

وزارة التعليم العالي والبحث العلمي

Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

BADJI MOKHTAR-ANNABA

UNIVERSITY

UNIVERSITE BADJI MOKHTAR
ANNABA



جامعة باجي مختار
- عنابة -

Faculté des Sciences

Département de Mathématiques

Année : 2023/2024



THÈSE

Présentée en vue de l'obtention du diplôme de Doctorat

**RÉGULARISATION ET APPROXIMATION NUMÉRIQUE D'UNE CLASSE DE
PROBLÈMES INVERSES DE TYPE FRACTIONNAIRE**

Filière
Mathématiques

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Mathématiques et Applications

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ANNABA



جامعة باجي مختار

- عنابة -

Faculty of Sciences

Departement of Mathematics

Year : 2023/2024



THESIS

Presented with a view to obtaining the doctorate degree

**REGULARIZATION AND NUMERICAL APPROXIMATION OF A CLASS OF
INVERSE PROBLEMS OF FRACTIONAL TYPE**

Stream

Mathematics

Speciality

Mathematics and Applications

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ملخص

تركز هذه الأطروحة على بعض المشاكل العكسية النابعة من المعادلات التفاضلية الجزئية الكسرية من النوع شبه المكافئ الذي يتضمن حدود بمتغيرات منحرفة. هذا البحث يطور استراتيجيات التعديل التي تنطبق بشكل جيد و منسجم على هذه الفئة من المشاكل المصنفة على أنها سيئة الطرح و معتلة بمهوم هادمار. تشمل هذه الدراسة المساهمات النظرية والأساليب الحسابية العملية المبنية على اساس متين و دقيق، مما يضمن أن النتائج المحصل عليها صارمة رياضياً وقابلة للتطبيق على اشكالات واقعية. تقدم الأفكار التي تم تطويرها في هذا النهج من البحث كل جانب بالتفصيل وتقدم فهما متعمقا للمشكلة التي تمت معالجتها، وتقترح خيارات مسبقة ولاحقة لوسيط التعديل، لتنفيذ مخططات عددية فعالة وناجعة.

الكلمات المفتاحية المسائل العكسية المعتلة، طرق التعديل شبه الانعكاسية، الطرق التكرارية، المعادلات التطورية الكسرية، المعادلات شبه القطعية، المتغيرات المنحرفة

Résumé

Cette thèse met l'accent sur quelques problèmes inverses engendrés par des EDPs fractionnaires de type pseudo-parabolique intervenant des termes d'involution (arguments déviés). Elle développe des stratégies de régularisation qui s'appliquent bien pour cette catégorie de problèmes qualifiés mal-posés au sens de Hadamard. Les contributions incluent des avancées théoriques et des méthodes de calculs pratiques, garantissant que les résultats sont à la fois rigoureux sur le plan mathématique et applicables à des problèmes réels. Plus précisément, on propose d'étendre des stratégies de régularisation (méthode de quasi-réversibilité modifiée, méthode QBV, méthode de Landweber) pour une classe de problèmes fractionnaires en temps non classiques.

Les idées développées dans cet axe de recherche explorent en détail chaque aspect et offrent une compréhension approfondie de la problématique abordée, et proposent des choix a priori et a posteriori du paramètre de régularisation, pour mettre en œuvre des schémas numériques efficaces et robustes.

MOTS CLÉS : *Problèmes mal posés, problèmes inverses, régularisation, méthodes itératives, méthode de quasi-réversibilité, équation pseudo-parabolique, involution, problème fractionnaire.*

Abstract

This thesis focuses on some inverse problems generated by pseudo-parabolic fractional PDEs involving involution terms (deviated arguments). It develops regularization strategies that apply well to this class of problems called ill-posed in the Hadamard sense. Contributions include theoretical advances and practical computational methods, ensuring that results are both mathematically rigorous and applicable to real problems. More precisely, we propose to extend regularization strategies (modified quasi-reversibility method, QBV method, Landweber method) to a class of non-classical fractional time problems. The ideas developed in this research axis explore in detail each aspect and provide a deep understanding of the problem addressed, and propose a priori and a posteriori choices of the regularization parameter, to implement efficient and robust numerical schemes.

KEYWORDS: *Ill-posed problems, inverse problems, regularization, iterative methods, quasi-reversibility method, pseudoparabolic equation, involution, fractional problems.*

Thanks

Avant tout je remercie **ALLAH** le tout puissant qui m'a donné la volonté, le courage, la force et la patience pour réaliser ce travail.

J'exprime toute ma reconnaissance à mes directeurs de thèse : **Mr. Boussetila Nadjib** (Prof. U. Guelma) et **Mr. Saker Hacene** (Prof. UBM. Annaba) pour leur aide, leur soutien, leur patience, leur compréhension, leurs conseils ainsi que la confiance qu'ils m'ont fait en acceptant de diriger mes recherches.

Je présente mes sincères remerciements à **Mr. Chorfi Lahcène** (Prof. UBM. Annaba) qui m'a fait l'honneur de présider le jury, ainsi que **Mme. Rebbani Faouzia** (Prof. ENSTI Annaba), **Mme. Zouyed Fairouz** (Prof. UBM. Annaba), **Mr. Fernane Khairedine** (Prof. U. Guelma) et **Mme. Atmania Rahima** (Prof. UBM. Annaba), pour avoir accepté de faire partie du jury et d'y avoir consacré une partie de leurs temps.

Je tiens également à remercier vivement **Mr. Meziani Sief Eddine** (MCA. ENSET Skikda) et **Mr. Lakhdari Abdelghani** (MCA. ENSTI Annaba) qui m'ont aidé beaucoup par leur soutien scientifique.

Je remercie chaleureusement tous les membres du Laboratoire LMA de l'UBM Annaba, la composante administrative du département de mathématiques, ainsi que le staff de la faculté des sciences, pour toute l'aide qui m'a été accordée.

Une grande reconnaissance et un grand remerciement à tous mes enseignants qui ont participé à ma formation, surtout les enseignants de Master et les enseignants de la formation doctorale de l'UBM Annaba.

Je remercie aussi ma famille, mes amis. Enfin, un grand merci à toutes les personnes qui ont contribué de près ou de loin à la réalisation de cette thèse

Dedication

Je dédie cette thèse à ceux qui m'ont appris la volonté et qui m'ont appris qu'une raison de succès vaut la raison du succès, aux personnes dont je suis sûr qu'elles seront plus heureuses et plus honorées que moi par ce travail.

To my parents and my family, for their love, support, and encouragement

Il reste toujours un peu de parfum à la main qui donne des roses.

CONFUCIUS (551 avant J.C., 479 avant J.C.) Philosophe chinois

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INTRODUCTION

Research review

PUBLICATIONS

The work of this thesis was the subject of two international publications: the first article which summarizes Chapter 2 is published in a **Q1** journal (Web of Science JCR 2023: 1.35), and the second paper which focuses on the ideas of Chapter 3 is accepted for publication in a **Q3** journal (SJR 2023 : 0.6).

1. Fares Benabbes, Nadjib Boussetila, Abdelghani Lakhdari, *The modified fractional-order quasi-reversibility method for a class of direct and inverse problems governed by time-fractional heat equations with involution perturbation*. Mathematical Methods in the Applied Sciences (2024).

DGRSDT Indexation[A]: No. 9459, MATHEMATICAL METHODS IN THE APPLIED SCIENCES WILEY, ISSN 0170-4214 eISSN 1099-1476.

2. Fares Benabbes, Nadjib Boussetila, Abdelghani Lakhdari, *Two regularization methods for a class of inverse fractional pseudo-parabolic equations with Involution perturbation* (accepted for publication in Fractional Differential Calculus Journal 2024).

DGRSDT Indexation[SCOPUS]: No. 94599076 Fractional Differential Calculus Element D.O.O. eISSN 18479677.

PRESENTATIONS AND COMMUNICATIONS

1. A variant of quasi-reversibility method for a class of relaxed heat equations with involution perturbation. Journées Jeunes Chercheurs en Mathématiques et Applications (JJCMA21), UBM. Annaba 2021.

2. Iterative regularization method for an inverse fractional pseudo-parabolic problem with involution. Journées Jeunes Chercheurs en Mathématiques et Applications (JJCMA22), UBM. Annaba 2022.

3. A variant of quasi-reversibility method for a class of fractional heat equations with involution perturbation. 1st International workshop on Applied Mathematics (1st-IWAM2022), December 6-8, 2022, University of Constantine.

The modified fractional-order quasi-reversibility method for a class of direct and inverse problems governed by time-fractional heat equations with involution perturbation

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Communicated by: N. Hyvonen

Funding information

No funding was received for this work.

This study aims to explore two classes of ill-posed problems governed by a non-classical fractional heat equation with an involution perturbation. To achieve a stable solution, we introduce a modified pseudo-parabolic regularization method that involves a correction term through a mixed fractional derivation operator, resulting in a sequence of well-posed problems that depend on a regularization parameter ε . We rigorously demonstrate that the resulting approximate problems are well-posed and present convergence results for this regularization approach.

KEYWORDS

fractional heat equation, ill-posed problems, involution perturbation, pseudo-parabolic regularization method

MSC CLASSIFICATION

35K20, 35R11, 35R25, 35R30, 47A52

1 | INTRODUCTION

1.1 | Direct problem for time-fractional heat equation with involution

We denote by $H = L^2((-1, 1), \mathbb{R})$ the Hilbert space equipped with the inner product $\langle u, v \rangle$ induced by the usual norm $\|\cdot\|$. That is,

$$\langle u, v \rangle := \int_{-1}^1 u(x)v(x)dx, \quad \|u\|^2 := \int_{-1}^1 |u(x)|^2 dx.$$

We consider the following nonclassical fractional heat problem in the rectangle $Q = (-1, 1) \times (0, T)$:

$$\begin{cases} D_t^\gamma u(x, t) - \alpha u_{xx}(x, t) - \beta u_{xx}(-x, t) = 0, & x \in (-1, 1), t \in (0, T), \\ u(-1, t) = u(1, t), u_x(1, t) = u_x(-1, t), & t \in (0, T), \\ u(x, 0) = f(x), & x \in (-1, 1), \end{cases} \quad (1)$$

where α is a positive real number, $\beta \in \mathbb{R}$ and D_t^γ is the Caputo derivative, defined for $0 < \gamma < 1$, by

$$D_t^\gamma u(x, t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{u_\tau(x, \tau)}{(t-\tau)^\gamma} d\tau, \quad 0 < \gamma < 1.$$

TWO REGULARIZATION METHODS FOR A CLASS OF INVERSE FRACTIONAL PSEUDO-PARABOLIC EQUATIONS WITH INVOLUTION PERTURBATION

FARES BENABBES, NADJIB BOUSSETILA*
AND ABDELGHANI LAKHDARI

(Communicated by R. Ashurov)

Abstract. In this study, we provide a theoretical analysis of an inverse problem governed by a time-fractional pseudo-parabolic equation with involution. The problem is characterized as ill-posed, meaning that the solution (if it exists) does not depend continuously on the measurable data. To address the inherent instability of this problem, we introduce two regularization strategies: the first employs a modified quasi-boundary value method, and the second utilizes a variant of the quasi-reversibility technique. We present convergence results under an a priori bound assumption and propose a practical a posteriori parameter selection rule.

1. Introduction

The study of inverse problems plays a pivotal role in the mathematical modeling of physical and engineering processes, where one seeks to determine unknown parameters or initial conditions based on observed data. Nevertheless, most inverse problems are inherently ill-posed, meaning that they lack unique solutions or are sensitive to small changes in data. This instability poses challenges in obtaining accurate and reliable solutions. To address this, regularization techniques are employed to introduce additional information and constraints, mitigating ill-posedness, and enhancing solution stability. Notable works such as Tikhonov's regularization, introduced by Tikhonov and Arsenin [44], and the truncated singular value decomposition (TSVD) method, as described by Hansen [21], exemplify classical regularization techniques. These techniques are indispensable for obtaining meaningful solutions in the presence of noise and limited data, making regularization a crucial component in solving ill-posed inverse problems across diverse domains. However, the landscape has seen the development of newer and more advanced methods that offer enhanced efficacy and performance in handling the intricacies of ill-posed inverse problems.

Mathematics subject classification (2020): 35R30, 35R25, 47A52, 35R11.

Keywords and phrases: Inverse problem, pseudo-parabolic problem, involution perturbation, modified quasi-boundary-value method, quasireversibility method.

* Corresponding author.

Thesis theme

Partial differential equations (PDEs) are fundamental to the mathematical modeling of numerous physical phenomena, ranging from fluid dynamics and electromagnetism to quantum mechanics and general relativity. Unlike ordinary differential equations (ODEs), which involve functions of a single variable, PDEs involve multivariable functions and their partial derivatives. This complexity allows PDEs to describe the behavior of systems influenced by multiple factors and varying in multiple dimensions.

A PDE typically requires not only the equation itself but also appropriate boundary and initial conditions to uniquely determine a solution. The nature of these conditions and the domain on which the PDE is defined significantly affect the characteristics of the solution. Classic examples of PDEs include the heat equation, describing the distribution of heat over time; the wave equation, modeling the propagation of waves; and the Laplace equation, essential in potential theory and steady-state problems.

The study of PDEs encompasses both theoretical and numerical approaches. Theoretically, it involves proving the existence, uniqueness, and stability of solutions. Numerically, it involves developing efficient algorithms to approximate solutions, often necessitating sophisticated computational techniques due to the complexity of the equations.

The rich interplay between theory and application makes PDEs a central topic in mathematics, with ongoing research continually uncovering new insights and applications.

In 1923, the French mathematician J. Hadamard wrote his famous book on partial differential equations and their physical significance [42]. This work marked the beginning of the concept of a well-posed problem in mathematical physics. A problem is said to be well-posed if its solution exists, is unique, and depends continuously on the data (stability). Initially, it was believed that problems not meeting Hadamard's conditions had no practical value and could not correctly model a physical phenomenon. However, the current understanding is quite different. There are numerous problems for which at least one of the three Hadamard conditions is not satisfied; these problems are termed ill-posed. Generally, the most significant challenge with such problems is instability, meaning that a slight perturbation in the data can lead to a significant change in the solution.

Ill-posed problems occur in many fields of science and technology, such as geophysics, non-destructive testing, corrosion, medical imaging (ultrasound, scanners, X-rays, etc.), energy (calculating oil flow in a reservoir with wells), chemistry (determining reaction con-

stants), radar and underwater acoustics (determining the shape of an obstacle), image processing (restoring blurred images), and other practical areas.

The general methods of mathematical analysis have been well adapted to solving well-posed problems. However, it was not clear in what sense ill-posed problems could have solutions. Several mathematicians, such as Tikhonov, John, Lavrentiev, Ivanov, and others, worked to develop the theory and methods to solve ill-posed problems. They were able to provide a precise mathematical definition of "approximate solutions" for a fairly large class of problems. Today, these problems constitute a very rich research area filled with mathematical questions. For more details on the study of ill-posed problems, one can refer to the two excellent books by D. Colton, H.W. Engel, A.K. Louis [24] and H.W. Engl, M. Hanke, A. Neubauer [34].

Among the situations that lead to an ill-posed problem, one can mention the problem of determining the past state of a physical system described by a differential equation from its present state, or that of determining the parameters of a system from experimental data. In both cases, these are referred to as inverse problems (see [35, 48, 55, 75]).

According to J.B. Keller [52], two problems are said to be inverses of each other if the formulation of one involves the other. A more operational definition is that an inverse problem consists of determining the causes of a phenomenon from the observation of its effects. Thus, this problem is the inverse of the so-called direct problem, which consists of seeking the effects from known causes. For example, locating the origin of an earthquake from measurements made by several seismographic stations distributed across the surface of the Earth is an inverse problem.

Inverse problems encompass a variety of categories. Examples include determining unknown boundary or initial conditions when direct measurement is not practically feasible, and estimating intrinsic parameters of a system from partial knowledge of its state. Additionally, some inverse problems involve identifying the geometry or domain where the phenomenon occurs. These problems are inherently diverse and find applications across numerous fields, including electromagnetics, geophysics, medical imaging, crack detection, non-destructive testing, structural mechanics, and more. The wide range of applications underscores the importance of developing robust methods to address the challenges posed by inverse problems.

From the definition of an inverse problem, it is clear that these problems may pose particular difficulties. Indeed, it is reasonable to demand that a direct problem be well-

posed: "the same causes produce the same effects". Conversely, it is easy to imagine that the same effects may result from different causes. This illustrates a difficulty in the study of inverse problems: they may have multiple solutions, requiring additional information to distinguish between them.

Another major difficulty in the study of inverse problems is the necessity of a thorough understanding of the associated direct problems. When identifying or calculating a physical quantity from observations (measurements), one is often led to invert an operator (the resolvent that gives the solution of the direct problem). This inversion is generally unstable and requires special treatment. These are called regularization techniques, which aim to make the problem well-posed and make its numerical implementation feasible by slightly perturbing it to eliminate the elements responsible for instability.

In mathematics, regularization is a procedure that consists of replacing an ill-posed problem with another problem that is close to it (in a certain sense) and that has good properties (well-posed), making its theoretical and numerical study easier.

In the mathematical literature, several regularization methods have been introduced and used to solve certain ill-posed Cauchy problems. Among them, we can cite:

- The quasi-reversibility method, introduced by Lattes and Lions (1969) [58], which consists of transforming the ill-posed second-order Cauchy problem into a well-posed higher-order (fourth-order) differential problem by perturbing the operator coefficient of the equation. This method has subsequently been used by several authors to solve the Cauchy problem, including Klivanov and Santosa [56] and more recently Bourgeois [13].
- The modified quasi-reversibility method, which was introduced by Gasjewski and developed by several authors, including N. Boussetila and F. Rebbani [15].
- Tikhonov's regularization method [95], the oldest regularization method, which consists of transforming the original ill-posed problem into a minimization problem.
- The iterative method of Kozlov et al. [57], based on an iterative procedure. It consists of solving an alternating sequence of well-posed problems with mixed boundary conditions until a certain stopping criterion is satisfied. The approximate solution converges, for compatible data, to the solution of the considered Cauchy problem.

- The regularization method by non-local conditions, also known as the "Quasi-Boundary-Value Method", introduced by Abdulkrimov [1]. The idea behind this method is to replace the ill-posed problem with a well-posed problem, in which the final condition is perturbed by replacing it with a non-local condition depending on a small parameter. It has been used by several authors, such as D.N. Hào [44] and Samarskii [87].

In the literature, many models have been proposed to describe the transport mechanism (diffusion) in homogeneous and nonhomogeneous media. At the microscopic level, diffusion is related to the random motion of individual particles. Following the work of Albert Einstein [33], the assumption that particle motion is a Gaussian process has been interpreted by using the Laplace operator and the first-order derivative in the canonical diffusion model. One major drawback of the Gaussian model (Fourier's laws) is that it does not adequately describe the diffusion phenomenon in materials with memory, such as viscoelastic materials, and heterogeneous media, such as soil, heterogeneous aquifers, and groundwater flow. Over the past two decades, a vast body of literature has shown that abnormal diffusion models in which the mean quadratic variance increases more rapidly (super-diffusion) or more slowly (sub-diffusion) than in a Gaussian process under certain circumstances can provide good modeling for physics and practical applications (see [50]).

In biological contexts, experimental results suggest that classical diffusion is not the best description in the case of complex biophysical transport. Instead, it has been demonstrated that abnormal diffusion occurs in various circumstances, potentially caused by underlying mechanisms such as active transport, macromolecular crowding in a tortuous extracellular or intracellular environment, where the geometry of the medium is complex. Recently, a fractional equation was introduced to describe diffusion in special types of porous media with fractal geometry, to extend the use of the Bloch equation to a wide range of experimental situations in nuclear magnetic resonance (NMR) (see [66]). In the same work, the author showed that the fractional-order wave equation governs the propagation of mechanical waves in viscoelastic media characterized by simple deformation and can provide a well-suited model for describing dynamic movements that occur in biological tissues.

Therefore, considerable interest has been devoted to fractional calculus over the past few decades; most authors will cite a particular date as the anniversary of fractional calculus. In a letter dated September 30, 1695, L'Hôpital wrote to Leibniz to inquire about the $\left(\frac{1}{2}\right)^{th}$ derivative of the function $f(x) = x$. Leibniz's response was, "An apparent paradox, from which

someday useful consequences will be drawn". However, little progress has been made in this field for three centuries. One reason is that the mathematical tools of fractional calculus were not available. Another reason is the lack of practical applications of this concept. More than 300 years later, we are only beginning to overcome these difficulties. Many mathematicians have studied this issue, especially Euler, Laplace, Fourier, Liouville, and Riemann, among others. Fractional calculus has become one of the most developed areas of mathematical analysis. It has had a rapid evolution and has emerged as a powerful tool in modeling some phenomena in several scientific fields such as physics, chemistry, biology, engineering, and finance, including possibly fractal phenomena. Due to its interesting properties and applications in various scientific fields, the numerical solutions of the problems treated have shown more consistency with experimental data than those produced by integer-order differential equations (see [12, 49]).

In a fractional model, there are a number of parameters, such as the fractional order, potential coefficients (when using a second-order elliptic operator in space), initial condition, source term, boundary conditions, and the geometry of the domain, which cannot be directly measured or characterized and must be indirectly deduced from measured data.

This has led to a wide variety of inverse problems for fractional differential equations (FDEs), which have begun to attract a lot of attention in recent years, since the famous work of Cheng et al. [21]. An interesting question is how non-local physics (from abnormal diffusion processes) will influence the behavior of inverse problems, such as uniqueness, stability, and the degree of ill-posedness. This degree is particularly important for the development of numerical reconstruction procedures.

In the realm of applied mathematics, pseudo-parabolic equations are notable for their ability to model phenomena that exhibit both diffusive and wave-like characteristics. These equations find application across various modeling scenarios, including two-phase flow filtration within porous media involving dynamic capillary pressure [9], isotropic material energy dynamics [23], wave phenomena [10], and the flow dynamics of a significant class of dilute polymer solutions, commonly known as Oldroyd-B fluids [39].

Numerous researchers have focused the study of inverse problems governed by pseudo-parabolic equations (see [47, 62, 63]). In the study presented in [104], the authors examined the asymptotic dynamics characterizing solutions pertinent to the inverse source problem associated with pseudo-parabolic equations. Furthermore, the investigation conducted in [5] delved into the realm of inverse problems concerning the determination of the right-hand

side in pseudo-parabolic equations incorporating a p-Laplacian operator and a nonlocal integral overdetermination condition.

On the other hand, an intriguing category of partial differential equations (PDEs) involves those subject to involution perturbations. In mathematical terms, an involution is an operation that, when applied twice, yields the original outcome. In the context of PDEs, an involution perturbation pertains to situations in which a PDE possesses a specific structure that is disrupted by a minor perturbation, leading to new and potentially unforeseen dynamics [103]. This concept traces its origins to the foundational contributions of Babbage [7], which were further expanded upon by Carleman [19] in 1932. Przewoerska-Rolewicz made substantial contributions to this field, engaging in probing examinations of multifaceted inquiries related to differential equations involving involutions through a series of seminal papers [77, 78, 80]. These investigations culminated in a cohesive consolidation of her findings within a dedicated monograph [79]. For a collection of works concerning the study of inverse problems containing an involution term, refer to [2, 3, 6] and the references therein. However, it is noteworthy that, apart from the recent contribution by Sassane et al. [89], there has been a discernible lack of consideration regarding regularization and approximation techniques for these problems.

Recently, problems governed by pseudo-parabolic equations with involution perturbation have been the subject of extensive research in various mathematical models, as evidenced by the works of researchers such as [83, 90]. These equations play a significant role in engineering applications and have been explored in-depth from both theoretical and physical perspectives in previous studies and monographs. For more details on this subject, we refer to [22, 25, 29, 46, 61, 62, 73, 101, 108].

Combining the concepts of inverse problems, fractional calculus, pseudo-parabolic dynamics, and involution terms, this thesis focuses on the regularization of certain inverse problems governed by pseudo-parabolic fractional differential equations with an involution perturbation. The contributions of this thesis include both theoretical advancements and practical computational methods, ensuring that the results are not only mathematically rigorous but also applicable to real-world scenarios. The following chapters will delve deeper into each of these aspects, providing a comprehensive understanding of the problems and the proposed solutions.

Content of the thesis

The thesis is composed of an introduction and three chapters. In Chapter 1, we recall some fundamental preliminary notions and necessary ingredients of functional analysis and fractional calculus essential for the study of the proposed problems. This foundational chapter aims to equip the reader with the requisite mathematical tools and concepts to facilitate a deeper understanding of the subsequent chapters.

Chapter 2 addresses ill-posed problems arising from fractional equations with involution. The primary objective here is to extend the quasi-reversibility method to certain classes of non-classical problems. By utilizing a modified variant of the pseudo-parabolic regularization method, we construct a family of well-posed pseudo-parabolic problems that approximate the original ill-posed problem. The chapter demonstrates the convergence of this regularization procedure, establishing that the solutions of the regularized problems converge to the solution of the original problem as the regularization parameter approaches zero.

Chapter 3 focuses on the regularization challenges in inverse problems associated with pseudo-parabolic equations with involution perturbations. It analyzes two regularization methodologies designed to enhance the accuracy and stability of numerical solutions. Additionally, this chapter extends an iterative method for solving inverse source problems governed by fractional pseudo-parabolic equations with involution. The theoretical analysis of errors is presented to demonstrate the convergence rates of the regularized solutions. This analysis offers valuable insights for researchers and practitioners in addressing complex inverse problems, ensuring that the regularized solutions are both accurate and stable.

Together, these chapters provide a comprehensive exploration of the regularization techniques for fractional pseudo-parabolic differential equations with involution, contributing both theoretical advancements and practical methodologies to the field.

Chapter 1

PRELIMINARIES

This chapter is dedicated to recalling certain concepts of functional analysis, elements of the theory of inverse problems, and some fundamental tools in fractional calculus.

1.1 Functional spaces

Let Ω denote a measurable subset of \mathbb{R}^n .

