UBM Annaba
Faculty of Sciences
Mathematical Department

Global existence, blow up and asymptotic decay for some hyperbolic systems

by

Djamel Ouchenane

Under the Supervision of

First Supervisor: Dr. Belkacem Said-Houari
Second Supervisor: Prof. Hocine Sissaoui

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ملخص

في هذه الرسالة ندرس إنفجار و تناقص الحل لبعض الجمل الزائدية؛ و قد قسمنا الرسالة إلى قسمين كل قسم منهما يحوي فصلين .

ندرس في القسم الأول إنفجار الحل لجملتين زائديتين حيث في الفصل الأول جملة معادلات تفاضلية جزئية غير خطية؛ أما في الفصل الثاني جملة معادلات تفاضلية جزئية غير خطية بإضافة حد يمثل اللزوجة .

في القسم الثاني ندرس وجود و وحدانية الحل و كذا الإستقرار الأسي للحل لجمل زائدية من نوع تيموشنكو ذات بعد واحد .

Résumé

Cette thèse est consacrée à l'étude de l'explosion dans le temps et la décroissance asymptotique de certains systèmes hyperboliques. Globalement, cette thèse est composée de 2 parties principales. La première partie est composée de deux chapitres 1 et 2. Dans le chapitre 1, on considère un système d'équations d'onde nonlinéaires, avec un terme d'amortissement dégénéré et un terme fort et nonlinéaire. On démontre ainsi qu'il y a explosion de la solution dans le temps. Dans le chapitre 2, on considère un système d'équations viscoélastiques nonlinéaires. On démontre cette fois qu'une solution globale d'un tel système n'existe pas.

La seconde partie, quant à elle, est composée de deux chapitres 3 et 4. Les deux sont simutanément consacrés à l'étude d'un système thermoélastique linéaire en dimension un de type Timoshenko, dans lequel le flux de chaleur est donné par la loi de Cattaneo, notons ici que l'itroduction du term de retard dans la contre-réaction ne concerne que le chapitre 4. Pour cette deuxième partie, on a obtenu des résultats relatifs à la décroissance exponentielle pour les solutions classiques et faibles. La preuve que nous avons établie est basées sur la construction d'une fonction de Lyapunov appropriée et équivalente à l'énergie de la solution considérée. Cette fonction vérifie une inequation différentielle menant au résultat de la décroissance désirée.

Mots clés : Dissipation nonlinéaire, dissipation forte, viscoélasticité, source non linéaire, solution locales, solution globales, décroissance exponentielle, décroissance polynômiale, explosion en temps fini.

Contents

Abstract

This thesis is devoted to study blow up and asymptotic decay for some hyperbolic systems. The first part of the thesis is composed of two chapters. In chapter 1, we consider a system of nonlinear wave equations with degenerate damping and strong nonlinear source terms. We prove that the solution blows up in time. In the chapter 2, we consider a system of nonlinear viscoelastic hyperbolic equations. We prove that the global solution of such a system at the same time is nonexisting.

The second part of the thesis contains two chapters 3 and 4. We studied a one-dimensional linear thermoelastic system of Timoshenko type, where the heat flux is given by Cattaneo's law, noting that in the chapter 4 we have introduced a delay term in the feedback. We established several exponential decay results for classical and weak solutions in one-dimensional. Our technics of proof is based on the construction of the appropriate Lyapunov function equivalent to the energy of the considered solution, and which satisfies a differential inequality leading to the desired decay.

Keywords: Nonlinear damping, Strong damping, Viscoelasticity, Nonlinear source, Locale solutions, Global solution, Exponential decay, Polynomial decay, Blow up.

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Djamel

Introduction

The main aim of this thesis is to study some hyperbolic systems with the presence of different mechanisms of damping, and under assumptions on initial data and boundary conditions. Our main goal is to investigate the existence of the solutions and their behavior when the time evolves. In fact, we prove that under some assumptions on the parameters in the systems and on the size of the initial data, the solutions can be proved to be either *global* in time or may *blow up* in finite time (i.e. some norms of the solution will be unbounded in finite time). If the solution are global in time, then the natural question is about their convergence to the steady state and the rate of the convergence. The system that we treated here are the following:

The damped wave equation

Let us consider the single wave equation with nonlinear source term and nonlinear damping term:

$$u_{tt} - \Delta u + a\,|u_t|^{m-2}\,u_t = b\,|u|^{p-2}\,u, \qquad \text{in } \Omega \times [0, T]\,, \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$, with a smooth boundary $\partial\Omega$ and a, b, m and p are positive constants. We consider the following initial conditions:

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x), \qquad x \in \Omega,\ t > 0, \tag{2}$$

and the boundary condition

$$u(x, t) = 0, \qquad x \in \partial\Omega,\ t > 0. \tag{3}$$

The modified energy associated to the problem (1)-(3) is defined as

$$E(u, u_t, t) = E(t) = \frac{1}{2}\int_\Omega |u_t(x, t)|^2 dx + \frac{1}{2}\int_\Omega |\nabla(x, t)|^2 dx - \frac{1}{p}\int_\Omega |u(x, t)|^p dx. \tag{4}$$

It is well known that in the absence of the source terms $|u|^{p-2}u$, then a uniform estimate of the form

$$\|u_t(t)\|_2 + \|\nabla u(t)\|_2 \leq C, \tag{5}$$

holds for any initial data $(u_0, u_1) = (u(0), u_t(0))$ in the energy space $H_0^1(\Omega) \times L^2(\Omega)$, where C is a generic positive constant independent of t. The estimate (5) shows that any local solution u of problem (1) can be continued in time when the condition (5) is verified. This result has been proved by several authors. See for example [32, 39]. On the other hand in the absence of the damping term $|u_t|^{m-2}u_t$, there exists a finite value T^* such that the solution of (1) ceases to exist and satisfies

$$\lim_{t\to T^*} \|u(t)\|_p = +\infty. \tag{6}$$

The reader is refereed to Ball [7] and Kalantarov and Ladyzhenskaya [36] for more details. When both terms are present in equation (1), the situation is more interesting. This case has been considered by Levine in [41, 42], where he investigated problem (1) in the linear damping case ($m = 2$) and showed that any local solution u of (1) can not be continued in $(0, \infty) \times \Omega$ whenever the initial data are large enough (negative initial energy). The main tool used in [41] and [42] is the "concavity method". This method has been a widely applicable tool to prove the blow up of solutions in finite time of some evolution equations. The basic idea of this method is to construct a positive functional $\theta(t)$ depending on certain norms of the solution and show that, for some $\gamma > 0$, the function $\theta^{-\gamma}(t)$ is a positive concave function of t. Thus there exists T^* such that $\lim_{t\to T^*} \theta^{-\gamma}(t) = 0$. Since then, the concavity method became a powerful and simple tool to prove blow up in finite time for other related problems. Unfortunately, this method is limited to the case of a linear damping. Georgiev and Todorova [20] extended Levine's result to the nonlinear damping case ($m > 2$). In their work, the authors considered the problem (1)-(3) and introduced a method different from the one known as the concavity method. They showed that solutions with negative energy continue to exist globally 'in time' if the damping term dominates the source one (i.e. $m \geq p$) and blow up in finite time in the other case (i.e. $p > m$) provide that the initial energy is sufficiently negative, that is $E(0) < -A$, for some constant $A > 0$. Their method is based on the constriction of an auxiliary function L which is a perturbation of the total energy (4) and satisfies the differential inequality

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t), \tag{7}$$

in $[0,\infty)$, where $\nu > 0$. Inequality (7) leads to a blow up of the solutions in finite time $t \geq L(0)^{-\nu}\xi^{-1}\nu^{-1}$, provided that $L(0) > 0$. However, the blow up result in [20] was not optimal in terms of the initial data causing the finite time blow up of solutions. Thus several improvement have been made to the result in [20] (see for example [40, 43]. In particular, Vitillaro in [87] combined the arguments in [20] and [40] to extend the result in [20] to situations where the damping is nonlinear and the solution has positive initial energy. That is $E(0) < d$, for some $d > 0$ known as the *potential well depth.*

The result of (1)-(3) have been extended to more general problems. For instance, in [89], Yang studied the problem

$$\begin{gathered} u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2}\nabla u\right) - div\left(|\nabla u_t|^{\beta-2}\nabla u_t\right) \\ +a\,|u_t|^{m-2}u_t = b|u|^{p-2}u, \end{gathered} \tag{8}$$

in $\Omega\times(0,T)$. with initial conditions and boundary condition of Dirichlet type. He showed that solutions blow up in finite time T^* under the condition $p > \max\{\alpha, m\}$, $\alpha > \beta$, and the initial energy is sufficiently negative (see condition (ii) in [89], Theorem 2.1). Messaoudi and Said-Houari [57] improved the result in [89] and showed that the blow up of solutions of problem (8) takes place for negative initial data only regardless of the size of Ω.

For the system of wave equations, Miranda and Medeiros [60] considered the following system:

$$\begin{cases} u_{tt} - \Delta u + u - |v|^{\rho+2}|u|^{\rho}u = f_1(x) \\ v_{tt} - \Delta v + v - |u|^{\rho+2}|v|^{\rho}v = f_2(x), \end{cases} \tag{9}$$

in $\Omega\times(0,T)$. By using the method of the potential well, the authors determined the existence of a global weak solutions of system (9). Agre and Rammaha [2] studied the following system:

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1}u_t = f_1(u,v), \\ v_{tt} - \Delta v + |v_t|^{r-1}v_t = f_2(u,v), \end{cases} \tag{10}$$

in $\Omega\times(0,T)$. With initial and boundary conditions of Dirichlet type and the nonlinear functions $f_1(u,v)$ and $f_2(u,v)$ satisfying

$$\begin{aligned} f_1(u,v) &= b_1|u+v|^{2(\rho+1)}(u+v) + b_2|u|^{\rho}u|v|^{(\rho+2)}, \\ f_2(u,v) &= b_1|u+v|^{2(\rho+1)}(u+v) + b_2|u|^{(\rho+2)}|v|^{\rho}v. \end{aligned} \tag{11}$$

They proved, under some appropriate conditions on $f_1(u,v)$, $f_2(u,v)$ and the initial data, several results on local and global existence, but no rate of decay has been discussed.

They also showed that any weak solution with negative initial energy blows up in finite time, using the same techniques as in [20]. Recently, the blow up result in [2] has been improved by Said-Houari [77] by considering certain class of initial data with positive initial energy. Subsequently, the paper [77] has been followed by [79], where the author proved that if the initial data are small enough, then the solution of (10) is global in time and decays with an exponential rate if $m = r = 1$ and with a polynomial rate like $t^{-2/(\max(m,r)-1)}$ if $\max(m, r) > 1$. See also the recent work by Rammaha and Sakuntasathien [74]. In [55], the authors considered a nonlinear viscoelastic system

$$\begin{cases} u_{tt} - \triangle u + \displaystyle\int_0^t g(t-s) \triangle u(x,s)\, ds + |u_t|^{m-1} u_t = f_1(u,v), \\ v_{tt} - \triangle v + \displaystyle\int_0^t h(t-s) \triangle v(x,s)\, ds + |v_t|^{r-1} v_t = f_2(u,v), \end{cases} \tag{12}$$

where $x \in \Omega$, $t > 0$ and $f_1(u, v)$ and $f_2(u, v)$ are as in (11). They proved a global nonexistence theorem for certain solutions with positive initial energy. As the main tool of the proof, they used the method applied in [77]. The global existence and decay rate of solution of system (12) has been investigated by Said-Houari, Messaoudi and Guesmia in [78], where they studied the nonlinear viscoelastic system (12). They proved a general decay rate of the solution under certain restrictions imposed on the linearity of damping, source terms and for certain class of relaxation functions. They found that the rate of decay of the total energy depends on the rate of decay of the relaxation functions.

We briefly mention that our main results in this part are as follows:

Chapter 1. Throughout this chapter we investigated a system of nonlinear wave equations with degenerate damping and source terms supplemented with the initial and Dirichlet boundary conditions. Our result extends the work in [74] and proved a blow up result by allowing our initial energy to have positive values whereas only negative values of initial energy have been treated in [74]. The main tool of the proof is based on a method used in [87] developed in [59, 77]. This result has been published in the paper [8].

Chapter 2. The whole chapter is devoted to the study of the system of nonlinear viscoelastic hyperbolic equations with degenerate damping and source terms supplemented with initial and Dirichlet boundary conditions. This has been investigated by Rammaha and Sakuntasathien [74] in the case where the initial energy is negative $E(0) < 0$. Moreover, a global nonexistence result on a solution with positive initial energy for a system of viscoelastic wave equation with nonlinear damping and source terms was obtained by Messaoudi and Said-Houari [55]. Our result extends these previous results where we proved a global nonexistence of the solutions of a system of wave equations with viscoelastic term, degenerate damping, and strong nonlinear sources acting in both equations at the same time, provided that the initial data are sufficiently large. The main tool of the proof was based on methods used by Vitillaro in [87] and developed by Said-Houari [77]. This result has been published in the paper [71].

The Timoshenko systems

The study of Timoshenko systems started in 1921 in the work of Timoshenko [86] where he considered the coupled hyperbolic equations

$$\begin{cases} \rho\varphi_{tt} = (K(\varphi_x - \psi))_x, & \text{in } (0, L) \times (0, +\infty) \\ I_\rho\psi_{tt} = (EI\psi_x)_x + K(\varphi_x - \psi), & \text{in } (0, L) \times (0, +\infty), \end{cases} \tag{13}$$

where t denotes the time variable, x is the space variable along the beam of length L, in its equilibrium configuration, φ is the transverse displacement of the beam and ψ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are, respectively, the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia a cross a section, and the shear modulus.
System (13), together with boundary conditions of the form

$$EI\varphi_x \mid_{x=0}^{x=L} = 0, \qquad K(u_x - \varphi) \mid_{x=0}^{x=L} = 0$$

is conservative, and thus the total energy is preserved, as time goes to infinity. Several authors introduced different types of dissipative mechanisms to stabilize system (13), and several results concerning uniform and asymptotic decay of energy have been established.

Kim and Renardy [37] considered (13) together with two boundary controls of the form

$$\begin{cases} K\psi(L,t) - K\varphi_x(L,t) = \alpha\varphi_t(L,t), & \forall t \geq 0 \\ EI\psi_x(L,t) = -\beta\varphi_t(L,t), & \forall t \geq 0 \end{cases}$$

and used the multiplier techniques to establish an exponential decay result for the total energy of (13). They also provided numerical estimates to the eigenvalues of the operator associated with system (13). Raposo *et al.* [75] treated the following system:

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x - \psi)_x + \varphi_t = 0, & \text{in } (0,L)\times(0,+\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x - \psi) + \psi_t = 0, & \text{in } (0,L)\times(0,+\infty) \end{cases} \tag{14}$$

with homogeneous Dirichlet boundary conditions and two linear frictional dampings, and proved that the associated energy decays exponentially. Soufyane and Wehbe [83] showed that it is possible to stabilize uniformly (13) by using a unique locally distributed feedback. They considered

$$\begin{cases} \rho\varphi_{tt} = (K(\varphi_x - \psi))_x, & \text{in } (0,L)\times(0,+\infty) \\ I_\rho\psi_{tt} = (EI\psi_x)_x + K(\varphi_x - \psi) - b\psi_t, & \text{in } (0,L)\times(0,+\infty) \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, \quad \forall t > 0, \end{cases} \tag{15}$$

where b is a positive and continuous function, which satisfies

$$b(x) \geq b_0 > 0, \quad \forall\ x\ \in\ [a_0, a_1] \subset [0, L].$$

In fact, they proved that the uniform stability of (15) holds if and only if the wave speeds are equal $\left(\frac{K}{\rho} = \frac{EI}{I_\rho}\right)$; otherwise only the asymptotic stability has been proved. Also, Muñoz Rivera and Racke [62] studied a nonlinear Timoshenko-type system of the form

$$\begin{cases} \rho_1\varphi_{tt} - \sigma_1(\varphi_x, \psi)_x = 0 \\ \rho_2\psi_{tt} - \chi(\psi_x)_x + \sigma_2(\varphi_x, \psi) + d\psi_t = 0 \end{cases}$$

in a one-dimensional bounded domain. The dissipation is produced here through a frictional damping which is only present in the equation for the rotation angle. The authors gave an alternative proof for a necessary and sufficient condition for exponential stability in the linear case and then proved a polynomial stability in general. Moreover, they investigated the global existence of small smooth solutions and exponential stability in the nonlinear case. Concerning the Timoshenko system with viscoelastic damping, Ammar-Khodja et *al.* [4] considered a linear Timoshenko-type system with memory of the form

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)ds + K(\varphi_x + \psi) = 0 \end{cases} \tag{16}$$

in $(0, L) \times (0, +\infty)$, together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system is uniformly stable if and only if the wave speeds are equal $\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$ and g decays uniformly. Precisely, they proved an exponential decay if g decays in an exponential rate and polynomially if g decays in a polynomial rate.

Messaoudi and Mustafa [47] improved the results of [4] and [25] by allowing more general decaying relaxation functions and showed that the rate of decay of the solution energy is exactly the rate of decay of the relaxation function. Also, Muñoz Rivera and Fernndez Sare [66], considered Timoshenko type system with past history acting only in one equation. More precisely they studied the following problem:

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \int_0^\infty g(t)\psi_{xx}(t-s, .)ds + K(\varphi_x + \psi) = 0, \end{cases} \tag{17}$$

together with homogenous boundary conditions, and showed that the dissipation given by the history term is strong enough to stabilize the system exponentially if and only if the wave speeds are equal. They also proved that the solution decays polynomially for the case of different wave speeds. This work was improved recently by Messaoudi and Said-Houari [46], where the authors considered system (17) for g decaying polynomially, and proved polynomial stability results for the equal and nonequal wave-speed propagation under conditions on the relaxation function weaker than those in [66]. The case of g having a general decay has been studied in [28–30] for Timoshenko-type and [27, 31] for abstract systems, where a general relation between the growth of g at infinity and the decay rate of solutions is explicitly found in terms of the growths at infinity.

For the Timoshenko systems in thermoelasticity of type III, we have the recent papers of Messaoudi and Said-Houari [48, 52] in which the authors proved several stability results. More precisely, in [48], they investigated the asymptotic behavior of the problem

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\theta_x = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - k\theta_{txx} = 0, \end{cases} \tag{18}$$

in $(0,+\infty)\times(0,1)$ and proved an exponential decay result similar to the one in [61]. We recall that the heat conduction in (18) is given by Green and Naghdi's theory. The same problem (18) with an additional damping of history type of the form

$$\int_0^\infty g(s)\,\psi_{xx}(x,\ t-s)\,ds, \tag{19}$$

acting in the second equation has been analyzed in [52]. The authors of [52] proved an exponential and polynomial stability results for the equal and nonequal wave-speed propagation under conditions on the relaxation function g weaker than those in [53] and [66]. For the general stability of (18) with (19), see [28].

For Timoshenko systems in classical thermoelasticity, Muñoz Rivera and Racke [61] considered

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x,\psi)_x = 0 & \text{in } (0,L)\times(0,+\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi) + \gamma\theta_x = 0 & \text{in } (0,L)\times(0,+\infty) \\ \rho_3\theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0 & \text{in } (0,L)\times(0,+\infty) \end{cases} \tag{20}$$

where φ,ψ and θ are functions of (x,t) which model the transverse displacement of the beam, the rotation angle of the filament, and the difference temperature respectively. Under appropriate conditions of σ,ρ_i,b,k,γ, they proved several exponential decay results for the linearized system and a non exponential stability result for the case of different wave speeds. We recall here that the heat flux given by Cattaneo's law is weaker than the one given by Fourier's law, for this motive, the removal of the paradox of infinite propagation speed rooted in Fourier's law by changing to the Cattaneo's law destroys some times the exponential stability property, see [18]. Consequently, in order to stabilize such a system, some dissipative mechanisms must be added to the system.

Modeling heat conduction with the so-called Fourier law (as in (20)), which assumes the flux q to be proportional to the gradient of the temperature θ at the same time t,

$$q+\kappa\nabla\theta, \qquad (\kappa>0),$$

leads to the phenomenon of infinite heat propagation speed. That is, any thermal disturbance at a single point has an instantaneous effect everywhere in the medium. To overcome this physical problem, a number of modifications of the basic assumption on the relation between the heat flux and the temperature have been made. The common property of these theories is that all lead to hyperbolic differential equation and the speed of propagation becomes limited. See [14] for more details. Among them Cattaneo's law,

$$\tau q_t + q + \kappa \nabla \theta = 0, \qquad (\tau > 0, \text{ relatively small}),$$

leading to the system with *second sound*, [49, 73, 84] and a suggestion by Green and Naghdi [22, 24], for thermoelastic systems introducing what is called thermoelasticity of type III, where the basal equations for the heat flux is characterized by

$$q + \kappa^* p_x + \tilde{\kappa} \nabla \theta = 0, \qquad (\tilde{\kappa} > \kappa > 0, \; p_t = \theta).$$

Messaoudi *et al.* [50] studied the following problem:

$$\begin{cases} \rho_1 \varphi_{tt} - \sigma(\varphi_x, \psi)_x + \mu \varphi_t = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta \theta_x = 0, \\ \rho_3 \theta_t + \gamma q_x + \delta \psi_{tx} = 0, \\ \tau_0 q_t + q + \kappa \theta_x = 0, \end{cases}$$

where $(x, t) \in (0, L) \times (0, \infty)$ and $\varphi = \varphi(x, t)$ is the displacement vector, $\psi = \psi(x, t)$ is the rotation angle of the filament, $\theta = \theta(x, t)$ is the temperature difference, $q = q(x, t)$ is the heat flux vector, ρ_1, ρ_2, ρ_3, b, k, γ, δ, κ, μ, τ_0 are positive constants. The nonlinear function σ is assumed to be sufficiently smooth and satisfy

$$\sigma_{\varphi_x}(0, 0) = \sigma_\psi(0, 0) = k$$

and

$$\sigma_{\varphi_x \varphi_x}(0, 0) = \sigma_{\varphi_x \psi}(0, 0) = \sigma_{\psi\psi} = 0.$$

Several exponential decay results for both linear and nonlinear cases have been established.

Concerning the Timoshenko system with delay, the investigation started with the paper [76] where the authors studied the following problem:

$$\begin{cases} \rho_1 \varphi_{tt}(x, t) - K(\varphi_x + \psi)_x (x, t) = 0, \\ \rho_2 \psi_{tt}(x, t) - b\psi_{xx}(x, t) + K(\varphi_x + \psi)(x, t) + \mu_1 \psi_t(x, t) + \mu_2 \psi_t(x, t - \tau) = 0. \end{cases} \tag{21}$$

Under the assumption $\mu_1 \geq \mu_2$ on the weights of the two feedbacks, they proved the well-posedness of the system. They also established for $\mu_1 > \mu_2$ an exponential decay result for the case of equal-speed wave propagation, i.e.

$$\frac{K}{\rho_1} = \frac{b}{\rho_2}. \tag{22}$$

Subsequently, the work in [76] has been extended to the case of time-varying delay of the form $\psi_t(x, t - \tau(t))$ by Kirane, Said-Houari and Anwar [38]. The case where the damping $\mu_1\psi_t$ is replaced by (19) (with either discrete delay $\mu_2\psi_t(t-\tau)$ or distributed one $\int_0^\infty f(s)\psi_t(t-s)\,ds$) has been treated in [30] (in case (22) and the opposite one), where several general decay estimates have been proved.

