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Study of a Fractional Problem of Volterra Type

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ملخص

يتم دراسة المعادلات التفاضلية الكسرية على نطاق واسع خلال السنوات الأخيرة بسبب وجود مجموعة واسعة من التطبيقات في مختلف مجالات العلوم والهندسة، كما تستخدم بشكل خاص لوصف العديد من الظواهر الفيزيائية التي تنطوي على تأثير الذاكرة. في هذه الأطروحة، نقوم بدراسة وجود الحلول من عدمها وتعدد الحلول الموجبة لبعض المعادلات التفاضلية الكسرية غير الخطية.

أولاً، نثبت بعض النتائج حول وجود حلول موجبة لمسألة قيم ذاتية كسرية غير خطية بشروط حدية من نوع ديريشلي (*Dirichlet*). هذه النتائج ترتبط أساساً بأول قيمة ذاتية للمسألة. إضافة إلى ذلك يتم تطوير طريقة الحلول الفوقية و التحتية لحل المسألة بشروط ديريشلي الحدية غير المتجانسة. بعد ذلك، ندرس مشكلتين حديتين أخريين لمعادلات تفاضلية كسرية غير خطية. تحت شروط معينة على سلوك تقارب الجزء غير الخطي نتحصل على عدة نتائج تخص الوجود وتعدد الحلول الموجبة حسب قيمة الوسيط الموجود في المعادلات.

ترتكز الإثباتات في هذه الأطروحة على تحويل و اختصار المسألة الحدية في معادلة فولتيرا (*Volterra*) التكاملية من نوع الاول ذات نواة G (دالة جرين (*Green*) المرتبطة بالمسألة الخطية)، استخراج الخصائص الرئيسية للدالة G التي ستكون أساسية في بناء المخروط المناسب في الفضاء الباناخي الملائم، ومن ثم تطبيق نظرية النقطة الصامدة لثيوكراسنوزلسكي (*Guo – Krasnoselskii*). بالاضافة الى ماسبق قد تم إدراج العديد من الأمثلة العددية لتوضيح النتائج الرئيسية.

Abstract

Fractional differential equations have been extensively investigated in the recent years, due to a wide range of applications in various fields of sciences and engineering. They are particularly used to describe many physical phenomena involving memory effect.

In this thesis we investigate the existence, nonexistence and multiplicity of positive solutions to some classes of nonlinear fractional differential equations.

First, we establish some existence results for solutions of a nonlinear fractional eigenvalue problem with Dirichlet boundary conditions. These results are mainly related to the first eigenvalue of the associated linear problem. Moreover, the method of lower and upper solutions is developed for the non-homogeneous Dirichlet problem. Next, we study two other integral boundary value problems (BVPs) for nonlinear fractional differential equations. Under suitable conditions on the asymptotic behavior of the nonlinearity, various existence and multiplicity results for positive solutions are derived depending on some equation's parameters.

The proof technic relies on reducing the considered nonlinear BVP to an integral equation of Volterra type, whose kernel $G(t, s)$ is the Green's function associated to the linear problem, extracting its main properties which will be fundamental in the construction of a suitable cone in an adequate Banach space, and then applying the Guo-Krasnoselskii fixed point theorem to get the existing result. Several examples are also included to illustrate the major obtained results.

Résumé

Les équations différentielles fractionnaires ont été largement étudiées ces dernières années, en raison d'une large gamme d'applications dans divers domaines de la science et de l'ingénierie. Elles sont particulièrement utilisées pour décrire de nombreux phénomènes physiques avec effet de mémoire. Dans cette thèse, nous étudions l'existence, la non existence et la multiplicité des solutions positives pour certaines classes d'équations différentielles fractionnaires non linéaires.

Tout d'abord, nous établissons certains résultats d'existence de solutions positives pour un problème de valeur propre fractionnaire non linéaire avec conditions aux limites de Dirichlet. Ces résultats sont principalement liés à la première valeur propre du problème linéaire associé. Par ailleurs, la méthode des sur et sous solutions a été développée pour le problème Dirichlet non homogène. Ensuite, nous étudions deux autres problèmes aux limites (BVP) pour des équations différentielles fractionnaires non linéaires avec des conditions intégrales. Sous des conditions convenables sur le comportement asymptotique de la non-linéarité, plusieurs résultats d'existence et de multiplicité de solutions positives sont obtenus selon des paramètres d'équations.

Nos démonstrations reposent sur la réduction du (BVP) non linéaire en une équation intégrale de Volterra de noyau $G(t,s)$ correspondant à la fonction de Green associée au problème, l'étude de ses principales propriétés et l'établissement des inégalités fondamentales à la construction d'un cône approprié dans un espace Banach adéquat. On appliquera enfin le théorème de point fixe de Guo-Krasnoselskii. Plusieurs exemples sont également inclus pour illustrer les principaux résultats obtenus.

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Introduction

”... This is an apparent paradox from which, one day, useful consequences will be drawn.”
Leibniz in a letter to l’Hospital on the meaning of the derivative of order 1/2.

The fractional calculus is a name attributed to the theory of integrals and derivatives of arbitrary order, which generalizes the notions of integer-order differentiation and n-fold integration. The subject of fractional calculus is almost as old as calculus itself, but for centuries this mathematical theory was considered unreasonably as a pure theoretical field of mathematics and useless in applications. However, the situation changed dramatically during the last few decades. This topic has blossomed and gained considerable importance, owing to its various particular applications in widespread fields of science and engineering. Perhaps one of the simplest approach to fractional integro-differentiation is through the generalization of Cauchy’s integral formula of n-fold integral operator:

$$I^n f(x) = \frac{1}{(n-1)!} \int_0^x (x-s)^{n-1} f(s) ds \quad n \in \mathbb{N}^*, x \in \mathbb{R}_+$$

with $I^0 f(x) = f(x)$. Once we replace the discrete factorial by the Euler’s Gamma function $\Gamma(n)$ one obtains, for the arbitrary order integration, the definition of Riemann-Liouville fractional integral

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s) ds \quad \alpha, x \in \mathbb{R}_+.$$

While the definition of an arbitrary order derivative of Riemann-Liouville type is simply given by the following composition

$$D^\alpha f(x) = D^n I^{n-\alpha} f(x) \quad \alpha \in \mathbb{R}_+, \alpha \leq n < \alpha + 1, n \in \mathbb{N}^*$$

where $D^0 f(x) = f(x)$. The Riemann-Liouville definition for the fractional integral and differential operators (I^α , D^α) is the most common and frequently used in applications.

Fractional differential equations are nowadays frequently used to describe many physical phenomena, especially when dealing with memory processes or viscoelastic and viscoplastic materials. This is due to the non-local character of fractional integro-differential operators: A fractional order (or integral) derivative of a function in a point t depends on the past values of this function from zero up to time t . This fact is one of the reasons for the recent interest in studying fractional differential equations. Several examples for fractional diffusion processes are given in [59] and recently in the papers [14, 24, 26, 54, 66]. Fractional models of viscoelasticity can be found in [8, 35, 61] and applications in the field of electrodynamics are discussed in [15, 27]. Additionally, a number of applications in various fields in physics and engineering were recently collected in the texbook by Vladimir V. Uchaikin in [67].

Integral boundary conditions constitute an interesting and important class of problems. They arise in different areas of applied mathematics and physics. Namely, problems in heat conduction, chemical engineering, underground water flow, and plasma physics can be reduced to nonlocal problems with integral boundary conditions [2, 32, 33, 74]. In the last years, several papers dealing with the existence of solutions for nonlinear fractional differential equations coupled with integral boundary conditions have appeared. In many of them the main results follow by means of suitable fixed point theorems [5, 12, 13, 23, 29, 62].

Brief Historical Background

The origin of fractional calculus concept can be traced back to Gottfried Wilhelm Leibniz (1646-1716). More precisely, in his letter to L'Hospital dated 3/8/1695, he gave a first answer to a curious question posed by L'Hospital about the significance of a non-integer order derivative especially the case 1/2: What would be the result if $n=1/2$ in the Leibniz notation $\frac{d^n}{dx^n}y(x)$? In his answer, Leibniz wrote "... This is an apparent paradox from which, one day, useful consequences will be drawn." The paradoxical aspects are due to the fact that there are several different (and non-equivalent) ways of generalizing the differentiation operator to non-integer orders.

Since that time, the subject of calculus of integrals and derivatives of non-integer order caught the attention of many well-known mathematicians. In 1738, Euler attempted to resolve the Leibniz paradox by the assistance of his famous invention that bore his name, namely, Euler's Gamma function $\Gamma(x)$. In [28] he defined the non-integer order derivative of a power function by

$$\frac{d^\alpha x^\beta}{dx^\alpha} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} x^{\beta - \alpha}$$

where α and β are arbitrary positive numbers.

Probably, the first elaborated definition of the non-integer order derivative by means of integral is given in 1812 by P.S. Laplace in his book [44]. If a function f can be represented by an integral of the form $\int_{-\infty}^{+\infty} e^{-xs} \varphi ds$, Laplace defined the non-integer order derivative by

$$\frac{d^\alpha}{dx^\alpha} f(x) = (-1)^\alpha \int_{-\infty}^{+\infty} s^\alpha \varphi e^{-xs} ds,$$

where α is an arbitrary real number.

A couple of years later, in 1822, J.B.J. Fourier [31] extended the formula for trigonometric functions

$$\frac{d^\alpha}{dx^\alpha} \cos(x) = \cos\left(x + \frac{\alpha\pi}{2}\right),$$

from $\alpha \in \mathbb{N}$ to $\alpha \in \mathbb{R}$. He suggested the following definition of fractional derivative

$$\frac{d^\alpha}{dx^\alpha} f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(s) ds \int_{-\infty}^{+\infty} p^\alpha \cos\left(p(x-s) + \frac{i\pi}{2}\right) dp, \quad \alpha \text{ non-integer.}$$

This was the first appropriate definition to any sufficiently "well-behaved" function, but it did not give a complete answer to L'Hospital's question. In 1823, N.H. Abel was the first mathematician who used fractional derivatives to solve a specific physical problem. Precisely, he solved a generalized version of the tautochrone problem (Abel's mechanical problem) which consists of the determination of a curve $f(y)$ such that the time $T(y)$ required for a particle to slide down the curve to its lowest point under gravity is independent of its initial position y_0 on the curve. Abel expressed the problem by the following integral equation

$$T(y) = \frac{1}{\sqrt{2g}} \int_0^y (y - y_0)^{-1/2} f(y_0) dy_0 := \frac{\Gamma(1/2)}{\sqrt{2g}} \frac{d^{-1/2}}{dy^{-1/2}} f(y),$$

where g is the gravitational acceleration. He obtained the solution by using the left-inverse of the fractional derivative

$$f(y) = \frac{\sqrt{2g}}{\Gamma(1/2)} \frac{d^{1/2}}{dy^{1/2}} T(y).$$

The tautochrone problem is a special case of Abel's mechanical problem when $T(y) := k$ is a constant,

$$f(y) = \frac{\sqrt{2gk}}{\pi\sqrt{y}}.$$

After Abel's application it was the turn of J. Liouville, who obtained a large number of theoretical and applied results on fractional calculus carried out in a series of papers [48]-[53]. Particularly, in [49, 50] he defined the derivative of non-integer order by the limits of difference quotients

$$\frac{d^\alpha f}{dx^\alpha} = (-1)^\alpha \lim_{h \rightarrow 0} \left\{ \frac{1}{h^\alpha} \sum_{m=0}^{\infty} \left[(-1)^m \binom{\alpha}{m} f(x + mh) \right] \right\}$$

where f is represented by an exponential series $\sum_k c_k \lambda_k^\alpha e^{\lambda_k x}$ and $\binom{\alpha}{m}$ is the generalized binomial coefficient. Later, this idea was developed by Grünwald and Letnikov. They suggested

$$\frac{d^\alpha f}{dx^\alpha} = \lim_{h \rightarrow 0} \left\{ \frac{1}{h^\alpha} \sum_{m=0}^{\infty} \left[(-1)^m \binom{\alpha}{m} f(x - mh) \right] \right\}$$

as definition of a non-integer order derivative, which today is called Grünwald-Letnikov fractional derivative. Liouville stated also that the ordinary fractional differential equation $d^\alpha \psi / dx^\alpha = 0$ has a complementary solution of the form $\psi_c = c_0 + c_1 x + c_2 x^2 + \dots + c_m x^m$ with undetermined number of arbitrary constants, which establishes an essential difference between the fractional and the ordinary differential equations. In 1847, G. F. B. Riemann sought a generalization of a Taylor's series expansion and derived the following definition for fractional integration:

$$D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-s)^{\alpha-1} f(s) ds + \psi(x)$$

where the function ψ corresponds to the complementary function mentioned in Liouville's work. After the contribution of a number of researchers such as N. Y. Sonin [64], A. V. Letnikov [47], H. Laurent [45] the definition of fractional integration became

$${}_c D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-s)^{\alpha-1} f(s) ds, \quad \alpha > 0$$

which is called today the Riemann-Liouville fractional integral. Moreover, the fractional derivative is defined by

$${}_c D_x^\alpha f(x) = {}_c D_x^{n-\beta} f(x) = {}_c D_x^n {}_c D_x^{-\beta} f(x) = \frac{d^n}{dx^n} \left(\frac{1}{\Gamma(\alpha)} \int_c^x (x-s)^{\beta-1} f(s) ds \right).$$

During the last four decades, a large amount of results and applications of fractional calculus was published by many authors including M. Caputo [20], in order to present more suitable definition with the physical settings, Caputo defined of an $(n-1)$ -absolutely continuous function by

$${}^C D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n f(s) ds,$$

where $\alpha \in \mathbb{R}_+^*$, $\alpha \leq n < \alpha+1$, $n \in \mathbb{N}^*$. Today it is named Caputo fractional derivative which is strongly connected to the Riemann-Liouville fractional derivative. The main advantage of Caputo's definition is that one can specify the initial conditions of fractional differential equations in classical form:

$$f^{(k)}(0) = b_k, \quad k = 0, 1, \dots, n-1.$$

Thesis Overview

In Chapter 1 we introduce some tools from functional analysis which we require to carry out the analysis in the subsequent chapters. We also recall some well known definitions and properties of special functions such as the Euler's gamma and Mittag-Leffler functions. Moreover, we give the definitions and some properties of fractional integrals and fractional derivatives of different Riemann-Liouville and Caputo kinds.

The Chapter 2 is concerned with the existence study of the following nonlinear fractional differential equation with Dirichlet boundary conditions

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + f(t, t^{2-\alpha} u(t)) = 0, & 0 < t < 1, \\ \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) = u(1) = 0, \end{cases} \quad (1)$$

where $1 < \alpha \leq 2$, $\lambda \in \mathbb{R}$, D^α is the Riemann-Liouville fractional derivative and f is a given continuous function. Our approach is based mainly upon the reduction of problem (1) to the equivalent first kind Volterra integral equation. In a first step, we obtain the exact expression of the Green's function related to the linear problem

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + y(t) = 0, & 0 < t < 1, \\ \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) = u(1) = 0. \end{cases} \quad (2)$$

Then, we study the values of the real parameter λ for which the Green's function has a constant sign on $(0, 1) \times (0, 1)$. More concisely, we prove that it is positive if and only if

$\lambda \in (\lambda_1, \infty)$, where $\lambda_1 < 0$ is the first eigenvalue of the linear homogeneous problem

$$\begin{cases} D^\alpha u(t) - \lambda u(t) = 0, & 0 < t < 1, \\ \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) = u(1) = 0. \end{cases} \quad (3)$$

For other values of the parameter, the Green's function changes sign on $(0, 1) \times (0, 1)$. In addition, we deduce additional useful inequalities satisfied by the Green's function. These properties will be fundamental in the construction of a suitable cone in an adequate Banach space. Under additional conditions on the behavior of the function f at $u = 0$ and $u = \infty$, we will be able to deduce the existence of positive solutions of problem (1) in a given sector of the cone. Moreover, we will develop the method of lower and upper solutions, considering, in this case, the non-homogeneous Dirichlet boundary conditions. The last section of the chapter contains some examples illustrating the obtained results.

