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NEUTRAL NONLINEAR FUNCTIONAL EQUATIONS WITH MULTIPLE DELAYS AND STABILITY BY THE FIXED POINT TECHNIQUE

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Neutral nonlinear functional equations with multiple delays and stability by the fixed point technique

A Doctoral Thesis,

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Contents

Dedication

This work is dedicated to

My dear father,

Mother,

Brother,

Freinds....

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Abstract

Study of the stability of solutions of delay differential equations and delay dynamic equations plays an important role in the qualitative analysis of differential equations and dynamic equations. This thesis is devoted to using the fixed-point technique to prove the existence and stability of solutions of a class of delay integro-differential equations and to apply the Bohl-Perron theorem to prove the exponential stability of integro-dynamic equations with delay.

Keywords: Delay intego-differential equations, Intego-dynamic equations, Fixed point, Stability, Time scales, Bohl-Perron.

Mathematics Subject Classification: 34K20, 34K30, 34k40, 34N05, 45J05, 45D05.

ملخص

دراسة استقرار حلول المعادلات التفاضلية ذات تأخر و المعادلات الديناميكية ذات تأخر تلعب دورا هاما في التحليل النوعي للمعادلات التفاضلية و المعادلات الديناميكية. هذه الأطروحة مكرسة لاستخدام تقنية النقطة الثابتة لإثبات وجود واستقرار الحلول لفئة من المعادلات التفاضلية التكاملية ذات تأخر ولتطبيق نظرية بوهل-بيرون لإثبات الاستقرار الأسّي للمعادلات الديناميكية التكاملية ذات تأخر.

الكلمات المفتاحية : المعادلات التفاضلية التكاملية ذات تأخر، المعادلات الديناميكية التكاملية، النقطة الثابتة، الاستقرار، الزمن السلمي، بوهل-بيرون.

Résumé

L'étude de la stabilité de solutions des équations différentielles à retard et des équations dynamiques à retard joue un rôle important dans l'analyse qualitative des équations différentielles et des équations dynamiques. Cette thèse est consacrée à utiliser la technique de point fixe pour prouver l'éxistence et la stabilité de solutions d'une classe des équations intégro-différentielles avec retard et d'employer la théoreme de Bohl-Perron pour prouver la stabilité exponentielle des équations intégro-dynamiques avec retard.

Mots-clés: Equations différentielles à retard, Equations intégro-dynamiques, Points fixe, Stabilité, Echelles de temps, Bohl-Perron.

Mathematics Subject Classification: 34K20, 34K30, 34k40, 34N05, 45J05, 45D05.

Chapter 1

Introduction

The theory of fixed point is one of the most powerful tools of modern mathematics. Theorem concerning the existence and properties of fixed points are known as fixed point theorem. Fixed point theory is a beautiful mixture of analysis, topology and geometry. In particular fixed point theorem has been applied in such field as mathematics engineering, physics, economics, game theory, biology and chemistry etc. Classical and major results in these areas are: Banach's fixed point theorem, Schauder's fixed point theorem and Krasnoselskii's fixed point theorem.

In 1886, Poincare [55] was the first to work in this field. Then Brouwer [21] in 1912, proved fixed point theorem for the solution of the equation $f(x) = x$. He also proved fixed point theorem for a square, a sphere and their n-dimensional counter parts which was further extended by Kakutani [45]. Meanwhile Banach principle came into existence which was considered as one of the fundamental principles in the field of functional analysis. In 1922, Banach [15] proved that a contraction mapping in the field of a complete metric space possesses a unique fixed point.

In 1932, Krasnoselskii [57] studied a paper of Schauder on partial differential equations and formulated the working hypothesis principle: the inversion of a perturbed differential operator yields the sum of a contraction and a compact map. Accordingly, he formulated an hybrid theorem known under its name.

The Bohl–Perron theorem test goes back to Bohl [32] and was formulated by Perron [53] in the following form: if for any bounded forcing term $f \in C([t_0, +\infty), \mathbb{R})$, the solution $x \in C^1([t_0, +\infty), \mathbb{R})$ of the differential equation

$$x'(t) + A(t)x(t) = f(t) \quad \text{for } t \in [t_0, +\infty)$$

is bounded together with its derivative x', then the trivial solution of the equation

$$x'(t) + A(t)x(t) = 0 \quad \text{for } t \in [t_0, +\infty)$$

is uniformly exponentially stable. After the publication of [53], there appeared a corresponding paper [48] by Li, in which similar methods were used to find analogous results for difference equations.

Halanay in [40], Dalec'kii and Krein in [32] are given a detailed discussion of this criterion. The result was extended to the case of delay differential equations by Halanay in [40].

In particular, the Bohl–Perron theorem was applied to deduce explicit stability conditions, where different techniques are used to study stability of delay differential equations.

The theory of time scales was first introduced by Stefan Hilger [41] in 1988 in order to unify continuous and discrete analysis. Many results concerning differential equations carry over easily to the corresponding results for difference equations, while others seem to be completely different in nature. However, some physical systems are modelled by what is called dynamic equations because they are either differential equations, difference equations or a combination of both. This talk will first give an introduction to the study of time scales, and in particular, dynamic equations on time scales, which are studied to unify and generalize many results for differential and difference equations. Hence, time scales calculus provides a generalization of differential and difference analysis.

The study of Levin-Nohel equations brings the traditional research areas of differential and difference equations. It allows one to handle these two research areas at the same time, hence shedding light on the reasons for their seeming discrepancies. In fact, many new results for the continuous and discrete cases have been obtained by studying the more general time scales case (for example [2, 3, 5, 9, 14], [46]-[49]).

The Lyapunov direct method has been very effective in establishing stability results and the existence of periodic solutions for wide variety of ordinary, functional and partial differential equations. Nevertheless, in the application of Lyapunov's direct method to problems of stability in delay differential equations, serious difficulties occur if the delay is unbounded or if the equation has unbounded terms. In recent years, several investigators have tried stability by using a new technique. Particularly, Burton and Furumochi [29], Zhang [59] and others began a study in which they noticed that some of this difficulties vanish or might be overcome by means of fixed point theory. The fixed point theory does not only solve the problem on stability but has other significant advantage over Lyapunov's direct method.

The thesis is organized in five chapter as follows:

The second chapter presents elementary results, which are needed in this project, in fixed point, delay differential equations, stability of delay differential equations, time scale calculus and a brief survey of Lesbegue Δ-integral on time scales.

The third chapter introduces the results published in [6] and relates to study asymptotic stability results about the zero solution using the contraction mapping theorem for the following linear neutral Levin-Nohel integro-differential equation

$$x'(t) + \int_{t-\tau(t)}^{t} a(t,s)x(s)ds + c(t)x'(t-\tau(t)) = 0.$$

An asymptotic stability theorem with a necessary and sufficient condition is proved. In addition, the case of the equation with several delays is studied. The results obtained here extend the work of Dung [36].

The fourth chapter provides the asymptotic stability and stability results about the zero solution using Krasnoselskii-Burton's fixed point theorem for the following nonlinear neutral Levin-Nohel integro-differential equation

$$x'(t) + \int_{t-\tau(t)}^{t} a(t,s)g\left(x(s)\right)ds + c(t)x'(t-\tau(t)) = 0.$$

The results obtained here extend the work of Mesmouli, Ardjouni and Djoudi [52].

In the last chapter, we use Bohl-Perron theorem to establish new conditions for the exponential stability of Levin-Nohel intego-dynamic equation with variable delay having the form

$$x^{\Delta}(t) + \int_{t-r(t)}^{t} a(t,s)x(s)\Delta s = 0, \text{ for } t \in \mathbb{T}^{+}$$

The results obtained here extend the work of Dung [38].

Chapter 2

Preliminaries

The aim of this chapter is to introduce the basic concepts, notations, and elementary results that are used throughout the thesis. Moreover, the results in this chapter may be found in most standard books on functional analysis, for example ([19], [23], [25], [57], [58]).

2.1 Fixed Point and Delay differential Equations

2.1.1 Fixed Point theorem

Definition 2.1 A pair (S, d) is a metric space if S is a set and $d : S \times S \to [0, +\infty)$ such that when y, z, and u are in S then

(a) $d(y, z) \geq 0$, $d(y, y) = 0$, and $d(y, z) = 0$ implies $y = z$,

(b) $d(y, z) = d(z, y)$, and

(c) $d(y, z) \leq d(y, u) + d(u, z)$.

The metric space is complete if every Cauchy sequense in (S, d) has a limit in S. A sequence $\{x_n\} \subset S$ is a Cauchy sequence if for each $\varepsilon > 0$ there exists $N > 0$ such that $n, m > N$ imply $d(x_n, x_m) < \varepsilon$.

Example 2.1 Let $\phi : [a, b] \to \mathbb{R}^n$ be continuous and let S be the set of bounded continuous functions $f : [a, \infty) \to \mathbb{R}^n$ with $f(t) = \phi(t)$ for $a \leq t \leq b$. For $f, g \in S$ define $d(f, g) = \sup_{a \leq t \leq b} |f(t) - g(t)| =: \|f - g\|$. Then $(S, \|.\|)$ is a complete metric space.

Let C be space of continuous functions. Let

$$M = \{\phi : [a, +\infty) \to \mathbb{R} \backslash\ \phi \in C, |\phi(t)| \leq 1, \phi(t) \to 0 \text{ as } t \to \infty\}$$

and

$$Q = \{\phi : [a, +\infty) \to \mathbb{R} \backslash\ \phi \in C, |\phi(t)| \leq 1\}.$$

Let $\|.\|$ be the supremum metric and $|.|_h$ be a weighted metric defined by $h : [0, \infty) \to [1, +\infty)$, $h(0) = 1$, $h(t) \to \infty$ monotonically, and for $\phi \in M$ or Q then

$$|\phi|_h = \sup_{t \geq 0} \frac{|\phi(t)|}{h(t)}.$$

Example 2.2 $(M, \|.\|)$ and $(Q, |.|_h)$ are a complete metric space.

Example 2.3 $(M, |.|_h)$ is not a complete metric space.

Proof. Let $\{\phi_n\}$ be the sequence of functions such that

$\phi_n(t) = 1$, $0 \leq t \leq n$,

$\phi_n(t) = 0$, $n + 1 \leq t < \infty$,

ϕ_n is linear and continuous on $[n, n + 1]$.

$\{\phi_n\}$ is a Cauchy sequence in this space.

Indeed, let $\varepsilon > 0$ be given. We must find N such that $n, m \geq N$, $t \in \mathbb{R}$ imply that

$$|\phi_n(t) - \phi_m(t)| \leq \varepsilon h(t). \tag{2.1}$$

Find T such that $\varepsilon h(T) > 2$. Clearly, for all n, m and all $t \geq T$ we have (2.1). Find $N > T$. For $n, m \geq N$ and $0 \leq t \leq N$ we have $\phi_n(t) = \phi_m(t) = 1$ so (2.1) holds. Then, $\{\phi_n\}$ is a Cauchy sequence. But, $\phi_n(t) \to 1$ as $n \to \infty$, and $1 \notin M$. ■

Definition 2.2 (compact) We say that the set L in a metric space (S, d) is compact if for each sequence $\{x_n\} \subset L$ has a subsequence with limit in L.

Definition 2.3 (uniformly bounded and Equicontinuous) Let U be an interval on $\mathbb{R}$ and let $\{f_n\}$ be a sequence of functions with $f_n : U \to \mathbb{R}^m$. Denote by $|.|$ any norm on $\mathbb{R}^m$.

(a) $\{f_n\}$ is uniformly bounded on U if there exists $M > 0$ such that $|f_n(t)| \leq M$ for all n and all $t \in U$.

(b) $\{f_n\}$ is equicontinuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $t_1,\, t_2 \in U$ and $|t_1 - t_2| < \delta$ imply $|f_n(t_1) - f_n(t_2)| < \varepsilon$ for all n.

Theorem 2.1 (Ascoli-Arzela, [22]) *if $\{f_n(t)\}$ is uniformly bounded and equicontinuous sequence of real functions on an interval $[a, b]$, then there is a subsequence which converges uniformly on $[a, b]$ to a continuous function.*

Definition 2.4 A vector space $(V, +, .)$ is a normed space if for each $x, y \in V$ there is a nonnegative real number $\|x\|$, called the norm of x, such

(i) $\|x\| = 0 \Leftrightarrow x = 0$,

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for $\alpha \in \mathbb{R}$, and

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

A normed space is a vector space and it is a metric space with $d(x, y) = \|x - y\|$. But a vector space with a metric is not always a normed space.

Example 2.4 The space $C([a, b], \mathbb{R}^n)$ consisting of all continuous function $f : [a, b] \to \mathbb{R}^n$ is a vector the reals.

Definition 2.5 A Banach space is a complete normed space.

Example 2.5 If $\|f\| = \max_{a \leq t \leq b} |f(t)|$, where $|.|$ is a norm in $\mathbb{R}^n$, then $(C, \|.\|)$ is a Banach space.

Theorem 2.2 (Contraction Mapping, [57]) *Let (S, d) be a complete metric space and let $P : S \to S$. if there is a constant $\alpha < 1$ such that for each pair $\phi_1, \phi_2 \in S$ we have*

$$d(P\phi_1, P\phi_2) \leq \alpha d(\phi_1, \phi_2),$$

then there is one and only one point $\phi \in S$ with $P\phi = \phi$.

Definition 2.6 (Large contraction) Let $(\mathbb{M}, d)$ be a metric space and $B : \mathbb{M} \to \mathbb{M}$. B is a large contraction if for each pair $\phi,\, \psi \in \mathbb{M}$ with $\phi \neq \psi$ then $d(B\phi, B\psi) < d(\phi, \psi)$ and if for each $\varepsilon > 0$ there exists $\delta < 1$ such that

$$[\phi, \psi \in \mathbb{M},\ d(\phi, \psi) \geq \varepsilon] \Rightarrow d(B\phi, B\psi) < \delta d(\phi, \psi).$$

Example 2.6 If $\|.\|$ is the supremum metric, if

$$\mathbb{M}=\left\{\phi:[0,\infty)\to\mathbb{R}\backslash\ \phi\in C,\ \ \|\phi\|\leq\frac{\sqrt{3}}{3}\right\},$$

and if $(H\phi)(t)=\phi(t)-\phi^{3}(t)$, then H is a large contraction of the set $\mathbb{M}$.

Proof. In the following computation, φ,ψ are evaluated at each t. We have $D:=|H\varphi-H\psi|=\left|\phi-\phi^{3}-\psi+\psi^{3}\right|=|\varphi-\psi|\left|1-\left(\varphi^{2}+\varphi\psi+\psi^{2}\right)\right|$. Then for

$$|\varphi-\psi|^{2}=\varphi^{2}-2\varphi\psi+\psi^{2}\leq2\left(\varphi^{2}+\psi^{2}\right)$$

and for $\varphi^{2}+\psi^{2}<1$ we have

$$\begin{aligned} D &= |\varphi-\psi|\left(1+|\varphi\psi|-\left(\varphi^{2}+\psi^{2}\right)\right)\\ &= |\varphi-\psi|\left(1+\frac{\varphi^{2}+\psi^{2}}{2}-\left(\varphi^{2}+\psi^{2}\right)\right)\\ &= |\varphi-\psi|\left(1-\frac{\varphi^{2}+\psi^{2}}{2}\right). \end{aligned}$$

For a given $\varepsilon\in(0,1)$, let $\varphi,\psi\in\mathbb{M}$ with $\|\varphi-\psi\|\geq\varepsilon$.

(a) Suppose that for some t we have

$$\frac{\varepsilon}{2}\leq|\varphi(t)-\psi(t)|$$

so that

$$\left(\frac{\varepsilon}{2}\right)^{2}\leq|\varphi(t)-\psi(t)|^{2}\leq2\left(\varphi^{2}(t)+\psi^{2}(t)\right)$$

or

$$\frac{\varepsilon^{2}}{8}\leq\varphi^{2}(t)+\psi^{2}(t).$$

For all such t we have

$$|(H\varphi)(t)-(H\psi)(t)|\leq|\varphi(t)-\psi(t)|\left(1-\frac{\varepsilon^{2}}{16}\right).$$

(b) Suppose that for some t we have $|\varphi(t)-\psi(t)|\leq\frac{\varepsilon}{2}$. Then

$$|(H\varphi)(t)-(H\psi)(t)|\leq|\varphi(t)-\psi(t)|\leq\frac{1}{2}\|\varphi-\psi\|.$$

Thus, for all t we have

$$|(H\varphi)(t)-(H\psi)(t)|\leq\min\left(\frac{1}{2},1-\frac{\varepsilon^{2}}{16}\right)\|\varphi-\psi\|.$$

■

Theorem 2.3 (Schauder, [22, 57, 58]) *Let $\mathbb{M}$ be a nonempty compat convex subset of a Banach space and let $P : \mathbb{M} \to \mathbb{M}$ be continuous. then P has a fixed point in $\mathbb{M}$.*

Theorem 2.4 *Let $\mathbb{M}$ be a nonempty convex subset of a normed space and let $P : \mathbb{M} \to K$ where K is a compact subset of $\mathbb{M}$. Then P has a fixed point in K.*

Krasnoselskii's found the solution by combining the two theorems of Banach and that of Schauder in one hybrid theorem which bears its name. In light, it establishes the following result [57].

Theorem 2.5 (Krasnoselskii, [57]) *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(S, \|.\|)$. Suppose that A and B map $\mathbb{M}$ into S such that*

(i) $Ax + By \in \mathbb{M}\,(\forall x, y \in \mathbb{M})$

(ii) A is continuous and $A\mathbb{M}$ is contained in a compact set,

(iii) B is a contraction with constant $\alpha < 1$.

Then there is a $z \in \mathbb{M}$ with $Az + Bz = z$.

Proof. By (iii) we have

$$\begin{aligned}
\|(I-B)\,x-(I-B)\,y\| &= \|(x-y)-(Bx-By)\| \\
&\leq \|x-y\|+\|Bx-By\| \\
&\leq \|x-y\|+\alpha\,\|x-y\| \\
&\leq (1+\alpha)\,\|x-y\|\,,
\end{aligned}$$

in the otherwise, we have

$$\begin{aligned}
\|(I-B)\,x-(I-B)\,y\| &= \|(x-y)-(Bx-By)\| \\
&\geq \|x-y\|-\|Bx-By\| \\
&\geq \|x-y\|-\alpha\,\|x-y\| \\
&\geq (1-\alpha)\,\|x-y\|\,.
\end{aligned}$$

Then

$$(1-\alpha)\,\|x-y\| \leq \|(I-B)\,x-(I-B)\,y\| \leq (1+\alpha)\,\|x-y\|\,.$$

This shows that $(I - B) : \mathbb{M} \to (I - B)\,\mathbb{M}$ is continuous and bijective. Thus, $(I - B)^{-1}$ exist and is continuous. Let $U := (I - B)^{-1} A$. It is clear that U is compact mapping, because U is a composition of a continuous mapping with a compact. Under the theorem of Schauder, U has a fixed point, i.e.

$$\exists z \in \mathbb{M} \text{ such that } (I - B)^{-1} Az = z.$$

This is equivalent to $z = Az + Bz$. ■

Remark 2.1 Note that if $A = 0$, the theorem becomes the theorem of Banach. If $B = 0$ then the theorem is not other than the Schauder theorem.