Definition 1.1.1. (Sobolev spaces $H^m(\Omega)$) Let $m \in \mathbb{N}$, we denote by $H^m(\Omega)$ the Sobolev space given by

$$H^m(\Omega) = \{u \in L^2(\Omega) : \forall \alpha, |\alpha| \leq m, \exists v_\alpha \in L^2(\Omega) \text{ such that: } v_\alpha = \partial^\alpha u \text{ in the weak sense}\}.$$

We introduce the scalar product on $H^m(\Omega)$ as follows:

$$\langle u, v \rangle_m = \sum_{|\alpha| \leq m} \langle \partial^\alpha u, \partial^\alpha v \rangle, \quad (1.1.1)$$

and the associated norm

$$\|u\|_{H^m} = \sqrt{\langle u, u \rangle_m}. \quad (1.1.2)$$

Theorem 1.1.1. Let Ω be an open set of \mathbb{R}^n and let $m \in \mathbb{N}$. The space $H^m(\Omega)$ endowed with the scalar product (1.1.1) is a separable Hilbert space.

In the case where $m = 1$, we use the density of $C_c^\infty(\Omega)$ in $H^1(\Omega)$ to define the following Sobolev space:

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \text{ such that } u = 0 \text{ on } \partial\Omega\}.$$

1.2 Elements of spectral theory

In what follows, we denote by H_i a Hilbert space on $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , endowed with the norm $\|u\|_{H_i}$ and the scalar product $\langle \cdot, \cdot \rangle_i$, ($i = 1, 2$).

1.2.1 Bounded linear operators

Definition 1.2.1. A linear operator is an application $A : \mathcal{D}(A) \subseteq H_1 \rightarrow H_2$ satisfying the following properties:

1. $\forall x, y \in \mathcal{D}(A), A(x + y) = A(x) + A(y),$
2. $\forall x \in \mathcal{D}(A) \text{ and } \lambda \in \mathbb{K}, A(\lambda x) = \lambda A(x),$

where $\mathcal{D}(A)$ is the domain of A , which is a vector subspace of H_1 and is generally assumed to be dense in H_1 .

Definition 1.2.2 (Bounded operators). A linear operator $A : \mathcal{D}(A) \subseteq H_1 \rightarrow H_2$ is said to be bounded, if

$$\|A\| = \sup \{ \|Au\|_{H_2} \mid u \in \mathcal{D}(A), \|u\|_{H_1} = 1 \} < \infty.$$

In this case A is a continuous linear map on $\mathcal{D}(A)$, and when $\mathcal{D}(A)$ is dense in H_1 , A uniquely extends to a bounded operator on H_1 .

We denote by $\mathcal{L}(H_1, H_2)$. (resp. $\mathcal{L}(H_1)$) the vector space of continuous linear operators from H_1 in H_2 (resp. of continuous endomorphisms of H_1) endowed with the topology of uniform convergence:

$$B \in \mathcal{L}(H_1, H_2), \|B\|_{\mathcal{L}(H_1, H_2)} = \sup_{u \in H_1 \setminus \{0\}} \frac{\|Bu\|_{H_2}}{\|u\|_{H_1}}.$$

Any operator A is completely defined by its graph $G(A)$ which is a vector subspace of $H_1 \times H_2$ defined by $G(A) = \{(v, Av), v \in \mathcal{D}(A)\}$.

For any linear operator $A : \mathcal{D}(A) \subseteq H_1 \rightarrow H_2$, we denote by:

$$\begin{aligned} N(A) &= \{h \in \mathcal{D}(A), Ah = 0\} \quad (\text{the kernel of } A), \\ R(A) &= \{h_2 = Ah_1, h_1 \in \mathcal{D}(A)\} \quad (\text{the range of } A). \end{aligned}$$

1.2.2 Inverse and adjoint of a bounded linear operator

Definition 1.2.3 (Inverse operators). We say that a continuous linear operator $S \in \mathcal{L}(H_1, H_2)$ is invertible if and only if there exists an operator $S' \in \mathcal{L}(H_2, H_1)$ such that:

$$S' \circ S = I_{H_1} \quad \text{and} \quad S \circ S' = I_{H_2},$$

where I_{H_1} (resp. I_{H_2}) is the identity operator of H_1 (resp. H_2). The operator S' , if it exists, is unique and we denote: $S' = S^{-1}$.

Theorem 1.2.1 (Banach isomorphism theorem [16]). *Any linear and continuous bijective operator $S \in \mathcal{L}(H_1, H_2)$ is invertible.*

Definition 1.2.4 (Adjoint operator). Let $T \in \mathcal{L}(H_1, H_2)$, there exists a unique operator noted $T^* \in \mathcal{L}(H_2, H_1)$ which verifies the relation:

$$\langle Th_1, h_2 \rangle_2 = \langle h_1, T^*h_2 \rangle_1, \quad \forall (h_1, h_2) \in H_1 \times H_2.$$

This operator T^* is called the adjoint of T .

Proposition 1.2.1. *Let $T, S \in \mathcal{L}(H_1, H_2)$ and $\alpha, \beta \in \mathbb{K}$, we have the following properties:*

- $\|T\| = \|T^*\|$,
- $T^{**} = (T^*)^* = T$,
- $(\alpha T + \beta S)^* = \alpha T^* + \beta S^*$,
- $(TS)^* = S^*T^*$,
- *If T is invertible, then T^* is also, and $(T^*)^{-1} = (T^{-1})^*$.*

Definition 1.2.5 (Self-adjoint operator). Let H be a Hilbert space. We say that $T \in \mathcal{L}(H)$ is a self-adjoint operator if $T = T^*$, in other words:

$$\langle Tx, y \rangle = \langle x, Ty \rangle, \quad \forall x, y \in H.$$

1.2.3 Spectrum of a linear operator and spectral decomposition

Definition 1.2.6. Let $A \in \mathcal{L}(H)$.

The **resolvent set** of A , denoted by $\rho(A)$, is defined as follows:

$$\rho(A) := \{\lambda \in \mathbb{C} \mid A_\lambda = (\lambda I - A) \text{ is invertible}\}.$$

The **resolvent** of A is given by:

$$R_\lambda(A) = (\lambda I - A)^{-1}.$$

The complement of $\rho(A)$ in the complex plane is called the **spectrum** of A and is denoted by $\sigma(A)$, where

$$\sigma(A) := \mathbb{C} \setminus \rho(A).$$

The **spectral radius** of the operator A , denoted $r(A)$, is defined by:

$$r(A) := \sup_{\lambda \in \sigma(A)} |\lambda|.$$

The spectrum of a bounded operator is a non-empty compact set.

The **point spectrum** of A , denoted $\sigma_p(A)$, is the set defined by:

$$\begin{aligned} \sigma_p(A) &= \{\lambda \in \mathbb{C} \mid N(\lambda I - A) \neq \{0\}\} \\ &= \{\lambda \in \mathbb{C} \mid \exists v \in H \setminus \{0\} : Av = \lambda v\}. \end{aligned}$$

The **residual spectrum** is the set:

$$\sigma_r(A) = \left\{ \lambda \in \mathbb{C} \mid N(\lambda I - A) = \{0\} \text{ and } \overline{R(\lambda I - A)} \neq H \right\}.$$

The **continuous spectrum** is the set $\sigma_c(A)$ defined by:

$$\sigma_c(A) = \left\{ \lambda \in \mathbb{C} \mid N(\lambda I - A) = \{0\} \text{ and } \overline{R(\lambda I - A)} = H \right\}.$$

Finally, we have the relationship:

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

1.2.4 Closed and unbounded operators

Definition 1.2.7 (Closed operators). An operator A is said to be closed if its graph $G(A)$ is closed in $H_1 \times H_2$, i.e., for any sequence $u_n \subset \mathcal{D}(A)$ such that $u_n \rightarrow u$ in H_1 and $Au_n \rightarrow v$ in H_2 , we have

$$u \in \mathcal{D}(A) \text{ and } v = Au.$$

Remark 1.2.1. The closed operator A can be considered as an operator bounded on its domain of definition $\mathcal{D}(A)$ endowed with the norm of the graph ($\|u\|_G := \|u\|_{H_1} + \|Au\|_{H_2}$) in H_1 .

Theorem 1.2.2 (Closed graph theorem [16]). *Let $H_1 \times H_2$ be two Banach spaces and $A : \mathcal{D}(A) \subset H_1 \rightarrow H_2$ a linear operator. If operator A is closed then it is bounded.*

Definition 1.2.8 (Unbounded operators). A linear operator $A : \mathcal{D}(A) \subset H_1 \rightarrow H_2$ is termed as unbounded if there exists a sequence $u_n \in \mathcal{D}(A)$ such that

$$\|u_n\|_{H_1} = 1 \text{ and } \|Au_n\|_{H_2} \rightarrow \infty, n \rightarrow \infty.$$

1.2.5 Adjoint of an unbounded operator

Definition 1.2.9. Let $A : \mathcal{D}(A) \subset H_1 \rightarrow H_2$ be an unbounded densely defined operator. The adjoint of A , denoted by A^* , is also an unbounded operator and is defined as follows:

$$\begin{aligned} A^* : \mathcal{D}(A^*) \subset H_2 &\rightarrow H_1 \\ v &\longmapsto A^*v = w, \end{aligned}$$

where:

$$\begin{aligned} \mathcal{D}(A^*) &= \{v \in H_2 : \exists c \geq 0 \text{ such that } |\langle Au, v \rangle_2| \leq c \|u\|_{H_1}, \forall u \in \mathcal{D}(A)\} \\ &= \{v \in H_2 : \exists w \in H_1 : \langle Au, v \rangle_2 = \langle u, w \rangle_1 = \langle u, A^*v \rangle_1, \forall u \in \mathcal{D}(A)\}. \end{aligned}$$

Definition 1.2.10 (Symmetric and self-adjoint operator). We say that the operator $A : \mathcal{D}(A) \subset H \rightarrow H$ is symmetric if

$$\forall u, v \in \mathcal{D}(A), \langle Au, v \rangle = \langle u, Av \rangle.$$

The operator $A : \mathcal{D}(A) \subset H \rightarrow H$ is said to be self-adjoint if $A = A^*$, i.e.,

$$\mathcal{D}(A) = \mathcal{D}(A^*) \text{ and } \langle Au, v \rangle = \langle u, Av \rangle, \forall u, v \in \mathcal{D}(A).$$

Remark 1.2.2. If $A : \mathcal{D}(A) \subset H_1 \rightarrow H_2$ is an unbounded densely defined operator, then A^* is closed.

Theorem 1.2.3 (Characterization of closed-image operators [16]). *Let $A : \mathcal{D}(A) \subset H_1 \rightarrow H_2$ be an unbounded, densely defined, and closed operator. Then, the following properties are*

equivalent:

- (i) $R(A)$ is closed, (ii) $R(A^*)$ is closed,
 (iii) $R(A) = N(A^*)^\perp$, (iv) $R(A^*) = N(A)^\perp$.

Theorem 1.2.4. *Let $A : \mathcal{D}(A) \subset H_1 \rightarrow H_2$ be an unbounded, densely defined, and closed operator. The following properties are then equivalent:*

1. A is surjective, i.e., $R(A) = H_2$,
2. there is a constant $k > 0$ such that:

$$|v| \leq k |A^*v|, \quad \forall v \in \mathcal{D}(A^*),$$

3. $N(A^*) = \{0\}$ and $R(A)$ is closed.

Corollary 1.2.1. *Let $A : \mathcal{D}(A) \subset H_1 \rightarrow H_2$ be an unbounded, closed operator with $\overline{\mathcal{D}(A)} = H_1$. The operator A admits a bounded inverse A^{-1} on H_2 if and only if there exist two constants m_1 and m_2 such that:*

$$|u| \leq m_1 |Au|, \quad \forall u \in \mathcal{D}(A),$$

$$|v| \leq m_2 |A^*v|, \quad \forall v \in \mathcal{D}(A^*).$$

1.2.6 Spectrum and resolvent of an unbounded operator

Let $A : \mathcal{D}(A) \subset H_1 \rightarrow H_2$ be an unbounded operator that we assume is closed and densely defined.

Proposition 1.2.2 ([20]). *We have the following assessments:*

- If $\lambda \in \rho(A)$, then the inverse operator $R(\lambda; A)$ is defined over the whole space and it is closed. By the closed graph theorem, this operator is bounded, i.e., $A_\lambda^{-1} \in \mathcal{L}(H)$ and it is called the resolvent of operator A .
- The set $\rho(A)$ is an open set of the complex plane.
- The map which associates to each $\lambda \in \rho(A)$ the operator $R(\lambda; A)$ is analytic on each connected component of $\rho(A)$.

- The resolvent verifies the following functional equation called "identity of the resolvent":

$$R(\lambda_1; A) - R(\lambda_2; A) = (\lambda_2 - \lambda_1) R(\lambda_1; A) R(\lambda_2; A).$$

- The spectrum of A is a closed set of \mathbb{C} , and if the operator A is bounded, then $\sigma(A)$ is a non-empty compact.

Theorem 1.2.5. Let $\mathcal{D}(A) \subset H \rightarrow H$ be a symmetric closed operator. A is self-adjoint if and only if $\sigma(A) \subset \mathbb{R}$.

1.3 Riesz–Fredholm theory

1.3.1 Diagonalization of compact self-adjoint operators

Definition 1.3.1 (compact operator). We say that an operator $A \in \mathcal{L}(H_1, H_2)$ is compact if it transforms any bounded subset of H_1 into a relatively compact subset of H_2 , i.e., $A(B_{H_1}(0, 1))$ is relatively compact for the strong topology.

We denote by $\mathcal{K}(H_1, H_2)$ the set of compact operators from H_1 to H_2 and we set $\mathcal{K}(H_1, H_2) = \mathcal{K}(H)$.

The compactness of an operator $T \in \mathcal{L}(H_1, H_2)$ is characterized as follows:

$$T \in \mathcal{K}(H_1, H_2) \iff [\forall (x_n) \subset H_1, x_n \rightarrow 0 \text{ (weakly)} \implies T x_n \rightarrow 0 \text{ (strongly)}].$$

Let E, F and G be three Banach spaces. If $A \in \mathcal{L}(E, F)$ and $B \in \mathcal{K}(F, G)$, then $AB \in \mathcal{K}(E, G)$.

Theorem 1.3.1 (Schauder's theorem). If A is a compact operator, then A^* is also compact and the converse is true.

Theorem 1.3.2 ([16]). Let $K \in \mathcal{K}(H)$ with $\dim(H) = \infty$. Then, we have:

- $0 \in \sigma(K)$,
- $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$.

In addition, we have one of the following situations:

1. $\sigma(K) = \{0\}$,
2. or else $\sigma(K) \setminus \{0\}$ is finite,
3. or else $\sigma(K) \setminus \{0\}$ is a sequence which tends to 0.

Theorem 1.3.3. *We assume that H is separable. Let $A \in \mathcal{K}(H)$ be a self-adjoint operator. Then, H admits a Hilbertian basis formed by eigenvectors of A :*

$$\forall x \in H, x = x_0 + \sum_{k \geq 0} (x, e_k) e_k, \quad x_0 \in N(A), \quad Ax = \sum_{k \geq 0} (x, e_k) \lambda_k e_k.$$

1.3.2 Spectral family and identity resolution

Discret version

Definition 1.3.2. Let $A : \mathcal{D}(A) \subset H \rightarrow H$ be an unbounded operator. Then, A is said to have a compact resolvent, if

$$\forall \lambda \in \rho(A), \quad R(\lambda; A) \in \mathcal{K}(H).$$

Theorem 1.3.4. *An operator $A : \mathcal{D}(A) \subset H \rightarrow H$ has compact resolvent if and only if there exists $\mu \in \rho(A)$ such that $R(\mu; A) \in \mathcal{K}(H)$.*

Theorem 1.3.5. *Let $A : \mathcal{D}(A) \subset H \rightarrow H$ be a self-adjoint operator. Then, we have*

1. $\sigma_r(A) = \emptyset$,
2. $\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \subseteq \mathbb{R}$,
3. $A \geq \theta \iff \sigma(A) \subset [\theta, \infty[$.

Theorem 1.3.6. *Let $A : \mathcal{D}(A) \subset H \rightarrow H$ be a lower bounded self-adjoint operator with compact resolvent. Then, A is diagonalizable, i.e., there exists a Hilbert basis in H , $(e_m)_{m \geq 1} \subset \mathcal{D}(A)$, and a sequence of real numbers $(\lambda_m)_{m \geq 1}$ such as :*

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m \rightarrow \infty, \quad Ae_m = \lambda_m e_m, \quad m = 1, 2, \dots$$

If $A : \mathcal{D}(A) \subset H \rightarrow H$ is a self-adjoint operator with $A \geq \theta > 0$ i.e. $0 \in \rho(A)$, and the injection $H_1 := (\mathcal{D}(A), \|\cdot\|_G) \hookrightarrow H$ is compact, then A has a compact resolvent and is therefore diagonalizable.

Continuous version

Definition 1.3.3. A family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ of orthogonal projections in H is called spectral family or even resolution of the identity if it satisfies the following conditions:

1. $E_\lambda E_\mu = E_{\inf(\lambda, \mu)}$, $\lambda, \mu \in \mathbb{R}$,
2. $E_{-\infty} = 0$, $E_{+\infty} = I$, where: $E_{-\infty}h = \lim_{\lambda \rightarrow -\infty} E_\lambda h$, et $E_{+\infty}h = \lim_{\lambda \rightarrow +\infty} E_\lambda h$, $h \in H$,
3. $E_{\lambda+0} = E_\lambda$ where: $E_{\lambda+0}h = \lim_{\varepsilon > 0, \varepsilon \rightarrow 0} E_{\lambda+\varepsilon}h$, $h \in H$.

The limits are taken in the sense of the norm of H .

Theorem 1.3.7. Let H be a Hilbert space and A a self-adjoint operator in H . Then there exists a spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ such that:

$$\langle Ax, y \rangle = \int_{\mathbb{R}} \lambda d \langle E_\lambda x, y \rangle, \quad Ax = \int_{\mathbb{R}} \lambda dE_\lambda x.$$

We note symbolically $A = \int_{\mathbb{R}} \lambda dE_\lambda$.

Theorem 1.3.8. Let $\lambda \mapsto f(\lambda)$ be a real-valued continuous function, and let $\mathcal{D} \subset H$ be defined by:

$$\mathcal{D} = \left\{ h \in H : \int_{\mathbb{R}} f(\lambda) d |E_\lambda h|^2 < +\infty \right\}.$$

Then, \mathcal{D} is dense in H , and we define a self-adjoint operator S in H by:

$$\langle Sx, y \rangle = \int_{\mathbb{R}} f(\lambda) d \langle E_\lambda x, y \rangle, \quad x \in \mathcal{D}, y \in H,$$

of domain $\mathcal{D}(S) = \mathcal{D}$.

Functions of a self-adjoint operator

Let A be a self-adjoint operator in the Hilbert space H defined by $A = \int_{\lambda_0}^{\infty} \lambda dE_\lambda$, where $\lambda_0 = \inf \sigma(A)$ and $\{E_\lambda\}$ is its resolution of the unit.

Definition 1.3.4. The powers of operator A are defined as follows:

$$A^r = \int_{\lambda_0}^{\infty} \lambda^r dE_\lambda, \quad r \in \mathbb{R}, \quad h \in \mathcal{D}(A^r) \iff \int_{\lambda_0}^{\infty} \lambda^{2r} d |E_\lambda h|^2 < +\infty,$$

and we have the following properties:

1. For all $r \leq 0$, we have: $A^r \in \mathcal{L}(H)$, and if $r = 0$, then $A^0 = I$.
2. For all $r \geq 0$ and $h \in \mathcal{D}(A^r)$, we have

$$\langle A^r h, h \rangle \geq \lambda_0^r |h|^2.$$

3. For all $r \geq 0$, $\mathcal{D}(A^r)$ endowed with the norm is a Hilbert space $\|h\|_r^2 = \|A^r h\|^2$ with $h \in \mathcal{D}(A^r)$.
4. If $0 \leq r_1 \leq r_2$, $\mathcal{D}(A^{r_2}) \hookrightarrow \mathcal{D}(A^{r_1})$ and $\overline{\mathcal{D}(A^{r_2})} = \mathcal{D}(A^{r_1})$.

Definition 1.3.5. If f is a continuous function on \mathbb{R} , then $f(A)$ is defined as follows:

$$f(A) = \int_{\lambda_0}^{\infty} f(\lambda) dE_{\lambda}, \quad r \in \mathbb{R}, \quad h \in \mathcal{D}(f(A)) \iff \int_{\lambda_0}^{\infty} |f(\lambda)|^2 d|E_{\lambda} h|^2 < +\infty.$$

1.4 Ill-Posed Problems

In his lectures published in [42], Hadamard claims that a mathematical model for a physical problem (he was thinking in terms of a boundary value problem for a partial differential equation) has to be properly-posed or wellposed in the sense that it has the following three properties:

1. There exists a solution of the problem (existence).
2. There is at most one solution of the problem (uniqueness).
3. The solution depends continuously on the data (stability).

Mathematically, the existence of a solution can be enforced by enlarging the solution space. The concept of distributional solutions of a differential equation is an example. If a problem has more than one solution, then information about the model is missing. In this case, additional properties, such as sign conditions, can be built into the model. The requirement of stability is the most important one. If a problem lacks the property of stability, then its solution is practically impossible to compute because any measurement or numerical computation is polluted by unavoidable errors: thus the data of a problem are always perturbed by noise! If the solution of a problem does not depend continuously on the

data, then in general, the computed solution will have no relation with the exact solution. Indeed, there is no way to overcome this difficulty unless additional information about the solution is available. Here, we remind the reader of the following statement:

- A lack of information cannot be remedied by any mathematical trickery !

Mathematically, we formulate the notion of well-posedness in the following way.

Definition 1.4.1. (well-posedness). Let H_1 and H_2 be two normed spaces, $K : H_1 \rightarrow H_2$ a (linear or nonlinear) mapping. The equation $Kx = y$ is called properly-posed or wellposed if the following holds:

1. Existence: For every $y \in H_2$ there is (at least one) $x \in H_1$ such that $Kx = y$.
2. Uniqueness: For every $y \in H_2$ there is at most one $x \in H_1$ with $Kx = y$.
3. Stability: The solution x depends continuously on y ; that is, for every sequence

$x_n \in H_1$ with $Kx_n \rightarrow Ky$ ($n \rightarrow \infty$), it follows that $x_n \rightarrow x$ ($n \rightarrow \infty$).

Equations for which (at least) one of these properties does not hold are called improperly-posed or ill-posed.

Example 1.4.1. (Backwards heat equation). Consider the one-dimensional heat equation

$$\frac{du(x, t)}{dt} = \frac{d^2u(x, t)}{dx^2},$$

with the following boundary conditions

$$u(0, t) = u(\pi, t) = 0, \quad t \geq 0,$$

and the initial condition

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq \pi.$$

Separation of variables leads to the following formal solution

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin(nx) \quad \text{with} \quad a_n = \frac{2}{\pi} \int_0^{\pi} \varphi(y) \sin(ny) dy. \quad (1.4.1)$$

The direct problem is to solve the classical initial boundary value problem: Given the initial temperature distribution φ and the final time T , determine $u(., T)$. In the inverse problem, one measures the final temperature distribution $u(., T)$ and tries to determine the temperature at earlier times $t < T$, for example, the initial temperature $u(., 0)$.

From formula (1.4.1), we see that we have to determine $\varphi := u(., 0)$ from the following integral equation:

$$K\varphi = \psi, \quad u(x, T) = \int_0^\pi k(x, y) \varphi(y) dy = \psi(x), \quad 0 \leq x \leq \pi, \quad (1.4.2)$$

with

$$k(x, y) := \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 T} \sin(nx) \sin(ny). \quad (1.4.3)$$

The integral operator K is of the Hilbert-Schmidt type (compact), hence K^{-1} is unbounded. This demonstrates the ill-posed nature of problem (1.4.2).

1.4.1 Regularization

Here, we assume that K is a compact injective operator. This is not a significant restriction because we can always replace the domain H_1 with the orthogonal complement of the kernel of K . We assume that there exists a solution $x \in H_1$ to the unperturbed equation $Kx = y$. In other words, we assume that $y \in \mathcal{R}(K)$. The injectivity of K implies that this solution is unique.

In practice, the right-hand side $y \in H_2$ is never known exactly but only up to an error $\delta > 0$. Therefore, we assume that the measured data $y^\delta \in H_2$ satisfies

$$\|y - y^\delta\| \leq \delta, \quad (1.4.4)$$

where $\delta > 0$ is the perturbation (noise).

Our goal is to "solve" the perturbed equation

$$Kx^\delta = y^\delta. \quad (1.4.5)$$

In general, equation (1.4.5) may not be solvable because we cannot assume that the measured data y^δ are in the range $\mathcal{R}(K)$ of K . Therefore, the best we can hope for is to determine an

approximation $x^\delta \in H_1$ to the exact solution x that is "not much worse" than the worst-case error $F(\delta, E, \|\cdot\|)$.

An additional requirement is that the approximate solution x^δ should depend continuously on the data y^δ . In other words, it is our aim to construct a suitable bounded approximation $R : H_2 \rightarrow H_1$ of the (unbounded) inverse operator $K^{-1} : \mathcal{R}(K) \rightarrow H_1$.

Definition 1.4.2. A regularization strategy is a family of linear and bounded operators

$$R_\alpha : H_2 \rightarrow H_1, \quad \alpha > 0,$$

such that

$$\lim_{\alpha \rightarrow 0} R_\alpha Kx = x \text{ for all } x \in H_1,$$

that is, the operators $R_\alpha K$ converge pointwise to the identity.

From this definition and the compactness of K , we conclude the following.

Theorem 1.4.1 ([55]). *Let R_α be a regularization strategy for a compact operator $K : H_1 \rightarrow H_2$ where $\dim H_1 = \infty$. Then, we have*

1. *The operators R_α are not uniformly bounded; that is, there exists a sequence (α_j) with $\|R_{\alpha_j}\| \rightarrow \infty$ for $j \rightarrow \infty$.*
2. *The sequence $(R_\alpha Kx)$ does not converge uniformly on bounded subsets of H_1 ; that is, there is no convergence $R_\alpha K$ to the identity I in the operator norm.*

The notion of a regularization is based on unperturbed data; that is, the regularizer $R_\alpha y$ converges to x for the exact right-hand side $y = Kx$.