In Chapter 3 we consider a kind of integral boundary conditions. As it has been stated in [23], this type of conditions appear in different real phenomena as in among others, blood flow problems, chemical engineering, thermo-elasticity or population dynamics, see [12, 25, 32, 67] and the references therein. To be concise, in this chapter we are concerned with the following nonlinear fractional differential equation with integral boundary value conditions

$$D^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad (4)$$

$$u(0) = u'(0) = 0, \quad u(1) = \lambda \int_0^1 u(s) ds. \quad (5)$$

where $2 < \alpha \leq 3$, $0 < \lambda$, $\lambda \neq \alpha$, D^α is the Riemann-Liouville fractional derivative and f is a known continuous function. First, we obtain the exact expression of the Green's function related to the linear problem

$$D^\alpha u(t) + y(t) = 0, \quad 0 < t < 1, \quad (6)$$

coupled with the integral boundary conditions (5). Once obtained, we study the values of λ for which the Green's function is positive on $(0, 1) \times (0, 1)$ and, moreover we deduce some suitable properties that relate the expression of $G(t, s)$ and $G(1, s)$. These properties will be crucial when constructing a suitable cone in the space of continuous functions. Essentially, we are based on reducing problem (4)–(5) to the following nonlinear Volterra

integral equation of the first kind:

$$u(t) = \int_0^1 G(t, s) f(s, u(s)) ds. \quad (7)$$

Under additional conditions on the behavior of function f at 0 and at ∞ , we deduce the existence of positive solutions of problem (4)–(5).

Since only positive solutions are meaningful in many applications, in Chapter 4 we discuss the existence and multiplicity of positive solutions of a nonlinear fractional differential equation with integral boundary conditions and parameter dependence. More precisely, we consider the following problem

$$\begin{cases} D^\alpha u(t) + \mu g(t) f(u(t)) = 0, & 0 \leq t \leq 1, \\ u(0) = u'(0) = 0, u(1) = \lambda \int_0^1 u(s) ds, & 0 < \lambda < \alpha, \end{cases} \quad (8)$$

depending on the real parameter $\mu > 0$. Here D^α denotes the Riemann-Liouville fractional derivative of order $\alpha \in (2, 3]$ and f and g are appropriate functions to be specified later. First, We show the main properties of the Green's function related to the linear problem. Then, we construct a suitable cone in the space of non-negative continuous functions. By means of the Guo-Krasnoselskii's fixed point theorem, we establish sufficient conditions for the existence of non-negative solutions and, under additional conditions on the behavior of function f at 0 and at ∞ , we prove the main results of existence and multiplicity of positive solutions for the problem (4.1). Finally, we illustrate our findings by an example.

We conclude the thesis with a brief synopsis of our key achievements and provide some ideas for future developments.

Chapter 1

Background

The aim of this chapter is to provide some necessary background for the research undertaking in this thesis. In section 1.1 we give the definitions and some properties of the Euler's gamma and Mittag-Leffler functions. Some useful results in functional analysis that are necessary for the study of fractional boundary value problems are stated in section 1.2, including the Guo-Krasnoselskii fixed point theorem. Finally, the section 1.3 contains definitions and some properties of fractional integrals and fractional derivatives of Riemann-Liouville and Caputo type.

1.1 Special Functions

1.1.1 Euler's Gamma Function

The Euler's gamma function $\Gamma(x)$ is defined by the so-called Euler integral of the second kind:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad (x > 0). \quad (1.1)$$

This integral is convergent for all real $x > 0$.

One of the basic properties of the gamma function is

$$\Gamma(x + 1) = x\Gamma(x) \quad (x > 0). \quad (1.2)$$

Obviously $\Gamma(1) = 1$. and using (1.2) we obtain for $n = 1, 2, 3, \dots$

$$\Gamma(n + 1) = n\Gamma(n) = n(n - 1)! = n!. \quad (1.3)$$

α	1/2	3/2	5/2	7/2	9/2
$\Gamma(\alpha)$	$\sqrt{\pi}$	$\frac{1}{2}\sqrt{\pi}$	$\frac{3}{4}\sqrt{\pi}$	$\frac{15}{8}\sqrt{\pi}$	$\frac{105}{16}\sqrt{\pi}$

Table 1.1: Euler's Gamma function values

The Beta function is defined by the Euler's integral of the first kind:

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt \quad (x > 0; y > 0). \quad (1.4)$$

This function is connected with the Euler's gamma functions by the relation

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (x > 0; y > 0). \quad (1.5)$$

With the help of the Beta function we can establish the following important relationship of the Euler's gamma function

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \quad (x \in \mathbb{R}^+, x \neq 0, 1, 2, \dots). \quad (1.6)$$

Taking $x = 1/2$ we obtain from (1.6) a useful particular value of the Gamma function:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

More often values for the Gamma function occurring in applications are given in Table 1.1.

The binomial coefficients are defined for $n, m \in \mathbb{N}$ by the formula

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}.$$

Such a relation can be extended from $n, m \in \mathbb{N}$ to arbitrary real $\alpha, \beta \in \mathbb{R}$ by

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)\Gamma(\beta+1)} \quad \alpha, \beta \in \mathbb{R}; \alpha \notin \mathbb{Z}^-,$$

where $\mathbb{Z}^- := \{-1, -2, -3, \dots\}$.

1.1.2 Mittag-Leffler Function

The function defined by

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad x \in \mathbb{R}; \alpha > 0. \quad (1.7)$$

has been introduced by G. M. Mittag-Leffler [55], it is known as the Mittag-Leffler function. Some important properties of this function have been established in Mittag-Leffler ([55],[56] and [57]) and in Wiman [72]. We present some of them below. In case $\alpha = 1$ and $\alpha = 2$, we have

$$E_1(x) = e^x, \quad E_2(x) = \cosh(\sqrt{x}).$$

When $\alpha = n \in \mathbb{N}$, the differentiation of the function $E_n(\lambda x^n)$ satisfies the following formula:

$$\left(\frac{d}{dx}\right)^n E_n(\lambda x^n) = \lambda E_n(\lambda x^n) \quad n \in \mathbb{N}; \lambda \in \mathbb{R}. \quad (1.8)$$

Let $\alpha, \beta > 0$, the function $E_{\alpha, \beta}$ defined by

$$E_{\alpha, \beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad (1.9)$$

whenever the series converges, is called the two-parameter Mittag-Leffler function with parameters α and β , it has first appeared in a paper by Wiman [72]. A number of properties of this function have been established in Hurnbert and Agarwal [38] by using the Laplace transform technique. We recall that for some special choices of parameters α and β , we can recover certain well known functions:

$$\begin{aligned} E_{1,1}(x) &= e^x, & E_{1,2}(x) &= \frac{e^x - 1}{x}, \\ E_{2,1}(x^2) &= \cosh(x), & E_{2,2}(x) &= \frac{\sinh(x)}{x}. \end{aligned}$$

Remark 1.1. *The one-parameter Mittag-Leffler functions may be defined in terms of their two-parameter counterparts via the relation $E_\alpha(x) = E_{\alpha,1}(x)$.*

Remark 1.2. *We should note that, $E_{\alpha, \beta}(x) > 0$ for all $x > 0$ and therefore the zeros of $E_{\alpha, \beta}(x)$ must be negative.*

The following differentiation formula generalizes the one in (1.8)

$$\left(\frac{d}{dx}\right)^n [x^{\beta-1} E_{n, \beta}(\lambda x^n)] = x^{\beta-n-1} E_{n, \beta-n}(\lambda x^n), \quad n \in \mathbb{N}; \lambda \in \mathbb{R}. \quad (1.10)$$

We consider the functions $E_\alpha(\lambda x^\alpha)$ and $x^{\beta-1} E_{\alpha, \beta}(\lambda x^\alpha)$ defined for $x \in \mathbb{R}^*$ and $\alpha, \beta, \lambda \in \mathbb{R}$.

The following differentiation formulas hold

$$\left(\frac{d}{dx}\right)^n E_\alpha(\lambda x^\alpha) = x^{-n} \lambda E_{\alpha, 1-n}(\lambda x^\alpha), \quad (1.11)$$

and

$$\left(\frac{d}{dx}\right)^n [x^{\beta-1} E_{\alpha, \beta}(\lambda x^\alpha)] = x^{\beta-n-1} E_{\alpha, \beta-n}(\lambda x^\alpha) \quad n \in \mathbb{N}. \quad (1.12)$$

1.2 Some Elements of Functional Analysis

In this section, we give an account of some basic functional spaces as well as some necessary fixed point theorems that will be used throughout this thesis.

Definition 1.1. Let $\Omega = [a, b]$ ($-\infty < a < b < \infty$) and $n \in \mathbb{N}$. We denote by $C^n(\Omega)$ the space of functions f that are n times continuously differentiable on Ω with the norm

$$\|f\|_{C^n} = \sum_{k=0}^n \|f^{(k)}\|_C = \sum_{k=0}^n \max_{x \in \Omega} |f^{(k)}(x)|, \quad n \in \mathbb{N}^*.$$

In particular, for $n = 0$, $C^0(\Omega) \equiv C(\Omega)$ is the space of continuous functions f on Ω with the norm

$$\|f\|_C = \max_{x \in \Omega} |f(x)|.$$

Definition 1.2. Let $\Omega = [a, b]$ ($-\infty \leq a < b \leq \infty$). We denote by $L^p(a, b)$ ($1 \leq p \leq \infty$) the set of Lebesgue measurable functions f on Ω for which $\|f\|_p < \infty$, where

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p} \quad (1 \leq p < \infty)$$

and

$$\|f\|_\infty = \text{ess sup}_{a \leq x \leq b} |f(x)|,$$

where $\text{ess sup}|f(x)|$ is the essential supremum of the function f .

Definition 1.3. Let $\Omega = [a, b]$ ($-\infty < a < b < \infty$) and $f \in AC(\Omega)$ the space of absolutely continuous functions on Ω . For $n \in \mathbb{N}^*$, we denote by $AC^n(\Omega)$ the space of functions f which have continuous derivatives up to order $(n - 1)$ on Ω such that $f^{(n-1)} \in AC(\Omega)$:

$$AC^n(\Omega) = \{f \in C^{n-1}(\Omega) : f^{(n-1)} \in AC(\Omega)\}$$

In particular, $AC^1(\Omega) = AC(\Omega)$.

A characterization of the space AC^n is given in Lemma 2.4 in [63].

Lemma 1.1. The space $AC^n(\Omega)$ consists of functions which can be represented in the following form

$$f(x) = (I_{a^+}^n \varphi)(x) + \sum_{k=0}^{n-1} c_k (x - a)^k$$

where $\varphi \in L^1(a, b)$, c_k ($k = 0, 1, \dots, n - 1$) are arbitrary constants, and

$$(I_{a^+}^n \varphi)(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} \varphi(t) dt, \quad a < x \leq b.$$

Definition 1.4. Let $\Omega = [a, b]$ ($-\infty < a < b < \infty$) and $\gamma \in \mathbb{R}$ ($0 \leq \gamma < 1$). The weighted space $C^\gamma(\Omega)$ denotes the space of functions f on $(a, b]$, such that the function $(x - a)^\gamma f(x) \in C(\Omega)$, and

$$\|f\|_{C_\gamma} = \|(x - a)^\gamma f(x)\|_C, \quad C_0(\Omega) = C(\Omega).$$

Definition 1.5. Let E be a real normed space. A nonempty convex closed set $P \subset E$ is called a cone if it satisfies the following two conditions:

- (i) $x \in P, \lambda > 0$ implies $\lambda x \in P$,
- (ii) $x \in P, -x \in P$ implies $x = 0$, where 0 denotes the zero element of E .

It is called solid if it contains interior points; $\overset{\circ}{P} \neq \emptyset$.

We recall the following results:

Theorem 1.1 (Guo-Krasnoselskii Fixed Point Theorem [37, 40]). Let $(E, \|\cdot\|)$ be a Banach space, and let $P \subset E$ be a cone. Assume that Ω_1, Ω_2 are open and bounded subsets of E with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow P$ be a completely continuous operator such that

- (i) $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_2$; or
- (ii) $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_2$.

Then, the operator T has at least one fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Definition 1.6. Let K be a compact metric space. We say that $F \subset C(K)$ is equicontinuous if for every $x \in K$ and every $\epsilon > 0$ there is a neighbourhood U_x of x such that for any $f \in F$ we have that $|f(x) - f(y)| < \epsilon$, for each $y \in U_x$.

Theorem 1.2 (Ascoli-Arzelà). Let $F \subset C(K)$. Then F is precompact in $C(K)$ (namely, \bar{F} is compact) if and only if F is equicontinuous and uniformly bounded.

Theorem 1.3 (Schauder Fixed Point Theorem). Let C be a closed convex subset of Banach space X . Suppose $f : C \rightarrow C$ is compact (i.e., bounded sets in C are mapped into relatively compact sets). Then, f has a fixed point in C .

1.3 Fractional Derivatives and Integrals

In this section, we present the definitions and some properties of fractional integrals and fractional derivatives of Riemann-Liouville as well as Caputo types.

Let $\Omega = [a, b]$, $(-\infty < a < b < \infty)$

Definition 1.7. Let $\alpha > 0$ and $f \in C(\Omega) \cap L^1(\Omega)$. The Riemann-Liouville fractional integral of order α is defined by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad a < t \leq b.$$

Here $\Gamma(\alpha)$ denotes the Euler's Gamma function.

In particular, when $\alpha = n \in \mathbb{N}$, the definition coincides with the n th integral:

$$I_a^n f(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s) ds, \quad a < t \leq b.$$

Definition 1.8. Let $\alpha > 0$ and $f \in AC^{[\alpha]}(\Omega)$. The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by

$$\begin{aligned} D_a^\alpha f(t) &:= \left(\frac{d}{dt} \right)^n (I^{n-\alpha} f)(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_a^t \frac{f(s) ds}{(t-s)^{\alpha-n+1}}, \quad n = [\alpha] + 1, \quad a < t \leq b, \end{aligned}$$

where $[\alpha]$ denotes the integer part of α .

In particular, when $\alpha = n \in \mathbb{N}$, we find

$$D_a^n f(t) = f^{(n)}(t); \quad D_a^0 f(t) = f(t), \quad t \in \Omega$$

Proposition 1.1. [39, Corollary 2.1] Let $\alpha > 0$ and $n = [\alpha] + 1$. The equation $D_a^\alpha f(t) = 0$ is valid if, and only if,

$$f(t) = \sum_{j=1}^n c_j (t-a)^{\alpha-j} \quad t \in \Omega$$

where $c_j \in \mathbb{R}$ are arbitrary constants.

Proposition 1.2. [39, Lemma 2.1] The fractional integration operators I_a^α with $\alpha > 0$ is linear and bounded in $L^p(\Omega)$, $1 \leq p \leq \infty$, with $f \in L^p(\Omega)$

$$\|I_a^\alpha f\|_p \leq K \|f\|_p,$$

where $K = \frac{|b-a|^\alpha}{\alpha \Gamma(\alpha)}$.

Proposition 1.3. [39, Lemma 2.2] Let $\alpha > 0$ and $n = [\alpha] + 1$. If $f \in AC^n(\Omega)$, then the fractional derivative D_a^α exists almost everywhere on Ω and can be expressed in the form

$$D_a^\alpha f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(1+k-\alpha)} (t-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s) ds}{(t-s)^{\alpha-n+1}}.$$

The following assertion gives an interesting result of the composition of fractional integral operator I_a^α with the fractional differential operator D_a^α .

Proposition 1.4. [39, Lemma 2.5] Let $\alpha > 0$, $n = [\alpha] + 1$ and let $f_{n-\alpha}(t) = I_a^{n-\alpha} f(t)$. If $f \in L^1(\Omega)$ and $f_{n-\alpha} \in AC^n(\Omega)$, then the equality

$$I_a^\alpha D_a^\alpha f(t) = f(t) - \sum_{j=1}^n \frac{f_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha-j+1)} (t-a)^{\alpha-j} \quad (1.13)$$

holds almost everywhere in Ω .

Proposition 1.5. [39, Lemma 2.9] Let $\alpha > 0$, $0 \leq \gamma < 1$, $n = [\alpha] + 1$ and let $f_{n-\alpha}(t) = I_a^{n-\alpha} f(t)$. If $f \in C(\Omega)$ (resp. $C_\gamma(\Omega)$) and $f_{n-\alpha} \in C^n(\Omega)$ (resp. $C_\gamma^n(\Omega)$), then the relation (1.13) holds at any point $t \in \Omega$ (resp. at any point $t \in (a, b]$).

The Riemann-Liouville fractional derivative of the Mittag-Leffler function with special parameters yields a function of the same kind.

