وزارة التعليم العالي والبحث العلمي

Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

BADJI MOKHTAR-ANNABA UNIVERSITY

جامعة باجي مختار

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-عنابة-

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Title

A Qualitative Study of Certain Classes of Differential Equations and Their Applications

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By

Bouaoud Safa

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Le Titre :

Étude qualitative de certaines classes d'équations différentielles et applications

Filière
Mathématiques Appliquées

Spécialité
Mathématiques Appliquées
Par
Bouaoud Safa

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Dedication

To *my **mother**, the heartbeat of my soul,*
*my **father**, the backbone of my strength,*
*my **husband**, the partner of my dreams and the calm in my storms,*
*my **children**, the reason my world keeps turning,*
*my **brothers** and **sisters,** my **friends**.*

You have all given me pieces of your hearts to help me build mine.

Through every late night of work, every moment of doubt, every small victory — you were there.
You believed in me when I struggled to believe in myself.

This modest work is not just mine — it is ours.
Every page carries a trace of your love, your patience, and your sacrifices.

Thank you for being my family, my foundation, and my greatest blessing.

I dedicate this work to you with all my love and eternal gratitude.

Contents

Abstract

It is well known that the theory of integral inequalities plays an essential role in the qualitative study of certain classes of differential equations. The aim of this thesis is, in a first step, to establish the qualitative analysis of perturbed dynamic equations on time scales. Secondly, we establish the practical stability of parametric time-scale systems via Lyapunov methods. Lastly, we study the existence, uniqueness and Ulam stability of certain classes of second-order nonlinear dynamic equations on time scales and applications.

Keywords: Existence, Time scales, Gronwall inequality, Practical h-stability, h-stability, Perturbed systems, Uniqueness.

Résumé

Il est bien connu que la théorie des inégalités intégrales joue un rôle essentiel dans l'étude qualitative de certaines classes d'équations différentielles.
L'objectif de cette thèse est, dans un premier temps, d'établir l'analyse qualitative d'équations dynamiques perturbées sur des échelles de temps. Dans un deuxième temps, nous établissons la stabilité pratique de systèmes paramétriques sur échelles de temps via des méthodes de Lyapunov. Enfin, nous étudions l'existence, l'unicité et la stabilité d'Ulam de certaines classes d'équations dynamiques non linéaires du second ordre sur des échelles de temps, ainsi que leurs applications.

Mots-clés : Existence, Échelles de temps, Inégalité de Gronwall, h-stabilité pratique, h-stabilité, Systèmes perturbés, Unicité.

الملخص

من المعروف أن نظرية المتباينات التكاملية تلعب دورًا أساسيًا في الدراسة النوعية لفئات معينة من المعادلات التفاضلية.

الهدف من هذه الأطروحة هو إجراء التحليل النوعي للمعادلات الديناميكية المضطربة على سلاسل الزمن (أو المقاييس الزمنية). نقوم بدراسة الاستقرار العملي للأنظمة البارامترية على سلاسل الزمن باستخدام طرق ليابونوف. وأخيرًا، ندرس الوجود والتفرد والاستقرار من نوع أولام لفئات معينة من المعادلات الديناميكية اللاخطية من الدرجة الثانية على سلاسل الزمن وتطبيقاتها.

الكلمات المفتاحية: الوجود، المقاييس الزمنية، متباينة جرونوال، الثبات العملي من النوعh ، الثبات من النوعh ، الأنظمة المضطربة، الوحدانية.

General Introduction

Unification and simplification are two key concepts in mathematics. A major approach to unifying continuous and discrete analysis is Time Scales Theory. This innovative theory was introduced by Stefan Hilger in his PhD dissertation in 1988. It provides a unified framework for studying differential and difference problems in dynamic equations.

Time Scales Theory is applicable in any field where dynamic processes are modeled by discrete, continuous time, or a combination of both. Examples include macroeconomic models, radioactive decay dynamics, and the agronomic cycle of seasonal plants. Researchers Martin Bohner and Allan Peterson further developed Hilger's foundational work, incorporating the results of many other contributors in this field. Their two influential books are comprehensive resources on this broad theory.

The mathematical theory of control has seen significant development, and its range of applications now spans numerous fields, including industry, economics, biology, and automation. This theory employs a variety of mathematical tools and is based on a wide range of models. These theoretical tools allow for both prediction and the practical application of its concepts to meet objectives that are directly related to it.

In other words, we have systems governed by partial differential equations, integral equations, or integro-differential equations when the characteristic variables of the process to be studied are functions of the spatial variable. Systems are mathematically represented by differential equations or finite difference equations when processes are described by a finite number of quantities. Systems are stochastic when random phenomena are involved.

One of the key themes of this theory is the verification of a system's stability, which is a crucial point in the development of the control law. To paraphrase, it even appears that all techniques for constructing control laws are closely linked to stability considerations. In the case of linear systems, this stability problem has been extensively studied, and a complete theory is available in the literature (see [56]). For nonlinear systems, the problem remains largely open, and results are still limited to particular classes of systems.

In general, a system's stability is verified by constructing a function $V : D \to \mathbb{R}$, positive definite and has a total time derivative that is negative definite, where D is a domain in $\mathbb{R}^n$. In concrete problems, although this condition is necessary and sufficient for asymptotic stability, it is often very difficult to find such a function. This motivates the establishment of sufficient (and necessary) conditions to guarantee asymptotic stability without requiring knowledge of a Lyapunov function.

In this thesis, we focus on the h-stability and practical h-stability of abstract dynamic equations on time scales by using Lyapunov's technique and the Gronwall–Bellman inequality.

The approach of Gronwall–Bellman inequality allows us to develop results that adhere to the following principle:

"If the original abstract equation behaves well and the perturbation is sufficiently small, then the perturbed abstract equation will also maintain good behavior."

This thesis is divided into four chapters as follows:

In Chapter 1, we introduce definitions, basic properties of time scales theory and some important inequalities.

The aim of Chapter 2 is to study the h-stability of the following perturbed dynamic equation:

$$
\begin{aligned}
x^{\Delta}(t) &= Ax(t) + f(t,x), \quad t \in \mathbb{T}^{+}_{t_0}, \\
x(t_0) &= x_0 \in D(A),
\end{aligned} \tag{1}
$$

where $f \in C_{rd}$ and A is the generator of a C_0-semigroup $\{T(t) : t \in \mathbb{T}\} \subset L(X)$ where X is a Banach space, $0 \in \mathbb{T} \subseteq \mathbb{R}_+$ is a time scale which is an additive semigroup with the property that $a - b \in \mathbb{T}$ for any $a, b \in \mathbb{T}$ such that $a > b$, and $D(A)$ is the domain of A.

Chapter 3 treats the concept of practical stability of dynamic equations on arbitrary time scales that incorporates a small perturbation parameter. By employing Lyapunov functions, we derive sufficient conditions ensuring practical stability of the perturbed system:

$$
x^{\Delta}(t) = Ax(t) + f(t, x(t), \varepsilon).
$$

In Chapter 4, we investigate the existence, uniqueness and Ulam stability of the following system:

$$
x^{\Delta^2}(t) = Q(t)x(t) + G\left(t, x(t), \int_a^t \mathcal{H}(t,s,x(s))\Delta s\right), \quad t \in \mathcal{J}^k, \tag{2}
$$

where $\mathcal{J} := [a,b] \cap \mathbb{T}$ with a time scale $\mathbb{T} \subset \mathbb{R}$, $a, b \in \mathbb{T}$ with $a < b$, $x : \mathbb{I} \to \mathbb{R}^n$ is an unknown function to be determined, $x^{\sigma} = x \circ \sigma$, x^{Δ} is the delta derivative of x, $Q : \mathbb{I} \to \mathbb{R}$ is regressive and rd-continuous, $G : \mathbb{I} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous in its first variable and continuous in its second and third variables, $\mathcal{H} : \mathbb{I} \times \mathbb{I} \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous in its first and second variables and continuous in its third variable. In addition, both G and $\mathcal{H}$ are assumed to be nonlinear functions.

Chapter 1

Preliminaries

In this chapter, we introduce fundamental concepts essential to the theory of time scales. We briefly recall key notions and definitions related to stability and semi-groups. Additionally, we outline some important inequalities that will be useful throughout this work. For further details, readers may consult references $[[20], [21], [19], [22]]$.

1.1 Basic Notions on Time Scales

The concept of time scales was initiated by Stefan Hilger $[[48], [49]]$ in his doctoral thesis in 1988 to develop a new theory that unifies discrete and continuous analysis. He introduced the concept of the Δ-derivative, which led to time scale equations, resembling differential equations or their approximations. For example, a first-order equation with the usual derivative (u') is replaced by the Δ-derivative (u^{Δ}).

We will see later that when $\mathbb{T} = \mathbb{R}$, the Δ-derivative coincides with the classical derivative, making time scale equations equivalent to differential equations. When $\mathbb{T} = \mathbb{Z}$, these become finite difference equations. The latter has gained significant interest in recent years for explaining various discrete phenomena, especially in economics, psychology, engineering, and computer science. The theory of time scale equations first seeks to unify results obtained in differential and difference equations. Secondly, it enables the study of phenomena that model both discrete and continuous behaviors simultaneously.

1.1.1 Terminology

Definition 1 *A time scale $\mathbb{T}$ is a non-empty closed subset of $\mathbb{R}$.*

Example 2 *Sets such as $\mathbb{R}$, $\mathbb{Z}^+$, $[-1,0]\cup[1,2]$, $\overline{q^{\mathbb{Z}}} = \{q^k : k \in \mathbb{Z},\ q > 1\}\cup\{0\}$, $[0,1]\cup\mathbb{N}$, and Cantor sets are examples of time scales.*

Example 3 *Sets like $\mathbb{Q}$, $\mathbb{R}\backslash\mathbb{Q}$, $\mathbb{C}$, and $(0,1)$ are not time scales.*

Remark 4 *The topology of $\mathbb{T}$ is induced by that of $\mathbb{R}$.*

1.1.2 Jump Operators and Point Classification

- **Forward Jump Operator σ:** For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}. \tag{1.1}$$

- **Backward Jump Operator ρ:** For $t \in \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}. \tag{1.2}$$

Using σ and ρ, points in a time scale $\mathbb{T}$ are classified as follows:

- **Right-dense (Left-dense):** t is right-dense (left-dense) if $\sigma(t) = t$ (respectively $\rho(t) = t$).
- **Dense:** t is dense if it is both right-dense and left-dense.
- **Right-scattered (Left-scattered):** t is right-scattered (left-scattered) if $\sigma(t) > t$ (respectively $\rho(t) < t$).

- **Isolated:** t is isolated if it is both right-scattered and left-scattered.

- **Graininess Function:** The graininess function $\mu : \mathbb{T} \to \mathbb{R}_+$ is defined by

$$\mu(t) = \sigma(t) - t. \tag{1.3}$$

The distance between consecutive time scale points is measured by the forward graininess operator μ. Also, we define the set of all non-degenerate points of $\mathbb{T}$ by

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases} \tag{1.4}$$

1.1.3 Derivatives on Time Scales

We start by recalling the definition of the Δ-derivative, also known as the Hilger derivative.

Definition 5 *Let $f : \mathbb{T} \to \mathbb{R}$. The function f is Δ-differentiable at $t \in \mathbb{T}^k$ if there exists a number $f^{\Delta}(t)$ such that for every $\varepsilon > 0$, there exists a neighborhood U of t such that*

$$\left| [f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s] \right| \leq \varepsilon \left| \sigma(t) - s \right|, \tag{1.5}$$

for all $s \in U$. If f is Δ-differentiable at every point $t \in \mathbb{T}^k$, then the function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is called the Δ-derivative of f on $\mathbb{T}^k$.

Theorem 6 *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^k$.*

(i) *If f is Δ-differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then f is Δ-differentiable at t, and moreover,*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}. \tag{1.6}$$

(iii) *If t is right-dense, then f is Δ-differentiable at t if and only if*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s} \tag{1.7}$$

exists and is finite. In this case,

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}. \tag{1.8}$$

(iv) *If f is differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{1.9}$$

- If $\mathbb{T} = \mathbb{R}$, then by (iii) of the previous theorem, the function $f : \mathbb{R} \to \mathbb{R}$ is Δ-differentiable at $t \in \mathbb{R}$ if and only if

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \tag{1.10}$$

  exists, and moreover, $f^{\Delta}(t) = f'(t)$.

- If $\mathbb{T} = \mathbb{Z}$, by (ii), the function $f : \mathbb{Z} \to \mathbb{R}$ is Δ-differentiable at $t \in \mathbb{Z}$ and we have

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{f(t+1) - f(t)}{1} = f(t+1) - f(t) = \Delta f(t). \tag{1.11}$$

Theorem 7 *If $f, g : \mathbb{T} \to \mathbb{R}$ are Δ-differentiable at $t \in \mathbb{T}^k$, then*

(i) *$f + g$ is Δ-differentiable at t, and*

$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t). \tag{1.12}$$

(ii) *αf is Δ-differentiable at t for all $\alpha \in \mathbb{R}$, and*

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t). \tag{1.13}$$

(iii) fg *is* Δ*-differentiable at* t*, and*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t)$$
$$= f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)). \tag{1.14}$$

(iv) *If* $f(t)f(\sigma(t)) \neq 0$*, then* $\frac{1}{f}$ *is* Δ*-differentiable at* t*, and*

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}. \tag{1.15}$$

(v) *If* $g(t)g(\sigma(t)) \neq 0$*, then* $\frac{f}{g}$ *is* Δ*-differentiable at* t*, and*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}. \tag{1.16}$$

Theorem 8 *Let* $g : \mathbb{T} \to \mathbb{R}$ *be* Δ*-differentiable on* $\mathbb{T}^k$ *and* $f : \mathbb{R} \to \mathbb{R}$ *be continuously differentiable. Then there exists* $c \in [t, \sigma(t)]$ *such that*

$$(f \circ g)^{\Delta}(t) = f'(g(c))\, g^{\Delta}(t). \tag{1.17}$$

Definition 9 *A function* $f : \mathbb{T} \to \mathbb{R}$ *is said to be* rd*-continuous if it is continuous at every right-dense point of* $\mathbb{T}$ *and its left-hand limit exists and is finite at every left-dense point of* $\mathbb{T}$*.*

Remark 10 *The set of all* rd*-continuous functions is denoted by* C_{rd} *or* $C_{rd}(\mathbb{T})$*.*

Remark 11 *The set of all functions that are both* Δ*-differentiable and* rd*-continuous is denoted by* C^1_{rd} *or* $C^1_{rd}(\mathbb{T})$*.*

1.1.4 Integration over Time Scales

Definition 12 *A function* $F : \mathbb{T} \to \mathbb{R}$ *is called a primitive of* $f : \mathbb{T} \to \mathbb{R}$ *if it satisfies* $F^{\Delta}(t) = f(t)$ *for all* $t \in \mathbb{T}^k$*.*

Theorem 13 *Every* rd*-continuous function* $f : \mathbb{T} \to \mathbb{R}$ *has a primitive* $F : \mathbb{T} \to \mathbb{R}$*, and we denote*

$$\int_s^r f(t)\,\Delta t = F(r) - F(s) \quad \text{for all } r, s \in \mathbb{T}. \tag{1.18}$$

Theorem 14 *If* $f \in C_{rd}(\mathbb{T})$ *and* $t \in \mathbb{T}^k$*, then*

$$\int_t^{\sigma(t)} f(\tau)\,\Delta\tau = \mu(t) f(t). \tag{1.19}$$

Theorem 15 *Let* $a, b, c \in \mathbb{T}$*,* $\lambda \in \mathbb{R}$*, and* $f, g \in C_{rd}(\mathbb{T})$*. Then*

(i) $\int_a^b [f(t) + g(t)]\,\Delta t = \int_a^b f(t)\,\Delta t + \int_a^b g(t)\,\Delta t$.