Now, let $y \in \mathcal{R}(K)$ be the exact right-hand side and $y^\delta \in H_2$ be the measured data with $\|y - y^\delta\| \leq \delta$. We define

$$x^{\alpha, \delta} := R_\alpha y^\delta,$$

as an approximation of the solution x of $Kx = y$. Then, the error splits into two parts by the following obvious application of the triangle inequality:

$$\begin{aligned} \|x^{\alpha, \delta} - x\| &\leq \|R_\alpha y^\delta - R_\alpha y\| + \|R_\alpha y - x\| \\ &\leq \|R_\alpha\| \|y^\delta - y\| + \|R_\alpha Kx - x\|, \end{aligned}$$

and thus

$$\|x^{\alpha, \delta} - x\| \leq \delta \|R_\alpha\| + \|R_\alpha Kx - x\|.$$

This is our fundamental estimate, which we use often in the following.

We observe that the error between the exact and computed solutions consists of two parts: The first term on the right-hand side describes the error in the data multiplied by the "condition number" $\|R_\alpha\|$ of the regularized problem. By Theorem 1.4.5, this term tends to infinity as α tends to zero. The second term denotes the approximation error $\|(R_\alpha - K^{-1})y\|$ at the exact right-hand side $y = Kx$. By the definition of a regularization strategy, this term tends to zero with α .

We need a strategy to choose $\alpha = \alpha(\delta)$ dependent on δ in order to keep the total error as small as possible. This means that we would like to minimize

$$\delta \|R_\alpha\| + \|R_\alpha Kx - x\|.$$

The procedure is the same in every concrete situation: One has to estimate the quantities $\|R_\alpha\|$ and $\|R_\alpha Kx - x\|$ in terms of α and then minimize this upper bound with respect to α . Before we carry out these steps for two model examples, we introduce the following notation.

Definition 1.4.3. A regularization strategy $\alpha = \alpha(\delta)$ is called admissible if $\alpha(\delta) \rightarrow 0$ and

$$\sup \left\{ \|R_{\alpha(\delta)} y^\delta - x\| : y^\delta \in H_2, \|Kx - y^\delta\| \leq \delta \right\} \rightarrow 0, \quad \delta \rightarrow 0,$$

for every $x \in H_1$.

Tikhonov Regularization

A common method to deal with overdetermined finite linear systems of the form $Kx = y$ is to determine the best fit in the sense that one tries to minimize the defect $\|Kx - y\|$ with respect to $x \in H_1$ for some norm in H_2 . If H_1 is infinite-dimensional and K is compact, this minimization problem is also ill-posed by the following lemma.

Lemma 1.4.1 ([55]). *Let H_1 and H_2 be Hilbert spaces, $K : H_1 \rightarrow H_2$ be linear and bounded, and $y \in H_2$. There exists $\hat{x} \in H_1$ with $\|K\hat{x} - y\| \leq \|Kx - y\|$ for all $x \in H_1$ if and only if $\hat{x} \in H_1$ solves the normal equation $K^*K\hat{x} = K^*y$. Here, $K^* : H_2 \rightarrow H_1$ denotes the adjoint of K .*

As a consequence of Lemma 1.4.1, we should penalize the defect (in the language of optimization theory) or replace the equation of the first kind $K^* K \hat{x} = K^* y$ with an equation of the second kind (in the language of integral equation theory). Both viewpoints lead to the following minimization problem.

Given the linear, bounded operator $K : H_1 \rightarrow H_2$ and $y \in H_2$, determine $x^\alpha \in H_1$ that minimizes the Tikhonov functional

$$J_\alpha(x) := \|Kx - y\|^2 + \alpha \|x\|^2 \quad \text{for } x \in H_1.$$

We prove the following theorem.

Theorem 1.4.2 ([55]). *Let $K : H_1 \rightarrow H_2$ be a linear and bounded operator between Hilbert spaces and $\alpha > 0$. Then, the Tikhonov functional J_α has a unique minimum $x^\alpha \in H_1$. This minimum x^α is the unique solution of the normal equation*

$$\alpha x^\alpha + K^* K x^\alpha = K^* y.$$

1.5 Fractional Calculus

Fractional calculus has transitioned from pure mathematical formulation to applications, being utilized to model numerous physical phenomena across various domains [69].

Fractional derivatives and integer-order derivatives are both linear operators. However, integer-order derivatives are local operators whereas fractional derivatives are non-local operators. This makes them a tool applicable for describing and modeling certain phenomena in physics, chemistry, biology, mechanical engineering, signal processing, parameter identification, electrical engineering, control theory, finance, and phase change dynamics [68, 74].

1.5.1 Some special Functions

The Gamma Function

The Euler's gamma function $\Gamma(z)$ is one of the basic functions of fractional calculus. It generalizes the factorial $z!$ to take also non-integers and complex values and it is defined as follows:

Definition 1.5.1. The gamma function $\Gamma(z)$ is defined as: for $z \in \mathbb{C}$ and $\Re(z) > 0$

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt,$$

where $t^{z-1} = e^{(z-1)\log(t)}$. This integral is convergent for all complex $z \in \mathbb{C}$, with $\Re(z) > 0$.

For this function the reduction formula

$$\Gamma(z+1) = z\Gamma(z),$$

holds. In particular, if $z = n \in \mathbb{N}$, then

$$\Gamma(n+1) = n! \quad (n \in \mathbb{N}),$$

with (as usual) $0! = 1$.

The Beta function

The Beta function is studied by Euler and Legendre, which is a kind of Euler integral. For complex numbers z and w , the function is defined by:

$$B(z, w) = \int_0^1 t^{z-1} (1-t)^{w-1} dt, \quad \Re(z) > 0, \quad \Re(w) > 0,$$

which is symmetric function. The relationship between beta function and gamma function

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

1.5.2 Fractional integrals and derivatives

Definition 1.5.2. The left and right Riemann-Liouville fractional integrals of order $0 < \gamma < 1$ of a function $f \in L^1([a; b], \mathbb{R})$ are defined respectively by

$$I_{a+}^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s) ds,$$

$$I_{b^-}^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} f(s) ds.$$

Definition 1.5.3. The left Riemann-Liouville's fractional derivative of order $0 < \gamma < 1$ of a continuous function $f : [a; b] \rightarrow \mathbb{R}$ is given by

$$D_{a^+}^\gamma f(t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_a^t (t-s)^{-\gamma} f(s) ds.$$

The right Riemann-Liouville's fractional derivative of order $0 < \gamma < 1$ terminating at b of x is defined by

$$D_{b^-}^\gamma f(t) = -\frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_t^b (s-t)^{-\gamma} f(s) ds.$$

Definition 1.5.4. The left Caputo fractional derivative of order $0 < \gamma < 1$ of an absolutely continuous function $f : [a; b] \rightarrow \mathbb{R}$ is given by

$${}^c D_{a^+}^\gamma f(t) = \frac{1}{\Gamma(1-\gamma)} \int_a^t (t-s)^{-\gamma} f'(s) ds.$$

The right Caputo fractional derivative of order $0 < \gamma < 1$ terminating at b of x is defined by

$${}^c D_{b^-}^\gamma f(t) = -\frac{1}{\Gamma(1-\gamma)} \int_t^b (s-t)^{-\gamma} f'(s) ds.$$

1.5.3 Mittag-Leffler functions: properties and applications

Definition 1.5.5. [30] Denote by $E_{\gamma,\beta}(z)$ the Mittag-Leffler function

$$E_{\gamma,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + \beta)}, \quad z \in \mathbb{C}, \gamma > 0, \beta \in \mathbb{R}.$$

For short, we also denote $E_{\gamma,1}(z) = E_\gamma(z)$.

Here are some properties of the Mittag-Leffler function [41].

Lemma 1.5.1. For $0 < \alpha < 1$ and $y > 0$, $E_\alpha(-y)$ is decreasing. Furthermore, for all $\lambda > 0$,

we have

$$D_y^\alpha E_\alpha(-\lambda y^\alpha) = -\lambda E_\alpha(-\lambda y^\alpha), \quad (1.5.1)$$

$$E_\alpha(0) = 1, \quad 0 \leq E_\alpha(-y) \leq 1, \quad \lim_{y \rightarrow +\infty} E_\alpha(-y) = 0, \quad (1.5.2)$$

and

$$\frac{d^m}{dy^m} (E_\alpha(-y)) = (-1)^m E_\alpha(-y), \quad m \in \mathbb{N}. \quad (1.5.3)$$

Now, we recall a fundamental result that will be extensively utilized throughout this work.

Lemma 1.5.2. [51] *Uniform estimates for the Mittag-Leffler function.* For every $0 < \alpha < 1$, the uniform estimate

$$\frac{1}{1 + c_1(\alpha)y} \leq E_\alpha(-y) \leq \frac{1}{1 + c_2(\alpha)y}, \quad (1.5.4)$$

holds over \mathbb{R}_+ , with the following optimal constants:

$$c_1(\alpha) = \Gamma(1 - \alpha), \quad \text{and} \quad c_2(\alpha) = \Gamma(1 + \alpha)^{-1}. \quad (1.5.5)$$

Lemma 1.5.3. [60] *Let $\gamma \in (0, 1)$. We have*

$$\frac{d}{dy} E_\gamma(y) = \frac{1}{\gamma y} E_{\gamma,0}(y), \quad y \in \mathbb{R}, \quad y \neq 0. \quad (1.5.6)$$

Lemma 1.5.4. [60] *For $0 < \gamma_0 < \gamma_1 < 1$, there exist positive constants $C_{1,\pm}, C_{2,\pm}, C_3, C_4$ and C_5 that depend only on γ_0, γ_1 such that*

$$\frac{C_{1,-}}{\gamma} e^{x^{\frac{1}{\gamma}}} \leq E_\gamma(x) \leq \frac{C_{1,+}}{\gamma} e^{x^{\frac{1}{\gamma}}}, \quad \text{for all } x \geq 0, \quad (1.5.7)$$

$$\frac{C_{2,-}}{\Gamma(1-\gamma)} \frac{1}{1-x} \leq E_\gamma(x) \leq \frac{C_{2,+}}{\Gamma(1-\gamma)} \frac{1}{1-x}, \quad \text{for all } x \leq 0, \quad (1.5.8)$$

$$|E_{\gamma,0}(x)| \leq \frac{C_3}{\gamma} (1+x)^{\frac{1}{\gamma}} e^{x^{\frac{1}{\gamma}}}, \quad \text{for all } x \geq 0. \quad |E_{\gamma,0}(x)| \leq \frac{C_4}{\Gamma(1-\gamma)} \frac{1}{1-x}, \quad \text{for all } x \leq 0, \quad (1.5.9)$$

$$\frac{|E_{\gamma,0}(x)|}{E_\gamma(x)} \leq C_5 (1+x)^{\frac{1}{\gamma}}, \quad \text{for all } x \geq 0. \quad (1.5.10)$$

These estimates hold uniformly for all γ in the interval $[\gamma_0, \gamma_1]$.

Lemma 1.5.5. [30] *If $\beta \in (-\alpha, \alpha)$, for any λ_k satisfying $\lambda_k \geq \lambda_1 > 0$, there exist positive*

constants \overline{C} and \underline{C} that depend on $\alpha, \beta, \gamma, T, \lambda_1$ such that

$$\frac{\underline{C}}{\lambda_k} \leq E_\gamma(-(\alpha \pm \beta)\lambda_k T^\gamma) \leq \frac{\overline{C}}{\lambda_k}.$$

Now we give the fundamental solution of the fractional problem with initial condition. Let x_0 and $\lambda \in \mathbb{R}$. The following problem

$$D_t^\theta x(t) + \lambda x(t) = 0, \quad x(0) = x_0, \quad 0 < \theta < 1, \quad (1.5.11)$$

admits a unique solution given by the formula

$$x(t) = E_{\theta,1}(t^\theta \lambda)x_0. \quad (1.5.12)$$

Chapter 2

THE MODIFIED FRACTIONAL-ORDER QUASI-REVERSIBILITY METHOD FOR A CLASS OF DIRECT AND INVERSE PROBLEMS GOVERNED BY TIME-FRACTIONAL HEAT EQUATIONS WITH INVOLUTION PERTURBATION

2.1 Direct problem for time-fractional heat equation with involution

We denote by $H = L^2((-1, 1); \mathbb{R})$ the Hilbert space equipped with the inner product $\langle u, v \rangle$ induced by the usual norm $\|\cdot\|$. That is,

$$\langle u, v \rangle := \int_{-1}^1 u(x) v(x) dx, \quad \|u\|^2 := \int_{-1}^1 |u(x)|^2 dx.$$

We consider the following nonclassical fractional heat problem in the rectangle $Q = (-1, 1) \times (0, T)$:

$$\begin{cases} D_t^\gamma u(x, t) - \alpha u_{xx}(x, t) - \beta u_{xx}(-x, t) = 0, & x \in (-1, 1), t \in (0, T), \\ u(-1, t) = u(1, t), u_x(1, t) = u_x(-1, t), & t \in (0, T), \\ u(x, 0) = f(x), & x \in (-1, 1), \end{cases} \quad (2.1.1)$$

where α is a positive real number, $\beta \in \mathbb{R}$ and D_t^γ is the Caputo derivative, defined for $0 < \gamma < 1$, by

$$D_t^\gamma u(x, t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{u_\tau(x, \tau)}{(t-\tau)^\gamma} d\tau, \quad 0 < \gamma < 1.$$

Fractional calculus is a branch of mathematics that deals with fractional order integrals and derivatives. It generalizes the classical calculus to non-integer orders and has important applications in physics, engineering, and other fields. For more details, we refer, for instance, to the books [41, 53].

On the other hand, involutive differential equations are a class of differential equations that exhibit alternating deviations. These equations were first introduced in [100] and have since become an integral part of the theory of functional differential equations. The study of involutive differential equations has gained increasing attention from researchers, as evidenced by recent research in [18, 81, 85].

Equations that contain terms such as $u(\sigma(x), t)$, $u_x(\sigma(x), t)$, and $u_{xx}(\sigma(x), t)$ are commonly referred to as equations with deviated variables. The function σ , also known as the Carleman translation or involution [18, 100], is defined by:

$$\sigma : [-1, 1] \longrightarrow [-1, 1] \text{ with } \sigma(\sigma(x)) = x.$$

A considerable number of researchers have studied equations with involutions, and there is a comprehensive bibliography on this topic in monographs [18, 103]. These equations find applications in various fields of engineering [83, 96]. For further details on the theoretical development of these equations and their physical motivations, interested readers can consult [78, 80].

The mathematical literature has shown a shortage of work dedicated to the study of inverse problems and ill-posed problems for models governed by PDEs with deviated variables, as well as their regularization methods. This is particularly true for numerical approaches that deal with this category of problems. Recent research has focused on inverse problems involving perturbations of the involution type, as seen in [2, 3, 8].

The authors in [2] investigated an inverse source problem when $\alpha = 1$ and $|\beta| < 1$. They provided results on the existence and uniqueness of solutions for various types of boundary conditions. However, the regularization procedure for the problem under consideration was not examined.

This study builds upon previous research [2, 89], and aims to investigate ill-posed problems related to a class of relaxed heat equations with involution perturbation. To the best of our knowledge, the present work is the second investigation devoted to the theoretical analysis for regularizing the ill-posed time-fractional heat equations with involution perturbation. We note here that when $\gamma = 1$ in (2.1.1), we find the classical model treated in [89], so this study is a generalization and extension of results obtained in the work [89]. Hence, this work constitutes a new contribution to the theme of ill-posed problems generated by PDEs involving involution terms, which is characterized by the lack of studies dealing with

their regularization aspects and their numerical approximation methods.

To regularize our problem (2.1.1), we propose a modified version of the pseudo-parabolic regularization method initially developed by Showalter [36] for solving ill-posed problems in parabolic equations, as discussed in [14, 15, 30, 36, 51, 87, 89, 92]. By assuming specific regularity conditions on the problem data, our regularization technique guarantees the convergence of the approximate solution and provides error estimates.

2.2 Notation and auxiliary results

In this section, we recall certain auxiliary materials that will serve as key tools in establishing our results.

The problem (2.1.1) can be expressed in the following form:

$$D_t^\gamma u(x, t) + (\alpha I + \beta S) Au(x, t) = 0, \quad 0 < t < T, \quad x \in (-1, 1), \quad u(x, 0) = f(x), \quad (2.2.1)$$

where $A : \mathcal{D}(A) \subset H \rightarrow H$ with

$$\mathcal{D}(A) = \{w \in H^2(-1, 1) : w(-1) = w(1), \quad w_x(1) = w_x(-1)\}, \quad Aw(x) = -w_{xx}(x),$$

and

$$S : H \rightarrow H, \quad w \rightarrow S(w) = v, \quad v(x) = w(-x).$$

Definition 2.2.1. Let $B : H \rightarrow H$ be a linear operator.

1. S is an involution if $B^2 = I$.
2. B is unitary if $BB^* = B^*B = I$, where B^* is the adjoint of B .

Proposition 2.2.1. [89] *The following properties hold.*

1. $S \in \mathcal{L}(H)$.
2. $SS^* = S^*S = I$.
3. $S^2 = I$.

Let us now focus on the spectral problem

$$-u_{xx}(x) = \lambda u(x), \quad x \in [-1, 1], \quad u(-1) = u(1), \quad u'(-1) = u'(1).$$

The eigenvalues of this problem can be shown to be as follows [89]:

$$\lambda_0 = 0, \quad \lambda_k = (k\pi)^2, \quad k = 1, 2, \dots,$$

and eigenfunctions

$$\psi_0(x) = \frac{1}{\sqrt{2}}, \quad \{\varphi_k(x) = \sin(k\pi x), \psi_k(x) = \cos(k\pi x)\}, \quad k = 1, 2, \dots$$

Let

$$\varphi_n(x) = \sin(n\pi x), \quad n \in \mathbb{N}^*, \quad \psi_0(x) = \frac{1}{\sqrt{2}}, \quad \psi_m(x) = \cos(m\pi x), \quad m \in \mathbb{N}^*,$$

and

$$\lambda_n = (n\pi)^2, \quad n \in \mathbb{N}^*, \quad \lambda_m = (m\pi)^2, \quad m \in \mathbb{N}.$$

2.3 Analysis of the problem (2.1.1)

In this section, we present results regarding the existence, uniqueness, and regularity of the solution for the direct problem. This aims to elucidate the topological nature of the considered problem.

We initiate the first step by providing the explicit solution of Problem (2.1.1) using the Fourier method.

It is well known that the set $B = \{\varphi_n, n \in \mathbb{N}^*\} \cup \{\psi_m, m \in \mathbb{N}\}$ forms a Hilbertian basis in the space $H = L^2((-1, 1), \mathbb{R})$ [89], then for $u \in H = H_s \oplus H_c$, where $H_s = \overline{\text{Vect}\{\varphi_n\}_{n=1}^{\infty}}$ and $H_c = \overline{\text{Vect}\{\psi_m\}_{m=0}^{\infty}}$, we have the decomposition:

$$u = u_s + u_c = \sum_{n=1}^{\infty} \langle u, \varphi_n \rangle \varphi_n + \sum_{m=0}^{\infty} \langle u, \psi_m \rangle \psi_m, \quad (2.3.1)$$

where

$$u_s = \sum_{n=1}^{\infty} \langle u, \varphi_n \rangle \varphi_n, \quad u_c = \sum_{m=0}^{\infty} \langle u, \psi_m \rangle \psi_m.$$

It is worth noting that equation (2.3.1) provides a means of decomposing problem

$$D_t^\gamma u + A_{\alpha,\beta} u = 0, \quad 0 < t < T, \quad u(0) = f, \quad (2.3.2)$$

into two distinct problems, the solutions to which can be added together to obtain the solution to problem (2.3.2). This can be accomplished using Fourier's method and the relationships given below:

$$A\varphi_n = \lambda_n \varphi_n, \quad A\psi_m = \lambda_m \psi_m,$$

$$A_{\alpha,\beta} \varphi_n = (\alpha I + \beta S) A\varphi_n = \lambda_n (\alpha I + \beta S) \varphi_n = \lambda_n (\alpha - \beta) \varphi_n = (\alpha - \beta) \lambda_n \varphi_n,$$

$$A_{\alpha,\beta} \psi_m = (\alpha I + \beta S) A\psi_m = \lambda_m (\alpha I + \beta S) \psi_m = \lambda_m (\alpha + \beta) \psi_m = (\alpha + \beta) \lambda_m \psi_m,$$

we can write,

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \varphi_n + \sum_{m=0}^{\infty} u_m(t) \psi_m,$$

and

$$f = \sum_{n=1}^{\infty} f_n(t) \varphi_n + \sum_{m=0}^{\infty} f_m(t) \psi_m.$$

Substituting these expressions into problem (2.3.2) yields the following problems:

$$\begin{cases} \sum_{n=1}^{\infty} (D_t^\gamma u_n(t) + (\alpha - \beta) \lambda_n u_n(t)) \varphi_n = 0, & 0 < t < T, \\ \sum_{n=1}^{\infty} u_n(0) \varphi_n = \sum_{n=1}^{\infty} f_n \varphi_n, \end{cases} \quad (2.3.3)$$

and

$$\begin{cases} \sum_{m=0}^{\infty} (D_t^\gamma u_m(t) + (\alpha + \beta) \lambda_m u_m(t)) \psi_m = 0, & 0 < t < T, \\ \sum_{m=0}^{\infty} u_m(0) \psi_m = \sum_{m=0}^{\infty} f_m \psi_m. \end{cases} \quad (2.3.4)$$

This gives us the system of equations :

$$\begin{cases} D_t^\gamma u_n(t) + (\alpha - \beta) \lambda_n u_n(t) = 0, & 0 < t < T, \\ u_n(0) = f_n, \end{cases} \quad (2.3.5)$$

for $n \in \mathbb{N}^*$, and

$$\begin{cases} D_t^\gamma u_m(t) + (\alpha + \beta) \lambda_m u_m(t) = 0, & 0 < t < T, \\ u_m(0) = f_m, \end{cases} \quad (2.3.6)$$

for $m \in \mathbb{N}$.

Remark 2.3.1. The problems (2.3.5) and (2.3.6) correspond to fractional order differential equations for any fixed value of n and m respectively. Thus, it follows that each of these problems has a unique solution that can be expressed using the Mittag-Leffler formula (see (1.5.12)).

Therefore, the unique solutions to these problems can be expressed as follow:

$$\text{odd component : } u_n(t) = E_\gamma(-(\alpha - \beta) \lambda_n t^\gamma) f_n, \quad n \in \mathbb{N}^*, \quad (2.3.7)$$

and

$$\text{even component : } u_m(t) = E_\gamma(-(\alpha + \beta) \lambda_m t^\gamma) f_m, \quad m \in \mathbb{N}. \quad (2.3.8)$$

Definition 2.3.1. [89] The set \mathcal{A} is considered admissible for problem (2.3.4) if, for all $f \in \mathcal{A}$, the problem (2.3.4) has at least one solution, meaning that the set of solutions \mathcal{S} associated with f is nonempty.

Now, we have reached a stage where we elucidate the nature of the considered problem based on the values of α and β . Two cases may be distinguished:

If $\beta \in [-\alpha, \alpha]$, then problem (2.3.2) is well-posed. furthermore, its solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} E_\gamma(-(\alpha - \beta) \lambda_n t^\gamma) f_n \varphi_n(x) + \sum_{m=0}^{\infty} E_\gamma(-(\alpha + \beta) \lambda_m t^\gamma) f_m \psi_m(x). \quad (2.3.9)$$

Furthermore, we have the continuity relation:

$$\|u(\cdot, t)\|^2 = \sum_{n=1}^{\infty} (E_\gamma(-(\alpha - \beta) \lambda_n t^\gamma))^2 |f_n|^2 + \sum_{m=0}^{\infty} (E_\gamma(-(\alpha + \beta) \lambda_m t^\gamma))^2 |f_m|^2 \leq \|f\|^2,$$

which implies that

$$\sup_{0 \leq t \leq T} \|u(\cdot, t)\| \leq \|f\|.$$

If $\beta \in]-\infty, -\alpha[\cup]\alpha, +\infty[$, then the problem (2.3.2) is considered ill-posed. In this case, the formal solution of the problem is provided as follows:

$$u(x, t) = \sum_{n=1}^{\infty} E_{\gamma}(-(\alpha - \beta)\lambda_n t^{\gamma}) f_n \varphi_n(x) + \sum_{m=0}^{\infty} E_{\gamma}(-(\alpha + \beta)\lambda_m t^{\gamma}) f_m \psi_m(x). \quad (2.3.10)$$

The reason for the instability of the solution is specifically linked to the high frequencies $\theta_n = E_{\gamma}(-(\alpha - \beta)\lambda_n t^{\gamma})$ when $\beta \in]\alpha, +\infty[$, $t \in]0, T[$:

$$\theta_n = E_{\gamma}(-(\alpha - \beta)\lambda_n t^{\gamma}) \rightarrow +\infty, \quad n \rightarrow +\infty, \quad (2.3.11)$$

and $\theta_m = E_{\gamma}(-(\alpha + \beta)\lambda_m t^{\gamma})$ for the case $\beta \in]-\infty, -\alpha[$, $t \in]0, T[$:

$$\theta_m = E_{\gamma}(-(\alpha + \beta)\lambda_m t^{\gamma}) \rightarrow +\infty, \quad m \rightarrow +\infty. \quad (2.3.12)$$

To ensure the existence of the solution to the ill-posed problem, we define classes of regularity that not only guarantee the existence of the solution but also offer error estimates. Specifically, we introduce the natural admissible class G_s^1 (resp. G_c^1) to ensure the existence of the solution and G_s^{θ} for $\theta > 1$ (resp. G_c^{θ}) to establish an error estimate between the exact solution and the regularized solution.