2.1.2 Delay differential Equations

Let a real number $\tau > 0$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|.|$, $C([a, b], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[a, b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. Suppose $C = C([-\tau, 0], \mathbb{R}^n)$ and designate the norm of an element ϕ in C by $|\phi| = \sup_{-\tau<\theta<0} |\phi(\theta)|$. Even though single bars are used for norms in different spaces, no confusion should arise. If

$$t_0 \in \mathbb{R},\ A \geq 0 \text{ and } x \in C([t_{0.} - \tau, t_0 + A], \mathbb{R}^n),$$

then for any $t \in [t_{0.}, t_0 + A]$, we let $x_t \in C$ defined by $x_t(\theta) = x(t + \theta)$ with $-\tau \leq \theta \leq 0$.

Let a function $g : \mathbb{R} \times C \to \mathbb{R}^n$. A functional differential equation is given by the following relation

$$\begin{cases} x'(t) = g(t, x_t),\ t \geq t_0, \\ x_{t_0} = \phi, \end{cases} \tag{2.2}$$

Definition 2.7 x is said to be a solution of (2.2) if there are $t_0 \in \mathbb{R}$, $A > 0$ such that $x \in C([t_{0.} - \tau, t_0 + A], \mathbb{R}^n)$ and x satisfies (2.2) for $t \in [t_0, t_0 + A]$. In such a case we say that x is a solution of (2.2) on $[t_{0.} - \tau, t_0 + A]$ for a given $t_0 \in \mathbb{R}$ and $\phi \in C$, we say that $x = x(t, t_0, \phi)$ is a solution of (2.2) with initial value at t_0 or simply a solution of (2.2) through (t_0, ϕ) if there is an $A > 0$ such that $x(t, t_0, \phi)$ is a solution of (2.2) on $[t_{0.} - \tau, t_0 + A]$ and $x_{t_0}(t, t_0, \phi) = \phi$.

Equation (2.2) is a very general type of equation and includes ordinary differential equations $(\tau = 0)$.

We say Equation (2.2) is linear if $g(t, x_t) = L(t, \phi) + h(t)$ where $L(t, \phi)$ is lenear in ϕ, is homogeneous if $h = 0$ and nonhomogeneous if $h \neq 0$. We claim Equation (2.2) is autonomous if $g(t, \phi) = f(\phi)$.

Example 2.7 Let the following delay differential equations:

$$x^{'}(t) = x(t) + 7x(t-2). \tag{2.3}$$

The equation (2.3) is an linear autonomous delay differential equation with constant $\tau = 2$.

$$x'(t) = \cos(t)\, x(t) + \sin(t)\, x'(t-1) + e^t. \tag{2.4}$$

The equation (2.4) is nonhomogeneous, linear nonautonomous delay functional differential equations.

$$x'(t) = \int_{-\tau}^{0} x(t+s)\, ds. \tag{2.5}$$

The equation (2.5) is a delay linear integro-differential equation.

Lemma 2.1 ([39]) *Let $t_0 \in \mathbb{R}$ and $\phi \in C$ be given and g be continuous on the product $\mathbb{R} \times C$. Then, finding a solution of equation (2.2) through (t_0, ϕ) is equivalent to solving*

$$x(t) = \psi(t_0) + \int_{t_0}^{t} g(s, x_s)\, ds, \quad t \geq t_0 \text{ and } x_{t_0} = \phi.$$

Lemma 2.2 ([39]) *If $x \in C([t_{0.} - \tau, t_0 + A], \mathbb{R}^n)$, then, x_t is a continuous function of t for $t \in [t_0, t_0 + A]$.*

Proof. Since x is continuous on $[t_{0.} - \tau, t_0 + A]$, it is uniformly continuous and thus $\forall \varepsilon > 0$, $\exists \eta > 0$, if $|t - s| < \eta$ then $|x(t) - x(s)| < \varepsilon$. Consequently for t, s in $[t_0, t_0 + A]$, $|t - s| < \eta$, we have $|x(t+\theta) - x(s+\theta)| < \varepsilon$, $\forall \theta \in [-\tau, 0]$. ■

Theorem 2.6 (Existence [39]) *Let $\mathbb{M}$ be an open subset of $\mathbb{R} \times C$ and $g : \mathbb{M} \to \mathbb{R}^n$ be continuous. For any $(t_0, \phi) \in \mathbb{M}$, there exists a solution of equation (2.2) through (t_0, ϕ).*

Theorem 2.7 (Existence and uniqueness, [39]) *Let* $\mathbb{M}$ *be an open subset of* $\mathbb{R} \times C$ *and suppose that* $g : \mathbb{M} \to \mathbb{R}^n$ *be continuous and* $g(t, \phi)$ *be lipschitzian with respect to* ϕ *in every compact subset of* $\mathbb{M}$*. If* $(t_0, \phi) \in \mathbb{M}$*, then equation (2.2) has a unique solution passing through* (t_0, ϕ) *.*

2.1.3 Neutral delay differential equations

In order to define a general class of neutral delay differential equations ($NDDEs$) (or neutral functional differential equations ($NFDEs$)), we need the definition of atomic.

Definition 2.8 Suppose $\Omega \subseteq \mathbb{R} \times C$ is open with elements (t, ϕ). A function $\Phi : \Omega \to \mathbb{R}^n$ is said to be atomic at β on Ω if Φ is continuous together with its first and second Fréchet derivatives with respect to ϕ and Φ_ϕ, the derivative with respect to ϕ, is atomic at β on Ω.

Definition 2.9 Suppose $\Omega \subseteq \mathbb{R} \times C$ is open, $f : \Omega \to \mathbb{R}^n$, $\Phi : \Omega \to \mathbb{R}^n$ are given continuous functions with Φ atomic at zero. The equation

$$\frac{d}{dt}\Phi(t, x_t) = f(t, x_t), \tag{2.6}$$

is called the neutral delay differential equation $NDDE(\Phi, f)$.

Definition 2.10 A function x is said to be a solution of the $NDDE(\Phi, f)$ or Eq.(2.6), if there are $t_0 \in \mathbb{R}$, $A > 0$, such that $x \in C([t_0 - \tau, t_0 + A), \mathbb{R}^n)$, $(t, x_t) \in \Omega$, $t \in [t_0, t_0 + A)$, $\Phi(t, x_t)$ is continuously differentiable and satisfies Eq.(2.6) on $[t_0, t_0 + A)$. For a given $t_0 \in \mathbb{R}$, $\phi \in C$, and $(t_0, \psi) \in \Omega$, we say $x(t_0, \phi)$ is a solution of Eq.(2.6) with initial value ϕ at t_0, or simply a solution through (t_0, ϕ) ,if there is an $A > 0$ such that $x(t_0, \phi)$, is a solution of (2.6) on $[t_0 - \tau, t_0 + A)$ and $x_{t_0}(t_0, \phi) = \phi$.

Theorem 2.8 *(Existence) if* Ω *is an open set in* $\mathbb{R} \times C$ *and* $(t_0, \phi) \in \Omega$*, then there exists a solution of the* $NDDE(\Phi, f)$ *through* (t_0, ϕ)*.*

Theorem 2.9 *(Uniqueness). If* $\Omega \subseteq \mathbb{R} \times C$ *is open and* $f : \Omega \to \mathbb{R}^n$ *as Lipschitz in* ϕ *on compact sets of* Ω*, then, for any* $(t_0, \phi) \in \Omega$*, there exists a unique solution of the* $NDDE(\Phi, f)$ *through* (t_0, ϕ)*.*

For example

$$\begin{aligned} x'(t) &= -x'(t-2), \\ x'(t) &= x'(t-1) + [x'(t-5)+2]^2, \end{aligned}$$

are neutral delay differential equations.

2.1.4 Method of Steps

The method of steps is an elementary method that can be used to solve some $DDEs$ analytically. This method is usually discarded as being too tedious, but in some cases the tedium can be removed by using computer algebra. Consider the following general DDE :

$$x'(t) = a_0 x(t) + a_1 x(t-w_1) + ... + a_m x(t-w_m), \tag{2.7}$$

where $x(t) = F(t)$ on the initial interval $-\max(w_i) \leq t \leq 0$. Let $b = \min(w_i)$. Then it is clear that the values of $x(t-w_m)$ are known in the interval $0 \leq t \leq b$. These values are $F(t-w_m)$. Thus, for the interval $0 \leq t \leq b$ we have

$$x'(t) = a_0 x(t) + a_1 F(t-w_1) + ... + a_m F(t-w_m),$$

and so

$$x(t) = \int_0^t (a_0 y(v) + a_1 F(v-w_1) + ... + a_m F(v-w_m))\, dv + y(0).$$

Now that we know $x(t)$ on $[0, b]$ we can repeat this procedure to obtain $x(t)$ on the interval $b \leq t \leq 2b$. This is given by:

$$x(t) = \int_b^t (a_0 y(v) + a_1 F(v-w_1) + ...)\, dv + y(b). \tag{2.8}$$

This process can be continued indefinitely, so long as the integrals that occur can be evaluated without too much effort. It is this last restriction that usually causes people to give up on this method, because the tedium and length of the method quickly overwhelms a human computer. However, it turns out that for certain classes of problems, where the phenomenon of "expression swell" is not too serious, we can take the method quite far, with a computer algebra system to automate the solution of the tedious sub-problems.

Example 2.8 For an example of this method we look first at a very simple DDE

$$x'(t) = -x(t-1),$$

with $y(t) = F(t) = 1$ for $-1 \leq t \leq 0$. The solution in the interval $0 \leq t \leq 1$ is given by:

$$x(t) = \int_0^t -F(x-1)dx + x(0) = 1 - t.$$

Now we can solve for the solution in the interval $1 \leq t \leq 2$. This solution is given by:

$$x(t) = \int_1^t -F(t-1)dx + x(1) = \frac{t^2}{2} - 2t + \frac{3}{2}.$$

This method can be programmed in Maple using a simple for loop.

2.1.5 Concrete exemple of delay differential equations

In this subsection we give an example used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey.

Predator-prey population models (Lotka-Voltera)

Let $x(t)$ be the population at time t of some species of animal called prey and let $y(t)$ be the population of a predator species which lives off these prey. We assume that $x(t)$ would increase at a rate proportional to $x(t)$ if the prey were left alone, i.e., we would have $x'(t) = a_1 x(t)$, where $a_1 > 0$. However the predators are hungry, and the rate at which each of them eats prey is limited only by his ability to find prey. (This seems like a reasonable assumption as long as there are not too many prey available.) Thus we shall assume that the activities of the predators reduce the growth rate of $x(t)$ by an amount proportional to the product $x(t)y(t)$, i.e.,

$$x'(t) = a_1 x(t) - b_1 x(t)y(t),$$

where b_1 is another positive constant.

Now let us also assume that the predators are completely dependent on the prey as their food supply. If there were no prey, we assume $y'(t) = -a_2 y(t)$, where $a_2 > 0$, i.e., the predator species would die out exponentially. However, given food the predators breed at a rate proportional to their number and to the amount of food available to them. Thus we consider the pair of equations

$$\begin{aligned} x'(t) &= a_1 x(t) - b_1 x(t) y(t), \\ y'(t) &= -a_2 y(t) + b_2 x(t) y(t), \end{aligned} \tag{2.9}$$

where a_1, a_2, b_1, and b_2 are positive constants. This well-known model was invented and studied by Lotka [1920], [1925] and Volterra [1928], [1931].

Vito Volterra was trying to understand the observed fluctuations in the sizes of populations $x(t)$ of commercially desirable fish and $y(t)$ of larger fish which fed on the smaller ones in the Adriatic Sea in the decade from 1914 to 1923 see [34].

The sunflower equation

Somolinos (1978) has considered the equation

$$x'' + (a/r)x' + (b/r)\sin x(t-r) = 0,$$

and has obtained interesting results on the existence of periodic solutions. The study of this problem goes back to the early 1800s and has attracted much attention. It involves the motion of a sunflower plant see [22].

2.1.6 Stability of delay differential equations

We consider the system

$$\begin{cases} x'(t) = f(t, x_t), \\ f(t, 0) = 0, \end{cases} \tag{2.10}$$

where $f : \mathbb{R} \times C \to \mathbb{R}^n$, with $C = C([-\tau, 0], \mathbb{R}^n)$ the Banach space of continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^n$, $\tau > 0$ with with the supremum norm $\|\phi\| = \sup_{-\tau \leq t \leq 0} |\phi|$. We suppose that f is continuous and is supposed to satisfy all the conditions which guarantee a solution and we define

$$S(t) = \{\phi : [t-\tau, t] \to \mathbb{R}^n,\ \phi \text{ is countinuous}\}.$$

Definition 2.11 ([25]) The solution $x(t) = 0$ of (2.10) is

(i) Stable (in the sense of Lyapunov) at $t = t_0$ if for any $\varepsilon > 0$ there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$[\phi \in S(t_0), \ \ \|\phi\| < \delta \text{ and } t \geq t_0] \Longrightarrow |x(t, t_0, \phi)| < \varepsilon,$$

(ii) Uniformly stable if δ of (i) is independent of t_0,

(iii) Asymptotically stable if it is stable and if, $\forall t_1 \geq t_0$, $\exists \eta > 0$ such that

$$[\phi \in S(t_1), \ \ \|\phi\| < \eta \text{ and } t \geq t_1] \Longrightarrow |x(t, t_1, \phi)| \to 0 \ \text{ as } \ t \to +\infty.$$

2.2 Time scale calculus

The aim of this chapter is to introduce the basic concepts, notations, and elementary results that are used throughout the thesis. Moreover, the results in this chapter may be found in most standard books on time scale, [17, 18].

Definition 2.12 A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers.

Example 2.9 The sets $\mathbb{R}$, $\mathbb{N}$, $[1, 2] \cup [3, 4]$, $[1, 2] \cup \mathbb{Z}$ and

$$h\mathbb{Z} = \{hk : k \in \mathbb{Z}, h > 0\}$$

are examples of time scales.

The sets $\mathbb{Q}$, $\mathbb{R}\backslash\mathbb{Q}$, $\mathbb{C}$ and $(1, 3)$ are not a time scales.

Definition 2.13 Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

Let $\varnothing$ denotes the empty set, we put $\inf \varnothing = \sup \mathbb{T}$ (i.e.,$\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \varnothing = \inf \mathbb{T}$ (i.e.,$\rho(t) = t$ if $\mathbb{T}$ has a minimum t).

Definition 2.14 we say that t is :

(a) right-scattered, if $\sigma(t) > t$,

(b) right-dense, if $\sigma(t) = t$,

(c) left-scattered, if $\rho(t) < t$,

(d) left-dense, if $\rho(t) = t$,

(e) isolated, if $\rho(t) < t < \sigma(t)$,

(f) dense, if $\rho(t) = t = \sigma(t)$.

Definition 2.15 If $\mathbb{T}$ has a left-scattered maximum m, define $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$. In summary,

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if} \quad \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if} \quad \sup \mathbb{T} = \infty. \end{cases}$$

Finally, if $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)).$$

Definition 2.16 The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

Example 2.10 (i) If $\mathbb{T} = \mathbb{R}$, then for any $t \in \mathbb{R}$, we have

$$\begin{aligned} \sigma(t) &= \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t, \\ \rho(t) &= \sup\{s \in \mathbb{R} : s < t\} = \sup(-\infty, t) = t. \end{aligned}$$

Hense every point $t \in \mathbb{R}$ is dense. The graininess function μ given by

$$\mu(t) = \sigma(t) - t = 0.$$

(ii) If $\mathbb{T} = h\mathbb{Z}$ with $h \in \mathbb{R}^+$, for any $t \in h\mathbb{Z}$, we have

$$\begin{aligned} \sigma(t) &= \inf\{s \in h\mathbb{Z} : s > t\} = \inf\{t + h, t + 2h, ...\} = t + h, \\ \rho(t) &= \sup\{s \in h\mathbb{Z} : s < t\} = \sup\{..., t - 2h, t - h\} = t - h. \end{aligned}$$

Hense every point $t \in h\mathbb{Z}$ is isolated. The graininess function μ is

$$\mu(t) = \sigma(t) - t = h.$$

(ii) If $\mathbb{T} = \{2^n : n \in \mathbb{Z}\} \cup \{0\}$, then for any $t \in \mathbb{T}$, we have

$$\begin{aligned}\sigma(t) &= \inf\{s \in \mathbb{R} : s > t\} = \inf\left\{2^{n+1}, 2^{n+2}, ...\right\} = 2t, \\ \rho(t) &= \sup\{s \in \mathbb{Z} : s < t\} = \sup\left\{..., 2^{n-2}, 2^{n-1}\right\} = \frac{t}{2}.\end{aligned}$$

Hense every point $t \in \mathbb{T}$ is isolated. The graininess function μ is

$$\mu(t) = \sigma(t) - t = t.$$

2.2.1 Differentiation

Now we consider a function $f : \mathbb{T} \to \mathbb{R}$ and define the so-called delta (or Hilger) derivative of f at a point $t \in \mathbb{T}^k$.

Definition 2.17 Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|(f(\sigma(t)) - f(s)) - f^{\Delta}(t)(\sigma(t) - s)\right| \leq \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U.$$

We call $f^{\Delta}(t)$ the delta (or Hilger) derivative of f at t.

Moreover, we say that f is delta (or Hilger) differentiable (or in short: differentiable) on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^k$.