(ii) $\int_a^b (\lambda f(t))\,\Delta t = \lambda \int_a^b f(t)\,\Delta t$.

(iii) $\int_a^b f(t)\,\Delta t = -\int_b^a f(t)\,\Delta t$.

(iv) $\int_a^b f(t)\,\Delta t = \int_a^c f(t)\,\Delta t + \int_c^b f(t)\,\Delta t$.

(v) $\int_a^a f(t)\,\Delta t = 0$.

(vi) *If* $|f(t)| \le g(t)$ *on* $[a, b]_{\mathbb{T}}$*, then* $\left|\int_a^b f(t)\,\Delta t\right| \le \int_a^b g(t)\,\Delta t$.

Proposition 16 • *For* $\mathbb{T} = \mathbb{R}$*,*

$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\,dt. \tag{1.20}$$

- *For* $\mathbb{T} = \mathbb{Z}$,

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum_{t=b}^{a-1} f(t) & \text{if } a > b. \end{cases} \tag{1.21}$$

- *If* $[a,b]$ *contains isolated points, then*

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{t\in[a,b[} \mu(t)f(t) & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum_{t\in[b,a[} \mu(t)f(t) & \text{if } a > b. \end{cases} \tag{1.22}$$

Proposition 17 *Integration by parts formulas:*

$$\int_a^b f^\sigma(t)g^\Delta(t)\,\Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(t)\,\Delta t,$$
$$\int_a^b f(t)g^\Delta(t)\,\Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)g^\sigma(t)\,\Delta t.$$

Lemma 18 *Let* $a \in \mathbb{T}^k$, $b \in \mathbb{T}$, *and* $L : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ *be continuous at* (t,t) *for* $t \in \mathbb{T}^k$, $t > a$, *and* $L^\Delta(t,\cdot)$ *be* rd*-continuous on* $[a,\sigma(t)]$. *Suppose that for any* $\varepsilon > 0$, *there exists a neighborhood* U *of* t *independent of* $\tau \in [a,\sigma(t)]$ *such that*

$$\left|L(\sigma(t),\tau) - L(s,\tau) - L^\Delta(t,\tau)(\sigma(t)-s)\right| \le \varepsilon\,|\sigma(t)-s|\,, \quad \forall s \in U, \tag{1.23}$$

where L^Δ *denotes the derivative with respect to the first variable. Then*

$$g(t) = \int_a^t L(t,\tau)\,\Delta\tau \implies g^\Delta(t) = \int_a^t L^\Delta(t,\tau)\,\Delta\tau + L(\sigma(t),t), \tag{1.24}$$

and

$$h(t) = \int_t^b L(t,\tau)\,\Delta\tau \implies h^\Delta(t) = \int_t^b L^\Delta(t,\tau)\,\Delta\tau - L(\sigma(t),t). \tag{1.25}$$

1.1.5 Exponential Function

Definition 19 *Let $h > 0$. The set of Hilger complex numbers is defined as*

$$\mathbb{C}_h = \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}. \tag{1.26}$$

Definition 20 *For $h > 0$, we define the set*

$$\mathbb{Z}_h = \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < \mathrm{Im}(z) \leq \frac{\pi}{h} \right\}, \tag{1.27}$$

and for $h = 0$, we set $\mathbb{Z}_0 = \mathbb{C}$.

Definition 21 *For $h > 0$, the cylindrical transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ is defined by*

$$\xi_h(z) = \frac{1}{h} \operatorname{Log}(1 + zh), \tag{1.28}$$

where Log is the principal logarithm. For $h = 0$, we set $\xi_0(z) = z$ for all $z \in \mathbb{C}$.

Definition 22 *Let $p : \mathbb{T} \to \mathbb{R}$. The function p is said to be regressive if it satisfies*

$$1 + \mu(t)p(t) \neq 0 \quad \textit{for all } t \in \mathbb{T}^k. \tag{1.29}$$

Remark 23 *The set of regressive and rd-continuous functions is denoted by $\Re$.*

Definition 24 *The set of all positive regressive functions is defined by*

$$\Re^+ = \left\{ p \in \Re : 1 + \mu(t)p(t) > 0 \ \textit{ for all } t \in \mathbb{T}^k \right\}. \tag{1.30}$$

Theorem 25 *Assume $p \in \Re$ and $t_0 \in \mathbb{T}$ is a fixed point. The initial value problem*

$$y^{\Delta}(t) = p(t)y(t), \quad y(t_0) = 1, \tag{1.31}$$

has a unique solution on $\mathbb{T}$.

Definition 26 *Let $p \in \Re$. The exponential function is defined as the solution of problem* (1.7) *and we have*

$$y(t) = \exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\,\Delta\tau\right), \quad t, t_0 \in \mathbb{T}, \tag{1.32}$$

which is often denoted as

$$e_p(t, t_0) = \exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\,\Delta\tau\right), \quad t, t_0 \in \mathbb{T}, \tag{1.33}$$

where ξ_μ is the cylindrical transformation.

Proposition 27 *Let $p, q \in \Re$ and $t, t_0, s \in \mathbb{T}$. Then*

1. $e_0(t, t_0) \equiv 1$ *and* $e_p(t, t) \equiv 1$,
2. $e_p(\sigma(t), t_0) = (1 + \mu(t)p(t))e_p(t, t_0)$,
3. $e_p(t, t_0)e_p(t_0, s) = e_p(t, s)$,
4. $e_p(t, t_0) = \dfrac{1}{e_p(t_0, t)}$,
5. $e_p(t, t_0)e_q(t, t_0) = e_{p\oplus q}(t, t_0)$, *where* $(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t)$ *for all* $t \in \mathbb{T}^k$.

For more details, the reader can refer to $[15, 16]$.

Example 28 *Let $p \in \Re$, $t, t_0 \in \mathbb{T}$ with $t > t_0$.*

- *If* $\mathbb{T} = \mathbb{R}$, *then* $e_p(t, t_0) = \exp\left(\int_{t_0}^{t} p(\tau)\,d\tau\right)$.
- *If* $\mathbb{T} = \mathbb{R}$ *and* $p(t) \equiv \alpha$, *then* $e_p(t, t_0) = e^{\alpha(t-t_0)}$.
- *If* $\mathbb{T} = \mathbb{Z}$, *then* $e_p(t, t_0) = \prod_{\tau=t_0}^{t-1}(1 + p(\tau))$.
- *If* $\mathbb{T} = h\mathbb{Z}$ *with* $h > 0$ *and* $p(t) \equiv \alpha$, *then* $e_p(t, t_0) = (1 + \alpha h)^{\frac{t-t_0}{h}}$.

1.2 Fundamental Notions of Stability

In mathematics, stability theory deals with the qualitative behavior of solutions of differential equations and trajectories of dynamical systems under small perturbations of initial conditions. The first study of stability on time scales was carried out in 1992 using the Lyapunov method, and the definitions of stability for systems on time scales are obtained by a slight modification of their standard definitions for ordinary differential equations.

In this section, we illustrate the different types of stability on time scales, in particular Lyapunov stability. The results that we will develop in Chapter 3 will rely on these concepts to demonstrate stability properties of the perturbed systems under study.

1.2.1 Definitions of Different Types of Stability

Hamza [44], **Dacunha** [33], **Choi** and **Koo** gave some generalized characterizations of various types of stability for solutions of dynamic systems on time scales.

Consider a dynamic system defined on a time scale $\mathbb{T}$

$$x^{\Delta}(t) = f(t, x), \qquad x(t_0) = x_0, \tag{1.34}$$

where $t \in \mathbb{T}$ and $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is an rd-continuous, regressive function, with $\|f(t,0)\| \leq f_0 < \infty$. Also, suppose that the conditions for the existence of a unique solution of system (1.34) on $\mathbb{T}_{t_0}^{+}$ are satisfied for every initial condition $(t_0, x_0) \in \mathbb{T} \times \mathbb{R}^n$. Denote this solution by $x(t) = x(t, t_0, x_0)$.

We note that the stability of any solution is closely related to the stability of the zero solution of the corresponding variational equation (1.34). Suppose first that $f(t, 0) = 0$; this indicates that the origin is an equilibrium point for system (1.34).

Definition 29 *The dynamic system* (1.34) *is said to be* ***stable*** *if for every* $t_0 \in \mathbb{T}$ *and every* $\varepsilon > 0$*, there exists a finite constant* $\delta = \delta(\varepsilon, t_0) > 0$ *such that every solution* $x(t)$

satisfies

$$\|x(t_0)\| < \delta \quad \Longrightarrow \quad \|x(t, t_0, x_0)\| < \varepsilon, \qquad t \geq t_0. \tag{1.35}$$

Definition 30 *The dynamic system* (1.34) *is said to be* ***uniformly stable*** *if there exists a finite constant* $\gamma > 0$ *such that for every* $t_0 \in \mathbb{T}$ *and* $x(t_0)$*, every solution* $x(t)$ *satisfies*

$$\|x(t)\| < \gamma \|x(t_0)\|, \qquad t \geq t_0. \tag{1.36}$$

Definition 31 *System* (1.34) *is said to be* ***exponentially stable*** *if there exists* $\lambda > 0$ *with* $-\lambda \in \Re^+$ *such that for every* $t_0 \in \mathbb{T}$*, there exists* $\gamma = \gamma(t_0) \geq 1$ *such that every solution* $x(t)$ *satisfies*

$$\|x(t)\| < \gamma \|x(t_0)\| e_{-\lambda}(t, t_0), \qquad \textit{for all } t \geq t_0. \tag{1.37}$$

Definition 32 *The dynamic system* (1.34) *is said to be* ***uniformly exponentially stable*** *if there exists* $\lambda > 0$ *with* $-\lambda \in \Re^+$ *and* $\gamma \geq 1$ *independent of the initial point* t_0*, such that for every* $t_0 \in \mathbb{T}$ *and* $x(t_0)$*, every solution* $x(t)$ *satisfies*

$$\|x(t)\| < \gamma \|x(t_0)\| e_{-\lambda}(t, t_0), \qquad \textit{for all } t \geq t_0. \tag{1.38}$$

Definition 33 *Let* $h : \mathbb{T} \to \mathbb{R}$ *be a positive and bounded function. Equation* (1.34) *is* ***h-stable*** *if there exists a constant* $\gamma \geq 1$ *such that any solution* $x(t) = x(t, t_0, x_0)$ *satisfies*

$$\|x(t, t_0, x_0)\| \leq \gamma \|x_0\| \, h(t) \, h(t_0)^{-1}, \qquad t \geq t_0, \tag{1.39}$$

where $h(t)^{-1} = \frac{1}{h(t)}$*.*

Definition 34 *Equation* (1.34) *is said to be* ***globally practically uniformly h-stable*** *if there exist* $\gamma \geq 1$ *and* $\rho > 0$ *such that for all* $t \geq t_0$ *and* $x_0 \in \mathbb{R}^n$*,*

$$\|x(t, t_0, x_0)\| \leq \gamma \|x_0\| \, h(t) \, h(t_0)^{-1} + \rho, \qquad t \geq t_0. \tag{1.40}$$

Remark 35 *If $X = \mathbb{R}^n$ and $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous in t, locally Lipschitz in x, and regressive (i.e., the function $I + \mu(t)f(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible), we obtain the following implications:*

$$
\begin{aligned}
\textit{globally practically uniformly } h\textit{-stable} &\in h\textit{-stability}\\
&\in \textit{uniform exponential stability}\\
&\in \textit{uniform Lipschitz stability}\\
&\in \textit{uniform stability.}
\end{aligned}
$$

For more details on the different types of stability, refer to [[**?**]]*. For more details on the different types of stability, refer to* [*44*]*.*

The stability analysis of non-autonomous linear systems can be fully characterized in terms of the resolvent of the studied system. Indeed, consider the problem

$$x^{\Delta}(t) = A(t)x(t), \quad x(t_0) = I_n, \tag{1.41}$$

where $x \in \mathbb{R}^n$, $t, t_0 \in \mathbb{T}$, and $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ is an rd-continuous, regressive matrix depending on t.

Definition 36 *The matrix function $A : \mathbb{T} \to M_n(\mathbb{R})$ is called regressive if for all $t \in \mathbb{T}$ the square matrix $I + \mu(t)A(t)$ of size $n \times n$ is invertible, where I is the identity matrix.*

The class of all regressive and rd-continuous functions A from $\mathbb{T}$ to $M_n(\mathbb{R})$ is denoted by $C_{rd}\Re(\mathbb{T}, M_n(\mathbb{R}))$.

Definition 37 *For $t_0 \in \mathbb{T}$, the solution of problem (1.41) is called the exponential of the matrix function, denoted by $\phi_A(t,t_0)$, where $A \in C_{rd}\Re(\mathbb{T}, M_n(\mathbb{R}))$.*

Consequently, the function $\phi_A(t,t_0)$ has the following two properties:

$$
\begin{aligned}
\phi_A^{\Delta}(t,t_0) &= A(t)\phi_A(t,t_0), \qquad (1.42)\\
\phi_A(t_0,t_0) &= I_n.
\end{aligned}
$$

This matrix function is called the transition matrix, and our assumption on $A(t)$ shows that the transition matrix exists and is unique.

Theorem 38 *Let $r, s, t \in \mathbb{T}$ and $A, B \in C_{rd}\Re(\mathbb{T}, M_n(\mathbb{R}))$. Then the transition matrix has the following properties:*

(i) $\phi_A(t, r)\phi_A(r, s) = \phi_A(t, s)$.

(ii) $\phi_A(\sigma(t), s) = (I + \mu(t)A(t))\phi_A(t, s)$.

(iii) *If $\mathbb{T} = \mathbb{R}$ and A is constant, then $\phi_A(t, s) = e_A(t, s) = e^{A(t-s)}$.*

(iv) *If $\mathbb{T} = h\mathbb{Z}$ with $h > 0$ and A is constant, then $\phi_A(t, s) = (I + hA)^{\frac{t-s}{h}}$.*

1.2.2 Lyapunov Stability

Lyapunov stability analysis consists of studying the trajectories of the system when the initial state is close to an equilibrium state. This subsection exhibits the general notions and mathematical tools that allow us to analyze Lyapunov stability for dynamical systems.

Definition 39 *(Class $\mathcal{K}$ function). A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to be of class $\mathcal{K}_\infty$ if, in addition, $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.*

Definition 40 *(Class $\mathcal{KL}$ function). A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{KL}$ if for every fixed s, the mapping $r \mapsto \beta(r, s)$ is of class $\mathcal{K}$, and for every fixed r, the mapping $s \mapsto \beta(r, s)$ is decreasing and $\beta(r, s) \to 0$ as $s \to \infty$.*

Definition 41 *A continuous function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is said to be:*

1. ***Positive definite** if there exists a function α of class $\mathcal{K}$ such that*

$$\forall t \geq 0,\ \forall x \in \mathbb{R}^n, \quad V(t, x) \geq \alpha(\|x\|). \tag{1.43}$$

2. ***Positive definite and radially unbounded (or proper)*** *if the preceding inequality holds for every $x \in \mathbb{R}^n$ with a function α of class $\mathcal{K}_\infty$.*

3. ***Bounded or decrescent*** *if there exists a function γ of class $\mathcal{K}$ such that*

$$\forall t \geq 0,\ \forall x \in \mathbb{R}^n, \quad V(t,x) \leq \gamma(\|x\|). \tag{1.44}$$

Definition 42 *(Lyapunov function). Consider system (1.34). Let $U(0)$ be a neighborhood of zero and $V : \mathbb{R}_+ \times U(0)$ a continuous and differentiable function on $U(0)$.*

- *We say that V is a* ***Lyapunov function in the broad sense*** *at 0 if it satisfies:*

  i. V is positive definite.

  ii. $\dot{V}(t,x) \leq 0$ for every $x \in U(0)$.