Definition 2.3.2. For $\beta \in]\alpha, +\infty[$, $\theta \geq 0$, we define the set

$$G_s^{\theta} = \left\{ h \in H : \sum_{n=1}^{\infty} E_{\gamma}^{2\theta}(-(\alpha - \beta)\lambda_n T^{\gamma}) |h_n|^2 < +\infty \right\}, \quad h_n = (h, \varphi_n), \quad (2.3.13)$$

and if $\beta \in]-\infty, -\alpha[$, we denote by G_c^{θ} the set given by

$$G_c^{\theta} = \left\{ h \in H : \sum_{m=0}^{\infty} E_{\gamma}^{2\theta}(-(\alpha + \beta)\lambda_m T^{\gamma}) |h_m|^2 < +\infty \right\}, \quad h_m = (h, \psi_m). \quad (2.3.14)$$

1. The sets G_s^{θ} and G_c^{θ} are guaranteed to be nonempty. This can be easily demonstrated by showing that the sequence $\{\varphi_n\}_{n=1}^{\infty}$ belongs to G_s^{θ} and the sequence $\{\psi_m\}_{m=0}^{\infty}$ belongs to G_c^{θ} .
2. $G_s^0 = H_s = \overline{\text{Vect}\{\varphi_n\}_{n=1}^{\infty}}$ and $G_c^0 = H_c = \overline{\text{Vect}\{\psi_m\}_{m=0}^{\infty}}$.
3. For $\theta_2 \geq \theta_1$, we have $G_s^{\theta_2} \subset G_s^{\theta_1}$ and $G_c^{\theta_2} \subset G_c^{\theta_1}$.

To simplify the notation, we put

$$a_1(\gamma) = \frac{C_{1,-}}{\gamma}, a_2(\gamma) = \frac{C_{1,+}}{\gamma},$$

and

$$b_1(\gamma) = \frac{C_{2,-}}{\Gamma(1-\gamma)}, b_2(\gamma) = \frac{C_{2,+}}{\Gamma(1-\gamma)}.$$

By using the estimate (1.5.7), we have the following estimates:

If $\beta \in]\alpha, +\infty[$, i.e., $(\beta - \alpha) > 0$, then

$$a_1(\gamma) e^{[(\beta-\alpha)\lambda_n]^{\frac{1}{\gamma}} t} \leq E_\gamma((\beta - \alpha)\lambda_n t) \leq a_1(\gamma) e^{[(\beta-\alpha)\lambda_n]^{\frac{1}{\gamma}} t},$$

and

$$a_1(\gamma)^{2\theta} e^{2\theta[(\beta-\alpha)\lambda_n]^{\frac{1}{\gamma}} t} \leq E_\gamma((\beta - \alpha)\lambda_n t)^{2\theta} \leq a_1(\gamma)^{2\theta} e^{2\theta[(\beta-\alpha)\lambda_n]^{\frac{1}{\gamma}} t}.$$

If $\beta \in]-\infty, -\alpha[$, i.e., $-(\beta + \alpha) > 0$, then

$$a_1(\gamma) e^{[-(\beta+\alpha)\lambda_n]^{\frac{1}{\gamma}} t} \leq E_\gamma(-(\beta + \alpha)\lambda_n t) \leq a_1(\gamma) e^{[-(\beta+\alpha)\lambda_n]^{\frac{1}{\gamma}} t},$$

and

$$a_1(\gamma)^{2\theta} e^{2\theta[-(\beta+\alpha)\lambda_n]^{\frac{1}{\gamma}} t} \leq E_\gamma(-(\beta + \alpha)\lambda_n t)^{2\theta} \leq a_1(\gamma)^{2\theta} e^{2\theta[-(\beta+\alpha)\lambda_n]^{\frac{1}{\gamma}} t}.$$

These equivalent estimates allow us to give a precise characterization of the regularity classes G_s^θ and G_c^θ as follows:

$$G_s^\theta = \left\{ h \in H : \mathbf{Q}_{s,\theta}^2(h) = \sum_{n=1}^{\infty} e^{2\theta[(\beta-\alpha)\lambda_n]^{\frac{1}{\gamma}} T} |h_n|^2 < +\infty \right\}, \quad h_n = (h, \varphi_n), \quad (2.3.15)$$

and

$$G_c^\theta = \left\{ h \in H : \mathbf{Q}_{c,\theta}^2(h) = \sum_{n=1}^{\infty} e^{2\theta[-(\beta+\alpha)\lambda_n]^{\frac{1}{\gamma}} T} |h_m|^2 < +\infty \right\}, \quad h_m = (h, \psi_m). \quad (2.3.16)$$

In the following, we provide key theorems, providing results related to the existence of a weak solution (strong generalized solution) and its regularity for the homogeneous case, as well as the existence of a weak solution for the nonhomogeneous case.

Theorem 2.3.1. *For $\beta \in]\alpha, +\infty[$, the problem (2.3.2) has a unique solution $u \in C([0, T]; H)$*

if and only if f belongs to the set G_s^1 .

Proof. The uniqueness of the solution to the problem (2.3.2) is due to the fact that the solutions to the problems (2.3.5) and (2.3.6) are also unique.

Furthermore, the statement that $u \in C([0, T]; H)$ is equivalent to the assertion that $\sup_{0 \leq t \leq T} \|u(\cdot, t)\|$ is finite. Indeed, we have

$$\begin{aligned} \sup_{0 \leq t \leq T} \|u(\cdot, t)\|^2 &= \sup_{0 \leq t \leq T} \left\{ \sum_{n=1}^{\infty} E_{\gamma}^2(-(\alpha - \beta) \lambda_n t^{\gamma}) |f_n|^2 + \sum_{m=0}^{\infty} E_{\gamma}^2(-(\alpha + \beta) \lambda_m t^{\gamma}) |f_m|^2 \right\} \\ &\leq \sum_{n=1}^{\infty} E_{\gamma}^2(-(\alpha - \beta) \lambda_n T^{\gamma}) |f_n|^2 + \sum_{m=0}^{\infty} |f_m|^2 < +\infty, \end{aligned}$$

if and only if $f \in G_s^1$. □

Similarly, we can prove the following theorem:

Theorem 2.3.2. For $\beta \in]-\infty, -\alpha[$, the problem (2.3.2) has a unique solution $u \in C([0, T], H)$ if and only if $f \in G_c^1$.

Remark 2.3.2. Let u_s denote the odd component of the solution u :

$$u_s(x, t) = \sum_{n=1}^{\infty} E_{\gamma}(-(\alpha - \beta) \lambda_n t^{\gamma}) f_n \varphi_n(x), \quad (2.3.17)$$

and u_c the even one:

$$u_c(x, t) = \sum_{m=0}^{\infty} E_{\gamma}(-(\alpha + \beta) \lambda_m t^{\gamma}) f_m \psi_m(x). \quad (2.3.18)$$

The same for f ,

$$f = f_s + f_c = \sum_{n=1}^{\infty} f_n \varphi_n + \sum_{m=0}^{\infty} f_m \psi_m.$$

It should be noted that for $\beta \in]\alpha, +\infty[$ (respectively, $\beta \in]-\infty, -\alpha[$), the instability of the solution is caused by the odd part u_s (respectively, u_c).

We assume here that $\beta \in]\alpha, +\infty[$. The case $\beta \in]-\infty, -\alpha[$ can be treated in the same way.

Theorem 2.3.3. *Let $f \in G_s^1$. Then, there exists a strong generalized solution $u \in \mathbf{V}_{s,c} = C([0, T]; H) \cap C((0, T); D(A))$ to Problem 2.1.1 given by (2.3.10) such that $D_t^\gamma u \in C((0, T); H)$ and $\lim_{t \rightarrow 0} \|u(t) - f\| = 0$. Moreover there exists a positive constant K such that*

$$\|u\|_{C([0,T];H)} = \sup_{0 \leq t \leq T} \|u(\cdot, t)\| \leq K [\mathbf{Q}_{s,1}(f) + \mathbf{Q}_{c,0}(f)], \quad (2.3.19)$$

and

$$\|D_t^\gamma u(\cdot, t)\| = \|Au(\cdot, t)\| \leq (\widehat{\kappa}_1 \mathbf{Q}_{s,1}(f) + \widehat{\kappa}_2 \mathbf{Q}_{c,0}(f)), \quad (2.3.20)$$

for $t \in]0, T[$.

Proof. We have

$$E_\gamma^2(-(\alpha - \beta) \lambda_n t^\gamma) \leq E_\gamma^2(-(\alpha - \beta) \lambda_n T^\gamma) \leq a_2(\gamma)^2 e^{2[(\beta - \alpha) \lambda_n]^{1/\gamma} T}, \quad (2.3.21)$$

$$0 \leq E_\gamma^2(-(\alpha + \beta) \lambda_n t^\gamma) \leq 1, \quad (2.3.22)$$

and

$$\|u(\cdot, t)\|^2 = \int_{-1}^1 |u(x, t)|^2 dx = \sum_{n=1}^{\infty} E_\gamma^2(-(\alpha - \beta) \lambda_n t^\gamma) |f_n|^2 + \sum_{m=0}^{\infty} E_\gamma^2(-(\alpha + \beta) \lambda_m t^\gamma) |f_m|^2. \quad (2.3.23)$$

By using (2.3.21) and (2.3.22), the quantity (2.3.23) can be estimated as follows

$$\begin{aligned} \|u(\cdot, t)\|^2 &\leq a_2(\gamma)^2 \sum_{n=1}^{\infty} e^{2[(\beta - \alpha) \lambda_n]^{1/\gamma} T} |f_n|^2 + \sum_{m=0}^{\infty} |f_m|^2 = a_2(\gamma)^2 \mathbf{Q}_{s,1}^2(f) + \mathbf{Q}_{c,0}^2(f) \quad (2.3.24) \\ &\leq \max(a_2(\gamma)^2, 1) [\mathbf{Q}_{s,1}^2(f) + \mathbf{Q}_{c,0}^2(f)] \end{aligned}$$

From (2.3.24), we obtain the desired estimate

$$\|u(\cdot, t)\|^2 \leq \max(a_2(\gamma), 1)^2 [\mathbf{Q}_{s,1}^2(f) + \mathbf{Q}_{c,0}^2(f)] \leq \max(a_2(\gamma), 1)^2 [\mathbf{Q}_{s,1}(f) + \mathbf{Q}_{c,0}(f)]^2, \quad (2.3.25)$$

with $K = \max(a_2(\gamma), 1)$.

In (2.3.25), since $u(x, t) = \sum_{n=1}^{\infty} E_\gamma(-(\alpha - \beta) \lambda_n t^\gamma) f_n \varphi_n(x) + \sum_{m=0}^{\infty} E_\gamma(-(\alpha + \beta) \lambda_m t^\gamma) f_m \psi_m(x)$ is convergent in $H = L^2((-1, 1); \mathbb{R})$ uniformly in $t \in [0, T]$, we see that $u \in C([0, T]; H)$.

Now, by using the discrete version of Lebesgue's dominated converge theorem, we prove that $\lim_{t \rightarrow 0} \|u(., t) - f\| = 0$. Indeed, we have

$$[E_\gamma(-(\alpha - \beta)\lambda_n t^\gamma) - 1]^2 \leq E_\gamma^2(-(\alpha - \beta)\lambda_n t^\gamma) \leq E_\gamma^2(-(\alpha - \beta)\lambda_n T^\gamma) \leq a_2(\gamma)^2 e^{2[(\beta - \alpha)\lambda_n]^{\frac{1}{\gamma}} T},$$

and

$$[E_\gamma(-(\alpha + \beta)\lambda_n t^\gamma) - 1]^2 \leq 1,$$

which implies that

$$\|u(., t) - f\|^2 \leq K^2 [\mathbf{Q}_{s,1}(f) + \mathbf{Q}_{c,0}(f)]^2 < \infty.$$

Furthermore, we have

$$\lim_{t \rightarrow 0} [E_\gamma(-(\alpha - \beta)\lambda_n t^\gamma) - 1]^2 = \lim_{t \rightarrow 0} [E_\gamma(-(\alpha + \beta)\lambda_n t^\gamma) - 1]^2 = 0,$$

for each $n \in \mathbb{N}^*$ and $m \in \mathbb{N}$. So, by the Lebesgue dominated convergence theorem, it follows the desired convergence result.

To continue the calculation, we define the function

$$M(\lambda_n) = \lambda_n \frac{E_\gamma(-(\alpha - \beta)\lambda_n t^\gamma)}{E_\gamma(-(\alpha - \beta)\lambda_n T^\gamma)}.$$

This function can be estimated as follows

$$M_\gamma(\lambda_n) \leq N_\gamma(\lambda_n) = \frac{a_2(\gamma)}{a_1(\gamma)} \lambda_n e^{(t-T)[(\beta-\alpha)\lambda_n]^{\frac{1}{\gamma}}}.$$

By a simple calculation, we show that

$$\sup_{n \geq 1} \lambda_n e^{(t-T)[(\beta-\alpha)\lambda_n]^{\frac{1}{\gamma}}} \leq b(t, T, \beta, \alpha, \gamma) = \frac{\gamma^\gamma}{(T-t)^\gamma (\beta-\alpha)} e^{-\gamma},$$

for $t \in [0, T[$. It results that

$$M_\gamma(\lambda_n) \leq \frac{a_2(\gamma)}{a_1(\gamma)} b(t, T, \beta, \alpha, \gamma) = \kappa_1(t, T, \beta, \alpha, \gamma). \quad (2.3.26)$$

We also have the inequality

$$\lambda_m E_\gamma (-(\alpha + \beta) \lambda_m t^\gamma) \leq b_2(\gamma) \frac{\lambda_m}{1 + (\alpha + \beta) \lambda_m t^\gamma} \leq \frac{b_2(\gamma)}{(\alpha + \beta) t^\gamma} = \kappa_2(t, \beta, \alpha, \gamma), \quad (2.3.27)$$

for $t \in]0, T]$.

By virtue of (2.3.26) and (2.3.27), we can write

$$\sum_{n=1}^{\infty} \left[\lambda_n \frac{E_\gamma (-(\alpha - \beta) \lambda_n t^\gamma)}{E_\gamma (-(\alpha - \beta) \lambda_n T^\gamma)} \right]^2 [E_\gamma (-(\alpha - \beta) \lambda_n T^\gamma)]^2 |f_n|^2 \leq [\kappa_1 a_2(\gamma)]^2 \mathbf{Q}_{s,1}^2(f),$$

and

$$\sum_{m=0}^{\infty} [\lambda_m E_\gamma (-(\alpha + \beta) \lambda_m t^\gamma)]^2 |f_m|^2 \leq \kappa_2^2 \mathbf{Q}_{c,0}^2(f).$$

By combining these two inequalities, we obtain

$$\|D_t^\gamma u(., t)\|^2 = \|-Au(., t)\|^2 = \|Au(., t)\|^2 \leq [\kappa_1 a_2(\gamma)]^2 \mathbf{Q}_{s,1}^2(f) + \kappa_2^2 \mathbf{Q}_{c,0}^2(f) < \infty, \quad (2.3.28)$$

for $t \in]0, T[$, which shows that $u \in C(]0, T[; D(A))$.

This completes the proof of Theorem 2.3.3. □

The nonhomogeneous case

Remark 2.3.3. If we consider the nonhomogeneous case:

$$\begin{cases} D_t^\gamma u(x, t) - \alpha u_{xx}(x, t) - \beta u_{xx}(-x, t) = g(x), & x \in (-1, 1), t \in (0, T), \\ u(-1, t) = u(1, t), u_x(1, t) = u_x(-1, t), & t \in (0, T), \\ u(x, 0) = f(x), & x \in (-1, 1), \end{cases} \quad (2.3.29)$$

we can impose reasonable conditions on g which guarantee the existence of the solution. In this case (see [86]), the unique solution is given by the formula

$$u(x, t; f, g) = u_s(x, t; f, g) + u_c(x, t; f, g), \quad (2.3.30)$$

where

$$u_s(x, t; f, g) = \sum_{n=1}^{\infty} \left[E_{\gamma}(-(\alpha - \beta)\lambda_n t^{\gamma}) f_n + \int_0^t G_{n,s}(t - \tau) g_{n,s}(\tau) d\tau \right] \varphi_n(x), \quad (2.3.31)$$

and

$$u_c(x, t; f, g) = \sum_{m=0}^{\infty} \left[E_{\gamma}(-(\alpha + \beta)\lambda_m t^{\gamma}) f_m + \int_0^t G_{m,c}(t - \tau) g_{m,c}(\tau) d\tau \right] \psi_m(x), \quad (2.3.32)$$

with

$$g_{n,s}(\tau) = (g(\cdot, \tau), \varphi_n), \quad G_{n,s}(t - \tau) = (t - \tau)^{\gamma-1} E_{\gamma,\gamma}(-(\alpha - \beta)\lambda_n(t - \tau)^{\gamma}),$$

and

$$g_{m,c}(\tau) = (g(\cdot, \tau), \psi_m), \quad G_{m,c}(t - \tau) = (t - \tau)^{\gamma-1} E_{\gamma,\gamma}(-(\alpha + \beta)\lambda_m(t - \tau)^{\gamma}).$$

To justify the convergence of expressions (2.3.31) and (2.3.32), we introduce the following regularity class (which represents a sufficient condition): for $r \in \mathbb{R}$, we define

$$\mathcal{G}_s^r = \left\{ g \in L^2([0, T]; H) : \mathbf{P}_{s,r}^2(g) = \sum_{n=1}^{\infty} \int_0^T [(\beta - \alpha)\lambda_n]^{2\frac{r}{\gamma}} e^{2[(\beta - \alpha)\lambda_n]^{\frac{1}{\gamma}}(t - \tau)} g_{n,s}(\tau)^2 d\tau < +\infty \right\}, \quad (2.3.33)$$

where $g_{n,s}(\tau) = (g(\cdot, \tau), \varphi_n)$.

The following two technical lemmas are necessary for our calculations.

Lemma 2.3.1. (cf. [27], Lemma 2.3) *Let $0 < \gamma_0 < \gamma_1 < 2$ and $\gamma \in [\gamma_0, \gamma_1]$. Then, there exists two constants $\mu_1(\gamma), \mu_2(\gamma) > 0$ such that*

$$\mu_1(\gamma)e^{z^{\frac{1}{\gamma}}} \leq E_{\gamma}(z) \leq \mu_2(\gamma)e^{z^{\frac{1}{\gamma}}}, \quad z \geq 0. \quad (2.3.34)$$

In addition, there exists two constants $\mu_3(\gamma), \mu_4(\gamma) > 0$ such that

$$\mu_3(\gamma)e^{z^{\frac{1}{\gamma}}} \leq z^{\frac{\gamma-1}{\gamma}} E_{\gamma,\gamma}(z) \leq \mu_4(\gamma)e^{z^{\frac{1}{\gamma}}}, \quad z > 0. \quad (2.3.35)$$

Lemma 2.3.2. (cf. [86], Lemma 3.2 and Lemma 3.3) *For $0 < \gamma < 1$ and $\lambda > 0$, the following*

are true.

- The function $t \mapsto E_\gamma(-\lambda t^\gamma)$ is continuous on $I = [0, T]$ and for each $j \in \mathbb{N}$, we have

$$\frac{d}{dt^j} E_\gamma(-\lambda t^\gamma) = -\lambda t^{\gamma-j} E_{\gamma, \gamma-j+1}(-\lambda t^\gamma), \quad t > 0. \quad (2.3.36)$$

- The function $t \mapsto t^{\gamma-1} E_{\gamma, \gamma}(-\lambda t^\gamma)$ belongs to $L^1(I)$ and we have

$$\int_0^T |t^{\gamma-1} E_{\gamma, \gamma}(-\lambda t^\gamma)| dt \leq \frac{1}{\lambda}. \quad (2.3.37)$$

We have

$$\|u_s(\cdot, t; f, g)\|^2 = \sum_{n=1}^{\infty} \left[E_\gamma(-(\alpha - \beta)\lambda_n t^\gamma) f_n + \int_0^t G_{n,s}(t - \tau) g_{n,s}(\tau) d\tau \right]^2 \leq 2 \sum_{n=1}^{\infty} \{A_n^2 + B_n^2\}, \quad (2.3.38)$$

where

$$A_n^2 = [E_\gamma(-(\alpha - \beta)\lambda_n t^\gamma) f_n]^2,$$

and

$$B_n^2 = \left[\int_0^t G_{n,s}(t - \tau) g_{n,s}(\tau) d\tau \right]^2.$$

By the Cauchy-Schwarz inequality, we can write

$$B_n^2 \leq \left[\int_0^t G_{n,s}^2(t - \tau) g_{n,s}^2(\tau) d\tau \right] \left[\int_0^t 1 d\tau \right] \leq T \int_0^t G_{n,s}^2(t - \tau) g_{n,s}^2(\tau) d\tau.$$

By using (2.3.35), we derive

$$B_n^2 \leq T \mu_4(\gamma)^2 \int_0^T [(\beta - \alpha)\lambda_n]^{2\frac{(1-\gamma)}{\gamma}} e^{2[(\beta - \alpha)\lambda_n]^{\frac{1}{\gamma}}(t-\tau)} g_{n,s}^2(\tau) d\tau. \quad (2.3.39)$$

This inequality gives

$$\sum_{n=1}^{\infty} B_n^2 \leq T\mu_4(\gamma)^2 \sum_{n=1}^{\infty} \int_0^T [(\beta - \alpha)\lambda_n]^{2\frac{(1-\gamma)}{\gamma}} e^{2[(\beta - \alpha)\lambda_n]^{\frac{1}{\gamma}}(t-\tau)} g_{n,s}(\tau)^2 d\tau = T\mu_4(\gamma)^2 \mathbf{P}_{s,(1-\gamma)}^2(g). \quad (2.3.40)$$

Now to estimate the quantity

$$\|u_c(\cdot, t; f, g)\|^2 = \sum_{m=0}^{\infty} \left[E_{\gamma}(-(\alpha + \beta)\lambda_m t^{\gamma}) f_m + \int_0^t G_{m,c}(t - \tau) g_{m,c}(\tau) d\tau \right]^2 \leq 2 \sum_{m=0}^{\infty} \{C_m^2 + D_m^2\}, \quad (2.3.41)$$

where

$$C_m^2 = [E_{\gamma}(-(\alpha + \beta)\lambda_m t^{\gamma}) f_m]^2,$$

and

$$D_m^2 = \left[\int_0^t G_{m,c}(t - \tau) g_{m,c}(\tau) d\tau \right]^2,$$

we use the following arguments:

$$|g_{m,c}(\tau)| \leq \|g(\cdot, \tau)\| \leq \sup_{0 \leq t \leq T} \|g(\cdot, t)\| = \|g\|_{\infty},$$

$$D_0^2 = \left[\int_0^t G_{0,c}(t - \tau) g_{0,c}(\tau) d\tau \right]^2 = \left[\int_0^t \frac{1}{\Gamma(\gamma)} (t - \tau)^{\gamma-1} g_{0,c}(\tau) d\tau \right]^2 \leq \left(\frac{T}{\gamma\Gamma(\gamma)} \right)^2 \|g\|_{\infty}^2,$$

$$D_m^2 \leq \left[\int_0^t G_{m,c}(t - \tau) |g_{m,c}(\tau)| d\tau \right]^2 \leq \left(\frac{1}{(\alpha + \beta)\lambda_m} \right)^2 \|g\|_{\infty}^2,$$

and

$$\sum_{m=1}^{\infty} \left(\frac{1}{(\alpha + \beta)\lambda_m} \right)^2 = \left(\frac{1}{(\alpha + \beta)\pi} \right)^2 \sum_{m=1}^{\infty} \frac{1}{m^4} = \left(\frac{1}{(\alpha + \beta)\pi} \right)^2 \frac{\pi^4}{90} = \left(\frac{1}{\alpha + \beta} \right)^2 \frac{\pi^2}{90} = M_2(\alpha, \beta)^2 < \infty.$$

we obtain

$$\sum_{m=0}^{\infty} D_m^2 \leq \max \left(M_2(\alpha, \beta)^2, \left(\frac{T}{\gamma\Gamma(\gamma)} \right)^2 \right) \|g\|_{\infty}^2. \quad (2.3.42)$$

Combining (2.3.24), (2.3.40) and (2.3.42), we obtain the following result.

Theorem 2.3.4. For $f \in G_s^1$ and $g \in \mathcal{G}_s^{1-\gamma} \cap C([0, T]; H)$, the nonhomogeneous Problem 2.3.29 admits a unique strong generalized solution $u \in C([0, T]; H)$ given by expressions (2.3.30), (2.3.31) and (2.3.32), and we have the conditional stability

$$\sup_{0 \leq t \leq T} \|u(\cdot, t; f, g)\| \leq \rho \{ \|g\|_\infty + \mathbf{P}_{s, (1-\gamma)}(g) + \mathbf{Q}_{s,1}(f) + \mathbf{Q}_{c,0}(f) \}. \quad (2.3.43)$$

2.4 Regularization method and error estimates: Problem (2.1.1)

This section constitutes the main contribution of the paper concerning Problem (2.1.1), in which we introduce a new modified pseudo-parabolic method of order $(\gamma, 2)$ to solve problem (2.3.2).

Consider the regularized problem where $u_\varepsilon(x, t)$ is the solution:

$$\begin{cases} L_\varepsilon u = 0, & (x, t) \in (-1, 1) \times (0, T), \\ u(-1, t) = u(1, t), & t \in (0, T), \\ u_x(1, t) = u_x(-1, t), & t \in (0, T), \\ u(x, 0) = f(x), & x \in (-1, 1), \end{cases} \quad (2.4.1)$$

where

$$L_\varepsilon u = D_t^\gamma u(x, t) - \alpha u_{xx}(x, t) - \beta u_{xx}(-x, t) - \varepsilon D_t^\gamma (u_{xx}(x, t)) + \varepsilon D_t^\gamma (u_{xx}(-x, t)), \quad (2.4.2)$$

and $\varepsilon > 0$ is a regularization parameter.