Our main results in this part can be summarized as follows:

Chapter 3. In this chapter we studied a one-dimensional linear thermoelastic system of Timoshenko type, where the heat flux is given by Cattaneo's law, see for example [50]. We consider damping terms acting on the second equation and we establish a general decay estimate without the usual assumption of the wave speeds. Our method of proof uses the energy method together with some properties of convex functions. The advantage here is that from our general estimates we can derive the exponential, polynomial or logarithmic decay rate. We also give some examples to illustrate our result.

Chapter 4. In this chapter we consider a one-dimonsional linear thermoelastic system of Timoshenko type with delay term in the feedback. The heat conduction in given by Cattaneo's law. Under an appropriate assumption between the weight of the delay and the weight of the damping, we proved a well-posedness result. Furthermore an exponential stability result has been shown without the usual assumption on the wave speeds. To achieve our goals, we made use of the semigroup method and the energy method. This work has been recently published in [70].

Part I

System of damped wave equations

Chapter 1

Blow up of positive initial-energy solutions to systems of nonlinear wave equations with degenerate damping and source term

1.1 Introduction

Some special cases of the single wave equations with nonlinear damping and nonlinear source terms in the form

$$u_{tt} - \triangle u + a\,|u_t|^{m-1}\,u_t = b\,|u|^{p-1}\,u \tag{1.1}$$

arise in quantum field theory which describe the motion of charged mesons in an electromagnetic filed. Equation (1.1) together with initial and boundary conditions of Dirichlet type has been extensively studied and results concerning existence, blow up and asymptotic behavior of smooth, as well as weak solutions, have been established by several authors over the near past decades. The study of single wave equation with the presence of different mechanisms of dissipation and for more general forms of nonlinearities has been extensively studied and results concerning existence, nonexistence and asymptotic behavior of solutions have been established by several authors and many results appeared in the literature over the past decades. See [1, 6, 20, 35, 40, 55, 59, 77] and references therein. In this work, we consider the following system of nonlinear wave equations with degenerate damping and strong nonlinear source terms:

$$\begin{cases} u_{tt} - \triangle u + \left(a_1 |u|^k + a_2 |v|^l\right) |u_t|^{m-1} u_t = f_1(u, v), \\ v_{tt} - \triangle v + \left(a_3 |v|^\theta + a_4 |u|^\vartheta\right) |v_t|^{r-1} v_t = f_2(u, v), \end{cases} \tag{1.2}$$

where $m, r > 0$, $k, l, \theta, \vartheta \geq 1$ and the two functions $f_1(u, v)$ and $f_2(u, v)$ are given by

$$\begin{cases} f_1(u, v) = a_5 |u + v|^{2(\rho+1)} (u + v) + a_6 |u|^\rho u |v|^{(\rho+2)} \\ f_2(u, v) = a_5 |u + v|^{2(\rho+1)} (u + v) + a_6 |u|^{(\rho+2)} v |v|^\rho, \end{cases} \tag{1.3}$$

with $\rho > -1$. In (1.2), $u = u(x, t)$, $v = v(x, t)$, where $x \in \Omega$ is a bounded domain of $\mathbb{R}^N$, $(N \geq 1)$ with a smooth boundary $\partial\Omega$ and $t > 0$, $a_i > 0$, $i = 1, ..., 6$. System (1.2) is supplemented with the following initial conditions:

$$(u(0), v(0)) = (u_0, v_0), \ (u_t(0), v_t(0)) = (u_1, v_1), \quad x \in \Omega \tag{1.4}$$

and boundary conditions

$$u(x) = v(x) = 0, \qquad x \in \partial\Omega. \tag{1.5}$$

To motivate our work, let us recall some related results. Concerning the system of wave equations, Miranda and Medeiros [60] considered the following system

$$\begin{cases} u_{tt} - \Delta u + u - |v|^{(\rho+2)} |u|^\rho u = f_1(x), \\ v_{tt} - \Delta v + v - |u|^{(\rho+2)} |v|^\rho v = f_2(x), \end{cases} \tag{1.6}$$

in $\Omega \times (0, T)$. Using the method of potential well, the authors determined the existence of weak solutions of system (1.6). In [2], Agre and Rammaha studied the following system:

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1} u_t = f_1(u, v), \\ v_{tt} - \Delta v + |v_t|^{r-1} v_t = f_2(u, v), \end{cases} \tag{1.7}$$

in $\Omega \times (0, T)$ with initial and boundary conditions and the nonlinear functions f_1 and f_2 satisfying appropriate conditions. They proved, under some restrictions on the parameters m, r and the initial data, the existence of a weak solution. They also showed that any weak solution with negative initial energy blows up in finite time using the same techniques as in [20].

In [77], Said-Houari considered the same problem treated in [2], and he improved the blow up result for a large class of initial data in which the initial energy can take positive values.

Recently, in [74] Rammaha and Sakuntasathien focused on the global well-posedness of the system of nonlinear wave equations (1.2). They proved that weak solutions blow up in finite time whenever the initial energy is negative and the exponent of the source term is greater than the exponents of both damping terms.

In this chapter, we prove that the blow up result can still be proved even for some positive values of the initial energy and we extended the result in [74].

In Section 1.2, we introduce and present some notations and prepare some material needed for our proof. In Section 1.3, we state and prove our main result, where we prove that, under some restrictions on the initial data and (with positive initial energy) for some conditions on the functions f_1 and f_2, the solution of problem (1.2)–(1.5) blows up in finite time as stated in Theorem 1.6.

1.2 Preliminaries

In this section, we introduce some notations and some technical lemmas to be used throughout this chapter. The constants c_i, $i = 0, 1, 2, ...$, used throughout this chapter are positive generic constants, which may be different in various occurrences. We introduce the "modified" energy functional E associated to our system

$$\begin{aligned} E(t) = E(t, u(t), v(t)) &= \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \frac{1}{2}J(u, v) - \int_\Omega F(u, v)\,dx. \\ &= \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \tilde{J}(u, v), \end{aligned} \tag{1.8}$$

where

$$J(u, v) = \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \tag{1.9}$$

and

$$\tilde{J}(u, v) = \frac{1}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) - \int_\Omega F(u, v)\,dx, \tag{1.10}$$

such that $\int_\Omega F(u, v)\,dx$ is defined in Lemma 1.1.

Let us point out that the integral $\int_\Omega F(u, v)\,dx$ in (1.8) makes sense because $H_0^1(\Omega) \hookrightarrow L^{2(\rho+2)}(\Omega)$ for

$$\begin{cases} -1 < \rho & \text{if } N = 1, 2, \\ -1 < \rho \le \frac{4-N}{N-2} & \text{if } N \ge 3. \end{cases} \tag{1.11}$$

It is very easy to prove this lemma

Lemma 1.1. *There exists a function* $F(u,v)$ *such that, for all* $(u,\ v) \in \mathbb{R}^2$,

$$\begin{aligned} F(u,v) &= \frac{1}{2(\rho+2)}\left[u f_1(u,v) + v f_2(u,v)\right], \\ &= \frac{1}{2(\rho+2)}\left[a_5 |u+v|^{2(\rho+2)} + 2a_6 |uv|^{\rho+2}\right] \geq 0, \end{aligned}$$

where

$$\frac{\partial F}{\partial u} = f_1(u,v), \frac{\partial F}{\partial v} = f_2(u,v).$$

Lemma 1.2. *[77] There exist two positive constants* c_0 *and* c_1 *such that*

$$\frac{c_0}{2(\rho+2)}\left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}\right) \leq F(u,v) \leq \frac{c_1}{2(\rho+2)}\left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}\right). \tag{1.12}$$

We take $a_i = 1\,(i = 1, ..., 6)$ for convenience. The following technical lemma will play an important role in the sequel and has been first proved in [77]:

Lemma 1.3. *Suppose that (1.11) holds. Then there exists* $\eta > 0$ *such that for any* $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *the inequality*

$$\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2} \leq \eta\,(J(u,v))^{\rho+2} \tag{1.13}$$

holds.

Proof. It is clear that by using the Minkowski inequality we get

$$\|u+v\|_{2(\rho+2)}^2 \leq 2(\|u\|_{2(\rho+2)}^2 + \|v\|_{2(\rho+2)}^2).$$

Also, Hőlder's and Young's inequalities give us

$$\|uv\|_{(\rho+2)} \leq \|u\|_{2(\rho+2)}\|v\|_{2(\rho+2)} \leq c(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)$$

A combination of the two last inequalities and the embedding $H_0^1(\Omega) \hookrightarrow L^{2(\rho+2)}(\Omega)$ give us (1.13). ◻

Lemma 1.4. *Let* $\nu > 0$ *be a real positive number and* L *be a solution of the ordinary differential inequality*

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t) \tag{1.14}$$

defined in $[0, \infty)$.
If $L(0) > 0$, *then the solution ceases to exist for* $t \geq L(0)^{-\nu}\xi^{-1}\nu^{-1}$.

Proof. Direct integration of (1.14) gives:

$$L^{-\nu}(0) - L^{-\nu}(t) \geq \xi \nu t,$$

Thus we obtain the following estimate:

$$L^{\nu}(t) \geq \left[L^{-\nu}(0) - \xi \nu t\right]^{-1}. \tag{1.15}$$

It is clear that the right-hand side of (1.15) is unbounded when

$$\xi \nu t = L^{-\nu}(0).$$

This completes the proof of Lemma 1.4. □

In the following lemma, we show the that the total energy of our system is a non-increasing function of t, thus, we have.

Lemma 1.5. *Suppose that (1.11) holds. Let* (u, v) *be the solution of the system (1.2)-(1.5), then the energy functional is a non-increasing function; that is, for all* $t > 0$,

$$\frac{dE(t)}{dt} = -\int_{\Omega} \left(|u(t)|^{k} + |v(t)|^{l}\right) |u_t(t)|^{m+1} dx - \int_{\Omega} \left(|v(t)|^{\theta} + |u(t)|^{\vartheta}\right) |v_t(t)|^{r+1} dx \tag{1.16}$$

The proof of the above lemma can be simply done by multiplying the first equation in (1.2) by u, the second equation by v, using the integration by parts and adding the obtained results, then (1.16) holds.

1.3 Blow up result

Our main result reads as follows:

Theorem 1.6. *Suppose that (1.11) holds. Assume further that*

$$2(\rho + 2) > \max(k + m + 1, l + m + 1, \theta + r + 1, \vartheta + r + 1). \tag{1.17}$$

Then any solution of problem (1.2)–(1.5) with initial data satisfying

$$\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)^{\frac{1}{2}} > \alpha_1, \qquad \text{and} \qquad E(0) < E_1$$

blows up in finite time, where the constants α_1 and E_1 are defined as

$$B = \eta^{\frac{1}{2(\rho+2)}}, \qquad \alpha_1 = B^{-\frac{\rho+2}{\rho+1}}, \qquad E_1 = \left(\frac{1}{2} - \frac{1}{2(\rho+2)}\right)\alpha_1^2, \tag{1.18}$$

where η is the optimal constant given in (1.13).

The following lemma is very useful to prove a blow up result. It is similar to the one in [77].

Lemma 1.7. *[77] Let the assumption (1.11) be fulfilled. Let (u, v) be a solution of (1.2)-(1.5). Assume further that $E(0) < E_1$ and*

$$\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)^{\frac{1}{2}} > \alpha_1. \tag{1.19}$$

Then there exists a constant $\alpha_2 > \alpha_1$ such that

$$\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\frac{1}{2}} > \alpha_2, \tag{1.20}$$

and

$$\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]^{\frac{1}{2(\rho+2)}} \geq B\alpha_2, \quad \forall t \geq 0. \tag{1.21}$$

Proof of Theorem 1.6. We suppose that the solution exists for all time and we reach to a contradiction. For this purpose, we put

$$H(t) = E_1 - E(t). \tag{1.22}$$

By using (1.8) and (1.22) we get

$$\begin{aligned} H'(t) = & \int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u_t(t)|^{m+1}\,dx \\ & + \int_\Omega \left(|v(t)|^\theta + |u(t)|^\vartheta\right)|v_t(t)|^{r+1}\,dx. \end{aligned}$$

From (1.16), it is clear that, for all $t \geq 0$, $H'(t) > 0$. Therefore

$$\begin{aligned} 0 &< H(0) \leq H(t) \\ &= E_1 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\ &\quad + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]. \end{aligned} \tag{1.23}$$

From (1.8) and (1.20), we obtain, for all $t \geq 0$,

$$\begin{aligned} & E_1 - \frac{1}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right] \\ < \; & E_1 - \frac{1}{2}\alpha_1^2 + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right] \\ = \; & -\frac{1}{2(\rho+2)}\alpha_1^2 + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right] \\ < \; & \frac{c_0}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]. \end{aligned}$$

Hence,

$$0 < H(0) \leq H(t) \leq \frac{c_1}{2(\rho+2)}\left[\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right], \; \forall t > 0. \tag{1.24}$$

We then define

$$L(t) = H^{1-\sigma}(t) + \varepsilon \int_\Omega (u.u_t + v.v_t)(x,t)\,dx, \tag{1.25}$$

for ε small to be chosen later and

$$\begin{aligned} 0 < \; & \sigma \leq \min\left\{\frac{2\rho+3-(k+m)}{2m(\rho+2)}, \frac{2\rho+3-(l+m)}{2m(\rho+2)},\right. \\ & \left.\frac{2\rho+3-(\vartheta+r)}{2r(\rho+2)}, \frac{2\rho+3-(\theta+r)}{2r(\rho+2)}, \frac{2\rho+2}{4(\rho+2)}\right\} \end{aligned} \tag{1.26}$$

Our goal is to show that $L(t)$ satisfies a differential inequality of the form

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t), \tag{1.27}$$

defined in $[0, \infty)$, with $v > 0$. By taking a derivative of (1.25) and equations (1.2), we obtain

$$\begin{aligned} L'(t) &= (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \varepsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\ &\quad -\varepsilon \int_\Omega u\left(|u(t)|^k + |v(t)|^l\right)|u_t|^{m-1} u_t dx \\ &\quad -\varepsilon \int_\Omega v\left(|v(t)|^\theta + |u(t)|^\vartheta\right)|v_t|^{r-1} v_t dx \\ &\quad -\varepsilon \int_\Omega \left(u f_1(u, v) + v f_2(u, v)\right) dx. \end{aligned} \tag{1.28}$$

By exploiting (1.8) and (1.22), the equation (1.28) takes the form

$$\begin{aligned} L'(t) &= (1-\sigma) H^{-\sigma}(t) H'(t) + 2\varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ &\quad -\varepsilon \int_\Omega u\left(|u(t)|^k + |v(t)|^l\right)|u_t|^{m-1} u_t dx \\ &\quad -\varepsilon \int_\Omega v\left(|v(t)|^\theta + |u(t)|^\vartheta\right)|v_t|^{r-1} v_t dx \\ &\quad +\varepsilon\left(1 - \frac{1}{(\rho+2)}\right)\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + \|uv\|_{\rho+2}^{\rho+2}\right) + 2\varepsilon H(t) - 2\varepsilon E_1. \end{aligned} \tag{1.29}$$

Then using (1.21), we obtain

$$\begin{aligned} L'(t) &\geq (1-\sigma) H^{-\sigma}(t) H'(t) + 2\varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ &\quad +\varepsilon c_3\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + \|uv\|_{\rho+2}^{\rho+2}\right) + 2\varepsilon H(t), \\ &\quad -\varepsilon \int_\Omega u\left(|u(t)|^k + |v(t)|^l\right)|u_t|^{m-1} u_t \\ &\quad -\varepsilon \int_\Omega v\left(|v(t)|^\theta + |u(t)|^\vartheta\right)|v_t|^{r-1} v_t dx \end{aligned} \tag{1.30}$$

where $c_3 = 1 - \dfrac{1}{\rho+2} - 2E_1 (B\alpha_2)^{-2(\rho+2)}$. From the definition of E_1, α_1 and since $\alpha_2 > \alpha_1$, we deduce $c_3 > 0$. That In order to estimate the last two terms in (1.30) we make use of the following Young's inequality:

$$XY \leq \frac{\delta^\alpha X^\alpha}{\alpha} + \frac{\delta^{-\beta} Y^\beta}{\beta},$$

where $X, Y \geq 0$, $\delta > 0$, and $\alpha, \beta > 0$ such that $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1$. Consequently, we get, for all $\delta_1 > 0$,

$$\left|u |u_t|^{m-1} u_t\right| \leq \frac{\delta_1^{m+1}}{m+1} |u|^{m+1} + \frac{m}{m+1} \delta_1^{-(m+1)/m} |u_t|^{m+1}$$

and therefore

$$\begin{aligned}
&\int_\Omega \left(|u(t)|^k + |v(t)|^l\right) \left|u |u_t|^{m-1} u_t\right| dx \\
\le\ & \frac{\delta_1^{m+1}}{m+1} \int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u|^{m+1} dx \\
&+ \frac{m}{m+1} \delta_1^{-(m+1)/m} \int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m+1} dx.
\end{aligned} \tag{1.31}$$

Similarly, for all $\delta_2 > 0$,

$$\left|v |v_t|^{r-1} v_t\right| \le \frac{\delta_2^{r+1}}{r+1} |v|^{r+1} + \frac{r}{r+1} \delta_2^{-(r+1)/r} |v_t|^{r+1},$$

which gives

$$\begin{aligned}
&\int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right) \left|v |v_t|^{r-1} v_t\right| dx \\
\le\ & \frac{\delta_2^{r+1}}{r+1} \int_\Omega \left(|v(t)|^\theta + |u(t)|^\vartheta\right) |v|^{r+1} dx \\
&+ \frac{r}{r+1} \delta_2^{-(r+1)/r} \int_\Omega \left(|v(t)|^\theta + |u(t)|^\vartheta\right) |v_t|^{r+1} dx.
\end{aligned} \tag{1.32}$$

Inserting the estimates (1.31), (1.32) into (1.30), we obtain

$$\begin{aligned}
L'(t) \ \ge\ & (1-\sigma) H^{-\sigma}(t) H'(t) + 2\varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\
&+ \varepsilon c_3 \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + \|uv\|_{\rho+2}^{\rho+2}\right) + 2\varepsilon H(t) \\
&- \varepsilon \frac{\delta_1^{m+1}}{m+1} \int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u|^{m+1} dx \\
&- \varepsilon \frac{m}{m+1} \delta_1^{-(m+1)/m} \int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m+1} dx \\
&- \varepsilon \frac{\delta_2^{r+1}}{r+1} \int_\Omega \left(|v(t)|^\theta + |u(t)|^\vartheta\right) |v|^{r+1} dx \\
&- \varepsilon \frac{r}{r+1} \delta_2^{-(r+1)/r} \int_\Omega \left(|v(t)|^\theta + |u(t)|^\vartheta\right) |v_t|^{r+1} dx.
\end{aligned} \tag{1.33}$$

Let us choose δ_1 and δ_2 such that

$$\delta_1^{-(m+1)/m} = M_1 H^{-\sigma}(t), \quad \delta_2^{-(r+1)/r} = M_2 H^{-\sigma}(t), \tag{1.34}$$

for M_1 and M_2 large constants to be fixed later. Thus, by using (1.34), the inequality (1.33) then takes the form

$$\begin{aligned} L^{'}(t) \geq\ & ((1-\sigma)-M\varepsilon)\, H^{-\sigma}(t)\, H^{'}(t) + 2\varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ & +\varepsilon c_4\left(\|u(t)\|_{2(\rho+2)}^{2(\rho+2)} + \|v(t)\|_{2(\rho+2)}^{2(\rho+2)}\right) + 2\varepsilon H(t) \\ & -\varepsilon M_1^{-m} H^{\sigma m}(t) \int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u|^{m+1}\, dx \\ & -\varepsilon M_2^{-r} H^{\sigma r}(t) \int_\Omega \left(|v(t)|^\theta + |u(t)|^\vartheta\right)|v|^{r+1}\, dx, \end{aligned} \tag{1.35}$$

where $M = m/(m+1)\, M_1 + r/(r+1)\, M_2$ and c_2 is a positive constants. Consequently we have

$$\int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u|^{m+1}\, dx = \|u\|_{k+m+1}^{k+m+1} + \int_\Omega |v|^l\, |u|^{m+1}\, dx \tag{1.36}$$

and

$$\int_\Omega \left(|v(t)|^\theta + |u(t)|^\vartheta\right)|v|^{r+1}\, dx = \|v\|_{\theta+r+1}^{\theta+r+1} + \int_\Omega |u|^\vartheta\, |v|^{r+1}\, dx. \tag{1.37}$$

Also by using Young's inequality, we have

$$\begin{aligned} \int_\Omega |v|^l\, |u|^{m+1} &\leq \frac{l}{l+m+1}\delta_1^{(l+m+1)/l}\, \|v\|_{l+m+1}^{l+m+1} + \frac{m+1}{l+m+1}\delta_1^{-(l+m+1)/(m+1)}\, \|u\|_{l+m+1}^{l+m+1} \\ \int_\Omega |u|^\vartheta\, |v|^{r+1} &\leq \frac{\vartheta}{\vartheta+r+1}\delta_2^{(\vartheta+r+1)/\vartheta}\, \|u\|_{\vartheta+r+1}^{\vartheta+r+1} + \frac{r+1}{\vartheta+r+1}\delta_2^{-(\vartheta+r+1)/(r+1)}\, \|v\|_{\vartheta+r+1}^{\vartheta+r+1}. \end{aligned}$$

Consequently,

$$\begin{aligned} & H^{\sigma m}(t) \int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u|^{m+1}\, dx \\ =\ & H^{\sigma m}(t)\, \|u\|_{k+m+1}^{k+m+1} + \frac{l}{l+m+1}\delta_1^{(l+m+1)/l} H^{\sigma m}(t)\, \|v\|_{l+m+1}^{l+m+1} \\ & + \frac{m+1}{l+m+1}\delta_1^{-(l+m+1)/(m+1)} H^{\sigma m}(t)\, \|u\|_{l+m+1}^{l+m+1} \end{aligned} \tag{1.38}$$

and

$$\begin{aligned} & H^{\sigma r}(t) \int_\Omega \left(|v(t)|^\theta + |u(t)|^\vartheta\right)|v|^{r+1}\, dx \\ =\ & H^{\sigma r}(t)\, \|v\|_{\theta+r+1}^{\theta+r+1} + \frac{\vartheta}{\vartheta+r+1}\delta_2^{\frac{\vartheta+r+1}{\vartheta}} H^{\sigma r}(t)\, \|u\|_{\vartheta+r+1}^{\vartheta+r+1} \\ & + \frac{r+1}{\vartheta+r+1}\delta_2^{-\frac{(\vartheta+r+1)}{r+1}} H^{\sigma r}(t)\, \|v\|_{\vartheta+r+1}^{\vartheta+r+1}. \end{aligned} \tag{1.39}$$

Since (1.17) holds. By using (1.26), we obtain

$$\begin{cases} H^{\sigma m}(t)\,\|u\|_{k+m+1}^{k+m+1} \le c_5\left(\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)+k+m+1} + \|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\,\|u\|_{k+m+1}^{k+m+1}\right), \\ H^{\sigma r}(t)\,\|v\|_{\theta+r+1}^{\theta+r+1} \le c_6\left(\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\theta+r+1} + \|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\,\|v\|_{\theta+r+1}^{\theta+r+1}\right). \end{cases} \tag{1.40}$$