- *We say that V is a* ***strict Lyapunov function*** *at 0 if it satisfies:*

  i. V is positive definite.

  ii. $\dot{V}(t,x) < 0$ for every $x \in U(0) \setminus \{0\}$.

The use of these functions provides criteria that allow us to conclude the stability of an equilibrium point without needing to integrate the equations of the considered system. For proofs and more details, the reader may refer to [56].

To apply these theorems to system (1.34), we assume that the time scale $\mathbb{T} = \mathbb{R}$.

Theorem 43 *Consider system (1.34). If the system admits a Lyapunov function in the broad sense on $U(0)$, then the origin $x = 0$ is a stable equilibrium point. If, in addition, V is decrescent, then $x = 0$ is a uniformly stable equilibrium point.*

Theorem 44 *Let $x = 0$ be an equilibrium point of system (1.34) and $U(0) = \{x \in \mathbb{R}^n : \|x\| < r\}$. Let $V : \mathbb{R}_+ \times U(0) \to \mathbb{R}$ be a C^1 function, and suppose there exist functions $\alpha_1, \alpha_2, \alpha_3$ of class $\mathcal{K}$ defined on $[0, r)$ such that for all $t \geq t_0$ and $x \in U(0)$,*

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|), \qquad \dot{V}(t,x) \leq -\alpha_3(\|x\|). \tag{1.45}$$

Then $x = 0$ is a uniformly asymptotically stable equilibrium point. If $U(0) = \mathbb{R}^n$ and α_1, α_2 are of class $\mathcal{K}_\infty$, then $x = 0$ is a globally uniformly asymptotically stable equilibrium point.

Theorem 45 *Consider system* (1.34)*. Suppose that this system admits a Lyapunov function $V(t,x)$ and that there exist constants $c_1, c_2, c_3, c_4 > 0$ such that for all $x \in U(0)$ and $t \geq t_0$,*

$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2, \quad \dot{V}(t,x) \leq -c_3\|x\|^2, \quad \left\|\frac{\partial V}{\partial x}(t,x)\right\| \leq c_4\|x\|. \tag{1.46}$$

Then $x = 0$ is an exponentially stable equilibrium point. If $U(0) = \mathbb{R}^n$, then the origin is a globally exponentially stable equilibrium point.

1.2.3 Lyapunov Stability on Time Scales

The stability of dynamical systems requires good tools for approximating solutions. Among the most effective techniques, we find the Lyapunov analysis approach to evaluate trajectory behaviors through an energy function. For more details see [[20], [79]].

Lyapunov Function

Let us present the Chain Rule theorem, which will be very useful for our study in Chapter 4. This theorem is due to **Christian Pötzsche** [79], who first derived it in 1998.

Theorem 46 *[20] Let $p : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $q : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $p \circ q : \mathbb{T} \to \mathbb{R}$ is delta differentiable, and*

$$[p(q(t))]^\Delta = \left(\int_0^1 p'(q(t) + h\mu(t)q^\Delta(t))\,dh\right) q^\Delta(t). \tag{1.47}$$

Definition 47 *We call $V : \mathbb{R}^n \to \mathbb{R}$ a "type I" Lyapunov function when*

$$V(x) = \sum_{i=1}^{n} V_i(x_i) = V_1(x_1) + \cdots + V_n(x_n), \tag{1.48}$$

where each $V_i : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

Definition 48 *Suppose that $V : \mathbb{R}^n \to \mathbb{R}$ is a "type I" Lyapunov function. Then the derivative of $V(x(t))$ is computed as follows*

$$\begin{aligned}
[V(x(t))]^{\Delta} &= \left[\sum_{i=1}^{n} V_i(x_i(t))\right]^{\Delta} = \sum_{i=1}^{n} [V_i(x_i(t))]^{\Delta} \\
&= \sum_{i=1}^{n} \int_0^1 V_i'(x_i(t) + h\mu(t) x_i^{\Delta}(t))\, dh\; x_i^{\Delta}(t) \\
&= \sum_{i=1}^{n} \int_0^1 V_i'(x_i(t) + h\mu(t) f_i(t, x(t)))\, dh\; f_i(t, x(t)) \\
&= \int_0^1 \nabla V(x(t) + h\mu(t) f(t, x(t))) \cdot f(t, x(t))\, dh,
\end{aligned}$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ is the gradient operator. This leads us to define $\dot{V} : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\dot{V}(t, x) = \int_0^1 \nabla V(x + h\mu(t) f(t, x)) \cdot f(t, x)\, dh. \tag{1.49}$$

If $\mu(t) = 0$, then we simply obtain

$$\dot{V}(t, x) = \nabla V(x) \cdot f(t, x). \tag{1.50}$$

Example 49 *Let $V(x) = \sum_{i=1}^{n} x_i^{1/2}$ for $x \in D$, where*

$$D = \{x \in \mathbb{R}^n : x_i > 0,\ x_i + \mu(t) f_i(t, x) \geq 0,\ i = 1, 2, \dots, n\}. \tag{1.51}$$

Then

$$\dot{V}(t, x) = \sum_{i=1}^{n} \frac{f_i(t, x)}{\sqrt{x_i + \mu(t) f_i(t, x)} + \sqrt{x_i}}. \tag{1.52}$$

Remark 50 *The scalar product on $\mathbb{R}^n$ is the function $\langle \cdot, \cdot \rangle$ defined by*

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n \quad \textit{for } x = (x_1, \ldots, x_n)^\top,\ y = (y_1, \ldots, y_n)^\top \in \mathbb{R}^n. \tag{1.53}$$

For each vector $x \in \mathbb{R}^n$, the norm of x is defined as

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x^\top x}. \tag{1.54}$$

1.3 C_0 Semigroups and Their Generators on Time Scales

In the sequel, we introduce some definitions of strongly continuous semigroups (C_0-semigroups) $T = \{T(t) : t \in \mathbb{T}\} \subset L(X)$ and their generator A, where $0 \in \mathbb{T} \subseteq \mathbb{R}_+$ is a time scale which is an additive semigroup with the property that $a - b \in \mathbb{T}$ for any $a, b \in \mathbb{T}$ such that $a > b$.

Definition 51 *Let $L(X)$ be the space of bounded linear operators on a Banach space X. A C_0-semigroup T on X is a family of bounded linear operators $\{T(t) : t \in \mathbb{T}\} \subset L(X)$ satisfying*

1. *$T(t + s) = T(t)T(s)$ for every $t, s \in \mathbb{T}$ (semigroup property),*

2. *$T(0) = I$ (I is the identity operator on X),*

3. *$\lim_{t \to 0^+} T(t)x = x$ (i.e., $T(\cdot)x : \mathbb{T} \to X$ is continuous at 0) for each $x \in X$.*

If in addition $\lim_{t \to 0^+} \|T(t) - I\| = 0$, then T is called a uniformly continuous semigroup.

Definition 52 *A linear operator A is called the generator of a C_0-semigroup T if*

$$Ax = \lim_{s \to 0^+} \frac{T(\mu(t))x - T(s)x}{\mu(t) - s}, \qquad x \in D(A), \tag{1.55}$$

where the domain $D(A)$ of A is the set of all $x \in X$ for which the above limit exists uniformly in t.

For more details about the properties of a C_0-semigroup T and its generator A, we refer the reader to $[[44], [43]]$.

Remark 53 *For more properties of a C_0-semigroup T and its generator A in the case $\mathbb{T} = \mathbb{R}_+$, we refer to $[74]$.*

1.4 Some Important Inequalities

In this section, we provide a brief reminder of some classes of functions as well as useful inequalities for this study.

Definition 54 *[8] A function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathfrak{F}$ if it satisfies the following conditions:*

$$g(x) > 0 \text{ is nondecreasing and continuous for } x \geq 0, \tag{1.56}$$
$$\frac{1}{a} g(x) \leq g\left(\frac{x}{a}\right) \quad \text{for } a > 0. \tag{1.57}$$

For example, if $g(x) = x^p$ with $p \geq 1$, then $g(x/a) = (x/a)^p \geq x^p/a = g(x)/a$ for $a \in (0, 1]$. The following nonlinear integral-type inequalities on time scales are useful in our study.

Definition 55 *[8] Let g be a positive, continuous function on $\mathbb{R}_+$. g is said to belong to class $\mathcal{T}$ if it satisfies*

$$\frac{1}{a} g(x) \geq g\left(\frac{x}{a}\right) \quad \text{for all } x \geq 0 \text{ and } a \geq 1. \tag{1.58}$$

Lemma 56 *Assume $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$. Then*

$$a^{q/p} \leq \frac{q}{p} k^{\frac{q-p}{p}} a + \frac{p-q}{p} k^{q/p} \quad \text{for all } k > 0. \tag{1.59}$$

Lemma 57 • *Let* $a, b \geq 0$ *and* $p \geq 1$. *Then*

$$(a+b)^p \leq 2^{p-1}(a^p + b^p). \tag{1.60}$$

• *Let* $a, b \geq 0$ *and* $p \leq 1$. *Then*

$$(a+b)^p \leq a^p + b^p. \tag{1.61}$$

Chapter 2

Qualitative Analysis of Perturbed Dynamic Equations on Time Scales

In this chapter, we study global practical h-stability for nonlinear perturbed abstract dynamic equations on time scales by using the Gronwall type inequality and Lyapunov functions.

The chapter is organized as follows: Section 2 provides a brief review of semigroup theory on time scales by using the concept of Laplace transform on time scales and we present some integral inequalities which play an important role in our analysis. Section 3 contains the statements and proofs of our main results. Section 4 shows the applicability of the theoretical results by numerical examples.

2.1 Statement of the Problem

In this chapter, we discuss practical h-stability for the following abstract dynamic equation on a time scale:

$$\begin{aligned} x^{\Delta}(t) &= Ax(t) + f(t,x), \quad t \in \mathbb{T},\ t > t_0, \qquad (2.1)\\ x(t_0) &= x_0 \in D(A), \end{aligned}$$

in terms of the h-stability of the homogeneous equation:

$$\begin{aligned} x^{\Delta}(t) &= Ax(t), \quad t \in \mathbb{T},\ t > t_0, \\ x(t_0) &= x_0 \in D(A), \end{aligned} \tag{2.2}$$

where $x \in X$, A is the generator of a C_0-semigroup T, $f : \mathbb{T} \times X \to X$ is rd-continuous with $f(t,0) = 0$ and x^{Δ} is the delta derivative of $x : \mathbb{T} \to \mathbb{X}$, X is a Banach space and assume $\sup \mathbb{T} = +\infty$. We derive sufficient conditions for practical h-stability notion of certain classes of abstract dynamic perturbed equations on time scales using time scale versions of some Gronwall type inequalities.

2.2 Some Important Inequalities

The following lemmas are very useful in our main results.

Lemma 2.2.1 *[8] Suppose that* $x, u, \delta, \vartheta, g \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$. *Then*

$$x(t) \leq \delta(t) + g(t) \int_{t_0}^{t} \left(\vartheta(s)x(s) + u(s)\right) \Delta s,\ t \in \mathbb{T}_{t_0}^{+},$$

implies

$$x(t) \leq \delta(t) + g(t) \int_{t_0}^{t} \left(\left(\delta(s)\vartheta(s) + u(s)\right) \exp\left(\int_{\sigma(s)}^{t} \vartheta(\tau)g(\tau)\Delta\tau \right) \right) \Delta s, \quad t \in \mathbb{T}_{t_0}^{+}.$$

Lemma 2.2.2 *[8] Suppose that* $x, u, \delta, \vartheta, g \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$. *If*

$$x(t) \leq \delta(t) + g(t) \int_{t_0}^{t} \left(\vartheta(s)x^{\alpha}(s) + u(s)\right) \Delta s, \quad t \in \mathbb{T}_{t_0}^{+},\ \alpha \in (0,1),$$

then we have

$$x(t) \leq \delta(t) + g(t) \int_{t_0}^{t} L(s) \exp\left(\int_{\sigma(s)}^{t} M(\tau)\Delta\tau \right) \Delta s, \quad t \in \mathbb{T}_{t_0}^{+},$$

where

$$
\begin{aligned}
L(t) &= \vartheta(t)\left(\alpha\delta(t)+1-\alpha\right)+u(t), \\
M\left(t\right) &= \alpha\vartheta(t)g(t).
\end{aligned}
$$

Lemma 2.2.3 *[22] (Stachurska's inequality) Assume f, q, p are rd-continuous and nonnegative on $\mathbb{T}$. Let $\alpha \in \mathbb{N}\backslash\{1\}$. If f/p is nondecreasing on $\mathbb{T}$, then each function x satisfying*

$$
x(t) \leq f(t) + p(t)\int_{t_0}^{t} q(s)x(s)^{\alpha}\Delta s \quad \text{for all } t \geq t_0
$$

satisfies

$$
x(t) \leq \frac{f(t)}{\left\{1+(\alpha-1)\int\limits_{t_0}^{t}(\ominus qpf^{\alpha-1})(s)\Delta s\right\}^{\frac{1}{\alpha-1}}} \tag{2.3}
$$

on $[t_0, t_m)$ where t_m is the first point for which the denominator on the right-hand side of (2.3) is nonpositive.

Now, we present the concepts of h-stability.

We consider the dynamic equation defined on a time scale $\mathbb{T}$ as below:

$$
\begin{aligned}
x^{\Delta}\left(t\right) &= f\left(t,x\right), \quad t \in \mathbb{T},\ t > t_0, \\
x\left(t_0\right) &= x_0 \in D(A) \subset X,
\end{aligned} \tag{2.4}
$$

where X is a Banach space and $f : \mathbb{T} \times X \to X$ is rd-continuous in the first argument.

Definition 2.2.4 *[13] Equation (2.4) is called globally uniformly h-stable if there exist a nonincreasing bounded rd-continuous function $h : \mathbb{T}_{t_0}^{+} \to \mathbb{R}_{+}^{*}$ and a constant $c \geq 1$ (independent of t_0) such that any solution $x\left(t\right) = x\left(t, t_0, x_0\right)$ of equation (2.4) satisfies*

$$
\|x\left(t,t_0,x_0\right)\| \leq c\|x_0\|h\left(t\right)\left(h\left(t_0\right)\right)^{-1}, \quad t \in \mathbb{T}_{t_0}^{+}, \tag{2.5}
$$

here $(h(t))^{-1} = \frac{1}{h(t)}$ *and* $\|h\|_\infty = \sup_{t\in\mathbb{T}} \mid h(t) \mid$.

A precise definition of the practical uniform h-stability is given as follows, which will be used in subsequent main result.