Property 1.1. Let $\alpha > 0$, $\beta > 0$, $\mu \geq 0$ and $\lambda \in \mathbb{R}$. Then the following equality holds

$$D_0^\alpha (t^{\beta-1} E_{\mu, \beta}(\lambda t^\mu)) = t^{\beta-\alpha-1} E_{\mu, \beta-\alpha}(\lambda t^\mu). \quad (1.14)$$

Now we present the definition and some properties of the Caputo fractional derivative by means of fractional Riemann-Liouville integral and differential operator.

Definition 1.9. The Caputo fractional derivative ${}^C D_a^\alpha u(t)$ of order $\alpha \in \mathbb{R}$ on Ω is defined via the Riemann-Liouville fractional derivative by

$${}^C D_a^\alpha u(t) := \left(D_a^\alpha \left[u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!} (t-a)^k \right] \right) \quad (1.15)$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}^*$; $n = \alpha$, for $\alpha \in \mathbb{N}^*$.

Remark 1.3. The Caputo fractional derivative ${}^C D_a^\alpha$ is defined for functions $u(t)$ for which the Riemann-Liouville fractional derivative exists. In particular, they are defined for $u(t)$ belonging to the space of absolutely continuous functions $AC^n(\Omega)$.

Theorem 1.4. [39, Theorem 2.2] Let $u \in C^n(\Omega)$. Then, the Caputo fractional derivative ${}^C D_a^\alpha u$ of order α ($\alpha \geq 0$) is continuous on Ω : ${}^C D_a^\alpha u(t) \in C(\Omega)$ and it is represented by

(a) If $\alpha \notin \mathbb{N}^*$,

$${}^C D_a^\alpha u(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{u^{(n)}(s) ds}{(t - s)^{\alpha - n + 1}} =: I_a^{n - \alpha} D_a^n u(t), \quad a < t \leq b.$$

Moreover,

$${}^C D_a^\alpha u(a) = 0.$$

(b) If $\alpha = n \in \mathbb{N}^*$, then

$${}^C D_a^n u(t) = u^{(n)}(t), \quad a < t \leq b.$$

In particular,

$${}^C D_a^0 u(t) = u(t), \quad a < t \leq b.$$

The next lemma shows that the Caputo derivative is not the right inverse of the Riemann-Liouville integral:

Proposition 1.6. [39, Lemma 2.22] Let $\alpha > 0$ and $n = [\alpha] + 1$. If $u \in AC^n(\Omega)$ or $u \in C^n(\Omega)$, then

$$I^\alpha {}^C D_a^\alpha u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!} (t - a)^k, \quad t \in \Omega.$$

Chapter 2

Existence results for nonlinear fractional Dirichlet problems on the right side of the first eigenvalue

2.1 Introduction

The aim of this Chapter concerns the existence of solutions of the following nonlinear fractional differential equation with Dirichlet boundary conditions

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + f(t, t^{2-\alpha} u(t)) = 0, & 0 < t < 1, \\ \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) = u(1) = 0, \end{cases} \quad (2.1)$$

where $1 < \alpha \leq 2$, $\lambda \in \mathbb{R}$, D^α is the Riemann-Liouville fractional derivative initiated at $a = 0$ and f is a given continuous function.

First, in Section 2.3, we obtain the exact expression of the Green's function related to the linear problem

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + y(t) = 0, & 0 < t < 1, \\ \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) = u(1) = 0, \end{cases} \quad (2.2)$$

Once we have obtained such an expression, we study the values of the real parameter λ for which the Green's function has a constant sign in $(0, 1) \times (0, 1)$. More precisely, we prove that it is positive if and only if $\lambda \in (\lambda_1, \infty)$, where $\lambda_1 < 0$ is the first eigenvalue of

the linear homogeneous problem

$$\begin{cases} D^\alpha u(t) - \lambda u(t) = 0, & 0 < t < 1, \\ \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) = u(1) = 0, \end{cases} \quad (2.3)$$

For other values of the parameter, the Green's function changes its sign in $I \times I$.

In addition, we deduce additional useful inequalities satisfied by the function $G(t, s)$. These properties will be fundamental in the construction, in Section 2.4, of a suitable cone in an adequate Banach space. So, under additional conditions on the behavior of the function f at $u = 0$ and $u = \infty$, we will be able to deduce the existence of positive solutions of problem (2.1) in a given sector of the cone. Moreover, in the next section, we will develop the method of lower and upper solutions, considering, in this case, the non homogeneous Dirichlet boundary conditions. The chapter ends with a section where some examples are presented in order to illustrate the obtained results.

The results presented in this chapter have been published in [20]

2.2 Preliminaries

In this section we introduce the results that we need for our purpose.

Theorem 2.1. [39, Theorem 5.1, page 284] *Let $n - 1 < \alpha \leq n$ ($n \in \mathbb{N}^*$) and $\lambda \in \mathbb{R}$. Then the functions*

$$u_j(t) = t^{\alpha-j} E_{\alpha, \alpha+1-j}(\lambda t^\alpha), \quad j = 1, \dots, n,$$

yield the fundamental system of solutions of the equation

$$D^\alpha u(t) - \lambda u(t) = 0, \quad t > 0.$$

Theorem 2.2. [39, Theorem 5.7, page 302] *Let $n - 1 < \alpha \leq n$ ($n \in \mathbb{N}^*$) and $\lambda \in \mathbb{R}$, and let $f(t)$ be a given real function defined on \mathbb{R} . Then the equation*

$$D^\alpha u(t) - \lambda u(t) = f(t), \quad t > 0,$$

is solvable and its general solution is given by

$$u(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}[\lambda(t-s)^\alpha] f(s) ds + \sum_{j=1}^n c_j t^{\alpha-j} E_{\alpha, \alpha+1-j}(\lambda t^\alpha), \quad t > 0,$$

with $c_j \in \mathbb{R}$, $j = 1, \dots, n$, are arbitrarily chosen.

As we will see along this chapter, we will look for solutions that belong to the set

$$C_{2-\alpha}[0, 1] = \{u(t) \in C(0, 1] : t^{2-\alpha}u(t) \in C[0, 1]\},$$

which is, for $\alpha \in (1, 2]$, a Banach space with respect to the norm

$$\|u\|_{2-\alpha} = \sup\{t^{2-\alpha}|u(t)| : t \in [0, 1]\}.$$

It is easy to verify that $C_{2-\alpha}[0, 1] \subset C(0, 1] \cap L^1(0, 1)$, for all $\alpha \in (1, 2]$.

2.3 Study of the Green's function

This section is devoted to the study of the main properties of the related Green's function to problem (2.2).

Along the chapter we will use the following notation

$$e_\alpha^{\lambda t} = t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha). \quad (2.4)$$

Before obtaining its exact expression, we determine the eigenvalues of problem (2.3). To this end, by Theorem 2.1, we know that the general solution of equation (2.3) is given by

$$u(t) = C_1 e_\alpha^{\lambda t} + C_2 t^{\alpha-2} E_{\alpha,\alpha-1}(\lambda t^\alpha). \quad (2.5)$$

As a consequence

$$t^{2-\alpha} u(t) = C_1 t E_{\alpha,\alpha}(\lambda t^\alpha) + C_2 E_{\alpha,\alpha-1}(\lambda t^\alpha).$$

Since

$$0 = \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) = C_2 E_{\alpha,\alpha-1}(0) = \frac{C_2}{\Gamma(\alpha-1)},$$

we conclude that $C_2 = 0$.

Moreover, $u(1) = 0$ implies that

$$C_1 E_{\alpha,\alpha}(\lambda) = 0.$$

So λ is an eigenvalue of problem (2.3) if and only if

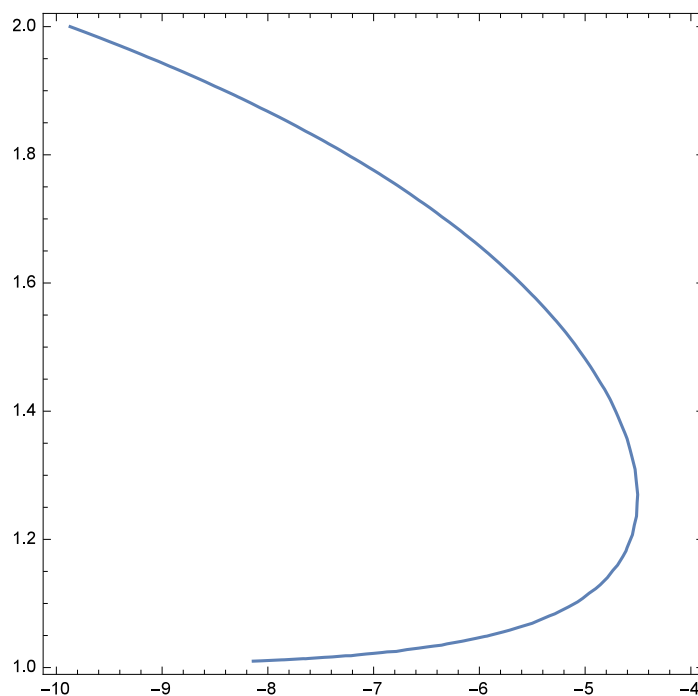
$$E_{\alpha,\alpha}(\lambda) = 0. \quad (2.6)$$

α	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
λ_1	-5.1005	-4.5672	-4.5183	-4.7080	-5.0754	-5.6096	-6.3223	-7.2403	-8.4042	π^2

Table 2.1: The eigenvalues λ_1 of Problem (2.3)

It is very well known that for $\alpha = 2$ there is a sequence of eigenvalues $\lambda_n = -(n\pi)^2$, $n = 1, 2, \dots$. However in [6] it is showed that equation (2.6) has a finite number of real roots for all $\alpha \in (1, 2)$. According to the definition of the Mittag-Leffler function $E_{\alpha,\beta}$, such zeros must be negative.

So, we denote by λ_1 the first negative eigenvalue of problem (2.3). The graph in Figure 2.1 shows the values of λ_1 for $1 < \alpha \leq 2$. Some particular values are compiled in Table 2.1.

Figure 2.1: Graph of λ_1 of problem (2.3) for $1 < \alpha \leq 2$.

In the sequel, we obtain the expression of the related Green's function to problem (2.2).

Theorem 2.3. *Given $y \in C(0, 1] \cap L^1(0, 1)$, $1 < \alpha \leq 2$ and $\lambda \in \mathbb{R}$, such that $E_{\alpha, \alpha}(\lambda) \neq 0$. Then problem (2.2) has a unique solution $u \in C_{2-\alpha}[0, 1]$, given by the expression*

$$u(t) = \int_0^1 G(t, s)y(s)ds,$$

where

$$G(t, s) = \frac{1}{e_\alpha^\lambda} \begin{cases} e_\alpha^{\lambda t} e_\alpha^{\lambda(1-s)}, & 0 \leq t \leq s \leq 1, \\ e_\alpha^{\lambda t} e_\alpha^{\lambda(1-s)} - e_\alpha^\lambda e_\alpha^{\lambda(t-s)}, & 0 \leq s \leq t \leq 1, \end{cases} \quad (2.7)$$

and $e_\alpha^{\lambda t}$ is defined in (2.4).

Proof. From Theorem 2.2, we have that the solutions of problem (2.2) are given by the following expression

$$u(t) = C_1 e_\alpha^{\lambda t} + C_2 t^{\alpha-2} E_{\alpha, \alpha-1}(\lambda t^\alpha) - \int_0^t e_\alpha^{\lambda(t-s)} y(s) ds, \quad (2.8)$$

for some adequate real constants C_1 and C_2 .

Now, since $\lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) = 0$ we have $C_2 = 0$. Moreover, the condition $u(1) = 0$ implies that

$$C_1 = \frac{1}{e_\alpha^\lambda} \int_0^1 e_\alpha^{\lambda(1-s)} y(s) ds$$

and the solution u satisfies

$$u(t) = \frac{e_\alpha^{\lambda t}}{e_\alpha^\lambda} \int_0^1 e_\alpha^{\lambda(1-s)} y(s) ds - \int_0^t e_\alpha^{\lambda(t-s)} y(s) ds.$$

As a consequence, we arrive at the following expression

$$\begin{aligned} u(t) &= \frac{1}{e_\alpha^\lambda} \int_0^t (e_\alpha^{\lambda t} e_\alpha^{\lambda(1-s)} - e_\alpha^\lambda e_\alpha^{\lambda(t-s)}) y(s) ds + \frac{1}{e_\alpha^\lambda} \int_t^1 e_\alpha^{\lambda t} e_\alpha^{\lambda(1-s)} y(s) ds \\ &= \int_0^1 G(t, s) y(s) ds. \end{aligned}$$

This completes the proof. □

Remark 2.1. *Notice that, in both expressions (2.5) and (2.8), since $C_2 = 0$ and $e_\alpha^{\lambda t}|_{t=0} = 0$, we get that u is a continuous function on $[0, 1]$ and $u(0) = 0$. In other words, the Dirichlet boundary conditions in (2.1), (2.2) or (2.3) are equivalent to $u(0) = u(1) = 0$ and we can look for solutions on $C[0, 1]$.*

The following result is a direct consequence of expression (2.7).

Lemma 2.1. *Let G be the Green's function related to problem (2.2), which is given by the expression (2.7). Then, for all $\alpha \in (1, 2]$ and $\lambda \in \mathbb{R}$ that does not satisfy (2.6), the following properties are fulfilled:*

- (i) $G(0, s) = G(1, s) = G(t, 0) = G(t, 1) = 0$ for all $t, s \in [0, 1]$.
- (ii) G is a continuous function on $[0, 1] \times [0, 1]$.

Notice that it is immediate to verify that, for all $0 \leq t \leq s \leq 1$, the following equality holds

$$\frac{\partial}{\partial t} G(t, s) = \frac{e_{\alpha}^{\lambda(1-s)}}{e_{\alpha}^{\lambda}} t^{\alpha-2} E_{\alpha, \alpha-1}(\lambda t^{\alpha}).$$

In particular

$$\frac{\partial}{\partial t} G(t, s)|_{t=0} = +\infty,$$

and the Green's function attains always positive values at some points of $[0, 1] \times [0, 1]$. In particular, there is no $\lambda \in \mathbb{R}$ for which the Green's function is negative on $(0, 1) \times (0, 1)$.

In the next result we characterize the set of the real parameters λ for which the Green's function has a constant sign.

Lemma 2.2. *Let G be the Green's function related to problem (2.2) and λ_1 be the first negative root of $E_{\alpha, \alpha}(\lambda) = 0$. Then, for all $\alpha \in (1, 2]$, the following property holds*

$$G(t, s) > 0 \quad \text{for all } t, s \in (0, 1).$$

if and only if $\lambda > \lambda_1$.

Proof. Assume, to the contrary, that there are $\lambda_2 > \lambda_1$ and $(t_0, s_0) \in (0, 1) \times (0, 1)$ such that

$$G_{\lambda_2}(t_0, s_0) = 0,$$

where G_{λ_2} denotes the Green's function related to problem (2.2) for $\lambda = \lambda_2$.

Next, we define function $v : [0, 1] \rightarrow \mathbb{R}$, as

$$v(t) = G_{\lambda_2}(t, s_0)$$

Suppose, first that $s_0 \geq t_0$. From expression (2.7), we know that $v(t) \not\equiv 0$ on the interval $[0, t_0]$.

From the definition of the Green's function, it is immediate to verify that v is a solution of problem

$$\begin{cases} D^\alpha v(t) - \lambda_2 v(t) = 0, & 0 < t < t_0, \\ v(0) = v(t_0) = 0. \end{cases} \quad (2.9)$$

Arguing as in the beginning of this section, one can verify that the eigenvalues $\tilde{\lambda}_n$ of problem (2.9) are characterized by the expression

$$E_{\alpha,\alpha}(\tilde{\lambda}_n t_0^\alpha) = 0.$$

Since λ_1 is the greatest negative zero of $E_{\alpha,\alpha}(\lambda)$, we deduce that

$$\tilde{\lambda}_1 := \frac{\lambda_1}{t_0^\alpha}$$

is the first eigenvalue of problem (2.9).

So, we conclude that (λ_2, v) is an eigenvalue-eigenvector couple of problem (2.9) and $\lambda_2 > \lambda_1 > \tilde{\lambda}_1$, which contradicts the fact that $\tilde{\lambda}_1$ is the first eigenvalue.

If $s_0 \leq t_0$, we know that $v(t) \not\equiv 0$ on the interval $[t_0, 1]$ and it satisfies

$$\begin{cases} D^\alpha v(t) - \lambda_2 v(t) = 0, & t_0 < t < 1, \\ v(t_0) = v(1) = 0. \end{cases} \quad (2.10)$$

In this case, the eigenvalues satisfy

$$E_{\alpha,\alpha}(\bar{\lambda}_n (1 - t_0)^\alpha) = 0$$

and

$$\bar{\lambda}_1 := \frac{\lambda_1}{(1 - t_0)^\alpha}.$$

The arguments hold in the same way.