This implies

$$\begin{aligned} &\tfrac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}} H^{\sigma m}(t)\,\|v\|_{l+m+1}^{l+m+1} \\ &\le c_7\tfrac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}\left(\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)+l+m+1} + \|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\,\|v\|_{l+m+1}^{l+m+1}\right), \\ &\tfrac{\vartheta}{\vartheta+r+1}\delta_2^{\frac{\vartheta+r+1}{\vartheta}} H^{\sigma r}(t)\,\|u\|_{\vartheta+r+1}^{\vartheta+r+1} \\ &\le c_8\tfrac{\vartheta}{\vartheta+r+1}\delta_2^{\frac{\vartheta+r+1}{\vartheta}}\left(\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\vartheta+r+1} + \|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\,\|u\|_{\vartheta+r+1}^{\vartheta+r+1}\right), \end{aligned} \tag{1.41}$$

for some positive constants c_5, c_6, c_7 and c_8. By using (1.26) and the algebraic inequality

$$z^{\nu} \le (z+1) \le \left(1+\frac{1}{a}\right)(z+a), \quad \forall z \ge 0,\ 0 < \nu \le 1,\ a > 0, \tag{1.42}$$

we have, for all $t \ge 0$,

$$\begin{cases} \|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)+k+m+1} \le d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + H(0)\right) \le d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + H(t)\right), \\ \|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\theta+r+1} \le d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + H(t)\right),\ \forall t \ge 0, \end{cases} \tag{1.43}$$

where $d = 1 + 1/H(0)$. Similarly,

$$\begin{cases} \|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)+l(m+1)} \le d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + H(0)\right) \le d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + H(t)\right), \\ \|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\varrho(r+1)} \le d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + H(t)\right),\ \forall t \ge 0. \end{cases} \tag{1.44}$$

Also, since

$$(X+Y)^s \le C\left(X^s + Y^s\right), \qquad X, Y \ge 0,\ s > 0, \tag{1.45}$$

by using (1.17), (1.26) and (1.43) we conclude

$$\begin{aligned} \|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|u\|_{k+m+1}^{k+m+1} &\leq |\Omega|^{\frac{2(\rho+2)-(k+m+1)}{2(\rho+2)}}\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|u\|_{2(\rho+2)}^{k+m+1} \quad (1.46)\\ &= |\Omega|^{\frac{2(\rho+2)-(k+m+1)}{2(\rho+2)}}\left(\|v\|_{2(\rho+2)}^{\sigma m}\|u\|_{2(\rho+2)}^{\frac{k+m+1}{2(\rho+2)}}\right)^{2(\rho+2)}\\ &\leq |\Omega|^{\frac{2(\rho+2)-(k+m+1)}{2(\rho+2)}}\left(c'\|v\|_{2(\rho+2)}^{\frac{2\sigma m(\rho+2)+k+m+1}{2(\rho+2)}}+c''\|u\|_{2(\rho+2)}^{\frac{2\sigma m(\rho+2)+k+m+1}{2(\rho+2)}}\right)^{2(\rho+2)}\\ &\leq c_9\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+\|u\|_{2(\rho+2)}^{2(\rho+2)}\right), \quad (1.47)\end{aligned}$$

where

$$c'=\frac{2\sigma m(\rho+2)}{2\sigma m(\rho+2)+k+m+1} \quad \text{and} \quad c''=\frac{k+m+1}{2\sigma m(\rho+2)+k+m+1}.$$

Similarly,

$$\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\|v\|_{\theta+r+1}^{\theta+r+1}\leq c_{10}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right), \quad (1.48)$$

$$\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|v\|_{l+m+1}^{l+m+1}\leq c_{11}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right) \quad (1.49)$$

and

$$\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\|u\|_{\vartheta+r+1}^{\vartheta+r+1}\leq c_{12}\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+\|u\|_{2(\rho+2)}^{2(\rho+2)}\right). \quad (1.50)$$

Taking into account (1.38)–(1.50), then, (1.35) takes the form

$$\begin{aligned} L'(t) \geq\ & ((1-\sigma)-M\varepsilon)H^{-\sigma}(t)H'(t)+2\varepsilon\left(\|u_t\|_2^2+\|v_t\|_2^2\right)\\ &+\varepsilon\left[2-CM_1^{-m}\left(1+\tfrac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}+\tfrac{m+1}{l+m+1}\delta_1^{-\frac{(l+m+1)}{m+1}}\right)\right.\\ &\left.-CM_2^{-r}\left(1+\tfrac{\varrho}{\varrho+r+1}\delta_2^{\frac{\vartheta+r+1}{\vartheta}}+\tfrac{r+1}{\vartheta+r+1}\delta_2^{-\frac{(\vartheta+r+1)}{r+1}}\right)\right]H(t)\\ &+\varepsilon\left[c_4-CM_1^{-m}\left(1+\tfrac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}+\tfrac{m+1}{l+m+1}\delta_1^{-\frac{(l+m+1)}{m+1}}\right)\right.\\ &\left.-CM_2^{-r}\left(1+\tfrac{\vartheta}{\vartheta+r+1}\delta_2^{\frac{\vartheta+r+1}{\vartheta}}+\tfrac{r+1}{\vartheta+r+1}\delta_2^{-\frac{(\vartheta+r+1)}{r+1}}\right)\right] \quad (1.51)\\ &\times\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right).\end{aligned}$$

At this point, and for large values of M_1 and M_2, we can find positive constants Λ_1 and Λ_2 such that (1.51) becomes

$$\begin{aligned} L'(t) \geq\ & ((1-\sigma)-M\varepsilon)H^{-\sigma}(t)H'(t)+2\varepsilon\left(\|u_t\|_2^2+\|v_t\|_2^2\right)\\ &+\varepsilon\Lambda_1\left(\|u(t)\|_{2(\rho+2)}^{2(\rho+2)}+\|v(t)\|_{2(\rho+2)}^{2(\rho+2)}\right)+\varepsilon\Lambda_2 H(t). \quad (1.52)\end{aligned}$$

Once M_1 and M_2 are fixed (hence, Λ_1 and Λ_2), we pick ε small enough so that $(1-\sigma)-M\varepsilon \geq 0$ and

$$L(0) = H^{1-\sigma}(0) + \varepsilon \int_\Omega [u_0.u_1 + v_0.v_1]\, dx > 0.$$

Consequently, there exists $\Gamma > 0$ such that (1.52) becomes

$$L'(t) \geq \varepsilon\Gamma \left(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} \right). \tag{1.53}$$

Thus, we have $L(t) \geq L(0) > 0$, for all $t \geq 0$. Next, by Holder's and Young's inequalities, we estimate

$$\begin{aligned} &\left(\int_\Omega u.u_t(x,t)\, dx + \int_\Omega v.v_t(x,t)\, dx \right)^{\frac{1}{1-\sigma}} \\ \leq\ & c_{13} \left(\|u\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}} + \|u_t\|_2^{\frac{s}{1-\sigma}} + \|v\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}} + \|v_t\|_2^{\frac{s}{1-\sigma}} \right), \end{aligned} \tag{1.54}$$

for $\frac{1}{\tau} + \frac{1}{s} = 1$. We takes $s = 2(1-\sigma)$ to get $\frac{\tau}{1-\sigma} = \frac{2}{1-2\sigma}$. By using (1.26) and (1.42) we get

$$\|u\|_{2(\rho+2)}^{\frac{2}{1-2\sigma}} \leq d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + H(t) \right),$$

and

$$\|v\|_{2(\rho+2)}^{\frac{2}{1-2\sigma}} \leq d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + H(t) \right), \ \forall t \geq 0.$$

Therefore, (1.54) becomes

$$\begin{aligned} &\left(\int_\Omega u.u_t(x,t)\, dx + \int_\Omega v.v_t(x,t)\, dx \right)^{\frac{1}{1-\sigma}} \\ \leq\ & c_{14} \left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u_t\|_2^2 + \|v_t\|_2^2 + H(t) \right), \ \forall \geq 0. \end{aligned} \tag{1.55}$$

Also, by noting that

$$\begin{aligned} L^{\frac{1}{1-\sigma}}(t) &= \left(H^{1-\sigma}(t) + \varepsilon \int_\Omega (u.u_t + v.v_t)(x,t)\, dx \right)^{\frac{1}{(1-\sigma)}} \\ &\leq c_{15} \left(H(t) + \left| \int_\Omega (u.u_t(x,t) + v.v_t(x,t))\, dx \right|^{\frac{1}{(1-\sigma)}} \right) \\ &\leq c_{16} \left[H(t) + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u_t\|_2^2 + \|v_t\|_2^2 \right], \ \forall t \geq 0, \end{aligned} \tag{1.56}$$

and combining with (1.56) and (1.53), we arrive at

$$L^{'}(t) \geq a_0 L^{\frac{1}{1-\sigma}}(t), \ \forall t \geq 0. \tag{1.57}$$

Finally, a simple integration of (1.57) gives the desired result. □

Chapter 2

Global nonexistence of solutions to system of nonlinear viscoelastic wave equations with degenerate damping and source terms

2.1 Introduction

In this chapter, we consider the following system of viscoelastic wave equations with degenerate damping and strong nonlinear source terms:

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g\,(t-s)\,\Delta u\,(x,s)\,ds + \left(a\,|u|^k + b\,|v|^l\right)|u_t|^{m-1}u_t &= f_1(u,v), \\ v_{tt} - \Delta v + \displaystyle\int_0^t h\,(t-s)\,\Delta v\,(x,s)\,ds + \left(c\,|v|^\theta + d\,|u|^\varrho\right)|v_t|^{r-1}v_t &= f_2(u,v), \end{cases} \tag{2.1}$$

where $m, r > 0$, $k, l, \theta, \varrho \geq 1$ and the two functions $f_1\,(u,v)$ and $f_2\,(u,v)$ given by

$$\begin{aligned} f_1(u,v) &= a_1|u+v|^{2(\rho+1)}(u+v) + b_1|u|^\rho u|v|^{(\rho+2)} \\ f_2(u,v) &= a_1|u+v|^{2(\rho+1)}(u+v) + b_1|u|^{(\rho+2)}|v|^\rho v,\ \ a_1, b_1 > 0, \end{aligned} \tag{2.2}$$

where $\rho > -1$. In (2.1), $u = u\,(x,t)$, $v = v\,(x,t)$, where $x \in \Omega$ is a bounded domain of $\mathbb{R}^N$, $(N \geq 1)$ with a smooth boundary $\partial\Omega$ and $t > 0$, $a, b, c, d > 0$.

System (2.1) is supplemented with the following initial conditions

$$(u(0), v(0)) = (u_0, v_0), (u_t(0), v_t(0)) = (u_1, v_1),\ x \in \Omega \tag{2.3}$$

and boundary conditions

$$u(x) = v(x) = 0,\ x \in \partial\Omega. \tag{2.4}$$

This type of problems arise usually in viscoelasticity and it has been considered first by Dafermos [15], where the general decay was discussed. A related problems to (2.1) have attracted a great deal of attention in the last two decades, and many results have been appeared on the existence and long time behavior of solutions. See in this directions ([9–12, 33, 53, 54, 72]) and references therein.

In the absence of viscoelastic term, some special cases of the single wave equations with nonlinear damping and nonlinear source terms in the form

$$u_{tt} - \Delta u + a|u_t|^{m-1}u_t = b|u|^{p-1}u \tag{2.5}$$

arise in quantum field theory which describe the motion of charged meson in an electromagnetic field. Equation (2.5) together with initial and boundary conditions of Dirichlet type has been extensively studied and results concerning existence, blow up and asymptotic behavior of smooth, as well as weak solutions, have been established by several authors over the past decades.

The study of single wave equation with the presence of different mechanisms of dissipation, damping and nonlinear sources has been extensively studied and results concerning existence, nonexistence and asymptotic behavior of solutions have been established by several authors and many results appeared in the literature over the past decades. See ([6, 20, 35, 40, 55, 58, 77]) and references therein.

In the work [55], authors considered the nonlinear viscoelastic system:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + |u_t|^{m-1}u_t = f_1(u,v), \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(x,s)ds + |v_t|^{r-1}v_t = f_2(u,v), \end{cases},\ x \in \Omega, t > 0 \tag{2.6}$$

where

$$\begin{aligned} f_1(u,v) &= a|u+v|^{2(\rho+1)}(u+v) + b|u|^{\rho}u|v|^{(\rho+2)} \\ f_2(u,v) &= a|u+v|^{2(\rho+1)}(u+v) + b|u|^{(\rho+2)}|v|^{\rho}v, \end{aligned} \tag{2.7}$$

and they proved a global nonexistence theorem for certain solutions with positive initial energy. The main tool of the proof is a method used in [77].

Concerning the study of decay of solutions of systems of evolution equations, let us mention the work of Said-Houari, Messaoudi and Guesmia, in [78], where they treated the nonlinear viscoelastic system in (2.6) and under some restrictions on the nonlinearity of the damping and the source terms, they proved that, for certain class of relaxation functions and for some restrictions on the initial data, the rate of decay of the total energy depends on those of the relaxation functions.

In this Chapter, we consider system (2.1) and proved a global nonexistence result of the solution. We extended to result in [55] to the more general problem (2.1).

2.2 Preliminaries

In this section, we introduce and present some notations and some technical lemmas to be used throughout this chapter.

We assume that the relaxation functions $g, h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are of class C^1 and nonincreasing differentiable satisfying:

$$\begin{cases} 1 - \int_0^\infty g(s)ds = l' > 0, \quad g(t) \geq 0, \quad g'(t) \leq 0, \\ 1 - \int_0^\infty h(s)ds = k' > 0, \quad h(t) \geq 0, \quad h'(t) \leq 0, \end{cases} \quad t \geq 0. \tag{2.8}$$

We introduce the ”*modified*” energy functional E associated to our system:

$$2E(t) = \|u_t\|_2^2 + \|v_t\|_2^2 + J(u,v) - 2\int_\Omega F(u,v)\,dx, \tag{2.9}$$

where $F(u,v)$ is defined in Lemma 1.1 and

$$\begin{aligned} J(u,v) &= \left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \left(1 - \int_0^t h(s)\,ds\right)\|\nabla v\|_2^2 \\ &\quad + (g \circ \nabla u) + (h \circ \nabla v), \end{aligned} \tag{2.10}$$

where

$$\begin{cases} (g \circ u)(t) = \int_0^t g(t-\tau) \|u(t) - u(\tau)\|_2^2 \, d\tau, \\ (h \circ v)(t) = \int_0^t h(t-\tau) \|v(t) - v(\tau)\|_2^2 \, d\tau. \end{cases} \tag{2.11}$$

As before, we assume that ρ satisfies

$$\begin{cases} -1 < \rho, & \text{if} \quad N = 1, 2, \\ -1 < \rho \leq \frac{4-N}{N-2} & \text{if} \quad N \geq 3. \end{cases} \tag{2.12}$$

Lemma 2.1. *Suppose that (2.12) holds. Then there exists $\eta > 0$ such that for any $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ the inequality*

$$2(\rho + 2) \int_\Omega F(u, v) \, dx \leq \eta \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right)^{\rho+2} \tag{2.13}$$

holds.

For the proof see [77].

2.3 Blow up result

Lemma 2.2. *Suppose that (2.12) holds. Let (u, v) be the solution of the system (2.1)–(2.4) then the energy functional is a non-increasing function; that is, for all $t \geq 0$,*

$$\begin{aligned} E'(t) &= -\int_\Omega \left(|u(t)|^k + |v(t)|^l \right) |u_t(t)|^{m+1} \, dx - \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho \right) |v_t(t)|^{r+1} \, dx \\ &\quad + \frac{1}{2} (g' \circ \nabla u) + \frac{1}{2} (h' \circ \nabla v) - \frac{1}{2} g(s) \|\nabla u\|_2^2 - \frac{1}{2} h(s) \|\nabla v\|_2^2 . \end{aligned} \tag{2.14}$$

The proof of Lemma 2.2 can be done as we explained previously after the statement of Lemma 1.5. We omit the details.

Our main result reads as follows

Theorem 2.3. *Suppose that (2.12) holds. Assume further that*

$$\rho > \max\left(\frac{k+m-3}{2}, \frac{l+m-3}{2}, \frac{\theta+r-3}{2}, \frac{\varrho+r-3}{2} \right), \tag{2.15}$$

and that there exists p *such that* $2 < p < 2(\rho + 2)$, *for which*

$$\max\left(\int_0^\infty g(s)\,ds,\ \int_0^\infty h(s)\,ds\right) < \frac{(p/2) - 1}{(p/2) - 1 + 1/(2p)}, \tag{2.16}$$

holds. Then any solution of problem (2.1)–(2.4), with initial data satisfying

$$\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 > \alpha_1^2, \qquad \text{and} \qquad E(0) < E_2 \tag{2.17}$$

blows up in finite time, where the constants α_1 *and* E_2 *are defined in (2.18).*

We take $a = b = c = d = 1$, $a_1 = b_1 = 1$ for convenience. We introduce the following:

$$\begin{aligned} B &= \eta^{\frac{1}{2(\rho+2)}}, \qquad \alpha_1 = B^{-\frac{\rho+2}{\rho+1}}, \qquad E_1 = \left(\frac{1}{2} - \frac{1}{2(\rho+2)}\right)\alpha_1^2, \\ E_2 &= \left(\frac{1}{p} - \frac{1}{2(\rho+2)}\right)\alpha_1^2, \end{aligned} \tag{2.18}$$

where η is the optimal constant in (2.13).

Lemma 2.4. *[77] Suppose that (2.12), (2.15) and (2.16) hold. Let* (u, v) *be a solution of (2.1)–(2.4). Assume further that* $E(0) < E_2$ *and*

$$\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 > \alpha_1^2. \tag{2.19}$$

Then there exists a constant $\alpha_2 > \alpha_1$ *such that*

$$J(t) > \alpha_2^2, \tag{2.20}$$

and

$$2(\rho + 2)\int_\Omega F(u, v)\,dx \geq (B\alpha_2)^{2(\rho+2)},\ \forall t \geq 0. \tag{2.21}$$

Proof of Theorem 2.3. The proof of this theorem is similar to the one given in [53] with the necessary modification imposed by the nature of our problem. We suppose that the solution exists for all time and we reach to a contradiction. For this purpose, we set

$$H(t) = E_2 - E(t). \tag{2.22}$$

By using the definition of $H(t)$, we get

$$\begin{aligned} H'(t) &= -E'(t) \\ &= \int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u_t(t)|^{m+1}\,dx + \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right)|v_t(t)|^{r+1}\,dx \\ &\quad -\frac{1}{2}\left(g' \circ \nabla u\right) - \frac{1}{2}\left(h' \circ \nabla v\right) + \frac{1}{2}g(s)\|\nabla u\|_2^2 + \frac{1}{2}h(s)\|\nabla v\|_2^2 \\ &\geq 0, \forall t \geq 0. \end{aligned} \tag{2.23}$$

Consequently, since E' is absolutely continuous

$$H(0) = E_2 - E(0) > 0. \tag{2.24}$$

Then,

$$\begin{aligned} 0 &< H(0) \leq H(t) \\ &= E_2 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \frac{J(t)}{2} \\ &\quad + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]. \end{aligned} \tag{2.25}$$

From (2.8) and (2.20), we obtain

$$\begin{aligned} E_2 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \frac{J(t)}{2} &< E_2 - \frac{1}{2}\alpha_2^2 \\ &< E_2 - \frac{1}{2}\alpha_1^2 \\ &< E_1 - \frac{1}{2}\alpha_1^2 \\ &= -\frac{1}{2(\rho+2)}\alpha_1^2 < 0, \quad \forall t \geq 0. \end{aligned} \tag{2.26}$$

Hence, by the above inequality and Lemma 1.2, we have, for all $t \geq 0$,

$$\begin{aligned} 0 &< H(0) \leq H(t) \leq \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right] \\ &\leq \frac{c_1}{2(\rho+2)}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right). \end{aligned} \tag{2.27}$$

Then we define the functional

$$M(t) = \frac{1}{2}\int_\Omega \left(u^2 + v^2\right)(x,t)\,dx. \tag{2.28}$$

We introduce

$$L(t) = H^{1-\sigma}(t) + \varepsilon M'(t), \tag{2.29}$$

for ε small to be chosen later and

$$\begin{aligned} 0 \quad < \quad \sigma \leq \min\left\{\frac{1}{2}, \frac{2\rho+3-(k+m)}{2(m+1)(\rho+2)}, \frac{2\rho+3-(l+m)}{2(m+1)(\rho+2)}, \right. \\ \left. \frac{2\rho+3-(\varrho+r)}{2(r+1)(\rho+2)}, \frac{2\rho+3-(\theta+r)}{2(r+1)(\rho+2)}, \frac{2\rho+2}{4(\rho+2)}\right\}. \end{aligned} \tag{2.30}$$

By taking a derivative of (2.29) and using (2.1), we obtain

$$\begin{aligned} L'(t) \quad = \quad & (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ & -\varepsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\ & -\varepsilon\int_\Omega u\left(|u(t)|^k + |v(t)|^l\right)|u_t|^{m-1}u_t dx \\ & -\varepsilon\int_\Omega v\left(|v(t)|^\theta + |u(t)|^\varrho\right)|v_t|^{r-1}v_t dx \\ & +\varepsilon\int_\Omega (uf_1(u,v) + vf_2(u,v))\,dx \\ & +\varepsilon\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(\tau)\,dxds \\ & +\varepsilon\int_\Omega \nabla v(t)\int_0^t h(t-s)\nabla v(\tau)\,dxds. \end{aligned} \tag{2.31}$$

Then,

$$\begin{aligned} L'(t) \quad = \quad & (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ & -\varepsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\ & -\varepsilon\int_\Omega u\left(|u(t)|^k + |v(t)|^l\right)|u_t|^{m-1}u_t dx \\ & -\varepsilon\int_\Omega v\left(|v(t)|^\theta + |u(t)|^\varrho\right)|v_t|^{r-1}v_t dx \\ & +\varepsilon\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\ & +\varepsilon\left(\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \left(\int_0^t h(s)\,ds\right)\|\nabla v\|_2^2 \\ & +\varepsilon\int_0^t g(t-s)\int_\Omega \nabla u(t).\left[\nabla u(\tau) - \nabla u(t)\right]dxds \\ & +\varepsilon\int_0^t h(t-s)\int_\Omega \nabla v(t).\left[\nabla v(\tau) - \nabla v(t)\right]dxds. \end{aligned} \tag{2.32}$$

By Cauchy-Schwarz and Young's inequalities, we estimate

$$\begin{aligned}
& \int_0^t g(t-s) \int_\Omega \nabla u(t) . [\nabla u(\tau) - \nabla u(t)]\, dxds \\
\leq\ & \int_0^t g(t-s) \|\nabla u\|_2 \|\nabla u(\tau) - \nabla u(t)\|_2\, d\tau \\
\leq\ & \lambda (g \circ \nabla u) + \frac{1}{4\lambda}\left(\int_0^t g(s)\, ds\right) \|\nabla u\|_2^2, \qquad \lambda > 0
\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}
& \int_0^t h(t-s) \int_\Omega \nabla v(t) . [\nabla v(\tau) - \nabla v(t)]\, dxds \\
\leq\ & \lambda (h \circ \nabla v) + \frac{1}{4\lambda}\left(\int_0^t h(s)\, ds\right) \|\nabla v\|_2^2, \qquad \lambda > 0.
\end{aligned} \tag{2.34}$$

