. *Course in Probability*, O.P.U., 1995.
14. *Schaum Series, Theory and Applications of Statistics*, 1991.
15. Yves Tillé. *Summary of the Descriptive Statistics Course*, 2010.
16. W. Rudin. *Principles of Mathematical Analysis*, 3rd Edition, McGraw-Hill, 1976.