Theorem 2.10 *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$.*

the following:

(i) *If f is differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) *If t is right-dense, then f is differentiable at t with*

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

Example 2.11 We define the function $f:\mathbb{T}\rightarrow\mathbb{R}$ by $f(t)=t^2$:

(1) Let $\mathbb{T}=h\mathbb{Z}$, then

$$f^{\Delta}(t)=\frac{f^{\sigma}(t)-f(t)}{\mu(t)}=\frac{(t+h)^2-t^2}{h}=2t+h.$$

(2) Let $q>1$ and $\mathbb{T}=q^{\mathbb{Z}}=\{q^n:n\in\mathbb{Z}\}$, then $\sigma(t)=qt$ and

$$f^{\Delta}(t)=\frac{f^{\sigma}(t)-f(t)}{\mu(t)}=\frac{(qt)^2-t^2}{qt-t}=t(q+1).$$

Theorem 2.11 *Assume $f,g:\mathbb{T}\rightarrow\mathbb{R}$ are differentiable at $t\in\mathbb{T}^k$. Then*

(i) The sum $f+g:\mathbb{T}\rightarrow\mathbb{R}$ is differentiable at $t\in\mathbb{T}^k$ with

$$(f+g)^{\Delta}(t)=f^{\Delta}(t)+g^{\Delta}(t).$$

(ii) For any constant α, $\alpha f:\mathbb{T}\rightarrow\mathbb{R}$ is differentiable at $t\in\mathbb{T}^k$ with

$$(\alpha f)^{\Delta}(t)=\alpha f^{\Delta}(t).$$

(iii) The product $fg:\mathbb{T}\rightarrow\mathbb{R}$ is differentiable at $t\in\mathbb{T}^k$ with

$$(fg)^{\Delta}(t)=f^{\Delta}(t)g(t)+f^{\sigma}(t)g^{\Delta}(t)=f(t)g^{\Delta}(t)+f^{\Delta}(t)g^{\sigma}(t).$$

(iv) If $f(t)f(\sigma(t))\neq 0$, then $\frac{1}{f}$ is differentiable at $t\in\mathbb{T}^k$ with

$$\left(\frac{1}{f}\right)^{\Delta}(t)=-\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

(v) If $g(t)g(\sigma(t))\neq 0$, then $\frac{f}{g}$ is differentiable at $t\in\mathbb{T}^k$ and

$$\left(\frac{f}{g}\right)^{\Delta}(t)=\frac{f^{\Delta}(t)g(t)-f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Example 2.12 Let $\mathbb{T}=h\mathbb{Z}$, $f(t)=t^2$ and $g(t)=\frac{1}{t}$ then

$$\begin{aligned} f^{\Delta}(t) &= 2t+h \\ g^{\Delta}(t) &= -\frac{1}{t(t+h)}. \end{aligned}$$

2.2.2 Integration

Definition 2.18 A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 2.19 A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will De denoted

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

Definition 2.20 Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F is called pre-antiderivative of f. We define the indefinite integral of a regulated function f by

$$\int f(t)\,\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f. We define the Cauchy integral by

$$\int_a^b f(t)\,\Delta t = F(b) + F(a) \quad \text{for all } a, b \in \mathbb{T}.$$

In this case the function F is called antiderivative of f.The infinite integrals are defined as

$$\int_a^\infty f(s)\Delta s = \lim_{t\to\infty} \int_a^t f(s)\Delta s.$$

Theorem 2.12 *If $f \in C_{rd}$ and $t \in \mathbb{T}^k$, then*

$$\int_t^{\sigma(t)} f(s)\,\Delta s = \mu(t)\, f(t).$$

Proof. There exists an antiderivative function F such that

$$\begin{aligned} \int_t^{\sigma(t)} f(s)\,\Delta s &= F(\sigma(t)) - F(t) \\ &= \mu(t)\, F^{\Delta}(t) \\ &= \mu(t)\, f(t). \end{aligned}$$

■

Theorem 2.13 *If* a*,* b*,*$c \in \mathbb{T}$*,* $\alpha \in \mathbb{R}$*, and* $f, g \in C_{rd}$*, then*

(*i*) $\int_a^b (f(t) + g(t))\,\Delta t = \int_a^b f(t)\,\Delta t + \int_a^b g(t)\,\Delta t$,

(*ii*) $\int_a^b (\alpha f(t))\,\Delta t = \alpha \int_a^b f(t)\,\Delta t$,

(*iii*) $\int_a^b f(t)\,\Delta t = -\int_b^a f(t)\,\Delta t$,

(*iv*) $\int_a^b f(t)\,\Delta t = \int_a^c f(t)\,\Delta t + \int_c^b f(t)\,\Delta t$,

(*v*) $\int_a^b f(\sigma(t))\, g^{\Delta}(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)\, g(t)\,\Delta t$,

(*vi*) $\int_a^b f(t)\, g^{\Delta}(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)\, g(\sigma(t))\,\Delta t$,

(*vii*) $\int_a^a f(t)\,\Delta t = 0$.

2.2.3 The Exponential Function

Definition 2.21 We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided

$$1 + \mu(t)\, p(t) > 0 \quad \text{for all } t \in \mathbb{T}^k.$$

The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).$$

We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall t \in \mathbb{T}\}.$$

The generalized exponential function $e_p(t, s)$ is defined next.

Definition 2.22 Let $p \in \mathcal{R}$, then the generalized exponential function e_p is defined as the unique solution of the initial value problem

$$x^{\Delta}(t) = p(t)x(t), \; x(s) = 1, \; \text{where } s \in \mathbb{T}.$$

An explicit formula for $e_p(t, s)$ is given by

$$e_p(t, s) = \exp\left(\int_s^t \zeta_{\mu(\tau)}(p(\tau))\Delta\tau\right), \quad \text{for all } \; s, t \in \mathbb{T},$$

with

$$\zeta_h(\tau) = \begin{cases} \frac{\log(1+h\tau)}{h} & \text{if } h \neq 0 \\ \tau & \text{if } h = 0 \end{cases},$$

where log is the principal logarithm function and $\zeta_h(\tau)$ is called cylinder transformation.

Lemma 2.3 *Let $p \in \mathcal{R}$, then*

(i) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$,

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))\, e_p(t,s)$,

(iii) $e_p^{\Delta}(t,s) = p(t)e_p(t,s)$,

(iv) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$ *with* $\ominus p = -\frac{p}{1+\mu p}$,

(v) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$,

(vi) $e_p(t,s)e_p(s,r) = e_p(t,r)$.

Lemma 2.4 *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(\tau)\Delta\tau\right),$$

for all $t \in [s,\infty)_{\mathbb{T}}$. It follows from Bernoulli's inequality that for any time scale, if the constant $\lambda \in \mathcal{R}^+$, then

$$0 < e_{\ominus\lambda}(t,t_0) \leq \frac{1}{1+\lambda(t-t_0)},\ t \geq t_0.$$

It follows that

$$\lim_{t\to\infty} e_{\ominus\lambda}(t,t_0) = 0.$$

Theorem 2.14 (Variation of Constants) *Let $t_0 \in \mathbb{T}$, $p \in \mathcal{R}$ and $x_0 \in \mathbb{R}$. The unique solution of the initial value problem*

$$x^{\Delta} = -p(t)\,x^{\sigma} + f(t)\,, \quad x(t_0) = x_0$$

is given by

$$x(t) = e_{\ominus p}(t,t_0)x_0 + \int_{t_0}^t e_{\ominus p}(t,s)f(s)\,\Delta s.$$

2.2.4 Lebesgue Δ-integral on time scales

First, denoting for every $x, y \in \mathbb{R}$, as $[x, y) = \{t \in \mathbb{R},\ x \leq t < y\}$, we define a countably additive measure m_1 on the set

$$\mathcal{F}_1 = \left\{ \left[\widetilde{a}, \widetilde{b}\right) \cap \mathbb{T} : \widetilde{a}, \widetilde{b} \in \mathbb{T}, \widetilde{a} \leq \widetilde{b} \right\},$$

that assigns to each interval $\left[\widetilde{a}, \widetilde{b}\right) \cap \mathbb{T} \in \mathcal{F}_1$ its length, that is,

$$m_1\left(\left[\widetilde{a}, \widetilde{b}\right)\right) = \widetilde{b} - \widetilde{a}.$$

Using m_1, they generate the outer measure m_1^* on $\mathcal{P}(\mathbb{T})$, defined for each $E \in \mathcal{P}(\mathbb{T})$ as

$$m_1^*(E) = \begin{cases} \inf_{\widetilde{R}} \sum_{i \in \widetilde{R}} \left(\widetilde{b} - \widetilde{a}\right) \in \mathbb{R}^+, & \text{if } b \notin E, \\ +\infty, & \text{if } b \in E, \end{cases}$$

with

$$\widetilde{R} = \left\{ \left\{ \left[\widetilde{a}, \widetilde{b}\right) \cap \mathbb{T} \in \mathcal{F}_1 \right\} : I_{\widetilde{R}} \subset \mathbb{N},\ E \subset \cup_{i \in \widetilde{R}} \left(\left[\widetilde{a}_i, \widetilde{b}_i\right) \cap \mathbb{T} \right) \right\}.$$

A set $A \subset \mathbb{T}$ is said to be Δ-measurable if the following equality

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T} \backslash A)),$$

holds for all subset E of $\mathbb{T}$.

Now, defining the family

$$\mathcal{M}(m_1^*) = \{A \subset \mathbb{T} : A \text{ is } \Delta\text{-measurable}\},$$

the Lebesgue Δ-measure, denoted by μ_Δ, is the restriction of m_1^* to $\mathcal{M}(m_1^*)$.

Proposition 2.1 ([31]) *The Lebesgue Δ-measure is defined over the Lebesgue measurable subsets of $\mathbb{T}$, moreover, it satisfies the following equality:*

$$\mu_\Delta = \begin{cases} \lambda + \sum_{i \in I} (\sigma(t_i) - t_i)\, \delta_{t_i} + \mu_M, & \textit{if } M \in \mathbb{T} \\ \lambda + \sum_{i \in I} (\sigma(t_i) - t_i)\, \delta_{t_i}, & \textit{if } M \notin \mathbb{T}, \end{cases}$$

where $\{t_i\}_{i \in I}$, $I \in \mathbb{N}$, is the set of all right-scattered points of $\mathbb{T}$, M is the supremum of $\mathbb{T}$, λ is the Lebesgue measure, δ_{t_i} is the Dirac measure concentrate at t_i, and μ_M is a degenerate measure defined as $\mu_M(A) = 0$ if $M \notin A$ and $\mu_M(A) = +\infty$ if $M \in A$.

Definition 2.23 ([31]) Let $\overline{\mathbb{R}} = [-\infty, +\infty]$ .We say that $f : \mathbb{T} \to \overline{\mathbb{R}}$ is Δ-musearable if for every $\alpha \in \mathbb{R}$, the set

$$f^{-1}([-\infty, \alpha)) = \{t \in \mathbb{T} : f(t) < \alpha\}$$

is Δ-musearable.

Definition 2.24 ([31]) We say that $S : \mathbb{T} \to \mathbb{R}$ is simple if it only takes a finite number of values $\alpha_1, ..., \alpha_n$, all of them different.

If $A_j = \{t \in \mathbb{T} : S(t) = \alpha_j\}$, then

$$S = \sum_{j=1}^{n} \alpha_j \mathcal{X}_{A_j},$$

with $\mathcal{X}_{A_j} : \mathbb{T} \to \mathbb{R}$ the characteristic function of A_j, i.e

$$\mathcal{X}_{A_j} = \begin{cases} 1, & \text{if } t \in A_j, \\ 0, & \text{if } t \in \mathbb{T} \backslash A_j. \end{cases}$$

Definition 2.25 ([31]) Let $E \subset \mathbb{T}$ be a Δ-measurable set and let $S : \mathbb{T} \to [0, +\infty)$ be a simple and Δ-measurable function with

$$S = \sum_{j=1}^{n} \alpha_j \mathcal{X}_{A_j}.$$

The Lebesgue Δ-integral of S on E it is defined as

$$\int_E S(s) \Delta s = \sum_{j=1}^{n} \alpha_j \mu_\Delta(A_j).$$

Definition 2.26 ([31]) Let $E \subset \mathbb{T}$ be a Δ-measurable set and let $f : \mathbb{T} \to [0, +\infty]$ be a Δ-measurable function The Lebesgue Δ-integral of f on E it is defined as

$$\int_E f(s) \Delta s = \sup \int_E S(s) \Delta s,$$

where the supremum is taken on all simple Δ-measurable functions S such that $0 \leq S \leq f$ in $\mathbb{T}$.

Definition 2.27 ([31]) Let $E \subset \mathbb{T}$ be a Δ-measurable set and let $f : \mathbb{T} \to \overline{\mathbb{R}}$ be a Δ-measurable function. We say that f is Lebesgue Δ-integrable on E if at least one of the elements

$$\int_E f^+(s)\,\Delta s \quad \text{or} \quad \int_E f^-(s)\,\Delta s,$$

is finite, where the positive and negative parts of f, where

$$f^+ \ : \ = \max\{f, 0\},$$
$$f^- \ : \ = \max\{-f, 0\}.$$

In which case, we define the Lebesgue Δ-integral of f on E as

$$\int_E f(s)\,\Delta s = \int_E f^+(s)\,\Delta s - \int_E f^-(s)\,\Delta s.$$

Moreover, we define the Lebesgue Δ-integrable of $|f|$ on E as

$$\int_E |f(s)|\,\Delta s = \int_E f^+(s)\,\Delta s + \int_E f^-(s)\,\Delta s.$$

Definition 2.28 ([31]) Let $E \subset \mathbb{T}$ be a Δ-measurable set and let $f : \mathbb{T} \to \overline{\mathbb{R}}$ be a Δ-measurable function. We say that f belongs to $L^1_\Delta(E)$ provided that

$$\int_{[a,b]\cap E} |f(s)|\,\Delta s < \infty.$$

Proposition 2.2 ([3]) *Let $E \subset \mathbb{T}$ be a Δ-measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is Δ-integrable on E, then*

$$\int_E f(s)\,\Delta s = \int_E f(s)\,ds + \sum_{i\in I_E} (\sigma(t_i) - t_i)\, f(t_i) + r(f, E),$$

where

$$r(f, E) = \begin{cases} \mu_M(E)\,.f(M), & \textit{if } M \in \mathbb{T} \\ 0, & \textit{if } M \notin \mathbb{T}, \end{cases}$$

$I_E := \{i \in I : t_i \in E\}$ *and* $\{t_i\}_{i\in I}$, $I \in \mathbb{N}$, *is the set of all right-scattered points of* $\mathbb{T}$.

Definition 2.29 ([3]) Let $A \subset \mathbb{T}$. A is called Δ-null set if $\mu_\Delta(A) = 0$. Say that a property P holds Δ-almost everywhere (Δ-a.e) on A, or for Δ-almost all (Δ-a.a) $t \in A$ if there is a Δ-null set $E \subset A$ such that P holds for all $t \in A\backslash E$.

Definition 2.30 ([3]) Let $E \subset \mathbb{T}$ be a Δ-measurable set and let $p \in \overline{\mathbb{R}}$ be such that $p \geq 1$ and let $f : \mathbb{T} \rightarrow \overline{\mathbb{R}}$ be a Δ-measurable function. Say that f belongs to $L^p_\Delta(E)$ provided that either

$$\int_E |f|^p \Delta s < \infty \quad \text{if} \quad p \in \mathbb{R},$$

or there exists a constant $C \in \mathbb{R}$ such that

$$|f| \leq C \quad \Delta\text{-a.e. on } E \quad \text{if} \quad p = +\infty.$$

Now, let $L^p_\Delta(J^0)$ spaces, where $J = [a, b] \cap \mathbb{T}$, $a, b \in \mathbb{T}$ with $a < b$, is an arbitrary closed subinterval of $\mathbb{T}$ and $J^0 = [a, b) \cap \mathbb{T}$.

Theorem 2.15 ([3]) *Let $p \in \mathbb{R}$ be such that $p \geq 1$. Then, the set $L^p_\Delta(J^0)$ is a Banach space together with the norm defined for every $f \in L^p_\Delta(J^0)$ as*

$$\|f\|_{L^p_\Delta} = \begin{cases} \left(\int_{J^0} |f|^p (s)\right)^{\frac{1}{p}}, & \text{if } p \in \mathbb{R}, \\ \inf\left\{C \in \mathbb{R} : |f| \leq C \quad \Delta\text{-a.e. on } J^0\right\}, & \text{if } p = +\infty. \end{cases}$$

Chapter 3

Stability in linear neutral Levin-Nohel integro-differential equations

Keywords. Fixed points, neutral integro-differential equations, stability.

In this chapter we present a work publiched in [6], namely, K. Ali Khelil, A. Ardjouni and A. Djoudi., Communications in Applied Analysis, 22, No. 1 (2018), 83-96.

We use, in this chapter, the contraction mapping theorem to obtain asymptotic stability results about the zero solution for the following linear neutral Levin-Nohel integro-differential equation

$$x'(t)+\int_{t-\tau(t)}^{t}a(t,s)x(s)ds+c(t)x'(t-\tau(t))=0.$$

An asymptotic stability theorem with a necessary and sufficient condition is proved. In addition, the case of the equation with several delays is studied. The results obtained here extend the work of Dung [36].

3.1 Introduction

We consider the following linear neutral Levin-Nohel integro-differential equation with variable delay

$$x'(t)+\int_{t-\tau(t)}^{t}a(t,s)x(s)ds+c(t)x'(t-\tau(t))=0,\ \ t\geq t_0, \tag{3.1}$$

with an assumed initial condition

$$x(t) = \phi(t), \ t \in [m(t_0), t_0],$$

where $\phi \in C([m(t_0), t_0], \mathbb{R})$ and

$$m(t_0) = \inf\{t - \tau(t) : t \in [t_0, \infty)\}.$$

Throughout this paper, we assume that

$$c \in C^1([t_0, \infty), \mathbb{R}),$$

$$a \in C([t_0, \infty) \times [m(t_0), \infty), \mathbb{R})$$

and

$$\tau \in C^2\left([t_0, \infty), \mathbb{R}^+\right)$$

with $t - \tau(t)$ as $t \to \infty$.

Our purpose here is to use the contraction mapping theorem (see [57]) to show the asymptotic stability of the zero solution for Eq. (3.1). An asymptotic stability theorem with a necessary and sufficient condition is proved. In addition, A study of the general form of (3.1) with several delays is given. In the special case $c = 0$, Dung [36] shows the zero solution of (3.1) is asymptotically stable with a necessary and sufficient condition by using the contraction mapping theorem.

3.2 Existence of solutions

For the convenience of the reader, let us recall the definition of asymptotic stability. For each t_0, we denote $C(t_0)$ the space of continuous functions on $[m(t_0), t_0]$ with the supremum norm $\|.\|_{t_0}$. For each $(t_0, \phi) \in [0, \infty) \times C(t_0)$, denoted by $x(t) = x(t, t_0, \phi)$ the unique solution of Eq.(3.1).

In order to be able to construct a new fixed mapping, we transform the Levin-Nohel equation into an equivalent equation. For this, we use the variation of parameter formula and the integration by parts.