Adding and substituting $pE(t)$ and using the definition of $H(t)$ and E_2 lead to

$$\begin{aligned}
L'(t) \geq\ & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon\left(1 + \frac{p}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\
& +\varepsilon\left(\frac{p}{2} - \lambda\right)[(g \circ \nabla u) + (h \circ \nabla v)] + p\varepsilon H(t) - p\varepsilon E_2 \\
& -\varepsilon \int_\Omega u\left(|u(t)|^k + |v(t)|^l\right)|u_t|^{m-1} u_t dx \\
& -\varepsilon \int_\Omega v\left(|v(t)|^\theta + |u(t)|^\varrho\right)|v_t|^{r-1} v_t dx \\
& +\varepsilon\left(1 - \frac{p}{2(\rho+2)}\right)\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\
& +\varepsilon\left[\left(\frac{p}{2} - 1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\lambda}\right)\int_0^\infty g(s)\, ds\right]\|\nabla u\|_2^2 \\
& +\varepsilon\left[\left(\frac{p}{2} - 1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\lambda}\right)\int_0^\infty h(s)\, ds\right]\|\nabla v\|_2^2,
\end{aligned} \tag{2.35}$$

for some λ such that

$$a_1 = \frac{p}{2} - \lambda > 0,$$

and

$$a_2 = \left[\left(\frac{p}{2} - 1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\lambda}\right)\max\left(\int_0^\infty g(s)\, ds,\ \int_0^\infty h(s)\, ds\right)\right] > 0.$$

Then, estimate (2.35) becomes

$$\begin{aligned} L'(t) \geq\ & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon\left(1+\frac{p}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ & + \varepsilon a_1 \left[(g \circ \nabla u) + (h \circ \nabla v)\right] + p\varepsilon H(t) - p\varepsilon E_2 \\ & - \varepsilon \int_\Omega u\left(|u(t)|^k + |v(t)|^l\right)|u_t|^{m-1} u_t dx \\ & - \varepsilon \int_\Omega v\left(|v(t)|^\theta + |u(t)|^\varrho\right)|v_t|^{r-1} v_t dx \\ & + \varepsilon\left(1 - \frac{p}{2(\rho+2)}\right)\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\ & + \varepsilon a_2\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right). \end{aligned} \tag{2.36}$$

By taking $c_3 = 1 - \dfrac{p}{\rho+2} - 2E_2(B\alpha_2)^{-2(\rho+2)} > 0$, since $\alpha_2 > B^{-\frac{2(\rho+2)}{\rho+1}}$. Therefore, (2.36) takes the form

$$\begin{aligned} L'(t) \geq\ & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon\left(1+\frac{p}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ & + \varepsilon a_1 \left[(g \circ \nabla u) + (h \circ \nabla v)\right] \\ & + \varepsilon a_2\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + p\varepsilon H(t) \\ & + \varepsilon c_3\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\ & - \varepsilon \int_\Omega u\left(|u(t)|^k + |v(t)|^l\right)|u_t|^{m-1} u_t dx \\ & - \varepsilon \int_\Omega v\left(|v(t)|^\theta + |u(t)|^\varrho\right)|v_t|^{r-1} v_t dx. \end{aligned} \tag{2.37}$$

We use the Young inequality as follows

$$XY \leq \frac{\delta^\alpha X^\alpha}{\alpha} + \frac{\delta^{-\beta} Y^\beta}{\beta}, \tag{2.38}$$

where $X, Y \geq 0$, $\delta > 0$, and $\alpha, \beta > 0$ such that $1/\alpha + 1/\beta = 1$, we get, for any $\delta_1 > 0$,

$$\left|u\,|u_t|^{m-1} u_t\right| \leq \frac{\delta_1^{m+1}}{m+1}|u|^{m+1} + \frac{m}{m+1}\delta_1^{-(m+1)/m}|u_t|^{m+1} \tag{2.39}$$

and

$$\begin{aligned} & \int_\Omega \left(|u(t)|^k + |v(t)|^l\right)\left|u\,|u_t|^{m-1} u_t\right| dx \\ \leq\ & \frac{\delta_1^{m+1}}{m+1}\int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u|^{m+1} dx \\ & + \frac{m}{m+1}\delta_1^{-(m+1)/m}\int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u_t|^{m+1} dx. \end{aligned} \tag{2.40}$$

Similarly, for any $\delta_2 > 0$,

$$\left| v \left| v_t \right|^{r-1} v_t \right| \leq \frac{\delta_2^{r+1}}{r+1} \left| v \right|^{r+1} + \frac{r}{r+1} \delta_2^{-(r+1)/r} \left| v_t \right|^{r+1}, \tag{2.41}$$

which gives

$$\begin{aligned} & \int_\Omega \left(\left| v(t) \right|^\theta + \left| u(t) \right|^\varrho \right) \left| v \left| v_t \right|^{r-1} v_t \right| dx \\ \leq \; & \frac{\delta_2^{r+1}}{r+1} \int_\Omega \left(\left| v(t) \right|^\theta + \left| u(t) \right|^\varrho \right) \left| v \right|^{r+1} dx \\ & + \frac{r}{r+1} \delta_2^{-(r+1)/r} \int_\Omega \left(\left| v(t) \right|^\theta + \left| u(t) \right|^\varrho \right) \left| v_t \right|^{r+1} dx. \end{aligned} \tag{2.42}$$

Then, we get

$$\begin{aligned} L'(t) \geq \; & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \left(1 + \frac{p}{2} \right) \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) \\ & + \varepsilon a_1 \left[(g \circ \nabla u) + (h \circ \nabla v) \right] \\ & + \varepsilon a_2 \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) + p \varepsilon H(t) \\ & + \varepsilon c_3 \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2 \|uv\|_{\rho+2}^{\rho+2} \right) \\ & - \varepsilon \frac{\delta_1^{m+1}}{m+1} \int_\Omega \left(\left| u(t) \right|^k + \left| v(t) \right|^l \right) \left| u \right|^{m+1} dx \\ & - \varepsilon \frac{m}{m+1} \delta_1^{-\frac{(m+1)}{m}} \int_\Omega \left(\left| u(t) \right|^k + \left| v(t) \right|^l \right) \left| u_t \right|^{m+1} dx \\ & - \varepsilon \frac{\delta_2^{r+1}}{r+1} \int_\Omega \left(\left| v(t) \right|^\theta + \left| u(t) \right|^\varrho \right) \left| v \right|^{r+1} dx \\ & - \varepsilon \frac{r}{r+1} \delta_2^{-\frac{(r+1)}{r}} \int_\Omega \left(\left| v(t) \right|^\theta + \left| u(t) \right|^\varrho \right) \left| v_t \right|^{r+1} dx. \end{aligned} \tag{2.43}$$

Let us choose δ_1 and δ_2 such that

$$\delta_1^{-\frac{(m+1)}{m}} = M_1 H(t)^{-\sigma}, \quad \delta_2^{-\frac{(r+1)}{r}} = M_2 H(t)^{-\sigma}, \tag{2.44}$$

for M_1 and M_2 large constants to be fixed later. Thus, by using (2.44), we get

$$\begin{aligned} L'(t) \geq\ & ((1-\sigma) - M\varepsilon) H^{-\sigma}(t) H'(t) + \varepsilon\left(1+\frac{p}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ & + \varepsilon a_1 \left[(g \circ \nabla u) + (h \circ \nabla v)\right] \\ & + \varepsilon a_2 \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + p\varepsilon H(t) \\ & + \varepsilon c_3 \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\ & - \varepsilon M_1^{-m} H^{\sigma m}(t) \int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u|^{m+1}\,dx \\ & - \varepsilon \frac{m}{m+1}\delta_1^{-\frac{(m+1)}{m}} \int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u_t|^{m+1}\,dx \\ & - \varepsilon M_2^{-r} H^{\sigma r}(t) \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right)|v|^{r+1}\,dx \\ & - \varepsilon \frac{r}{r+1}\delta_2^{-\frac{(r+1)}{r}} \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right)|v_t|^{r+1}\,dx, \end{aligned} \tag{2.45}$$

where $M = m/(m+1)M_1 + r/(r+1)M_2$. Consequently, we have

$$\int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u|^{m+1}\,dx = \|u\|_{k+m+1}^{k+m+1} + \int_\Omega |v|^l\,|u|^{m+1}\,dx \tag{2.46}$$

and

$$\int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right)|v|^{r+1}\,dx = \|v\|_{\theta+r+1}^{\theta+r+1} + \int_\Omega |u|^\varrho\,|v|^{r+1}\,dx. \tag{2.47}$$

Also by using Young's inequality, we have

$$\begin{aligned} \int_\Omega |v|^l\,|u|^{m+1} &\leq \frac{l}{l+m+1}\delta_1^{(l+m+1)/l}\|v\|_{l+m+1}^{l+m+1} + \frac{m+1}{l+m+1}\delta_1^{-(l+m+1)/(m+1)}\|u\|_{l+m+1}^{l+m+1}, \\ \int_\Omega |u|^\varrho\,|v|^{r+1} &\leq \frac{\varrho}{\varrho+r+1}\delta_2^{(\varrho+r+1)/\varrho}\|u\|_{\varrho+r+1}^{\varrho+r+1} + \frac{r+1}{\varrho+r+1}\delta_2^{-(\varrho+r+1)/(r+1)}\|v\|_{\varrho+r+1}^{\varrho+r+1}. \end{aligned}$$

Consequently,

$$\begin{aligned} & H^{\sigma m}(t)\int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u|^{m+1}\,dx \\ =\ & H^{\sigma m}(t)\|u\|_{k+m+1}^{k+m+1} + \frac{l}{l+m+1}\delta_1^{(l+m+1)/l} H^{\sigma m}(t)\|v\|_{l+m+1}^{l+m+1} \\ & + \frac{m+1}{l+m+1}\delta_1^{-(l+m+1)/(m+1)} H^{\sigma m}(t)\|u\|_{l+m+1}^{l+m+1} \end{aligned} \tag{2.48}$$

and

$$
\begin{aligned}
& H^{\sigma r}(t)\int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right)|v|^{r+1}\,dx \\
= \; & H^{\sigma r}(t)\,\|v\|_{\theta+r+1}^{\theta+r+1} + \frac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}} H^{\sigma r}(t)\,\|u\|_{\varrho+r+1}^{\varrho+r+1} \\
& + \frac{r+1}{\varrho+r+1}\delta_2^{-\frac{(\varrho+r+1)}{r+1}} H^{\sigma r}(t)\,\|v\|_{\varrho+r+1}^{\varrho+r+1}.
\end{aligned} \tag{2.49}
$$

Since (2.15) holds, we obtain by using (2.30)

$$
\begin{cases}
H^{\sigma m}(t)\,\|u\|_{k+m+1}^{k+m+1} \le c_5\left(\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)+k+m+1} + \|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|u\|_{k+m+1}^{k+m+1}\right), \\
H^{\sigma r}(t)\,\|v\|_{\theta+r+1}^{\theta+r+1} \le c_6\left(\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\theta+r+1} + \|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\|v\|_{\theta+r+1}^{\theta+r+1}\right).
\end{cases} \tag{2.50}
$$

This implies

$$
\begin{aligned}
& \tfrac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}} H^{\sigma m}(t)\,\|v\|_{l+m+1}^{l+m+1} \\
& \le c_7 \tfrac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}\left(\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)+l+m+1} + \|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|v\|_{l+m+1}^{l+m+1}\right)
\end{aligned} \tag{2.51}
$$

and

$$
\begin{aligned}
& \tfrac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}} H^{\sigma r}(t)\,\|u\|_{\varrho+r+1}^{\varrho+r+1} \\
& \le c_8 \tfrac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}\left(\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\varrho+r+1} + \|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\|u\|_{\varrho+r+1}^{\varrho+r+1}\right).
\end{aligned} \tag{2.52}
$$

By using (2.30) and the algebraic inequality

$$
z^\nu \le (z+1) \le \left(1+\frac{1}{a}\right)(z+a), \quad \forall z \ge 0,\; 0<\nu\le 1,\; a>0, \tag{2.53}
$$

we have, for all $t \ge 0$,

$$
\begin{cases}
\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)+k+m+1} \le d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + H(0)\right) \le d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + H(t)\right), \\
\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\theta+r+1} \le d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + H(t)\right), \; \forall t \ge 0,
\end{cases} \tag{2.54}
$$

where $d = 1 + 1/H(0)$. Similarly

$$
\begin{cases}
\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)+l(m+1)} \le d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + H(0)\right) \le d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + H(t)\right), \\
\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\varrho(r+1)} \le d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + H(t)\right), \; \forall t \ge 0.
\end{cases} \tag{2.55}
$$

Also, since

$$(X+Y)^s \le C\left(X^s+Y^s\right), \qquad X,Y\ge 0,\ s>0, \tag{2.56}$$

by using (2.30) and (2.53) we conclude

$$\begin{aligned} \|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|u\|_{k+m+1}^{k+m+1} &\le c_9\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+\|u\|_{k+m+1}^{2(\rho+2)}\right)\\ &\le c_{10}\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+\|u\|_{2(\rho+2)}^{2(\rho+2)}\right), \end{aligned} \tag{2.57}$$

similarly

$$\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\|v\|_{\theta+r+1}^{\theta+r+1} \le c_{11}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right), \tag{2.58}$$

$$\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|v\|_{l+m+1}^{l+m+1} \le c_{12}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right) \tag{2.59}$$

and

$$\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\|u\|_{\varrho+r+1}^{\varrho+r+1} \le c_{13}\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+\|u\|_{2(\rho+2)}^{2(\rho+2)}\right). \tag{2.60}$$

Taking into account (2.48)-(2.60), then (2.45) takes the form

$$\begin{aligned} L'(t) \ \ge\ & \left((1-\sigma)-M\varepsilon\right)H^{-\sigma}(t)H'(t)+2\varepsilon\left(\|u_t\|_2^2+\|v_t\|_2^2\right)\\ &+\varepsilon\left[2-CM_1^{-m}\left(1+\frac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}+\frac{m+1}{l+m+1}\delta_1^{-\frac{(l+m+1)}{m+1}}\right)\right.\\ &\left.-CM_2^{-r}\left(1+\frac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}+\frac{r+1}{\varrho+r+1}\delta_2^{-\frac{(\varrho+r+1)}{r+1}}\right)\right]H(t)\\ &+\varepsilon\left[c_4-CM_1^{-m}\left(1+\frac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}+\frac{m+1}{l+m+1}\delta_1^{-\frac{(l+m+1)}{m+1}}\right)\right.\\ &\left.-CM_2^{-r}\left(1+\frac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}+\frac{r+1}{\varrho+r+1}\delta_2^{-\frac{(\varrho+r+1)}{r+1}}\right)\right]\\ &\times\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right). \end{aligned} \tag{2.61}$$

At this point, and for large values of M_1 and M_2, we can find positive constants Λ_1 and Λ_2 such that (2.61) becomes

$$\begin{aligned} L'(t) \ \ge\ & \left((1-\sigma)-M\varepsilon\right)H^{-\sigma}(t)H'(t)+2\varepsilon\left(\|u_t\|_2^2+\|v_t\|_2^2\right)\\ &+\varepsilon\Lambda_1\left(\|u(t)\|_{2(\rho+2)}^{2(\rho+2)}+\|v(t)\|_{2(\rho+2)}^{2(\rho+2)}\right)+\varepsilon\Lambda_2 H(t). \end{aligned} \tag{2.62}$$

Once M_1 and M_2 are fixed (hence, Λ_1 and Λ_2), we pick ε small enough so that $((1-\sigma)-M\varepsilon)\ge 0$ and

$$L(0)=H^{1-\sigma}(0)+\varepsilon\int_\Omega\left[u_0.u_1+v_0.v_1\right]dx>0. \tag{2.63}$$

Consequently, there exists $\Gamma > 0$ such that (2.62) becomes

$$L^{'}(t) \geq \varepsilon\Gamma\left(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right). \tag{2.64}$$

Thus, we have $L(t) \geq L(0) > 0$, for all $t \geq 0$. Next, by Holder's and Young's inequalities, we estimate

$$\begin{aligned}
&\left(\int_\Omega u.u_t(x,t)\,dx + \int_\Omega v.v_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\
\leq\ & C\left(\|u\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}} + \|u_t\|_2^{\frac{s}{1-\sigma}} + \|v\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}} + \|v_t\|_2^{\frac{s}{1-\sigma}}\right),
\end{aligned} \tag{2.65}$$

for $1/\tau + 1/s = 1$. We takes $s = 2(1-\sigma)$, to get $\dfrac{\tau}{1-\sigma} = \dfrac{2}{1-2\sigma}$. By using (2.22) and (2.53) we get

$$\|u\|_{2(\rho+2)}^{\frac{2}{1-2\sigma}} \leq d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + H(t)\right) \tag{2.66}$$

and

$$\|v\|_{2(\rho+2)}^{\frac{2}{1-2\sigma}} \leq d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + H(t)\right),\ \forall t \geq 0. \tag{2.67}$$

Therefore, (2.65) becomes

$$\left(\int_\Omega uu_t(x,t)\,dx + \int_\Omega vv_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \tag{2.68}$$

$$\leq\ c_{14}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u_t\|_2^2 + \|v_t\|_2^2 + H(t)\right),\ \forall t \geq 0. \tag{2.69}$$

Also, by noting that

$$\begin{aligned}
L^{\frac{1}{1-\sigma}}(t) &= \left(H^{1-\sigma}(t) + \varepsilon\int_\Omega (u.u_t + v.v_t)(x,t)\,dx\right)^{\frac{1}{(1-\sigma)}} \\
&\leq c_{15}\left(H(t) + \left|\int_\Omega (u.u_t(x,t) + v.v_t(x,t))\,dx\right|^{\frac{1}{(1-\sigma)}}\right) \\
&\leq c_{16}\left[H(t) + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u_t\|_2^2 + \|v_t\|_2^2\right],\ \forall t \geq 0,
\end{aligned} \tag{2.70}$$

and combining with (2.70) and (2.64), we arrive at

$$L^{'}(t) \geq a_0 L^{\frac{1}{1-\sigma}}(t),\ \forall t \geq 0. \tag{2.71}$$

Finally, a simple integration of (2.71) gives the desired result. □

Part II

Stability result of a Timoshenko system

Chapter 3

Stability result of a Timoshenko system in thermoelasticity of second sound

3.1 Introduction

This chapter aims at investigating long-term behavior of solutions to the following system:

$$\begin{cases} \rho_1\varphi_{tt} - K\left(\varphi_x + \psi\right)_x + \mu\varphi_t = 0 \\ \rho_2\psi_{tt} - \bar{b}\psi_{xx} + \displaystyle\int_0^t g\left(t-s\right)\left(a\left(x\right)\psi_x\left(s\right)\right)_x ds + K\left(\varphi_x + \psi\right) + b\left(x\right)h\left(\psi_t\right) + \gamma\theta_x = 0 \\ \rho_3\theta_t + \kappa q_x + \gamma\psi_{tx} = 0 \\ \tau_0 q_t + \delta q + \kappa\theta_x = 0. \end{cases} \tag{3.1}$$

where $t \in (0, \infty)$ denotes the time variable and $x \in (0,1)$ is the space variable, the function φ and ψ are the displacement of the solid elastic material, the function θ is the temperature difference, $q = q(x,t) \in \mathbb{R}$ is the heat flux, and $\rho_1, \rho_2, \rho_3, \gamma, \tau_0, \delta, \kappa, \bar{b}$ and K are positive constants and $\mu > 0$. We consider the following initial conditions:

$$\begin{aligned} \varphi(.,0) &= \varphi_0(x),\ \varphi_t(.,0) = \varphi_1(x),\ \psi(.,0) = \psi_0(x) \\ \psi_t(.,0) &= \psi_1(x),\ \theta(.,0) = \theta_0\left(x\right),\ q(.,0) = q_0(x), \end{aligned} \tag{3.2}$$

and boundary conditions

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = q(0,t) = q(1,t) = 0,\ \forall t \geq 0. \tag{3.3}$$

In system (3.1)−(3.3) the heat conduction given by Cattaneo's law instead of the usual Fourier's one. We should note here that dissipative effects of heat conduction induced by Cattaneo's law are usually weaker than those induced by Fourier's law. This justifies the presence of the extra damping term in the second equation of (3.1). In fact if $a = b = 0$, Fernández Sare and Racke [18] have proved recently that (3.1)−(3.3) is no longer exponentially stable even in the case of equal propagation speed ($\rho_1/\rho_2 = K/\bar{b}$).

In this chapter, we show a general decay result of the total energy of system (3.1)−(3.3) (Theorem 3.5 below). To prove this result, we followed very carefully the method used by Guesmia and Messaoudi [25, 26] and Guesmia [29].

In order to state and prove our result, we formulate the following assumptions:

- **(H1)** a, b: $[0,1] \to \mathbb{R}^+$ are such that

$$\begin{aligned} a &\in C^1([0,1]), \qquad b \in L^\infty([0,1]) \\ a &= 0 \text{ or } a(0) + a(1) > 0, \qquad \inf_{x\in[0,1]} \{a(x) + b(x)\} > 0. \end{aligned}$$

- **(H2)** $h : \mathbb{R} \to \mathbb{R}$ is a differentiable nondecreasing function such that there exist constants $\varepsilon', c_1, c_2 > 0$ and a convex and increasing function $H : \mathbb{R}^+ \to \mathbb{R}^+$ of class $C^1(\mathbb{R}^+) \cap C^2((0,\infty))$ satisfying $H(0) = 0$ and H is linear on $[0,\varepsilon']$ or $H'(0) = 0$ and $H'' > 0$ on $(0,\varepsilon']$ such that

$$\begin{cases} c_1|s| \le h(s) \le c_2|s| & \text{if } |s| \ge \varepsilon' \\ s^2 + h^2(s) \le H^{-1}(sh(s)) & \text{if } |s| \le \varepsilon'. \end{cases}$$

- **(H3)** $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function such that

$$g(0) > 0, \qquad 1 - \|a\|_\infty \int_0^\infty g(s)\,ds = l > 0.$$

- **(H4)** There exists a non-increasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$g'(s) \le -\xi(s)\,g(s), \qquad \forall s \ge 0.$$

Throughout this chapter, we use the following notations

$$\begin{aligned}(\phi * \psi)(t) &:= \int_0^t \phi(t-\tau)\psi(\tau)\,d\tau \\ (\phi \diamond \psi)(t) &:= \int_0^t \phi(t-\tau)\left|\psi(t)-\psi(\tau)\right|d\tau \\ (\phi \circ \psi)(t) &:= \int_0^t \phi(t-\tau)\int_\Omega \left|\psi(t)-\psi(\tau)\right|^2 dx d\tau.\end{aligned}$$

The following lemma was introduced in [63].

Lemma 3.1. *For any function $\phi \in C^1(\mathbb{R})$ and any $\psi \in H^1(0,1)$, we have*

$$\begin{aligned}(\phi * \psi)(t)\psi_t(t) &= -\frac{1}{2}\phi(t)\left|\psi(t)\right|^2 + \frac{1}{2}(\phi' \diamond \psi)(t) \\ &\quad -\frac{1}{2}\frac{d}{dt}\left\{(\phi \diamond \psi)(t) - \left(\int_0^t \phi(\tau)\,d\tau\right)\left|\psi(t)\right|^2\right\}.\end{aligned}$$

Now, we are going to prepare some materials in order to state two lemmas due to Cavalcanti and Oquendo [13]. See also [26] for the proof.

By using the fact that $a(0) > 0$ and since a is continuous, then there exists $\varepsilon > 0$ such that $\inf_{x\in[0,\varepsilon]} a(x) \geq \varepsilon$. Let us denote

$$d = \min\left\{\varepsilon, \inf_{x\in[0,1]}\{a(x)+b(x)\}\right\} > 0$$

and let $\alpha \in C^1([0,1])$ be such that $0 \leq \alpha \leq a$ and

$$\begin{cases} \alpha(x) = 0 & \text{if } a(x) \leq \frac{d}{4} \\ \alpha(x) = a(x) & \text{if } a(x) \geq \frac{d}{2} \end{cases} \tag{3.4}$$

To simplify the notations we introduce the following

$$g \odot v = \int_0^1 \alpha(x)\int_0^t g(s)(v(t)-v(s))\,ds dx$$

for all $v \in L^2(0,1)$. Here and in the sequel, we denote various generic positive constants by C or c.