Definition 2.2.5 *[34] Equation (2.4) is called globally practically uniformly h-stable if there exist* $c \geq 1$ *and* $\rho > 0$ *such that for all* $t \in \mathbb{T}^+_{t_0}$ *and* $x_0 \in \mathbb{R}^n$ *we have*

$$\|x(t)\| \leq \frac{c\,\|x_0\|\,h(t)}{h(t_0)} + \rho. \tag{2.6}$$

The stability behavior of the origin as an equilibrium point for the linear time-varying system (3.2) can be completely characterized in terms of $T(t-t_0)$:

$$x(t) = T(t-t_0)x_0, \quad t \geq t_0,\ t \in \mathbb{T}.$$

Definition 2.2.6 *[68] If* $f \in C_{rd}$*, then any solution of (2.1) with initial value* $x_0 \in D(A)$ *is given by:*

$$x(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T(t-\sigma(s))f(s,x(s))\Delta s.$$

2.3 Main results

Now, we obtain some results about the practical h-stability of perturbed equation in the following theorems.

Theorem 2.3.1 *Assume that the homogeneous equation (3.2) is globally uniformly h-stable and suppose that*

$$\|f(t,x(t))\| \leq \beta(t)\,\|x(t)\| + p(t), \quad t \in \mathbb{T}, \tag{2.7}$$

where $\beta, p \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$ *such that*

$$\begin{aligned} \int_{t_0}^{+\infty} \frac{p(s)}{h(\sigma(s))}\Delta s &\leq M_1, \quad M_1 > 0, \\ \int_{t_0}^{+\infty} \frac{h(s)\beta(s)}{h(\sigma(s))}\Delta s &\leq M_2, \quad M_2 > 0, \quad t \in \mathbb{T}. \end{aligned} \tag{2.8}$$

Then the perturbed equation (2.1) is globally practically uniformly h-stable.

Proof 1 *Let $x(t)$ be the solution of equation (2.1). Then*

$$x(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T(t-\sigma(s))f(s,x(s))\Delta s,$$

where $T(t-t_0)x_0$ is the solution of the homogeneous equation (3.2).
Thus, from the global uniform h-stability of system (3.2) and the condition (2.7), we obtain

$$\|x(t)\| \leq c\frac{\|x_0\|}{h(t_0)}h(t) + ch(t)\int_{t_0}^{t} \frac{\beta(s)\,\|x(s)\| + p(s)}{h(\sigma(s))}\Delta s.$$

Put $\vartheta(t) = \frac{\|x(t)\|}{h(t)}$. We get

$$\vartheta(t) \leq c\vartheta(t_0) + c\int_{t_0}^{t} \left(\frac{h(s)\beta(s)}{h(\sigma(s))}\vartheta(s) + \frac{p(s)}{h(\sigma(s))}\right)\Delta s.$$

By applying Lemma (2.2.1), we obtain for all $t \geq t_0$

$$\vartheta(t) \leq c\vartheta(t_0) + c\int_{t_0}^{t} \left(c\vartheta(t_0)\frac{h(s)\beta(s)}{h(\sigma(s))} + \frac{p(s)}{h(\sigma(s))}\right)\exp\left(c\int_{\sigma(s)}^{t} \frac{h(\tau)\beta(\tau)}{h(\sigma(\tau))}\Delta\tau\right)\Delta s.$$

From conditions (2.8), we have

$$\vartheta(t) \leq (c\vartheta(t_0)(1+cM_2) + cM_1)\exp(cM_2),$$

we deduce that, for all $t \in \mathbb{T}_{t_0}^{+}$ and all $x_0 \in \mathbb{R}^n$, the solution of system (2.1) satisfies

$$\|x(t)\| \leq \frac{k\,\|x_0\|\,h(t)}{h(t_0)} + \rho,$$

where $k = c(1 + cM_2)\exp(cM_2) \geq 1$ and $\rho = cM_1 \exp(cM_2) \|h\|_\infty > 0$. Therefore the perturbed equation (2.1) is globally practically uniformly h-stable. ∎

Theorem 2.3.2 *Assume that the perturbation term f of equation (2.1) satisfies the following condition:*

$$\|f(t, x(t))\| \leq g\left(\beta(t)\|x(t)\|^\alpha + p(t)\right), \quad \forall t \in \mathbb{T}, \quad 0 < \alpha < 1, \tag{2.9}$$

where β and p are non-negative rd-continuous functions on $\mathbb{T}$, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable increasing function on $(0, +\infty)$ with continuous nonincreasing first derivative g' on $(0, +\infty)$ with

$$\begin{aligned} \int_{t_0}^{+\infty} \frac{g'(p(s))\beta(s)h(s)}{h(\sigma(s))}\Delta s &\leq M_1, \quad M_1 > 0, \qquad (2.10) \\ \int_{t_0}^{+\infty} \frac{(1-\alpha)g'(p(s))\beta(s) + g(p(s))}{h(\sigma(s))}\Delta s &\leq M_2, \quad M_2 > 0, \quad t \in \mathbb{T}. \end{aligned}$$

If the homogeneous equation (3.2) is globally uniformly h-stable, then the perturbed equation (2.1) is globally practically uniformly h-stable.

Proof 2 *Let $x(t)$ be the solution of perturbed equation (2.1). Then*

$$x(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t - \sigma(s))f(s, x(s))\Delta s,$$

where $T(t - t_0)x_0$ is the solution of the homogeneous equation (3.2). Thus, from the global uniform h-stability of the homogeneous equation (3.2) and the condition (2.9), we obtain

$$\|x(t)\| \leq c\frac{\|x_0\|}{h(t_0)}h(t) + ch(t)\int_{t_0}^{t} \frac{g\left(\beta(s)\|x(s)\|^\alpha + p(s)\right)}{h(\sigma(s))}\Delta s. \tag{2.11}$$

Applying the Mean Value Theorem for the function g, then for any $y \geq x > 0$; there exists $k \in (x, y)$; such that

$$g(y) - g(x) = g'(k)(y - x) \leq g'(x)(y - x),$$

then

$$g\left(\beta(s)\left\|x(s)\right\|^{\alpha}+p(s)\right)\leq g'(p(s))\beta(s)\left\|x(s)\right\|^{\alpha}+g(p(s)), \tag{2.12}$$

hence, from (2.11) and (2.12), one gets that

$$\|x(t)\|\leq c\frac{\|x_0\|}{h(t_0)}h(t)+ch(t)\int_{t_0}^{t}\left(\frac{g'(p(s))\beta(s)}{h(\sigma(s))}\left\|x(s)\right\|^{\alpha}+\frac{g(p(s))}{h(\sigma(s))}\right)\Delta s. \tag{2.13}$$

By applying Lemma (2.2.2), we obtain for all $t\geq t_0$

$$\begin{aligned}&\|x(t)\|\\ &\leq c\frac{\|x_0\|}{h(t_0)}h(t)+ch(t)\int_{t_0}^{t}\left(\frac{g'(p(s))\beta(s)}{h(\sigma(s))}\left(c\alpha\frac{\|x_0\|}{h(t_0)}h(s)+1-\alpha\right)+\frac{g(p(s))}{h(\sigma(s))}\right)\\ &\times\exp\left(c\alpha\int_{\sigma(s)}^{t}\frac{g'(p(\tau))\beta(\tau)h(\tau)}{h(\sigma(\tau))}\Delta\tau\right)\Delta s,\end{aligned}$$

from conditions (2.10), we obtain

$$\|x(t)\|\leq c\frac{\|x_0\|}{h(t_0)}h(t)\left(1+c\alpha M_1\right)\exp(c\alpha M_1)+c(1-\alpha)M_2\exp(c\alpha M_1)\left\|h\right\|_{\infty},$$

we deduce that, for all $t\geq t_0$ *and all* $x_0\in\mathbb{R}^n$*, the solution of the perturbed equation (2.1) satisfies*

$$\|x(t)\|\leq\frac{k\left\|x_0\right\|h(t)}{h(t_0)}+\rho,$$

where $k=c\left(1+c\alpha M_1\right)\geq 1$ *and* $\rho=c(1-\alpha)M_2\exp(c\alpha M_1)\left\|h\right\|_{\infty}>0$*. We can see that the perturbed equation (2.1) is globally practically uniformly h-stable. The proof is complete.* ■

Theorem 2.3.3 *Assume that the perturbation term* f *of system (2.1) satisfies the following condition:*

$$\|f(t,x(t))\|\leq\eta(t)\left\|x(t)\right\|^{\alpha}+\varrho(t),\quad\forall t\geq 0,\quad\alpha>1, \tag{2.14}$$

where η and ϱ are non-negative rd-continuous functions on $\mathbb{T}$ such that

$$\int_{t_0}^{t} \frac{\varrho(s)}{h(\sigma(s))}\Delta s \leq M_1, \quad M_1 > 0, \tag{2.15}$$

and

$$\int_{t_0}^{t} \frac{h(s)\eta(s)}{h(\sigma(s))}\Delta s \leq \frac{1}{(\alpha-1)c^{\alpha+1}\left(\frac{\|x_0\|}{h(t_0)} + M_1\right)^{\alpha-1}}. \tag{2.16}$$

If the homogeneous equation (3.2) is globally uniformly h-stable, then the perturbed equation (2.1) is globally practically uniformly h-stable.

Proof 3 *Let $x(t)$ be the solution of the perturbed equation (2.1). Then*

$$x(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T(t-\sigma(s))f(s,x(s))\Delta s,$$

where $T(t-t_0)x_0$ is the solution of the homogeneous equation (3.2). Thus, from the global uniform h-stability of (3.2) and the conditions (2.14), (2.15), we obtain

$$\|x(t)\| \leq \frac{c\,\|x_0\|\,h(t)}{h(t_0)} + ch(t)\int_{t_0}^{t} \frac{\eta(s)\,\|x(s)\|^{\alpha} + \varrho(s)}{h(\sigma(s))}\Delta s.$$

Then

$$\|x(t)\| \leq ch(t)\left(\frac{\|x_0\|}{h(t_0)} + M_1\right) + ch(t)\int_{t_0}^{t} \frac{\eta(s)}{h(\sigma(s))}\|x(s)\|^{\alpha}\Delta s.$$

By applying Lemma (2.2.3), we have for all $t \geq t_0$

$$\|x(t)\| \leq \frac{ch(t)\left(\frac{\|x_0\|}{h(t_0)} + M_1\right)}{\left\{1 + (\alpha-1)\int\limits_{t_0}^{t}\left(\ominus\frac{ch(s)\eta(s)}{h(\sigma(s))}\left(ch(t)\left(\frac{\|x_0\|}{h(t_0)} + M_1\right)\right)^{\alpha-1}\right)\Delta s\right\}^{\frac{1}{\alpha-1}}}.$$

Since $\ominus r = \frac{-r}{1+\mu r} \geq -r$ for all $r \geq 0$ and by condition (2.16), we obtain that

$$\|x(t)\| \leq \frac{ch(t)\left(\frac{\|x_0\|}{h(t_0)} + M_1\right)}{\left(1 - \frac{1}{c}\right)^{\frac{1}{\alpha-1}}},$$

which implies that

$$\|x(t)\| \leq k\frac{\|x_0\|}{h(t_0)}h(t) + k\|h\|_\infty M_1,$$

where $k = \left(\frac{c^\alpha}{c-1}\right)^{\frac{1}{\alpha-1}} \geq 1,$ *with* $\|h\|_\infty = \sup_{t\in\mathbb{T}}\{h(t)\}$*. We deduce that, for all* $t \geq t_0$ *and all* $x_0 \in \mathbb{R}^n$*, the solution of perturbed equation (2.1) satisfies*

$$\|x(t)\| \leq \frac{k\|x_0\|h(t)}{h(t_0)} + \rho,$$

where $\rho = kM_1\|h\|_\infty > 0$*. This finishes the proof.* ■

Remark 1 *In [33], the authors discussed the practical h-stability of the system (1) in the case* $\mathbb{T} = \mathbb{R}$*.*

2.4 Applications

In this section, we give two illustrative examples to highlight the utility of our results.

Example 2.4.1 *Let us consider the following perturbed equation:*

$$\begin{aligned} x^\Delta(t) &= Ax(t) + e_{-\frac{4}{3}}(t,0)x(t) + \frac{e_{-\frac{2}{3}}(t,0)}{(t+1)^2}, \quad t, t_0 \in \mathbb{T} = \frac{1}{2}\mathbb{Z}_+,\ t \geq t_0, \qquad (2.17)\\ x(t_0) &= x_0, \end{aligned}$$

where $\mathbb{Z}_+$ *is the set of positive or zero integers,* $x \in C_{rd}(\mathbb{T}^+_{t_0}, \mathbb{R}^2)$, $A = \begin{pmatrix} -\frac{4}{3} & -4 \\ 0 & -2 \end{pmatrix} \in M_2(\mathbb{R})$ *and* $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}^2$ *such that*

$$f(t, x(t)) = e_{-\frac{4}{3}}(t,0)x(t) + \frac{e_{-\frac{2}{3}}(t,0)}{(t+1)^2}.$$

It is easy to see that the matrix A *is the generator of the following* C_0 *semigroup:*

$$T(t) = (I + \mu A)^{2t} = \left(I + \frac{1}{2}A\right)^{2t},$$

then

$$T(t) = \begin{pmatrix} \frac{1}{3} & -2 \\ 0 & 0 \end{pmatrix}^{2t}$$

and

$$T(t-t_0) = \begin{pmatrix} \frac{1}{3} & -2 \\ 0 & 0 \end{pmatrix}^{2(t-t_0)} = \left(\frac{1}{3}\right)^{2(t-t_0)} \begin{pmatrix} 1 & -6 \\ 0 & 0 \end{pmatrix}^{2(t-t_0)}.$$

Therefore,

$$\|T(t-t_0)\| = \sqrt{37}\left(\frac{1}{3}\right)^{2(t-t_0)} = \sqrt{37} e_{-\frac{4}{3}}(t, t_0),$$

by taking $c \geq \sqrt{37}$ *and* $h(t) = e_{-\frac{4}{3}}(t, 0)$*, we get*

$$\|T(t-t_0)\| \leq \sqrt{37} e_{-\frac{4}{3}}(t, 0) e_{-\frac{4}{3}}(0, t_0) = ch(t)h(t_0)^{-1},$$

which ensures the h-stability of the homogeneous equation of the form $x^{\Delta}(t) = Ax(t)$. *Furthermore, we have*

$$h(\sigma(t)) = e_{-\frac{4}{3}}(\sigma(t), 0) = \left(1 + \frac{1}{2}\left(-\frac{4}{3}\right)\right) e_{-\frac{4}{3}}(t, 0) = \frac{1}{3} e_{-\frac{4}{3}}(t, 0).$$

Let us now verify the other two conditions of Theorem (2.3.1). It is clear that

$$\|f(t, x(t))\| \leq e_{-\frac{4}{3}}(t, 0)\,\|x(t)\| + \frac{e_{-\frac{2}{3}}(t, 0)}{(t+1)^2},$$

also

$$\begin{aligned} \int_{t_0}^{+\infty} \frac{e_{-\frac{2}{3}}(s, 0) e_{-\frac{4}{3}}(s, 0)}{\frac{1}{3} e_{-\frac{2}{3}}(s, 0)} \Delta s &= 3 \int_{t_0}^{+\infty} e_{-\frac{4}{3}}(s, 0) \Delta s \\ &\leq \frac{3}{2} \sum_{s \in \mathbb{Z}_+} \left(\frac{1}{3}\right)^{2s} < \infty. \end{aligned}$$

$$\int_{t_0}^{+\infty} \frac{\frac{e_{-\frac{2}{3}}(s,0)}{(s+1)^2}}{\frac{2}{3}e_{-\frac{2}{3}}(s,0)}\Delta s = \frac{3}{2}\int_{t_0}^{+\infty} \frac{1}{(s+1)^2}\Delta s$$
$$\leq \frac{3}{2}\sum_{s\in\mathbb{Z}_+} \frac{1}{(s+1)^2}\Delta s < \infty.$$

All statements of Theorem (2.3.1) are approved. So, we deduce that equation (2.17) is globally practically uniformly h-stable.