We know that $u_\varepsilon(x, t)$ has the following form

$$u_\varepsilon(x, t) = \sum_{n=1}^{\infty} u_{\varepsilon n}(t) \varphi_n + \sum_{m=0}^{\infty} u_{\varepsilon m}(t) \psi_m.$$

By injecting this formula into the expression of L_ε , we get the following problems:

$$\begin{cases} \sum_{n=1}^{\infty} ((1 + 2\varepsilon\lambda_n) D_t^\gamma u_{\varepsilon n}(t) + (\alpha - \beta) \lambda_n u_{\varepsilon n}(t)) \varphi_n = 0, & 0 < t < T, \\ \sum_{n=1}^{\infty} u_{\varepsilon n}(0) \varphi_n = \sum_{n=1}^{\infty} f_n \varphi_n, \end{cases} \quad (2.4.3)$$

and

$$\begin{cases} \sum_{m=0}^{\infty} (D_t^\gamma u_{\varepsilon m}(t) + (\alpha + \beta) \lambda_m u_{\varepsilon m}(t)) \psi_m = 0, & 0 < t < T, \\ \sum_{m=0}^{\infty} u_{\varepsilon m}(0) \psi_m = \sum_{m=0}^{\infty} f_m \psi_m. \end{cases} \quad (2.4.4)$$

We thus obtain the family of differential equations

$$\begin{cases} (1 + 2\varepsilon\lambda_n) D_t^\gamma u_{\varepsilon n}(t) + (\alpha - \beta) \lambda_n u_{\varepsilon n}(t) = 0, & 0 < t < T, \\ u_n(0) = f_n, \end{cases} \quad (2.4.5)$$

for $n \in \mathbb{N}^*$, and

$$\begin{cases} D_t^\gamma u_{\varepsilon m}(t) + (\alpha + \beta) \lambda_m u_{\varepsilon m}(t) = 0, & 0 < t < T, \\ u_{\varepsilon m}(0) = f_m, \end{cases} \quad (2.4.6)$$

for $m \in \mathbb{N}$.

Thus, we can express the solutions to problems (2.4.5) and (2.4.6) as unique families given respectively by:

$$\text{odd component : } u_{\varepsilon n}(t) = E_\gamma \left(-\frac{(\alpha - \beta)}{1 + 2\varepsilon\lambda_n} \lambda_n t^\gamma \right) f_n, \quad n \in \mathbb{N}^*,$$

and

$$\text{even component : } u_{\varepsilon m}(t) = E_\gamma (-(\alpha + \beta) \lambda_m t^\gamma) f_m, \quad m \in \mathbb{N}.$$

The solution of the perturbed problem (2.4.1) is obtained by adding the solutions of the two previous problems. Therefore,

$$u_\varepsilon(x, t) = \sum_{n=1}^{\infty} E_\gamma \left(-\frac{(\alpha - \beta)}{1 + 2\varepsilon\lambda_n} \lambda_n t^\gamma \right) f_n \varphi_n(x) + \sum_{m=0}^{\infty} E_\gamma (-(\alpha + \beta) \lambda_m t^\gamma) f_m \psi_m(x). \quad (2.4.7)$$

Now, we present a result on the stability of the regularized solution.

Theorem 2.4.1. For $\beta \in]\alpha, +\infty[$, the problem (2.4.1) has a unique solution $u_\varepsilon \in C([0, T], H)$.

Proof. The uniqueness of the solution to the problem (2.4.1) is due to the fact that the solutions to the problems (2.4.5) and (2.4.6) are also unique.

Here, we have

$$\begin{aligned}
\|u_\varepsilon(\cdot, t)\|^2 &= \sum_{n=1}^{\infty} E_\gamma^2\left(-\frac{(\alpha-\beta)}{1+2\varepsilon\lambda_n}\lambda_n t^\gamma\right) |f_n|^2 + \sum_{m=0}^{\infty} E_\gamma^2(-(\alpha+\beta)\lambda_m t^\gamma) |f_m|^2 \\
&\leq \sum_{n=1}^{\infty} E_\gamma^2\left(-\frac{(\alpha-\beta)}{1+2\varepsilon\lambda_n}\lambda_n t^\gamma\right) |f_n|^2 + \sum_{m=0}^{\infty} |f_m|^2 \\
&\leq \sum_{n=1}^{\infty} \left(\frac{C_{1,+}}{\gamma}\right)^2 e^{2\left(-\frac{(\alpha-\beta)}{1+2\varepsilon\lambda_n}\lambda_n T^\gamma\right)^{\frac{1}{\gamma}}} |f_n|^2 + \sum_{m=0}^{\infty} |f_m|^2 \\
&\leq \sum_{n=1}^{\infty} \left(\frac{C_{1,+}}{\gamma}\right)^2 e^{2\left(-\frac{(\alpha-\beta)}{2\varepsilon}T^\gamma\right)^{\frac{1}{\gamma}}} |f_n|^2 + \sum_{m=0}^{\infty} |f_m|^2 \\
&\leq \left(\frac{C_{1,+}}{\gamma}\right)^2 e^{2\left(-\frac{(\alpha-\beta)}{2\varepsilon}T^\gamma\right)^{\frac{1}{\gamma}}} \sum_{n=1}^{\infty} |f_n|^2 + \sum_{m=0}^{\infty} |f_m|^2 \\
&\leq \tilde{C} e^{2\left(-\frac{(\alpha-\beta)}{2\varepsilon}T^\gamma\right)^{\frac{1}{\gamma}}} \|f\|^2.
\end{aligned} \tag{2.4.8}$$

□

In the following, we provide convergence results and error estimate under certain regularity assumptions.

Theorem 2.4.2. *Setting $\Delta_\varepsilon(x, t) = u(x, t) - u_\varepsilon(x, t)$. If $\beta \in]\alpha, +\infty[$, it follows that*

$$\|\Delta_\varepsilon(\cdot, t)\|^2 \leq \|\Delta_\varepsilon(T)\|^2 \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

Proof. We have

$$\begin{aligned}
\Delta_\varepsilon(x, t) &= u(x, t) - u_\varepsilon(x, t) \\
&= \sum_{n=1}^{\infty} \left\{ E_\gamma(-(\alpha-\beta)\lambda_n t^\gamma) - E_\gamma\left(-\frac{(\alpha-\beta)}{1+2\varepsilon\lambda_n}\lambda_n t^\gamma\right) \right\} f_n \varphi_n(x).
\end{aligned}$$

Thus,

$$\|\Delta_\varepsilon(\cdot, t)\|^2 = \sum_{n=1}^{\infty} \underbrace{\left\{ E_\gamma(-(\alpha-\beta)\lambda_n t^\gamma) - E_\gamma\left(-\frac{(\alpha-\beta)}{1+2\varepsilon\lambda_n}\lambda_n t^\gamma\right) \right\}^2}_{G(\lambda_n)} |f_n|^2.$$

So we can write

$$G(\lambda_n) = E_\gamma^2(-(\alpha - \beta)\lambda_n t^\gamma) \left\{ 1 - \frac{E_\gamma\left(-\frac{(\alpha - \beta)}{1 + 2\varepsilon\lambda_n}\lambda_n t^\gamma\right)}{E_\gamma(-(\alpha - \beta)\lambda_n t^\gamma)} \right\}^2.$$

As $\|\Delta_\varepsilon(\cdot, t)\|^2 \leq \|\Delta_\varepsilon(\cdot, T)\|^2$, we just need to estimate $\Delta_\varepsilon(\cdot, T)$. Indeed, we have

$$\|\Delta_\varepsilon(\cdot, T)\|^2 = \sum_{n=1}^{\infty} E_\gamma^2(-(\alpha - \beta)\lambda_n T^\gamma) \left\{ 1 - \frac{E_\gamma\left(-\frac{(\alpha - \beta)}{1 + 2\varepsilon\lambda_n}\lambda_n T^\gamma\right)}{E_\gamma(-(\alpha - \beta)\lambda_n T^\gamma)} \right\}^2 |f_n|^2.$$

Note that condition $f \in G_s^1$ is equivalent to $\sum_{n=1}^{\infty} E_\gamma^2(-(\alpha - \beta)\lambda_n T^\gamma) |f_n|^2 < +\infty$. Hence, for any $\eta > 0$, there exists an integer $N \in \mathbb{N}$ such that:

$$\sum_{n=N+1}^{\infty} E_\gamma^2(-(\alpha - \beta)\lambda_n T^\gamma) |f_n|^2 < \frac{\eta^2}{2}.$$

So,

$$\begin{aligned} \|\Delta_\varepsilon(\cdot, T)\|^2 &= \sum_{n=1}^N E_\gamma^2(-(\alpha - \beta)\lambda_n T^\gamma) \left\{ 1 - \frac{E_\gamma\left(-\frac{(\alpha - \beta)}{1 + 2\varepsilon\lambda_n}\lambda_n T^\gamma\right)}{E_\gamma(-(\alpha - \beta)\lambda_n T^\gamma)} \right\}^2 |f_n|^2 \\ &\quad + \sum_{n=N+1}^{\infty} E_\gamma^2(-(\alpha - \beta)\lambda_n T^\gamma) \left\{ 1 - \frac{E_\gamma\left(-\frac{(\alpha - \beta)}{1 + 2\varepsilon\lambda_n}\lambda_n T^\gamma\right)}{E_\gamma(-(\alpha - \beta)\lambda_n T^\gamma)} \right\}^2 |f_n|^2. \end{aligned}$$

We have

$$\begin{aligned} A_2 &= \sum_{n=N+1}^{\infty} E_\gamma^2(-(\alpha - \beta)\lambda_n T^\gamma) \left\{ 1 - \frac{E_\gamma\left(-\frac{(\alpha - \beta)}{1 + 2\varepsilon\lambda_n}\lambda_n T^\gamma\right)}{E_\gamma(-(\alpha - \beta)\lambda_n T^\gamma)} \right\}^2 |f_n|^2 \\ &\leq \sum_{n=N+1}^{\infty} E_\gamma^2(-(\alpha - \beta)\lambda_n T^\gamma) |f_n|^2 < \frac{\eta^2}{2}. \end{aligned}$$

On the other hand, by Lemma 1.5.3, we have $\frac{d}{dy} E_\gamma(y) = \frac{1}{\gamma y} E_{\gamma,0}(y)$. Therefore, there exist

constants $\xi_n \in \left(-\frac{(\alpha - \beta)}{1 + 2\varepsilon\lambda_n} \lambda_n T^\nu, -(\alpha - \beta) \lambda_n T^\nu \right)$, $n \geq 1$, such that

$$E_\gamma(-(\alpha - \beta) \lambda_n T^\nu) - E_\gamma\left(-\frac{(\alpha - \beta)}{1 + 2\varepsilon\lambda_n} \lambda_n T^\nu\right) = \frac{E_{\gamma,0}(\xi_n)}{\gamma\xi_n} \left(\frac{-2(\alpha - \beta)}{1 + 2\varepsilon\lambda_n} \varepsilon \lambda_n^2 T^\nu \right).$$

We have

$$\begin{aligned} A_1 &= \sum_{n=1}^N E_\gamma^2(-(\alpha - \beta) \lambda_n T^\nu) \left\{ 1 - \frac{E_\gamma\left(-\frac{(\alpha - \beta)}{1 + 2\varepsilon\lambda_n} \lambda_n T^\nu\right)}{E_\gamma(-(\alpha - \beta) \lambda_n T^\nu)} \right\}^2 |f_n|^2 \\ &= \sum_{n=1}^N E_\gamma^2(-(\alpha - \beta) \lambda_n T^\nu) \left\{ \frac{E_\gamma(-(\alpha - \beta) \lambda_n T^\nu) - E_\gamma\left(-\frac{(\alpha - \beta)}{1 + 2\varepsilon\lambda_n} \lambda_n T^\nu\right)}{E_\gamma(-(\alpha - \beta) \lambda_n T^\nu)} \right\}^2 |f_n|^2 \\ &\leq \sum_{n=1}^N E_\gamma^2(-(\alpha - \beta) \lambda_n T^\nu) \frac{(E_{\gamma,0}(\xi_n))^2}{(\gamma\xi_n)^2} \left\{ \frac{-2(\alpha - \beta) \lambda_n T^\nu - \frac{-(\alpha - \beta)}{1 + 2\varepsilon\lambda_n} \lambda_n T^\nu}{E_\gamma(-(\alpha - \beta) \lambda_n T^\nu)} \right\}^2 |f_n|^2 \\ &\leq \sum_{n=1}^N E_\gamma^2(-(\alpha - \beta) \lambda_n T^\nu) \frac{(E_{\gamma,0}(\xi_n))^2}{E_\gamma(-(\alpha - \beta) \lambda_n T^\nu)^2} \left\{ \frac{-2(\alpha - \beta) \varepsilon \lambda_n^2 T^\nu}{\gamma\xi_n} \right\}^2 |f_n|^2 \\ &\leq \sum_{n=1}^N E_\gamma^2(-(\alpha - \beta) \lambda_n T^\nu) \left(\frac{E_{\gamma,0}(-(\alpha - \beta) \lambda_n T^\nu)}{E_\gamma(-(\alpha - \beta) \lambda_n T^\nu)} \right)^2 \left\{ \frac{-2(\alpha - \beta) \varepsilon \lambda_n^2 T^\nu}{1 + 2\varepsilon\lambda_n} \right\}^2 |f_n|^2 \\ &\leq \sum_{n=1}^N E_\gamma^2(-(\alpha - \beta) \lambda_n T^\nu) \left(C_5 (1 - (\alpha - \beta) \lambda_n T^\nu)^{\frac{1}{\nu}} \right)^2 \left(\frac{2}{\gamma} \varepsilon \lambda_n \right)^2 |f_n|^2 \\ &\leq \left(C_6 \lambda_N (1 - (\alpha - \beta) \lambda_N T^\nu)^{\frac{1}{\nu}} \right)^2 (2\varepsilon)^2 \sum_{n=1}^N E_\gamma^2(-(\alpha - \beta) \lambda_n T^\nu) |f_n|^2 \\ &\leq 4 \left(C_6 \lambda_N (1 - (\alpha - \beta) \lambda_N T^\nu)^{\frac{1}{\nu}} \right)^2 F_1^2 \varepsilon^2. \end{aligned}$$

We choose ε such that

$$4 \left(C_6 \lambda_N (1 - (\alpha - \beta) \lambda_N T^\nu)^{\frac{1}{\nu}} \right)^2 F_1^2 \varepsilon^2 \leq \frac{\eta^2}{2},$$

as a result, we get

$$\|\Delta_\varepsilon(T)\|^2 = A_1 + A_2 \leq \eta^2.$$

For any ε satisfying $\varepsilon \leq \frac{\eta}{2\sqrt{2}C_6\lambda_N(1 - (\alpha - \beta)\lambda_N T^\nu)^{\frac{1}{\nu}}F_1}$, and positive real number η , we

have $\|\Delta_\varepsilon(\cdot, T)\|^2 \leq \eta^2$, this shows that

$$\|\Delta_\varepsilon(\cdot, T)\|^2 \rightarrow 0, \varepsilon \rightarrow 0.$$

□

Now, let us assume that

$$\sum_{n=1}^{\infty} E_\gamma^2(-(\alpha - \beta)\lambda_n T^\gamma) \left(\lambda_n (1 - (\alpha - \beta)\lambda_n T^\gamma)^{\frac{1}{\gamma}} \right)^{2r} |f_n|^2 \leq E_1^2 < \infty.$$

Theorem 2.4.3. *Under the aforementioned condition, the following estimate holds:*

$$\sup_{0 \leq t \leq T} \|\Delta_\varepsilon(t)\|^2 \leq \|\Delta_\varepsilon(\cdot, T)\|^2 \leq k^2 E_1^2 \varepsilon^{2r},$$

where $0 < r \leq 1$.

Proof. We have

$$\begin{aligned} \|\Delta_\varepsilon(\cdot, T)\|^2 &= \sum_{n=1}^{\infty} E_\gamma^2(-(\alpha - \beta)\lambda_n T^\gamma) \left\{ 1 - \frac{E_\gamma\left(-\frac{(\alpha-\beta)}{1+2\varepsilon\lambda_n}\lambda_n T^\gamma\right)}{E_\gamma(-(\alpha - \beta)\lambda_n T^\gamma)} \right\}^2 |f_n|^2 \\ &\leq \sum_{n=1}^{\infty} E_\gamma^2(-(\alpha - \beta)\lambda_n T^\gamma) \left\{ 1 - \frac{E_\gamma\left(-\frac{(\alpha-\beta)}{1+2\varepsilon\lambda_n}\lambda_n T^\gamma\right)}{E_\gamma(-(\alpha - \beta)\lambda_n T^\gamma)} \right\}^{2r} |f_n|^2 \\ &\leq \sum_{n=1}^{\infty} E_\gamma^2(-(\alpha - \beta)\lambda_n T^\gamma) \left(\lambda_n (1 - (\alpha - \beta)\lambda_n T^\gamma)^{\frac{1}{\gamma}} \right)^{2r} (2C_6\varepsilon)^{2r} |f_n|^2 \\ &\leq (2C_6)^{2r} E_1^2 \varepsilon^{2r} \\ &\leq k^2 E_1^2 \varepsilon^{2r}, \end{aligned} \tag{2.4.9}$$

where $k = (2C_6)^r$. Hence, we obtain the desired estimation. □

Let $u_\varepsilon^\delta(x, t)$ be the approximate solution associated to inexact data $f^\delta(x)$ such that

$$\|f^\delta - f\| \leq \delta, \tag{2.4.10}$$

where $\delta > 0$ is the noise level.

Theorem 2.4.4. Suppose that u is the solution of (2.1.1) and u_ε is the solution of problem (2.4.1), and let

$$\sum_{n=1}^{\infty} E_\gamma^2 (-(\alpha - \beta) \lambda_n T^\gamma) \left(\lambda_n (1 - (\alpha - \beta) \lambda_n T^\gamma)^{\frac{1}{\gamma}} \right)^{2r} |f_n|^2 \leq E_1^2 < \infty.$$

By choosing the parameter ε as follows

$$\varepsilon = \frac{(\beta - \alpha) T^\gamma}{2 (\ln (\delta^{\theta-1}))^\gamma}, \quad 0 < \theta < 1,$$

then for any fixed $t \in [0, T]$, the following estimate holds

$$\sup_{0 \leq t \leq T} \|u_\varepsilon^\delta(., t) - u(., t)\| \leq \tilde{C} \delta^\theta + k \left(\frac{(\beta - \alpha) T^\gamma}{2 (\ln (\delta^{\theta-1}))^\gamma} \right)^r E_1,$$

where \tilde{C}, k are positive constants depending on $r, \gamma, T, \alpha, \beta$ and λ_1 .

Proof. First, we have

$$\|u_\varepsilon^\delta(., t) - u(., t)\| \leq \|u_\varepsilon^\delta(., t) - u_\varepsilon(., t)\| + \|u_\varepsilon(., t) - u(., t)\|.$$

From (2.4.8) and (2.4.10), we get

$$\begin{aligned} \|u_\varepsilon^\delta(., t) - u_\varepsilon(., t)\|^2 &= \sum_{n=1}^{\infty} \left(E_\gamma \left(-\frac{(\alpha - \beta)}{1 + \varepsilon \lambda_n} \lambda_n t^\gamma \right) \right)^2 |f_n^\delta - f_n|^2 \\ &\quad + \sum_{m=0}^{\infty} (E_\gamma (-(\alpha + \beta) \lambda_m t^\gamma))^2 |f_m^\delta - f_m|^2 \\ &\leq \left(\tilde{C} e^{(-\frac{\alpha - \beta}{2\varepsilon} T^\gamma)^{\frac{1}{\gamma}}} \right)^2 \|f^\delta - f\|^2 \\ &\leq \left(\tilde{C} e^{(-\frac{\alpha - \beta}{2\varepsilon} T^\gamma)^{\frac{1}{\gamma}}} \right)^2 \delta^2. \end{aligned} \tag{2.4.11}$$

By choosing $\varepsilon = \frac{(\beta - \alpha) T^\gamma}{2 (\ln (\delta^{\theta-1}))^\gamma}$, for $0 < \theta < 1$, and using (2.4.11), we obtain

$$\|u_\varepsilon^\delta(., t) - u_\varepsilon(., t)\| \leq \tilde{C} \delta^\theta. \tag{2.4.12}$$

On the other hand, we have

$$\|u_\varepsilon(., t) - u(., t)\|^2 \leq \|\Delta_\varepsilon(., T)\|^2.$$

Then, using (2.4.9), we get

$$\|u_\varepsilon(., t) - u(., t)\| \leq k \left(\frac{(\beta - \alpha) T^\gamma}{2 (\ln(\delta^{\theta-1}))^\gamma} \right)^r E_1, \quad 0 < r \leq 1. \quad (2.4.13)$$

By combining (2.4.12) and (2.4.13), we obtain

$$\|u_\varepsilon^\delta(., t) - u(., t)\| \leq \tilde{C} \delta^\theta + k \left(\frac{(\beta - \alpha) T^\gamma}{2 (\ln(\delta^{\theta-1}))^\gamma} \right)^r E_1.$$

The proof is completed. \square

2.5 Inverse problem for time-fractional heat equation with involution

This section addresses a novel problem related to backward time-fractional heat problem with involution perturbation.

We consider the following nonclassical fractional heat problem:

$$\begin{cases} D_t^\gamma v(x, t) - \alpha v_{xx}(x, t) - \beta v_{xx}(-x, t) = 0, & x \in (-1, 1), t \in (0, T), \\ v(-1, t) = v(1, t), u_x(1, t) = v_x(-1, t), & t \in (0, T), \\ v(x, T) = g(x), & x \in (-1, 1), \end{cases} \quad (2.5.1)$$

where D_t^γ is the Caputo derivative for $0 < \gamma < 1$. Here, α is a positive real number and $\beta \in (-\alpha, \alpha)$.

The problem (2.5.1) can be written as follows

$$D_t^\gamma v(x, t) + (\alpha I + \beta S) Av(x, t) = 0, \quad 0 < t < T, x \in (-1, 1), \quad v(x, T) = g(x), \quad (2.5.2)$$

where $A : \mathcal{D}(A) \subset H \rightarrow H$ with

$$\mathcal{D}(A) = \{w \in H^2(-1, 1) : w(-1) = w(1), w_x(1) = w_x(-1)\}, Aw(x) = -w_{xx}(x),$$

and

$$S : H \rightarrow H, w \rightarrow S(w) = v, v(x) = w(-x).$$

Define

$$H^r = \mathcal{D}(A^r) = \left\{ h \in H : \sum_{n=1}^{\infty} \lambda_n^{2r} |(h, \varphi_n)|^2 + \sum_{m=0}^{\infty} \lambda_m^{2r} |(h, \psi_m)|^2 < \infty \right\}.$$

It is readily deduced that the space $\mathcal{D}(A^r)$ forms a Hilbert space, equipped with the norm defined by the expression

$$\|h\|_{\mathcal{D}(A^r)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2r} |(h, \varphi_n)|^2 + \sum_{m=0}^{\infty} \lambda_m^{2r} |(h, \psi_m)|^2 \right)^{\frac{1}{2}}.$$

With the same methodology employed to ensure the existence and regularity of the solution of problem discussed in Section 2.3, we define classes of regularity that not only guarantee the existence of the solution of Problem (2.5.1) but also offer error estimates \mathcal{F}^θ .

Definition 2.5.1. For $\theta \geq 0$, we define the set

$$\mathcal{F}^\theta = \left\{ h \in H : \sum_{n=1}^{\infty} \frac{1}{E_\gamma^{2\theta}(-(\alpha - \beta)\lambda_n T^\gamma)} |h_n|^2 + \sum_{m=0}^{\infty} \frac{1}{E_\gamma^{2\theta}(-(\alpha + \beta)\lambda_m T^\gamma)} |h_m|^2 < +\infty \right\},$$

$$h_n = (h, \varphi_n), h_m = (h, \psi_m).$$

1. The set \mathcal{F}^θ is nonempty. In fact, it is easy to see that $\{\varphi_n\}_{n=1}^{\infty} \cup \{\psi_m\}_{m=1}^{\infty} \subset \mathcal{F}^\theta$.
2. If $\theta = 0$, then $\mathcal{F}^0 = H$.
3. If $\theta_2 \geq \theta_1$, then $\mathcal{F}^{\theta_2} \subset \mathcal{F}^{\theta_1}$.

Remark 2.5.1. From Lemma 1.5.5, we deduce that $\frac{1}{E_\gamma(-(\alpha \pm \beta)\lambda_n T^\gamma)} \approx \lambda_n$. Consequently, we have $H^\theta = \mathcal{F}^\theta$.

For $v \in H$, we have the decomposition:

$$v = v_s + v_c = \sum_{n=1}^{\infty} \langle v, \varphi_n \rangle \varphi_n + \sum_{m=0}^{\infty} \langle v, \psi_m \rangle \psi_m, \quad (2.5.3)$$

where

$$v_s = \sum_{n=1}^{\infty} \langle v, \varphi_n \rangle \varphi_n, \quad v_c = \sum_{m=0}^{\infty} \langle v, \psi_m \rangle \psi_m.$$

Note that the equation (2.5.3) enables us to decompose the problem

$$D_t^\gamma v + A_{\alpha, \beta} v = 0, \quad 0 < t < T, \quad v(T) = g, \quad (2.5.4)$$

we can write

$$v(x, t) = \sum_{n=1}^{\infty} v_n(t) \varphi_n + \sum_{m=0}^{\infty} v_m(t) \psi_m,$$

and

$$g = \sum_{n=1}^{\infty} g_n \varphi_n + \sum_{m=0}^{\infty} g_m \psi_m.$$

By substituting these expressions into the problem (2.5.4), we can obtain the following problems:

$$\left\{ \begin{array}{l} \sum_{n=1}^{\infty} (D_t^\gamma v_n(t) + (\alpha - \beta) \lambda_n v_n(t)) \varphi_n = 0, \quad 0 < t < T, \\ \sum_{n=1}^{\infty} v_n(T) \varphi_n = \sum_{n=1}^{\infty} g_n \varphi_n, \end{array} \right. \quad (2.5.5)$$

and

$$\left\{ \begin{array}{l} \sum_{m=0}^{\infty} (D_t^\gamma v_m(t) + (\alpha + \beta) \lambda_m v_m(t)) \psi_m = 0, \quad 0 < t < T, \\ \sum_{m=0}^{\infty} v_m(T) \psi_m = \sum_{m=0}^{\infty} g_m \psi_m. \end{array} \right. \quad (2.5.6)$$

Consequently, we obtain the following family of equations:

$$\left\{ \begin{array}{l} D_t^\gamma v_n(t) + (\alpha - \beta) \lambda_n v_n(t) = 0, \quad 0 < t < T, \\ v_n(T) = g_n, \end{array} \right. \quad (2.5.7)$$

for $n \in \mathbb{N}^*$, and

$$\begin{cases} D_t^\gamma v_m(t) + (\alpha + \beta) \lambda_m v_m(t) = 0, & 0 < t < T, \\ v_m(T) = g_m, \end{cases} \quad (2.5.8)$$

for $m \in \mathbb{N}$.