Thus for any $\lambda > \lambda_1$ the Green's function G has no zeros in $(0, 1) \times (0, 1)$.

The fact that it is strictly positive holds from the continuity of function G with respect to the parameter λ and the fact that for $\lambda = 0$,

$$G_0(s, s) = \frac{1}{\Gamma(\alpha)} s^{\alpha-1} (1 - s)^{\alpha-1} > 0$$

for all $s \in (0, 1)$.

Consider now $\lambda < \lambda_1$ such that $E_{\alpha,\alpha}(\lambda) \neq 0$. In particular the Green's function exists. Let s_0 be small enough such that

$$\lambda(1 - s_0)^\alpha < \lambda_1,$$

then we have that $E_{\alpha,\alpha}(\lambda(1 - s)^\alpha)$ changes its sign on $[s_0, 1]$. As a result, for $t = t_0 < s_0$, the function $G_\lambda(t_0, s)$ changes its sign on $[s_0, 1]$.

The result is proved. □

As a direct consequence of the previous result we deduce the following property of the Green's function of a Dirichlet problem on any arbitrary bounded interval $[a, b]$.

Corollary 2.1. *Let $a < b$ and $\alpha \in (1, 2]$ be arbitrarily fixed real values, and λ_1 be the first negative root of $E_{\alpha,\alpha}(\lambda) = 0$. Let $y \in C(a, b] \cap L^1(a, b)$ and G be the Green's function related to problem*

$$\begin{cases} D_a^\alpha u(t) - \lambda u(t) + y(t) = 0, & a < t < b, \\ \lim_{t \rightarrow a^-} (t - a)^{2-\alpha} u(t) = u(b) = 0. \end{cases}$$

Then

$$G(t, s) > 0 \quad \text{for all } t, s \in (a, b).$$

if and only if

$$\lambda > \frac{\lambda_1}{(b - a)^\alpha}.$$

Now, we prove two additional inequalities for the Green's function G . Such properties, together with the previous ones given above, will be fundamental to ensure the existence of solutions, of the nonlinear problem (2.1), that will be proven in the next section.

Lemma 2.3. *Fix $1 < \alpha \leq 2$ and $\lambda > \lambda_1$. Let $G(t, s)$ be the Green's function related to problem (2.2), given by expression (2.7). Then there exist a positive constant M and a function $m \in C[0, 1]$ such that $m(t) \geq 0$, for all $t \in (0, 1)$, $m(t) > 0$ on $(0, \delta)$, for some $\delta > 0$ and $m(0) = m(1) = 0$, for which the following inequalities hold:*

$$m(t) \leq \frac{t^{2-\alpha} G(t, s)}{s e_\alpha^{\lambda(1-s)}} \leq M, \quad \text{for all } t, s \in (0, 1). \quad (2.11)$$

Proof. It is clear that the function

$$h(t, s) := \frac{t^{2-\alpha}G(t, s)}{se_\alpha^{\lambda(1-s)}}$$

is continuous on $[0, 1] \times (0, 1)$, moreover, for all $t \in [0, 1]$ the limits

$$\begin{aligned} l_1(t) = \lim_{s \rightarrow 1} h(t, s) &= \lim_{s \rightarrow 1} \frac{t^{2-\alpha} e_\alpha^{\lambda t} e_\alpha^{\lambda(1-s)}}{e_\alpha^\lambda s e_\alpha^{\lambda(1-s)}} \\ &= t \frac{E_{\alpha, \alpha}(\lambda t^\alpha)}{E_{\alpha, \alpha}(\lambda)} \end{aligned}$$

and

$$\begin{aligned} l_2(t) = \lim_{s \rightarrow 0} h(t, s) &= \lim_{s \rightarrow 0} \frac{t^{2-\alpha} (e_\alpha^{\lambda t} e_\alpha^{\lambda(1-s)} - e_\alpha^\lambda e_\alpha^{\lambda(t-s)})}{e_\alpha^\lambda s e_\alpha^{\lambda(1-s)}} \\ &= \frac{t^{2-\alpha} e_\alpha^\lambda t^{\alpha-2} E_{\alpha, \alpha-1}(\lambda t^\alpha) - E_{\alpha, \alpha-1}(\lambda) e_\alpha^{\lambda t}}{(e_\alpha^\lambda)^2} \\ &= \frac{E_{\alpha, \alpha}(\lambda) E_{\alpha, \alpha-1}(\lambda t^\alpha) - t E_{\alpha, \alpha-1}(\lambda) E_{\alpha, \alpha}(\lambda t^\alpha)}{(E_{\alpha, \alpha}(\lambda))^2}, \end{aligned}$$

exist and are finite, so h has removable discontinuities at $s = 0, 1$, and we can extend it continuously to a function \tilde{h} on $[0, 1] \times [0, 1]$.

As a consequence

$$m(t) = \min_{s \in [0, 1]} \{\tilde{h}(t, s)\}$$

is a continuous function, such that

$$0 \leq m(t) \leq \tilde{h}(t, s) \leq M,$$

for all $t, s \in [0, 1]$ and $m(0) = m(1) = 0$, where

$$M = \max_{(t, s) \in [0, 1] \times [0, 1]} \tilde{h}(t, s).$$

Since $l_1(t) > 0$, for all $t \in (0, 1]$ and l_2 is a continuous function such that

$$l_2(0) = \frac{1}{E_{\alpha, \alpha} \Gamma(\alpha - 1)} > 0,$$

we can ensure that there exists $\delta > 0$ for which $m(t) > 0$ on $(0, \delta)$.

□

2.4 Existence of Positive Solutions

This section is devoted to the proof of the existence of a positive solution to the nonlinear boundary value problem (2.1). To this end, we use a new existence theorem given in [30], which is obtained by applying the Guo-Krasnoselskii fixed point theorem [37] for a certain cone structure.

Let E be an ordered Banach space with an order cone E_+ . An ordered interval is defined as

$$[x, y] = \{z \in E : x \leq z \leq y\}.$$

For any $r > 0$, we denote $\Omega_r = \{u \in E : \|u\| < r\}$. We define now the subcone P_{u_0} in the Banach space E as follows

$$P_{u_0} = \{u \in E_+, u \geq \|u\|u_0\}, \quad (2.12)$$

where $u_0 \in E_+$ is such that $\|u_0\| \leq 1$.

The following theorem will be used to prove our results.

Theorem 2.4. [30, Theorem 2.1] *Assume E be an order Banach space with the order cone E_+ . Let $0 \leq u_0 \leq \varphi$ be such that $\|u_0\| \leq 1$ and $\|\varphi\| = 1$, satisfying the following condition:*

$$\text{if } u \in E_+, \|u\| \leq 1 \text{ then } u \leq \varphi. \quad (2.13)$$

If there exist positive numbers $0 < a < b$ such that $T : P_{u_0} \cap (\bar{\Omega}_b \setminus \Omega_a) \rightarrow P_{u_0}$ is a completely continuous operator and the conditions

$$\|Tu\|_{u \in [au_0, a\varphi]} \leq a \quad \text{and} \quad \|Tu\|_{u \in [bu_0, b\varphi]} \geq b$$

or

$$\|Tu\|_{u \in [au_0, a\varphi]} \geq a \quad \text{and} \quad \|Tu\|_{u \in [bu_0, b\varphi]} \leq b$$

are fulfilled, then the operator T has at least one fixed point $u \in [au_0, b\varphi]$.

Define now the operator $T : C_{2-\alpha}[0, 1] \rightarrow C_{2-\alpha}[0, 1]$ as

$$Tu(t) := \int_0^1 G(t, s)f(s, s^{2-\alpha}u(s))ds, \quad (2.14)$$

with G defined in (2.7).

Considering the following assumption:

(f) $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is a continuous function.

It is clear, from Theorem 2.3, that the fixed points of operator T coincide with the solutions of problem (2.1). That is to say, the existence of a solution to the nonlinear Volterra integral equation

$$u(t) = \int_0^1 G(t, s) f(s, s^{2-\alpha} u(s)) ds, \quad (2.15)$$

coincide with the existence of a solution to the problem (2.1).

Let $E := C_{2-\alpha}[0, 1]$ be the Banach space endowed with the norm $\|\cdot\|_{2-\alpha}$. Now define

$$u_0(t) := t^{\alpha-2} \frac{m(t)}{M},$$

where $m(t)$ and M were introduced in Lemma 3.2. We have $u_0 \in E$ and

$$\|u_0\|_{2-\alpha} \leq 1,$$

therefore the cone $P_{u_0} \subset C_{2-\alpha}[0, 1]$ takes the form

$$P_{u_0} = \left\{ u \in C_{2-\alpha}[0, 1], u(t) \geq 0 \text{ for all } t \in (0, 1], \quad t^{2-\alpha} u(t) \geq \frac{m(t)}{M} \|u\|_{2-\alpha}, \text{ for all } t \in [0, 1] \right\}. \quad (2.16)$$

Set

$$f_0 = \lim_{u \rightarrow 0^+} \left\{ \min_{t \in [0, 1]} \frac{f(t, u)}{u} \right\}, \quad f_\infty = \lim_{u \rightarrow \infty} \left\{ \min_{t \in [0, 1]} \frac{f(t, u)}{u} \right\},$$

and

$$f^0 = \lim_{u \rightarrow 0^+} \left\{ \max_{t \in [0, 1]} \frac{f(t, u)}{u} \right\}, \quad f^\infty = \lim_{u \rightarrow \infty} \left\{ \max_{t \in [0, 1]} \frac{f(t, u)}{u} \right\}.$$

Now, we are in a position to prove the main result of this section.

Theorem 2.5. *Assume that condition (f) holds coupled with one of the two following conditions:*

(i) (sublinear case) $f_0 = \infty$ and $f^\infty = 0$.

(ii) (superlinear case) $f^0 = 0$, $f_\infty = \infty$.

Then, for all $\alpha \in (1, 2]$ and $\lambda > \lambda_1$, problem (2.1) has a positive solution that belongs to the cone P_{u_0} defined in (2.16). Moreover $u \in C[0, 1]$.

Proof. Firstly, we prove that $T : P_{u_0} \rightarrow P_{u_0}$ is completely continuous.

From the continuity and the non negativity of functions G and f on their domains of definition, we have that, if $u \in P_{u_0}$ then $Tu \in C_{2-\alpha}[0, 1]$, and $Tu(t) \geq 0$, for all $t \in [0, 1]$.

Let's prove that $T(P_{u_0}) \subset P_{u_0}$.

Take $u \in P_{u_0}$, then, for all $t \in [0, 1]$, by using Lemmas 2.1 and 2.3, the following inequalities are satisfied

$$\begin{aligned} t^{2-\alpha} Tu(t) &= \int_0^1 t^{2-\alpha} G(t, s) f(s, s^{2-\alpha}u(s)) ds \\ &\geq m(t) \int_0^1 s e_\alpha^{\lambda(1-s)} f(s, s^{2-\alpha}u(s)) ds \\ &\geq \frac{m(t)}{M} \int_0^1 \max_{t \in [0,1]} \{t^{2-\alpha} G(t, s)\} f(s, s^{2-\alpha}u(s)) ds \\ &\geq \frac{m(t)}{M} \max_{t \in [0,1]} \left\{ t^{2-\alpha} \int_0^1 G(t, s) f(s, s^{2-\alpha}u(s)) ds \right\} = \frac{m(t)}{M} \|Tu\|_{2-\alpha}. \end{aligned}$$

In view of the continuity of functions G and f , the operator $T : P_{u_0} \rightarrow P_{u_0}$ is continuous.

Let $\Omega \subset P_{u_0}$ be bounded, i.e., there exists a positive constant $N > 0$ such that $\|u\|_{2-\alpha} \leq N$, for all $u \in \Omega$.

Define now

$$L = \max_{0 \leq t \leq 1, 0 \leq x \leq N} |f(t, x)| + 1.$$

Then, for all $u \in \Omega$, it is satisfied that

$$|t^{2-\alpha} Tu(t)| \leq L \int_0^1 t^{2-\alpha} G(t, s) ds \leq L N \int_0^1 s e_\alpha^{\lambda(1-s)} ds \equiv R < \infty, \quad (2.17)$$

that is, the set $T(\Omega)$ is bounded in $C_{2-\alpha}[0, 1]$.

Now we prove that $\{Tu : u \in \Omega\}$ is an equicontinuous family of $C_{2-\alpha}[0, 1]$.

For each $t_1, t_2 \in [0, 1]$, and $u \in \Omega$, we have

$$\begin{aligned} \left| t_1^{2-\alpha} Tu(t_1) - t_2^{2-\alpha} Tu(t_2) \right| &= \left| \int_0^1 t_1^{2-\alpha} G(t_1, s) f(s, s^{2-\alpha}u(s)) - t_2^{2-\alpha} G(t_2, s) f(s, s^{2-\alpha}u(s)) ds \right| \\ &\leq L \int_0^1 \left| t_1^{2-\alpha} G(t_1, s) - t_2^{2-\alpha} G(t_2, s) \right| ds. \end{aligned}$$

The result holds from the uniform continuity of function $t^{2-\alpha} G(t, s)$ on $[0, 1] \times [0, 1]$.

Now, from the Ascoli-Arzelà Theorem we conclude that $\overline{T(\Omega)}$ is compact, i.e. $T : P_{u_0} \rightarrow P_{u_0}$ is a completely continuous operator.

Let $\varphi(t) = t^{\alpha-2}$, it is clear that

$$0 \leq u_0 \leq \varphi, \|\varphi\|_{2-\alpha} = 1$$

and that the condition (2.13) is fulfilled. Consider now the first situation

(i) *Sublinear case* ($f_0 = \infty$ and $f^\infty = 0$).

Since $f_0 = \infty$, then there exists a constant $\rho_1 > 0$ such that

$$f(t, u) \geq \delta_1 u$$

for all $0 \leq u \leq \rho_1$, where $\delta_1 > 0$ satisfies

$$\frac{\delta_1}{M} \max_{t \in [0,1]} \left\{ t^{2-\alpha} \int_0^1 m(s) G(t, s) ds \right\} \geq 1. \quad (2.18)$$

Take $a = \rho_1$, then for $u \in [au_0, a\varphi]$ and from the expression (2.18), we deduce the following inequalities

$$\begin{aligned} \|Tu\|_{2-\alpha} &= \max_{t \in [0,1]} \left\{ t^{2-\alpha} \int_0^1 G(t, s) f(s, s^{2-\alpha}u(s)) ds \right\} \\ &\geq \delta_1 \max_{t \in [0,1]} \left\{ t^{2-\alpha} \int_0^1 G(t, s) s^{2-\alpha}u(s) ds \right\} \\ &\geq \delta_1 \frac{a}{M} \max_{t \in [0,1]} \left\{ t^{2-\alpha} \int_0^1 m(s) G(t, s) ds \right\} \\ &\geq a. \end{aligned}$$

Moreover, from the continuity of the function f , we can define the following function:

$$\tilde{f}(t, u) = \max_{z \in [0, u]} \{f(t, z)\}.$$

Clearly $\tilde{f}(t, \cdot)$ is a nondecreasing function on $[0, \infty)$ for any $t \in [0, 1]$ fixed.

Moreover, since $f^\infty = 0$, it is obvious, see [68], that

$$\lim_{u \rightarrow \infty} \left\{ \max_{t \in [0,1]} \frac{\tilde{f}(t, u)}{u} \right\} = 0,$$

and then, there exists a constant $\rho_2 > \rho_1 > 0$ such that

$$\tilde{f}(t, u) \leq \delta_2 u$$

for all $u \geq \rho_2$, where $\delta_2 > 0$ satisfies

$$\delta_2 \max_{t \in [0,1]} \left\{ t^{2-\alpha} \int_0^1 G(t, s) ds \right\} \leq 1. \quad (2.19)$$

Let $b = \rho_2 \frac{M}{\tilde{M}}$, where $\tilde{M} = \max_{s \in [0,1]} m(s) > 0$.