Lemma 3.1 *Suppose that*

$$\tau'(t) \neq 1, \ \forall t \in [t_0, \infty) . \tag{3.2}$$

Then x is a solution of equation (3.1) if and only if

$$\begin{aligned} x(t) &= (\phi(t_0) + \gamma(t_0)\phi(t_0 - \tau(t_0)))\, e^{-\int_{t_0}^{t} A(z)dz} - \gamma(t)x(t-\tau(t)) \\ &\quad - \int_{t_0}^{t} \left[L_x(s) - \mu(s)x(s-\tau(s))\right] e^{-\int_{s}^{t} A(z)dz} ds, \ t \geq t_0, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} L_x(t) \ &= \ \int_{t-\tau(t)}^{t} a(t,s) \left(\int_{s}^{t} \left(\int_{u-\tau(u)}^{u} a(u,v)x(v)dv - r(u)x(u-\tau(u)) \right) du \right. \\ &\quad \left. + \gamma(t)x(t-\tau(t)) - \gamma(s)x(s-\tau(s)) \right) ds \end{aligned} \tag{3.4}$$

$$r(t) = \frac{c'(t)(1-\tau'(t)) + \tau''(t)c(t)}{(1-\tau'(t))^2}, \ \gamma(t) = \frac{c(t)}{1-\tau'(t)}, \tag{3.5}$$

and

$$\mu(t) = \frac{(c'(t) + c(t)A(t))(1-\tau'(t)) + \tau''(t)c(t)}{(1-\tau'(t))^2}, \ A(t) = \int_{t-\tau(t)}^{t} a(t,s)ds. \tag{3.6}$$

Proof. Obviously, we have

$$x(s) = x(t) - \int_{s}^{t} x'(u)du.$$

Inserting this relation into (3.1), we get

$$x'(t) + \int_{t-r(t)}^{t} a(t,s) \left(x(t) - \int_{s}^{t} x'(u)du \right) ds + c(t)x'(t-\tau(t)) = 0, \ t \geq t_0,$$

or equivalently

$$x'(t) + x(t)\int_{t-\tau(t)}^{t} a(t,s)ds - \int_{t-\tau(t)}^{t} a(t,s) \left(\int_{s}^{t} x'(u)du \right) ds + c(t)x'(t-\tau(t)) = 0, \ t \geq t_0.$$

After substituting x' from (3.1), we obtain

$$\begin{aligned} & x'(t) + x(t)\int_{t-\tau(t)}^{t} a(t,s)ds \\ & + \int_{t-\tau(t)}^{t} a(t,s) \left(\int_{s}^{t} \left(\int_{u-\tau(u)}^{u} a(u,v)x(v)dv + c(u)x'(u-\tau(u)) \right) du \right) ds \\ & + c(t)x'(t-\tau(t)) \ = \ 0, \ t \ \geq \ t_0. \end{aligned} \tag{3.7}$$

By performing the integration by parts, we have

$$\begin{aligned}&\int_s^t c(u)x'(u-\tau(u))du \\ &= \int_s^t \frac{c(u)}{1-\tau'(u)} dx(u-\tau(u)) \\ &= \gamma(t)\, x(t-\tau(t)) - \gamma(s)\, x(s-\tau(s)) - \int_s^t r(u)x(u-\tau(u))du, \end{aligned} \tag{3.8}$$

where r and γ are given by (3.5). After substituting (3.8) into (3.7), we have

$$x'(t) + A(t)x(t) + L_x(t) + c(t)x'(t-\tau(t)) = 0,\ t \geq t_0,$$

where A and L_x are given by (3.6) and (3.4), respectively. By the variation of constants formula, we get

$$x(t) = \phi(t_0)e^{-\int_{t_0}^t A(z)dz} - \int_{t_0}^t \left[L_x(s) + c(s)x'(s-\tau(s))\right] e^{-\int_s^t A(z)dz} ds,\ t \geq t_0. \tag{3.9}$$

Letting

$$\int_{t_0}^t c(s)x'(s-\tau(s))e^{-\int_s^t A(z)dz} ds = \int_{t_0}^t \frac{c(s)}{1-\tau'(s)} e^{-\int_s^t A(z)dz} dx(s-\tau(s)).$$

By using the integration by parts, we obtain

$$\begin{aligned}&\int_{t_0}^t c(s)x'(s-\tau(s))e^{-\int_s^t A(z)dz} ds \\ &= \frac{c(t)}{1-\tau'(t)} x(t-\tau(t)) - \frac{c(t_0)}{1-\tau'(t_0)} x(t_0-\tau(t_0))e^{-\int_{t_0}^t A(z)dz} \\ &- \int_{t_0}^t \mu(s)x(s-\tau(s))e^{-\int_s^t A(z)dz} ds, \end{aligned} \tag{3.10}$$

where μ is given by (3.6). Finally, we obtain (3.3) by substituting (3.10) in (3.9). Since each step is reversible, the converse follows easily. This completes the proof. ■

3.3 Asymptotic stablility

Theorem 3.1 *Let (3.2) holds and suppose that the following two conditions hold:*

$$\liminf_{t\to\infty} \int_0^t A(z)dz > -\infty, \tag{3.11}$$

$$\sup_{t\geq 0}\left(|\gamma(t)| + \int_0^t \omega(s)e^{-\int_s^t A(z)dz}ds\right) = \alpha < 1, \tag{3.12}$$

where

$$\omega(s) = \int_{s-\tau(s)}^{s} |a(s,w)| \left(\int_w^s \left(\int_{u-\tau(u)}^{u} |a(u,v)|\, dv + |r(u)|\right) du + |\gamma(s)| + |\gamma(w)|\right) dw + |\mu(s)|\,.$$

Then the zero solution of (3.1) is asymptotically stable if and only if

$$\int_0^t A(z)dz \to \infty \text{ as } t \to \infty. \tag{3.13}$$

Proof. Sufficient condition. Suppose that (3.13) holds. Denoted by C the space of continuous bounded functions $x : [m(t_0), \infty) \to \mathbb{R}$ such that $x(t) = \phi(t)$, $t \in [m(t_0), t_0]$. It is known that C is a complete metric space endowed with a metric $\|x\| = \sup_{t\geq m(t_0)} |x(t)|$. Define the operator P on C by $(Px)(t) = \phi(t)$, $t \in [m(t_0), t_0]$ and

$$\begin{aligned}(Px)(t) &= (\phi(t_0) + \gamma(t_0)\phi(t_0 - \tau(t_0)))\, e^{-\int_{t_0}^t A(z)dz} - \gamma(t)x(t - \tau(t)) \\ &\quad - \int_{t_0}^t [L_x(s) - \mu(s)x(s - \tau(s))]\, e^{-\int_s^t A(z)dz}ds,\ t \geq t_0.\end{aligned}$$

Obviously, Px is continuous for each $x \in C$. Moreover, it is a contraction operator. Indeed, let $x, y \in C$

$$\begin{aligned}&|(Px)(t) - (Py)(t)| \\ &\leq |\gamma(t)|\, |x(t - \tau(t)) - y(t - \tau(t))| \\ &+ \int_{t_0}^t [|L_x(s) - L_y(s)| + |\mu(s)|\, |x(s - \tau(s)) - y(s - \tau(s))|]\, e^{-\int_s^t A(z)dz}ds.\end{aligned}$$

Since $x(t) = y(t) = \phi(t)$ for all $t \in [m(t_0), t_0]$, this implies that

$$\begin{aligned}&|L_x(s) - L_y(s)| \\ &\leq \left(\int_{s-\tau(s)}^{s} |a(s,w)| \left(\int_w^s \left(\int_{u-\tau(u)}^{u} |a(u,v)|\, dv + |r(u)|\right) du + |\gamma(s)| + |\gamma(w)|\right) dw\right) \|x - y\|\,.\end{aligned}$$

Consequently, it holds for all $t \geq t_0$ that

$$\begin{aligned}&|(Px)(t) - (Py)(t)| \\ &\leq \Bigg[|\gamma(t)| + \int_{t_0}^t \left(\int_{s-\tau(s)}^{s} |a(s,w)| \left(\int_w^s \left(\int_{u-\tau(u)}^{u} |a(u,v)|\, dv + |r(u)|\right) du\right.\right. \\ &\left.\left. + |\gamma(s)| + |\gamma(w)|\right) dw + |\mu(s)|\right) e^{-\int_s^t A(z)dz}ds\Bigg]\, \|x - y\|\,.\end{aligned}$$

Hence, it follows from (3.12) that

$$|(Px)(t) - (Py)(t)| \le \alpha \|x - y\|, \ t \ge t_0.$$

Thus P is a contraction operator on C.

We now consider a closed subspace S of C that is defined by

$$S = \{x \in C : |x(t)| \to 0 \text{ as } t \to \infty\}.$$

We will show that $P(S) \subset S$. To do this, we need to point out that for each $x \in S$, $|(Px)(t)| \to 0$ as $t \to \infty$. Let $x \in S$, by the definition of P we have

$$\begin{aligned}(Px)(t) &= (\phi(t_0) + \gamma(t_0)\phi(t_0 - \tau(t_0)))\, e^{-\int_{t_0}^t A(z)dz} - \gamma(t)x(t - \tau(t)) \\ &\quad - \int_{t_0}^t [L_x(s) - \mu(s)x(s - \tau(s))]\, e^{-\int_s^t A(z)dz} ds, \\ &= I_1 + I_2 + I_3, \ t \ge t_0.\end{aligned}$$

The first term I_1 tends to 0 by (3.13) and I_2 tends to 0 by $t - \tau(t) \to \infty$ as $t \to \infty$. For any $T \in (t_0, t)$, we have the following estimate for the third term

$$\begin{aligned} I_3 &\le \left| \int_{t_0}^T [L_x(s) - \mu(s)x(s - \tau(s))]\, e^{-\int_s^t A(z)dz} ds \right| \\ &+ \left| \int_T^t [L_x(s) - \mu(s)x(s - \tau(s))]\, e^{-\int_s^t A(z)dz} ds \right| \\ &\le \int_{t_0}^T \left(\int_{s-\tau(s)}^s |a(s,w)| \left(\int_w^s \left(\int_{u-\tau(u)}^u |a(u,v)| \, \|x\| \, dv + |r(u)| \, \|\phi\|_{t_0} \right) du \right.\right. \\ &+ |\gamma(s)| \, \|\phi\|_{t_0} + |\gamma(w)| \, \|\phi\|_{t_0} \Big) dw + |\mu(s)| \, \|\phi\|_{t_0} \Big) \, e^{-\int_s^t A(z)dz} ds \\ &+ \int_T^t \left(\int_{s-\tau(s)}^s |a(s,w)| \left(\int_w^s \left(\int_{u-\tau(u)}^u |a(u,v)| \, |x(v)| \, dv + |r(u)| \, |x(u - \tau(u))| \right) du \right.\right. \\ &+ |\gamma(s)| \, |x(s - \tau(s))| + |\gamma(w)| \, |x(w - \tau(w))| \Big) dw + |\mu(s)| \, |x(s - \tau(s))| \Big) \, e^{-\int_s^t A(z)dz} ds \\ &\le \left[\int_{t_0}^T \left(\int_{s-\tau(s)}^s |a(s,w)| \left(\int_w^s \left(\int_{u-\tau(u)}^u |a(u,v)| \, dv + |r(u)| \right) du \right.\right.\right. \\ &+ |\gamma(s)| + |\gamma(w)| \Big) dw + |\mu(s)| \Big) \, e^{-\int_s^t A(z)dz} ds \Big] \left(\|x\| + \|\phi\|_{t_0} \right) \\ &+ \int_T^t \left(\int_{s-\tau(s)}^s |a(s,w)| \left(\int_w^s \left(\int_{u-\tau(u)}^u |a(u,v)| \, |x(v)| \, dv + |r(u)| \, |x(u - \tau(u))| \right) du \right.\right. \\ &+ |\gamma(s)| \, |x(s - \tau(s))| + |\gamma(w)| \, |x(w - \tau(w))| \Big) dw + |\mu(s)| \, |x(s - \tau(s))| \Big) \, e^{-\int_s^t A(z)dz} ds \\ &= I_{31} + I_{32}. \end{aligned}$$

Since $t-\tau(t)\to\infty$ as $t\to\infty$, this implies that $u-\tau(u)\to\infty$ as $T\to\infty$. Thus, from the fact $|x(v)|\to 0,\ v\to\infty$ we can infer that for any $\varepsilon>0$ there exists $T_1=T>t_0$ such that

$$\begin{aligned} I_{32} &< \frac{\varepsilon}{2\alpha}\int_{T_1}^{t}\left(\int_{s-\tau(s)}^{s}|a(s,w)|\left(\int_{w}^{s}\left(\int_{u-\tau(u)}^{u}|a(u,v)|\,dv+|r(u)|\right)du\right.\right.\\ &\quad \left.\left.+|\gamma(s)|+|\gamma(w)|\right)dw+|\mu(s)|\right)e^{-\int_s^t A(z)dz}ds, \end{aligned}$$

and hence, $I_{32}<\frac{\varepsilon}{2}$ for all $t\geq T_1$. On the other hand, $\|x\|<\infty$ because $x\in S$. This combined with (3.13) yields $I_{31}\to 0$ as $t\to\infty$. As a consequence, there exists $T_2\geq T_1$ such that $I_{31}<\frac{\varepsilon}{2}$ for all $t\geq T_2$. Thus, $I_3<\varepsilon$ for all $t\geq T_2$, that is, $I_3\to 0$ as $t\to\infty$. So $P(S)\subset S$.

By the Contraction Mapping Principle, P has a unique fixed point x in S which is a solution of (3.1) with $x(t)=\phi(t)$ on $[m(t_0),t_0]$ and $x(t)=x(t,t_0,\phi)\to 0$ as $t\to\infty$.

To obtain the asymptotic stability, we need to show that the zero solution of (3.1) is stable. By condition (3.11), we can define

$$K=\sup_{t\geq 0} e^{-\int_0^t A(z)dz}<\infty. \tag{3.14}$$

Using the formula (3.3) and condition (3.12), we can obtain

$$|x(t)|\leq K\left(1+|\gamma(t_0)|\right)\|\phi\|_{t_0}e^{\int_0^{t_0}A(z)dz}+\alpha(\|x\|+\|\phi\|_{t_0}),\ t\geq t_0,$$

which leads us to

$$\|x\|\leq\frac{K\left(1+|\gamma(t_0)|\right)e^{\int_0^{t_0}A(z)dz}+\alpha}{1-\alpha}\|\phi\|_{t_0}. \tag{3.15}$$

Thus for every, $\varepsilon>0$, we can find $\delta>0$ such that $\|\phi\|_{t_0}<\delta$ implies that $\|x\|<\varepsilon$. This shows that the zero solution of (3.1) is stable and hence, it is asymptotically stable.

Necessary condition. Suppose that the zero solution of (3.1) is asymptotically stable and that the condition (3.13) fails. It follows from (3.11) that there exists a sequence $\{t_n\}$, $t_n\to\infty$ as $n\to\infty$ such that

$$\lim_{n\to\infty}\int_0^{t_n}A(z)dz \text{ exists and is finite.}$$

Hence, we can choose a positive constant L satisfying

$$-L\leq\lim_{n\to\infty}\int_0^{t_n}A(z)dz\leq L,\ \forall n\geq 1. \tag{3.16}$$

Then, condition (3.12) gives us

$$c_n = \int_0^{t_n} \omega(s) e^{\int_0^s A(z)dz} ds \le \alpha e^{\int_0^{t_n} A(z)dz} < e^L.$$

The sequence $\{c_n\}$ is increasing and bounded, so it has a finite limit. For any $\delta_0 > 0$, there exists $n_0 > 0$ such that

$$\int_{t_{n_0}}^{t_n} \omega(s) e^{\int_0^s A(z)dz} ds < \frac{\delta_0}{2K}, \ \forall n \ge n_0, \tag{3.17}$$

where K is as in (3.14). We choose δ_0 such that $\delta_0 < \frac{1-\alpha}{K(1+|\gamma(t_0)|)e^L+1}$ and consider the solution $x(t) = x(t, t_n, \phi)$ of (3.1) with the initial data $\phi(t_{n_0}) = \delta_0$ and $|\phi(s)| \le \delta_0$, $s \le t_{n_0}$. It follows from (3.15) that

$$|x(t)| \le 1 - \delta_0, \ \forall t \ge t_{n_0}. \tag{3.18}$$

Applying the fundamental inequality $|a - b| \ge |a| - |b|$ and then using (3.18), (3.17) and (3.16), we get

$$\begin{aligned}
&|x(t_n) + \gamma(t_n)x(t_n - \tau(t_n))| \\
&\ge \delta_0 e^{-\int_{t_{n_0}}^{t_n} A(z)dz} - \int_{t_{n_0}}^{t_n} \omega(s) e^{-\int_s^{t_n} A(z)dz} ds \\
&\ge e^{-\int_{t_{n_0}}^{t_n} A(z)dz} \left(\delta_0 - e^{-\int_0^{t_{n_0}} A(z)dz} \int_{t_{n_0}}^{t_n} \omega(s) e^{\int_0^s A(z)dz} ds \right) \\
&\ge e^{-\int_{t_{n_0}}^{t_n} A(z)dz} \left(\delta_0 - K \int_{t_{n_0}}^{t_n} \omega(s) e^{\int_0^s A(z)dz} ds \right) \\
&\ge \frac{1}{2}\delta_0 e^{-\int_{t_{n_0}}^{t_n} A(z)dz} \ge \frac{1}{2}\delta_0 e^{-2L} > 0,
\end{aligned}$$

which is a contradiction because $x(t_n) + \gamma(t_n)x(t_n - \tau(t_n)) \to 0$ as $t_n \to \infty$. The proof is complete. ■

Let $c(t) = 0$ we get the following corollary.

Corollary 3.1 *Suppose that the following two conditions hold*

$$\liminf_{t\to\infty} \int_0^t A_0(z)dz > -\infty, \tag{3.19}$$

$$\sup_{t\ge 0} \int_0^t \left(\int_{s-\tau(s)}^s |a(s, w)| \int_w^s \int_{u-\tau(u)}^u |a(u, v)| \, dvdudw \right) e^{-\int_s^t A_0(z)dz} ds = \alpha < 1, \tag{3.20}$$

where

$$A_0(z) = \int_{z-\tau(z)}^{z} a(z,s)ds.$$

Then the zero solution of

$$x'(t) + \int_{t-\tau(t)}^{t} a(t,s)x(s)ds = 0,$$

is asymptotically stable if and only if

$$\int_0^t A_0(z)dz \to \infty \text{ as } t \to \infty. \tag{3.21}$$

We give an example to illustrate the applications of Theorem 3.1.

Example 3.1 Consider the following linear neutral Levin-Nohel integro-differential equation with variable delay

$$x'(t) + \int_{t-\tau(t)}^{t} a(t,s)x(s)ds + c(t)x'(t-\tau(t)) = 0, \tag{3.22}$$

where $a(t,s) = \frac{10}{t^2+1}$, $c(t) = 0.01$, $\tau(t) = 0.2t$. Then the zero solution of (3.22) is asymptotically stable.