Lemma 3.2. *The function α is not identically zero and satisfies*

$$\inf_{x\in[0,1]}\{\alpha(x)+b(x)\} \geq \frac{d}{2}.$$

Lemma 3.3. *There exists a positive constant c such that*

$$(g \odot v)^2 \leq c g \circ v_x,$$

for all $v \in H_0^1(0,1)$.

In order to make this chapter self contained, we state, without proof, a local existence result. The proof can be established by the classical Galerkin method.

Theorem 3.4. *Let* (φ_0, φ_1), $(\psi_0, \psi_1) \in H_0^1(0,1) \times L^2(0,1)$ *and* $(\theta_0, q_0) \in L^2(0,1) \times L^2(0,1)$ *be given. Assume that (**H1**)−(**H4**) are satisfied, then problem (3.1)−(3.3) has a unique global (weak) solution satisfying*

$$\begin{aligned} \varphi, \psi &\in C\left(\mathbb{R}_+; H_0^1(0,1)\right) \cap C^1\left(\mathbb{R}_+; L^2(0,1)\right) \\ \theta, q &\in C\left(\mathbb{R}_+; L^2(0,1)\right). \end{aligned}$$

3.2 Stability result

In this section, we show the uniform decay property of the solution of the system (3.1)−(3.3). In order to use the Poincaré inequality for θ, we introduce, as in [73],

$$\bar{\theta}(x,t) = \theta(x,t) - \int_0^1 \theta_0(x)dx.$$

Then, by the third equation in (3.1) we easily verify that

$$\int_0^1 \bar{\theta}(x,t)\, dx = 0,$$

for all $t \geq 0$. In this case the Poincaré inequality is applicable for $\bar{\theta}$. On the other hand, $(\varphi, \psi, \bar{\theta}, q)$ satisfies the same system (3.1) and the boundary conditions (3.3). So, in the sequel, we shall work with $\bar{\theta}$ but we write θ for simplicity.

The first-order energy, associated to (3.1)−(3.3), is then given by

$$\begin{aligned} E\left(t, \varphi, \psi, \bar{\theta}, q\right) &= \frac{1}{2}\int_0^1 \left\{\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \left(\bar{b} - a(x)\int_0^t g(s)\, ds\right)\psi_x^2\right\} dx \\ &+ \frac{1}{2}\int_0^1 \left\{K(\varphi_x + \psi)^2 + \rho_3 \theta^2 + \tau_0 q^2\right\} dx + \frac{1}{2}(g \circ \psi_x). \end{aligned} \tag{3.5}$$

In what follows, we denote $E(t) = E\left(t, \varphi, \psi, \bar{\theta}, q\right)$ and $E(0) = E\left(0, \varphi_0, \psi_0, \bar{\theta}_0, q_0\right)$ for simplicity. The main result of this chapter is given by the following theorem:

Theorem 3.5. *Let* $(\varphi_0, \varphi_1), (\psi_0, \psi_1), \in H_0^1(0,1) \times L^2(0,1)$ *and* $(\theta_0, q_0) \in L^2(0,1) \times L^2(0,1)$ *be given. Assume that* ***(H1)–(H4)*** *are satisfied, then there exist positive constants* c', c'' *and* ε_0 *for which the (weak) solution of problem (3.1)–(3.3) satisfies*

$$E(t) \leq c'' H_1^{-1}\left(c' \int_0^t \xi(s)\, ds\right), \qquad \forall t \geq 0, \tag{3.6}$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2}(s)\, ds$$

and

$$H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } \left[0, \varepsilon'\right] \\ tH'(\varepsilon_0 t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on} \left(0, \varepsilon'\right] \end{cases} \tag{3.7}$$

and $\xi = 1$ *if* $a = 0$.

Remark 3.6. The result of Theorem 3.5 holds true without any assumption on the wave speeds of the first two equations in (3.1).

Remark 3.7. The result of Theorem 3.5 is more general than the one obtained in ([50] Theorem 2). For $a = b = 0$, the result of Theorem 3.5 is the same as that in [50, Theorem 2] .

To prove Theorem 3.5, we will use the energy method to produce a suitable Lyapunov functional. This will be established through several lemmas. A starting point is, as usual, the dissipativity inequality which states that the energy E of the entire system (3.1)-(3.3) is a non-increasing function. Of course this fact is a necessary preliminary step of stability analysis. More precisely, we have the following result:

Lemma 3.8. *Let* $(\varphi, \psi, \theta, q)$ *be the solution of (3.1)–(3.3), then the energy* E *is non-increasing function and satisfies, for all* $t \geq 0$,

$$\begin{aligned} \frac{dE(t)}{dt} &= -\delta \int_0^1 q^2 dx - \frac{1}{2} g(t) \int_0^1 a(x) \psi_x^2 dx - \int_0^1 b(x) \psi_t h(\psi_t)\, dx \\ &\quad + \frac{1}{2}(g' \circ \psi_x) - \mu \int_0^1 \varphi_t^2 dx, \\ &\leq -\delta \int_0^1 q^2 dx - \int_0^1 b(x) \psi_t h(\psi_t)\, dx + \frac{1}{2}(g' \circ \psi_x) - \mu \int_0^1 \varphi_t^2 dx \leq 0. \end{aligned} \tag{3.8}$$

Proof. Multiplying the first equation in (3.1) by φ_t, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \rho_1\varphi_t^2 dx + K\int_0^1 \varphi_{tx}\varphi_x dx + K\int_0^1 \varphi_{tx}\psi dx = -\mu\int_0^1 \varphi_t^2 dx. \tag{3.9}$$

Similarly, multiplying the second equation in (3.1) by ψ_t, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \rho_2\psi_t^2 dx + \bar{b}\int_0^1 \psi_x\psi_{tx} dx + \int_0^1 \psi_t\int_0^t g(t-s)(a(x)\psi_x(s))_x\, ds dx \\ &+K\int_0^1 \psi_t\varphi_x dx + K\int_0^1 \psi_t\psi dx - \gamma\int_0^1 \psi_{tx}\theta dx \\ = &-\int_0^1 b(x)\psi_t h(\psi_t)\, dx. \end{aligned} \tag{3.10}$$

Also, multiplying the third equation in (3.1) by θ, we find

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \rho_3\theta^2 dx + \kappa\int_0^1 q_x\theta dx + \gamma\int_0^1 \psi_{tx}\theta dx = 0. \tag{3.11}$$

Finally, multiplying the fourth equation in (3.1) by q, we deduce

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \tau_0 q^2 dx - \kappa\int_0^1 \theta q_x dx = -\delta\int_0^1 q^2 dx. \tag{3.12}$$

Now, using Lemma 3.1, to handle the last term in first line of (3.10) and summing up (3.9)–(3.12), then (3.8) holds. □

Let us now define the functional I_1 as follows:

$$\begin{aligned} I_1(t) := &-\int_0^1 \rho_2\alpha(x)\psi_t\int_0^t g(t-s)(\psi(t)-\psi(s))\, ds dx \\ &+\frac{\gamma\tau_0}{\kappa}\int_0^1 \alpha(x)\, q\int_0^t g(t-s)(\psi(t)-\psi(s))\, ds dx, \end{aligned}$$

for simplicity we write

$$I_1(t) := \chi_1(t) + \chi_2(t). \tag{3.13}$$

Then, we have the following result:

Lemma 3.9. *Let $(\varphi, \psi, \theta, q)$ be the solution of (3.1)–(3.3). Assume that* ***(H1)–(H4)*** *hold. Then we have, for any ε_1, $\varepsilon_1' > 0$,*

$$\begin{aligned} \frac{dI_1}{dt} \leq & -\left(\rho_2 \int_0^t g(s)\,ds - \varepsilon_1 \left(\rho_2^2 + \int_0^t g(s)\,ds\right)\right) \int_0^1 \alpha(x)\,\psi_t^2 dx \\ & +\varepsilon_1' K^2 \int_0^1 (\varphi_x + \psi)^2\,dx + \varepsilon_1 \int_0^1 b(x) h^2(\psi_t) dx \\ & +\varepsilon_1' \left(2\bar{b}^2 + 1\right) \int_0^1 \psi_x^2 dx + \left(c\varepsilon_1 + \frac{1}{\varepsilon_1} \int_0^t g(s)\,ds\right) \int_0^1 q^2 dx \\ & +c\left(\varepsilon_1' + \frac{1}{\varepsilon_1'}\right) g \circ \psi_x + c\left(\varepsilon_1 + \frac{1}{\varepsilon_1}\right) g \circ \psi_x - \frac{c}{\varepsilon_1} g' \circ \psi_x \end{aligned} \tag{3.14}$$

Proof. Differentiating χ_1 with respect to t to obtain

$$\begin{aligned} \chi_1'(t) = & -\int_0^1 \rho_2 \alpha(x) \psi_{tt} \int_0^t g(t-s)(\psi(t) - \psi(s))\,ds dx \\ & -\int_0^1 \rho_2 \alpha(x) \psi_t \int_0^t g'(t-s)(\psi(t) - \psi(s))\,ds dx \\ & -\int_0^1 \rho_2 \alpha(x) \psi_t^2 \int_0^t g(s)\,ds dx. \end{aligned} \tag{3.15}$$

Now, using the second equation in (3.1), we get

$$\begin{aligned} & -\int_0^1 \rho_2 \alpha(x) \psi_{tt} \int_0^t g(t-s)(\psi(t) - \psi(s))\,ds dx \\ & = \int_0^1 \bar{b} \alpha(x) \psi_x \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))\,ds dx \\ & + \int_0^t K\alpha(x)(\varphi_x + \psi) \int_0^t g(t-s)(\psi(t) - \psi(s))\,ds dx \\ & - \int_0^1 \alpha(x) a(x) \left(\int_0^t g(t-s) \psi_x(s)\,ds\right)\left(\int_0^t g(t-s)(\psi_x(t) - \psi_x(s))\,ds\right) dx \\ & + \int_0^1 b(x) h(\psi_t) \left(\int_0^t g(t-s)(\psi(t) - \psi(s))\,ds\right) dx \\ & + \int_0^1 \alpha(x) \gamma \theta_x \left(\int_0^t g(t-s)(\psi(t) - \psi(s))\,ds\right) dx \\ & + \int_0^1 \alpha'(x) \left(\bar{b}\psi_x - a(x) \int_0^t g(s) \psi_x(s)\,ds\right)\left(\int_0^t g(t-s)(\psi(t) - \psi(s))\,ds\right) dx. \end{aligned} \tag{3.16}$$

Next, we will estimate the second term in the right-hand side of (3.15). So, by using Lemma 3.3, we have, for any $\varepsilon_1 > 0$

$$
\begin{aligned}
&-\int_0^1 \rho_2 \alpha(x) \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))\, ds dx \\
&\leq \varepsilon_1 \rho_2^2 \int_0^1 \alpha(x) \psi_t^2 dx - \frac{c}{\varepsilon_1} g' \circ \psi_x.
\end{aligned}
\tag{3.17}
$$

Also, as above we have

$$
\begin{aligned}
\chi_2'(t) &= \frac{\gamma\tau_0}{\kappa} \int_0^1 \alpha(x) q_t \int_0^t g(t-s)(\psi(t)-\psi(s))\, ds dx \\
&\quad + \frac{\gamma\tau_0}{\kappa} \int_0^1 \alpha(x) q \int_0^t g'(t-s)(\psi(t)-\psi(s))\, ds dx \\
&\quad + \frac{\gamma\tau_0}{\kappa} \int_0^1 \alpha(x) q\psi_t \int_0^t g(s)\, ds.
\end{aligned}
$$

Using the fourth equation in (3.1), we get

$$
\begin{aligned}
\chi_2'(t) &= -\frac{\gamma\delta}{\kappa} \int_0^1 \alpha(x) q \int_0^t g(t-s)(\psi(t)-\psi(s))\, ds dx \\
&\quad - \int_0^1 \alpha(x) \gamma\theta_x \left(\int_0^t g(t-s)(\psi(t)-\psi(s))\, ds \right) dx \\
&\quad + \frac{\gamma\tau_0}{\kappa} \int_0^1 \alpha(x) q \int_0^t g'(t-s)(\psi(t)-\psi(s))\, ds dx \\
&\quad + \frac{\gamma\tau_0}{\kappa} \left(\int_0^t g(s)\, ds \right) \int_0^1 \alpha(x) q\psi_t dx.
\end{aligned}
\tag{3.18}
$$

Similarly to (3.17), by exploiting Young's inequality, we estimate the terms in the right-hand side of (3.16) as follows:

$$
\begin{aligned}
&\int_0^1 \bar{b} \alpha(x) \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\, ds dx \\
&\leq \varepsilon_1' \bar{b}^2 \int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon_1'} g \circ \psi_x.
\end{aligned}
\tag{3.19}
$$

Similarly,

$$
\begin{aligned}
&\int_0^t K \alpha(x)(\varphi_x + \psi) \int_0^t g(t-s)(\psi(t)-\psi(s))\, ds dx \\
&\leq \varepsilon_1' K^2 \int_0^1 (\varphi_x + \psi)^2 dx + \frac{c}{\varepsilon_1'} g \circ \psi_x.
\end{aligned}
\tag{3.20}
$$

By the same method used in [26], we have the following estimates:

$$\begin{aligned} &-\int_0^1 \alpha(x)\, a(x) \left(\int_0^t g(s)\, \psi_x(s)\, ds\right)\left(\int_0^t g(t-s)\, (\psi_x(t) - \psi_x(s))\, ds\right) dx \\ &\leq \varepsilon_1' \int_0^1 \psi_x^2 dx + c\left(\varepsilon_1' + \frac{1}{\varepsilon_1'}\right) g \circ \psi_x \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} &\int_0^1 b(x)\, h(\psi_t) \left(\int_0^t g(t-s)\, (\psi(t) - \psi(s))\, ds\right) dx \\ &\leq \varepsilon_1 \int_0^1 b(x) h^2(\psi_t) dx + c\left(\varepsilon_1 + \frac{1}{\varepsilon_1}\right) g \circ \psi_x. \end{aligned} \tag{3.22}$$

Finally,

$$\begin{aligned} &\int_0^1 \alpha'(x) \left(\bar{b}\psi_x - a(x) \int_0^t g(s)\, \psi_x(s)\, ds\right)\left(\int_0^t g(t-s)\, (\psi(t) - \psi(s))\, ds\right) dx \\ &\leq \varepsilon_1' \bar{b}^2 \int_0^1 \psi_x^2 dx + c\left(\varepsilon_1' + \frac{1}{\varepsilon_1'}\right) g \circ \psi_x. \end{aligned} \tag{3.23}$$

As in (3.17), it is easy to prove

$$\begin{aligned} &\frac{\gamma\tau_0}{\kappa} \int_0^1 \alpha(x)\, q \int_0^t g'(t-s)\, (\psi(t) - \psi(s))\, ds dx \\ &\leq \varepsilon_1 \int_0^1 q^2 dx - \frac{c}{\varepsilon_1} g' \circ \psi_x. \end{aligned} \tag{3.24}$$

Also, we estimate the first term in the right-hand side of (3.18) as follows:

$$\begin{aligned} &-\frac{\gamma\delta}{\kappa} \int_0^1 \alpha(x)\, q \int_0^t g(t-s)\, (\psi(t) - \psi(s))\, ds dx \\ &\leq \left(\frac{\gamma\delta}{\kappa}\right)^2 \varepsilon_1 \int_0^1 q^2 dx + \frac{c}{\varepsilon_1} g \circ \psi_x \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} &\frac{\gamma\tau_0}{\kappa} \left(\int_0^t g(s)\, ds\right) \int_0^1 \alpha(x)\, q\psi_t dx \\ \leq\ &\left(\int_0^t g(s)\, ds\right) \frac{1}{\varepsilon_1} \int_0^1 q^2 dx + \left(\int_0^t g(s)\, ds\right) c\varepsilon_1 \int_0^1 \psi_t^2 dx. \end{aligned} \tag{3.26}$$

Consequently, by combining all the above estimates (3.15)−(3.26), the assertion of Lemma 3.9 is fulfilled. □

Now, as in [61], let w be the solution of

$$\begin{cases} -w_{xx} = \psi_x, \\ w(0) = w(1) = 0. \end{cases} \tag{3.27}$$

Then, we have the following inequalities:

Lemma 3.10. *The solution of (3.27) satisfies*

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx$$

and

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

Proof. We multiply Equation (3.27) by w, integrate by parts and use the Cauchy-Schwarz inequality to obtain

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx.$$

Next, we differentiate (3.27) with respect to t and by the same procedure, we obtain

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

This completes the proof of Lemma 3.10. □

Let w be the solution of (3.27). We introduce the following functional:

$$I_2(t) := \int_0^1 \left(\rho_2 \psi_t \psi + \rho_1 \varphi_t w - \frac{\gamma \tau_0}{\kappa} \psi q \right) dx. \tag{3.28}$$

Then, we have the following estimate:

Lemma 3.11. *Let $(\varphi, \psi, \theta, q)$ be the solution of (3.1)–(3.3). Assume that **(H1)–(H4)** hold. Then we have, for any $\varepsilon_2 > 0$*

$$\begin{aligned} \frac{dI_2}{dt} \leq & -\left(\bar{b} + \frac{c\mu\varepsilon_2}{2} - 2c\varepsilon_2 - \frac{\delta\gamma\varepsilon_2}{2\kappa} \right) \int_0^1 \psi_x^2 dx + \left(\frac{\rho_1}{2\varepsilon_2} + \frac{\mu}{2\varepsilon_2} \right) \int_0^1 \varphi_t^2 dx \\ & + \left(\rho_2 + \frac{\gamma\tau_0\varepsilon_2}{2\kappa} + \frac{\rho_1\varepsilon_2}{2} \right) \int_0^1 \psi_t^2 dx + \frac{c}{\varepsilon_2} g \circ \psi_x \\ & + \left(\frac{\gamma\tau_0}{2\kappa\varepsilon_2} + \frac{\delta\gamma}{2\kappa\varepsilon_2} \right) \int_0^1 q^2 dx + \frac{1}{2\varepsilon_2} \int_0^1 b(x) h^2(\psi_t) dx. \end{aligned} \tag{3.29}$$

Proof. By taking the derivative of I_2 with respect to t we get

$$\begin{aligned} I_2'(t) &= \int_0^1 \left(\rho_2\psi_{tt}\psi + \rho_2\psi_t^2\right)dx + \int_0^1 \left(\rho_1\varphi_{tt}w + \rho_1\varphi_t w_t\right)dx \\ &\quad -\frac{\gamma\tau_0}{\kappa}\int_0^1 \left(\psi_t q + \psi q_t\right)dx \\ &:= J_1 + J_2 + J_3. \end{aligned} \tag{3.30}$$

Next, using the first and the fourth equations in (3.1) we get

$$\begin{aligned} J_2 + J_3 &= -K\int_0^1 \varphi\psi_x dx + K\int_0^1 w_x^2 dx + \rho_1\int_0^1 \varphi_t w_t dx \\ &\quad -\frac{\gamma\tau_0}{\kappa}\int_0^1 \psi_t q dx + \frac{\delta\gamma}{\kappa}\int_0^1 \psi q dx + \gamma\int_0^1 \psi\theta_x dx. \end{aligned} \tag{3.31}$$

Next, using the second equation in (3.1), we get

$$\begin{aligned} J_1 &= -\bar{b}\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx + \int_0^1 \psi_x\int_0^t g(t-s)\,a(x)\,\psi_x(s)\,ds dx \\ &\quad -K\int_0^1 \psi^2 dx - K\int_0^1 \varphi_x\psi dx - \int_0^1 b(x)\,\psi h(\psi_t)\,dx - \int_0^1 \gamma\psi\theta_x dx. \end{aligned} \tag{3.32}$$

From (3.31), (3.32) and by using Lemma 3.10, we deduce

$$\begin{aligned} I_2'(t) &\leq -\mu\int_0^1 \varphi_t w dx + \rho_1\int_0^1 \varphi_t w_t dx - \frac{\gamma\tau_0}{\kappa}\int_0^1 \psi_t q dx + \frac{\delta\gamma}{\kappa}\int_0^1 \psi q dx \\ &\quad -\bar{b}\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx - \int_0^1 b(x)\,\psi h(\psi_t)\,dx \\ &\quad +\int_0^1 a(x)\,\psi_x\int_0^t g(t-s)\,\psi_x(s)\,ds dx. \end{aligned} \tag{3.33}$$

By exploiting the inequality

$$|ab| \leq \frac{\nu}{2}a^2 + \frac{1}{2\nu}b^2, \qquad a, b \in \mathbb{R},\ \nu > 0,$$

we easily find, for any $\varepsilon_2 > 0$,

$$
\begin{aligned}
I_2'(t) \leq\; & -\bar{b}\int_0^1 \psi_x^2 dx + \frac{\mu}{2}\int_0^1 \left(\frac{1}{\varepsilon_2}\varphi_t^2 + \varepsilon_2 w^2\right) + \frac{\rho_1}{2}\int_0^1 \left(\frac{1}{\varepsilon_2}\varphi_t^2 + \varepsilon_2 w_t^2\right) dx \\
& + \frac{\gamma\tau_0}{2\kappa}\int_0^1 \left(\varepsilon_2\psi_t^2 + \frac{1}{\varepsilon_2}q^2\right) dx + \frac{\delta\gamma}{2\kappa}\int_0^1 \left(\varepsilon_2\psi^2 + \frac{1}{\varepsilon_2}q^2\right) dx \\
& + \rho_2\int_0^1 \psi_t^2 dx - \int_0^1 b(x)\,\psi h(\psi_t)\, dx \\
& + \int_0^1 a(x)\,\psi_x \int_0^t g(t-s)\,\psi_x(s)\, ds dx.
\end{aligned} \tag{3.34}
$$

We now proceed to the evaluation of the last two terms in the right-hand side of (3.34). First, by Young's and Poicaré's inequalities we have

$$
\left|\int_0^1 b(x)\,\psi h(\psi_t)\, dx\right| \leq \varepsilon_2 c\int_0^1 \psi_x^2 dx + \frac{1}{2\varepsilon_2}\int_0^1 b(x)\, h^2(\psi_t)\, dx. \tag{3.35}
$$

Furthermore, by Young's and Cauchy-Schwartz inequalities we have

$$
\left|\int_0^1 a(x)\,\psi_x \int_0^t g(t-s)\,\psi_x(s)\, ds dx\right| \leq \varepsilon_2 c\int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon_2} g\circ\psi_x. \tag{3.36}
$$

Then, plugging (3.35) and (3.36) into (3.34) and using the second inequality in Lemma 3.10, there fore the assertion of Lemma 3.11 holds. □

Now, following [50], we define the functional I_3 as follows:

$$
I_3(t) := \int_0^1 \rho_1\varphi_t\varphi dx + \frac{\mu}{2}\int_0^1 \varphi^2 dx. \tag{3.37}
$$

Then, we have the following estimate:

Lemma 3.12. *Let* $(\varphi, \psi, \theta, q)$ *be the solution of (3.1)–(3.3). Then, for any* $\varepsilon_3 > 0$, *we have*

$$
I_3'(t) \leq \left(\frac{K\varepsilon_3}{2} - K\right)\int_0^1 \varphi_x^2 dx + \frac{K}{2\varepsilon_3}\int_0^1 \psi_x^2 dx + \rho_1\int_0^1 \varphi_t^2 dx. \tag{3.38}
$$