If we take $u \in [bu_0, b\varphi]$ then $b \geq \|u\|_{2-\alpha} \geq \rho_2$ and, from the expression (2.19), we deduce the following inequalities

$$\begin{aligned} \|Tu\|_{2-\alpha} &\leq \max_{t \in [0,1]} \left\{ t^{2-\alpha} \int_0^1 G(t,s) \tilde{f}(s, s^{2-\alpha}u(s)) ds \right\} \\ &\leq \max_{t \in [0,1]} \left\{ t^{2-\alpha} \int_0^1 G(t,s) \tilde{f}(s, \|u\|_{2-\alpha}) ds \right\} \\ &\leq \delta_2 \|u\|_{2-\alpha} \max_{t \in [0,1]} \left\{ t^{2-\alpha} \int_0^1 G(t,s) ds \right\} \\ &\leq b. \end{aligned}$$

Thus, by the first part of Theorem 2.4, we conclude that problem (2.1) has at least one positive solution u such that $u \in [au_0, b\varphi]$.

Consider now the second case (ii).

Let $\delta_2 > 0$ be given as in equation (2.19). Since $f^0 = 0$, then there exists a constant $r_1 > 0$ such that

$$f(t, u) \leq \delta_2 u$$

for all $0 \leq u \leq r_1$. Take $b = r_1$, then for $u \in [bu_0, b\varphi]$ we deduce the following inequalities

$$\begin{aligned} \|Tu\|_{2-\alpha} &= \max_{t \in [0,1]} \left\{ t^{2-\alpha} \int_0^1 G(t,s) f(s, s^{2-\alpha}u(s)) ds \right\} \\ &\leq \delta_2 \|u\|_{2-\alpha} \max_{t \in [0,1]} \left\{ t^{2-\alpha} \int_0^1 G(t,s) ds \right\} \\ &\leq b. \end{aligned}$$

Now since $m(t) > 0$ on $(0, \delta)$ we can ensure the existence of the constants c_1, c_2, δ_3 such that $0 < c_1 < c_2 < 1$, satisfying

$$\frac{\delta_3}{M} \max_{t \in [0,1]} \left\{ t^{2-\alpha} \int_{c_1}^{c_2} m(s) G(t,s) ds \right\} \geq 1. \quad (2.20)$$

Since $f_\infty = \infty$ then there exists a constant $r_2 > r_1 > 0$ such that

$$f(t, u) \geq \delta_3 u$$

for all $u \geq r_2$. Let $a = r_2 \frac{M}{\check{M}}$, where $\check{M} = \min_{s \in [c_1, c_2]} m(s) > 0$. If we take $u \in [au_0, a\varphi]$ we deduce the following inequalities

$$\begin{aligned}
 \|Tu\|_{2-\alpha} &= \max_{t \in [0,1]} \left\{ t^{2-\alpha} \int_0^1 G(t,s) f(s, s^{2-\alpha} u(s)) ds \right\} \\
 &\geq \max_{t \in [0,1]} \left\{ t^{2-\alpha} \int_{c_1}^{c_2} G(t,s) f(s, s^{2-\alpha} u(s)) ds \right\} \\
 &\geq \delta_3 \max_{t \in [0,1]} \left\{ t^{2-\alpha} \int_{c_1}^{c_2} G(t,s) s^{2-\alpha} u(s) ds \right\} \\
 &\geq \delta_3 \frac{a}{M} \max_{t \in [0,1]} \left\{ t^{2-\alpha} \int_{c_1}^{c_2} m(s) G(t,s) ds \right\} \\
 &\geq a.
 \end{aligned}$$

Thus, by the second part of Theorem 2.4, we conclude that problem (2.1) has at least one positive solution u such that $u \in [au_0, b\varphi]$. \square

Remark 2.2. *In order to improve our results in Theorem 2.5, we can replace (respectively) the conditions (i) and (ii) by the following ones:*

$$(\tilde{i}) \quad f_0 > \delta_1 \text{ and } f^\infty < \delta_2.$$

$$(\tilde{ii}) \quad f^0 < \delta_2 \text{ and } f_\infty > \delta_3.$$

2.5 Lower and Upper Solutions

This section is devoted to the employment of the lower and upper solutions method for the study of the non-homogeneous Dirichlet problem

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + f(t, t^{2-\alpha} u(t)) = 0, & 0 < t < 1, \\ \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) = A, \quad u(1) = B, \end{cases} \quad (2.21)$$

for $\lambda > \lambda_1$, $1 < \alpha \leq 2$, f a continuous function and $A, B \in \mathbb{R}$.

This method is very well known for Ordinary and Partial Differential Equations (see [16, 17, 41]) and allow us to ensure the existence and location of at least one solution of the considered problem. We introduce the concept of such functions as follows.

Definition 2.1. Let $\gamma \in E$ be such that $D^\alpha \gamma \in C(0, 1] \cap L^1(0, 1)$. We say that γ is a lower solution of problem (2.21) if it satisfies

$$\begin{cases} D^\alpha \gamma(t) - \lambda \gamma(t) + f(t, t^{2-\alpha} \gamma(t)) \geq 0, & 0 < t < 1, \\ \lim_{t \rightarrow 0^+} t^{2-\alpha} \gamma(t) \leq A, & \gamma(1) \leq B \end{cases} \quad (2.22)$$

where δ will be an upper solution of problem (2.21) if the regularity assumptions are fulfilled and the reverse inequalities hold.

The existence result is the following one

Theorem 2.6. Suppose that there is $\gamma \leq \delta$ a pair of lower and upper solutions of problem (2.21). Then problem (2.21) has at least one solution lying between γ and δ on $(0, 1]$.

Proof. Consider the following modified problem:

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + f(t, p(t, t^{2-\alpha} u(t))) = 0, & 0 < t < 1, \\ \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) = A = u(1) = B, \end{cases} \quad (2.23)$$

where $p(t, x) = \max \{t^{2-\alpha} \gamma(t), \min \{x, t^{2-\alpha} \delta(t)\}\}$.

We point out that u , γ and δ are continuous functions on $(0, 1]$.

Let u be any solution of problem (2.23). By the definition of γ and δ we obtain that

$$\gamma(1) \leq u(1) \leq \delta(1)$$

and

$$\lim_{t \rightarrow 0^+} t^{2-\alpha} \gamma(t) \leq \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) \leq \lim_{t \rightarrow 0^+} t^{2-\alpha} \delta(t).$$

If there exists $t_0 \in (0, 1)$ such that $u(t_0) < \gamma(t_0)$ then, by the continuity of both functions in $(0, 1]$, there exists a sub-interval $(c, d) \subset (0, 1)$ where $\gamma > u$ on (c, d) and

$$(\gamma - u)(c) = (\gamma - u)(d) = 0$$

.

In such a case, using the linearity of the Riemman-Liouville derivative, denoting by $v = \gamma - u$, we have that

$$p(t, t^{2-\alpha} u(t)) = t^{2-\alpha} \gamma(t)$$

for all $t \in (c, d)$ and, as consequence

$$0 \leq D^\alpha v(t) - \lambda v(t) + f(t, t^{2-\alpha} \gamma(t)) - f(t, t^{2-\alpha} p(t, u(t))) = D^\alpha v(t) - \lambda v(t), \quad c < t < d.$$

It is important to note that, due to the regularity imposed in function γ , we have that $D^\alpha v - \lambda v$ is a continuous function in $[c, d]$. Now, since $v(c) = v(d) = 0$, and $\lambda > \lambda_1 > \lambda_1/(d-c)^\alpha$ we have, by Corollary 2.1, that

$$v \geq 0 \quad \text{on } (c, d),$$

which contradicts the existence of t_0 .

So, arguing in a similar way with δ , we conclude that any solution u of the truncated problem (2.23) is lying between γ and δ on $(0, 1]$.

So, to conclude the proof, we only must verify that problem (2.23) has at least one solution in E . To this end, we take into account that the solutions of such a problem coincide with the fixed points of the operator $\mathcal{F} : E \rightarrow E$, defined as

$$\mathcal{F}u(t) = \int_0^1 G(t, s) f(s, s^{2-\alpha} u(s)) ds + A v_1(t) + B v_2(t), \quad 0 < t \leq 1. \quad (2.24)$$

Here G is the Green's function related to problem (2.2) and defined in (2.7), v_1 is the unique solution of problem

$$\begin{cases} D^\alpha v_1(t) - \lambda v_1(t) = 0, & 0 < t < 1, \\ \lim_{t \rightarrow 0^+} t^{2-\alpha} v_1(t) = v_1(1) = 1, \end{cases} \quad (2.25)$$

and v_2 is the unique solution of problem

$$\begin{cases} D^\alpha v_2(t) - \lambda v_2(t) = 0, & 0 < t < 1, \\ \lim_{t \rightarrow 0^+} t^{2-\alpha} v_2(t) = v_2(1) = 1, \end{cases} \quad (2.26)$$

Using Theorem 2.1, we see that v_1 and v_2 follow the general expression (2.5). So, substituting on such expressions in the non-homogeneous boundary conditions, and arguing as at the beginning of Section 2.3, we have that

$$v_1(t) = -\Gamma(\alpha - 1) \frac{E_{\alpha, \alpha-1}(\lambda)}{E_{\alpha, \alpha}(\lambda)} e_\alpha^{\lambda t} + \Gamma(\alpha - 1) t^{\alpha-2} E_{\alpha, \alpha-1}(\lambda t^\alpha)$$

and

$$v_2(t) = \frac{e_\alpha^{\lambda t}}{E_{\alpha, \alpha}(\lambda)}.$$

It is obvious that there is a constant $K > 0$ such that

$$\max\{A \|v_1\|_{2-\alpha} + B \|v_2\|_{2-\alpha}\} \leq K,$$

moreover, since u , γ and δ belong to E , then $p(t, t^{2-\alpha}u(t))$ is a continuous and bounded function on $[0, 1]$. As a consequence, the continuity of the function f tell us that we can define a positive constant $L > 0$ such that

$$L = \max_{0 \leq t \leq 1, u \in C_{2-\alpha}[0,1]} |f(t, t^{2-\alpha}u(t))| + 1.$$

So, arguing as in the proof of Theorem 2.5, we conclude that operator \mathcal{F} is a compact operator that maps the space E in to the closed ball centered at 0 and radius $R + K$. The result holds from the Schauder's fixed point Theorem [76]. \square

2.6 Examples

In this section we present examples in order to illustrate our results. The first example and the second one are chosen such that the conditions (i) and (ii) are satisfied, respectively. The last one is delivered to the construction of a lower and an upper solution for the considered problem.

Example 2.1. *Let consider the fractional differential equation (2.1) with*

$$f(t, u(t)) = \sqrt{u(t)} + t.$$

It is clear that for all $u > 0$

$$\min_{t \in [0,1]} \frac{f(t, u)}{u} = \frac{\sqrt{u}}{u}$$

and

$$\max_{t \in [0,1]} \frac{f(t, u)}{u} = \frac{\sqrt{u} + 1}{u}$$

.

By a direct calculation, we obtain $f_0 = \infty$ and $f^\infty = 0$. From the first part of Theorem 2.5, we conclude that the problem (2.1) has a positive solution if and only if $\lambda > \lambda_1$.

Example 2.2. *Consider the fractional differential equation (2.1) with*

$$f(t, u(t)) = u^2(t) + tu^3(t).$$

Clearly, for every $u > 0$ it is verified that

$$\min_{t \in [0,1]} \frac{f(t, u)}{u} = u$$

and

$$\max_{t \in [0,1]} \frac{f(t, u)}{u} = u + u^2$$

Obviously, $f^0 = 0$ and $f_\infty = \infty$. From the second part of Theorem 2.5, we conclude that the problem (2.1) has a positive solution if and only if $\lambda > \lambda_1$.

Example 2.3. Consider the fractional differential equation (2.21) with

$$f(t, x) = \frac{c}{2} t E_{\alpha, \alpha}(\lambda t^\alpha) - x,$$

$A = 0$ and $B \geq E_{\alpha, \alpha}(\lambda)$, where

$$c \geq \max\left\{\frac{B}{E_{\alpha, \alpha}(\lambda)}, 2\right\}$$

Now let $\gamma = e_\alpha^{\lambda t}$ and $\delta = ce_\alpha^{\lambda t}$, it is very easy to verify that

$$D^\alpha \gamma(t) - \lambda \gamma(t) + f(t, t^{2-\alpha} \gamma(t)) = \left(\frac{c}{2} - 1\right) t E_{\alpha, \alpha}(\lambda t^\alpha) \geq 0$$

and

$$D^\alpha \delta(t) - \lambda \delta(t) + f(t, t^{2-\alpha} \delta(t)) = -\frac{c}{2} t E_{\alpha, \alpha}(\lambda t^\alpha) \leq 0,$$

then obviously $\gamma \leq \delta$ are lower and upper solutions of the problem (2.21). Hence, by Theorem 2.6, the problem (2.21) has at least one solution lying between γ and δ on $[0, 1]$.

Chapter 3

Nonlinear fractional differential equations with integral boundary value conditions

3.1 Introduction

To be concise, in this chapter we are concerned with the study of the existence of solutions of the following nonlinear fractional differential equations with integral boundary value conditions

$$D^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad (3.1)$$

$$u(0) = u'(0) = 0, \quad u(1) = \lambda \int_0^1 u(s) ds. \quad (3.2)$$

where $2 < \alpha \leq 3$, $0 < \lambda$, $\lambda \neq \alpha$, D^α is the Riemann-Liouville fractional derivative and $f : [0, 1] \times \mathbb{R}^* \rightarrow \mathbb{R}$ is a continuous function.

In a first moment we obtain the exact expression of the Green's function related to the linear problem

$$D^\alpha u(t) + y(t) = 0, \quad 0 < t < 1, \quad (3.3)$$

coupled with the integral boundary conditions (3.2).

Once we have obtained such an expression, we study the values of λ for which the Green's function is positive in $(0, 1) \times (0, 1)$. Moreover we deduce some suitable properties that relate the expression of $G(t, s)$ and $G(1, s)$. These properties will be fundamental to construct a

suitable cone in the spaces of the continuous functions and so, under additional conditions on the behavior of a function f at 0 and at ∞ , we deduce the existence of positive solutions of problem (3.1)–(3.2).

In some sense, the results given in this work follow similar steps to the ones obtained in [23] for the problem

$$\begin{cases} {}^C D^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u''(0) = 0, \quad u(1) = \lambda \int_0^1 u(s) ds, \end{cases}$$

where $2 < \alpha < 3$, $0 < \lambda < 2$, ${}^C D^\alpha$ is the Caputo fractional derivative and $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is a continuous function.

The results presented in this chapter have been published in [22]

3.2 Study of the Green's function

In this section, we obtain the exact expression of the Green's function related to the linear fractional differential equation with integral boundary value conditions (3.3)–(3.2). The result is the following

Theorem 3.1. *Let $2 < \alpha \leq 3$ and $\lambda \neq \alpha$. Assume $y \in C[0, 1]$, then the problem (3.3)–(3.2) has a unique solution $u \in C^1[0, 1]$, given by the expression*

$$u(t) = \int_0^1 G(t, s)y(s)ds,$$

where

$$G(t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (\alpha-\lambda)(t-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(\alpha-\lambda)\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (3.4)$$

Proof. It is very well known, see [39, Lemma 2.5], that the equation (3.3) is equivalent to the following integral equation

$$u(t) = - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}.$$

Since $u(0) = u'(0) = 0$, we deduce that

$$u(t) = - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + c_1 t^{\alpha-1}.$$

Finally, condition $u(1) = \lambda \int_0^1 u(s) ds$ implies that

$$c_1 = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \lambda \int_0^1 u(s) ds.$$

Hence, we have the following form

$$u(t) = - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + t^{\alpha-1} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \lambda t^{\alpha-1} \int_0^1 u(s) ds. \quad (3.5)$$

Let $\int_0^1 u(s) ds = A$, then, from the previous equality, we deduce that

$$\begin{aligned} A &= \int_0^1 u(t) dt \\ &= - \int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds dt + \int_0^1 \int_0^1 \frac{t^{\alpha-1} (1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds dt + \lambda A \int_0^1 t^{\alpha-1} dt \\ &= - \int_0^1 \frac{(1-s)^\alpha}{\alpha \Gamma(\alpha)} y(s) ds + \frac{1}{\alpha} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \frac{\lambda}{\alpha} A. \end{aligned} \quad (3.6)$$

So, conditions (3.2) imply that

$$A = - \frac{1}{\alpha - \lambda} \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha)} y(s) ds + \frac{1}{\alpha - \lambda} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds.$$

Replacing this value in (3.2), we arrive at the following expression for function u :

$$\begin{aligned} u(t) &= - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + t^{\alpha-1} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds - \frac{\lambda}{\alpha - \lambda} t^{\alpha-1} \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha)} y(s) ds \\ &\quad + \frac{\lambda}{\alpha - \lambda} t^{\alpha-1} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ &= - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + t^{\alpha-1} \int_0^1 \frac{(1-s)^{\alpha-1} (\alpha + \lambda (s-1))}{(\alpha - \lambda) \Gamma(\alpha)} y(s) ds \\ &= \int_0^t \frac{t^{\alpha-1} (1-s)^{\alpha-1} (\alpha + \lambda (s-1)) - (\alpha - \lambda) (t-s)^{\alpha-1}}{(\alpha - \lambda) \Gamma(\alpha)} y(s) ds \\ &\quad + \int_t^1 \frac{t^{\alpha-1} (1-s)^{\alpha-1} (\alpha + \lambda (s-1))}{(\alpha - \lambda) \Gamma(\alpha)} y(s) ds \\ &= \int_0^1 G(t, s) y(s) ds. \end{aligned}$$

This complete the proof. □

Example 3.1. Consider the problem

$$\begin{cases} D^{5/2}u(t) - \frac{12}{\sqrt{\pi}}\sqrt{t} = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = \frac{3}{2} \int_0^1 u(s)ds, \end{cases}$$

By using Theorem 3.1, we obtain that $u(t) = t^3$ is the unique solution of this problem.