Remark 2.5.2. For any fixed integer n (resp. m), problem (2.5.7) (resp. (2.5.8)) represents a fractional-order differential equation.

Indeed, through straightforward computations, it can be shown that

$$v(x, t) = \sum_{n=1}^{\infty} \frac{E_\gamma(-(\alpha - \beta) \lambda_n t^\gamma)}{E_\gamma(-(\alpha - \beta) \lambda_n T^\gamma)} g_n \varphi_n(x) + \sum_{m=0}^{\infty} \frac{E_\gamma(-(\alpha + \beta) \lambda_m t^\gamma)}{E_\gamma(-(\alpha + \beta) \lambda_m T^\gamma)} g_m \psi_m(x). \quad (2.5.9)$$

We distinguish here two cases:

If $0 < t \leq T$, then problem (2.5.4) is well-posed. furthermore, its solution is given by

$$v(x, t) = \sum_{n=1}^{\infty} \frac{E_\gamma(-(\alpha - \beta) \lambda_n t^\gamma)}{E_\gamma(-(\alpha - \beta) \lambda_n T^\gamma)} g_n \varphi_n(x) + \sum_{m=0}^{\infty} \frac{E_\gamma(-(\alpha + \beta) \lambda_m t^\gamma)}{E_\gamma(-(\alpha + \beta) \lambda_m T^\gamma)} g_m \psi_m(x), \quad (2.5.10)$$

we have also the following continuity relation

$$\begin{aligned} & \|v(\cdot, t)\|^2 \\ &= \sum_{n=1}^{\infty} \left(\frac{E_\gamma(-(\alpha - \beta) \lambda_n t^\gamma)}{E_\gamma(-(\alpha - \beta) \lambda_n T^\gamma)} \right)^2 |g_n|^2 + \sum_{m=0}^{\infty} \left(\frac{E_\gamma(-(\alpha + \beta) \lambda_m t^\gamma)}{E_\gamma(-(\alpha + \beta) \lambda_m T^\gamma)} \right)^2 |g_m|^2 \\ &\leq \sum_{n=1}^{\infty} \left(\frac{C_{2,+} 1 + (\alpha - \beta) \lambda_n T^\gamma}{C_{2,-} 1 + (\alpha - \beta) \lambda_n t^\gamma} \right)^2 |g_n|^2 + \sum_{m=0}^{\infty} \left(\frac{C_{2,+} 1 + (\alpha + \beta) \lambda_m T^\gamma}{C_{2,-} 1 + (\alpha + \beta) \lambda_m t^\gamma} \right)^2 |g_m|^2 \\ &\leq \widehat{C}^2 \sum_{n=1}^{\infty} \left(\frac{T}{t} \right)^{2\gamma} |g_n|^2 + \widehat{C}^2 \sum_{m=0}^{\infty} \left(\frac{T}{t} \right)^{2\gamma} |g_m|^2 \leq \widehat{C}^2 \left(\frac{T}{t} \right)^{2\gamma} \|g\|^2, \end{aligned}$$

which implies that

$$\|v(\cdot, t)\| \leq \widehat{C} \left(\frac{T}{t} \right)^\gamma \|g\|. \quad (2.5.11)$$

Remark 2.5.3. From (2.5.11), we observe that the solution $u(\cdot, t)$ is stable for $t \in [\tau_0, T]$, $\tau_0 > 0$.

If $t = 0$, we obtain

$$v(x, 0) = \sum_{n=1}^{\infty} \frac{1}{E_{\gamma}(-(\alpha - \beta)\lambda_n T^{\gamma})} g_n \varphi_n(x) + \sum_{m=0}^{\infty} \frac{1}{E_{\gamma}(-(\alpha + \beta)\lambda_m T^{\gamma})} g_m \psi_m(x). \quad (2.5.12)$$

We can observe from this representation that $v(x, 0)$ is unstable, which can be attributed to the high frequencies $\vartheta_n = \frac{1}{E_{\gamma}(-(\alpha - \beta)\lambda_n T^{\gamma})}$:

$$\vartheta_n = \frac{1}{E_{\gamma}(-(\alpha - \beta)\lambda_n T^{\gamma})} \rightarrow +\infty, \quad n \rightarrow +\infty, \quad (2.5.13)$$

and $\vartheta_m = \frac{1}{E_{\gamma}(-(\alpha + \beta)\lambda_m T^{\gamma})}$:

$$\vartheta_m = \frac{1}{E_{\gamma}(-(\alpha + \beta)\lambda_m T^{\gamma})} \rightarrow +\infty, \quad m \rightarrow +\infty. \quad (2.5.14)$$

The following theorem is a key result summarizing the findings mentioned in Remark 2.5.2 and Remark 2.5.3.

Theorem 2.5.1. *The problem (2.5.4) has a unique solution $v \in C([0, T], H)$ if and only if $g \in H^1$.*

Proof. The uniqueness of the solution to the problem (2.5.4) is due to the fact that the solutions to the problems (2.5.7) and (2.5.8) are also unique.

Now, the property $v \in C([0, T], H)$ is equivalent to $\sup_{0 \leq t \leq T} \|v(\cdot, t)\| < +\infty$. We have

$$\begin{aligned} \sup_{0 \leq t \leq T} \|v(\cdot, t)\|^2 &= \sup_{0 \leq t \leq T} \left\{ \sum_{n=1}^{\infty} \left(\frac{E_{\gamma}(-(\alpha - \beta)\lambda_n t^{\gamma})}{E_{\gamma}(-(\alpha - \beta)\lambda_n T^{\gamma})} \right)^2 |g_n|^2 + \sum_{m=0}^{\infty} \left(\frac{E_{\gamma}(-(\alpha + \beta)\lambda_m t^{\gamma})}{E_{\gamma}(-(\alpha + \beta)\lambda_m T^{\gamma})} \right)^2 |g_m|^2 \right\} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{E_{\gamma}^2(-(\alpha - \beta)\lambda_n T^{\gamma})} |g_n|^2 + \sum_{m=0}^{\infty} \frac{1}{E_{\gamma}^2(-(\alpha + \beta)\lambda_m T^{\gamma})} |g_m|^2 < +\infty, \end{aligned} \quad (2.5.15)$$

if and only if $g \in H^1$. □

2.6 Regularization method and error estimates: Problem (2.5.1)

This section introduces the second primary contribution related to the inverse problem generated by the time-fractional heat equation with final data and an involution term. Here, we introduce a modified pseudo-parabolic procedure of order $(\gamma, 2s)$ to regularize problem (2.5.4), where s is the relaxation parameter, playing a role in enhancing the quality of the regularized solution.

Let $v_\varepsilon(x, t)$ be the solution of the following regularized problem

$$\begin{cases} L_\varepsilon^s v = 0, & (x, t) \in (-1, 1) \times (0, T), \\ v(-1, t) = v(1, t), & t \in (0, T), \\ v_x(1, t) = v_x(-1, t), & t \in (0, T), \\ v(x, T) = g(x), & x \in (-1, 1), \end{cases} \quad (2.6.1)$$

where

$$L_\varepsilon^s v = D_t^\gamma v(x, t) - \alpha \frac{\partial^2 v}{\partial x^2}(x, t) - \beta \frac{\partial^2 v}{\partial x^2}(-x, t) + \varepsilon (-1)^s D_t^\gamma \left(\frac{\partial^{2s} v}{\partial x^{2s}}(x, t) \right), \quad s \in \mathbb{N}^*, \quad (2.6.2)$$

and $\varepsilon > 0$ is a regularization parameter.

By inserting the formula

$$v_\varepsilon(x, t) = \sum_{n=1}^{\infty} v_{\varepsilon n}(t) \varphi_n + \sum_{m=0}^{\infty} v_{\varepsilon m}(t) \psi_m,$$

in the expression of L_ε^s , we get the following problems:

$$\begin{cases} \sum_{n=1}^{\infty} ((1 + \varepsilon \lambda_n^s) D_t^\gamma v_{\varepsilon n}(t) + (\alpha - \beta) \lambda_n v_{\varepsilon n}(t)) \varphi_n = 0, & 0 < t < T, \\ \sum_{n=1}^{\infty} v_{\varepsilon n}(T) \varphi_n = \sum_{n=1}^{\infty} g_n \varphi_n, \end{cases} \quad (2.6.3)$$

and

$$\begin{cases} \sum_{m=0}^{\infty} ((1 + \varepsilon \lambda_m^s) D_t^\gamma v_{\varepsilon m}(t) + (\alpha + \beta) \lambda_m v_{\varepsilon m}(t)) \psi_m = 0, & 0 < t < T, \\ \sum_{m=0}^{\infty} v_{\varepsilon m}(T) \psi_m = \sum_{m=0}^{\infty} g_m \psi_m. \end{cases} \quad (2.6.4)$$

We thus obtain the family of differential equations

$$\begin{cases} (1 + \varepsilon \lambda_n^s) D_t^\gamma v_{\varepsilon n}(t) + (\alpha - \beta) \lambda_n v_{\varepsilon n}(t) = 0, & 0 < t < T, \\ v_n(T) = g_n, \end{cases} \quad (2.6.5)$$

for $n \in \mathbb{N}^*$, and

$$\begin{cases} (1 + \varepsilon \lambda_m^s) D_t^\gamma v_{\varepsilon m}(t) + (\alpha + \beta) \lambda_m v_{\varepsilon m}(t) = 0, & 0 < t < T, \\ v_{\varepsilon m}(T) = g_m, \end{cases} \quad (2.6.6)$$

for $m \in \mathbb{N}$.

Therefore, the unique family of solutions of problems (2.6.5) and (2.6.6) are given, respectively, by

$$v_{\varepsilon n}(t) = \frac{E_\gamma \left(-\frac{(\alpha - \beta)}{1 + \varepsilon \lambda_n^s} \lambda_n t^\gamma \right)}{E_\gamma \left(-\frac{(\alpha - \beta)}{1 + \varepsilon \lambda_n^s} \lambda_n T^\gamma \right)} g_n, \quad n \in \mathbb{N}^*, \quad (2.6.7)$$

and

$$v_{\varepsilon m}(t) = \frac{E_\gamma \left(-\frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \lambda_m t^\gamma \right)}{E_\gamma \left(-\frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \lambda_m T^\gamma \right)} g_m, \quad m \in \mathbb{N}. \quad (2.6.8)$$

The following lemma provides a result on the existence and uniform stability of the regularized solution.

Lemma 2.6.1. *Problem (2.6.1) admits a unique solution*

$$v_\varepsilon(x, t) = \sum_{n=1}^{\infty} \frac{E_\gamma \left(-\frac{(\alpha - \beta)}{1 + \varepsilon \lambda_n^s} \lambda_n t^\gamma \right)}{E_\gamma \left(-\frac{(\alpha - \beta)}{1 + \varepsilon \lambda_n^s} \lambda_n T^\gamma \right)} g_n \varphi_n(x) + \sum_{m=0}^{\infty} \frac{E_\gamma \left(-\frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \lambda_m t^\gamma \right)}{E_\gamma \left(-\frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \lambda_m T^\gamma \right)} g_m \psi_m(x). \quad (2.6.9)$$

Furthermore, there exist positive constants C_7 and C_8 such that

$$\|v_\varepsilon(\cdot, \cdot)\| \leq \left(C_7^2 \left(1 + \varepsilon^{-\frac{1}{s}}\right)^2 + 1 + C_8^2 \left(1 + \varepsilon^{-\frac{1}{s}}\right)^2 \right) \|g\|.$$

Proof. Formula (2.6.9) is obtained by direct calculations.

$$\|v_\varepsilon(\cdot, t)\|^2 \leq \sum_{n=1}^{\infty} \frac{|g_n|^2}{E_\gamma^2\left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma\right)} + \sum_{m=0}^{\infty} \frac{|g_m|^2}{E_\gamma^2\left(-\frac{(\alpha+\beta)}{1+\varepsilon\lambda_m^s}\lambda_m T^\gamma\right)}.$$

From Lemma 1.5.4, we obtain

$$\begin{aligned} \|v_\varepsilon(\cdot, t)\|^2 &\leq \left(\sum_{n=1}^{\infty} \left(\frac{\Gamma(1-\gamma)}{C_{2,-}} \right)^2 \left(1 + \frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s} \lambda_n T^\gamma \right)^2 |g_n|^2 \right. \\ &\quad \left. + |g_0|^2 + \sum_{m=1}^{\infty} \left(\frac{\Gamma(1-\gamma)}{C_{2,-}} \right)^2 \left(1 + \frac{(\alpha+\beta)}{1+\varepsilon\lambda_m^s} \lambda_m T^\gamma \right)^2 |g_m|^2 \right). \end{aligned} \quad (2.6.10)$$

For $s > 1$, we have

$$1 + \varepsilon\lambda_n^s \geq \frac{s-1}{s} \cdot 1^{\frac{s}{s-1}} + \frac{1}{s} \left(\varepsilon^{\frac{1}{s}} \lambda_n \right)^s \geq \varepsilon^{\frac{1}{s}} \lambda_n,$$

$$1 + \varepsilon\lambda_n^s \geq \varepsilon^{\frac{1}{s}} \lambda_n \text{ for all } s \geq 1.$$

Therefore

$$\frac{\lambda_n}{1 + \varepsilon\lambda_n^s} \leq \varepsilon^{-\frac{1}{s}}, \text{ and } \frac{\lambda_m}{1 + \varepsilon\lambda_m^s} \leq \varepsilon^{-\frac{1}{s}}.$$

It follows from this inequality and (2.6.10) that there exist constants $C_7, C_8 > 0$ such that

$$\|v_\varepsilon(\cdot, t)\|^2 \leq \left(C_7^2 \left(1 + \varepsilon^{-\frac{1}{s}} \right)^2 + 1 + C_8^2 \left(1 + \varepsilon^{-\frac{1}{s}} \right)^2 \right) \|g\|^2.$$

□

We are now at the point of announcing the main result of this regularization strategy.

Theorem 2.6.1. *Let $\Delta_\varepsilon^s(0) = v(x, 0) - v_\varepsilon(x, 0)$. If $g \in H^1$, it follows that*

$$\|\Delta_\varepsilon^s(0)\|^2 = \|\Delta_{\varepsilon s}^s(0)\|^2 + \|\Delta_{\varepsilon c}^s(0)\|^2 \rightarrow 0, \varepsilon \rightarrow 0.$$

Proof. We have

$$\begin{aligned} \Delta_\varepsilon^s(0) &= v(x, 0) - v_\varepsilon(x, 0) \\ &= \sum_{n=1}^{\infty} \left\{ \frac{1}{E_\gamma(-(\alpha-\beta)\lambda_n T^\gamma)} - \frac{1}{E_\gamma\left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma\right)} \right\} g_n \varphi_n(x) \end{aligned}$$

$$+ \sum_{m=0}^{\infty} \left\{ \frac{1}{E_{\gamma}(-(\alpha + \beta)\lambda_m T^{\gamma})} - \frac{1}{E_{\gamma}\left(-\frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \lambda_m T^{\gamma}\right)} \right\} g_m \psi_m(x).$$

Thus,

$$\begin{aligned} \|\Delta_{\varepsilon}(0)\|^2 &= \sum_{n=1}^{\infty} \underbrace{\left\{ \frac{1}{E_{\gamma}(-(\alpha - \beta)\lambda_n T^{\gamma})} - \frac{1}{E_{\gamma}\left(-\frac{(\alpha - \beta)}{1 + \varepsilon \lambda_n^s} \lambda_n T^{\gamma}\right)} \right\}^2}_{T(\lambda_n)} |g_n|^2 \\ &\quad + \sum_{m=1}^{\infty} \underbrace{\left\{ \frac{1}{E_{\gamma}(-(\alpha + \beta)\lambda_m T^{\gamma})} - \frac{1}{E_{\gamma}\left(-\frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \lambda_m T^{\gamma}\right)} \right\}^2}_{T(\lambda_m)} |g_m|^2. \end{aligned}$$

So, we can write

$$\begin{aligned} T(\lambda_n) &= \frac{1}{E_{\gamma}^2(-(\alpha - \beta)\lambda_n T^{\gamma})} \left(1 - \frac{E_{\gamma}(-(\alpha - \beta)\lambda_n T^{\gamma})}{E_{\gamma}\left(-\frac{(\alpha - \beta)}{1 + \varepsilon \lambda_n^s} \lambda_n T^{\gamma}\right)} \right)^2, \\ T(\lambda_m) &= \frac{1}{E_{\gamma}^2(-(\alpha + \beta)\lambda_m T^{\gamma})} \left(1 - \frac{E_{\gamma}(-(\alpha + \beta)\lambda_m T^{\gamma})}{E_{\gamma}\left(-\frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \lambda_m T^{\gamma}\right)} \right)^2. \end{aligned}$$

As $\|\Delta_{\varepsilon}^s(0)\|^2 = \|\Delta_{\varepsilon s}^s(0)\|^2 + \|\Delta_{\varepsilon c}^s(0)\|^2$, we just need to estimate $\Delta_{\varepsilon s}^s(0)$ and $\Delta_{\varepsilon c}^s(0)$. Indeed, we have

$$\|\Delta_{\varepsilon s}^s(0)\|^2 = \sum_{n=1}^{\infty} \frac{1}{E_{\gamma}^2(-(\alpha - \beta)\lambda_n T^{\gamma})} \left(1 - \frac{E_{\gamma}(-(\alpha - \beta)\lambda_n T^{\gamma})}{E_{\gamma}\left(-\frac{(\alpha - \beta)}{1 + \varepsilon \lambda_n^s} \lambda_n T^{\gamma}\right)} \right)^2 |g_n|^2.$$

Note that condition $g \in H^1$, which means $\sum_{n=1}^{\infty} \frac{1}{E_{\gamma}^2(-(\alpha - \beta)\lambda_n T^{\gamma})} |g_n|^2 < +\infty$, then for any $\eta > 0$, there is an integer $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} \frac{1}{E_{\gamma}^2(-(\alpha - \beta)\lambda_n T^{\gamma})} |g_n|^2 < \frac{\eta^2}{4}.$$

Hence,

$$\begin{aligned} \|\Delta_{\varepsilon s}(0)\|^2 &= \sum_{n=1}^N \frac{1}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha-\beta)\lambda_n T^\gamma)}{E_\gamma\left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma\right)}\right)^2 |g_n|^2 \\ &\quad + \sum_{n=N+1}^{\infty} \frac{1}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha-\beta)\lambda_n T^\gamma)}{E_\gamma\left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma\right)}\right)^2 |g_n|^2. \end{aligned}$$

We have

$$\begin{aligned} B_2 &= \sum_{n=N+1}^{\infty} \frac{1}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha-\beta)\lambda_n T^\gamma)}{E_\gamma\left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma\right)}\right)^2 |g_n|^2 \\ &\leq \sum_{n=N+1}^{\infty} \frac{1}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} |g_n|^2 < \frac{\eta^2}{4}. \end{aligned}$$

On the other hand, by Lemma 1.5.3, we have $\frac{d}{dy} E_\gamma(y) = \frac{1}{\gamma y} E_{\gamma,0}(y)$. Therefore, there exist constants $\rho_n \in \left(-(\alpha-\beta)\lambda_n T^\gamma, -\frac{(\alpha-\beta)}{1+2\varepsilon\lambda_n}\lambda_n T^\gamma\right)$, $n \geq 1$, such that

$$E_\gamma\left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma\right) - E_\gamma(-(\alpha-\beta)\lambda_n T^\gamma) = \frac{E_{\gamma,0}(\rho_n)}{\gamma\rho_n} \left(\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\varepsilon\lambda_n^{s+1}T^\gamma\right).$$

We have

$$\begin{aligned} B_1 &= \sum_{n=1}^N \frac{1}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha-\beta)\lambda_n T^\gamma)}{E_\gamma\left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma\right)}\right)^2 |g_n|^2 \\ &= \sum_{n=1}^N \frac{1}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} \left(\frac{E_\gamma\left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma\right) - E_\gamma(-(\alpha-\beta)\lambda_n T^\gamma)}{E_\gamma\left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma\right)}\right)^2 |g_n|^2 \\ &\leq \sum_{n=1}^N \frac{1}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} \frac{(E_{\gamma,0}(\rho_n))^2}{(\gamma\rho_n)^2} \left(\frac{-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma + (\alpha-\beta)\lambda_n T^\gamma}{E_\gamma\left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma\right)}\right)^2 |g_n|^2 \\ &\leq \sum_{n=1}^N \frac{1}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} C_9^2 \frac{\left(1 + \frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma\right)^2}{(\gamma\rho_n)^2 (1-\rho_n)^2} \left(\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\varepsilon\lambda_n^{s+1}T^\gamma\right)^2 |g_n|^2 \\ &\leq \sum_{n=1}^N \frac{1}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} \left(C_{10} \left(\frac{1+\varepsilon\lambda_n^s}{(\alpha-\beta)\lambda_n T^\gamma}\right) \frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\varepsilon\lambda_n^{s+1}T^\gamma\right)^2 |g_n|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{n=1}^N \frac{1}{E_\gamma^2(-(\alpha - \beta)\lambda_n T^\gamma)} (C_{10}\varepsilon\lambda_n^s)^2 |g_n|^2 \\
&\leq (C_{10}\varepsilon\lambda_N^s)^2 \sum_{n=1}^N \frac{1}{E_\gamma^2(-(\alpha - \beta)\lambda_n T^\gamma)} |g_n|^2 \\
&\leq (C_{10}\varepsilon\lambda_N^s)^2 F_2^2.
\end{aligned}$$

We choose ε such that

$$(C_{10}\lambda_N^s)^2 F_2^2 \varepsilon^2 \leq \frac{\eta^2}{4},$$

as a result, we have

$$\|\Delta_{\varepsilon^s}^s(0)\|^2 = B_1 + B_2 \leq \frac{\eta^2}{2},$$

and

$$\|\Delta_{\varepsilon^c}^s(0)\|^2 = \sum_{m=1}^{\infty} \frac{1}{E_\gamma^2(-(\alpha + \beta)\lambda_m T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha + \beta)\lambda_m T^\gamma)}{E_\gamma\left(-\frac{(\alpha + \beta)}{1 + \varepsilon\lambda_m^s}\lambda_m T^\gamma\right)}\right)^2 |g_m|^2.$$

We assume that $g \in H^1$, is equivalent to $\sum_{m=1}^{\infty} \frac{1}{E_\gamma^2(-(\alpha + \beta)\lambda_m T^\gamma)} |g_m|^2 < +\infty$, then for any $\eta > 0$, there is an integer $M \in \mathbb{N}$ such that

$$\sum_{m=M+1}^{\infty} \frac{1}{E_\gamma^2(-(\alpha + \beta)\lambda_m T^\gamma)} |g_m|^2 < \frac{\eta^2}{4}.$$

Hence,

$$\begin{aligned}
\|\Delta_{\varepsilon^c}^s(0)\|^2 &= \sum_{m=1}^M \frac{1}{E_\gamma^2(-(\alpha + \beta)\lambda_m T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha + \beta)\lambda_m T^\gamma)}{E_\gamma\left(-\frac{(\alpha + \beta)}{1 + \varepsilon\lambda_m^s}\lambda_m T^\gamma\right)}\right)^2 |g_m|^2 \\
&\quad + \sum_{m=M+1}^{\infty} \frac{1}{E_\gamma^2(-(\alpha + \beta)\lambda_m T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha + \beta)\lambda_m T^\gamma)}{E_\gamma\left(-\frac{(\alpha + \beta)}{1 + \varepsilon\lambda_m^s}\lambda_m T^\gamma\right)}\right)^2 |g_m|^2.
\end{aligned}$$

We have

$$B_4 = \sum_{m=M+1}^{\infty} \frac{1}{E_\gamma^2(-(\alpha + \beta)\lambda_m T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha + \beta)\lambda_m T^\gamma)}{E_\gamma\left(-\frac{(\alpha + \beta)}{1 + \varepsilon\lambda_m^s}\lambda_m T^\gamma\right)}\right)^2 |g_m|^2$$

$$\leq \sum_{m=M+1}^{\infty} \frac{1}{E_{\gamma}^2(-(\alpha + \beta)\lambda_m T^{\gamma})} |g_m|^2 < \frac{\eta^2}{4}.$$

On the other hand, by Lemma 1.5.3, we have $\frac{d}{dy} E_{\gamma}(y) = \frac{1}{\gamma y} E_{\gamma,0}(y)$. Therefore, there exist constants $\rho_m \in \left(-(\alpha + \beta)\lambda_m T^{\gamma}, -\frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \lambda_m T^{\gamma}\right)$, $m \geq 1$, such that

$$E_{\gamma}\left(-\frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \lambda_m T^{\gamma}\right) - E_{\gamma}(-(\alpha + \beta)\lambda_m T^{\gamma}) = \frac{E_{\gamma,0}(\rho_m)}{\gamma \rho_m} \left(\frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \varepsilon \lambda_m^{s+1} T^{\gamma}\right).$$

We have

$$\begin{aligned} B_3 &= \sum_{m=0}^M \frac{1}{E_{\gamma}^2(-(\alpha + \beta)\lambda_m T^{\gamma})} \left(1 - \frac{E_{\gamma}(-(\alpha + \beta)\lambda_m T^{\gamma})}{E_{\gamma}\left(-\frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \lambda_m T^{\gamma}\right)}\right)^2 |g_m|^2 \\ &= \sum_{m=1}^M \frac{1}{E_{\gamma}^2(-(\alpha + \beta)\lambda_m T^{\gamma})} \left(\frac{E_{\gamma}\left(-\frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \lambda_m T^{\gamma}\right) - E_{\gamma}(-(\alpha + \beta)\lambda_m T^{\gamma})}{E_{\gamma}\left(-\frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \lambda_m T^{\gamma}\right)}\right)^2 |g_m|^2 \\ &\leq \sum_{m=1}^M \frac{1}{E_{\gamma}^2(-(\alpha + \beta)\lambda_m T^{\gamma})} \frac{(E_{\gamma,0}(\rho_m))^2}{(\gamma \rho_m)^2} \left(\frac{-\frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \lambda_m T^{\gamma} + (\alpha + \beta)\lambda_m T^{\gamma}}{E_{\gamma}\left(-\frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \lambda_m T^{\gamma}\right)}\right)^2 |g_m|^2 \\ &\leq \sum_{m=1}^M \frac{1}{E_{\gamma}^2(-(\alpha + \beta)\lambda_m T^{\gamma})} C_{11}^2 \frac{\left(1 + \frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \lambda_m T^{\gamma}\right)^2}{(\gamma \rho_m)^2 (1 - \rho_m)^2} \left(\frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \varepsilon \lambda_m^{s+1} T^{\gamma}\right)^2 |g_m|^2 \\ &\leq \sum_{m=1}^M \frac{1}{E_{\gamma}^2(-(\alpha + \beta)\lambda_m T^{\gamma})} \left(C_{12} \left(\frac{1 + \varepsilon \lambda_m^s}{(\alpha + \beta)\lambda_m T^{\gamma}}\right) \frac{(\alpha + \beta)}{1 + \varepsilon \lambda_m^s} \varepsilon \lambda_m^{s+1} T^{\gamma}\right)^2 |g_m|^2 \\ &\leq \sum_{m=1}^M \frac{1}{E_{\gamma}^2(-(\alpha + \beta)\lambda_m T^{\gamma})} (C_{12} \varepsilon \lambda_m^s)^2 |g_m|^2 \\ &\leq (C_{12} \varepsilon \lambda_M^s)^2 \sum_{m=1}^M \frac{1}{E_{\gamma}^2(-(\alpha + \beta)\lambda_m T^{\gamma})} |g_m|^2 \\ &\leq (C_{12} \varepsilon \lambda_M^s)^2 F_2^2. \end{aligned}$$

We choose ε such that

$$(C_{12} \varepsilon \lambda_M^s)^2 F_2^2 \varepsilon^2 \leq \frac{\eta^2}{4},$$

as a result, we have

$$\|\Delta_{\varepsilon c}^s(0)\|^2 = B_3 + B_4 \leq \frac{\eta^2}{2}.$$

For any ε satisfying $\varepsilon \leq \frac{\eta}{2C_{10}\lambda_N^s F_2}$ and $\varepsilon \leq \frac{\eta}{2C_{12}\lambda_M^s F_2}$, and η is an arbitrary positive real number, we have $\|\Delta_\varepsilon^s(0)\|^2 \leq \eta^2$, this shows that

$$\|\Delta_\varepsilon^s(0)\|^2 \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

□

Let $v_\varepsilon^\delta(x, t)$ be the approximate solution associated to inexact data $g^\delta(x)$ such that

$$\|g^\delta - g\| \leq \delta, \quad (2.6.11)$$

where $\delta > 0$ is a noise level.