Example 2.4.2 *Let us consider the following perturbed problem:*

$$\begin{aligned} x^{\Delta}(t) &= Ax(t) + \ln\left(e_{-\frac{4}{3}}(t,0)\,(x(t))^{\frac{1}{4}} + \frac{e_{-\frac{2}{3}}(t,0)}{(t+1)^2} + 1\right), \qquad (2.18)\\ & \quad t, t_0 \in \mathbb{T} = \frac{1}{2}\mathbb{Z}_+, \\ x(t_0) &= x_0, \quad t \geq t_0. \end{aligned}$$

where $\mathbb{Z}_+$ is the set of positive or zero integers, $x \in C_{rd}(\mathbb{T}_{t_0}^+, \mathbb{R}^2), A = \begin{pmatrix} 1 & -6 \\ 0 & -2 \end{pmatrix} \in M_2(\mathbb{R})$ *and* $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}^2$ *such that*

$$f(t, x(t)) = \ln\left(e_{-\frac{4}{3}}(t,0)\,(x(t))^{\frac{1}{4}} + \frac{e_{-\frac{2}{3}}(t,0)}{(t+1)^2} + 1\right).$$

It is easy to see that the matrix A is the generator of the following C_0 semigroup:

$$T(t) = (I + \mu A)^{2t} = \left(I + \frac{1}{2}A\right)^{2t},$$

then

$$T(t) = \begin{pmatrix} \frac{3}{2} & -3 \\ 0 & 0 \end{pmatrix}^{2t}$$

and

$$T(t-t_0) = \begin{pmatrix} \frac{3}{2} & -3 \\ 0 & 0 \end{pmatrix}^{2(t-t_0)} = \left(-\frac{3}{2}\right)^{2(t-t_0)} \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}^{2(t-t_0)}.$$

Therefore,

$$\|T(t-t_0)\| = \sqrt{5}\left(\frac{3}{2}\right)^{2(t-t_0)} = \sqrt{5}e_1(t,t_0),$$

by taking $c \geq \sqrt{5}$ *and* $h(t) = e_1(t,0)$*, we get*

$$\|T(t-t_0)\| \leq \sqrt{5}e_1(t,0)e_1(0,t_0) = ch(t)h(t_0)^{-1},$$

which ensures the h-stability of the homogeneous equation of the form $x^{\Delta}(t) = Ax(t)$. *Furthermore, we have*

$$h(\sigma(t)) = e_1(\sigma(t),0) = \left(1 + \frac{1}{2}\times 1\right) e_{-\frac{2}{3}}(t,0) = \frac{3}{2}e_1(t,0).$$

The perturbation satisfies condition (2.10) of Theorem (2.3.2) with $g(x) = \ln(1+x), x > 0$*, which is a differentiable increasing function on* $(0,\infty)$ *with continuous nonincreasing first derivative.*

$$\begin{aligned} \|f(t,x(t))\| &= \ln\left(e_{-\frac{4}{3}}(t,0)\,\|x(t)\|^{\frac{1}{4}} + e_{-\frac{2}{3}}(t,0) + 1\right) \\ &= g\left(\beta(t)\,\|x(t)\|^{\alpha} + p(t)\right), \\ \beta(t) &= e_{-\frac{4}{3}}(t,0), \quad p(t) = e_{-\frac{2}{3}}(t,0), \quad \alpha = \frac{1}{4}. \end{aligned} \tag{2.19}$$

We have

$$\begin{aligned} \int_{t_0}^{+\infty} \frac{g'(p(s))\beta(s)h(s)}{h(\sigma(s))}\Delta s &\leq \int_{t_0}^{+\infty}\left(\frac{e_{-\frac{4}{3}}(s,0)e_1(s,0)}{\frac{3}{2}e_1(s,0)(1+e_{-\frac{2}{3}}(s,0))}\right)\Delta s \\ &\leq \frac{2}{3}\int_{t_0}^{+\infty} e_{-\frac{4}{3}}(s,0)\Delta s \leq \frac{2}{3}\sum_{s\in\frac{1}{2}\mathbb{Z}_+}\left(\frac{1}{3}\right)^{2s} < \infty. \end{aligned}$$

$$
\begin{aligned}
&\int_{t_0}^{+\infty} \frac{(1-\alpha)g'(p(s))\beta(s)+g(p(s))}{h(\sigma(s))}\Delta s \\
&= \int_{t_0}^{+\infty} \frac{\frac{\frac{3}{4}e_{-\frac{4}{3}}(s,0)}{(1+e_{-\frac{2}{3}}(s,0))} + \ln(e_{-\frac{2}{3}}(s,0)+1)}{\frac{3}{2}e_1(s,0)}\Delta s \\
&\leq \int_{t_0}^{+\infty} \left(\frac{3}{4}e_{-\frac{4}{3}}(s,0) + \frac{e_{-\frac{2}{3}}(s,0)}{e_1(s,0)}\right)\Delta s \\
&= \sum_{s\in\frac{1}{2}\mathbb{Z}_+} \left[\frac{3}{4}\left(\frac{1}{3}\right)^{2s} + \frac{e_1(s,0)e_{-\frac{10}{9}}(s,0)}{e_1(s,0)}\right] \\
&\leq \sum_{s\in\mathbb{Z}_+} \left(\frac{3}{4}\left(\frac{1}{3}\right)^{2s} + \left(\frac{4}{9}\right)^{2s}\right) < \infty.
\end{aligned}
$$

All statements of Theorem (2.3.2) are approved. So, we deduce that equation (2.18) is globally practically uniformly h-stable.

Chapter 3

Practical Stability of Parametric Time-Scale Systems via Lyapunov Methods

This chapter is devoted to the study of stability criteria for parametric dynamic systems defined on time scales. In this context, particular attention is given to the influence of parametric perturbations on the qualitative behavior of solutions. Small variations in system parameters may significantly affect stability properties, making their analysis essential for both theoretical understanding and practical applications. To address this problem, we adopt the Lyapunov functional approach, which provides a powerful and systematic method for investigating stability without requiring explicit solutions of the system.

3.1 Statement of the Problem

Let $\mathbb{T}$ be a time scale. The delta derivative $x^{\Delta}(t)$ generalizes the derivative in continuous-time and forward difference in discrete-time.

Consider the time scale dynamic equation:

$$x^{\Delta}(t) = A(t)x(t) + f(t, x(t), \epsilon), \quad t \in \mathbb{T}, \tag{3.1}$$

in terms of the h-stability of the homogeneous equation:

$$\begin{aligned} x^{\Delta}(t) &= Ax(t), \quad t \in \mathbb{T},\ t > t_0, \\ x(t_0) &= x_0, \end{aligned} \tag{3.2}$$

where $x(t) \in \mathbb{R}^n$, $A(t) \in \mathbb{R}^{n \times n}$ is rd-continuous and uniformly regressive (i.e., $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}$), $f : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is rd-continuous, and $\epsilon > 0$ is a small scalar parameter.

3.2 Some Important Inequalities

The following lemmas are very useful in our main results.

Lemma 3.2.1 *Let $\mathbb{T}$ be a time scale with $t_0 \in \mathbb{T}$. Suppose $a \in \mathbb{R}_+$, and $u, b \in C_{\mathrm{rd}}([t_0, t_1]_{\mathbb{T}}, \mathbb{R}_+)$ (rd-continuous, nonnegative functions).*

If

$$u(t) \le a + \int_{t_0}^{t} b(s)u(s)\,\Delta s, \quad t \in [t_0, t_1]_{\mathbb{T}},$$

then

$$u(t) \le a\, e_b(t, t_0), \quad t \in [t_0, t_1]_{\mathbb{T}},$$

where $e_b(t, t_0)$ is the time-scale exponential function.

Now, we present the concepts of h-stability.

Consider the dynamic equation defined on a time scale $\mathbb{T}$ as follows:

$$\begin{cases} x^{\Delta}(t) = A(t)x(t), \\ x(t_0) = x_0, \end{cases} \tag{3.3}$$

where $x \in \mathbb{R}^n$, $t, t_0 \in \mathbb{T}$, and $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ is an rd-continuous, regressive matrix depending on t.

Definition 3.2.2 *For $t_0 \in \mathbb{T}$, the solution of the problem (3.3) is called the exponential of the matrix function, denoted by $\phi_A(t, t_0)$, where $A \in C_{\text{rd}}\mathbb{R}(\mathbb{T}, M_n(\mathbb{R}))$. Consequently, the function $\phi_A(t, t_0)$ possesses the following two properties:*

$$\begin{cases} \phi_A^{\Delta}(t, t_0) = A(t)\phi_A(t, t_0), \\ \phi_A(t_0, t_0) = I_n. \end{cases} \tag{3.4}$$

This matrix function is called the transition matrix, and our assumption on $A(t)$ shows that the transition matrix exists and is unique.

Definition 3.2.3 *The system (3.3) is* uniformly exponentially stable *if there exist constants $M > 0$, $\lambda > 0$ such that its transition matrix $\Phi_A(t, t_0)$ satisfies:*

$$\|\Phi_A(t, t_0)\| \leq Me_{-\lambda}(t, t_0), \quad \forall t \geq t_0 \in \mathbb{T},$$

where $e_{-\lambda}(t, t_0)$ denotes the exponential function on time scales.

Definition 3.2.4 *The system (3.3) is said to be **locally uniformly exponentially stable** if there exist $\gamma \geq 1$, independent of t_0, $\lambda > 0$ with $-\lambda \in \mathbb{R}^+$, and $\omega > 0$ such that:*

$$\|x_0\| \leq \delta \implies \|x(t)\| \leq \gamma e_{-\lambda}(t, t_0)^{\omega}\|x_0\|, \quad \forall t \in \mathbb{T}. \tag{3.5}$$

Definition 3.2.5 *The system (3.3) is said to be **uniformly exponentially stable** if relation (3.5) holds $\forall x_0 \in \mathbb{R}^n$.*

Definition 3.2.6 *The system (3.3) is said to be **uniformly practically exponentially stable** if there exist $\gamma \geq 1$, $\lambda > 0$ with $-\lambda \in \mathbb{R}^+$, $\omega > 0$, and $r \geq 0$ such that:*

$$\|x(t)\| \leq \gamma e_{-\lambda}(t, t_0)^{\omega}\|x_0\| + r, \quad \forall t \in \mathbb{T}. \tag{3.6}$$

Definition 3.2.7 *The system (3.3) is said to be* ***uniformly practically stable*** *if for every $\eta > 0$, there exists $\delta > 0$ such that if $\|x(t_0)\| < \delta$, then $\|x(t)\| < \eta$ for all $t \geq t_0 \in \mathbb{T}$, provided ϵ is sufficiently small.*

3.3 Main Result

3.3.1 Practical Stability via Lyapunov Method

Theorem 3.3.1 *Consider the system* (3.1)*. Suppose that:*

- The matrix function $A(t)$ is uniformly regressive and rd-continuous.

- The unperturbed system (3.2) is uniformly exponentially stable.

- There exists a Lyapunov function $V : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$, and class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$, and a positive function $\alpha_4(\epsilon)$ with $\lim_{\epsilon \to 0} \alpha_4(\epsilon) = 0$, such that:

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|),$$

$$V^\Delta(t,x) \leq -\alpha_3(\|x\|) + \alpha_4(\epsilon).$$

Proof 4 *Let $\eta > 0$ be given. Since α_3 is class $\mathcal{K}_\infty$, it is strictly increasing. Choose $r > 0$ such that:*

$$\alpha_3(r) = 2\alpha_4(\epsilon).$$

Then for any $\|x\| > r$,

$$V^\Delta(t,x) \leq -\alpha_3(\|x\|) + \alpha_4(\epsilon) < -\alpha_4(\epsilon) < 0,$$

which means that the Lyapunov function decreases whenever $\|x(t)\| > r$. Hence, trajectories are attracted to and remain inside the ball of radius r centered at the origin.

Now, for any initial condition $\|x(t_0)\| < \delta$ small enough such that $\alpha_2(\delta) < \alpha_1(r)$, we *have:*

$$V(t_0, x(t_0)) < \alpha_1(r),$$

which implies that $\|x(t)\| < r$ *for all* $t \geq t_0$*.*

By choosing $r < \eta$*, we ensure* $\|x(t)\| < \eta$ *for all* $t \geq t_0$*, i.e., practical stability is achieved.* ■

The above theorem applies to both continuous-time systems (when $\mathbb{T} = \mathbb{R}$) and discrete-time systems (when $\mathbb{T} = \mathbb{Z}$). It is particularly useful for hybrid and switched systems modeled naturally by time scale calculus.

We have established a Lyapunov-based practical stability result for dynamic systems on arbitrary time scales under small perturbations. The method provides a unified tool applicable to both continuous and discrete systems, highlighting the power and flexibility of time scale theory in the study of real-world dynamical systems.

Now, we study the practical stability of dynamic systems on time scales under Lipschitz continuous perturbations. Assuming the linear part is uniformly exponentially stable, we prove that the perturbed system remains practically stable if the Lipschitz constant is sufficiently small. The result is proved using a Lyapunov-like approach and the Gronwall inequality on time scales, thereby generalizing known continuous and discrete results.

3.3.2 Practical Stability with Lipschitz Perturbation

In this section, we discuss practical stability with Lipschitz perturbation equation on a time scale:

$$\begin{aligned} x^{\Delta}(t) &= Ax(t) + f(t, x), \quad t \in \mathbb{T},\ t > t_0, \\ x(t_0) &= x_0, \end{aligned} \tag{3.7}$$

in terms of the h-stability of the homogeneous equation:

$$\begin{aligned} x^{\Delta}(t) &= Ax(t), \quad t \in \mathbb{T},\ t > t_0, \\ x(t_0) &= x_0, \end{aligned} \tag{3.8}$$

where $x \in \mathbb{R}^n$. The matrix $A(t) \in \mathbb{R}^{n\times n}$ is rd-continuous and uniformly regressive. The perturbation $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous.