Proof. We have

$$A(t) = \int_{0.8t}^{t} a(t,s)ds = \int_{0.8t}^{t} \frac{10}{t^2+1}ds = \frac{2t}{t^2+1},$$

$$\int_0^t A(z)dz = \ln(t^2+1),\ \gamma(t) = 0.0125,\ r(t) = 0,\ \mu(t) = \frac{0.025t}{t^2+1},$$

$$\begin{aligned}\omega(s) &= \int_{0.8s}^{s} \frac{10}{s^2+1}\left(\int_w^s \left(\int_{0.8u}^{u} \frac{10}{u^2+1}dv\right) du + 0.025\right) dw + \frac{0.025s}{s^2+1} \\ &= \frac{1}{s^2+1}\left(-20\arctan s + 20\arctan 0.8s - 8s\ln\left(s^2+1.0\right) + 8s\ln\left(0.64s^2+1.0\right) + 4.075s\right),\end{aligned}$$

and

$$\begin{aligned}&\int_0^t \omega(s)e^{-\int_s^t A(z)dz}ds \\ &= \frac{1}{t^2+1}\int_0^t \left(-20\arctan s + 20\arctan 0.8s - 8s\ln\left(s^2+1\right) + 8s\ln\left(0.64s^2+1\right) + 4.075s\right) ds \\ &= \frac{1}{t^2+1}\left(6.0\ln\left(t^2+1\right) - 6.25\ln\left(t^2+1.5625\right) - 20t\arctan t + 20t\arctan 0.8t\right. \\ &\left.-4t^2\ln\left(t^2+1\right) + 4t^2\ln\left(0.64t^2+1\right) + 2.0375t^2 + 2.7893\right) \\ &\le 0.253.\end{aligned}$$

Then

$$\sup_{t\geq 0}\left(|\gamma(t)|+\int_0^t \omega(s)e^{-\int_s^t A(z)dz}ds\right)\leq 0.265<1.$$

It is easy to see that all the conditions of Theorem 3.1 hold for $\alpha = 0.265 < 1$. Thus Theorem 3.1 implies that the zero solution of (3.22) is asymptotically stable. ■

3.4 Neutral Levin-Nohel integro-differential equations with several delays

Next we turn our attention to the following neutral Levin-Nohel integro-differential equations with several delays

$$x'(t)+\sum_{k=1}^{M}\int_{t-\tau_k(t)}^{t} a_k(t,s)x(s)ds+\sum_{k=1}^{M} c_k(t)x'(t-\tau_k(t))=0,\ t\geq t_0, \tag{3.23}$$

where $c_k \in C^1([t_0,\infty),\mathbb{R})$, $a_k \in C([t_0,\infty)\times[m(t_0),\infty),\mathbb{R})$ and $\tau_k \in C^2([t_0,\infty),\mathbb{R}^+)$ with $t-\tau_k(t)$ as $t\to\infty$, $1\leq k\leq M$.

Lemma 3.2 *Suppose that*

$$\tau_k'(t)\neq 1,\ \forall t\in[t_0,\infty),\ 1\leq k\leq M. \tag{3.24}$$

Then x is a solution of equation (3.23) if and only if

$$\begin{aligned} x(t)=&\left(\phi(t_0)+\sum_{k=1}^{M}\gamma_k(t_0)\phi(t_0-\tau_k(t_0))\right)e^{-\int_{t_0}^t \overline{A}(z)dz}-\sum_{k=1}^{M}\gamma_k(t)x(t-\tau_k(t))\\ &-\int_{t_0}^{t}\left[\overline{L}_x(s)-\sum_{k=1}^{M}\mu_k(s)x(s-\tau_k(s))\right]e^{-\int_s^t \overline{A}(z)dz}ds,\ t\geq t_0, \end{aligned}$$

where

$$\begin{aligned} \overline{L}_x(t)=&\sum_{k=1}^{M}\int_{t-\tau_k(t)}^{t} a_k(t,s)\left(\int_s^t\left(\sum_{i=1}^{M}\int_{u-\tau_i(u)}^{u} a_i(u,v)x(v)dv-\sum_{i=1}^{M} r_i(u)x(u-\tau_i(u))\right)du\right.\\ &\left.+\sum_{i=1}^{M}\gamma_i(t)x(t-\tau_i(t))-\sum_{i=1}^{M}\gamma_i(s)x(s-\tau_i(s))\right)ds, \end{aligned}$$

$$r_k(t)=\frac{c_k'(t)(1-\tau_k'(t))+\tau_k''(t)c_k(t)}{(1-\tau_k'(t))^2},\ \gamma_k(t)=\frac{c_k(t)}{1-\tau_k'(t)},$$

and

$$\mu_k(t) = \frac{(c_k'(t) + c_k(t)\overline{A}(t))(1-\tau_k'(t)) + \tau_k''(t)c(t)}{(1-\tau_k'(t))^2}, \ \overline{A}(t) = \sum_{k=1}^{M}\int_{t-\tau_k(t)}^{t} a_k(t,s)ds.$$

The proof follows along the lines of Lemma 3.1, and hence we omit it.

Theorem 3.2 *Let (3.24) holds and Suppose that the following two conditions hold*

$$\liminf_{t\to\infty}\int_0^t \overline{A}(z)dz > -\infty,$$

and

$$\sup_{t\geq 0}\left(\sum_{k=1}^{M}|\gamma_k(t)| + \int_0^t \overline{\omega}(s)e^{-\int_s^t \overline{A}(z)dz}ds\right) = \alpha < 1,$$

where

$$\begin{aligned}\overline{\omega}(s) &= \sum_{k=1}^{M}\int_{s-\tau_k(s)}^{s}|a_k(s,w)|\left(\int_w^s\left(\sum_{i=1}^{M}\int_{u-\tau_i(u)}^{u}|a_i(u,v)|\,dv + \sum_{i=1}^{M}|r_i(u)|\right)du\right.\\ &\quad \left.+\sum_{k=1}^{M}|\gamma_k(s)| + \sum_{k=1}^{M}|\gamma_k(t)|\right)dw + \sum_{k=1}^{M}|\mu_k(s)|.\end{aligned}$$

Then the zero solution of (3.23) is asymptotically stable if and only if

$$\int_0^t \overline{A}(z)dz \to \infty \ \text{as}\ t\to\infty.$$

The proof is similar to that of Theorem 3.1, and hence, we omit it.

Chapter 4

Stability in nonlinear neutral Levin-Nohel integro-differential equations with delay

Keywords. Fixed points, neutral integro-differential equations, stability.

In this chapter we present a work publiched in [7], namely, K. Ali Khelil, A. Ardjouni and A. Djoudi, Korean J. Math. 25 (2017), No. 3, pp. 303-321.

In this chapter we use the Krasnoselskii-Burton's fixed point theorem to obtain asymptotic stability and stability results about the zero solution for the following nonlinear neutral Levin-Nohel integro-differential equation

$$x'(t) + \int_{t-\tau(t)}^{t} a(t,s)g\left(x(s)\right)ds + c(t)x'(t-\tau(t)) = 0.$$

The results obtained here extend the work of Mesmouli, Ardjouni and Djoudi [52].

4.1 Introduction

We consider the following nonlinear neutral Levin-Nohel integro-differential equation with variable delay

$$x'(t) + \int_{t-\tau(t)}^{t} a(t,s)g\left(x(s)\right)ds + c(t)x'(t-\tau(t)) = 0,\ t \geq t_0, \tag{4.1}$$

with an assumed initial condition

$$x\left(t\right) = \phi\left(t\right),\ t \in \left[m\left(t_0\right), t_0\right],$$

where $\phi \in C\left(\left[m\left(t_0\right), t_0\right], \mathbb{R}\right)$ and

$$m\left(t_0\right) = \inf\left\{t - \tau\left(t\right) : t \in \left[t_0, \infty\right)\right\}.$$

Throughout this chapter, we assume that $c \in C^1\left(\left[t_0, \infty\right), \mathbb{R}\right)$, $a \in C(\left[t_0, \infty\right) \times \left[m\left(t_0\right), \infty\right), \mathbb{R}_+)$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous with respect to its argument. We assume that $g\left(0\right) = 0$ and $\tau \in C^2\left(\left[t_0, \infty\right), \mathbb{R}^+\right)$ such that

$$\tau'\left(t\right) \neq 1,\ t \in \left[t_0, \infty\right). \tag{4.2}$$

Our purpose here is to use the Krasnoselskii-Burton's fixed point theorem to show the asymptotic stability and stability of the zero solution for (4.1). In the special case $c = 0$, Mesmouli, Ardjouni and Djoudi [52] show the zero solution of (4.1) is asymptotically stable with a necessary and sufficient condition by using the contraction mapping theorem in a weighted Banach space.

4.2 The inversion and the fixed point theorem

One crucial step in the investigation of an equation using fixed point theory involves the construction of a suitable fixed point mapping. For that end we must invert (4.1) to obtain an equivalent integral equation from which we derive the needed mapping. During the process, an integration by parts has to be performed on the neutral term $x'(t - \tau(t))$. Unfortunately, when doing this, a derivative $\tau'(t)$ of the delay appears on the way, and so we have to support it.

Lemma 4.1 *Suppose that (4.2) holds. Then x is a solution of equation (4.1) if and only if*

$$\begin{aligned} x(t) &= \left(\phi(t_0) + \gamma(t_0)\phi(t_0 - \tau(t_0))\right) e^{-\int_{t_0}^t A(z)dz} \\ &+ \int_{t_0}^t \left(\int_{s-\tau(s)}^s a(s,u)\left(Gx\right)(u)du\right) e^{-\int_s^t A(z)dz} ds - \gamma(t)x(t - \tau(t)) \\ &- \int_{t_0}^t \left[L_x(s) - \mu(s)x(s - \tau(s))\right] e^{-\int_s^t A(z)dz} ds,\ t \geq t_0, \end{aligned} \tag{4.3}$$

where

$$L_x(t) = \int_{t-\tau(t)}^{t} a(t,s) \left(\int_s^t \left(\int_{u-\tau(u)}^{u} a(u,v)x(v)dv - r(u)x(u-\tau(u)) \right) du \right.$$
$$\left. + \gamma(t)x(t-\tau(t)) - \gamma(s)x(s-\tau(s)) \right) ds \tag{4.4}$$

$$r(t) = \frac{c'(t)(1-\tau'(t)) + \tau''(t)c(t)}{(1-\tau'(t))^2}, \ \gamma(t) = \frac{c(t)}{1-\tau'(t)}, \tag{4.5}$$

$$(Gx)(t) = x(t) - g(x(t)), \tag{4.6}$$

and

$$\mu(t) = \frac{(c'(t) + c(t)A(t))(1-\tau'(t)) + \tau''(t)c(t)}{(1-\tau'(t))^2}, \ A(t) = \int_{t-\tau(t)}^{t} a(t,s)ds. \tag{4.7}$$

Proof. Let x be a solution of (4.1). Rewrite (4.1) as

$$x'(t) + \int_{t-\tau(t)}^{t} a(t,s)x(s)ds$$
$$- \int_{t-\tau(t)}^{t} a(t,s) \left(x(s) - g(x(s)) \right) ds + c(t)x'(t-\tau(t)) = 0, \ t \geq t_0.$$

Obviously, we have

$$x(s) = x(t) - \int_s^t x'(u)du.$$

Inserting this relation into (4.1), we get

$$x'(t) + \int_{t-r(t)}^{t} a(t,s) \left(x(t) - \int_s^t x'(u)du \right) ds$$
$$- \int_{t-\tau(t)}^{t} a(t,s)(Gx)(s)\, ds + c(t)x'(t-\tau(t)) = 0, \ t \geq t_0,$$

or equivalently

$$x'(t) + x(t) \int_{t-\tau(t)}^{t} a(t,s)ds - \int_{t-\tau(t)}^{t} a(t,s) \left(\int_s^t x'(u)du \right) ds$$
$$- \int_{t-\tau(t)}^{t} a(t,s)(Gx)(s)\, ds + c(t)x'(t-\tau(t)) = 0, \ t \geq t_0.$$

After substituting x' from (4.1), we obtain

$$x'(t) + x(t) \int_{t-\tau(t)}^{t} a(t,s)ds$$
$$+ \int_{t-\tau(t)}^{t} a(t,s) \left(\int_s^t \left(\int_{u-\tau(u)}^{u} a(u,v)x(v)dv + c(u)x'(u-\tau(u)) \right) du \right) ds$$
$$- \int_{t-\tau(t)}^{t} a(t,s)(Gx)(s)\, ds + c(t)x'(t-\tau(t)) = 0, \ t \geq t_0. \tag{4.8}$$

4.2. The inversion and the fixed point theorem

By performing the integration by parts, we have

$$\begin{aligned}
&\int_s^t c(u)x'(u-\tau(u))du \\
&= \int_s^t \frac{c(u)}{1-\tau'(u)} dx(u-\tau(u)) \\
&= \gamma(t)\, x(t-\tau(t)) - \gamma(s)\, x(s-\tau(s)) - \int_s^t r(u)x(u-\tau(u))du, \qquad (4.9)
\end{aligned}$$

where r and γ are given by (4.5). After substituting (4.9) into (4.8), we have

$$\begin{aligned}
&x'(t) + A(t)x(t) + L_x(t) \\
&- \int_{t-\tau(t)}^t a(t,s)(Gx)(s)\, ds + c(t)x'(t-\tau(t)) \;=\; 0,\; t \geq t_0,
\end{aligned}$$

where A and L_x are given by (4.7) and (4.4), respectively. By the variation of constants formula, we get

$$\begin{aligned}
x(t) = \phi(t_0)e^{-\int_{t_0}^t A(z)dz} &+ \int_{t_0}^t \left(\int_{s-\tau(s)}^s a(s,u)\,(Gx)\,(u)du\right) e^{-\int_s^t A(z)dz} ds \\
&- \int_{t_0}^t \left[L_x(s) + c(s)x'(s-\tau(s))\right] e^{-\int_s^t A(z)dz} ds,\; t \geq t_0. \qquad (4.10)
\end{aligned}$$

Letting

$$\int_{t_0}^t c(s)x'(s-\tau(s))e^{-\int_s^t A(z)dz} ds = \int_{t_0}^t \frac{c(s)}{1-\tau'(s)} e^{-\int_s^t A(z)dz} dx(s-\tau(s)).$$

By using the integration by parts, we obtain

$$\begin{aligned}
&\int_{t_0}^t c(s)x'(s-\tau(s))e^{-\int_s^t A(z)dz} ds \\
&= \frac{c(t)}{1-\tau'(t)} x(t-\tau(t)) - \frac{c(t_0)}{1-\tau'(t_0)} x(t_0-\tau(t_0))e^{-\int_{t_0}^t A(z)dz} \\
&- \int_{t_0}^t \mu(s)x(s-\tau(s))e^{-\int_s^t A(z)dz} ds, \qquad (4.11)
\end{aligned}$$

where μ is given by (4.7). Finally, we obtain (4.3) by substituting (4.11) in (4.10). Since each step is reversible, the converse follows easily. This completes the proof. ■

Burton studied the theorem of Krasnoselskii and observed (see [27]) that Krasnoselskii result can be more interesting in applications with certain changes and formulated the Theorem 4.2 below (see [27] for its proof).

Theorem 4.1 ([22]) *Let* $(\mathbb{M}, d)$ *be a complete metric space and* F *be a large contraction. Suppose there is* $x \in M$ *and* $\rho > 0$ *such that* $d(x, F^n x) \leq \rho$ *for all* $n \geq 1$*. Then* F *has a unique fixed point in* $\mathbb{M}$*.*

Below, we state Krasnoselskii-Burton's hybrid fixed point theorem which enables us to establish a stability result of the trivial solution of (4.1). For more details on Krasnoselskii's captivating theorem we refer to Smart [57] or [25].

Theorem 4.2 (Krasnoselskii-Burton) *Let* $\mathbb{M}$ *be a closed bounded convex nonempty subset of a Banach space* $(S, \|.\|)$*. Suppose that* $\mathcal{A}$*,* $\mathcal{B}$ *map* $\mathbb{M}$ *into* $\mathbb{M}$ *and that*

(i) *for all* $x, y \in \mathbb{M} \Rightarrow \mathcal{A}x + \mathcal{B}y \in \mathbb{M}$*,*

(ii) $\mathcal{A}$ *is continuous and* $\mathcal{A}\mathbb{M}$ *is contained in a compact subset of* $\mathbb{M}$*,*

(iii) $\mathcal{B}$ *is a large contraction.*

Then there is $z \in \mathbb{M}$ *with* $z = \mathcal{A}z + \mathcal{B}z$*.*

Here we manipulate function spaces defined on infinite t-intervals. So for compactness, we need an extension of Arzela-Ascoli theorem. This extension is taken from [[25], Theorem 1.2.2, p. 20] and is as follows.

Theorem 4.3 *Let* $q : \mathbb{R}_+ \to \mathbb{R}_+$ *be a continuous function such that* $q(t) \to 0$ *as* $t \to \infty$*. If* $\{\varphi_n(t)\}$ *is an equicontinuous sequence of* $\mathbb{R}^m$*-valued functions on* $\mathbb{R}_+$ *with* $|\varphi_n(t)| \leq q(t)$ *for* $t \in \mathbb{R}_+$*, then there is a subsequence that converges uniformly on* $\mathbb{R}_+$ *to a continuous function* $\varphi(t)$ *with* $|\varphi(t)| \leq q(t)$ *for* $t \in \mathbb{R}_+$*, where* $|.|$ *denotes the Euclidean norm on* $\mathbb{R}^m$*.*

4.3 Stability by Krasnoselskii-Burton's theorem

From the existence theory which can be found in [25], we conclude that for each continuous initial function $\phi : [m_0, t_0] \to \mathbb{R}$, there exists a continuous solution $x(t, t_0, \phi)$ which satisfies (4.1) on an interval $[0, \sigma)$ for some $\sigma > 0$ and $x(t, t_0, \phi) = \phi(t)$ for $t \in [m_0, t_0]$.

We need the following stability definitions taken from [25].

Definition 4.1 The zero solution of (4.1) is said to be stable at $t = t_0$ if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\phi : [m_0, t_0] \to (-\delta, \delta)$ implies that $|x(t, t_0, \phi)| < \varepsilon$ for all

$t \geq m_0$.

Definition 4.2 The zero solution of (4.1) is said to be asymptotically stable if it is stable at $t = t_0$ and $\delta > 0$ exists such that for any contunuos function $\phi : [m_0, t_0] \to (-\delta, \delta)$ the solution $x(t, t_0, \phi)$ with $x(t, t_0, \phi) = \phi(t)$ on $[m_0, t_0]$ tends to zero as $t \to \infty$.

To apply Theorem 4.2, we have to choose carefully a Banach space depending on the initial function ϕ and construct two mappings, a large contraction and a compact operator which obey the conditions of the theorem. So let S be the Banach space of continuous bounded functions $\varphi : [m_0, \infty] \to \mathbb{R}$ with the supremum norm $\|.\|$. Let $L > 0$ and define the set

$$\begin{aligned} S_\phi &= \{\varphi \in S : \varphi \text{ is Lipschitzian, } |\varphi(t)| \leq L,\ t \in [m_0, \infty), \\ &\qquad \varphi(t) = \phi(t) \text{ if } t \in [m_0, t_0] \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\}. \end{aligned}$$

Clearly, if $\{\varphi_n\}$ is a sequence of k-Lipschitzian functions converging to a function φ then

$$\begin{aligned} |\varphi(u) - \varphi(v)| &\leq |\varphi(u) - \varphi_n(u)| + |\varphi_n(u) - \varphi_n(v)| + |\varphi_n(v) - \varphi(v)| \\ &\leq \|\varphi - \varphi_n\| + k|u - v| + \|\varphi - \varphi_n\|. \end{aligned}$$

Consequently, as $n \to \infty$, we see that φ is k-Lipschitzian. It is clear that S_ϕ is convex, bounded and complete endowed with $\|.\|$.