Theorem 2.6.2. For $s \geq 1$, problem (2.6.1) is well-posed. Moreover, if the solution u of problem (2.5.1) satisfies

$$\|v(\cdot, 0)\|_{\mathcal{D}(A^r)} \leq E_2, \quad r > 0, \quad E_2 > \delta,$$

then the following statements hold:

(1) If $0 < r < s$ and choose $\varepsilon = \left(\frac{\delta}{E_2}\right)^{\frac{s}{r+1}}$, we have the following convergence estimate

$$\|v_\varepsilon^\delta(\cdot, 0) - v(\cdot, 0)\| \leq \bar{K} E_2^{\frac{1}{r+1}} \delta^{\frac{r}{r+1}}.$$

(2) If $r \geq s$ and choose $\varepsilon = \left(\frac{\delta}{E_2}\right)^{\frac{s}{s+1}}$, we have the following convergence estimate

$$\|v_\varepsilon^\delta(\cdot, 0) - v(\cdot, 0)\| \leq \tilde{K} E_2^{\frac{1}{s+1}} \delta^{\frac{s}{s+1}}.$$

where \bar{K}, \tilde{K} are positive constants depending on $r, \gamma, T, \alpha, \beta$ and λ_1 .

Proof. By the triangle inequality, we have

$$\|v_\varepsilon^\delta(\cdot, 0) - v(\cdot, 0)\| \leq \|v_\varepsilon^\delta(\cdot, 0) - v_\varepsilon(\cdot, 0)\| + \|v_\varepsilon(\cdot, 0) - v(\cdot, 0)\|.$$

We firstly give an estimate for the first term. From (2.6.9) and (2.6.11), we have

$$\|v_\varepsilon^\delta(\cdot, 0) - v_\varepsilon(\cdot, 0)\|^2 = \sum_{n=1}^{\infty} \left(\frac{g_n^\delta - g_n}{E_\gamma \left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s} \lambda_n T^\gamma \right)} \right)^2$$

$$\begin{aligned}
& + \sum_{m=0}^{\infty} \left(\frac{g_m^\delta - g_m}{E_\gamma \left(-\frac{(\alpha+\beta)}{1+\varepsilon\lambda_m^s} \lambda_m T^\gamma \right)} \right)^2 \\
& \leq \left(\left(\sup_{n \geq 1} A(n) \right)^2 + 1 + \left(\sup_{m \geq 1} A(m) \right)^2 \right) \delta^2, \tag{2.6.12}
\end{aligned}$$

where

$$A(n) = \frac{1}{E_\gamma \left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s} \lambda_n T^\gamma \right)},$$

and

$$A(m) = \frac{1}{E_\gamma \left(-\frac{(\alpha+\beta)}{1+\varepsilon\lambda_m^s} \lambda_m T^\gamma \right)}.$$

By Lemma 2.6.1, we get

$$A(n) \leq C_7 \left(1 + \varepsilon^{-\frac{1}{s}} \right),$$

and

$$A(m) \leq C_8 \left(1 + \varepsilon^{-\frac{1}{s}} \right).$$

Then, by using (2.6.12), we obtain

$$\left\| v_\varepsilon^\delta(\cdot, 0) - v_\varepsilon(\cdot, 0) \right\| \leq \sqrt{\left(C_7 \left(1 + \varepsilon^{-\frac{1}{s}} \right) \right)^2 + 1 + \left(C_8 \left(1 + \varepsilon^{-\frac{1}{s}} \right) \right)^2} \delta. \tag{2.6.13}$$

On the other hand, we have

$$\|v_\varepsilon(\cdot, 0) - v(\cdot, 0)\|^2 = \|\Delta_{\varepsilon s}^s(0)\|^2 + \|\Delta_{\varepsilon c}^s(0)\|^2,$$

where

$$\|\Delta_{\varepsilon s}^s(0)\|^2 = \sum_{n=1}^{\infty} \frac{1}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha-\beta)\lambda_n T^\gamma)}{E_\gamma\left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma\right)} \right)^2 |g_n|^2,$$

and

$$\|\Delta_{\varepsilon c}^s(0)\|^2 = \sum_{m=1}^{\infty} \frac{1}{E_\gamma^2(-(\alpha+\beta)\lambda_m T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha+\beta)\lambda_m T^\gamma)}{E_\gamma\left(-\frac{(\alpha+\beta)}{1+\varepsilon\lambda_m^s}\lambda_m T^\gamma\right)} \right)^2 |g_m|^2.$$

We have

$$\begin{aligned} \|\Delta_{\varepsilon^s}^s(0)\|^2 &= \sum_{n=1}^{\infty} \frac{1}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha-\beta)\lambda_n T^\gamma)}{E_\gamma\left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma\right)}\right)^2 |g_n|^2 \\ &= \sum_{n=1}^{N_1-1} \frac{\lambda_n^{2r}\lambda_n^{-2r}}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha-\beta)\lambda_n T^\gamma)}{E_\gamma\left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma\right)}\right)^2 |g_n|^2 \\ &\quad + \sum_{n=N_1}^{\infty} \frac{\lambda_n^{2r}\lambda_n^{-2r}}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha-\beta)\lambda_n T^\gamma)}{E_\gamma\left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma\right)}\right)^2 |g_n|^2, \end{aligned}$$

where $N_1 = \min\{n : \lambda_n \geq \varepsilon^{-\frac{1}{s}}\}$.

$$\begin{aligned} \|\Delta_{\varepsilon^s}^s(0)\|^2 &= \sum_{n=1}^{\infty} \frac{1}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha-\beta)\lambda_n T^\gamma)}{E_\gamma\left(-\frac{(\alpha-\beta)}{1+\varepsilon\lambda_n^s}\lambda_n T^\gamma\right)}\right)^2 |g_n|^2 \\ &\leq \sum_{n=1}^{N_1-1} \frac{\lambda_n^{2r}\lambda_n^{-2r}}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} \lambda_n^{2s} (C_{10}\varepsilon)^2 |g_n|^2 + \sum_{n=N_1}^{\infty} \frac{\lambda_n^{2r}\lambda_n^{-2r}}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} |g_n|^2 \\ &\leq \sum_{n=1}^{N_1-1} \frac{\lambda_n^{2r}}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} (C_{10}\lambda_n^{s-r}\varepsilon)^2 |g_n|^2 + \sum_{n=N_1}^{\infty} \frac{\lambda_n^{2r}}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} \varepsilon^{\frac{2r}{s}} |g_n|^2. \end{aligned}$$

If $r < s$, then $\lambda_n^{s-r} \leq \varepsilon^{\frac{r-s}{s}}$, for all $n \leq N_1 - 1$. We obtain

$$\begin{aligned} &\|\Delta_{\varepsilon^s}^s(0)\|^2 \\ &\leq \sum_{n=1}^{N_1-1} \frac{\lambda_n^{2r}}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} (C_{10}\varepsilon^{\frac{r}{s}})^2 |g_n|^2 + \sum_{n=N_1}^{\infty} \frac{\lambda_n^{2r}}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} \varepsilon^{\frac{2r}{s}} |g_n|^2 \\ &\leq (C_{10})^2 E^2 \varepsilon^{\frac{2r}{s}} + E^2 \varepsilon^{\frac{2r}{s}} \\ &\leq k_1^2 E_2^2 \varepsilon^{\frac{2r}{s}}. \end{aligned} \tag{2.6.14}$$

If $r \geq s$, then $\lambda_n^{s-r} \leq \lambda_1^{s-r}$. We get

$$\begin{aligned} \|\Delta_{\varepsilon^s}^s(0)\|^2 &\leq \sum_{n=1}^{\infty} \frac{\lambda_n^{2r}}{E_\gamma^2(-(\alpha-\beta)\lambda_n T^\gamma)} (C_{10}\lambda_1^{s-r}\varepsilon)^2 |g_n|^2 \\ &\leq (C_{13})^2 E_2^2 \varepsilon^2. \end{aligned} \tag{2.6.15}$$

On the other hand, we have

$$\begin{aligned} \|\Delta_{\varepsilon c}^s(0)\|^2 &= \sum_{m=1}^{\infty} \frac{1}{E_\gamma^2(-(\alpha+\beta)\lambda_m T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha+\beta)\lambda_m T^\gamma)}{E_\gamma\left(-\frac{(\alpha+\beta)}{1+\varepsilon\lambda_m^s}\lambda_m T^\gamma\right)}\right)^2 |g_m|^2 \\ &= \sum_{m=1}^{M_1-1} \frac{\lambda_m^{2r}\lambda_m^{-2r}}{E_\gamma^2(-(\alpha+\beta)\lambda_m T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha+\beta)\lambda_m T^\gamma)}{E_\gamma\left(-\frac{(\alpha+\beta)}{1+\varepsilon\lambda_m^s}\lambda_m T^\gamma\right)}\right)^2 |g_m|^2 \\ &\quad + \sum_{m=M_1}^{\infty} \frac{\lambda_m^{2r}\lambda_m^{-2r}}{E_\gamma^2(-(\alpha+\beta)\lambda_m T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha+\beta)\lambda_m T^\gamma)}{E_\gamma\left(-\frac{(\alpha+\beta)}{1+\varepsilon\lambda_m^s}\lambda_m T^\gamma\right)}\right)^2 |g_m|^2, \end{aligned}$$

where $M_1 = \min\{m : \lambda_m \geq \varepsilon^{-\frac{1}{s}}\}$.

$$\begin{aligned} \|\Delta_{\varepsilon c}^s(0)\|^2 &= \sum_{m=1}^{\infty} \frac{1}{E_\gamma^2(-(\alpha+\beta)\lambda_m T^\gamma)} \left(1 - \frac{E_\gamma(-(\alpha+\beta)\lambda_m T^\gamma)}{E_\gamma\left(-\frac{(\alpha+\beta)}{1+\varepsilon\lambda_m^s}\lambda_m T^\gamma\right)}\right)^2 |g_m|^2 \\ &\leq \sum_{m=1}^{M_1-1} \frac{\lambda_m^{2r}\lambda_m^{-2r}}{E_\gamma^2(-(\alpha+\beta)\lambda_m T^\gamma)} \lambda_m^{2s} (C_{12}\varepsilon)^2 |g_m|^2 + \sum_{m=M_1}^{\infty} \frac{\lambda_m^{2r}\lambda_m^{-2r}}{E_\gamma^2(-(\alpha+\beta)\lambda_m T^\gamma)} |g_m|^2 \\ &\leq \sum_{m=1}^{M_1-1} \frac{\lambda_m^{2r}}{E_\gamma^2(-(\alpha+\beta)\lambda_m T^\gamma)} (C_{12}\lambda_m^{s-r}\varepsilon)^2 |g_m|^2 + \sum_{m=M_1}^{\infty} \frac{\lambda_m^{2r}}{E_\gamma^2(-(\alpha+\beta)\lambda_m T^\gamma)} \varepsilon^{\frac{2r}{s}} |g_m|^2. \end{aligned}$$

If $r < s$, then $\lambda_m^{s-r} \leq \varepsilon^{\frac{r-s}{s}}$, for all $1 \leq m \leq M_1 - 1$. We obtain

$$\begin{aligned} \|\Delta_{\varepsilon c}^s(0)\|^2 &\leq \sum_{m=1}^{M_1-1} \frac{\lambda_m^{2r}}{E_\gamma^2(-(\alpha+\beta)\lambda_m T^\gamma)} (C_{12}\varepsilon^{\frac{r}{s}})^2 |g_m|^2 + \sum_{m=M_1}^{\infty} \frac{\lambda_m^{2r}}{E_\gamma^2(-(\alpha+\beta)\lambda_m T^\gamma)} \varepsilon^{\frac{2r}{s}} |g_m|^2 \\ &\leq (C_{12})^2 E_2^2 \varepsilon^{\frac{2r}{s}} + E^2 \varepsilon^{\frac{2r}{s}} \\ &\leq k_2^2 E_2^2 \varepsilon^{\frac{2r}{s}}. \end{aligned} \tag{2.6.16}$$

If $r \geq s$, then $\lambda_m^{s-r} \leq \lambda_1^{s-r}$. We get

$$\begin{aligned} \|\Delta_{\varepsilon c}^s(0)\|^2 &\leq \sum_{m=1}^{\infty} \frac{\lambda_m^{2r}}{E_\gamma^2(-(\alpha+\beta)\lambda_m T^\gamma)} (C_{12}\lambda_1^{s-r}\varepsilon)^2 |g_m|^2 \\ &\leq (C_{14})^2 E_2^2 \varepsilon^2, \end{aligned} \tag{2.6.17}$$

where $k_1^2 = (C_{10})^2 + 1$ and $k_2^2 = (C_{12})^2 + 1$. We obtain

$$\|v_\varepsilon(., 0) - v(., 0)\| \leq \begin{cases} \sqrt{k_1^2 + k_2^2} E_2 \varepsilon^{\frac{r}{s}}, & 0 < r < s, \\ \sqrt{(C_{13})^2 + (C_{14})^2} E_2 \varepsilon, & r \geq s. \end{cases} \quad (2.6.18)$$

By combining (2.6.18) and (2.6.13), we obtain

$$\begin{aligned} \|v_\varepsilon^\delta(., 0) - v(., 0)\| &\leq \sqrt{\left(C_7 \left(1 + \varepsilon^{-\frac{1}{s}}\right)\right)^2 + 1 + \left(C_8 \left(1 + \varepsilon^{-\frac{1}{s}}\right)\right)^2} \delta \\ &+ \begin{cases} \sqrt{k_1^2 + k_2^2} E_2 \varepsilon^{\frac{r}{s}}, & 0 < r < s, \\ \sqrt{(C_{13})^2 + (C_{14})^2} E_2 \varepsilon, & r \geq s. \end{cases} \end{aligned}$$

For $E_2 > \delta$, we have $\delta \leq E_2^{\frac{1}{r+1}} \delta^{\frac{r}{r+1}}$ and $\delta \leq E_2^{\frac{1}{s+1}} \delta^{\frac{s}{s+1}}$.

Choosing the regularization parameter ε by

$$\varepsilon = \begin{cases} \left(\frac{\delta}{E_2}\right)^{\frac{s}{r+1}}, & 0 < r < s, \\ \left(\frac{\delta}{E_2}\right)^{\frac{s}{s+1}}, & r \geq s, \end{cases}$$

then, we have

$$\|v_\varepsilon^\delta(., 0) - v(., 0)\| \leq \begin{cases} \bar{K} E_2^{\frac{1}{r+1}} \delta^{\frac{r}{r+1}}, & 0 < r < s, \\ \tilde{K} E_2^{\frac{1}{s+1}} \delta^{\frac{s}{s+1}}, & r \geq s. \end{cases}$$

The proof is completed. \square

2.7 Numerical experiments

In this section, we present the numerical results for an academic example in a one-dimensional case to validate the proposed method. Numerical tests are conducted for Section 2.5 (Inverse problem for time-fractional heat equation with involution). For the calculations in this numerical test, we utilize the Matlab software, employing the Matlab trapz code to compute the integral via the Trapezoidal rule in the interval $[a, b]$. Additionally, we incorporate the Mittag-Leffler function by *Roberto Garrappa*, which can be downloaded from <https://www.mathworks.com/matlabcentral/fileexchange/48154-the-mittag-leffler-function>.

We consider the following one-dimensional problem

$$\begin{cases} D_t^\gamma v(x, t) - \alpha v_{xx}(x, t) - \beta v_{xx}(-x, t) = 0, & x \in (-1, 1), t \in (0, T), \\ v(-1, t) = v(1, t), u_x(1, t) = v_x(-1, t), & t \in (0, T), \\ v(x, 0) = f(x), & x \in (-1, 1), \end{cases} \quad (2.7.1)$$

with

$$T = 1, \alpha = 1, \beta = 0.5, \gamma = \frac{1}{2}, f(x) = \cos(\pi x) + \sin(2\pi x).$$

Therefore, the solution $v(x, t)$ of (2.7.1) is given by

$$v(x, t) = E_{0.5}(-0.5(2\pi)^2 t^{0.5}) \sin(2\pi x) + E_{0.5}(-1.5(\pi)^2 t^{0.5}) \cos(\pi x). \quad (2.7.2)$$

From (2.7.2), we generate the final data

$$g(x) = v(x, 1) = E_{0.5}(-0.5(4\pi)^2) \sin(2\pi x) + E_{0.5}(-1.5(\pi)^2) \cos(\pi x). \quad (2.7.3)$$

Now, we add noise to the data g using a random perturbation (obtained by the MATLAB command `randn`), we obtain the vector g_δ :

$$\begin{aligned} noise &= randn(size(g)); \\ noise &= \delta \times noise \times norm(g)/norm(noise); \\ g_\delta &= g + noise; \end{aligned}$$

where δ denotes the noise level in the information provided by the measurement. The function "`randn(.)`" generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$, and standard deviation $\sigma = 1$. "`randn(size(ψ))`" returns an array of random entries that is the same size as g .

Our inverse problem is to reconstruct the initial data f_δ from the final data g_δ , and to give an estimate of the difference $\|f - f_\delta\|$, where f is the exact initial data. This question leads us to the resolution of the problem

$$\begin{cases} D_t^{0.5} v(x, t) - v_{xx}(x, t) - 0.5v_{xx}(-x, t) = 0, & x \in (-1, 1), t \in (0, 1), \\ v(-1, t) = v(1, t), u_x(1, t) = v_x(-1, t), & t \in (0, 1), \\ v(x, 1) = g_\delta(x), & x \in (-1, 1), \end{cases} \quad (2.7.4)$$

By computing the regularizing solution with noisy data $\delta = 0.1$, $\epsilon = 10^{-10}$ (the regularization parameter), $s = 1, 2, 3, 4$ (the relaxation parameter) and $M = N = 50$ (cutoff frequency in the Fourier series), we have

$$v_\epsilon^\delta(x, t) = \sum_{n=1}^{50} \frac{E_{0.5}\left(-\frac{(0.5)}{1+\epsilon\lambda_n^s}\lambda_n t^{0.5}\right)}{E_{0.5}\left(-\frac{(0.5)}{1+\epsilon\lambda_n^s}\lambda_n\right)} g_n^\delta \varphi_n(x) + \sum_{m=0}^{50} \frac{E_{0.5}\left(-\frac{(1.5)}{1+\epsilon\lambda_m^s}\lambda_m t^{0.5}\right)}{E_{0.5}\left(-\frac{(1.5)}{1+\epsilon\lambda_m^s}\lambda_m\right)} g_m^\delta \psi_m(x), \quad (2.7.5)$$

end

$$f_\epsilon^\delta(x) = v_\epsilon^\delta(x, 0) = \sum_{n=1}^{50} \frac{1}{E_{0.5}\left(-\frac{(0.5)}{1+\epsilon\lambda_n^s}\lambda_n\right)} g_n^\delta \varphi_n(x) + \sum_{m=0}^{50} \frac{1}{E_{0.5}\left(-\frac{(1.5)}{1+\epsilon\lambda_m^s}\lambda_m\right)} g_m^\delta \psi_m(x). \quad (2.7.6)$$

The results of this section are presented in Figs. 2.1-2.4. We show the error estimation between the exact data $v(x, 0) = f(x)$ and the regularized solution $v_\epsilon^\delta(x, 0) = f_\epsilon^\delta(x)$. We can see that: if the regularization parameter ϵ is small, the regularized solution converges to the exact solution when ϵ tends to zero.

Conclusion

In this study, we have investigated two classes of ill-posed problems arising from fractional equations containing involution terms. To neutralize the instability associated with such problems, we employed a regularization strategy based on a modified pseudo-parabolic regularization method (fractional-order quasi-reversibility) and the Fourier series expansion technique. We rigorously established the convergence of this regularization approach and derived error estimates under specific assumptions regarding the regularity of the sought solution.

It is noteworthy that the regularization effect of this new procedure is particularly pronounced for $s = 3$ and $s = 4$ even in the presence of a very high noise level ($\delta = 10\%$).

This study can be extended to two-dimensional case with respect to the spatial variable, provided that the Fourier method can be employed.

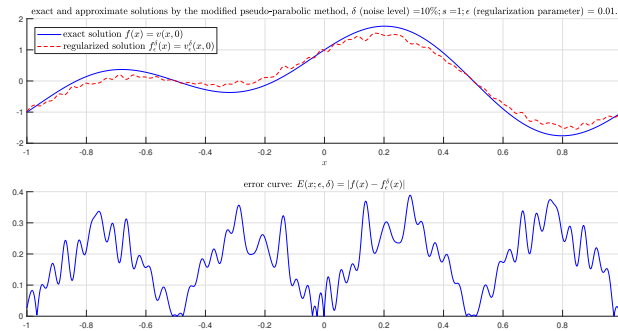


Figure 2.1: The reconstruction result at $t = 0$ from noisy data g_δ .

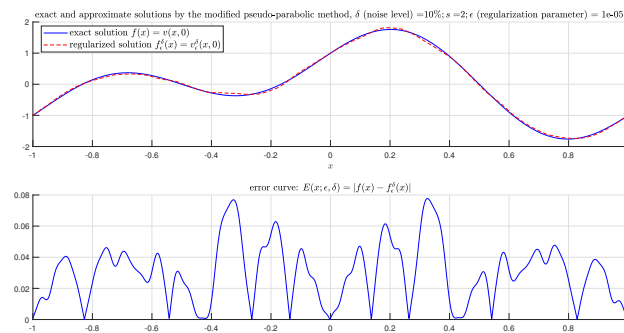


Figure 2.2: The reconstruction result at $t = 0$ from noisy data g_δ .

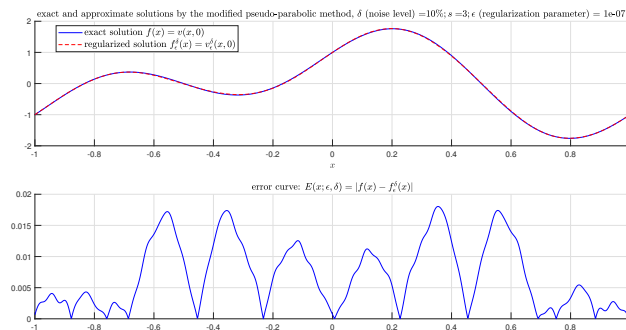


Figure 2.3: The reconstruction result at $t = 0$ from noisy data g_δ .

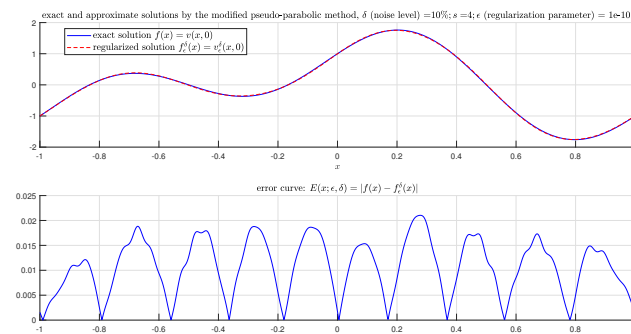


Figure 2.4: The reconstruction result at $t = 0$ from noisy data g_δ .