Theorem 3.3.2 *If $f \in C_{rd}$, then any solution of (3.7) with initial value $x_0 \in \mathbb{R}^n$ is given by:*

$$x(t) = \Phi_A(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi_A(t, \sigma(s))f(s, x(s))\Delta s. \tag{3.9}$$

Theorem 3.3.3 *Assume that the homogeneous equation (3.8) is* ***uniformly exponentially stable*** *and the perturbation $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous and satisfies a* ***Lipschitz condition****:*

$$\|f(t, x)\| \leq L\|x\|, \quad \forall t \in \mathbb{T},\ x \in \mathbb{R}^n,$$

for some constant $L \geq 0$, and further assume:

$$L < \frac{\lambda}{M}.$$

Then the perturbed equation (3.7) is ***uniformly practically stable****.*

Proof 5 *Let $x(t)$ be a solution of the system. By the variation of constants formula on time scales:*

$$x(t) = \Phi_A(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi_A(t, \sigma(s))f(s, x(s))\Delta s.$$

Taking norms and applying assumptions:

$$\|x(t)\| \leq \|\Phi_A(t, t_0)\|\|x(t_0)\| + \int_{t_0}^{t} \|\Phi_A(t, \sigma(s))\| \cdot \|f(s, x(s))\|\Delta s.$$

Using:

$$\|\Phi_A(t,t_0)\| \le Me_{-\lambda}(t,t_0), \quad \|f(s,x(s))\| \le L\|x(s)\|,$$

we get:

$$\|x(t)\| \le Me_{-\lambda}(t,t_0)\|x(t_0)\| + ML\int_{t_0}^{t} e_{-\lambda}(t,\sigma(s))\|x(s)\|\Delta s.$$

Define $y(t) := \|x(t)\|$. *Then:*

$$y(t) \le Me_{-\lambda}(t,t_0)y(t_0) + ML\int_{t_0}^{t} e_{-\lambda}(t,\sigma(s))y(s)\Delta s.$$

*By the **Gronwall inequality on time scales**, we obtain:*

$$y(t) \le Me_{-\lambda}(t,t_0)y(t_0) \cdot e_{ML}(t,t_0).$$

This simplifies to:

$$y(t) \le Me_{-(\lambda-ML)}(t,t_0)y(t_0).$$

Since $\lambda - ML > 0$*, the solution decays exponentially. Hence:*

$$\|x(t)\| \le Me_{-(\lambda-ML)}(t,t_0)\|x(t_0)\|.$$

Let $\eta > 0$ *be given. Choose:*

$$\delta := \frac{\eta}{M},$$

then $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \eta$*, for all* $t \ge t_0$*, proving practical stability.* ■

The key condition $L < \lambda/M$ ensures that the perturbation is not strong enough to destabilize the system. This result generalizes the robustness of exponentially stable systems under Lipschitz perturbations to the setting of time scales.

3.4 Numerical example

Consider the scalar time scale system

$$x^{\Delta}(t) = -2x(t) + \epsilon \sin(t)x(t),$$

where $\epsilon > 0$ is a small parameter.

The unperturbed system $x^{\Delta} = -2x$ is exponentially stable with $M = 1$ and decay rate $\lambda = 2$.

The perturbation satisfies

$$|f(t,x)| = |\epsilon \sin(t)||x| \leq \epsilon|x|,$$

so the Lipschitz constant is $L = \epsilon$.

The practical stability theorem requires

$$\epsilon < \frac{\lambda}{M} = 2.$$

For $\epsilon = 3 \times 10^{-8}$, solutions decay exponentially, demonstrating practical stability under the perturbation.

Remark: Numerical simulation can be performed using standard ODE solvers for continuous time $\mathbb{T} = \mathbb{R}$ or difference equations for discrete time $\mathbb{T} = \mathbb{Z}$, confirming the theoretical result.

Application

The main theorem can be applied to model the concentration of a drug in a patient's bloodstream, where the dosage is administered at specific, potentially irregular, times (a hybrid system). This is a perfect scenario for time-scale analysis.

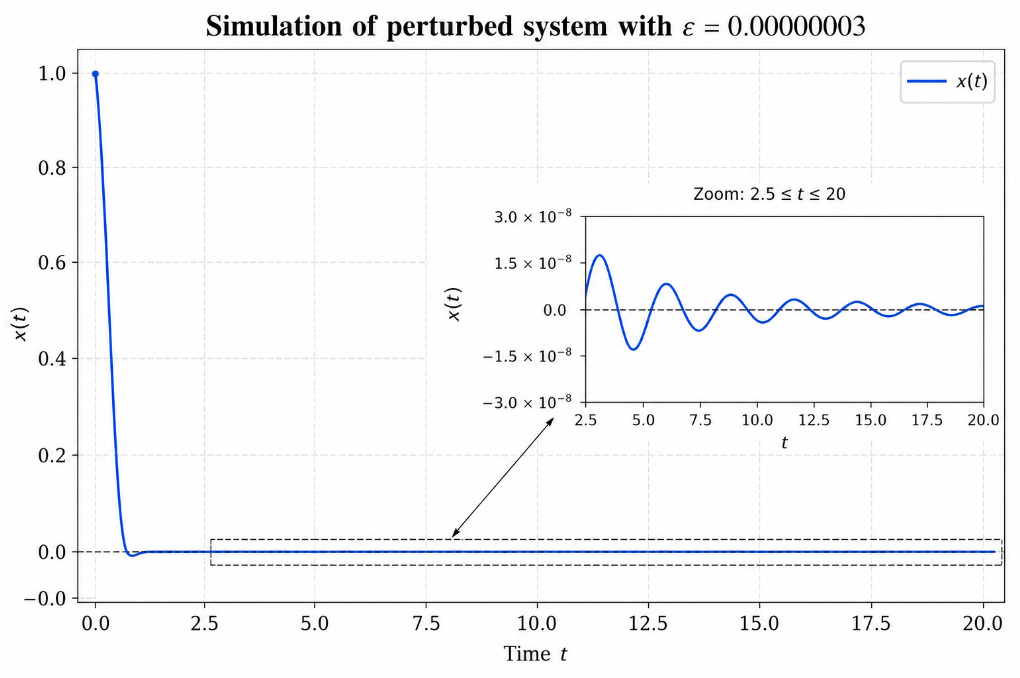

Figure 3.1: Solution evolution for perturbed system

A patient is prescribed a medication that is administered via intravenous (IV) injection at discrete times $t_0, t_1, t_2, \ldots$, forming a time scale $\mathbb{T}$. Between these doses, the drug is metabolized and cleared from the body continuously.

Let $x(t)$ represent the concentration of the drug in the bloodstream at time t.

The Unperturbed System ($x^\Delta = A(t)x$): This models the body's natural process of metabolizing and clearing the drug. Between doses (on continuous intervals), the concentration decays exponentially: $x'(t) = -\alpha x(t)$, where $\alpha > 0$ is the clearance rate. At each dosing time t_k (a right-dense point), an impulse is applied, instantly increasing the concentration: $x(t_k^+) = x(t_k^-) + d_k$, where d_k is the dose size.

We can model this hybrid process on a time scale $\mathbb{T}$ that is a union of closed intervals connecting the dosing times. The system $x^\Delta(t) = A(t)x(t)$ can be defined such that it captures both the continuous decay and the impulsive jumps. If the dosing schedule and clearance rate are consistent, this unperturbed system can be designed to be uniformly exponentially stable (meaning the drug concentration eventually returns to zero if dosing stops), with calculated constants $M \geq 1$ and $\lambda > 0$.

The Perturbation ($f(t, x(t))$): This represents a real-world uncertainty or variation. A classic example is the impact of another drug or food (e.g., grapefruit juice) that inhibits the metabolism of the primary drug. This inhibition effectively reduces the clearance rate in a way that depends on the current concentration of the inhibitor, which itself may vary with time. We can model this as a state-dependent perturbation:

$$f(t, x(t)) = \epsilon(t, x) \cdot x(t)$$

Here, $\epsilon(t, x)$ is a function that represents the time-varying and concentration-dependent inhibition effect. For instance, it could be $\epsilon(t, x) = \gamma \cdot \frac{I(t)}{K+I(t)}$, where $I(t)$ is the concentration of the inhibitor.

Let us verify the conditions of the theorem for this example:

Uniform Exponential Stability of Nominal System: As stated, by properly designing the dosing schedule $\{t_k\}$ and doses $\{d_k\}$ relative to the clearance rate α, we can ensure the unperturbed system (without the inhibitor) is uniformly exponentially stable. This means the drug concentration does not blow up and is effectively cleared between doses. We assume this has been established, giving us constants M and λ.

Lipschitz Condition on Perturbation: The perturbation is $f(t, x) = \epsilon(t, x)x$. We need to check if its norm is bounded by $L\|x\|$.

$$\|f(t, x)\| = |\epsilon(t, x)| \cdot \|x\|$$

If the inhibitory effect is bounded—meaning there is a physiological maximum to how much the metabolism can be slowed—then there exists a constant $L > 0$ such that $|\epsilon(t, x)| \leq L$ for all t and x. For example, if grapefruit juice can at most reduce the clearance rate by 30%, then $L = 0.3\alpha$. This directly gives us:

$$\|f(t, x)\| \leq L\|x\|$$

The perturbation satisfies the required Lipschitz condition.

The Key Stability Condition ($L < \lambda/M$): The theorem now requires that the maximum possible perturbation is small enough relative to the stability margin of the original system.

$$L < \frac{\lambda}{M}$$

In our pharmacological context, this translates to:

The maximum possible inhibition of drug metabolism (L) must be less than the inherent stability margin (λ/M) of the prescribed dosage regimen.

If the condition $L < \lambda/M$ holds, the theorem guarantees that the perturbed system (the patient taking the drug along with the inhibitor) is uniformly practically stable.

Practical Stability in this Context: It means the drug concentration $x(t)$ will not exhibit dangerous blow-ups or toxic accumulations. While the concentration might be higher than expected due to the inhibited clearance, it will remain bounded within a safe, predictable range η as long as the initial concentration was within a safe range δ.

Clinical Implication: A doctor, knowing the stability properties of their dosing regimen (λ/M) and the maximum potential metabolic inhibition from a drug interaction (L), can use this theorem's condition as a safety criterion.

If $L < \lambda/M$, the interaction is likely manageable, and the regimen is robust. If $L \geq \lambda/M$, the interaction is a serious risk, potentially leading to toxic drug levels. The doctor must then adjust the dose (d_k) or the dosing frequency (changing $\mathbb{T}$, which affects M and λ) to re-establish a safe stability margin.

This example demonstrates how the abstract theorem provides a rigorous framework for ensuring robustness in a critical, real-world hybrid system.

Chapter 4

Existence, Uniqueness and Ulam Stability of Certain Classes of Second-Order Nonlinear Dynamic Equations on Time Scales and Applications

In this chapter, we study a nonlinear integro-dynamic equation on time scales with local initial condition. The purpose of this chapter is to prove existence and uniqueness of solutions and to investigate qualitative properties of solutions of this equation such as Ulam stability. The analysis is based on the Krasnoselskii fixed point theorem and Gronwall-type dynamic inequalities.

The chapter is organized as follows: Section 2 contains existence and uniqueness of solutions of the nonlinear equation. Section 3 contains the Ulam stability of the equation. Section 4 shows the applicability of the theoretical results by numerical examples.

4.1 Statement of the Problem

We investigate sufficient conditions for Hyers–Ulam and Hyers–Ulam– Rassias stability of second-order nonlinear dynamic equations on time scales of the form

$$x^{\Delta^2}(t) = Q(t)x(t) + G\left(t, x(t), \int_a^t \mathcal{H}(t, s, x(s))\Delta s\right), \quad t \in \mathbb{I}^k, \tag{4.1}$$

subject to the initial condition

$$x^{\Delta^i}(a) = c_i \in \boldsymbol{X}, \quad i = 0, 1, \tag{4.2}$$

where $\mathbb{I} := [a, b] \cap \mathbb{T}$ with a time scale $\mathbb{T} \subset \mathbb{R}$, $a, b \in \mathbb{T}$ with $a < b$, $x : \mathbb{I} \to \mathbb{R}^n$ is an unknown function to be determined, $x^\sigma = x \circ \sigma$, x^Δ is the delta derivative of x, $Q : \mathbb{I} \to \mathbb{R}$ is regressive and rd-continuous, $G : \mathbb{I} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous in its first variable and continuous in its second and third variables, $\mathcal{H} : \mathbb{I} \times \mathbb{I} \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous in its first and second variables and continuous in its third variable. In addition, both G and $\mathcal{H}$ are assumed to be nonlinear functions.

4.2 Sufficient Conditions for Existence and Uniqueness of Solutions

Theorem 58 *Let $K(t, x) : \mathbb{I} \times \boldsymbol{X} \to \boldsymbol{X}$ be rd-continuous in $t \in \mathbb{I}$ for every $x \in \boldsymbol{X}$, and continuous in x for every $t \in \mathbb{I}$. Then x is a solution of*

$$x^{\Delta^2}(t) = K(t, x(t)), \quad t \in \mathbb{I}^k,\ x^{\Delta^i}(a) = c_i \in \boldsymbol{X},\ i = 0, 1, \tag{4.3}$$

if and only if x solves the integral equation

$$x(t) = c_0 + c_1(t - a) - \int_a^t (s - t + \mu(s))K(s, x(s))\Delta s, \quad t \in \mathbb{I}, \tag{4.4}$$

for some constants $c_0, c_1 \in \boldsymbol{X}$.

Definition 59 *A function* $G : \mathbb{T} \times \mathbb{X}^K \to \mathbb{X}$ *is said to satisfy the Lipschitz condition with constant* $\beta > 0$ *if*

$$|G(t, x_1, x_2, \dots, x_n) - G(t, y_1, y_2, \dots, y_n)| \leq \beta \sum_{i=1}^{n} |x_i - y_i|$$

for all $x_i, y_i \in \mathbb{X}$ *and all* $t \in \mathbb{T}$.

As usual, a function $\mathcal{H} : \mathbb{X}^K \to \mathbb{X}$ *is said to satisfy the Lipschitz condition with constant* $\gamma > 0$ *if*

$$\|\mathcal{H}(x_1, x_2, \dots, x_n) - \mathcal{H}(x_1, x_2, \dots, x_n)\| \leq \gamma \sum_{i=1}^{n} |x_i - y_i| \,.$$

Proof 6 *Assume that* x *satisfies the integral Equation* (4.4). *We denote by*

$$M(t) = -\int_a^t (s - t + \mu(s))K(s, x(s))\Delta s.$$

By Theorem 1.117(i) in [*20*], *we conclude that*

$$M^{\Delta}(t) = \int_a^t k(s, x(s))\Delta s$$

and

$$M^{\Delta^2}(t) = k(t, x(t)).$$

This implies that $x^{\Delta^2}(t) = k(t, x(t))$. *To prove the other direction, assume* x *is a solution of Equation* (4.3). *We denote by*

$$G(t) = \int_a^t k(s, x(s))\Delta s$$

and

$$\mathcal{L}(t) = \int_a^t G(s)\Delta s.$$

By integrating both sides of (4.3) *twice, we have*

$$x(t) = c_0 + c_1(t - t_0) + \mathcal{L}(t).$$

Here $c_i = x^{\Delta^i}(a), i = 0, 1$*. It is readily seen that* $M(t) = \mathcal{L}(t)$ *for every* t*. Indeed, we have*

$$\begin{aligned}
\mathcal{L}^{\Delta}(t) &= G(t) \\
&= \int_a^t k(s, x(s))\Delta s \\
&= M^{\Delta}(t).
\end{aligned}$$

Consequently, $M(t) = \mathcal{L}(t) + g \in [a, b]_{\boldsymbol{T}}$*. We have* $g = M(a) - \mathcal{L}(a) = 0$*. Therefore,* x *satisfies Equation* (4.4)*.* ■

Corollary 60 x *is a solution of Equation* (4.1) *if and only if* x *solves the integral equation*

$$x(t) = c_0 + c_1(t - t_0) - \int_a^t (s - t + \mu(s)) \left(Q(s)x(s) + G\left(t, x(t), \int_a^t \mathcal{H}(s, \tau, x(\tau))\Delta\tau\right)\right) \Delta s, \quad t \in \mathbb{I}, \tag{4.5}$$

for some constants $c_0, c_1 \in \boldsymbol{X}$.