A careful analysis of the Green's function G allows us to deduce the following result:

Lemma 3.1. Let G be the Green's function related to problem (3.3)–(3.2), which is given by the expression (3.4). Then, for all $\alpha \in (2, 3]$ and $\lambda \geq 0$, the following properties are fulfilled:

- (i) $G(0, s) = G(t, 1) = 0$, for all $t, s \in [0, 1]$ and $\lambda \neq \alpha$.
- (ii) $G(1, s) = 0$, for all $s \in (0, 1)$, if and only if $\lambda = 0$.
- (iii) $(\alpha - \lambda)G(1, s) > 0$, for all $s \in (0, 1)$, if and only if $\lambda \neq \alpha$.
- (iv) $G(t, 0) = 0$, for all $t \in [0, 1]$.
- (v) $G(t, s) \leq \frac{\alpha}{(\alpha - \lambda)\Gamma(\alpha)}$, for all $t, s \in [0, 1]$ and $\lambda \in [0, \alpha)$.
- (vi) $G(t, s)$ is a continuous function, for all $t, s \in [0, 1]$ and $\lambda \neq \alpha$.

Now, we prove two additional inequalities of the Green's function G . Such properties, together with the previous ones given above, will be of fundamental interest to ensure the existence of solutions of problem (3.1)–(3.2) that will be proved in the next section.

Lemma 3.2. Fix $2 < \alpha \leq 3$ and $0 < \lambda < \alpha$. Let $G(t, s)$ be the Green's function related to problem (3.3)–(3.2) given by the expression (3.4). Then the following inequalities hold:

$$t^{\alpha-1}G(1, s) \leq G(t, s) \leq \frac{\alpha}{\lambda}G(1, s), \quad \text{for all } t, s \in (0, 1). \quad (3.7)$$

Proof. Assume in a first moment that $0 < t \leq s < 1$. In such a case:

$$h(t, s) \equiv \frac{G(t, s)}{G(1, s)} = \frac{t^{\alpha-1}(\alpha - \lambda(1 - s))}{\lambda s}, \quad \text{for all } 0 < t \leq s < 1.$$

Now, it is immediate to verify the following inequalities:

$$t^{\alpha-1} < t^{\alpha-1} \left(1 + \frac{\alpha - \lambda}{\lambda s} \right) = h(t, s) \leq t^{\alpha-1} \frac{\alpha}{\lambda s} \leq t^{\alpha-2} \frac{\alpha}{\lambda} < \frac{\alpha}{\lambda}, \quad \text{for all } 0 < t \leq s < 1.$$

On the other hand, if $0 < s \leq t < 1$ we have that

$$h(t, s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha - \lambda(1-s)) - (\alpha - \lambda)(t-s)^{\alpha-1}}{\lambda s(1-s)^{\alpha-1}}, \quad \text{for all } 0 < s \leq t < 1,$$

and since $s \geq t s$ we obtain

$$\begin{aligned} h(t, s) &\geq \frac{t^{\alpha-1}(1-s)^{\alpha-1}[\alpha - \lambda(1-s) - (\alpha - \lambda)]}{\lambda s(1-s)^{\alpha-1}} \\ &= t^{\alpha-1}. \end{aligned}$$

As in the previous case, it is not difficult to verify that $h(t, s) \leq \alpha/\lambda$ whenever $0 < s \leq t < 1$. Now, by Lemma 3.1, (iii), the inequalities (3.7) are fulfilled. \square

As a corollary of the previous result and Lemma 3.1, we deduce the following:

Corollary 3.1. *Let G be the Green's function related to problem (3.3)–(3.2), which is given by the expression (3.4). Then, for all $\alpha \in (2, 3]$ and $\lambda \geq 0$, the following property holds:*

$$G(t, s) > 0, \text{ for all } t, s \in (0, 1) \text{ and all } \lambda \in [0, \alpha).$$

3.3 Existence of Positive Solutions

This section is devoted to prove the existence of a positive solution of the nonlinear boundary value problem (3.1) – (3.2). To this end, we will use the Guo-Krasnoselskii fixed point theorem [37].

Define now the operator $T : C[0, 1] \rightarrow C[0, 1]$ as

$$Tu(t) := \int_0^1 G(t, s)f(s, u(s))ds, \quad (3.8)$$

with G defined in (3.4).

It is clear, from Theorem 3.1, that the fixed points of operator T which are solutions to the nonlinear Volterra integral equation

$$u(t) = \int_0^1 G(t, s)f(s, u(s))ds, \quad (3.9)$$

coincide with the solutions of problem (3.1)–(3.2).

Let $E = C[0, 1]$ be the Banach space endowed with the usual supremum norm $\|\cdot\|$, and suppose the following assumption:

(f) $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is a continuous function.

Define now the cone $P \subset E$ as follows

$$P = \left\{ u \in E, u(t) \geq 0 \text{ for all } t \in [0, 1], u(t) \geq \frac{t^{\alpha-1} \lambda}{\alpha} \|u\|, \text{ for all } t \in \left[\frac{1}{2}, 1\right] \right\}. \quad (3.10)$$

Set

$$f_0 = \lim_{u \rightarrow 0^+} \left\{ \min_{t \in [0, 1]} \frac{f(t, u)}{u} \right\}, \quad f_\infty = \lim_{u \rightarrow \infty} \left\{ \min_{t \in [0, 1]} \frac{f(t, u)}{u} \right\},$$

and

$$f^0 = \lim_{u \rightarrow 0^+} \left\{ \max_{t \in [0, 1]} \frac{f(t, u)}{u} \right\}, \quad f^\infty = \lim_{u \rightarrow \infty} \left\{ \max_{t \in [0, 1]} \frac{f(t, u)}{u} \right\}.$$

Now, we are in position to prove the main result of this chapter.

Theorem 3.2. *Assume that condition (f) holds coupled with one of the two following conditions:*

(i) (sublinear case) $f_0 = \infty$ and $f^\infty = 0$.

(ii) (superlinear case) $f^0 = 0$, $f_\infty = \infty$.

Then, for all $\alpha \in (2, 3]$ and $\lambda \in (0, \alpha)$, the problem (3.1)–(3.2) has a positive solution that belongs to the cone P defined in (3.10).

Proof. Firstly, we prove that $T : P \rightarrow P$ is completely continuous.

From the continuity and the non negativeness of functions G and f on their domains of definition, we have that if $u \in P$, then $Tu \in E$ and $Tu(t) \geq 0$, for all $t \in [0, 1]$.

Let's prove that $T(P) \subset P$.

Take $u \in P$, then, for all $t \in [0, 1]$, by using Lemmas 3.1 and 3.2, the following inequalities are satisfied

$$\begin{aligned}
 Tu(t) &= \int_0^1 G(t, s) f(s, u(s)) ds \\
 &\geq t^{\alpha-1} \int_0^1 G(1, s) f(s, u(s)) ds \\
 &\geq \frac{t^{\alpha-1} \lambda}{\alpha} \int_0^1 \max_{t \in [0, 1]} \{G(t, s)\} f(s, u(s)) ds \\
 &\geq \frac{t^{\alpha-1} \lambda}{\alpha} \max_{t \in [0, 1]} \left\{ \int_0^1 G(t, s) f(s, u(s)) ds \right\} \\
 &= \frac{t^{\alpha-1} \lambda}{\alpha} \|Tu\|.
 \end{aligned}$$

In view of the continuity of functions G and f , the operator $T : P \rightarrow P$ is continuous.

Let $\Omega \subset P$ be bounded, that is to say there exists a positive constant $M > 0$ such that $\|u\|_\infty \leq M$, for all $u \in \Omega$.

Define now

$$L = \max_{0 \leq t \leq 1, 0 \leq u \leq M} |f(t, u)| + 1.$$

Then, for all $u \in \Omega$, it is satisfied that

$$|Tu(t)| \leq L \int_0^1 G(t, s) ds \leq \frac{L\alpha}{(\alpha - \lambda)\Gamma(\alpha)}, \quad \text{for all } t \in [0, 1], \quad (3.11)$$

that is, the set $T(\Omega)$ is bounded in E .

For each $u \in \Omega$, we have

$$\begin{aligned}
 |(Tu)'(t)| &= \left| - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, u(s)) ds + \int_0^1 \frac{t^{\alpha-2}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(\alpha-\lambda)\Gamma(\alpha-1)} f(s, u(s)) ds \right| \\
 &\leq \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} |f(s, u(s))| ds \\
 &\quad + \frac{1}{(\alpha-\lambda)\Gamma(\alpha-1)} \int_0^1 t^{\alpha-2}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) |f(s, u(s))| ds \\
 &\leq \frac{L}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} ds + \frac{\alpha L}{(\alpha-\lambda)\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-1} ds \\
 &\leq \frac{L}{\Gamma(\alpha)} + \frac{\alpha L}{(\alpha-\lambda)\Gamma(\alpha)} := N.
 \end{aligned}$$

As a consequence, for all $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, we have

$$|(Tu)(t_2) - (Tu)(t_1)| \leq \int_{t_1}^{t_2} |(Tu)'(s)| ds \leq N(t_2 - t_1),$$

and the set $T(\Omega)$ is equicontinuous in E .

Now, from the Arzelà-Ascoli Theorem we conclude that $\overline{T(\Omega)}$ is compact, i.e. $T : P \rightarrow P$ is a completely continuous operator.

Consider now the first cas:

(i) *Sublinear case* ($f_0 = \infty$ and $f^\infty = 0$).

Since $f_0 = \infty$, then there exists a constant $\rho_1 > 0$ such that $f(t, u) \geq \delta_1 u$, for all $0 < u \leq \rho_1$, where $\delta_1 > 0$ satisfies

$$\delta_1 \frac{\lambda}{\alpha} \max_{t \in [0,1]} \left\{ \int_{\frac{1}{2}}^1 s^{\alpha-1} G(t, s) ds \right\} \geq 1. \quad (3.12)$$

Take $u \in P$, such that $\|u\| = \rho_1$, then, from the expression (3.12), we deduce the following inequalities

$$\begin{aligned} \|Tu\| &= \max_{t \in [0,1]} \left\{ \int_0^1 G(t, s) f(s, u(s)) ds \right\} \\ &\geq \delta_1 \max_{t \in [0,1]} \left\{ \int_{\frac{1}{2}}^1 G(t, s) u(s) ds \right\} \\ &\geq \delta_1 \|u\| \frac{\lambda}{\alpha} \max_{t \in [0,1]} \left\{ \int_{\frac{1}{2}}^1 s^{\alpha-1} G(t, s) ds \right\} \\ &\geq \|u\|. \end{aligned}$$

Since $f(t, \cdot)$ is a continuous function on $[0, \infty)$, we can define the following function:

$$\tilde{f}(t, u) = \max_{z \in [0, u]} \{f(t, z)\}.$$

Clearly $\tilde{f}(t, \cdot)$ is nondecreasing on $[0, \infty)$, moreover, since $f^\infty = 0$ it is obvious that (see [68]).

$$\lim_{u \rightarrow \infty} \left\{ \max_{t \in [0,1]} \frac{\tilde{f}(t, u)}{u} \right\} = 0.$$

Choose $\delta_2 > 0$ satisfying the following property:

$$\frac{\delta_2}{(\alpha - \lambda) \Gamma(\alpha - 1)} \leq 1. \quad (3.13)$$

Therefore, there exists a constant $\rho_2 > \rho_1 > 0$ such that $\tilde{f}(t, u) \leq \delta_2 u$, for all $u \geq \rho_2$.

Let now $u \in P$ be such that $\|u\| = \rho_2$, then, from the definition of \tilde{f} , equation (3.13) and property (v) in Lemma 3.1, we attain at the following inequalities:

$$\begin{aligned}
 \|Tu\| &= \max_{t \in [0,1]} \left\{ \int_0^1 G(t,s) f(s, u(s)) ds \right\} \\
 &\leq \max_{t \in [0,1]} \left\{ \int_0^1 G(t,s) \tilde{f}(s, \|u\|) ds \right\} \\
 &\leq \delta_2 \|u\| \max_{t \in [0,1]} \left\{ \int_0^1 G(t,s) ds \right\} \\
 &\leq \frac{\delta_2}{(\alpha - \lambda) \Gamma(\alpha - 1)} \|u\| \\
 &\leq \|u\|.
 \end{aligned}$$

Thus, by the first part of Guo-Krasnoselskii fixed point theorem, we conclude that problem (3.1)–(3.2) has at least one positive solution u such that

$$\rho_1 \leq \|u\| \leq \rho_2.$$

Consider now the second case (ii).

Let $\delta_2 > 0$ be given as in equation (3.13). Since $f^0 = 0$, there exists a constant $r_1 > 0$ such that $f(t, u) \leq \delta_2 u$, for $0 \leq u \leq r_1$.

Take $u \in P$, such that $\|u\| = r_1$. Then we have

$$\|Tu\| = \max_{t \in [0,1]} \left\{ \int_0^1 G(t,s) f(s, u(s)) ds \right\} \tag{3.14}$$

$$\begin{aligned}
 &\leq \delta_2 \|u\| \max_{t \in [0,1]} \left\{ \int_0^1 G(t,s) ds \right\} \\
 &\leq \frac{\delta_2}{(\alpha - \lambda) \Gamma(\alpha - 1)} \|u\| \tag{3.15}
 \end{aligned}$$

$$\leq \|u\|. \tag{3.16}$$

Consider now $\delta_3 > 0$ satisfying

$$\delta_3 \frac{\lambda}{2^{\alpha-1} \alpha} \max_{t \in [0,1]} \left\{ \int_{\frac{1}{2}}^1 G(t,s) ds \right\} \geq 1, \tag{3.17}$$

The fact that $f_\infty = \infty$ says us that there exists a constant $r_2 > r_1 > 0$ with $r_2 \alpha 2^{\alpha-1} > r_1 \lambda$ such that $f(t, u) \geq \delta_3 u$, for all $u \geq r_2$.

Let now $u \in P$ be such that $\|u\| = r_2 \frac{\alpha}{\lambda} 2^{\alpha-1}$. Notice that from the definition of the cone P , we have that $u(t) \geq r_2$, for all $t \in [1/2, 1]$.