Proof. By exploiting the first equation in (3.1) and using Young's inequality, we get

$$\begin{aligned} I_3'(t) &= \int_0^1 \rho_1\varphi_{tt}\varphi dx + \rho_1 \int_0^1 \varphi_t^2 dx + \mu \int_0^1 \varphi_t\varphi dx \\ &= \int_0^1 K\varphi\left(\varphi_{xx} + \psi_x\right) dx + \rho_1 \int_0^1 \varphi_t^2 dx \\ &= -K \int_0^1 \varphi_x^2 dx + K \int_0^1 \varphi\psi_x dx + \rho_1 \int_0^1 \varphi_t^2 dx \\ &\leq -K \int_0^1 \varphi_x^2 dx + \frac{K}{2} \int_0^1 \left(\varepsilon_3\varphi^2 + \frac{1}{\varepsilon_3}\psi_x^2\right) dx + \rho_1 \int_0^1 \varphi_t^2 dx. \end{aligned}$$

A simple use of Poincaré's inequality completes the proof of Lemma 3.12. □

Now, in order to obtain negative terms of $\int_0^1 \theta^2 dx$ we introduce the following functional:

$$I_4(t) := -\tau_0\rho_3 \int_0^1 q(t,x)\left(\int_0^x \theta(t,y)dy\right) dx. \tag{3.39}$$

Then we have the following estimate:

Lemma 3.13. *Let $(\varphi, \psi, \theta, q)$ be the solution of (3.1)–(3.3). Then, for any $\varepsilon_4 > 0$, we have*

$$\begin{aligned} I_4'(t) \leq & \left(-\rho_3\kappa + \frac{\varepsilon_4\rho_3\delta c}{2}\right) \int_0^1 \theta^2 dx + \frac{\varepsilon_4\tau_0\gamma}{2} \int_0^1 \psi_t^2 dx \\ & + \left(\tau_0\kappa + \frac{\rho_3\delta}{2\varepsilon_4} + \frac{\tau_0\gamma}{2\varepsilon_4}\right) \int_0^1 q^2 dx. \end{aligned} \tag{3.40}$$

Proof. By using the fourth equation in (3.1), we get

$$\begin{aligned} I_4'(t) &= -\rho_3 \int_0^1 \tau_0 q_t \left(\int_0^x \theta dy\right) dx - \tau_0 \int_0^1 q\left(\int_0^x \rho_3\theta_t dy\right) dx \\ &= -\rho_3 \int_0^1 (-\delta q - \kappa\theta_x)\left(\int_0^x \theta dy\right) dx - \tau_0 \int_0^1 q\left(\int_0^x (-\kappa q_x - \gamma\psi_{tx})\, dy\right) dx \\ &= \rho_3\delta \int_0^1 q\left(\int_0^x \theta dy\right) dx + \rho_3\kappa \int_0^1 \theta_x\left(\int_0^x \theta dy\right) dx \\ &\quad + \tau_0\kappa \int_0^1 q\left(\int_0^x q_x dy\right) dx + \tau_0\gamma \int_0^1 q\left(\int_0^x \psi_{tx} dy\right) dx. \end{aligned}$$

That is

$$
\begin{aligned}
I_4'(t) \leq\ & \frac{\rho_3\delta}{2}\int_0^1\left(\varepsilon_4\left(\int_0^x \theta^2 dy\right)^2+\frac{1}{\varepsilon_4}q^2\right)dx-\rho_3\kappa\int_0^1 \theta^2 dx \\
& +\tau_0\kappa\int_0^1 q^2 dx+\frac{\tau_0\gamma}{2}\int_0^1\left(\varepsilon_4\psi_t^2+\frac{1}{\varepsilon_4}q^2\right)dx. \qquad (3.41)
\end{aligned}
$$

Consequently, the assertion of Lemma 3.13 immediately follows. □

Proof of Theorem 3.5. For $N, N_1, N_2 > 0$, we can define an auxiliary functional $\mathcal{F}$ by

$$
\mathcal{F}(t) := NE(t) + N_1 I_1 + N_2 I_2 + I_3 + I_4 \qquad (3.42)
$$

and let $t_0 > 0$, and $g_0(t) = \int_0^t g(s)ds > 0$. By combining (3.8), (3.14), (3.29), (3.38) and (3.41), and by using the inequality

$$
(\varphi_x+\psi)^2 \leq 2\varphi_x^2 + 2\psi^2
$$

and Poincaré's inequality, we arrive at

$$
\begin{aligned}
\frac{d\mathcal{F}(t)}{dt} \leq\ & -N_1\left(\rho_2 g_0-\varepsilon_1\left(\rho_2^2+g_0\right)\right)\int_0^1 (\alpha(x)+b(x))\psi_t^2 dx \\
& +\left(N_2\left(\rho_2+\frac{\gamma\tau_0\varepsilon_2}{2\kappa}+\frac{\rho_1\varepsilon_2}{2}\right)+\frac{\tau_0\gamma\varepsilon_4}{2}\right)\int_0^1 \psi_t^2 dx - N\int_0^1 b(x)\psi_t h(\psi_t)\,dx \\
& +\left(N_2\left(\frac{\rho_1}{2\varepsilon_2}+\frac{\mu}{2\varepsilon_2}\right)+\rho_1-N\mu\right)\int_0^1 \varphi_t^2 dx+\left(N_1\varepsilon_1+\frac{N_2}{2\varepsilon_2}\right)\int_0^1 b(x)h^2(\psi_t)dx \\
& +N_1\left(\rho_2 g_0-\varepsilon_1\left(\rho_2^2+g_0\right)\right)\int_0^1 b(x)\psi_t^2 dx \\
& +\left\{N_1\varepsilon_1'\left(2\bar{b}^2+1+2K^2\right)-N_2\left(\bar{b}-2c\varepsilon_2-\frac{\delta\gamma\varepsilon_2}{2\kappa}\right)+\frac{K}{2\varepsilon_3}\right\}\int_0^1 \psi_x^2 dx \\
& +\left(2N_1\varepsilon_1' K^2+\frac{K\varepsilon_3}{2}-K\right)\int_0^1 \varphi_x^2 dx+\left(-\rho_3\kappa+\frac{\varepsilon_4\rho_3\delta c}{2}\right)\int_0^1 \theta^2 dx \\
& +\left\{cN_1\left(\varepsilon_1+\frac{1}{\varepsilon_1}\right)+cN_1\left(\varepsilon_1'+\frac{1}{\varepsilon_1'}\right)+\frac{N_2 c}{\varepsilon_2}\right\}g\circ\psi_x+\left(\frac{N}{2}-\frac{cN_1}{\varepsilon_1}\right)g'\circ\psi_x \\
& +\left\{N_1\left(c\varepsilon_1+\frac{g_0}{\varepsilon_1}\right)+N_2\left(\frac{\gamma\tau_0}{2\kappa\varepsilon_2}+\frac{\delta\gamma}{2\kappa\varepsilon_2}\right)\right. \\
& \left.+\left(\tau_0\kappa+\frac{\rho_3\delta}{2\varepsilon_4}+\frac{\tau_0\gamma}{2\varepsilon_4}\right)-\delta N\right\}\int_0^1 q^2 dx
\end{aligned}
$$

for all $t \geq t_0$. At this point, we have to choose our constants very carefully. First, let us take $\varepsilon_3 < 1$, $\varepsilon_1, \varepsilon_2$ and ε_4 small enough such that

$$\begin{aligned}\varepsilon_1 &\leq \min\left\{\left(\frac{\rho_2 g_0}{2}\right)/\left(\rho_2^2 + g_0\right), \frac{1}{4K}\right\},\\ \varepsilon_2 &\leq \left(\frac{\bar{b}}{2}\right)/\left(2c + \frac{\delta\gamma}{2\kappa}\right)\end{aligned}$$

and

$$\varepsilon_4 \leq \frac{\kappa}{\delta c}.$$

After that, we pick N_2 large enough so that

$$N_2 \geq \frac{2K\bar{b}}{\varepsilon_3}.$$

Now, by using Lemma 3.2, and choosing N_1 large enough such that

$$\frac{N_1\rho_2 g_0}{2} > \left(N_2\left(\rho_2 + \frac{\gamma\tau_0\varepsilon_2}{2k} + \frac{\rho_1\varepsilon_2}{2}\right) + \frac{\tau_0\gamma\varepsilon_4}{2}\right)\frac{2}{d}$$

then, we can select $\varepsilon_1^{'}$ small enough such that

$$\varepsilon_1^{'} \leq \min\left\{\frac{1}{4N_1K}, \left(\frac{N_2\bar{b}}{4}\right)/N_1\left(2\bar{b}^2 + 1 + 2K^2\right)\right\}. \tag{3.43}$$

Finally, we choose N large enough so that, there exist positive constants η, η_1, and η_2 such that, for $t \geq t_0$,

$$\begin{aligned}\frac{d\mathcal{F}(t)}{dt} \leq\ & -\eta\left\{\int_0^1 (\alpha(x) + b(x))\psi_t^2 dx + \int_0^1 \varphi_t^2 dx\right.\\ & \left.+ \int_0^1 \theta^2 dx + \int_0^1 q^2 dx\right\} - \eta_1\int_0^1 \psi_x^2 dx - \eta_2\int_0^1 \varphi_x^2 dx\\ & + cg\circ\psi_x + c\int_0^1 b(x)\left(\psi_t^2 + h^2(\psi_t)\right)dx.\end{aligned}$$

By the same method as in [50] (see inequality (25) in [50]), we can find $\eta_3 > 0$ such that, for $t \geq t_0$,

$$\begin{aligned}\frac{d\mathcal{F}(t)}{dt} \leq & -\eta_3 \left\{ \int_0^1 (\alpha(x) + b(x)) \psi_t^2 dx + \int_0^1 \varphi_t^2 dx + \int_0^1 \psi_x^2 dx \right. \\ & \left. + \int_0^1 (\varphi_x + \psi)^2 dx + \int_0^1 \theta^2 dx + \int_0^1 q^2 dx \right\} \\ & + cg \circ \psi_x + c \int_0^1 b(x) \left(\psi_t^2 + h^2(\psi_t) \right) dx. \end{aligned} \tag{3.44}$$

Moreover, we have the following: there exist two positive constants β_1 and β_2 depending on N, N_1, N_2, such that

$$\beta_1 E(t) \leq \mathcal{F}(t) \leq \beta_2 E(t), \qquad \forall t \geq 0. \tag{3.45}$$

This can be seen simply from estimate (3.8), (3.13), (3.28), (3.37), (3.40), (3.42), Young's and Poincaré's inequalities, that

$$|\mathcal{F}(t) - NE(t)| \leq CE(t), \qquad \forall t \geq 0.$$

Consequently, we can choose N large enough such that $\beta_1 = N - C > 0$ and (3.44) therefore (3.45) holds true. Our goal now is to estimate the last term in the right-hand side of (3.44). Following the method presented in [26], we consider the following partition of the interval $(0, 1)$:

$$\Omega^+ = \left\{ x \in (0,1) : |\psi_t| > \varepsilon' \right\} \text{ and } \Omega^- = \left\{ x \in (0,1) : |\psi_t| \leq \varepsilon' \right\} \tag{3.46}$$

where ε' is defined in (H2). By using the hypothesis (H2) , we have $|\psi_t| \leq c_1^{-1} \psi_t h(\psi_t)$ on Ω^+ and therefore taking into account the estimate (3.8), we arrive at

$$\begin{aligned}\int_{\Omega^+} b(x) \left(\psi_t^2 + h^2(\psi_t) \right) dx & \leq c \int_{\Omega^+} b(x) \psi_t h(\psi_t) dx \\ & \leq c \int_0^1 b(x) \psi_t h(\psi_t) dx \\ & \leq -cE'(t). \end{aligned} \tag{3.47}$$

According to (H2) , we distinguish two cases:

Case 1: H is linear on $\left[0, \varepsilon'\right]$. Consequently, there exist two positive constants c_1' and c_2' such that $c_1' |s| \leq |h(s)| \leq c_2' |s|$, for all $s \in \mathbb{R}_+$, therefore the above inequality (3.47) holds on $(0, 1)$. Now, from (3.44) and (3.47), we arrive at

$$\begin{aligned} \frac{d}{dt}\left(\mathcal{F}(t) + cE(t)\right) &\leq -cE(t) + cg \circ \psi_x \\ &= -cH_2(E(t)) + cg \circ \psi_x, \qquad \forall t \geq t_0, \end{aligned} \tag{3.48}$$

where the function H_2 is defined by (3.7).

Case 2: $H'(0) = 0$ and $H''(0) > 0$ on $\left[0, \varepsilon'\right]$. Let H^* denote the dual of H in the sense of Young, then we have (see [66] for more details)

$$H^*(s) = s\left(H'\right)^{-1}(s) - H\left[\left(H'\right)^{-1}(s)\right], \qquad \forall s \in \mathbb{R}_+.$$

By using Jensen's inequality, we deduce

$$\begin{aligned} \int_{\Omega^-} b(x)\left(\psi_t^2 + h^2(\psi_t)\right) dx &\leq c\int_{\Omega^-} b(x) H^{-1}(\psi_t h(\psi_t))\, dx \\ &\leq c\int_{\Omega^-} H^{-1}(b(x)\psi_t h(\psi_t))\, dx \\ &\leq cH^{-1}\left(\int_{\Omega^-} b(x)\psi_t h(\psi_t)\, dx\right) \\ &\leq cH^{-1}\left(-cE'(t)\right). \end{aligned} \tag{3.49}$$

Thus, it follows from (3.44), (3.47) and (3.49) that

$$\mathcal{F}'(t) \leq -cE(t) + cH^{-1}\left(-cE'(t)\right) - cE'(t) + cg \circ \psi_x, \qquad \forall t \geq t_0.$$

By using Young's inequality and the fact that

$$H^*(s) \leq s(H')(s),\ E'(t) \leq 0,\ H'' \geq 0,$$

we obtain by the same method as in [26] (we omit the details)

$$H'(\varepsilon_0 E(t))\left(\mathcal{F}'(t) + cE'(t) + c_0 E'(t)\right) \leq -cH_2(E(t)) + cg \circ \psi_x \tag{3.50}$$

where ε_0 is a small positive constant and c_0 is a large positive constant. Now, let us define the following functional:

$$\mathcal{L}(t) = \begin{cases} \mathcal{F}(t) + cE(t) & \text{if } H \text{ is linear on } \left[0, \varepsilon'\right] \\ H'(\varepsilon_0 E(t))\left(\mathcal{F}(t) + cE(t)\right) + c_0 E(t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on } (0, \varepsilon']. \end{cases}$$

We can easily show that

$$\mathcal{L} \sim E.$$

On the other hand, by making use of (3.48) and (3.50), we easily deduce that the following inequality

$$\mathcal{L}^{'}(t) \leq -cH_2(E(t)) + cg \circ \psi_x$$

holds for all $t \geq t_0$. By using (3.8) and (H4), we obtain

$$\begin{aligned} (\xi(t)\mathcal{L}(t))^{'} &= \xi^{'}(t)\mathcal{L}(t) + \xi(t)\mathcal{L}^{'}(t) \\ &\leq -c\xi(t)H_2(E(t)) - cE^{'}(t). \end{aligned}$$

Next, let $\mathcal{K}(t) = \varepsilon(\xi(t)\mathcal{L}(t) + cE(t))$, where $0 < \varepsilon < \bar{\varepsilon}$ and $\bar{\varepsilon}$ is a positive constant satisfying

$$\xi(t)\mathcal{L}(t) + cE(t) \leq \frac{1}{\bar{\varepsilon}}E(t), \quad \forall t \geq 0.$$

We can also show that

$$\mathcal{K} \sim E$$

and, for $t \geq t_0$,

$$\mathcal{K}^{'}(t) \leq -c\varepsilon\xi(t)H_2(\mathcal{K}(t)).$$

A simple integration of the above inequality over (t_0, t) yields

$$\mathcal{K}(t) \leq H_1^{-1}\left(c\varepsilon\int_0^t \xi(s)\,ds + H_1(\mathcal{K}(t_0)) - c\varepsilon\int_0^{t_0}\xi(s)\,ds\right), \forall t \geq t_0,$$

where $H_1(t) = \int_t^1 \left(\frac{1}{H_2(t)}\right)ds$. Since $\lim_{t\to 0^+} H_1(t) = \infty$ and

$$0 \leq \mathcal{K}(t_0) \leq \frac{\varepsilon}{\bar{\varepsilon}}E(t_0) \leq \frac{\varepsilon}{\bar{\varepsilon}}E(0).$$

We may choose ε small enough such that

$$H_1(F(t_0)) - c\varepsilon\int_0^{t_0}\xi(s)\,ds \geq 0.$$

Therefore, $\mathcal{K}(t) \leq H_1^{-1}\left(c\varepsilon\int_0^t \xi(s)\,ds\right)$, for $t \geq t_0$. Consequently, there exist two positive constants $c^{'}$, and $c^{''}$ for which

$$\mathcal{K}(t) \leq c^{''}H_1^{-1}\left(c^{'}\int_0^t \xi(s)\,ds\right), \ \forall t \geq 0,$$

since $\mathcal{K}$ is bounded, which gives (3.6).

This completes the proof of the Theorem 3.5 □

3.3 Examples

In this section, we give some examples to illustrate our results.

Example 3.1. *Let us first assume that the function h has a polynomial growth at the origin, i.e.*

$$c_1^{'} |s|^q \leq |h(s)| \leq c_2^{'} |s|^{\frac{1}{q}}, \quad on \left[-\varepsilon^{'}, \varepsilon^{'}\right]$$

where $c_1^{'}$ and $c_2^{'}$ are two positive constants and $q \geq 1$. As in [26], we obtain the following decay rate of the energy:

$$\begin{cases} E(t) \leq c^{''} e^{-c^{'} \int_0^t \xi(s)ds} & if\, q = 1 \\ E(t) \leq \left(c^{'} \int_0^t \xi(s)\, ds + c^{''}\right)^{-\frac{2}{q-1}} & if\, q > 1 \end{cases}. \tag{3.51}$$

In the next example, and from the general assumptions ($\boldsymbol{H}4$), we obtain several decay rates in which the exponential and polynomial rates are only particular cases.

Example 3.2. *Here we consider some examples of the function g:*

• *Let $a, b, \nu > 0$,*

$$g(t) = ae^{-b(1+t)^{\nu}}.$$

then it's clear that ($\boldsymbol{H}4$) holds for $\xi(t) = b\nu(1+t)^{\min\{0,\, \nu-1\}}$. Consequently, applying the first inequality in (3.51), we obtain the following exponential decay:

$$E(t) \leq c^{''} e^{-c^{'} b(1+t)^{\min\{1,\, \nu\}}}.$$

•*If, for $a, b > 0$ and $\nu > 1$,*

$$g(t) = ae^{-b[\ln(1+t)]^{\nu}},$$

then, for

$$\xi(t) = \frac{b\nu\, [\ln(1+t)]^{\nu-1}}{1+t},$$

the first inequality in (3.51), gives

$$E(t) \leq c^{''} e^{-c^{'} b[\ln(1+t)]^{\nu}}.$$

• *If*

$$g(t) = \frac{a}{(2+t)^{\nu}\left[\ln(2+t)\right]^{b}},$$

where

$$a > 0 \text{ and } \begin{cases} \nu > 1 \text{ and } b \in \mathbb{R} \\ \text{or} \\ \nu = 1 \text{ and } b > 1 \end{cases}.$$

Therefor

$$\xi(t) = \frac{\nu\left(\ln(2+t)\right)+b}{(2+t)\left[\ln(2+t)\right]^{b}},$$

and we obtain from the first inequality in (3.51)

$$E(t) \leq \frac{c''}{\left[(2+t)^{\nu}\left[\ln(2+t)\right]^{b}\right]^{c'}}.$$

Example 3.3. *Let us now suppose that the function h has an exponential growth at the origin, i.e.*

$$h_0(|s|) \leq |h(s)| \leq h_0^{-1}(|s|), \quad \text{on } \left[-\varepsilon', \varepsilon'\right]$$

where $h_0 = (1/s)\, e^{-s^{-\gamma_1}}$ *and* $\gamma_1 > 0$. *Then we get the same decay rate of [26], i.e.*

$$E(t) \leq c'''\left(\ln\left(c'\int_0^t \xi(s)\,ds + c''\right)\right)^{-2/\gamma_1}.$$

Remark 3.14. We can also prove the same decay results for the following boundary conditions:

$$\varphi_x(0, t) = \varphi_x(1, t) = \psi(0,t) = \psi(1,t) = q(0,t) = q(1,t) = 0.$$

Chapter 4

A stability result of a Timoshenko system in thermoelasticity of second sound with a delay term in the internal feedback

4.1 Introduction

This chapter is devoted to the investigation of the effect of time delay on the behavior of the solution to the following system:

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) + \mu_1 \varphi_t(x,t) + \mu_2 \varphi_t(x,t-\tau) = 0, \\ \rho_2 \psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) + \gamma \theta_x(x,t) = 0, \\ \rho_3 \theta_t(x,t) + \kappa q_x(x,t) + \gamma \psi_{tx}(x,t) = 0, \\ \tau_0 q_t(x,t) + \delta q(x,t) + \kappa \theta_x(x,t) = 0, \end{cases} \tag{4.1}$$

where $t \in (0, \infty)$ denotes the time variable, $x \in (0, 1)$ is the space variable, the functions φ and ψ are respectively, the transverse displacement of the solid elastic material and the rotation angle, the function θ is the temperature difference, $q = q(x,t) \in \mathbb{R}$ is the heat flux, and $\rho_1,\ \rho_2,\ \rho_3,\ \gamma,\ \tau_0,\ \delta,\ \kappa,\ \mu_1,\ \mu_2$ and K are positive constants and $\tau > 0$ represents the time delay. We consider the following initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), & \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \\ \theta(x,0) = \theta_0(x), & q(x,0) = q_0(x), \quad \varphi_t(x,t-\tau) = f_0(x,t-\tau), \end{cases} \tag{4.2}$$

where $x \in (0,1)$ and $t \in (0,\tau)$. We consider the boundary conditions

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = q(0,t) = q(1,t) = 0, \qquad \forall t \geq 0. \tag{4.3}$$

Our main interest in this work is to prove the existence and the asymptotic behavior of the solution to problem (4.1)–(4.3). Before going on, let us first review some related results which seem to us interesting.

Questions related to stability/instability of wave equations with delay have attracted considerable attention in recent years and many authors have shown that delays can destabilize a system that is asymptotically stable in the absence of delays (see [17] for more details).

As it has been proved by Datko [16, Example 3.5], systems of the form

$$\begin{cases} w_{tt} - w_{xx} - aw_{xxt} = 0, & x \in (0,1),\ t > 0, \\ w(0,t) = 0, \qquad w_x(1,t) = -kw_t(1,t-\tau), & t > 0, \end{cases}$$

where a, k and τ are positive constants become unstable for any arbitrarily small values of τ and any values of a and k.

Nicaise and Pignotti [67] examined the problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \\ u(x,-t) = u_0(x,t), \qquad u_t(x,0) = u_1(x), & x \in \Omega,\ t \geq 0 \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega,\ t \in (0,\tau). \end{cases} \tag{4.4}$$

Using an observability inequality obtained with a Carleman estimate, they proved that, under the assumption

$$\mu_2 < \mu_1, \tag{4.5}$$

the energy is exponentially stable. On the contrary, if (4.5) does not hold, they found a sequence of delays for which the corresponding solution of (4.4) is unstable. The same results were shown if both the damping and the delay act in the boundary of the domain.