Throughout the rest of the paper, we use the following conditions.

(A) $Q \in \mathbb{R}$.

(B) G and $\mathcal{H}$ satisfy the Lipschitz conditions with constants β and γ, respectively.

(C) For any $c_0, c_1 \in \boldsymbol{X}$, (4.1) has a solution x satisfying $x^{\Delta^i}(a) = c_i, i = 0, 1$.

(D) There is $\alpha \in (0, 1)$ such that

$$(b - a)\left(\sup_{t \in \mathbb{I}} \int_a^t |Q(s)|\, \Delta s + \beta(1 + \gamma)\,(b - a)\right) < \alpha.$$

Theorem 61 *Assume* $(A), (B)$, *and* (D). *If* $c_0, c_1 \in \boldsymbol{X}$, *then* (4.1) *has a unique solution* x *satisfying* $x^{\Delta^i}(a) = c_i, i = 0, 1.$

Proof 7 *Fix* $c_0, c_1 \in \boldsymbol{X}$. *Define the operator* $T : C_{rd}(\mathbb{I}, \boldsymbol{X}) \to C_{rd}(\mathbb{I}, \boldsymbol{X})$ *by*

$$Tx(t) = c_0 + c_1(t - t_0) - \int_a^t (s - t + \mu(s)) \left(Q(s)x(s) + G\left(t, x(t), \int_a^s \mathcal{H}(s, \tau, x(\tau))\Delta\tau\right)\right) \Delta s, \quad t \in \mathbb{I}.$$

For $x_1, x_2 \in C_{rd}(\mathbb{I}, \boldsymbol{X})$, *we have*

$$\begin{aligned} |Tx_1(t) - Tx_2(t)| \ \leq\ & \int_a^t |s - t + \mu(s)|\, Q(s) \left\| x_1(s) - x_2(s) \right\| \\ & + \left\| G\left(t, x_1(t), \int_a^s \mathcal{H}(s, \tau, x_1(\tau))\Delta\tau\right) - G\left(t, x_2(t), \int_a^s \mathcal{H}(s, \tau, x_2(\tau))\Delta\tau\right) \right\| \Delta s. \end{aligned} \tag{4.6}$$

It follows from (B) *and* (D) *that*

$$\begin{aligned} |Tx_1(t) - Tx_2(t)| \ \leq\ & \int_a^t |s - t + \mu(s)| \left[|Q(s)| \left\| x_1(s) - x_2(s) \right\| \right. \\ & \left. + \beta \left(\left\| x_1(s) - x_2(s) \right\| + \left\| \int_a^s \left(\mathcal{H}(s, \tau, x_1(\tau))\Delta\tau - \mathcal{H}(s, \tau, x_2(\tau)) \right) \Delta\tau \right\| \right) \right] \Delta s \\ \leq\ & \int_a^t |s - t + \mu(s)| \left[|Q(s)| \left\| x_1(s) - x_2(s) \right\| \right. \\ & \left. + \beta \left(\left\| x_1(s) - x_2(s) \right\| + \gamma \int_a^s \left\| x_1(\tau) - x_2(\tau) \right\| \Delta\tau \right) \right] \Delta s \\ \leq\ & (b - a) \left\| x_1(s) - x_2(s) \right\|_\infty \left(\int_a^t |Q(s)|\, \Delta s + \beta\,(1 + \gamma)\,(b - a) \right) \\ \leq\ & \alpha \left\| x_1 - x_2 \right\|_\infty . \end{aligned}$$

This implies that T *is a contraction. Therefore,* T *has a unique fixed point* x, *which is the unique solution of the integral Equation* (4.5). *By Corollary* (60), x *is the unique solution of* (4.1) *satisfying the initial conditions.* ■

4.3 Hyers–Ulam Stability Results

In this section, we shall investigate Ulam stability for NIDE (4.1). For this, first we shall introduce the following definitions.

Definition 62 *We say that NIDE* (4.1) *has Hyers–Ulam stability if there exists a real number* $L > 0$ *such that for each* $\epsilon > 0$ *and for each* $y \in C^2_{rd}(\mathbb{I}, \mathbb{R}^n)$ *satisfying*

$$\left\| y^{\Delta^2}(t) - Q(t)\sigma(t) - G\left(t, y(t), \int_a^t H(t,s,y(s))\Delta s\right)\right\|_n \leq \epsilon \quad \textit{for all } t \in \mathbb{I}^k, \tag{4.7}$$

there exists a solution $x \in C^2_{rd}(\mathbb{I}, \mathbb{R}^n)$ *of* (4.1) *with*

$$\|y(t) - x(t)\|_n \leq \epsilon L \quad \textit{for all } t \in \mathbb{I}.$$

Here L *is a so-called HUS constant.*

Lemma 63 [20], *(Grönwall's Inequality) Suppose that* $x, f \in C_{rd}(T, \mathbb{R}_+), g \in R^+$*. If*

$$x(t) \leq f(t) + \int_a^t g(s)x(s)\Delta s \quad \textit{for all } t \in T^+_{t0},$$

then

$$x(t) \leq f(t) + \int_a^t e_g(t, \sigma(s))f(s)g(s)\Delta s \quad \textit{for all } t \in T^+_{t0}.$$

Theorem 64 *If* $(A), (B)$*, and* (C) *hold, then* (4.1) *has Hyers–Ulam stability with HUS constant*

$$L := (b-a)^2 \, e_{(b-a)[|Q(s)| + \beta(1+\gamma)]}\,(b,a)\,. \tag{4.8}$$

Proof 8 *Let* $y \in C^2_{rd}(\mathbb{I}, \mathbb{R}^n)$ *satisfy* (4.7)*. We have*

$$\mathcal{G}(t) = y^{\Delta^2}(t) - Q(t)y(t) - G\left(t, y(t), \int_a^t H(t,s,y(s))\Delta s\right) \quad \textit{for all } t \in \mathbb{I}^k, \tag{4.9}$$

and

$$y^{\Delta^2}(t) = Q(t)y(t) + G\left(t, y(t), \int_a^t H(t,s,y(s))\Delta s\right) + \mathcal{G}(t) \quad \textit{for all } t \in \mathbb{I}^k.$$

Now, (4.5) *implies that*

$$y(t)=c_0+c_1(t-a)-\int_a^t(s-t+\mu(s))\left[Q(s)y(s)+G\left(t,y(t),\int_a^s\mathcal{H}(s,\tau,y(\tau))\Delta\tau\right)+\mathcal{G}(s)\right]\Delta s. \tag{4.10}$$

By (C)*, there exists a solution* x *of* (4.1) *with* $x^{\Delta^i}(a)=c_i, i=0,1$*, that is*

$$x(t)=c_0+c_1(t-a)-\int_a^t(s-t+\mu(s))\left[Q(s)x(s)+G\left(t,x(t),\int_a^s\mathcal{H}(s,\tau,x(\tau))\Delta\tau\right)\right]\Delta s. \tag{4.11}$$

Subtracting (4.11) *from* (4.10)*, we find, for all* $t\in\mathbb{I}$*,*

$$\begin{aligned}\|x(t)-y(t)\| &\le \int_a^t|s-t+\mu(s)|\left[|Q(s)|\,|x(s)-y(s)|+\left|G\left(t,x(t),\int_a^s\mathcal{H}(s,\tau,x(\tau))\Delta\tau\right)\right.\right.\\ &\left.\left.-G\left(t,y(t),\int_a^s\mathcal{H}(s,\tau,y(\tau))\Delta\tau\right)\right|+|\mathcal{G}(s)|\right]\Delta s. \end{aligned}\tag{4.12}$$

Taking into account (B)*, we get*

$$\begin{aligned}\|x(t)-y(t)\| &\le (b-a)\int_a^t\left[|Q(s)|\,|x(s)-y(s)|+\beta\,|x(s)-y(s)|+\right.\\ &\left.+\beta\int_a^s|\mathcal{H}(s,\tau,x(\tau))-\mathcal{H}(s,\tau,y(\tau))|\,\Delta\tau+|\mathcal{G}(s)|\right]\Delta s\\ &\le (b-a)\int_a^t[|Q(s)|\,\|x(s)-y(s)\|+\beta\,\|x(s)-y(s)\|+\\ &+\beta\gamma\,\|x(s)-y(s)\|+\|\mathcal{G}(s)\|]\,\Delta s. \end{aligned}\tag{4.13}$$

Hence,

$$\|x(t)-y(t)\|\le(b-a)\int_a^t[(|Q(s)|+\beta(1+\gamma))\,\|x(s)-y(s)\|+\|\mathcal{G}(s)\|]\,\Delta s. \tag{4.14}$$

Since $\|\mathcal{G}(s)\|\le\epsilon$ *holds for* $t\in\mathbb{I}$*, we have*

$$\begin{aligned}\|x(t)-y(t)\| &\le (b-a)\int_a^t\|\mathcal{G}(s)\|\,\Delta s+(b-a)\int_a^t[|Q(s)|+\beta(1+\gamma)]\,\|x(s)-y(s)\|\,\Delta s\\ &\le (b-a)^2\epsilon+(b-a)\int_a^t[|Q(s)|+\beta(1+\gamma)]\,\|x(s)-y(s)\|\,\Delta s.\end{aligned}$$

Thus, by Gronwall's inequality, Lemma (63)*, we conclude that*

$$\|x(t)-y(t)\| \leq (b-a)^2 \epsilon e_{(b-a)[\|Q(s)|+\beta(1+\gamma)]}(b,a).$$

Therefore, (4.1) *has Hyers–Ulam stability with HUS constant* L *given in* (4.8)*.* ■

Theorem 65 *If* $(A),(B)$*, and* (D) *hold, then* (4.1) *has Hyers–Ulam stability with HUS constant*

$$L := \frac{(b-a)^2}{1-\alpha}. \tag{4.15}$$

Proof 9 *Let* $\epsilon > 0$ *and* $y \in C^2_{rd}(\mathbb{I}, \boldsymbol{X})$ *such that* (4.7) *holds. Set* $\mathcal{G}(t)$ *as in* (4.9)*. Then* y *satisfies* (4.10)*. Let* $y^{\Delta^i}(a) = c_i, i = 0,1$*. By Theorem* (58)*,* (4.10) *holds. By Theorem* (61)*, there exists a unique solution* x *of* (4.1) *with* $x^{\Delta^i}(a) = c_i, i = 0,1$*. By Corollary* (60)*,* $x(t)$ *is as in* (4.12)*. By subtracting* (4.11) *from* (4.9) *and as in the proof of Theorem* (64)*, we get*

$$\begin{aligned}
\|x(t)-y(t)\| &\leq \epsilon (b-a)^2 + \int_a^t (b-a)\left[\|Q(s)\| + \beta(1+\gamma)\right]\|x(s)-y(s)\|\,\Delta s \\
&\leq \epsilon (b-a)^2 + (b-a)\|x-y\|_\infty \int_a^t \left[\|Q(s)\| + \beta(1+\gamma)\right]\Delta s \\
&\leq \epsilon (b-a)^2 + \alpha \|x-y\|_\infty .
\end{aligned}$$

This implies that

$$\|x-y\|_\infty \leq \frac{(b-a)^2}{1-\alpha}\epsilon. \tag{4.16}$$

Therefore, (4.1) *has Hyers–Ulam stability with HUS constant* L *given in* (4.15)*.* ■

4.4 Hyers–Ulam–Rassias Stability

In this section, we introduce the Hyers–Ulam–Rassias Stability of (4.1).

Definition 66 *(Hyers–Ulam–Rassias stability). Let* N *be a family of positive rd-continuous functions on* $\mathbb{I}$*. We say that Equation* (4.1) *has Hyers–Ulam–Rassias stability of type*

N if there exists a constant $L > 0$, a so-called HURS$_N$ constant, with the following property. For any $w \in N$, if $y \in C^2_{rd}(\mathbb{I}, \mathbb{R}^n)$ is such that

$$\|\mathcal{G}(t)\| = \left\| y^{\Delta^2}(t) - Q(t)y(t) - G\left(t, y(t), \int_a^t \mathcal{H}(t, s, y(s))\Delta s\right)\right\| \leq w(t) \quad \textit{for all } t \in \mathbb{I}^k, \tag{4.17}$$

there exists a solution $x \in C^2_{rd}(\mathbb{I}, \mathbb{R}^n)$ of (4.1) with

$$\|y(t) - x(t)\|_n \leq Lw(t) \quad \textit{for all } t \in \mathbb{I}.$$

We note that Hyers–Ulam–Rassias stability yields Hyers–Ulam stability when

$$N = \{l_\epsilon : \epsilon > 0\}, \tag{4.18}$$

where $l_\epsilon(t) = \epsilon, t \in \mathbb{I}$. We use the notation (4.9):

$$N^* := \{w \in C_{rd}(\mathbb{I}, (0, \infty)) : w \textit{ is nondecreasing}\}, \tag{4.19}$$

and for $\Lambda \geq 1, \delta > 0$:

$$N^*_\Lambda := \left\{w \in C_{rd}(\mathbb{I}, (0, \infty)) : \int_a^t w^\Lambda(s) \leq \delta w^\Lambda(t) \textit{ for all } t \in \mathbb{I}\right\}. \tag{4.20}$$

The following theorem is concerned with Hyers–Ulam–Rassias stability.