So, condition (ii) gives us the following properties:

$$\begin{aligned}
 \|Tu\| &= \max_{t \in [0,1]} \left\{ \int_0^1 G(t,s) f(s, u(s)) ds \right\} \\
 &\geq \max_{t \in [0,1]} \left\{ \int_{\frac{1}{2}}^1 G(t,s) f(s, u(s)) ds \right\} \\
 &\geq \delta_3 \max_{t \in [0,1]} \left\{ \int_{\frac{1}{2}}^1 G(t,s) u(s) ds \right\} \\
 &\geq \delta_3 \frac{\lambda}{2^{\alpha-1}\alpha} \|u\| \max_{t \in [0,1]} \left\{ \int_{\frac{1}{2}}^1 G(t,s) ds \right\} \\
 &\geq \|u\|.
 \end{aligned}$$

Therefore, by the second part of Guo-Krasnoselskii fixed point theorem, we can conclude that problem (3.1)–(3.2) has at least one positive solution. \square

Remark 3.1. *It is important to point out that, since $G(0, s) = 0$, in order to ensure the existence of r_2 in case (ii) of the previous theorem, it is necessary to reduce the interval of definition $[0, 1]$ to the smaller one $[1/2, 1]$. In fact, given two real constants $0 < a \leq b \leq 1$, by redefining*

$$f_0 = \lim_{u \rightarrow 0^+} \left\{ \min_{t \in [a,b]} \frac{f(t, u)}{u} \right\} \quad \text{and} \quad f_\infty = \lim_{u \rightarrow \infty} \left\{ \min_{t \in [a,b]} \frac{f(t, u)}{u} \right\},$$

it is immediate to verify that Theorem 3.2 remains true in the cone

$$P = \left\{ u \in E, u(t) \geq 0 \quad \text{for all } t \in [0, 1], \quad u(t) \geq \frac{t^{\alpha-1} \lambda}{\alpha} \|u\|, \quad \text{for all } t \in [a, b] \right\}. \quad (3.18)$$

3.4 Generalization of results for arbitrary α

In this section, we consider more general fractional boundary value problems of arbitrary order and give a generalization of our results. The proofs are analogous to those already given, so we will omit them. The results are the following:

Theorem 3.3. *Given $y \in C^n(0, 1)$, $1 \leq n < \alpha \leq n + 1$, and $\lambda \neq \alpha$, the nonlinear fractional differential equations with integral boundary value conditions*

$$\begin{cases} D^\alpha u(t) + y(t) = 0, & 0 < t < 1, \\ u^{(i)}(0) = 0, \quad 0 \leq i \leq n - 1, \quad u(1) = \lambda \int_0^1 u(s) ds, \end{cases}$$

where D^α is the Riemann-Liouville fractional derivative of order α has a unique solution given by

$$u(t) = \int_0^1 G(t, s)y(s)ds,$$

where

$$G(t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (\alpha-\lambda)(t-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(\alpha-\lambda)\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (3.19)$$

The only difference between the Green's function from (3.19) and the Green's function from Theorem 3.1 is in the value set for α . Because of that, the following theorem is obvious, as a consequence of Theorem 3.2 and Remark 3.1.

Theorem 3.4. *Given two real constants $0 < a \leq b \leq 1$. Assume that one of the following hypotheses is satisfied:*

- (i) $f_0 = \lim_{u \rightarrow 0^+} \left\{ \min_{t \in [a, b]} \frac{f(t, u)}{u} \right\}$ and $f^\infty = 0$
- (ii) $f^0 = 0$ and $f_\infty = \lim_{u \rightarrow \infty} \left\{ \min_{t \in [a, b]} \frac{f(t, u)}{u} \right\} = 0$.

Then, for $\lambda \in (0, \alpha)$, the problem

$$\begin{cases} D^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u^{(i)}(0) = 0, \quad 0 \leq i \leq n-1, \quad u(1) = \lambda \int_0^1 u(s)ds, \end{cases}$$

has at least one solution that belongs to cone

$$P = \left\{ u \in E, u(t) \geq 0 \text{ for all } t \in [0, 1], \quad u(t) \geq \frac{t^{\alpha-1} \lambda}{\alpha} \|u\|, \text{ for all } t \in [a, b] \right\}. \quad (3.20)$$

3.5 Examples

We now give two examples to illustrate our results. The first example and the second one are chosen such that the conditions (i) and (ii) are satisfied, respectively.

Example 3.2. Let consider the fractional differential equation (3.1)–(3.2) with

$$f(t, u(t)) = \sqrt{u(t)} + \log(tu^2(t) + 2).$$

One can easily see that for all $u > 0$

$$\min_{t \in [0,1]} \frac{f(t, u)}{u} = \frac{\sqrt{u} + \log 2}{u},$$

and

$$\max_{t \in [0,1]} \frac{f(t, u)}{u} = \frac{\sqrt{u} + \log(u^2 + 2)}{u}.$$

By a direct calculation, we obtain $f_0 = \infty$ and $f^\infty = 0$. From the first part of Theorem 3.2, we conclude that the problem (3.1)–(3.2) has a positive solution.

Example 3.3. Consider the fractional differential equation (3.1)–(3.2) with

$$f(t, u(t)) = u^2(t) - u(t) + t(e^{u(t)} - 1).$$

Clearly, for every $u > 0$ it is verified that

$$\min_{t \in [0,1]} \frac{f(t, u)}{u} = u - 1,$$

and

$$\max_{t \in [0,1]} \frac{f(t, u)}{u} = u - 1 + \frac{e^u - 1}{u}.$$

Obviously, $f^0 = 0$ and $f_\infty = \infty$. From the second part of Theorem 3.2, we conclude that the problem (3.1)–(3.2) has at least one positive solution.

Chapter 4

Multiplicity results for integral boundary value problems of fractional order with parametric dependence

4.1 Introduction

Since only positive solutions are meaningful in many applications, in this chapter we discuss the existence and multiplicity of positive solutions of a nonlinear fractional differential equation with integral boundary conditions and parameter dependence. More precisely, we consider the following problem

$$\begin{cases} D^\alpha u(t) + \mu g(t) f(u(t)) = 0, & 0 \leq t \leq 1, \\ u(0) = u'(0) = 0, u(1) = \lambda \int_0^1 u(s) ds, & 0 < \lambda < \alpha, \end{cases} \quad (4.1)$$

depending on the real parameter $\mu > 0$. Here D^α denotes the Riemann-Liouville fractional derivative of order $\alpha \in (2, 3]$ and f and g are appropriate functions to be specified later.

In some sense, the results given in this chapter follow similar steps to the ones obtained in [18] and [19] for second order differential and difference equations with periodic boundary value conditions. The main difference in our approach is that in this work the Green's function vanishes at $t = 0$ and $s = 0, 1$, which makes the main properties of the Green's function used in those references not valid here.

The results presented in this chapter have been published in [21]

4.2 Preliminaries

In this section we recompile the fundamental properties of the Green's function related to problem (3.3)–(3.2). Such properties has been proven in [22].

Theorem 4.1. [22, Theorem 2.1] *Let $2 < \alpha \leq 3$ and $\lambda \neq \alpha$. Assume $y \in C[0, 1]$, then problem (3.3)–(3.2) has a unique solution $u \in C^1[0, 1]$, given by the expression*

$$u(t) = \int_0^1 G(t, s)y(s)ds,$$

where

$$G(t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (\alpha-\lambda)(t-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(\alpha-\lambda)\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (4.2)$$

In the next result, are showed the fundamental properties of the Green's function G , that will be used to prove our main results.

Lemma 4.1. [22, Lemmas 2.2 and 2.3] *Let G be the Green's function related to problem (3.3)–(3.2), which is given by the expression (4.2). Then, for all $\alpha \in (2, 3]$ and $\lambda \geq 0$, the following properties are fulfilled:*

- (i) $G(0, s) = G(t, 1) = 0$ for all $t, s \in [0, 1]$ and $\lambda \neq \alpha$.
- (ii) $G(1, s) = 0$ for all $s \in (0, 1)$ if and only if $\lambda = 0$.
- (iii) $(\alpha - \lambda)G(1, s) > 0$ for all $s \in (0, 1)$ if and only if $\lambda \neq \alpha$.
- (iv) $G(t, 0) = 0$ for all $t \in [0, 1]$.
- (v) $G(t, s) \leq \frac{1}{(\alpha-\lambda)\Gamma(\alpha-1)}$ for all $t, s \in [0, 1]$ and $\lambda \in [0, \alpha)$.
- (vi) $G(t, s)$ is a continuous function for all $t, s \in [0, 1]$ and $\lambda \neq \alpha$.
- (vii) $t^{\alpha-1}G(1, s) \leq G(t, s) \leq \frac{\alpha}{\lambda}G(1, s)$, for all $t, s \in (0, 1)$.

As a corollary of the previous result and Lemma 4.1, we deduce the following:

Corollary 4.1. *Let G be the Green's function related to problem (3.3)–(3.2), which is given by the expression (4.2). Then, for all $\alpha \in (2, 3]$ and $\lambda \geq 0$, the following property holds:*

$$G(t, s) > 0 \text{ for all } t, s \in (0, 1) \text{ and all } \lambda \in [0, \alpha).$$

4.3 Main result

In this section we consider the nonlinear problem with parameter dependence (4.1). We will prove some existence, nonexistence and multiplicity results for some suitable values of the positive real parameter μ . In order to do so, we use the well known Guo-Krasnoselskii fixed point Theorem [37] in cones. We recall that a cone P is a subset of a Banach space which is nonempty, nontrivial, convex and closed. Moreover $\rho P \subset P$, for all $\rho > 0$, and $P \cap (-P) = \{0\}$.

Let $E = C[0, 1]$ be the Banach space of continuous functions endowed with the usual supremum norm $\|\cdot\|$, and we assume the following hypotheses:

$$(H1) \quad \mu > 0,$$

$$(H2) \quad g \in L^1[0, 1], \quad g(t) \geq 0 \text{ for a.e. } t \in [0, 1] \text{ and } \int_{1/2}^1 g(s) ds > 0,$$

$$(H3) \quad f : [0, \infty) \rightarrow [0, \infty) \text{ is continuous. Moreover } f(x) > 0 \text{ for all } x > 0.$$

In view of Theorem 4.1, we define the operator $T : C[0, 1] \rightarrow C[0, 1]$ as follows,

$$T_\mu u(t) := \mu \int_0^1 G(t, s) g(s) f(u(s)) ds, \quad (4.3)$$

with G defined in (4.2).

It is clear that the fixed points of operator T_μ coincide with the solutions of problem (4.1).

Define now the cone $P \subset E$ as follows,

$$P = \left\{ u \in E, \quad u(t) \geq 0 \text{ for all } t \in [0, 1], \quad u(t) \geq \frac{t^{\alpha-1} \lambda}{\alpha} \|u\|, \text{ for all } t \in \left[\frac{1}{2}, 1 \right] \right\}. \quad (4.4)$$

Let's prove that $T_\mu(P) \subset P$.

It is obvious that for $T_\mu(u) \geq 0$ for all $u \in P$ and $\mu > 0$.

If $u \in P$ then, due to Lemma 4.1 and the fact that $G(t, s)$ is continuous, the following inequalities are satisfied, for all $t \in [0, 1]$:

$$\begin{aligned}
T_\mu u(t) &= \mu \int_0^1 G(t, s) g(s) f(u(s)) ds \\
&\geq \mu t^{\alpha-1} \int_0^1 G(1, s) g(s) f(u(s)) ds \\
&\geq \mu \frac{t^{\alpha-1} \lambda}{\alpha} \int_0^1 \max_{t \in [0, 1]} \{G(t, s)\} g(s) f(u(s)) ds \\
&\geq \frac{t^{\alpha-1} \lambda}{\alpha} \max_{t \in [0, 1]} \left\{ \mu \int_0^1 G(t, s) g(s) f(u(s)) ds \right\} \\
&= \frac{t^{\alpha-1} \lambda}{\alpha} \|T_\mu u\|.
\end{aligned}$$

It is not difficult to show that $T_\mu : P \rightarrow P$ is completely continuous. Next, in order to apply the Guo-Krasnoselskii's fixed point theorem, we introduce a series of preliminary results depending on the value of the positive parameter μ .

We denote

$$f_0 = \lim_{x \rightarrow 0^+} \frac{f(x)}{x}$$

and

$$f_\infty = \lim_{x \rightarrow \infty} \frac{f(x)}{x}.$$

We have

Lemma 4.2. *Assume that the hypotheses (H1), (H2) and (H3) hold. Then, for every $R > 0$, there exists $\mu_0(R) > 0$ such that for every $0 < \mu \leq \mu_0(R)$, we have*

$$\|T_\mu u\| \leq \|u\|, \text{ for all } u \in P \text{ with } \|u\| = R.$$

Proof. Fix $R > 0$, and let $u \in P$ with $\|u\| = R$. Let $\mu > 0$ be such that

$$0 < \mu \leq \mu_0(R) = \frac{R(\alpha - \lambda)\Gamma(\alpha - 1)}{\max_{u \in [0, R]} \{f(u)\} \int_0^1 g(s) ds}. \quad (4.5)$$

Notice that from (H3) we know that $\max_{u \in [0, R]} \{f(u)\} > 0$ for all $R > 0$. Moreover (H2) implies that

$$\int_0^1 g(s) ds > 0.$$

Then, for all $t \in [0, 1]$, the following inequalities hold:

$$\begin{aligned}
T_\mu u(t) &= \mu \int_0^1 G(t, s) g(s) f(u(s)) ds \\
&\leq \frac{\mu}{(\alpha - \lambda) \Gamma(\alpha - 1)} \int_0^1 g(s) f(u(s)) ds \\
&\leq \frac{\mu}{(\alpha - \lambda) \Gamma(\alpha - 1)} \max_{u \in [0, R]} \{f(u)\} \int_0^1 g(s) ds \\
&\leq R = \|u\|,
\end{aligned}$$

and so $\|T_\mu u\| \leq \|u\|$. □

Lemma 4.3. *Assume that the hypotheses (H1), (H2) and (H3) hold. Then, for each $r > 0$, there exists $\mu_0(r) > 0$ such that, for every $\mu \geq \mu_0(r)$, we have*

$$\|T_\mu u\| \geq \|u\|, \text{ for } u \in P \text{ with } \|u\| = r.$$

Proof. Fix $r > 0$ and let $u \in P$ with $\|u\| = r$. Let

$$\mu_0(r) := \frac{r}{\min_{u \in [\frac{\lambda}{\alpha 2^{\alpha-1}} r, r]} \{f(u)\} \int_{\frac{1}{2}}^1 G(1, s) g(s) ds} > 0. \quad (4.6)$$

We point out that conditions (H2) and (H3) warrant that $\mu_0(r)$ is well defined.

Thus

$$\begin{aligned}
\|T_\mu u\| \geq T_\mu u(1) &= \mu \int_0^1 G(1, s) g(s) f(u(s)) ds \\
&\geq \mu \int_{\frac{1}{2}}^1 G(1, s) g(s) f(u(s)) ds.
\end{aligned}$$

Now, since $u \in \mathcal{P}$ and $s \in [1/2, 1]$, the last expression is greater than or equal to

$$\mu \min_{u \in [\frac{\lambda}{\alpha 2^{\alpha-1}} r, r]} \{f(u)\} \int_{\frac{1}{2}}^1 G(1, s) g(s) ds \geq r = \|u\|,$$

i.e., $\|T_\mu u\| \geq \|u\|$. □

Lemma 4.4. *Assume that conditions (H1), (H2) and (H3) are satisfied. Then if $f_0 = \infty$, there exists $r_0(\mu) > 0$ such that, for every $0 < r < r_0(\mu)$, we have*

$$\|T_\mu u\| \geq \|u\|, \text{ for } u \in P \text{ with } \|u\| = r.$$

Proof. Since $f_0 = \infty$, then there exists a constant $r_0(\mu) > 0$ such that $f(u) \geq Lu$, for all $0 < u \leq r_0(\mu)$, where $L = L(\mu) > 0$ is defined by

$$L = L(\mu) = \frac{\alpha}{\mu \lambda \int_{\frac{1}{2}}^1 G(1, s) g(s) s^{\alpha-1} ds},$$

L it is well defined by virtue of the condition (H2).

Fix $0 < r < r_0(\mu)$, and let $u \in P$ be such that $\|u\| = r$. Then, for all $t \in [0, 1]$ we have that

$$T_\mu u(t) \geq t^{\alpha-1} \mu \int_0^1 G(1, s) g(s) f(u(s)) ds.$$

As a consequence

$$\begin{aligned} \|T_\mu u\| &\geq \max_{t \in [0,1]} \left\{ t^{\alpha-1} \mu \int_0^1 G(1, s) g(s) f(u(s)) ds \right\} \\ &= \mu \int_0^1 G(1, s) g(s) f(u(s)) ds \\ &\geq \mu L \int_0^1 G(1, s) g(s) u(s) ds \\ &\geq \mu L \int_{\frac{1}{2}}^1 G(1, s) g(s) u(s) ds \\ &\geq \mu L \frac{\lambda}{\alpha} \|u\| \int_{\frac{1}{2}}^1 G(1, s) g(s) s^{\alpha-1} ds \\ &= \|u\|. \end{aligned}$$

□

Lemma 4.5. *Assume that the hypotheses (H1), (H2) and (H3) hold. Then, if $f_0 = 0$, there exists $r_0(\mu) > 0$ such that, for every $0 < r < r_0(\mu)$, we have*

$$\|T_\mu u\| \leq \|u\|, \text{ for } u \in P \text{ with } \|u\| = r.$$

Proof. Since $f_0 = 0$, for

$$\epsilon = \epsilon(\mu) = \frac{(\alpha - \lambda) \Gamma(\alpha - 1)}{\mu \int_0^1 g(s) ds} > 0,$$

there exists $r_0(\mu) > 0$ such that $f(u) \leq \epsilon u$, for each $0 < r \leq r_0(\mu)$.