Said-Houari and Laskri [76] considered the following Timoshenko system with a delay term in the internal feedback:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \mu_1\psi_t(x,t) + \mu_2\psi_t(x,t-\tau) = 0. \end{cases} \tag{4.6}$$

Under the assumption $\mu_1 \geq \mu_2$ on the weights of the two feedbacks, they proved the well-posedness of the system. They also established, for $\mu_1 > \mu_2$ an exponential decay result for the case of equal-speed wave propagation, i.e.

$$\frac{K}{\rho_1} = \frac{b}{\rho_2}. \tag{4.7}$$

The work in [76] has been extended to the case of time-varying delay of the form $\psi_t(x, t-\tau(t))$ by Kirane, Said-Houari and Anwar [38]. First, by using the variable norm technique of Kato, and under some restriction on the parameters μ_1, μ_2 and on the delay function $\tau(t)$, the system has been shown to be well-posed. Second, under a hypothesis between the weight of the delay term in the feedback, the weight of the term without delay and the wave speeds, an exponential decay result of the total energy has been proved.

As a consequence of what we have said before, the following questions naturally arise:

- Condition (4.7) is significant only from the mathematical point of view since in practice the velocities of waves propagations are always different, see [45] for more details. It is well known that this condition can be avoided by considering two linear damping terms of the form φ_t and ψ_t in the left-hand side in the first and the second equation in system (4.6) (with $\mu_1 = \mu_2 = 0$). This result has been shown in [75] and others. But is it possible to get an exponential decay result when the damping is weak and without the assumption (4.7)?

- In [50], the authors have shown that the heat conduction given by Cattaneo's law together with a linear damping term of the form $\mu\varphi_t$ acting on the first equation in yield an exponential stability of the total energy, without the assumption (4.7).

Under what condition system (4.1) remains exponentially stable independently of (4.7) when a delay term in the feedback is considered?

One of the main purpose of this chapter is to answer the above two questions.

The plan of the chapter is as follows. In Section 4.2, and under the assumption $\mu_1 \geq \mu_2$, we prove the global existence of the solution of problem (4.1). While Section 4.3 is devoted to the exponential decay result for $\mu_1 > \mu_2$. We make use of the energy method to build a Lyapunov functional that leads to the desired result.

4.2 Well-posedness of the problem

In order to prove the well-posedness result, we proceed as in [67], see also [69, 76]. Let us introduce the following new dependent variable:

$$z(x,\rho,t) = \varphi_t(x, t-\tau\rho), \ \ x \in (0,1), \ \ \rho \in (0,1), t > 0.$$

Then, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \ \ (x,\rho,t) \ \text{in} \ (0,1)\times(0,1)\times(0,\infty).$$

Therefore, problem (4.1) is equivalent to

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) + \mu_1\varphi_t(x,t) + \mu_2 z(x,1,t) = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x+\psi) + \gamma\theta_x = 0 \\ \rho_3\theta_t + \kappa q_x + \gamma\psi_{tx} = 0 \\ \tau_0 q_t + \delta q + \kappa\theta_x = 0 \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \end{cases} \tag{4.8}$$

where $x \in (0,1), \rho \in (0,1)$, and $t > 0$. The above system subjected to the following initial conditions:

$$\begin{cases} \varphi(x,0) = \varphi_0(x), & \varphi_t(x,0) = \varphi_1(x), \ \ x \in (0,1) \\ \psi(x,0) = \psi_0(x), & \psi_t(x,0) = \psi_1(x), \ \ x \in (0,1) \\ \theta(x,0) = \theta_0(x), & q(x,0) = q_0(x), \ \ x \in (0,1) \\ z(x,0,t) = \varphi_t(x,t), & x \in (0,1), \ \ t > 0 \\ z(x,1,t) = f_0(x,t-\tau), & (x,t) \in (0,1)\times(0,\tau). \end{cases} \tag{4.9}$$

and boundary conditions

$$\varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=q(0,t)=q(1,t)=0,\quad \forall t\geq 0 \tag{4.10}$$

The main question to be asked here is whether problem (4.8)−(4.10) is well posed. Our main goal In this section is to give a positive answer to this question. In other words, we give the sufficient conditions that guarantee the well-posedness of problem (4.8)−(4.10). To prove this, we adapt the steps used in the recent paper [76] in which a Thimoshenko problem with a frictional damping has been investigated. In order to use the semigroup approach, we rewrite system (4.8)−(4.10) as a first-order system. To this end, let $U=(\varphi,\varphi_t,\psi,\psi_t,\theta,q,z)^T$ and rewrite (4.8)−(4.10) as

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = U_0 = (\varphi_0,\varphi_1,\psi_0,\psi_{1,}\theta,q,f_0(.,-.\tau))^T, \end{cases} \tag{4.11}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} \varphi \\ u \\ \psi \\ v \\ \theta \\ q \\ z \end{pmatrix} = \begin{pmatrix} u \\ K/\rho_1(\varphi_{xx}+\psi_x) - \mu_1/\rho_1 u - \mu_2/\rho_1 z(.,1) \\ v \\ b/\rho_2\psi_{xx} - K/\rho_2(\varphi_x+\psi) - \gamma/\rho_2\theta_x \\ -\kappa/\rho_3 q_x - \gamma/\rho_3 v_x \\ -\delta/\tau_0 q - \kappa/\tau_0\theta_x \\ -\frac{1}{\tau}z_\rho \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \left\{(\varphi,u,\psi,v,\theta,q,z)^T \in H : z_\rho \in L^2\left((0,1);L^2(0,1)\right),\ u\equiv z(.,0),\ \text{in}\ (0,1)\right\}, \tag{4.12}$$

where

$$\begin{aligned} H\ :\ &= \left(H^2(0,1)\cap H_0^1(0,1)\right)\times H_0^1(0,1)\times\left(H^2(0,1)\cap H_0^1(0,1)\right)\times H_0^1(0,1) \\ &\times H^1(0,1)\times H_0^1(0,1)\times L^2\left((0,1);L^2(0,1)\right). \end{aligned}$$

Now, the energy space

$$\begin{aligned}\mathcal{H} \; : \; &= H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times L^2(0,1) \\ &\times L^2(0,1) \times L^2\Big((0,1)\,; L^2(0,1)\Big).\end{aligned}$$

For $U = (\varphi, u, \psi, v, \theta, q, z)^T$, $\bar{U} = \left(\bar{\varphi}, \bar{u}, \bar{\psi}, \bar{v}, \bar{\theta}, \bar{q}, \bar{z}\right)^T$ and for ξ a positive constant satisfying

$$\tau\mu_2 \leq \xi \leq \tau(2\mu_1 - \mu_2), \tag{4.13}$$

we define the following inner product in $\mathcal{H}$:

$$\begin{aligned}\langle U, \bar{U}\rangle_{\mathcal{H}} &= \int_0^1 \left\{\rho_1 u\bar{u} + \rho_2 v\bar{v} + K(\varphi_x + \psi)(\bar{\varphi}_x + \bar{\psi}) + b\psi_x\bar{\psi}_x + \rho_3\theta\bar{\theta}\right\} dx \\ &+ \int_0^1 \tau_0 q\bar{q}dx + \xi \int_0^1 \int_0^1 z(x,\rho)\,\bar{z}(x,\rho)\,d\rho dx.\end{aligned}$$

Our existence and uniqueness result reads as follows.

Theorem 4.1. *Assume that $\mu_2 \leq \mu_1$, then for any $U_0 \in \mathcal{H}$, there exists a unique solution $U \in C([0,\infty), \mathcal{H})$ of problem (4.8)–(4.10). Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty), \mathcal{H}).$$

Proof. In order to prove Theorem 4.1, we use the semigroup approach. That is, we show that the operator $\mathcal{A}$ generates a C_0-semigroup in $\mathcal{H}$. In this step, we prove that the operator $\mathcal{A}$ is dissipative. For $U = (\varphi, u, \psi, v, \theta, q, z)^T \in D(\mathcal{A})$, we have

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= -\delta \int_0^1 q^2 dx - \mu_1 \int_0^1 u^2(x)\,dx - \mu_2 \int_0^1 z(x,1)\,u(x)\,dx \\ &\quad -\frac{\xi}{\tau} \int_0^1 \int_0^1 z(x,\rho)\,z_\rho(x,\rho)\,d\rho dx.\end{aligned} \tag{4.14}$$

Looking now at the last term of the right-hand side of Eq. (4.14), we have

$$\begin{aligned}\int_0^1 \int_0^1 z(x,\rho)\,z_\rho(x,\rho)\,d\rho dx &= \int_0^1 \int_0^1 \frac{1}{2}\frac{\partial}{\partial\rho} z^2(x,\rho)\,d\rho dx \\ &= \frac{1}{2} \int_0^1 \left\{z^2(x,1) - z^2(x,0)\right\} dx.\end{aligned} \tag{4.15}$$

Consequently, (4.14) becomes

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= -\delta\int_0^1 q^2 dx - \mu_1\int_0^1 u^2(x)\,dx - \mu_2\int_0^1 z(x,1)\,u(x)\,dx \\ &\quad -\frac{\xi}{2\tau}\int_0^1 z^2(x,1)\,dx + \frac{\xi}{2\tau}\int_0^1 u^2(x)\,dx. \end{aligned}\tag{4.16}$$

By using Young's inequality we obtain, from (4.16),

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &\leq -\delta\int_0^1 q^2 dx + \left(-\mu_1 + \frac{\mu_2}{2} + \frac{\xi}{2\tau}\right)\int_0^1 u^2(x)\,dx \\ &\quad + \left(\frac{\mu_2}{2} - \frac{\xi}{2\tau}\right)\int_0^1 z^2(x,1)\,dx.\end{aligned}$$

Keeping in mind condition (4.13), we observe that

$$-\mu_1 + \frac{\mu_2}{2} + \frac{\xi}{2\tau} \leq 0,\ \frac{\mu_2}{2} - \frac{\xi}{2\tau} \leq 0.$$

Consequently, the operator $\mathcal{A}$ is dissipative.

Now, we will prove that the operator $\lambda I - \mathcal{A}$ is surjective for $\lambda > 0$. For this purpose, let $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7) \in \mathcal{H}$, we seek $U = (\varphi, u, \psi, v, \theta, q, z)^T \in D(\mathcal{A})$, solution to the equation

$$\lambda U - \mathscr{A}U = F \tag{4.17}$$

or equivalently

$$\begin{cases} \lambda\varphi - u = f_1, \\ \lambda u - \dfrac{K}{\rho_1}(\varphi_{xx} + \psi_x) + \dfrac{\mu_1}{\rho_1}u + \dfrac{\mu_2}{\rho_1}z(.,1) = f_2, \\ \lambda\psi - v = f_3, \\ \lambda v - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x + \psi) + \dfrac{\gamma}{\rho_2}\theta_x = f_4, \\ \lambda\theta + \dfrac{\kappa}{\rho_3}q_x + \dfrac{\gamma}{\rho_3}v_x = f_5, \\ \lambda q + \dfrac{\delta}{\tau_0}q + \dfrac{\kappa}{\tau_0}\theta_x = f_6, \\ \lambda z + \dfrac{1}{\tau}z_\rho = f_7. \end{cases}\tag{4.18}$$

Suppose that we have found φ and ψ with the appropriate regularity. Therefore, the first and the third equations in (4.18) yields

$$\begin{cases} u = \lambda\varphi - f_1, \\ v = \lambda\psi - f_3. \end{cases} \tag{4.19}$$

It is clear that $u \in H_0^1(0,1)$ and $v \in H_0^1(0,1)$. Furthermore, we can find z as

$$z(x,0) = u(x), \text{ for } x \in (0,1). \tag{4.20}$$

Following the same approach as in [67] we obtain, by using the last equation in (4.18),

$$z(x,\rho) = u(x)e^{-\lambda\rho\tau} + \tau e^{-\lambda\rho\tau}\int_0^\rho f_7(x,\sigma)e^{\lambda\sigma\tau}d\sigma.$$

From (4.19), we obtain

$$z(x,\rho) = \lambda\varphi(x)e^{-\lambda\rho\tau} - f_1 e^{-\lambda\rho\tau} + \tau e^{-\lambda\rho\tau}\int_0^\rho f_7(x,\sigma)e^{\lambda\sigma\tau}d\sigma. \tag{4.21}$$

From (4.21), we have

$$z(x,1) = \lambda\varphi(x)e^{-\lambda\tau} + z_0(x),$$

where $x \in (0,1)$ and

$$z_0(x) = -f_1 e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^\rho f_7(x,\sigma)e^{\lambda\sigma\tau}d\sigma. \tag{4.22}$$

It is clear from the above formula that z_0 depends only on f_1 and f_1.

By using (4.18) and (4.19) the functions φ, ψ, θ and q satisfy the system

$$\begin{cases} \left(\lambda^2 + \frac{\mu_1}{\rho_1}\lambda + \lambda e^{-\lambda\tau}\frac{\mu_2}{\rho_1}\right)\varphi - \frac{K}{\rho_1}(\varphi_{xx} + \psi_x) = f_2 + \left(\lambda + \frac{\mu_1}{\rho_1}\right)f_1 - \frac{\mu_2}{\rho_1}z_0(x), \\ \lambda^2\psi - \frac{b}{\rho_2}\psi_{xx} + \frac{K}{\rho_2}(\varphi_x + \psi) + \frac{\gamma}{\rho_2}\theta_x = f_4 + \lambda f_3, \\ \lambda\theta + \frac{\kappa}{\rho_3}q_x + \frac{\gamma\lambda}{\rho_3}\psi_x = f_5 + \frac{\gamma}{\rho_3}f_{3x}, \\ \lambda q + \frac{\delta}{\tau_0}q + \frac{\kappa}{\tau_0}\theta_x = f_6. \end{cases} \tag{4.23}$$

Solving system (4.23) is equivalent to find $(\varphi,\psi,\theta,q)\in\left(H^2(0,1)\cap H_0^1(0,1)\right)\times\left(H^2(0,1)\cap H_0^1(0,1)\right)$ $\times H^1(0,1)\times H_0^1(0,1)$ such that

$$\begin{cases}\int_0^1\left(\left(\lambda^2\rho_1+\mu_1\lambda+\lambda e^{-\lambda\tau}\mu_2\right)\varphi w+K(\varphi_x+\psi)w_x\right)dx=\int_0^1(\rho_1 f_2+(\lambda\rho_1+\mu_1)f_1-\mu_2 z_0(x))\,wdx,\\ \int_0^1\left(\rho_2\lambda^2\psi\chi+b\psi_x\chi_x+K(\varphi_x+\psi)\chi+\gamma\theta_x\chi\right)dx=\int_0^1\rho_2(f_4+\lambda f_3)\chi dx,\\ \int_0^1(\rho_3\lambda\theta w_1+\kappa q_x w_1+\gamma\lambda\psi_x w_1)\,dx=\int_0^1(\rho_3 f_5+\gamma f_{3x})\,w_1 dx,\\ \int_0^1((\tau_0\lambda+\delta)\,q\chi_1+\kappa\theta_x\chi_1)dx=\int_0^1\tau_0 f_6\chi_1 dx,\end{cases}\tag{4.24}$$

for all $(w,\chi,w_1,\chi_1)\in H_0^1(0,1)\times H_0^1(0,1)\times H^1(0,1)\times H_0^1(0,1)$.

Consequently, problem (4.24) is equivalent to the equation

$$\zeta\left((\varphi,\psi,\theta,q),(w,\chi,w_1,\chi_1)\right)=l(w,\chi,w_1,\chi_1),\tag{4.25}$$

where the bilinear from $\zeta:\left(H_0^1(0,1)\times H_0^1(0,1)\times H^1(0,1)\times H_0^1(0,1)\right)^2\to\mathbb{R}$ and the linear form $l:H_0^1(0,1)\times H_0^1(0,1)\times H^1(0,1)\times H_0^1(0,1)\to\mathbb{R}$ are defined by

$$\begin{aligned}\zeta\left((\varphi,\psi,\theta,q),(w,\chi,w_1,\chi_1)\right) &= \int_0^1\left(\left(\lambda^2\rho_1+\mu_1\lambda+\lambda e^{-\lambda\tau}\mu_2\right)\varphi w+K(\varphi_x+\psi)(w_x+\chi)\right)dx\\ &\quad+\int_0^1\left(\rho_2\lambda^2\psi\chi+b\psi_x\chi_x+\gamma\theta_x w_{1x}\right)dx\\ &\quad+\int_0^1(\rho_3\lambda\theta w_1+\kappa q_x\chi_{1x}+\gamma\lambda\psi_x\chi_x)\,dx\\ &\quad+\int_0^1((\tau_0\lambda+\delta)\,q\chi_1+\kappa\theta_x\chi_{1x})\,dx\end{aligned}$$

and

$$\begin{aligned}l(w,\chi,w_1,\chi_1) &= \int_0^1(\rho_1 f_2+(\rho_1\lambda+\mu_1)f_1-\mu_2 z_0(x))\,wdx\\ &\quad+\int_0^1\rho_2(f_4+\lambda f_3)\chi dx+\int_0^1(\rho_3 f_5+\gamma f_{3x})\,w_1 dx\\ &\quad+\int_0^1\tau_0 f_6\chi_1 dx,\end{aligned}\tag{4.26}$$

where $z_0(x)$ satisfies the equation in (4.22).

It is easy to verify that ζ is continuous and coercive, and l is continuous, so, applying the Lax-Milgram theorem, we deduce that, for all $(w,\chi,w_1,\chi_1) \in H_0^1(0,1)\times H_0^1(0,1)\times H^1(0,1)\times H_0^1(0,1)$, problem (4.25) admits a unique solution $(\varphi,\psi,\theta,q) \in H_0^1(0,1)\times H_0^1(0,1)\times H^1(0,1)\times H_0^1(0,1)$. Applying the classical elliptic regularity, it follows from (4.24) that $(\varphi,\psi,\theta,q) \in H^2(0,1)\times H^2(0,1)\times H^1(0,1)\times H_0^1(0,1)$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. Consequently, the result of Theorem 4.1 follows from the Hille-Yosida theorem. □

4.3 Exponential stability for $\mu_1 > \mu_2$

In this section, we show that, under the assumption $\mu_1 > \mu_2$, the solution of problem (4.8)–(4.10) decay exponentially, independently of the wave speed assumption. To achieve our goal we use the energy method to produce a suitable Lyapunov functional which leads to an exponential decay result.

In order to use the Poincaré inequality for θ, we introduce, as in [73],

$$\bar{\theta}(x,t) = \theta(x,t) - \int_0^1 \theta_0(x)\,dx.$$

Then by the third equation in (4.1) we easily verify that

$$\int_0^1 \bar{\theta}(x,t)\,dx = 0,$$

for all $t \geq 0$. In this case the Poincaré inequality is applicable for $\bar{\theta}$. On the other hand $\left(\varphi,\psi,\bar{\theta},q,z\right)$ satisfies the same system (4.8) and the boundary conditions (4.10). For ξ satisfying

$$\tau\mu_2 < \xi < \tau(2\mu_1 - \mu_2), \tag{4.27}$$

we define the functional energy of the solution of problem (4.8)–(4.10) as

$$\begin{aligned} E(t) &= E(t,z,\varphi,\psi,\theta,q) \\ &= \frac{1}{2}\int_0^1 \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right)dx + \frac{1}{2}\int_0^1 \left\{K(\varphi_x+\psi)^2 + b\psi_x^2 + \rho_3\theta^2\right\}dx \\ &\quad + \frac{1}{2}\int_0^1 \tau_0 q^2\,dx + \frac{\xi}{2}\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho\,dx. \end{aligned} \tag{4.28}$$

We multiply the first equation in (4.8) by φ_t, the second equation by ψ_t, the third by θ, and the fourth by q, use integration by parts to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\left(\rho_1\varphi_t^2+\rho_2\psi_t^2\right)dx+\frac{1}{2}\frac{d}{dt}\int_0^1\left\{K\left(\varphi_x+\psi\right)^2+b\psi_x^2+\rho_3\theta^2+\tau_0 q^2\right\}dx\\
=\;& -\delta\int_0^1 q^2dx-\mu_1\int_0^1\varphi_t^2\left(x,t\right)dx-\mu_2\int_0^1\varphi_t\left(x,t\right)z\left(x,1,t\right)dx. \qquad (4.29)
\end{aligned}$$

Now, multiplying the last equation in (4.8) by $(\xi/\tau)\,z$, and integrating the result over $(0,1)\times(0,1)$ with respect to ρ and x, we obtain

$$\begin{aligned}
&\frac{\xi}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2\left(x,\rho,t\right)d\rho dx\\
=\;& -\frac{\xi}{\tau}\int_0^1\int_0^1 zz_\rho\left(x,\rho,t\right)d\rho dx=-\frac{\xi}{2\tau}\int_0^1\int_0^1\frac{\partial}{\partial\rho}z^2\left(x,\rho,t\right)d\rho dx\\
=\;& \frac{\xi}{2\tau}\int_0^1\left(z^2\left(x,0,t\right)-z^2\left(x,1,t\right)\right)dx. \qquad (4.30)
\end{aligned}$$

From (4.28), (4.29) and (4.30), we get

$$\begin{aligned}
\frac{dE\left(t\right)}{dt} =\;& -\delta\int_0^1 q^2dx-\left(\mu_1-\frac{\xi}{2\tau}\right)\int_0^1\varphi_t^2\left(x,t\right)dx\\
&-\frac{\xi}{2\tau}\int_0^1 z^2\left(x,1,t\right)dx-\mu_2\int_0^1\varphi_t\left(x,t\right)z\left(x,1,t\right)dx. \qquad (4.31)
\end{aligned}$$

Now, using Young's inequality, (4.31) can be rewritten as

$$\begin{aligned}
\frac{dE\left(t\right)}{dt} \leq\;& -\delta\int_0^1 q^2dx-\left(\mu_1-\frac{\xi}{2\tau}-\frac{\mu_2}{2}\right)\int_0^1\varphi_t^2\left(x,t\right)dx\\
&-\left(\frac{\xi}{2\tau}-\frac{\mu_2}{2}\right)\int_0^1 z^2\left(x,1,t\right)dx.
\end{aligned}$$

Then, by using (4.27), there exists a positive constant C such that

$$\frac{dE\left(t\right)}{dt}\leq-\delta\int_0^1 q^2dx-C\left\{\int_0^1\varphi_t^2\left(x,t\right)dx+\int_0^1 z^2\left(x,1,t\right)dx\right\}. \qquad (4.32)$$

Then we obtain E is a non-increasing function.

Let us now state our main result.

Theorem 4.2. *Assume that $\mu_2 < \mu_1$. Then there exist two positive constants C and γ independent of t such that, for any solution of problem (4.8)–(4.10), we have*

$$E(t) \leq Ce^{-\gamma t}, \quad \forall t \geq 0. \tag{4.33}$$

Remark 4.3. The exponential result in Theorem 4.2 holds without any assumption on the wave speeds of the first and the second equations in (4.1). In the absence of the delay term and linear damping term, i.e. $\mu_1 = \mu_2 = 0$ in (4.1), it has been proved by Fernandez Sare and Racke [18] that the assumption on the wave speeds of the first and the second equations does not yield an exponential stability when the heat conduction is given by the Cattaneo law. However, it has been proved recently that there is a new number depending on τ_0 that leads to an exponential stability of the system. See [80] for more details. On the other hand, the heat conduction given by Fourier's law stabilizes the whole system exponentially when the wave speeds are the same, see [61] for more details.

Remark 4.4. It is an interesting open problem to look whether or not the heat conduction is strong enough to stabilize system (4.1) (at least polynomially) in the case when $\mu_2 \geq \mu_1$.

To derive the exponential decay of the solution, it is enough to construct a functional $\mathcal{L}$, equivalent to the energy E, satisfying

$$\frac{d\mathcal{L}(t)}{dt} \leq -\Lambda\mathcal{L}(t), \quad \forall t \geq 0$$

where Λ is a positive constant.