For $\varphi \in S_\phi$ and $t \geq t_0$, define the maps $\mathcal{A}$, $\mathcal{B}$ and H on S_ϕ as follows

$$\begin{aligned} (\mathcal{A}\varphi)(t) &= -\gamma(t)\varphi(t - \tau(t)) - \int_{t_0}^{t} L_\varphi(s) e^{-\int_s^t A(z)dz} ds \\ &\quad + \int_{t_0}^{t} \mu(s)\varphi(s - \tau(s)) e^{-\int_s^t A(z)dz} ds, \end{aligned} \tag{4.12}$$

$$\begin{aligned} (\mathcal{B}\varphi)(t) &= (\phi(t_0) + \gamma(t_0)\phi(t_0 - \tau(t_0)))\, e^{-\int_{t_0}^t A(z)dz} \\ &\quad + \int_{t_0}^{t} \left(\int_{s-\tau(s)}^{s} a(s,u)\,(G\varphi)(u)du \right) e^{-\int_s^t A(z)dz} ds, \end{aligned} \tag{4.13}$$

and

$$(H\varphi)(t) = (\mathcal{A}\varphi)(t) + (\mathcal{B}\varphi)(t). \tag{4.14}$$

If we are able to prove that H possesses a fixed point φ on the set S_ϕ, then $x(t,t_0,\phi) = \varphi(t)$ for $t \geq t_0$, $x(t,t_0,\phi) = \phi(t)$ on $[m_0, t_0]$, $x(t,t_0,\phi)$ satisfies (4.1) when its derivative exists and $x(t,t_0,\phi) \to 0$ as $t \to \infty$.

Let

$$\omega(t) = \int_{t-\tau(t)}^{t} |a(t,s)| \left(\int_s^t \left(\int_{u-\tau(u)}^{u} |a(u,v)|\, dv + |r(u)| \right) du + |\gamma(t)| + |\gamma(s)| \right) ds,$$

and assume that there are constants $k_1, k_2, k_3 > 0$ such that for $t_0 \leq t_1 \leq t_2$,

$$\left| \int_{t_1}^{t_2} A(z)dz \right| \leq k_1 |t_2 - t_1|, \tag{4.15}$$

$$|\tau(t_2) - \tau(t_1)| \leq k_2 |t_2 - t_1|, \tag{4.16}$$

and

$$|\gamma(t_2) - \gamma(t_1)| \leq k_3 |t_2 - t_1|. \tag{4.17}$$

Suppose for $t \geq t_0$,

$$|\mu(t)| \leq \delta A(t), \tag{4.18}$$

$$\omega(t) \leq \lambda A(t), \tag{4.19}$$

$$\sup_{t \geq t_0} |\gamma(t)| = \alpha_0, \tag{4.20}$$

and that

$$J(\alpha_0 + \lambda + \delta) < 1, \tag{4.21}$$

$$\max\left(|G(-L)|, |G(L)|\right) \leq \frac{2L}{J}, \tag{4.22}$$

where α_0, δ, λ, J are positive constants with $J > 3$.

Choose $\rho > 0$ small enough and such that

$$(1 + \gamma(t_0))\rho + \frac{3L}{J} \leq L. \tag{4.23}$$

The chosen ρ in the relation (4.23) is used below in Lemma 4.3 to show that if $\varepsilon = L$ and if $\|\phi\| < \rho$, then the solutions satisfy $x(t,t_0,\phi) < \varepsilon$.

Assume further that

$$t - \tau(t) \to \infty \text{ as } t \to \infty \text{ and } \int_0^t A(z)dz \to \infty \text{ as } t \to \infty, \tag{4.24}$$

$$\gamma(t) \to 0 \text{ as } t \to \infty, \tag{4.25}$$

$$\frac{\mu(t)}{A(t)} \to 0 \text{ as } t \to \infty, \tag{4.26}$$

and

$$\frac{\omega(t)}{A(t)} \to 0 \text{ as } t \to \infty. \tag{4.27}$$

We begin by showing that G given by (4.6) is a large contraction on the set S_ϕ. So, we suppose that $g : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions.

$(H1)$ $g : \mathbb{R} \to \mathbb{R}$ is continuous on $[-L, L]$ and differentiable on $(-L, L)$,

$(H2)$ the function g is strictly increasing on $[-L, L]$,

$(H3)$ $\sup_{t\in(-L,L)} g'(t) \leq 1$.

Theorem 4.4 ([1]) *Let $g : \mathbb{R} \to \mathbb{R}$ be a function satisfying $(H1) - (H3)$. Then the mapping G in (4.6) is a large contraction on the set S_ϕ.*

By step we will prove the fulfillment of (i), (ii) and (iii) in Theorem 4.2.

Lemma 4.2 *Suppose that (4.18)–(4.21) and (4.24) hold. For $\mathcal{A}$ defined in (4.12), if $\varphi \in S_\phi$, then $|(\mathcal{A}\varphi)(t)| \leq L/J \leq L$. Moreover, $(\mathcal{A}\varphi)(t) \to 0$ as $t \to \infty$.*

Proof. Using the conditions (4.18)–(4.21) and the expression (4.12) of the map $\mathcal{A}$, we get

$$\begin{aligned}
|(\mathcal{A}\varphi)(t)| &\leq |\gamma(t)| |\varphi(t-\tau(t))| + \int_{t_0}^{t} |L_\varphi(s)| e^{-\int_s^t A(z)dz} ds \\
&\quad + \int_{t_0}^{t} |\mu(s)| |\varphi(s-\tau(s))| e^{-\int_s^t A(z)dz} ds \\
&\leq \alpha_0 L + L \int_{t_0}^{t} \omega(s) e^{-\int_s^t A(z)dz} ds + L \int_{t_0}^{t} |\mu(s)| e^{-\int_s^t A(z)dz} ds \\
&\leq \alpha_0 L + \lambda L \int_{t_0}^{t} A(s) e^{-\int_s^t A(z)dz} ds + \delta L \int_{t_0}^{t} A(s) e^{-\int_s^t A(z)dz} ds \\
&\leq (\alpha_0 + \lambda + \delta) L \leq \frac{L}{J} < L.
\end{aligned}$$

So AS_ϕ is bounded by L as required.

Let $\varphi \in S_\phi$ be fixed. We will prove that $(\mathcal{A}\varphi)(t) \to 0$ as $t \to \infty$. Due to the conditions $t - \tau(t) \to \infty$ as $t \to \infty$ in (4.24) and (4.20), it is obvious that the first term on the right hand side of $\mathcal{A}$ tends to 0 as $t \to \infty$. That is

$$|\gamma(t)\varphi(t-\tau(t))| \leq \alpha_0 |\varphi(t-\tau(t))| \to 0 \text{ as } t \to \infty.$$

It is left to show that the two remaining integral terms of $\mathcal{A}$ go to zero as $t \to \infty$. Let $\varepsilon > 0$ be given. Find T such that $|\varphi(t-\tau(t))| < \varepsilon$ for $t \geq T$. Then we have

$$\begin{aligned}
&\left| \int_{t_0}^{t} L_\varphi(s) e^{-\int_s^t A(z)dz} ds \right| \\
&\leq \int_{t_0}^{T} |L_\varphi(s)| e^{-\int_s^t A(z)dz} ds + \int_{T}^{t} |L_\varphi(s)| e^{-\int_s^t A(z)dz} ds \\
&\leq L e^{-\int_T^t A(z)dz} \int_{t_0}^{T} \omega(s) e^{-\int_s^T A(z)dz} ds + \varepsilon \int_{T}^{t} \omega(s) e^{-\int_s^t A(z)dz} ds \\
&\leq L\lambda e^{-\int_T^t A(z)dz} + \varepsilon\lambda,
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int_{t_0}^{t} \mu(s)\varphi(s-\tau(s)) e^{-\int_s^t A(z)dz} ds \right| \\
&\leq \int_{t_0}^{T} |\mu(s)| |\varphi(s-\tau(s))| e^{-\int_s^t A(z)dz} ds \\
&+ \int_{T}^{t} |\mu(s)| |\varphi(s-\tau(s))| e^{-\int_s^t A(z)dz} ds \\
&\leq L e^{-\int_T^t A(z)dz} \int_{t_0}^{T} |\mu(s)| e^{-\int_s^T A(z)dz} ds + \varepsilon \int_{T}^{t} |\mu(s)| e^{-\int_s^t A(z)dz} ds \\
&\leq L\delta e^{-\int_T^t A(z)dz} + \varepsilon\delta.
\end{aligned}$$

The terms $L\lambda e^{-\int_T^t A(z)dz}$ and $L\delta e^{-\int_T^t A(z)dz}$ are arbitrarily smalls as $t \to \infty$, because of (4.24). This ends the proof. ■

Lemma 4.3 *Let (4.18)–(4.22) and (4.24) hold. For $\mathcal{A}$, $\mathcal{B}$ defined in (4.12) and (4.13), if $\varphi, \psi \in S_\phi$ are arbitrary, then*

$$\|\mathcal{A}\varphi + \mathcal{B}\psi\| \leq L.$$

Moreover, $\mathcal{B}$ is a large contraction on S_ϕ with a unique fixed point in S_ϕ and $(\mathcal{B}\psi)(t) \to 0$ as $t \to \infty$.

Proof. Using the definitions (4.12), (4.13) of $\mathcal{A}$ and $\mathcal{B}$ and applying (4.18)–(4.22), we obtain

$$\begin{aligned}
&\left|(\mathcal{A}\varphi)(t)+(\mathcal{B}\psi)(t)\right| \\
&\leq \left|(\mathcal{A}\varphi)(t)\right|+\left|(\mathcal{B}\psi)(t)\right| \\
&\leq \alpha_0 L+\lambda L\int_{t_0}^{t}A(s)e^{-\int_s^t A(z)dz}ds+L\int_{t_0}^{t}\left|\mu(s)\right|e^{-\int_s^t A(z)dz}ds \\
&+(1+\gamma(t_0))\left\|\phi\right\|e^{-\int_{t_0}^t A(z)dz}+\frac{2L}{J}\int_{t_0}^{t}A(s)e^{-\int_s^t A(z)dz}ds \\
&\leq (1+\gamma(t_0))\left\|\phi\right\|+(\alpha_0+\lambda+\delta)L+\frac{2L}{J} \\
&\leq (1+\gamma(t_0))\left\|\phi\right\|+\frac{L}{J}+\frac{2L}{J},
\end{aligned}$$

by the monotonicity of the mapping G. So from the above inequality, by choosing the initial function ϕ having small norm, say $\|\phi\|\leq\rho$, then, and referring to (4.23), we obtain

$$\left\|\mathcal{A}\varphi+\mathcal{B}\psi\right\|\leq(1+\gamma(t_0))\rho+\frac{3L}{J}\leq L.$$

Since $0\in S_\phi$, we have also proved that $|(\mathcal{B}\psi)(t)|\leq L$. The proof that $\mathcal{B}\psi$ is Lipschitzian is similar to that of the map $\mathcal{A}\varphi$ below. To see that $\mathcal{B}$ is a large contraction on S_ϕ with a unique fixed point, we know from Theorem 4.4 that $G(\varphi)=\varphi-g(\varphi)$ is a large contraction within the integrand. Thus, for any ε, from the proof of that Theorem 4.4, we have found $\eta<1$ such that

$$\begin{aligned}
&\left|(\mathcal{B}\varphi)(t)-(\mathcal{B}\psi)(t)\right| \\
&\leq\int_{t_0}^{t}\left(\int_{s-\tau(s)}^{s}\left|a(s,u)\right|\left|(G\varphi)(u)-(G\psi)(u)\right|du\right)e^{-\int_s^t A(z)dz}ds \\
&\leq\eta\int_{t_0}^{t}\left(\int_{s-\tau(s)}^{s}a(s,u)\left\|\varphi-\psi\right\|du\right)e^{-\int_s^t A(z)dz}ds \\
&\leq\eta\int_{t_0}^{t}A(s)\left\|\varphi-\psi\right\|e^{-\int_s^t A(z)dz}ds \\
&\leq\eta\left\|\varphi-\psi\right\|.
\end{aligned}$$

To prove that $(\mathcal{B}\psi)(t)\to 0$ as $t\to\infty$, we use (4.24) for the first term, and for the second term, we argue as above for the map $\mathcal{A}$. ■

4.3. Stability by Krasnoselskii-Burton's theorem

Lemma 4.4 *Suppose (4.18)–(4.21) hold. Then the mapping $\mathcal{A}$ is continuous on S_ϕ.*

Proof. Let $\varphi, \psi \in S_\phi$, then

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| \\
&\leq \alpha_0 |\varphi(t-\tau(t)) - \psi(t-\tau(t))| + \int_{t_0}^{t} |L_\varphi(s) - L_\psi(s)| e^{-\int_s^t A(z)dz} ds \\
&+ \int_{t_0}^{t} |\mu(s)| |\varphi(s-\tau(s)) - \psi(s-\tau(s))| e^{-\int_s^t A(z)dz} ds \\
&\leq \alpha_0 \|\varphi - \psi\| + \|\varphi - \psi\| \int_{t_0}^{t} \omega(s) e^{-\int_s^t A(z)dz} ds \\
&+ \|\varphi - \psi\| \int_{t_0}^{t} |\mu(s)| e^{-\int_s^t A(z)dz} ds \\
&\leq \alpha_0 \|\varphi - \psi\| + \lambda \|\varphi - \psi\| \int_{t_0}^{t} A(s) e^{-\int_s^t A(z)dz} ds \\
&+ \delta \|\varphi - \psi\| \int_{t_0}^{t} A(s) e^{-\int_s^t A(z)dz} ds \\
&\leq (\alpha_0 + \lambda + \delta) \|\varphi - \psi\| \leq \frac{1}{J} \|\varphi - \psi\|.
\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Define $\eta = \varepsilon J$. Then for $\|\varphi - \psi\| \leq \eta$, we obtain

$$\|\mathcal{A}\varphi - \mathcal{A}\psi\| \leq \frac{1}{J} \|\varphi - \psi\| \leq \varepsilon.$$

Therefore, $\mathcal{A}$ is continuous. ■

Lemma 4.5 *Let (4.15)–(4.20) and (4.25)–(4.27) hold. The function $\mathcal{A}\varphi$ is Lipschitzian and the operator $\mathcal{A}$ maps S_ϕ into a compact subset of S_ϕ.*

Proof. Let $\varphi \in S_\phi$ and let $0 \leq t_1 < t_2$. Then

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \\
&\leq |\gamma(t_2)\varphi(t_2 - \tau(t_2)) - \gamma(t_1)\varphi(t_1 - \tau(t_1))| \\
&+ \left| \int_{t_0}^{t_2} L_\varphi(s) e^{-\int_s^{t_2} A(z)dz} ds - \int_{t_0}^{t_1} L_\varphi(s) e^{-\int_s^{t_1} A(z)dz} ds \right| \\
&+ \left| \int_{t_0}^{t_2} \mu(s)\varphi(s-\tau(s)) e^{-\int_s^{t_2} A(z)dz} ds - \int_{t_0}^{t_1} \mu(s)\varphi(s-\tau(s)) e^{-\int_s^{t_1} A(z)dz} ds \right|. \qquad (4.28)
\end{aligned}$$

By hypotheses (4.16)–(4.17), we have

$$\begin{aligned}
&|\gamma(t_2)\varphi(t_2-\tau(t_2))-\gamma(t_1)\varphi(t_1-\tau(t_1))| \\
&\leq |\gamma(t_2)|\,|\varphi(t_2-\tau(t_2))-\varphi(t_1-\tau(t_1))|+|\varphi(t_1-\tau(t_1))|\,|\gamma(t_2)-\gamma(t_1)| \\
&\leq \alpha_0 k\,|(t_2-t_1)-(\tau(t_2)-\tau(t_1))|+Lk_3\,|t_2-t_1| \\
&\leq (\alpha_0 k+\alpha_0 k k_2+Lk_3)\,|t_2-t_1|\,,
\end{aligned} \tag{4.29}$$

where k is the Lipschitz constant of φ. By hypotheses (4.15) and (4.18), we have

$$\begin{aligned}
&\left|\int_{t_0}^{t_2}\mu(s)\varphi(s-\tau(s))e^{-\int_s^{t_2}A(z)dz}ds-\int_{t_0}^{t_1}\mu(s)\varphi(s-\tau(s))e^{-\int_s^{t_1}A(z)dz}ds\right| \\
&\leq \left|\int_{t_0}^{t_1}\mu(s)\varphi(s-\tau(s))e^{-\int_s^{t_1}A(z)dz}\left(e^{-\int_{t_1}^{t_2}A(z)dz}-1\right)ds\right. \\
&\left.+\int_{t_1}^{t_2}\mu(s)\varphi(s-\tau(s))e^{-\int_s^{t_2}A(z)dz}ds\right| \\
&\leq L\left|e^{-\int_{t_1}^{t_2}A(z)dz}-1\right|\int_{t_0}^{t_1}\delta A(s)e^{-\int_s^{t_1}A(z)dz}ds+L\int_{t_1}^{t_2}|\mu(s)|\,e^{-\int_s^{t_2}A(z)dz}ds \\
&\leq L\delta\int_{t_1}^{t_2}A(s)ds+L\int_{t_1}^{t_2}e^{-\int_s^{t_2}A(z)dz}d\left(\int_{t_1}^{s}|\mu(v)|\,dv\right) \\
&\leq L\delta\int_{t_1}^{t_2}A(s)ds+L\left\{\left[e^{-\int_s^{t_2}A(z)dz}\int_{t_1}^{s}|\mu(v)|\,dv\right]_{t_1}^{t_2}\right. \\
&\left.+\int_{t_1}^{t_2}A(s)e^{-\int_s^{t_2}A(z)dz}\int_{t_1}^{s}|\mu(v)|\,dvds\right\} \\
&\leq L\delta\int_{t_1}^{t_2}A(s)ds+L\int_{t_1}^{t_2}|\mu(v)|\,dv\left(1+\int_{t_1}^{t_2}A(s)e^{-\int_s^{t_2}A(z)dz}ds\right) \\
&\leq L\delta\int_{t_1}^{t_2}A(s)ds+2L\int_{t_1}^{t_2}|\mu(v)|\,dv \\
&\leq L\delta\int_{t_1}^{t_2}A(s)ds+2L\delta\int_{t_1}^{t_2}A(v)dv \\
&\leq 3L\delta k_1\,|t_2-t_1|\,.
\end{aligned} \tag{4.30}$$