Fix $0 < r < r_0(\mu)$, and let $u \in P$ with $\|u\| = r$. Then, for every $t \in [0, 1]$, it is fulfilled

$$\begin{aligned}
T_\mu u(t) &= \mu \int_0^1 G(t, s) g(s) f(u(s)) ds \\
&\leq \frac{\mu}{(\alpha - \lambda) \Gamma(\alpha - 1)} \int_0^1 g(s) f(u(s)) ds \\
&\leq \frac{\mu}{(\alpha - \lambda) \Gamma(\alpha - 1)} \int_0^1 g(s) \epsilon u(s) ds \\
&\leq \frac{\mu}{(\alpha - \lambda) \Gamma(\alpha - 1)} \epsilon \|u\| \int_0^1 g(s) ds \\
&= \|u\|.
\end{aligned}$$

As consequence, we conclude that $\|T_\mu u\| \leq \|u\|$. \square

Lemma 4.6. *Assume that conditions (H1), (H2) and (H3) are satisfied. Then if $f_\infty = \infty$, there exists $R_0(\mu) > 0$ such that, for every $R \geq R_0(\mu)$, we have*

$$\|T_\mu u\| \geq \|u\|, \text{ for } u \in P \text{ with } \|u\| = R.$$

Proof. Since $f_\infty = \infty$, for

$$L = L(\mu) = \frac{\alpha}{\mu \lambda \int_{\frac{1}{2}}^1 G(1, s) g(s) s^{\alpha-1} ds} > 0$$

there exists $R_1(\mu) > 0$ such that $f(u) \geq Lu$, for each $u \geq R_1(\mu)$.

Define now

$$R_0(\mu) := \frac{2^{\alpha-1} \alpha}{\lambda} R_1(\mu) > R_1(\mu)$$

Fix $R \geq R_0(\mu)$, and let $u \in P$ with $\|u\| = R$. Then, for all $t \in [0, 1]$, the following inequalities hold:

$$\begin{aligned}
\|T_\mu u\| &\geq \max_{t \in [0, 1]} \left\{ t^{\alpha-1} \mu \int_0^1 G(1, s) g(s) f(u(s)) ds \right\} \\
&= \mu \int_0^1 G(1, s) g(s) f(u(s)) ds \\
&\geq \mu \int_{\frac{1}{2}}^1 G(1, s) g(s) f(u(s)) ds.
\end{aligned}$$

Note that, for all $s \in [\frac{1}{2}, 1]$ we have

$$\begin{aligned} u(s) &\geq s^{\alpha-1} \frac{\lambda}{\alpha} \|u\| \geq \left(\frac{1}{2}\right)^{\alpha-1} \frac{\lambda}{\alpha} \|u\| \\ &= \left(\frac{1}{2}\right)^{\alpha-1} \frac{\lambda}{\alpha} R \geq \left(\frac{1}{2}\right)^{\alpha-1} \frac{\lambda}{\alpha} R_0 = R_1, \end{aligned}$$

and so

$$\begin{aligned} \|T_\mu u\| &\geq \mu L \int_{\frac{1}{2}}^1 G(1, s) g(s) u(s) ds \\ &\geq \mu L \frac{\lambda}{\alpha} \|u\| \int_{\frac{1}{2}}^1 G(1, s) g(s) s^{\alpha-1} ds \\ &= \|u\|. \end{aligned}$$

□

Lemma 4.7. *Assume that conditions (H1), (H2) and (H3) are satisfied. Then if $f_\infty = 0$ then, there exists $R_0(\mu) > 0$ such that for every $R \geq R_0(\mu)$, we have*

$$\|T_\mu u\| \leq \|u\|, \text{ for } u \in P \text{ with } \|u\| = R.$$

Proof. Since $f_\infty = 0$, for

$$\epsilon(\mu) = \frac{(\alpha - \lambda) \Gamma(\alpha - 1)}{2\mu \int_0^1 g(s) ds}$$

there exists $R_1(\mu) > 0$ such that $f(u) \leq \epsilon u$ for each $u \geq R_1(\mu)$.

Let

$$R_0(\mu) := \max \left\{ \frac{2^{\alpha-1} \alpha}{\lambda} R_1(\mu), 2\gamma \right\},$$

where

$$\gamma = \frac{\mu}{(\alpha - \lambda) \Gamma(\alpha - 1)} \max_{u \in [0, R_1]} \{f(u)\} \int_0^1 g(s) ds$$

Note that $R_0(\mu) > R_1(\mu)$.

Fix $R \geq R_0(\mu)$, and let $u \in P$ with $\|u\| = R$. Then, for any $t \in [0, 1]$ we obtain

$$\begin{aligned}
T_\mu u(t) &= \mu \int_0^1 G(t, s) g(s) f(u(s)) ds \\
&\leq \frac{\mu}{(\alpha - \lambda) \Gamma(\alpha - 1)} \max_{u \in [0, R]} \{f(u)\} \int_0^1 g(s) ds \\
&\leq \frac{\mu}{(\alpha - \lambda) \Gamma(\alpha - 1)} \left(\max_{u \in [0, R_1(\mu)]} \{f(u)\} + \max_{u \in [R_1(\mu), R]} \{f(u)\} \right) \int_0^1 g(s) ds \\
&\leq \frac{\mu}{(\alpha - \lambda) \Gamma(\alpha - 1)} \left(\max_{u \in [0, R_1(\mu)]} \{f(u)\} + \epsilon R \right) \int_0^1 g(s) ds \\
&\leq \gamma + \frac{R}{2} < \frac{R}{2} + \frac{R}{2} \\
&= R = \|u\|,
\end{aligned}$$

and thus $\|T_\mu u\| \leq \|u\|$. □

Now, after these technical lemmas, we are in a position to prove the following existence, non existence and multiplicity result for problem (4.1). We refer to a positive solution of problem (4.1) any solution u of such a problem with $u(t) > 0$, for all $t \in (0, 1]$.

Theorem 4.2. *Assume that conditions (H1), (H2) and (H3) are satisfied. Then, the following results hold:*

1. *If $f_0 > 0$ and $f_\infty > 0$ then there exists $\mu_0 > 0$ such that problem (4.1) has no positive solutions for all $\mu > \mu_0$.*
2. *If $f_0 = \infty$ or $f_\infty = \infty$, then there exists $\mu_0 > 0$ such that problem (4.1) has a positive solution for every $0 < \mu < \mu_0$.*
3. *If $f_\infty = 0$, then there exists $\mu_0 > 0$ such that problem (4.1) has a positive solution for every $\mu > \mu_0$.*
4. *If $f_0 = \infty$ and $f_\infty = 0$, then problem (4.1) has a positive solution for every $\mu > 0$.*
5. *If $f_0 = \infty$ and $f_\infty = \infty$, then there exists $\mu_0 > 0$ such that problem (4.1) has two positive solutions for every $0 < \mu < \mu_0$.*
6. *If $f_0 = 0$ and $f_\infty = \infty$, then problem (4.1) has a positive solution for every $\mu > 0$.*

7. If $f_0 = 0$ and $f_\infty = 0$, then there exists $\mu_0 > 0$ such that problem (4.1) has two positive solutions for every $\mu > \mu_0$.

Proof. First, note that any solution u of problem (4.1) satisfies, in virtue of Lemma 4.1 (vii), the following inequalities for all $t \in [0, 1]$:

$$\begin{aligned} u(t) = T_\mu u(t) &= \mu \int_0^1 G(t, s) g(s) f(u(s)) ds \\ &\geq \mu t^{\alpha-1} \int_0^1 G(1, s) g(s) f(u(s)) ds \\ &= t^{\alpha-1} T_\mu u(1) = t^{\alpha-1} u(1). \end{aligned}$$

On the other hand, if $u \in \mathcal{P}$, then

$$u(1) \geq \lambda \|u\|/\alpha.$$

In particular, any solution $u \in \mathcal{P}$ of problem (4.1) satisfies

$$u(t) \geq t^{\alpha-1} \lambda \|u\|/\alpha, \quad \text{for all } t \in [0, 1],$$

and, if it is nontrivial on $[0, 1]$, then it is strictly positive on $(0, 1]$.

As a consequence, since in all the considered situations in the statements, the solutions of problem (4.1) are obtained as nontrivial fixed points of operator T_μ in the cone \mathcal{P} , we deduce that all of them are strictly positive in $(0, 1]$.

Let's prove the first assertion of the result.

Since $f_0 > 0$ and $f_\infty > 0$ there exists $L > 0$ such that $f(u) \geq L u$, for all $u > 0$.

Define

$$\bar{\mu} := \frac{\alpha}{L \lambda \int_{\frac{1}{2}}^1 G(1, s) g(s) s^{\alpha-1} ds}. \quad (4.7)$$

Consider $\mu > \mu_0$. If there is a nontrivial solution $u \in \mathcal{P}$ of problem (4.1) we have that

$$\begin{aligned}
\|u\| = \|T_\mu u\| &= \max_{t \in [0,1]} \left\{ \mu \int_0^1 G(t, s) g(s) f(u(s)) ds \right\} \\
&\geq \max_{t \in [0,1]} \left\{ \mu t^{\alpha-1} \int_0^1 G(1, s) g(s) f(u(s)) ds \right\} \\
&= \mu \int_0^1 G(1, s) g(s) f(u(s)) ds \\
&\geq \mu L \int_0^1 G(1, s) g(s) u(s) ds \\
&\geq \mu L \int_{\frac{1}{2}}^1 G(1, s) g(s) u(s) ds \\
&\geq \mu L \frac{\lambda}{\alpha} \int_{\frac{1}{2}}^1 G(1, s) g(s) s^{\alpha-1} \|u\| ds \\
&> \|u\|,
\end{aligned}$$

which is a contradiction.

The rest of the cases follow by combining Lemmas 4.2 - 4.7 together with Theorem 1.1 one or twice. The proofs use similar arguments to the ones given in [18, Theorem 2.14] and we omit them. \square

Remark 4.1. *It is immediate to verify that, instead of condition (H2), we can assume the weaker one:*

(H2') $g \in L^1[0, 1]$, $g(t) \geq 0$ for a.e. $t \in [0, 1]$ and there is a $a \in (0, 1)$ such that $\int_a^1 g(s) ds > 0$.

The proofs follow by a direct adaptation of the ones given here. In this case the solutions belong to the cone:

$$\mathcal{P}' = \left\{ u \in E, u(t) \geq 0 \text{ for all } t \in [0, 1], u(t) \geq \frac{t^{\alpha-1} \lambda}{\alpha} \|u\|, \text{ for all } t \in [a, 1] \right\}.$$

4.4 Examples

To illustrate our results we shall give three examples where we apply Theorem 4.2. We point out that we are able to get some estimates of the values μ_0 mentioned in Theorem 4.2. In all those cases it is immediate to verify that conditions (H1), (H2) and (H3) are fulfilled.

Example 4.1. Consider the following integral boundary value problem

$$\begin{cases} D^{\frac{5}{2}}u(t) + \mu g(t)f(u(t)) = 0 & 0 \leq t \leq 1 \\ u(0) = u'(0) = 0, u(1) = 2 \int_0^1 u(s)ds, \end{cases} \quad (4.8)$$

where

$$f(x) = \sqrt{x} + \log(x^2 + 1)$$

and

$$g(t) = t^2.$$

By a direct calculation, we obtain $f_0 = \infty$ and $f_\infty = 0$. From Theorem 4.2, part 4, we conclude that problem (4.8) has a positive solution for every $\mu > 0$.

Example 4.2. Consider the fractional differential equation (4.8) with

$$f(x) = e^{-1/x}(\sqrt{x} + \sin x)$$

and

$$g(t) = 10\sqrt{t}.$$

It is not difficult to verify that $f_0 = 0$ and $f_\infty = 0$. In consequence, Theorem 4.2, part 7, is applicable and we can ensure the existence of two positive solutions of this problem for μ large enough.

Moreover, we can deduce sharper results, in the sense that it is possible to get estimates on the values of the parameters for which the multiplicity result holds. To this end, let $r_1 = 1$ and $r_2 = 2$. By evaluating such values in (4.6) we obtain that

$$\mu_0(1) \approx 38.1972 \quad \text{and} \quad \mu_0(2) \approx 4.10508.$$

Now, Lemma 4.3 tell us that

$$\|T_\mu u\| \geq \|u\|, \quad \text{if } \|u\| = r_i \quad \text{for all } \mu \geq \mu_0(r_i), \quad i = 1, 2.$$

Now, from Lemma 4.5 we have that for any $\mu > 0$ given, there is $r_0(\mu) > 0$ for which

$$\|T_\mu u\| \leq \|u\|, \quad \text{if } \|u\| = r \quad \text{for all } r \leq r_0(\mu).$$

In particular, from Theorem 1.1, we have that for any $\mu \geq 38.1972$ there is a positive solution u_1 of problem (4.8) that satisfies

$$0 < \|u_1\| \leq 1.$$

Using now Lemma 4.7 we deduce that if $\mu \geq 4.10508$ then problem (4.8) has a positive solution u_2 such that

$$2 \leq \|u_2\|.$$

Obviously we can ensure the existence of two solutions for $\mu \geq 38.1972$.

Example 4.3. Consider the integral boundary value problem (4.8) with

$$f(x) = e^x$$

and

$$g(t) = \frac{t^2}{2}.$$

Clearly, $f_0 = \infty$ and $f_\infty = \infty$.

As consequence, part 1 of Theorem 4.2 ensures that for μ large enough there is no solution of the considered problem. Moreover Theorem 4.2, part 4 implies that this problem has two positive solutions for positive and small enough μ .

Since

$$f(x)/x \geq e \quad \text{for all } x > 0,$$

we obtain that the expression (4.7) remains $\bar{\mu} \approx 25.3927$. As a consequence, we have that for $\mu > 25.3927$ problem (4.8) has no positive solution.

On the other hand, let $R_1 = 1$ and $R_2 = 2$. So

$$\mu_0(1) \approx 0.978074 \quad \text{and} \quad \mu_0(2) \approx 0.719627$$

in (4.5). Thus, from Theorem 1.1 and Lemma 4.4, we deduce, for all $\mu \leq \mu_0(1)$, the existence of a positive solution u_1 such that

$$0 < \|u_1\| \leq 1.$$

Analogously, from Lemma 4.6 we have that if $\mu \leq \mu_0(2)$ then problem (4.8) has a positive solution u_2 satisfying

$$2 \leq \|u_2\|.$$

As consequence, we can ensure the existence of the two positive solutions u_1 and u_2 for all $\mu \leq 0.719627$.

Conclusion

In this thesis we have developed new theoretical results on the existence of solutions for some classes of nonlinear fractional differential equations. In Chapter 1 we introduced necessary tools from functional analysis that enabled us to carry out our study.

In Chapter 2 we considered a class of nonlinear fractional differential equations coupled with Dirichlet boundary conditions. The left-hand side of the equation is linear and depends on a parameter λ . First, we studied the sign of the Green's function of the associated linear problem; namely, we have proved that it is positive if and only if the parameter λ is in the interval (λ_1, ∞) , where λ_1 is the first eigenvalue of the linear homogeneous problem. The proof is not elementary because in this case the Green's function is expressed in terms of Mittag-Leffler functions. Next, we proved more additional inequalities for the Green's function. Such properties are fundamental in the construction of a suitable cone in the space of the solutions. Finally, under two different conditions on the behavior of the function f at $u = 0$ and $u = \infty$, we proved the existence of at least one positive solution via a particular fixed point theorem [30]. In the case of non-homogeneous Dirichlet problem, we used the method of lower and upper solutions, to prove that if there are $\gamma \leq \delta$ a pair of lower and upper solutions, then the non-homogeneous Dirichlet problem has at least one solution lying between γ and δ .

In Chapter 3 we considered a nonlinear boundary value problem of fractional differential equation with integral boundary conditions. In this problem the related Green's function is more complex than in the case of usual boundary conditions. Besides, it depends on the parameter of the integral boundary condition. We made an exhaustive study of the properties of the Green's function. These properties enabled us to define the suitable cone

for which we applied the Guo-Krasnoselskii's fixed point theorem. The existence result was obtained under some usual conditions.

The considered problem in Chapter 4 is similar to the previous one, except that the fractional differential equation in this problem depends on the parameter μ . With almost similar arguments. Various existence and multiplicity results for positive solutions were derived depending on different values of the parameter μ .

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