In order to obtain such a functional $\mathcal{L}$, we need several Lemmas. Let us define the following functional:

$$I_1(t) := \int_0^1 \rho_1\varphi_t\varphi dx + \frac{\mu_1}{2}\int_0^1 \varphi^2 dx, \tag{4.34}$$

then we have the following estimate:

Lemma 4.5. *Let* $(\varphi, \psi, \theta, q, z)$ *be the solution of (4.8)–(4.10), then, for any* $\varepsilon_1 > 0$*, we have*

$$\begin{aligned} \frac{dI_1}{dt} \leq & \left(-K + \varepsilon_1 \left(\frac{K}{2} + \frac{\mu_2 c}{2}\right)\right) \int_0^1 \varphi_x^2 dx + \frac{K}{2\varepsilon_1} \int_0^1 \psi_x^2 dx \\ & + \frac{\mu_2}{2\varepsilon_1} \int_0^1 z^2 (x, 1, t)\, dx + \rho_1 \int_0^1 \varphi_t^2 dx, \end{aligned} \tag{4.35}$$

where $c = 1/\pi^2$ *is the Poincaré constant.*

Proof. Differentiating I_1 with respect to t, we conclude

$$\frac{dI_1}{dt} = \int_0^1 \rho_1 \varphi_{tt} \varphi dx + \rho_1 \int_0^1 \varphi_t^2 dx + \mu_1 \int_0^1 \varphi \varphi_t dx. \tag{4.36}$$

Then, by using the first equation in (4.8), we find

$$\frac{dI_1}{dt} = K \int_0^1 (\varphi_x + \psi)_x \varphi dx - \mu_2 \int_0^1 \varphi z (x, 1, t)\, dx + \rho_1 \int_0^1 \varphi_t^2 dx.$$

Consequently, we arrive at

$$\frac{dI_1}{dt} = -K \int_0^1 (\varphi_x + \psi) \varphi_x dx - \mu_2 \int_0^1 \varphi z (x, 1, t)\, dx + \rho_1 \int_0^1 \varphi_t^2 dx.$$

Applying Young's inequality and Poincaré's inequality, we find (4.35). □

Now, let w be the solution of

$$-w_{xx} = \psi_x, w(0) = w(1) = 0. \tag{4.37}$$

Then we get

$$w(x,t) = -\int_0^x \psi(y,t)\, dy + x \left(\int_0^1 \psi(y,t)\, dy \right).$$

We have the following inequalities:

Lemma 4.6. *The solution of (4.37) satisfies*

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx$$

and

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

Proof. See of Lemma 3.10. □

We introduce the following functional:

$$I_2(t) := \int_0^1 \left(\rho_2 \psi_t \psi + \rho_1 \varphi_t w - \frac{\gamma \tau_0}{\kappa} \psi q \right) dx. \tag{4.38}$$

Then we have the following estimate:

Lemma 4.7. *Let* $(\varphi, \psi, \theta, q, z)$ *be the solution of* (4.8)–(4.10). *Then we have, for any* $\varepsilon_2 > 0$,

$$\begin{aligned} \frac{dI_2(t)}{dt} \leq & \left(-b + \frac{c\mu_1\varepsilon_2}{2} + \frac{c\mu_2\varepsilon_2}{2} + \frac{\delta\gamma\varepsilon_2 c}{2\kappa} \right) \int_0^1 \psi_x^2 dx + \frac{\mu_2}{2\varepsilon_2} \int_0^1 z^2(x,1,t) \\ & + \left(\rho_2 + \frac{\gamma\tau_0\varepsilon_2}{2\kappa} + \frac{\rho_1\varepsilon_2}{2} \right) \int_0^1 \psi_t^2 dx + \left(\frac{\mu_1}{2\varepsilon_2} + \frac{\rho_1}{2\varepsilon_2} \right) \int_0^1 \varphi_t^2 dx \\ & + \left(\frac{\gamma\tau_0}{2\kappa\varepsilon_2} + \frac{\delta\gamma}{2\kappa\varepsilon_2} \right) \int_0^1 q^2 dx. \end{aligned} \tag{4.39}$$

Proof. By taking the derivative of (4.38), we conclude

$$\begin{aligned} \frac{dI_2(t)}{dt} = & -b \int_0^1 \psi_x^2 dx + K \int_0^1 \varphi \psi_x dx - K \int_0^1 \psi^2 dx + \rho_2 \int_0^1 \psi_t^2 dx - K \int_0^1 \varphi_x w_x dx \\ & -K \int_0^1 \psi w_x dx - \mu_1 \int_0^1 \varphi_t w dx - \mu_2 \int_0^1 z(x,1,t) w dx + \rho_1 \int_0^1 \varphi_t w_t dx \\ & -\frac{\gamma\tau_0}{\kappa} \int_0^1 \psi_t q dx + \frac{\delta\gamma}{\kappa} \int_0^1 \psi q dx. \end{aligned}$$

By using (4.37) and the first inequality in Lemma 4.6, we get

$$\begin{aligned} \frac{dI_2(t)}{dt} \leq & -b \int_0^1 \psi_x^2 dx + K \int_0^1 \varphi \psi_x dx - K \int_0^1 \psi^2 dx + \rho_2 \int_0^1 \psi_t^2 dx + K \int_0^1 \varphi_x \psi dx \\ & +K \int_0^1 \psi^2 dx - \mu_1 \int_0^1 \varphi_t w dx - \mu_2 \int_0^1 z(x,1,t) w dx + \rho_1 \int_0^1 \varphi_t w_t dx \\ & -\frac{\gamma\tau_0}{\kappa} \int_0^1 \psi_t q dx + \frac{\delta\gamma}{\kappa} \int_0^1 \psi q dx. \end{aligned}$$

We apply Young's inequality and Poincaré's inequality and using the inequalities in Lemma 4.7, we find (4.39). □

Now, let us introduce the following functional, used by [76]:

$$I_3(t) := \int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\, d\rho dx. \tag{4.40}$$

Then the following result holds:

Lemma 4.8. *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (4.8)–(4.10), then we have*

$$\frac{dI_3(t)}{dt} \leq -I_3(t) - \frac{c_1}{2\tau}\int_0^1 z^2(x,1,t)\,dx + \frac{1}{2\tau}\int_0^1 \psi_t^2(x,t)\,dx, \tag{4.41}$$

where c_1 is a positive constant.

Proof. Differentiating (4.40) with respect to t and using the last equation in (4.8), we have

$$\begin{aligned}
\frac{d}{dt}\left(\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho dx\right) &= -\frac{1}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho}zz_\rho(x,\rho,t)\,d\rho dx \\
&= -\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho dx \\
&\quad -\frac{1}{2\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau\rho}z^2(x,\rho,t)\right)d\rho dx
\end{aligned}$$

The above formula implies that there exists a positive constant c_1 such that (4.41) holds. □

In order to obtain a negative term of $\int_0^1 \psi_t^2 dx$, we introduce, the following functional used by [50]:

$$I_4(t) := \rho_2\rho_3\int_0^1\left(\int_0^x \theta(t,y)dy\right)\psi_t(t,x)\,dx. \tag{4.42}$$

Then we have the following estimate:

Lemma 4.9. *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (4.8)–(4.10). Then, for any $\varepsilon_4, \varepsilon_4^{'} > 0$, we have*

$$\begin{aligned}
\frac{d}{dt}I_4(t) \leq\ & \left(-\gamma\rho_2 + \frac{\varepsilon_4\rho_2\kappa}{2}\right)\int_0^1 \psi_t^2 dx + \left(\frac{\varepsilon_4^{'}\rho_3}{2}(b+\kappa c)\right)\int_0^1 \psi_x^2 dx \\
&+\frac{\varepsilon_4^{'}\kappa\rho_3 c}{2}\int_0^1 \varphi_x^2 dx + \left(\gamma\rho_3 + \frac{\rho_3}{2\varepsilon_4^{'}}(b+2\kappa)\right)\int_0^1 \theta^2 dx \\
&+\frac{\rho_2\kappa}{2\varepsilon_4}\int_0^1 q^2 dx.
\end{aligned} \tag{4.43}$$

Proof. Differentiating (4.42) and using the third equation in (4.8), we have

$$\begin{aligned}
\frac{d}{dt}I_4(t) &= \int_0^1 \left(\int_0^x \rho_3\theta_t dy\right)\rho_2\psi_t dx + \int_0^1 \left(\int_0^x \rho_3\theta dy\right)\rho_2\psi_{tt} dx \\
&= -\int_0^1 \left(\int_0^x (\kappa q_x + \gamma\psi_{tx})\, dy\right)\rho_2\psi_t dx \\
&\quad + \int_0^1 \left(\int_0^x \rho_3\theta dy\right)(b\psi_{xx} - \kappa(\varphi_x + \psi) - \gamma\theta_x)\, dx, \\
&= -\gamma\rho_2 \int_0^1 \psi_t^2 dx - \rho_2\kappa \int_0^1 q\psi_t dx - b\rho_3 \int_0^1 \theta\psi_x dx \\
&\quad + \kappa\rho_3 \int_0^1 \theta\varphi dx - \kappa\rho_3 \int_0^1 \left(\int_0^x \theta dy\right)\psi dx + \gamma\rho_3 \int_0^1 \theta^2 dx.
\end{aligned}$$

By using Young's inequality, we obtain

$$\begin{aligned}
\frac{d}{dt}I_4(t) &\leq -\gamma\rho_2 \int_0^1 \psi_t^2 dx + \frac{\kappa\rho_3}{2}\int_0^1 \left(\varepsilon_4'\psi^2 + \frac{1}{\varepsilon_4'}\left(\int_0^x \theta dy\right)^2\right) dx \\
&\quad + \frac{\rho_2\kappa}{2}\int_0^1 \left(\varepsilon_4\psi_t^2 + \frac{1}{\varepsilon_4}q^2\right) dx + \frac{b\rho_3}{2}\int_0^1 \left(\varepsilon_4'\psi_x^2 + \frac{1}{\varepsilon_4'}\theta^2\right) dx \\
&\quad + \gamma\rho_3 \int_0^1 \theta^2 dx,
\end{aligned}$$

consequently, we find (4.43). □

Now, in order to obtain negative terms of $\int_0^1 \theta^2 dx$ we introduce the following functionals, used by [50]:

$$I_5(t) := -\tau_0\rho_3 \int_0^1 q(t,x)\left(\int_0^x \theta(t,y)dy\right) dx. \tag{4.44}$$

Then we have the following estimate:

Lemma 4.10. *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (4.8)–(4.10). Then, for any $\varepsilon_5, \varepsilon_5' > 0$, we have*

$$\begin{aligned}
\frac{dI_5(t)}{dt} &\leq \left(-\rho_3\kappa + \frac{\varepsilon_5\rho_3\delta c}{2}\right)\int_0^1 \theta^2 dx + \frac{\varepsilon_5'\tau_0\gamma}{2}\int_0^1 \psi_t^2 dx \\
&\quad + \left(\tau_0\kappa + \frac{\rho_3\delta}{2\varepsilon_5} + \frac{\tau_0\gamma}{2\varepsilon_5'}\right)\int_0^1 q^2 dx.
\end{aligned} \tag{4.45}$$

Proof. By using the fourth equation in (4.8), we get

$$
\begin{aligned}
\frac{dI_5(t)}{dt} &= -\rho_3 \int_0^1 \tau_0 q_t \left(\int_0^x \theta dy \right) dx - \tau_0 \int_0^1 q \left(\int_0^x \rho_3 \theta_t dy \right) dx \\
&= -\rho_3 \int_0^1 (-\delta q - \kappa \theta_x) \left(\int_0^x \theta dy \right) dx - \tau_0 \int_0^1 q \left(\int_0^x -\kappa q_x - \gamma \psi_{tx} dy \right) dx \\
&= \rho_3 \delta \int_0^1 q \left(\int_0^x \theta dy \right) dx + \rho_3 \kappa \int_0^1 \theta_x \left(\int_0^x \theta dy \right) dx \\
&\quad + \tau_0 \kappa \int_0^1 q \left(\int_0^x q_x dy \right) dx + \tau_0 \gamma \int_0^1 q \left(\int_0^x \psi_{tx} dy \right) dx.
\end{aligned}
$$

By using Young's inequality, we get

$$
\begin{aligned}
\frac{dI_5(t)}{dt} &\le \frac{\rho_3 \delta}{2} \int_0^1 \left(\varepsilon_5 \left(\int_0^x \theta dy \right)^2 + \frac{1}{\varepsilon_5} q^2 \right) dx - \rho_3 \kappa \int_0^1 \theta^2 dx \\
&\quad + \tau_0 \kappa \int_0^1 q^2 dx + \frac{\tau_0 \gamma}{2} \int_0^1 \left(\varepsilon_5' \psi_t^2 + \frac{1}{\varepsilon_5'} q^2 \right) dx.
\end{aligned}
\tag{4.46}
$$

Consequently, the assertion of Lemma (4.45) immediately follows. □

Proof of Theorem 4.2. To prove Theorem 4.2, we can define, for $N, N_2, N_4, N_5 > 0$, an auxiliary functional $\mathcal{L}$ by

$$
\mathcal{L}(t) := NE(t) + I_1(t) + N_2 I_2(t) + I_3(t) + N_4 I_4(t) + N_5 I_5(t). \tag{4.47}
$$

By combining (4.32), (4.35), (4.39), (4.41), (4.43), and (4.45), we get

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) &\le \left[\frac{K}{2\varepsilon_1} + N_2 \left(-b + \frac{c\mu_1 \varepsilon_2}{2} + \frac{c\mu_2 \varepsilon_2}{2} + \frac{\delta \gamma \varepsilon_2 c}{2\kappa} \right) + N_4 \left(\frac{\varepsilon_4' \rho_3}{2} (b + \kappa c) \right) \right] \int_0^1 \psi_x^2 dx \\
&\quad + \left\{ -K + \varepsilon_1 \left(\frac{K}{2} + \frac{\mu_2 c}{2} \right) + N_4 \frac{\varepsilon_4' \kappa \rho_3 c}{2} \right\} \int_0^1 \varphi_x^2 dx - I_3(t) \\
&\quad + \left[-NC + \frac{\mu_2}{2\varepsilon_1} + N_2 \frac{\mu_2}{2\varepsilon_1} - \frac{c_1}{2\tau} \right] \int_0^1 z^2(x, 1, t) dx \\
&\quad + \left[-NC + N_2 \left(\frac{\mu_1}{2\varepsilon_2} + \frac{\rho_1}{2\varepsilon_2} \right) + \rho_1 \right] \int_0^1 \varphi_t^2 dx \\
&\quad + \left[N_2 \left(\rho_2 + \frac{\gamma \tau_0 \varepsilon_2}{2\kappa} + \frac{\rho_1 \varepsilon_2}{2} \right) + \frac{1}{2\tau} + N_4 \left(-\gamma \rho_2 + \frac{\varepsilon_4 \rho_2 \kappa}{2} \right) + N_5 \frac{\varepsilon_5' \tau_0 \gamma}{2} \right] \int_0^1 \psi_t^2 dx \\
&\quad + \left[-N\delta + N_2 \left(\frac{\gamma \tau_0}{2\kappa \varepsilon_2} + \frac{\delta \gamma}{2\kappa \varepsilon_2} \right) + N_4 \frac{\rho_2 \kappa}{2\varepsilon_4} + N_5 \left(\tau_0 \kappa + \frac{\rho_3 \delta}{2\varepsilon_5} + \frac{\tau_0 \gamma}{2\varepsilon_5'} \right) \right] \int_0^1 q^2 dx \\
&\quad + \left[N_4 \left(\gamma \rho_3 + \frac{\rho_3}{2\varepsilon_4'} (b + 2\kappa) \right) + N_5 \left(-\rho_3 \kappa + \frac{\varepsilon_5 \rho_3 \delta c}{2} \right) \right] \int_0^1 \theta^2 dx.
\end{aligned}
\tag{4.48}
$$

At this point, we have to choose our constants very carefully. First, choosing ε_1, ε_2, ε_4 and ε_5 small enough such that

$$\begin{aligned} \varepsilon_2\left(\frac{c\mu_1}{2}+\frac{c\mu_2}{2}+\frac{\delta\gamma c}{2\kappa}\right) &\leq \frac{b}{2},\ \varepsilon_1\left(\frac{K}{2}+\frac{\mu_2 c}{2}\right)\leq\frac{K}{2},\\ \varepsilon_4 &\leq \frac{\gamma}{\kappa},\ \varepsilon_5\leq\frac{\kappa}{\delta c}. \end{aligned}$$

After that, we can choose N_2 large enough such that

$$N_2 \geq \frac{2K}{b\varepsilon_1}.$$

Moreover, we pick N_4 large enough so that

$$N_4\frac{\gamma\rho_2}{4} \geq N_2\left(\rho_2+\frac{\gamma\tau_0\varepsilon_2}{2\kappa}+\frac{\rho_1\varepsilon_2}{2}\right)+\frac{1}{2\tau}.$$

Once N_2 and N_4 are fixed, we take $\varepsilon_4^{'}$ small enough such that

$$\varepsilon_4^{'} \leq \min\left\{\frac{N_2 b}{4N_4\rho_3\left(b+\kappa c\right)},\frac{K}{2N_4\kappa\rho_3 c}\right\}.$$

Next, let N_5 be large enough such that

$$\frac{N_5\rho_3\kappa}{4} \geq N_4\left(\gamma\rho_3+\frac{\rho_3}{2\varepsilon_4^{'}}\left(b+2\kappa\right)\right).$$

After that we fix $\varepsilon_5^{'}$ small enough such that

$$\varepsilon_5^{'} \leq \frac{N_4\gamma\rho_2}{4N_5\tau_0\gamma}.$$

Finally, once all the above constants are fixed, we choose N large enough such that

$$\begin{cases} \dfrac{CN}{2} \geq \max\left\{\dfrac{\mu_2}{2\varepsilon_1}+N_2\dfrac{\mu_2}{2\varepsilon_1},N_2\left(\dfrac{\mu_1}{2\varepsilon_2}+\dfrac{\rho_1}{2\varepsilon_2}\right)+\rho_1\right\},\\ \dfrac{N\delta}{2} \geq N_2\left(\dfrac{\gamma\tau_0}{2\kappa\varepsilon_2}+\dfrac{\delta\gamma}{2\kappa\varepsilon_2}\right)+N_4\dfrac{\rho_2\kappa}{2\varepsilon_4}+N_5\left(\tau_0\kappa+\dfrac{\rho_3\delta}{2\varepsilon_5}+\dfrac{\tau_0\gamma}{2\varepsilon_5^{'}}\right). \end{cases}$$

Consequently, there exists a positive constant η_1 such that (4.48) becomes

$$\frac{d}{dt}\mathcal{L}(t) \leq -\eta_1\int_0^1(\psi_t^2+\psi_x^2+\varphi_t^2+(\varphi_x+\psi)^2+\theta^2+q^2)dx-\eta_1\int_0^1\int_0^1 z^2\left(x,\rho,t\right)d\rho dx, \quad (4.49)$$

which implies by (4.28) that there exists also η_2, such that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\eta_2 E(t), \quad \forall t \geq 0. \tag{4.50}$$

Moreover, we have the following:

Lemma 4.11. *For N large enough, there exists two positive constants β_1 and β_2 depending on $N, N_1, N_2, N_4, N_5, \varepsilon_1, \varepsilon_2, \varepsilon_4, \varepsilon_4^{'}, \varepsilon_5$ and $\varepsilon_5^{'}$ such that*

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t), \quad \forall t \geq 0. \tag{4.51}$$

Proof. We consider the functional

$$H(t) = I_1(t) + N_2 I_2(t) + I_3(t) + N_4 I_4(t) + N_5 I_5(t)$$

and show that

$$|H(t)| \leq CE(t), \quad C > 0.$$

From (4.34), (4.38), (4.40), (4.42) and (4.44) we obtain

$$\begin{aligned} |H(t)| \leq & \left|\int_0^1 \rho_1\varphi_t\varphi dx + \frac{\mu_1}{2}\int_0^1 \varphi^2 dx\right| + N_2\left|\int_0^1 \left(\rho_2\psi_t\psi + \rho_1\varphi_t w - \frac{\gamma\tau_0}{\kappa}\psi q\right)dx\right| \\ & + \left|\int_0^1\int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\,d\rho dx\right| + N_4\left|\rho_2\rho_3\int_0^1\left(\int_0^x \theta(t,y)dy\right)\psi_t(t,x)\,dx\right| \\ & + N_5\left|-\tau_0\rho_3\int_0^1 q(t,x)\left(\int_0^x \theta(t,y)dy\right)dx\right|. \end{aligned}$$

By using, the trivial relation

$$\int_0^1 \varphi^2 dx \leq 2c\int_0^1 (\varphi_x + \psi)^2\,dx + 2c\int_0^1 \psi_x^2 dx,$$

Young's and Poincaré's inequalities, we get

$$\begin{aligned} |H(t)| \leq & \ \alpha_1\int_0^1 \varphi_t^2 dx + \alpha_2\int_0^1 \psi_t^2 dx + \alpha_3\int_0^1 (\varphi_x + \psi)^2\,dx + \alpha_4\int_0^1 \psi_x^2 dx + \alpha_5\int_0^1 \theta^2 dx \\ & + \alpha_6\int_0^1 q^2 dx + \int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx, \end{aligned} \tag{4.52}$$

where the positive constants $\alpha_1, ..., \alpha_6$ are determined as follows:

$$\begin{cases} \alpha_1 := \frac{1}{2}\left(\rho_1 + N_2\rho_1\right), \\ \alpha_2 := \frac{1}{2}\left(N_2\rho_2 + N_4\rho_2\rho_3\right), \\ \alpha_3 = \rho_1 c, \\ \alpha_4 := \frac{1}{2}\left(\frac{N_2\gamma\tau_0 c}{\kappa} + N_2\rho_1 c^2 + N_2\rho_2 c\right), \\ \alpha_5 := \frac{1}{2}\left(N_4\rho_2\rho_3 c + N_5\tau_0\rho_3 c\right), \\ \alpha_6 := \frac{1}{2}\left(N_2\frac{\gamma\tau_0}{\kappa} + N_5\tau_0\rho_3\right). \end{cases}$$

According to (4.52), we have

$$|H(t)| \leq \widehat{C}E(t),$$

for

$$\widehat{C} = \frac{\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}}{\min\{\rho_1, \rho_2, \rho_3, K, b, \kappa, \gamma, \delta, \tau_0\}}$$

Therefor we obtain

$$|\mathcal{L}(t) - NE(t)| \leq \widehat{C}E(t).$$

So, we can choose N large enough so that $\beta_1 = N - \widehat{C} > 0$. Then (4.51) holds true for $\beta_2 = N + \widehat{C} > 0$, this concludes the proof of the Lemma. □

Combining now (4.50) and (4.51), we conclude that there exists some $\Lambda > 0$ such that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\Lambda\mathcal{L}(t), \quad \forall t \geq 0. \tag{4.53}$$

A simple integration of (4.53) leads to.

$$\mathcal{L}(t) \leq \mathcal{L}(0)\, e^{-\Lambda t}, \quad \forall t \geq 0. \tag{4.54}$$

Again, the use of (4.51) and (4.54) yields the desired result (4.33). This completes the proof of theorem 4.2. □

Conclusion

In this thesis we study nonexistence/existence and asymptotic behavior of solutions of some nonlinear hyperbolic systems. My goal in the future it is the study exponential decay for $\mu = 0$ at the problem studied in chapter 3, and in chapter 4 it is an interesting open problem to look whether or not the heat conduction is strong enough to stabilize system (at least polynomially) in the case when $\mu_1 \leq \mu_2$.

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