Similarly, by (4.15) and (4.19), we deduce

$$\begin{aligned}
&\left|\int_{t_0}^{t_2} L_\varphi(s)e^{-\int_s^{t_2} A(z)dz}ds - \int_{t_0}^{t_1} L_\varphi(s)e^{-\int_s^{t_1} A(z)dz}ds\right| \\
&= \left|\int_{t_0}^{t_1} L_\varphi(s)e^{-\int_s^{t_1} A(z)dz}\left(e^{-\int_{t_1}^{t_2} A(z)dz} - 1\right)ds + \int_{t_1}^{t_2} L_\varphi(s)e^{-\int_s^{t_2} A(z)dz}ds\right| \\
&\le L\left|e^{-\int_{t_1}^{t_2} A(z)dz} - 1\right|\int_{t_0}^{t_1} \omega(s)e^{-\int_s^{t_1} A(z)dz}ds + L\int_{t_1}^{t_2} \omega(s)e^{-\int_s^{t_2} A(z)dz}ds \\
&\le L\left|e^{-\int_{t_1}^{t_2} A(z)dz} - 1\right|\int_{t_0}^{t_1} \lambda A(s)e^{-\int_s^{t_1} A(z)dz}ds + L\int_{t_1}^{t_2} \omega(s)e^{-\int_s^{t_2} A(z)dz}ds \\
&\le \lambda L\int_{t_1}^{t_2} A(z)dz + L\int_{t_1}^{t_2} e^{-\int_s^{t_2} A(z)dz}d\left(\int_{t_1}^{s}\omega(v)dv\right) \\
&\le \lambda L\int_{t_1}^{t_2} A(z)dz + L\left\{\left[e^{-\int_s^{t_2} A(z)dz}\int_{t_1}^{s}\omega(v)dv\right]_{t_1}^{t_2}\right. \\
&\left. + \int_{t_1}^{t_2} A(s)e^{-\int_s^{t_2} A(z)dz}\int_{t_1}^{s}\omega(v)dvds\right\} \\
&\le \lambda L\int_{t_1}^{t_2} A(z)dz + L\int_{t_1}^{t_2}\omega(v)dv\left(1 + \int_{t_1}^{t_2} A(s)e^{-\int_s^{t_2} A(z)dz}ds\right) \\
&\le \lambda L\int_{t_1}^{t_2} A(z)dz + 2L\int_{t_1}^{t_2}\omega(v)dv \\
&\le \lambda L\int_{t_1}^{t_2} A(z)dz + 2L\lambda\int_{t_1}^{t_2} A(v)dv \\
&\le 3\lambda L k_1\left|t_2 - t_1\right|. \qquad (4.31)
\end{aligned}$$

Thus, by substituting (4.29)–(4.31) in (4.28), we obtain

$$\begin{aligned}
&\left|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)\right| \\
&\le (\alpha_0 k + \alpha_0 k k_2 + L k_3)\left|t_2 - t_1\right| + 3L\delta k_1\left|t_2 - t_1\right| + 3L\lambda k_1\left|t_2 - t_1\right| \\
&\le K\left|t_2 - t_1\right|, \qquad (4.32)
\end{aligned}$$

for a constant $K > 0$. This shows $\mathcal{A}\varphi$ that is Lipschitzian if φ is and that $\mathcal{A}S_\phi$ is equicontinuous. Next, we notice that for arbitrary $\varphi \in S_\phi$, we have

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t)| \\
&\le |\gamma(t)\varphi(t-\tau(t))| + \int_{t_0}^{t} |L_\varphi(s)|\, e^{-\int_s^t A(z)dz} ds \\
&+ \int_{t_0}^{t} |\mu(s)|\, |\varphi(s-\tau(s))|\, e^{-\int_s^t A(z)dz} ds \\
&\le L\,|\gamma(t)| + L\int_{t_0}^{t} \omega(s) e^{-\int_s^t A(z)dz} ds + L\int_{t_0}^{t} |\mu(s)|\, e^{-\int_s^t A(z)dz} ds \\
&\le L\,|\gamma(t)| + L\int_{t_0}^{t} A(s)\frac{\omega(s)}{A(s)} e^{-\int_s^t A(z)dz} ds + L\int_{t_0}^{t} A(s)\frac{|\mu(s)|}{A(s)} e^{-\int_s^t A(z)dz} ds \\
&:= q(t),
\end{aligned}$$

because of (4.25)–(4.27). Using a method like the one used for the map $\mathcal{A}$, we see that $q(t) \to 0$ as $t \to \infty$. By Theorem 4.3, we conclude that the set $\mathcal{A}S_\phi$ resides in a compact set. ■

Theorem 4.5 *Let $L > 0$. Suppose that the conditions $(H1) - (H3)$, (4.2) and (4.25)–(4.27) hold. If ϕ is a given initial function which is sufficiently small, then there is a solution $x(t, t_0, \phi)$ of (4.1) with $|x(t, t_0, \phi)| \le L$ and $x(t, t_0, \phi) \to 0$ as $t \to \infty$.*

Proof. From Lemmas 4.2 and 4.5 we have $\mathcal{A}$ is bounded by L, Lipschitzian and $(\mathcal{A}\varphi)(t) \to 0$ as $t \to \infty$. So $\mathcal{A}$ maps S_ϕ into S_ϕ. From Lemmas 4.3 and 4.5 for arbitrary, we have $\varphi, \psi \in S_\phi$, $\mathcal{A}\varphi + \mathcal{B}\psi$ since both $\mathcal{A}\varphi$ and $\mathcal{B}\psi$ are Lipschitzian bounded by L and $(\mathcal{B}\psi)(t) \to 0$ as $t \to \infty$. From Lemmas 4.4 and 4.5, we have proved that $\mathcal{A}$ is continuous and $\mathcal{A}S_\phi$ resides in a compact set. Thus, all the conditions of Theorem 4.2 are satisfied. Therefore, there exists a solution of (4.1) with $|x(t, t_0, \phi)| \le L$ and $x(t, t_0, \phi) \to 0$ as $t \to \infty$. ■

4.4 Stability in weighted Banach spaces

Referring to Burton [25], except for the fixed point method, we know of no other way proving that solutions of (4.1) converge to zero. Nevertheless, if all we need is stability and not asymptotic stability, then we can avoid conditions (4.25)–(4.27) and still use Krasnoselskii-Burton's theorem on a Banach space endowed with a weighted norm.

Let $h : [m_0, \infty) \to [1, \infty)$ be any strictly increasing and continuous function with $h(m_0) = 1$, $h(s) \to \infty$ as $s \to \infty$. Let $(S, |.|_h)$ be the Banach space of continuous $\varphi : [m_0, \infty) \to \mathbb{R}$ for which

$$|\varphi|_h = \sup_{t \geq m_0} \left| \frac{\varphi(t)}{h(t)} \right| < \infty,$$

exists. We continue to use $\|.\|$ as the supremum norm of any $\varphi \in S$ provided φ bounded. Also, we use $\|\phi\|$ as the bound of the initial function. Further, in a similar way as Theorem 4.4, we can prove that the function $G(\varphi) = \varphi - g(\varphi)$ is still a large contraction with the norm $|.|_h$.

Theorem 4.6 *If the conditions of Theorem 4.5 hold, except for (4.25)–(4.27), then the zero solution of (4.1) is stable.*

Proof. We prove the stability starting at t_0. Let $\varepsilon > 0$ be given such that $0 < \varepsilon < L$, then for $|x| \leq \varepsilon$, find α^* with $|x - g(x)| \leq \alpha^*$ and choose a number α such that

$$\alpha + \alpha^* + \frac{\varepsilon}{J} \leq \varepsilon. \tag{4.33}$$

In fact, since $x - g(x)$ is increasing on $(-L, L)$, we may take $\alpha^* = \frac{2\varepsilon}{J}$. Thus, inequality (4.33) allows $\alpha > 0$. Now, remove the condition $\varphi(t) \to 0$ as $t \to \infty$ from S_ϕ defined previously and consider the set

$$\begin{aligned} E_\phi &= \{\varphi \in S : \varphi \text{ Lipshitzian, } |\varphi(t)| \leq \varepsilon,\, t \in [m_0, \infty) \\ &\quad \text{and } \varphi(t) = \phi(t) \text{ for } t \in [m_0, t_0]\}. \end{aligned}$$

Define $\mathcal{A}$, $\mathcal{B}$ on E_ϕ as before by (4.12), (4.13). We easily check that if $\varphi \in E_\phi$, then $|(\mathcal{A}\varphi)(t)| \leq \varepsilon$, and $\mathcal{B}$ is a large contraction on E_ϕ. Also, by choosing $\|\phi\| \leq \alpha$ and referring to (4.33), we verify that for $\varphi, \psi \in E_\phi$, $|(\mathcal{A}\varphi)(t) + (\mathcal{B}\psi)(t)| \leq \varepsilon$ and $|(\mathcal{B}\psi)(t)| \leq \varepsilon$. $\mathcal{A}E_\phi$ is an equicontinuous set. According to [[25], Theorem 4.0.1], in the space $(S, |.|_h)$ the set $\mathcal{A}E_\phi$ resides in a compact subset of E_ϕ. Moreover, the operator $\mathcal{A} : E_\phi \to E_\phi$ is countinuous. Indeed, for $\varphi, \psi \in S_\phi$,

$$\begin{aligned}
&\frac{|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)|}{h(t)} \\
&\leq \frac{1}{h(t)} \{|\gamma(t)| |\varphi(t-\tau(t)) - \psi(t-\tau(t))| \\
&+ \left| \int_{t_0}^{t} (L_\varphi(s) - L_\psi(s)) e^{-\int_s^t A(z)dz} ds \right| \\
&+ \left| \int_{t_0}^{t} \mu(s) (\varphi(s-\tau(s)) - \psi(s-\tau(s))) e^{-\int_s^t A(z)dz} ds \right| \Bigg\} \\
&\leq \alpha_0 |\varphi - \psi|_h + |\varphi - \psi|_h \int_{t_0}^{t} \omega(s) \frac{h(s)}{h(t)} e^{-\int_s^t A(z)dz} ds \\
&+ |\varphi - \psi|_h \int_{t_0}^{t} |\mu(s)| \frac{h(s-\tau(s))}{h(t)} e^{-\int_s^t A(z)dz} ds \\
&\leq \alpha_0 |\varphi - \psi|_h + \lambda |\varphi - \psi|_h \int_{t_0}^{t} A(s) e^{-\int_s^t A(z)dz} ds \\
&+ \delta |\varphi - \psi|_h \int_{t_0}^{t} A(s) e^{-\int_s^t A(z)dz} ds \\
&\leq (\alpha_0 + \lambda + \delta) |\varphi - \psi|_h \leq \frac{1}{J} |\varphi - \psi|_h .
\end{aligned}$$

■

Chapter 5

Exponential stability of linear Levin-Nohel integro-dynamic equations on time scales

Keywords. Exponential stability, Levin-Nohel integro-dynamic equations, Bohl-Perron theorem, time scales.

In this chapter we present a work publiched in [8], namely, K. Ali Khelil, F. Bouchelaghem and L. Bouzettouta, Int. J. Appl. Math. Stat., 56 (2017), No. 6, pp. 138-149.

In chapter, we use Bohl-Perron theorem to establish new conditions for the exponential stability of Levin-Nohel intego-dynamic equation with variable delay. The results obtained here extend the work of Dung [38].

5.1 Introduction

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above and below; i.e., it is a time scale interval of the form $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$ and we denote $\mathbb{T}^{+} = [t_0, \infty)_{\mathbb{T}}$.

In [38], Dung has used the Bohl-Perron theorem to study the exponential stability of Levin-Nohel integro-differential equation with delay having the form

$$x'(t) + \int_{t-r(t)}^{t} a(t,s)x(s)ds = 0,\ t \in [t_0, \infty). \tag{5.1}$$

The aim of this paper is to extend the theory established in [38] to Levin-Nohel integro-dynamic equation on time scales. More precisely, we consider the equation

$$x^{\Delta}(t) + \int_{t-r(t)}^{t} a(t,s)x(s)\Delta s = 0, \text{ for } t \in \mathbb{T}^{+} \tag{5.2}$$

with the following conditions,

(A_1). The delay $r(t)$ is a Lebesgue Δ-measurable function satisfying $r(t) \geq 0$ and $\tau = \sup_{t\in[t_0,\infty)_{\mathbb{T}}} r(t) < \infty$.

(A_2). The Kernel $a : [t_0, +\infty)_{\mathbb{T}} \times [-\tau, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}$ is Lebesgue Δ-measurable such that $a(t,.)$ are locally Δ-integrables for any t and the function $t \longmapsto \int_{t-r(t)}^{t} |a(t,s)| (\sigma(t) - s) \Delta s$ is bounded on $\mathbb{T}^{+}$.

The $^{\Delta}$ is called Hilger-derivative, when $\mathbb{T} = \mathbb{R}$ we have the usual derivative and the equation (5.2) becomes (5.1). If $\mathbb{T} = \mathbb{Z}$, we have the difference operator $\Delta x(t) = x(t+1) - x(t)$ and then the equation (5.2) becomes

$$\Delta x(t) + \sum_{s=t-r(t)}^{t-1} a(t,s)x(s) = 0, \ t \in [t_0, \infty)_{\mathbb{Z}}.$$

The goal of this study is to employ Bohl-Perron theorem to show that under suitable conditions the trivial solution of (5.2) is exponentially stable.

5.2 Existence and exponential stability

For each $t_0 \geq 0$, we consider the Levin-Nohel equation

$$x^{\Delta}(t) + \int_{t-r(t)}^{t} a(t,s)x(s)\Delta s = 0 \tag{5.3}$$

and the initial data

$$x(t) = \varphi(t), \ t < t_0, \ x(t_0) = x_0, \tag{5.4}$$

where $\varphi : (-\infty, t_0)_{\mathbb{T}} \rightarrow \mathbb{R}$ is a Borel Δ-measurable bounded function.

For the convenience we recall some fundamental concepts.

Definition 5.1 An absolutely rd-continuous function $x : \mathbb{T}^{+} \rightarrow \mathbb{R}$ is called a solution of Eq. (5.3) with initial data (5.4) if it satisfies equalities (5.4) for $t \leq t_0$ and (5.3) for all most $t \geq t_0$.

Definition 5.2 Let $t_0 \geq 0$ and $x_0 \in \mathbb{R}$. We say that the zero solution of the (5.3) and (5.4) is exponentially stable if there exists a constant $K \in \mathbb{R}^+$ and a $\lambda \in \mathcal{R}^+$such that for any solution of (5.3) and (5.4) one has the estimate

$$|x(t)| \leq K \left(\sup_{s \in (-\infty, t_0)_{\mathbb{T}}} |\varphi(s)| + |x(t_0)| \right) e_{\ominus\lambda}(t, t_0) \;\text{ for all } t \in \mathbb{T}^+.$$

(We notice that K, λ do not depend on t_0.)

Denote by $L^\infty_\Delta(\mathbb{T}^+)$ the space of all Δ-measurable essentially bounded functions on $\mathbb{T}^+$ with the essential supremum norm $\|x\|_{L^\infty_\Delta} = ess\sup_{t \geq t_0} |x(t)|$, and by $BC_{rd}(\mathbb{T}^+)$ the space of all rd-continuous bounded functions on $\mathbb{T}^+$ with the supremum norm $\|x\|_{BC_{rd}} = \sup_{t \geq t_0} |x(t)|$, then

$$BC_{rd}\left(\mathbb{T}^+\right) = BC_{rd}\left(\mathbb{T}^+, \mathbb{R}\right) = \left\{ x \in C_{rd}\left(\mathbb{T}^+, \mathbb{R}\right) : \sup_{t \in \mathbb{T}^+} |x(t)| < \infty \right\}.$$

Let $f \in L^\infty_\Delta(\mathbb{T}^+)$, consider the equation

$$x^\Delta(t) + \int_{t-r(t)}^t a(t,s)x(s)\Delta s = f(t),\; t \in \mathbb{T}^+, \tag{5.5}$$

with the zero initial data, i.e., $x(t) = 0$ for $t \leq t_0$. The following lemma is a Bohl-Perron type theorem.

Lemma 5.1 ([20]) *Suppose that there exists $t_0 \geq 0$ such that for any $f \in L^\infty_\Delta(\mathbb{T}^+)$, the solution of Eq.(5.5) with the zero initial data belongs to $BC_{rd}(\mathbb{T}^+)$. Then, Eq. (5.2) is exponentially stable.*

Theorem 5.1 *Suppose assumptions (A_1) and (A_2) hold. Suppose that the ordinary differential equation $x^\Delta(t) + A(t)x^\sigma(t) = 0$ is exponentially stable and that*

$$\limsup_{u \to \infty} \left(\sup_{t \geq u} \int_u^t \omega(s) e_{\ominus A}(t,s)\Delta s \right) < 1, \tag{5.6}$$

where $A(z) = \int_{z-r(z)}^z a(z,s)\Delta s$ with $A \in \mathcal{R}^+$ and

$$\omega(s) = \int_{s-r(s)}^s |a(s,w)| \int_w^{\sigma(s)} \left(\int_{u-r(u)}^u |a(u,v)|\,\Delta v \right) \Delta u \Delta w.$$

Then, Eq. (5.2) is exponentially stable.

5.2. Existence and exponential stability

Proof. For any $t_0 \geq 0$, let $x(t)$ be the solution of (5.5). Obviously, we always have the relation

$$x(s) = x^\sigma(t) - \int_s^{\sigma(t)} x^\Delta(u)\Delta u$$

Inserting this relations into (5.5), we get

$$x^\Delta(t) + \int_{t-r(t)}^t a(t,s)\left(x^\sigma(t) - \int_s^{\sigma(t)} x^\Delta(u)\Delta u\right)\Delta s = f(t),\ t \geq t_0.$$

After substituting x^Δ from (5.5), we obtain

$$\begin{aligned} &x^\Delta(t) + x^\sigma(t)\int_{t-r(t)}^t a(t,s)\Delta s \\ &+ \int_{t-r(t)}^t a(t,s)\left(\int_s^{\sigma(t)}\left(\int_{u-r(u)}^u a(u,v)x(v)\Delta v\right)\Delta u\right)\Delta s \\ &= f(t) + \int_{t-r(t)}^t a(t,s)\left(\int_s^{\sigma(t)} f(u)\Delta u\right)\Delta s. \end{aligned} \tag{5.7}$$

For the simplicity, we put

$$g(s) := f(s) + \int_{s-r(s)}^s a(s,u)\left(\int_u^{\sigma(s)} f(v)\Delta v\right)\Delta u$$

and

$$h(t) := \int_{t_0}^t g(s)e_{\ominus A}(t,s)\Delta s.$$

Then, Eq. (5.7) can be rewritten as follows

$$x^\Delta(t) + A(t)x^\sigma(t) + \int_{t-r(t)}^t a(t,s)\left(\int_s^{\sigma(t)}\left(\int_{u-r(u)}^u a(u,v)x(v)\Delta v\right)\Delta u\right)\Delta s = g(t)$$

which, by the variation of constants formula, gives us

$$x(t)+\int_{t_0}^t e_{\ominus A}(t,s)\left[\int_{s-r(s)}^s a(s,w)\left(\int_w^{\sigma(s)}\left(\int_{u-r(u)}^u a(u,v)x(v)\Delta v\right)\Delta w\right)\right]\Delta s = h(t),\ t \geq t_0. \tag{5.8}$$

Consider the operator $\mathcal{N}$ defined by $(\mathcal{N}x)(t) = 0$, $t \leq t_0$ and for $t \geq t_0$,

$$(\mathcal{N}x)(t) = \int_{t_0}^t e_{\ominus A}(t,s)\left[\int_{s-r(s)}^s a(s,w)\left(\int_w^{\sigma(s)}\left(\int_{u-r(u)}^u a(u,v)x(v)\Delta v\right)\Delta w\right)\right]\Delta s.$$

5.2. Existence and exponential stability

Then, we can rewrite (5.8) as follows

$$(I+\mathcal{N})\,x = h, \tag{5.9}$$

where I is the unit operator.