Lemma 67 [*20*], *If $p \in R$ and $a, b, c \in T$:*

$$[e_p(c, .)]^\Delta = -p\,[e_p(c, .)]^\sigma$$

and

$$\int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c, a) - e_p(c, b).$$

Theorem 68 *If $(A), (B)$, and (C) hold, then (4.1) has Hyers–Ulam–Rassias stability* *of type $\mathcal{N}^*$ with HURS$_{\mathcal{N}^*}$ constant*

$$L := (b-a)^2 \, e_{(b-a)[|Q|+\beta(1+\gamma)]}(b,a). \tag{4.21}$$

Proof 10 *Let $w \in \mathcal{N}^*$ and $x \in C^2_{rd}(\mathbb{I}, \boldsymbol{X})$ such that (4.17) holds. Set $\mathcal{G}(t)$ as in (4.9). Then y satisfies (4.10). Let $y^{\Delta^i}(a) = c_i, i = 0, 1$. By Theorem (58), (4.10) holds. By Theorem (61), there exists a unique solution x of (4.1) with $x^{\Delta^i}(a) = c_i, i = 0, 1$. By subtracting (4.11) from (4.10) we obtain inequality (4.12), and by taking into account (B), we get inequality (4.13).*

Since $\|\mathcal{G}(s)\| \leq w(t)$ for all $t \in \mathbb{I}$, we get

$$\|x(t) - y(t)\| \leq (b-a)^2 \, w(t) + (b-a) \int_a^t [|Q(s)| + \beta(1+\gamma)] \, \|x(s) - y(s)\| \, \Delta s.$$

Thus, by Gronwall's inequality Lemma (63) and Lemma (67), we conclude that

$$\begin{aligned}
\|x(t) - y(t)\| &\leq (b-a)^2 \, w(t) \\
&\quad + \int_a^t (b-a) \, e_{(b-a)[|Q|+\beta(1+\gamma)]}(t, \sigma(s)) w(s) \, (b-a) \, [\|Q(s)\| + \beta \, (1+\gamma)] \, \Delta s \\
&\leq (b-a)^2 \, w(t) \\
&\quad + w(t) \, (b-a)^2 \left(\int_a^t e_{(b-a)[|Q|+\beta(1+\gamma)]}(t, \sigma(s)) \, [\|Q(s)\| + \beta \, (1+\gamma)] \, \Delta s \right) \\
&\leq (b-a)^2 \, w(t) \\
&\quad + w(t) \, (b-a)^2 \left(e_{(b-a)[|Q|+\beta(1+\gamma)]}(t, a) - e_{(b-a)[|Q|+\beta(1+\gamma)]}(t, t) \right) \\
&\leq (b-a)^2 \, e_{(b-a)[|Q|+\beta(1+\gamma)]}(b, a) w(t).
\end{aligned}$$

Therefore, (4.1) has Hyers–Ulam–Rassias stability of type $\mathcal{N}^*$ with a constant L defined by (4.21). ■

Theorem 69 *If $(A), (B)$, and (C) hold, then (4.1) has Hyers–Ulam–Rassias stability of type $\mathcal{N}^* \cap \mathcal{N}_1^\delta$ with HURS$_{\mathcal{N}^* \cap \mathcal{N}_1^\delta}$ constant*

$$L := \delta \, (b-a) \, e_{(b-a)[|Q|+\beta(1+\gamma)]}(b,a). \tag{4.22}$$

Proof 11 *Let* $w \in \mathcal{N}^* \cap \mathcal{N}_1^\delta$ *and* $x \in C_{rd}^2(\mathbb{I}, \boldsymbol{X})$ *such that* (4.17) *holds. Set* $\mathcal{G}(t)$ *as in* (4.9)*. Then* y *satisfies* (4.10)*. Let* $y^{\Delta^i}(a) = c_i, i = 0, 1$*. By Theorem* (58)*,* (4.10) *holds. By Theorem* (61)*, there exists a unique solution* x *of* (4.1) *with* $x^{\Delta^i}(a) = c_i, i = 0, 1$*. By subtracting* (4.11) *from* (4.10) *we obtain inequality* (4.12)*, and by taking into account* (B)*, we get inequality* (4.13)*.*

Since $\|\mathcal{G}(s)\| \leq w(t)$ *for all* $t \in \mathbb{I}$*, we get*

$$\begin{aligned}
\|x(t) - y(t)\| &\leq (b-a)\int_a^t w(s)\Delta s + (b-a)\int_a^t \left[|Q(s)| + \beta(1+\gamma)\right]\|x(s) - y(s)\|\,\Delta s \\
&\leq \delta\,(b-a)\,w(t) + (b-a)\int_a^t \left[|Q(s)| + \beta\,(1+\gamma)\right]\|x(s) - y(s)\|\,\Delta s.
\end{aligned}$$

Thus, by Gronwall's inequality Lemma (63) *and by Lemma* (67)*, we conclude that*

$$\begin{aligned}
\|\psi(t) - \phi(t)\| &\leq \delta\,(b-a)\,w(t) \\
&\quad + \int_a^t e_{(b-a)[|Q|+\beta(1+\gamma)]}(t, \sigma(s))w(s)\delta\,(b-a)\left[|Q(s)| + \beta\,(1+\gamma)\right]\Delta s \\
&\leq \delta\,(b-a)\,w(t) \\
&\quad + w(s)\delta\,(b-a)\int_a^t e_{(b-a)[|Q|+\beta(1+\gamma)]}(t, \sigma(s))\,(b-a)\left[|Q(s)| + \beta\,(1+\gamma)\right]\Delta s \\
&\leq \delta\,(b-a)\,w(t) \\
&\quad + w(s)\delta\,(b-a)\left(e_{(b-a)[|Q|+\beta(1+\gamma)]}(t, a) - e_{(b-a)[|Q|+\beta(1+\gamma)]}(t, t)\right) \\
&\leq \delta\,(b-a)\,e_{(b-a)[|Q|+\beta(1+\gamma)]}(b, a)w(t).
\end{aligned}$$

Therefore, (4.1) *has Hyers–Ulam–Rassias stability of type* $\mathcal{N}^* \cap \mathcal{N}_1^\delta$ *with a constant* L *given in* (4.22)*.* ■

Theorem 70 *Let* $\Lambda > 1$ *and* $\Gamma := \Lambda/(\Lambda - 1)$*. If* $(A), (B)$*, and* (C) *hold, then* (4.1) *has Hyers–Ulam–Rassias stability of type* $\mathcal{N}^* \cap \mathcal{N}_\Lambda^\delta$ *with HURS*$_{\mathcal{N}^* \cap \mathcal{N}_\Lambda^\delta}$ *constant*

$$L := \delta^{\frac{1}{\lambda}}\,(b-a)^{\frac{\Gamma+1}{\Gamma}}\,e_{(b-a)[|Q|+\beta(1+\gamma)]}(b, a). \tag{4.23}$$

Proof 12 *Let* $w \in \mathcal{N}^* \cap \mathcal{N}_\Lambda^\delta$ *and* $x \in C_{rd}^2(\mathbb{I}, \boldsymbol{X})$ *such that* (4.17) *holds. Set* $\mathcal{G}(t)$ *as in* (4.9)*. Then* y *satisfies* (4.10)*. Let* $y^{\Delta^i}(a) = c_i, i = 0, 1$*. By Theorem* (58)*,* (4.10) *holds.*

By Theorem (61)*, there exists a unique solution* x *of* (4.1) *with* $x^{\Delta^i}(a) = c_i, i = 0, 1$*. By subtracting* (4.11) *from* (4.10) *we obtain inequality* (4.12)*, and by taking into account* (B)*, we get inequality* (4.13)*.*

Since $\|\mathcal{G}(s)\| \leq w(t)$ *for all* $t \in \mathbb{I}$*, we get*

$$\begin{aligned}
\|x(t) - y(t)\| &\leq (b-a)\int_a^t w(s)\Delta s + (b-a)\int_a^t [|Q(s)| + \beta(1+\gamma)]\,\|x(s) - y(s)\|\,\Delta s \\
&\leq (b-a)\sqrt[\Gamma]{t-\alpha}\,\sqrt[\Lambda]{\int_a^t w^{\Lambda}(s)\Delta s} + (b-a)\int_a^t [|Q(s)| + \beta\,(1+\gamma)]\,\|x(s) - y(s)\|\,\Delta s \\
&\leq \delta^{\frac{1}{\Lambda}}\,(b-a)^{\frac{\Gamma+1}{\Gamma}}\,w(t) + (b-a)\int_a^t [|Q(s)| + \beta\,(1+\gamma)]\,\|x(s) - y(s)\|\,\Delta s.
\end{aligned}$$

Thus, by Gronwall's inequality Lemma (63) *and by Lemma* (67)*, we conclude that*

$$\begin{aligned}
\|x(t) - y(t)\| &\leq \delta^{\frac{1}{\Lambda}}\,(b-a)^{\frac{\Gamma+1}{\Gamma}}\,w(t) \\
&\quad + \int_a^t e_{(b-a)[|Q|+\beta(1+\gamma)]}(t,\sigma(s))w(s)\delta^{\frac{1}{\Lambda}}\,(b-a)^{\frac{\Gamma+1}{\Gamma}}\,[|Q(s)| + \beta\,(1+\gamma)]\,\Delta s \\
&\leq \delta^{\frac{1}{\Lambda}}\,(b-a)^{\frac{\Gamma+1}{\Gamma}}\,w(t) \\
&\quad + w(t)\delta^{\frac{1}{\Lambda}}\,(b-a)^{\frac{\Gamma+1}{\Gamma}}\int_a^t e_{(b-a)[|Q|+\beta(1+\gamma)]}(t,\sigma(s))\,(b-a)\,[|Q(s)| + \beta\,(1+\gamma)]\,\Delta s \\
&\leq \delta^{\frac{1}{\Lambda}}\,(b-a)^{\frac{\Gamma+1}{\Gamma}}\,w(t) \\
&\quad + w(t)\delta^{\frac{1}{\Lambda}}\,(b-a)^{\frac{\Gamma+1}{\Gamma}}\left(e_{(b-a)[|Q|+\beta(1+\gamma)]}(t,a) - e_{(b-a)[|Q|+\beta(1+\gamma)]}(t,t)\right) \\
&\leq \delta^{\frac{1}{\Lambda}}\,(b-a)^{\frac{\Gamma+1}{\Gamma}}\,e_{(b-a)[|Q|+\beta(1+\gamma)]}(b,a)w(t).
\end{aligned}$$

Therefore, Equation (4.1) *has Hyers–Ulam–Rassias stability of type* $\mathcal{N}^* \cap \mathcal{N}_\Lambda^\delta$ *with constant* L *defined in* (4.23)*.* ■

Theorem 71 *Let* $\Lambda > 1$ *and* $\Gamma := \Lambda/(\Lambda - 1)$*. If* $(A), (B)$*, and* (C) *hold, then* (4.1) *has Hyers–Ulam–Rassias stability of type* $\mathcal{N}_\Lambda^\delta$ *with HURS*$_{\mathcal{N}_\Lambda^\delta}$ *constant*

$$L := \delta^{\frac{1}{\lambda}}\,(b-a)^{\frac{\Gamma+1}{\Gamma}}\left(1 + \delta^{\frac{1}{\lambda}}\,(b-a)^{\frac{\Gamma+1}{\Gamma}}\,[|Q(s)| + \beta\,(1+\gamma)]\,e_{(b-a)[|Q|+\beta(1+\gamma)]}(b,a)\right). \quad (4.24)$$

Proof 13 *Let* $w \in \mathcal{N}_\Lambda^\delta$ *and* $x \in C_{rd}^2(\mathbb{I}, \boldsymbol{X})$ *such that* (4.17) *holds. Set* $\mathcal{G}(t)$ *as in* (4.9)*. Then* y *satisfies* (4.10)*. Let* $y^{\Delta^i}(a) = c_i, i = 0, 1$*. By Theorem* (58)*,* (4.10) *holds. By*

Theorem (61)*, there exists a unique solution* x *of* (4.1) *with* $x^{\Delta^i}(a) = c_i, i = 0, 1$ *. By subtracting* (4.11) *from* (4.10) *we obtain inequality* (4.12)*, and by taking into account* (B)*, we get inequality* (4.13)*.*

Since $\|\mathcal{G}(s)\| \leq w(t)$ *for all* $t \in \mathbb{I}$*, we get*

$$\begin{aligned} \|x(t) - y(t)\| &\leq (b-a) \int_a^t w(s)\Delta s + (b-a) \int_a^t [|Q(s)| + \beta(1+\gamma)] \|x(s) - y(s)\| \Delta s \\ &\leq (b-a) \sqrt[\Gamma]{t-\alpha} \sqrt[\Lambda]{\int_a^t w^\Lambda(s)\Delta s} + (b-a) \int_a^t [|Q(s)| + \beta(1+\gamma)] \|x(s) - y(s)\| \Delta s \\ &\leq \delta^{\frac{1}{\Lambda}} (b-a)^{\frac{\Gamma+1}{\Gamma}} w(t) + \int_a^t (b-a) [|Q(s)| + \beta(1+\gamma)] \|x(s) - y(s)\| \Delta s, \end{aligned}$$

where we have applied the Hölder inequality Lemma (67)*. Thus, by using Gronwall's inequality Lemma* (63) *we get*

$$\begin{aligned} \|x(t) - y(t)\| &\leq \delta^{\frac{1}{\Lambda}} (b-a)^{\frac{\Gamma+1}{\Gamma}} w(t) \\ &\quad + \int_a^t e_{(b-a)[|Q|+\beta(1+\gamma)]}(t, \sigma(s)) w(s) \delta^{\frac{1}{\Lambda}} (b-a)^{\frac{\Gamma+1}{\Gamma}} [|Q(s)| + \beta(1+\gamma)] \Delta s \\ &\leq \delta^{\frac{1}{\Lambda}} (b-a)^{\frac{\Gamma+1}{\Gamma}} w(t) \\ &\quad + \delta^{\frac{1}{\Lambda}} (b-a)^{\frac{2\Gamma+1}{\Gamma}} [\|Q(s)\|_\infty + \beta(1+\gamma)] e_{(b-a)[|Q|+\beta(1+\gamma)]}(b, a) (b-a) \int_a^t w(s)\Delta s \\ &\leq \delta^{\frac{1}{\Lambda}} (b-a)^{\frac{\Gamma+1}{\Gamma}} w(t) \\ &\quad + \delta^{\frac{1}{\Lambda}} (b-a)^{\frac{2\Gamma+1}{\Gamma}} [\|Q(s)\|_\infty + \beta(1+\gamma)] e_{(b-a)[|Q|+\beta(1+\gamma)]}(b, a) (b-a) \left(\int_a^t w^\Lambda(s)\Delta s\right)^{\frac{1}{\Lambda}} \\ &\leq \delta^{\frac{1}{\Lambda}} (b-a)^{\frac{\Gamma+1}{\Gamma}} w(t) + \delta^{\frac{2}{\Lambda}} (b-a)^{\frac{2\Gamma+2}{\Gamma}} [\|Q(s)\|_\infty + \beta(1+\gamma)] e_{(b-a)[|Q|+\beta(1+\gamma)]}(b, a) w(t) \\ &= Lw(t). \end{aligned}$$

Therefore, Equation (4.1) *has Hyers–Ulam–Rassias stability of type* $\mathcal{N}_\Lambda^\delta$ *with constant* L *given in* (4.24)*.* ■

Example 72 *Now, we give an example for which conditions* $(A), (B)$ *and* (D) *are satisfied. Let* $\boldsymbol{T} := \cup_{k=0}^{\infty}[2k, 2k+1]$*. Let*

$$m \in \boldsymbol{N}, a = 0 \text{ and } b = 2m+1. \text{ Fix } \alpha \in (0,1) \text{ and } \beta \in \left(0, \frac{1}{(5e)^{m+1}(m+1)(4m+1)}\right).$$

Assume

$$G\left(t, x(t), \int_a^t \mathcal{H}(t,s,x(s))\Delta s\right) = \beta \sin\left(x(t) + \int_a^t (3\sin x(s) - \cos x(s) + 5t)\Delta s\right),$$

Take

$$Q(t) = \frac{t+\sigma(t)+2}{(t+1)^2}.$$

Equation (4.1) *takes the form*

$$x^{\Delta^2}(t) = \frac{t+\sigma(t)+2}{(t+1)^2}x(t) + \beta \sin\left(x(t) + \int_a^t (3\sin x(s) - \cos x(s) + 5t)\Delta s\right).$$

Clearly, condition (A) *holds. Additionally, condition* (B) *is true, since* G *and* $\mathcal{H}$ *satisfy Lipschitz conditions with constants* β *and* $\gamma = 1$*, respectively. Finally, we check that* (D) *holds. Indeed,*

$$\begin{aligned}
\int_0^b Q(s)\Delta s &= \int_0^b \frac{s+\sigma(s)+2}{(s+1)^2(\sigma(s)+1)^2}\Delta s \\
&= \frac{-1}{(t+1)^2}\Big|_0^b \\
&= \frac{-1}{(b+1)^2} + 1 \\
&\leq 1.
\end{aligned}$$

Conclusions

- Time scales theory is a **perfect language**, which can be applied to any field in which the dynamics of processes are described by models in **discrete time** or **continuous time**, or a **combination of both**.

- The problem of **stability** and **stabilization** of certain classes of nonlinear systems is well developed on time scales thanks to the **integral inequality** approach.

- We have also found results on **practical stability** and h**-stability** by developing some conditions on the perturbed term and using appropriate **integral inequalities** on arbitrary time scales.

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