By the definition of the function g, we have

$$\begin{aligned} g(t) &\le \|f\|_{L^\infty_\Delta} + \|f\|_{L^\infty_\Delta} \int_{t-r(t)}^{t} |a(t,s)|\,(\sigma(t)-s)\,\Delta s \\ &= \|f\|_{L^\infty_\Delta}\left(1+\int_{t-r(t)}^{t} |a(t,s)|\,(\sigma(t)-s)\,\Delta s\right),\ t\in\mathbb{T}^+. \end{aligned}$$

Hence, assumption (A_2) implies that $\|g\|_{L^\infty_\Delta} < \infty$ or $g \in L^\infty_\Delta(\mathbb{T}^+)$. Moreover, the equation $x^\Delta(t) + A(t)x^\sigma(t) = 0$ is exponentially stable. This ensures that $h \in L^\infty_\Delta(\mathbb{T}^+)$. Indeed, we have

$$\begin{aligned} h(t) &\le \left(\sup_{t\ge t_0}\int_{t_0}^{t} e_{\ominus A}(t,s)\Delta s\right)\|g\|_{L^\infty_\Delta} \\ &\le \left(\sup_{t\ge t_0}\int_{t_0}^{t} K e_{\ominus\lambda}(t,s)\Delta s\right)\|g\|_{L^\infty_\Delta} \\ &= \left(\sup_{t\ge t_0}\frac{K}{\lambda}\left(1-e_{\ominus\lambda}(t,t_0)\right)\right)\|g\|_{L^\infty_\Delta} \\ &\le \frac{K}{\lambda}\|g\|_{L^\infty_\Delta} < \infty. \end{aligned}$$

For the operator $\mathcal{N}$, we have the following estimate

$$\begin{aligned} |(\mathcal{N}x)(t)| &\le \int_{t_0}^{t} e_{\ominus A}(t,s)\left[\int_{s-r(s)}^{s} a(s,w)\left(\int_{w}^{\sigma(s)}\int_{u-r(u)}^{u} a(u,v)\Delta v\Delta w\right)\right]\Delta s\,\|x\|_{L^\infty_\Delta} \\ &= \left(\int_{t_0}^{t}\omega(s)e_{\ominus A}(t,s)\Delta s\right)\|x\|_{L^\infty_\Delta} \\ &\le \left(\sup_{t\ge t_0}\int_{t_0}^{t}\omega(s)e_{\ominus A}(t,s)\Delta s\right)\|x\|_{L^\infty_\Delta},\ t\in\mathbb{T}^+, \end{aligned}$$

or equivalently

$$\|\mathcal{N}x\|_{L^\infty_\Delta} = \left(\sup_{t\ge t_0}\int_{t_0}^{t}\omega(s)e_{\ominus A}(t,s)\Delta s\right)\|x\|_{L^\infty_\Delta}.$$

Condition (5.6) means that there exists $t_0 \ge 0$ such that

$$\sup_{t\ge t_0}\int_{t_0}^{t}\omega(s)e_{\ominus A}(t,s)\Delta s := \alpha < 1.$$

Consequently, $\|\mathcal{N}\| \leq \alpha < 1$ in the space $L^{\infty}_{\Delta}(\mathbb{T}^{+})$, and hence, $(I+\mathcal{N})^{-1}$ exists and its norm is bounded by $\dfrac{1}{1-\alpha}$. We, therefore, can conclude that Eq. (5.9) has a unique solution which is given by

$$x = (I+\mathcal{N})^{-1} h.$$

It is clear that this solution is rd-continuous and $\|x\|_{L^{\infty}_{\Delta}} \leq \frac{1}{1-\alpha} \|\mathcal{N}\|_{L^{\infty}_{\Delta}} < \infty$. So $x \in BC_{rd}(\mathbb{T}^{+})$. ■

Now, we consider a special form of (5.2) that is the convolution equation of the form

$$x^{\Delta}(t) + \int_{t-r}^{t} a(t-s)\, x(s)\, \Delta s = 0, \tag{5.10}$$

where $r > 0$, $a : [0,r]_{\mathbb{T}} \to \mathbb{R}$ is Lebesgue Δ-measurable and we introduce the condition $\mu^{*} := \sup_{t\geq 0} \mu(t) < \infty$. We have

$$A(z) = \int_{z-r(z)}^{z} a(z-s)\, \Delta s = \int_{z-r}^{z} a(z-s)\, \Delta s = \int_{0}^{r} a(s)\, \Delta s := a_0 = const.$$

Hence, the equation $x^{\Delta}(t) + A(t)x^{\sigma}(t) = 0$ is exponentially stable only if $a_0 \in \mathcal{R}^{+}$. In this case, the left hand side of (5.6) reduces to

$$\begin{aligned}
&\int_{0}^{t} \left(\int_{s-r(s)}^{s} |a(s,w)| \int_{w}^{\sigma(s)} \left(\int_{u-r(u)}^{u} |a(u,v)|\, \Delta v \right) \Delta u \Delta w \right) e_{\ominus A}(t,s) \Delta s \\
=\; & \int_{0}^{t} \left(\int_{s-r}^{s} |a(s-w)| \int_{w}^{s} \left(\int_{u-r}^{u} |a(u-v)|\, \Delta v \right) \Delta u \Delta w \right) e_{\ominus A}(t,s) \Delta s \\
&+ \int_{0}^{t} \left(\int_{s-r}^{s} |a(s-w)| \int_{s}^{\sigma(s)} \left(\int_{u-r}^{u} |a(u-v)|\, \Delta v \right) \Delta u \Delta w \right) e_{\ominus A}(t,s) \Delta s \\
=\; & \int_{0}^{t} \left(\int_{s-r}^{s} |a(s-w)| \int_{w}^{s} \left(\int_{u-r}^{u} |a(u-v)|\, \Delta v \right) \Delta u \Delta w \right) e_{\ominus a_0}(t,s) \Delta s \\
&+ \int_{0}^{t} \left(\int_{s-r}^{s} |a(s-w)|\, \mu(s) \left(\int_{s-r}^{s} |a(s-v)|\, \Delta v \right) \Delta w \right) e_{\ominus a_0}(t,s) \Delta s \\
=\; & \left(\int_{0}^{r} |a(v)|\, \Delta v \right) \left(\int_{0}^{r} |a(w)|\, w \Delta w \right) \int_{0}^{t} e_{\ominus a_0}(t,s) \Delta s \\
&+ \left(\int_{0}^{r} |a(v)|\, \Delta v \right)^{2} \int_{0}^{t} \mu(s)\, e_{\ominus a_0}(t,s) \Delta s \\
\leq\; & \frac{1}{a_0} \left(\int_{0}^{r} |a(v)|\, \Delta v \right) \left(\int_{0}^{r} |a(w)|\, w \Delta w + \mu^{*} \int_{0}^{r} |a(v)|\, \Delta v \right).
\end{aligned}$$

By applying Theorem 4.1, we get the following corollary.

Corollary 5.1 *Suppose that the kernel* $a : [0, r]_{\mathbb{T}} \to \mathbb{R}$ *is Lebesgue* Δ*-measurable such that*

$$0 < \int_0^r a(s)\Delta s < \infty \text{ and } \left(\int_0^r |a(s)|\,\Delta s\right)\left(\int_0^r |a(s)s|\,\Delta s + \mu^* \int_0^r |a(s)|\,\Delta s\right) < \int_0^r |a(s)|\,\Delta s. \tag{5.11}$$

Then, the zero solution of (5.10) is exponentially stable.

5.3 Levin-Nohel integro-dynamic equations with impulsive effects

As an application of Theorem 5.1 we now establish a sufficient condition for exponential stability of Levin-Nohel integro-dynamic equations with impulsive effects. Consider the equation

$$\begin{cases} x^{\Delta}(t) + \int_{t-r(t)}^{t} a_0(t,s)x(s)\Delta s = 0,\ t \geq t_0,\ t \neq t_k \\ x(t_k^+) - x(t_k) = \lambda_k x(t_k),\ k = 1,\ 2, ... \end{cases} \tag{5.12}$$

where $\lambda_k,\ k = 1, 2, ...$are constants, the impulsive moments satisfy

$$0 = t_0 < t_1 < t_2 < ..., \lim_{k\to\infty} t_k = \infty \text{ and } x(t_k^+) = \lim_{t\to t_k^+} x(t).$$

The kernel a_0 satisfies the following assumption.

(A_2') The kernel $a_0 : [t_0, \infty)_{\mathbb{T}} \times [-\tau, \infty)_{\mathbb{T}} \to \mathbb{R}$ is Lebesgue Δ-measurable such that $a_0(t,s) \prod\limits_{s\leq t_k<t} (1+\lambda)^{-1}$ are locally integrable in s for any t and the function

$$t \to \left|\int_{t-r(t)}^{t} a_0(t,s) \prod_{s\leq t_k<t} (1+\lambda)^{-1}\right| (\sigma(t) - s)\,\Delta s$$

is bounded on $\mathbb{T}^+$.

If we denote by t_k the first moment of time such that $\lambda_k = -1$, then $x(t_k^+) = 0$ and so we can infer that $x(t) = 0$ for all $t > t_k$. Thus, in this case, the solution is exponentially stable. In the sequel, it is more interesting to consider the case where $\lambda_k \neq -1$ for all $k = 1, 2, ...$

In order to be able to apply Theorem 5.1, we will transform Eq. (5.12) into an equation without impulses. So, consider the substitution

$$x(t) = \prod_{0<t_k<t} (1+\lambda_k) y(t),$$

where we assume that a product equals unity if the number of factors is zero. We have

$$y(t) = \prod_{0<t_k<t} (1+\lambda_k)^{-1} x(t),$$

and

$$y^{\Delta}(t) = \prod_{0<t_k<t} (1+\lambda_k)^{-1} x^{\Delta}(t) = -\prod_{0<t_k<t} (1+\lambda_k)^{-1} \int_{t-r(t)}^{t} a_0(t,s) \prod_{0<t_k<s} (1+\lambda_k) y(s)\Delta s,$$

or equivalently

$$y^{\Delta}(t) + \int_{t-r(t)}^{t} \left[a_0(t,s) \prod_{s\le t_k<t} (1+\lambda_k)^{-1} \right] y(s)\Delta s = 0, \ t \in \mathbb{T}^{+}. \tag{5.13}$$

By applying Theorem 5.1 to Eq. (5.13), we get the following theorem.

Theorem 5.2 *Let assumptions (A_1) and (A_2') hold. Suppose that there exists a finite number $M > 0$ such that*

$$\left| \prod_{0<t_k<t} (1+\lambda_k) \right| \le M, \ t \in \mathbb{T}^{+},$$

the ordinary differential equation $x^{\Delta}(t) + A_{\lambda}(t) x^{\sigma}(t) = 0$ is exponentially stable and that

$$\limsup_{u\to\infty} \left(\sup_{t\ge u} \int_u^t \omega_{\lambda}(s) e_{\ominus A_{\lambda}}(t,s)\Delta s \right) < 1, \tag{5.14}$$

where

$$A_{\lambda}(z) = \int_{z-r(z)}^{z} a_0(t,s) \prod_{s\le t_k<z} (1+\lambda_k)^{-1} \Delta s$$

and

$$\omega_{\lambda}(s) =$$
$$\int_{s-r(s)}^{s} \left| a_0(s,w) \prod_{w\le t_k<s} (1+\lambda_k)^{-1} \right| \int_{w}^{\sigma(s)} \left(\int_{u-r(u)}^{u} \left| a_0(u,v) \prod_{v\le t_k<u} (1+\lambda_k)^{-1} \right| \Delta v \right) \Delta u \Delta w.$$

Then, Eq. (5.12) is exponentially stable.

Proof. The result follows from the facts that Eq. (5.13) is exponentially stable and that $|x(t)| \le M |y(t)|$ for all $t \in \mathbb{T}^{+}$. ■

5.3. Levin-Nohel integro-dynamic equations with impulsive effects

5.4 The Levin-Nohel equatons with several delays

The aim of this section is to investigate the exponential stability of Levin-Nohel equations with several delays of the form

$$x^{\Delta}(t) + \int_{t-r(t)}^{t} a(t,s)x(s)\Delta s + \sum_{i=1}^{n} \int_{t-r_i(t)}^{t} b_i(t,s)x(s)\Delta s = 0,\ t \in \mathbb{T}^{+}. \tag{5.15}$$

We make use of the following assumptions:

(B_1). The delays $r_i(t)$, $i = 1, ..., n$ are Lebesgue Δ-measurable functions satisfying $r_i(t) \geq 0$ and $\tau_i(t) := \sup_{t\geq t_0} r_i(t) < \infty$.

(B_2). The kernels $b_i : [t_0, \infty)_{\mathbb{T}} \times [-\tau_i, \infty)_{\mathbb{T}} \to \mathbb{R}$ are Lebesgue Δ-measurable such that $b_i(t, .)$ are locally Δ-integrable for any t.

We first observe that Eq. (5.15) can be rewritten as follows

$$x^{\Delta}(t) + \int_{t-R(t)}^{t} A(t,s)x(s)\Delta s = 0,\ t \in \mathbb{T}^{+}, \tag{5.16}$$

where $A(t,s) = a(t,s)\mathbf{1}_{\{s\geq t-r(t)\}}(s) + \sum_{i=1}^{n} b_i(t,s)\mathbf{1}_{\{s\geq t-r_i(t)\}}(s)$ with $\mathbf{1}$ is the indicator function and $R(t) = \max\{r(t), r_1(t), ..., r_n(t)\}$.

Consequently, applying Theorem 5.1 to (5.16) gives us a stability condition for (5.15). In the sequel, we try to provide a different condition for exponential stability of the solution. To end this, we need to introduce some concepts and well known results.

Definition 5.3 For each $s \geq t_0$, the solution $Z(t,s)$ of the equation,

$$x(t) = 0,\ t \in (-\infty, s)_{\mathbb{T}},\ x(s) = 1,\ x^{\Delta}(t) + \int_{t-r(t)}^{t} a(t,s)x(s)\Delta s = 0,\ [s,\infty)_{\mathbb{T}}, \tag{5.17}$$

is called the fundamental solution of (5.3).

The next lemmas show the role of the fundamental solution. The reader can consult [2] for more details.

Lemma 5.2 *If Eq. (5.3) is uniformly exponentially stable, then there exist* $K_0 > 0, \lambda_0 \in \mathcal{R}^{+}$ *such that*

$$|Z(t,s)| \leq K_0 e_{\ominus\lambda_0}(t,s),\ t \geq s \geq 0 \tag{5.18}$$

for any fundamental solution $Z(t,s)$.

The main result of this section is formulated in the below theorem.

Theorem 5.3 *Let assumptions (B_1) and (B_2) hold. Suppose that Eq. (5.2) is exponentially stable, its fundamental solution satisfies estimate (5.18) and that there exist $t_0 \geq 0$ such that*

$$\sup_{t\geq t_0} \int_{t_0}^{t} \left(K_0 e_{\ominus\lambda_0}(t,s) \sum_{i=1}^{n} \int_{s-r_i(s)}^{s} b_i(s,u)\Delta u \right) \Delta s := \alpha < 1. \tag{5.19}$$

Then, Eq. (5.15) is exponentially stable.

Proof. Let $f \in L_\Delta^\infty(\mathbb{T}^+)$, consider the equation

$$x^\Delta(t) + \int_{t-r(t)}^{t} a(t,s)x(u)\Delta s + \sum_{i=1}^{n} \int_{t-r_i(t)}^{t} b_i(t,s)x(s)\Delta s = f(t), \ t \in \mathbb{T}^+, \tag{5.20}$$

with the zero initial conditions. Using Lemma 5.1, it is enough to verify that the solution of Eq.(5.20) belongs to $BC_{rd}(\mathbb{T}^+)$.

By Lemma 5.2 and by variation of parameters formula we have

$$x(t) + \int_{t_0}^{t} Z(t,s) \left(\sum_{i=1}^{n} \int_{s-r_i(s)}^{s} b_i(s,u)x(u)\Delta u \right) \Delta s = \int_{t_0}^{t} Z(t,s)f(s)\Delta s, \ t \in \mathbb{T}^+,$$

which possesses the following form

$$(I + \mathcal{P})x = k,$$

with $k(t) = \int_{t_0}^{t} Z(t,s)f(s)\Delta s$ and the operator $\mathcal{P}$ is defined by $(\mathcal{P}x)(t) = 0$, $t \leq t_0$ and

$$(\mathcal{P}x)(t) = \int_{t_0}^{t} Z(t,s) \left(\sum_{i=1}^{n} \int_{s-r_i(s)}^{s} b_i(s,u)x(u)\Delta u \right) \Delta s \ \text{ for } \ t \in \mathbb{T}^+.$$

Since the fundamental solution $Z(t,s)$ satisfies estimate (5.18) and $f \in L_\Delta^\infty(\mathbb{T}^+)$, those imply $k \in L_\Delta^\infty(\mathbb{T}^+)$. Indeed, we have

$$\begin{aligned} |k(t)| &\leq \|f\|_{L_\Delta^\infty} \int_{t_0}^{t} K_0 e_{\ominus\lambda_0}(t,s)\Delta s = \frac{K_0}{\lambda_0}(1 - e_{\ominus\lambda_0}(t,t_0))\|f\|_{L_\Delta^\infty} \\ &\leq \frac{K_0}{\lambda_0}\|f\|_{L_\Delta^\infty} < \infty, \ t \in \mathbb{T}^+. \end{aligned}$$

Moreover, we have

$$|(\mathcal{P}x)(t)| \leq \left(\int_{t_0}^{t} K_0 e_{\ominus\lambda_0}(t,s) \left(\sum_{i=1}^{n} \int_{s-r_i(s)}^{s} |b_i(s,u)|\Delta u \right) \Delta s \right) \|x\|_{L_\Delta^\infty}, \ t \in \mathbb{T}^+.$$

It follows from (5.19) that the operator norm of $\mathcal{P}$ in the space $L^{\infty}_{\Delta}(\mathbb{T}^{+})$ is such that

$$\|\mathcal{P}\| \leq \alpha < 1.$$

The remaining of the proof is similar to one of Theorem 5.1. So we omit it here. ■

5.5 Conclusion

We give certain sufficient conditions on time scale, which guarantee the exponential stability of linear Levin-Nohel integro-dynamic equations. By the defining a suitable Bohl-Perron theorem, we prove a result on the topic. The method in this chapter can be applied to prove the exponential stability of solutions of some other similar integro-dynamic equations.

General Conclusion

In this work we prove the stability of Levin-Nohel integro-differential equations by fixed points theorems and the stability of Levin-Nohel integro-dynamic equations by Bohl-Perron theorem in functional spaces of infinite dimension.

This study can be used to show the stability of solutions of some problems contain stochastic terms.

The research in time scales calculus is a new idea that we allow to generalize some problems in delay differential equations, delay fractional differential equations,....

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