Chapter 3

SOME REGULARIZATION METHODS FOR A CLASS OF FRACTIONAL PSEUDO-PARABOLIC INVERSE PROBLEMS WITH INVOLUTION

3.1 Position of the problem

This chapter delves into a specific class of inverse problems generated by fractional pseudo-parabolic equation, characterized by the presence of an involution term. Even though previous studies have shed light on the theoretical development and physical motivations behind these problems, there exists a significant void in the literature regarding regularization techniques and numerical approximations for such type of problems. To fill this research gap and offer a thorough examination, this study employs Three distinct methodologies to address the following fractional inverse source problem governed by a pseudo-parabolic equation with involution:

$$\begin{cases} D_t^\gamma [u(x, t) - u_{xx}(x, t) + \varepsilon u_{xx}(\pi - x, t)] - u_{xx}(x, t) + \varepsilon u_{xx}(\pi - x, t) = f(x), & (x, t) \in Q, \\ u(0, t) = u(\pi, t) = 0, & t \in (0, T), \\ u(x, 0) = 0, & x \in (0, \pi), \\ u(x, T) = g(x), & x \in (0, \pi), \end{cases} \quad (3.1.1)$$

where $Q = (0, \pi) \times (0, T)$, ε is a real number and D_t^γ is the Caputo derivative for $0 < \gamma < 1$ defined by (see [53])

$$D_t^\gamma u(x, t) = \begin{cases} \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{u_\tau(x, \tau)}{(t-\tau)^\gamma} d\tau, & 0 < \gamma < 1, \\ \frac{d}{dt} u(x, t) = u_t(x, t), & \gamma = 1. \end{cases}$$

Here we specify that the problem (3.1.1) entails determining the source term $f(x)$ from the final state $g(x)$.

The first method proposed in this study uses a modified version of the quasi-boundary value method. The quasi-boundary value method, also called the non-local auxiliary bound-

ary condition, was pioneered and advanced by Showalter [93]. This regularization technique involves substituting the final or boundary condition with a nonlocal condition, ensuring that the perturbed problem is well-posed. This approach has found application in resolving certain ill-posed problems associated with parabolic, hyperbolic, and elliptic equations. For further insights, refer to [37, 38, 54, 103] and the associated references. Instead of the traditional approach of using $u(x, T) = g(x)$ to address the unknown source problem, we adopt the approach of using $u_\alpha^\delta(x, T) + \alpha L f_\alpha^\delta(x) = g^\delta(x)$ (see [99]). This modified method exhibits a convergence rate of $O(\delta^{\frac{2}{3}})$ under an a-priori choice of the regularization parameter, where δ represents the noise level.

The second method introduced in this study is the quasi-reversibility method, which was originally introduced by Lattes and Lions [58] and has found application in tackling diverse categories of ill-posed problems, including inverse source problems (see [31, 99, 105]). The underlying principle of this approach involves the introduction of regularization terms resembling reversible behaviors. These terms counteract instability and enhance solution accuracy (see [31] and the recent work [32]). In this context, we provide a convergence rate under an a priori bound assumption of the exact solution and introduce a posteriori parameter choice rule. This rule yields a corresponding convergence rate estimate, that is more practical in real-world scenarios, as discussed in [99].

The Third method introduced in this study is expand upon the iterative method presented in [100] for solving the inverse source problem, which is governed by a fractional pseudo-parabolic equation with an involution term. In addition, we conduct an error theoretical analysis to determine the convergence rates of the regularized solution generated using the proposed method. For works on the use of the Landweber regularization method, we advise readers to consult [98, 99, 106, 107].

The objective of this study is to assess the effectiveness and suitability of these Three distinct methods in addressing the inverse source problem. By implementing these diverse strategies, we intend to strengthen the underlying theoretical framework and offer insightful perspectives relevant to real-world engineering applications.

3.2 Analysis of the problem

Throughout the paper, we denote by $H = L^2((0, \pi); \mathbb{R})$ the Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$, which are defined as follows:

$$\langle u, v \rangle := \int_0^\pi u(x) v(x) dx, \quad \|u\|^2 := \int_0^\pi |u(x)|^2 dx.$$

Let $H^2(0, \pi) = W^{2,2}(0, \pi)$ be the usual Sobolev space equipped with the norm $\|h\|_2^2 = \|h\|^2 + \|h'\|^2 + \|h''\|^2$.

Problem (3.1.1) can be written in the form

$$\begin{cases} D_t^\gamma [u(x, t) + Lu(x, t)] + Lu(x, t) = f(x), & (x, t) \in Q, \\ u(x, 0) = 0, & x \in (0, \pi), \\ u(x, T) = g(x), & x \in (0, \pi), \end{cases} \quad (3.2.1)$$

where $L : D(L) \subset H \rightarrow H$ with

$$\begin{cases} D(L) = H_0^1(0, \pi) \cap H^2(0, \pi) = \{u \in H^2(0, \pi) : u(0) = u(\pi) = 0\}, \\ Lu(x) = -u_{xx}(x) + \varepsilon u_{xx}(\pi - x), \quad x \in (0, \pi), \end{cases} \quad (3.2.2)$$

and $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \varepsilon_0 < 1$. It can be readily verified that the introduced operator is self-adjoint. For all $|\varepsilon| < 1$, the nonlocal problem (3.2.2) has the following eigenvalues, as detailed in reference [90]:

$$\lambda_{2k+1} = (1 - \varepsilon)(2k + 1)^2, \quad k \in \mathbb{N}, \quad \lambda_{2k} = (1 + \varepsilon)(2k)^2, \quad k \in \mathbb{N}^*, \quad (3.2.3)$$

with the following corresponding normalized eigenfunctions

$$\varphi_{2k+1}(x) = \sqrt{\frac{2}{\pi}} \sin((2k + 1)x), \quad k \in \mathbb{N}, \quad \varphi_{2k}(x) = \sqrt{\frac{2}{\pi}} \sin(2kx), \quad k \in \mathbb{N}^*. \quad (3.2.4)$$

We have

$$L\varphi_{2k+1}(x) = \lambda_{2k+1}\varphi_{2k+1}(x), \quad k \in \mathbb{N}, \quad L\varphi_{2k}(x) = \lambda_{2k}\varphi_{2k}(x), \quad k \in \mathbb{N}^*,$$

with

$$0 < \lambda_1 \leq \lambda_3 \leq \dots \leq \lambda_{2k+1} \leq \dots, \lim_{k \rightarrow \infty} \lambda_{2k+1} = +\infty,$$

$$0 < \lambda_2 \leq \lambda_4 \leq \dots \leq \lambda_{2k} \leq \dots, \lim_{k \rightarrow \infty} \lambda_{2k} = +\infty,$$

Under this notation, the system $\mathcal{B} = \{\varphi_{2k}\}_{k=1}^{\infty} \cup \{\varphi_{2k+1}\}_{k=0}^{\infty}$ constitutes an orthonormal basis in H .

For $r \geq 0$, let

$$\mathbb{H}_r = \mathcal{D}(L^r) = \left\{ h \in H : \sum_{k=0}^{\infty} \lambda_{2k+1}^{2r} |(h, \varphi_{2k+1})|^2 + \sum_{k=1}^{\infty} \lambda_{2k}^{2r} |(h, \varphi_{2k})|^2 < \infty \right\},$$

denotes the Hilbert space (sub-space of H) with the norm

$$\|h\|_r = \left(\sum_{k=0}^{\infty} \lambda_{2k+1}^{2r} |(h, \varphi_{2k+1})|^2 + \sum_{k=1}^{\infty} \lambda_{2k}^{2r} |(h, \varphi_{2k})|^2 \right)^{\frac{1}{2}}.$$

The inverse problem focuses on reconstructing the function $f(x)$ from the noisy measured data $g^\delta(x)$. We assume that:

$$\|g^\delta(x) - g(x)\| \leq \delta, \quad (3.2.5)$$

where $\delta > 0$ represents the level of noise.

For the sake of simplicity, we adopt the following notations:

$$z_{2k+1}(t) = \left(1 - E_{\gamma,1} \left(-\frac{\lambda_{2k+1}}{1 + \lambda_{2k+1}} t^\gamma \right) \right), \quad k = 0, 1, 2, \dots, \quad (3.2.6)$$

$$z_{2k}(t) = \left(1 - E_{\gamma,1} \left(-\frac{\lambda_{2k}}{1 + \lambda_{2k}} t^\gamma \right) \right), \quad k = 1, 2, \dots, \quad (3.2.7)$$

We start by establishing the existence and uniqueness of the solution of Problem (3.2.1). Given that the eigenfunctions (3.2.4) constitute an orthonormal basis within H , we search the functions $u(x, t)$ and $f(x)$ in the following forms:

$$u(x, t) = \sum_{k=0}^{\infty} w_k(t) \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} v_k(t) \varphi_{2k}(x), \quad (3.2.8)$$

and

$$f(x) = \sum_{k=0}^{\infty} f_{1k} \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} f_{2k} \varphi_{2k}(x). \quad (3.2.9)$$

where $w_k(t)$, $v_k(t)$, f_{1k} and f_{2k} are unknown.

By substituting equations (3.2.8) and (3.2.9) into equation (3.1.1), we derive the scalar equations for the functions $w_k(t)$, $v_k(t)$ and the constants f_{1k} , f_{2k} :

$$D_t^\gamma w_k(t) + \frac{\lambda_{2k+1}}{1 + \lambda_{2k+1}} w_k(t) = \frac{f_{1k}}{1 + \lambda_{2k+1}}, \quad D_t^\gamma v_k(t) + \frac{\lambda_{2k}}{1 + \lambda_{2k}} v_k(t) = \frac{f_{2k}}{1 + \lambda_{2k}}. \quad (3.2.10)$$

Solving these equations, we obtain

$$w_k(t) = \frac{f_{1k}}{\lambda_{2k+1}} + C_{1k} E_{\gamma,1} \left(-\frac{\lambda_{2k+1}}{1 + \lambda_{2k+1}} t^\gamma \right), \quad v_k(t) = \frac{f_{2k}}{\lambda_{2k}} + C_{2k} E_{\gamma,1} \left(-\frac{\lambda_{2k}}{1 + \lambda_{2k}} t^\gamma \right). \quad (3.2.11)$$

To determine these constants, we apply the conditions outlined in Equation (3.2.1). Let

$$g_{1k} = (g, \varphi_{2k+1}), \quad g_{2k} = (g, \varphi_{2k}). \quad (3.2.12)$$

We first find C_{1k} :

$$\begin{aligned} w_k(0) &= \frac{f_{1k}}{\lambda_{2k+1}} + C_{1k} = 0, \\ w_k(T) &= \frac{f_{1k}}{\lambda_{2k+1}} + C_{1k} E_{\gamma,1} \left(-\frac{\lambda_{2k+1}}{1 + \lambda_{2k+1}} T^\gamma \right) = g_{1k}. \end{aligned} \quad (3.2.13)$$

Then, we have

$$C_{1k} = \frac{-g_{1k}}{1 - E_{\gamma,1} \left(-\frac{\lambda_{2k+1}}{1 + \lambda_{2k+1}} T^\gamma \right)},$$

The constant f_{1k} is represented as

$$f_{1k} = -C_{1k} \lambda_{2k+1} = \frac{\lambda_{2k+1} g_{1k}}{1 - E_{\gamma,1} \left(-\frac{\lambda_{2k+1}}{1 + \lambda_{2k+1}} T^\gamma \right)}.$$

Now we find C_{2k} :

$$v_k(0) = \frac{f_{2k}}{\lambda_{2k}} + C_{2k} = 0,$$

$$v_k(T) = \frac{f_{2k}}{\lambda_{2k}} + C_{2k} E_{\gamma,1} \left(-\frac{\lambda_{2k}}{1 + \lambda_{2k}} T^\gamma \right) = g_{2k}. \quad (3.2.14)$$

Then, we obtain

$$C_{2k} = \frac{-g_{2k}}{1 - E_{\gamma,1} \left(-\frac{\lambda_{2k}}{1 + \lambda_{2k}} T^\gamma \right)}.$$

For the constant f_{2k} we find

$$f_{2k} = -C_{2k} \lambda_{2k} = \frac{\lambda_{2k} g_{2k}}{1 - E_{\gamma,1} \left(-\frac{\lambda_{2k}}{1 + \lambda_{2k}} T^\gamma \right)}.$$

Substituting $w_k(t)$, $v_k(t)$, f_{1k} and f_{2k} into equations (3.2.8) and (3.2.9), we obtain

$$u(x, t) = \sum_{k=0}^{\infty} \frac{f_{1k}}{\lambda_{2k+1}} z_{2k+1}(t) \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{f_{2k}}{\lambda_{2k}} z_{2k}(t) \varphi_{2k}(x), \quad (3.2.15)$$

Then, we get

$$u(x, t) = \sum_{k=0}^{\infty} \frac{z_{2k+1}(t)}{z_{2k+1}(T)} g_{1k} \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{z_{2k}(t)}{z_{2k}(T)} g_{2k} \varphi_{2k}(x), \quad (3.2.16)$$

From this representation, we find

$$f(x) = \sum_{k=0}^{\infty} \frac{\lambda_{2k+1}}{z_{2k+1}(T)} g_{1k} \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{\lambda_{2k}}{z_{2k}(T)} g_{2k} \varphi_{2k}(x). \quad (3.2.17)$$

We define the linear operator $\mathbf{K} : L^2(0, \pi) \longrightarrow L^2(0, \pi)$ as follows.

$$\mathbf{K}f(x) = \sum_{k=0}^{\infty} \frac{f_{2k+1}}{\lambda_{2k+1}} z_{2k+1}(T) \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{f_{2k}}{\lambda_{2k}} z_{2k}(T) \varphi_{2k}(x).$$

We can also write this equation in the form of an integral equation of the first kind:

$$\mathbf{K}f(x) := \int_0^\pi k(x, \zeta) f(y) dy = g(x), \quad (3.2.18)$$

where the kernel $k(x, y) = \sum_{k=0}^{\infty} \theta_{2k+1} \varphi_{2k+1}(x) \varphi_{2k+1}(y) + \sum_{k=1}^{\infty} \theta_{2k} \varphi_{2k}(x) \varphi_{2k}(y)$.

It is clear that the kernel $k(., .)$ is real-valued and symmetric, i.e., $k(x, y) = \overline{k(y, x)}$,

which allows us to conclude that \mathbf{K} is a compact self-adjoint operator. Moreover, \mathbf{K} is an injective operator, with eigenpairs (θ_j, φ_j) :

$$\mathbf{K}\varphi_{2k} = \theta_{2k}\varphi_{2k}, \quad \mathbf{K}\varphi_{2k+1} = \theta_{2k+1}\varphi_{2k+1},$$

and

$$\theta_{2k} = \frac{z_{2k}(T)}{\lambda_{2k}}, \quad \theta_{2k+1} = \frac{z_{2k+1}(T)}{\lambda_{2k+1}}.$$

Remark 3.2.1. In equation (3.2.18), the solution f is given by $f = \mathbf{K}^{-1}g$. In this situation, the algebraic inversion does not pose a problem, because the operator \mathbf{K} is injective. On the other hand, this inversion is not continuous (unstable) because \mathbf{K} is compact. So, we are concerned with an ill-posed problem which requires a regularization procedure.

We note that the proofs of our results are technical. To simplify, we introduce the following technical lemmas.

Lemma 3.2.1. *For all $\lambda_j > 0$, $j = 2k$ and $j = 2k + 1$, there exist positive constants a_1 and a_2 such that:*

$$z_{2k}(T) \geq a_1 > 0, \quad z_{2k+1}(T) \geq a_2 > 0. \quad (3.2.19)$$

Proof. For $T > 0$ and $\lambda_{2k} > 0$, we have

$$\frac{1}{T^\gamma} \leq \frac{1 + \lambda_{2k}}{\lambda_{2k}T^\gamma} \leq \frac{1 + \lambda_2}{\lambda_2T^\gamma} = \frac{1 + 4(1 + \varepsilon)}{4(1 + \varepsilon)T^\gamma} = m_1(\varepsilon).$$

We suppose that $|\varepsilon| \leq \varepsilon_0 < 1$. The function $m_1(\varepsilon)$ is positive and decreasing on the interval $[\varepsilon_0; \varepsilon_0]$. It follows that

$$m_1(\varepsilon) \leq m_1(-\varepsilon_0) = \frac{1 + 4(1 - \varepsilon_0)}{4(1 - \varepsilon_0)T^\gamma}.$$

From these remarks and (1.5.4), we can write

$$\begin{aligned} z_{2k}(T) &\geq 1 - \frac{1}{1 + \frac{\lambda_{2k}}{1 + \lambda_{2k}}T^\gamma\Gamma(1 + \gamma)^{-1}} = \frac{\Gamma(1 + \gamma)^{-1}}{\frac{1 + \lambda_{2k}}{\lambda_{2k}T^\gamma} + \Gamma(1 + \gamma)^{-1}} \\ &\geq \frac{\Gamma(1 + \gamma)^{-1}}{m_1(-\varepsilon_0) + \Gamma(1 + \gamma)^{-1}} = a_1(T, \gamma, \varepsilon_0) = a_1 > 0. \end{aligned}$$

In the same way, for $\lambda_{2k+1} > 0$, we can write

$$\frac{1}{T^\gamma} \leq \frac{1 + \lambda_{2k+1}}{\lambda_{2k+1} T^\gamma} \leq \frac{1 + \lambda_1}{\lambda_1 T^\gamma} = \frac{1 + (1 - \varepsilon)}{(1 - \varepsilon) T^\gamma} = m_2(\varepsilon) \leq m_2(\varepsilon_0) = \frac{2 - \varepsilon_0}{(1 - \varepsilon_0) T^\gamma},$$

which implies that

$$z_{2k+1}(T) \geq \frac{\Gamma(1 + \gamma)^{-1}}{m_2(\varepsilon_0) + \Gamma(1 + \gamma)^{-1}} = a_2(T, \gamma, \varepsilon_0) = a_2 > 0.$$

□

By a direct calculation of the minimum and maximum of a real function with the help of (3.2.19), we show the following.

For $s \geq 0$ and $\alpha, \beta > 0$, we have

$$A(s) = \frac{s}{\alpha s^2 + \beta} \leq \frac{1}{2\sqrt{\beta}} \frac{1}{\sqrt{\alpha}}, \quad (3.2.20)$$

$$A_{2k+1} = \frac{\lambda_{2k+1}}{\alpha \lambda_{2k+1}^2 + z_{2k+1}(T)} \leq \frac{\lambda_{2k+1}}{\alpha \lambda_{2k+1}^2 + a_1} \leq \frac{1}{2\sqrt{a_1}} \frac{1}{\sqrt{\alpha}}, \quad (3.2.21)$$

and

$$A_{2k} = \frac{\lambda_{2k}}{\alpha \lambda_{2k}^2 + z_{2k}(T)} \leq \frac{\lambda_{2k}}{\alpha \lambda_{2k}^2 + a_1} \leq \frac{1}{2\sqrt{a_2}} \frac{1}{\sqrt{\alpha}}. \quad (3.2.22)$$

Remark 3.2.2. By virtue of (1.5.2) and (3.2.19), we have

$$a_1 \leq z_{2k}(T) \leq 1, \quad a_2 \leq z_{2k+1}(T) \leq 1, \quad 1 \leq \frac{1}{z_{2k}(T)} \leq \frac{1}{a_1}, \quad 1 \leq \frac{1}{z_{2k+1}(T)} \leq \frac{1}{a_2},$$

which implies that

$$\|g\|_1^2 = \sum_{k=1}^{\infty} \lambda_{2k}^2 |g_{2k}|^2 + \sum_{k=0}^{\infty} \lambda_{2k+1}^2 |g_{2k+1}|^2 \leq \|f\|^2,$$

and

$$\|f\|^2 \leq \kappa^2 \left(\sum_{k=1}^{\infty} \lambda_{2k}^2 |g_{2k}|^2 + \sum_{k=0}^{\infty} \lambda_{2k+1}^2 |g_{2k+1}|^2 \right) = \kappa^2 \|g\|_1^2,$$

where $\kappa = \max\left(\frac{1}{a_1}, \frac{1}{a_2}\right)$.

From this Remark, we conclude that $f \in H$ if and only if $g \in \mathbb{H}_1$.

Lemma 3.2.2. [102] For any $\alpha, p, \beta > 0$ and $s \geq \lambda_j, j = 1, 2$, we have:

(i) If $s \geq \lambda_1 = (1 - \varepsilon) > 0$, then

$$F(s) = \frac{\alpha s^{2-\frac{p}{2}}}{\alpha s^2 + \beta} \leq \begin{cases} b_1 \alpha^{\frac{p}{4}}, & 0 < p < 4, \\ b_2 \alpha, & p \geq 4. \end{cases} \quad (3.2.23)$$

(ii) If $s \geq \lambda_2 = 4(1 + \varepsilon) > 0$, then

$$F(s) = \frac{\alpha s^{2-\frac{p}{2}}}{\alpha s^2 + \beta} \leq \begin{cases} c_1 \alpha^{\frac{p}{4}}, & 0 < p < 4, \\ c_2 \alpha, & p \geq 4. \end{cases} \quad (3.2.24)$$

Here,

$$b_2 = \frac{1}{\beta \lambda_1^{\frac{p-4}{2}}}, \quad c_2 = \frac{1}{\beta \lambda_2^{\frac{p-4}{2}}} \quad \text{and} \quad b_1 = c_1 = \frac{\left(\frac{\beta(4-p)}{p}\right)^{\frac{4-p}{p}}}{\frac{\beta(4-p)}{p} + \beta} \leq 1.$$

Lemma 3.2.3. [99] For any $\alpha, p > 0$ and $s \geq \lambda_j, j = 1, 2$, we have:

(i) If $s \geq \lambda_1 = (1 - \varepsilon) > 0$, then

$$\hat{F}(s) = \frac{\alpha s^{\frac{2-p}{2}}}{\alpha s + 1} \leq \begin{cases} \hat{b}_1 \alpha^{\frac{p}{2}}, & 0 < p < 2, \\ \hat{b}_2 \alpha, & p \geq 2. \end{cases} \quad (3.2.25)$$

(ii) If $s \geq \lambda_2 = 4(1 + \varepsilon) > 0$, then

$$\hat{F}(s) = \frac{\alpha s^{\frac{2-p}{2}}}{\alpha s + 1} \leq \begin{cases} \hat{c}_1 \alpha^{\frac{p}{2}}, & 0 < p < 2, \\ \hat{c}_2 \alpha, & p \geq 2. \end{cases} \quad (3.2.26)$$

Here,

$$\hat{b}_2 = \frac{1}{\lambda_1^{\frac{p-2}{2}}}, \quad \hat{c}_2 = \frac{1}{\lambda_2^{\frac{p-2}{2}}} \quad \text{and} \quad \hat{b}_1 = \hat{c}_1 = \frac{\left(\frac{2}{p}\right)^{\frac{2-p}{2}}}{\frac{2}{p} + 1} \leq 1.$$

Lemma 3.2.4. [102] For any $\alpha, p, \beta > 0$ and $s \geq \lambda_j, j = 1, 2$, we have:

(i) If $s \geq \lambda_1 = (1 - \varepsilon) > 0$, then

$$\tilde{F}(s) = \frac{\alpha s^{\frac{2-p}{2}}}{\alpha s^2 + \beta} \leq \begin{cases} \tilde{b}_1 \alpha^{\frac{2+p}{4}}, & 0 < p < 2, \\ \tilde{b}_2 \alpha, & p \geq 2. \end{cases} \quad (3.2.27)$$

(ii) If $s \geq \lambda_2 = 4(1 + \varepsilon) > 0$, then

$$\tilde{F}(s) = \frac{\alpha s^{\frac{2-p}{2}}}{\alpha s^2 + \beta} \leq \begin{cases} \tilde{c}_1 \alpha^{\frac{2+p}{4}}, & 0 < p < 2, \\ \tilde{c}_2 \alpha, & p \geq 2. \end{cases} \quad (3.2.28)$$

Here,

$$\tilde{b}_2 = \frac{1}{\beta \lambda_1^{\frac{p-2}{2}}}, \quad \tilde{c}_2 = \frac{1}{\beta \lambda_2^{\frac{p-2}{2}}} \quad \text{and} \quad \tilde{b}_1 = \tilde{c}_1 = \frac{\left(\frac{\beta(2-p)}{2+p}\right)^{\frac{2-p}{4}}}{\frac{\beta(2-p)}{2+p} + \beta} \leq 1.$$

Putting

$$B_{2k+1} = \frac{\alpha \lambda_{2k+1}^{2-\frac{p}{2}}}{\alpha \lambda_{2k+1}^2 + z_{2k+1}(T)},$$

and

$$B_{2k} = \frac{\alpha \lambda_{2k}^{2-\frac{p}{2}}}{\alpha \lambda_{2k}^2 + z_{2k}(T)}.$$

By virtue of (3.2.19), (3.2.23) and (3.2.24), we have

$$B_{2k} \leq \frac{\alpha \lambda_{2k}^{2-\frac{p}{2}}}{\alpha \lambda_{2k}^2 + a_1} \leq \begin{cases} b_1 \alpha^{\frac{p}{4}}, & 0 < p < 4, \\ b_2 \alpha, & p \geq 4, \end{cases} \quad (3.2.29)$$

and

$$B_{2k+1} \leq \frac{\alpha \lambda_{2k+1}^{2-\frac{p}{2}}}{\alpha \lambda_{2k+1}^2 + a_2} \leq \begin{cases} c_1 \alpha^{\frac{p}{4}}, & 0 < p < 4, \\ c_2 \alpha, & p \geq 4. \end{cases} \quad (3.2.30)$$

From these inequalities, we have

$$\sup_{k \geq 1} B_{2k} \leq \begin{cases} b_1 \alpha^{\frac{p}{4}}, & 0 < p < 4, \\ b_2 \alpha, & p \geq 4, \end{cases} \quad (3.2.31)$$

and

$$\sup_{k \geq 0} B_{2k+1} \leq \begin{cases} c_1 \alpha^{\frac{p}{4}}, & 0 < p < 4, \\ c_2 \alpha, & p \geq 4. \end{cases} \quad (3.2.32)$$

Lemma 3.2.5. [100] For $0 < \lambda < 1$, we define $p_m(\lambda) = \sum_{i=0}^{m-1} (1-\lambda)^i$ and $r_m(\lambda) = 1 - \lambda p_m(\lambda) = (1-\lambda)^m$. Under these circumstances, we have:

$$p_m(\lambda) \lambda^\mu \leq k^{1-\mu}, \quad \text{for all } 0 \leq \mu \leq 1,$$

$$r_m(\lambda) \lambda^\nu \leq \theta_\nu (m+1)^{-\nu},$$

where

$$\theta = \begin{cases} 1, & 0 \leq \nu \leq 1, \\ \nu^\nu, & \nu > 1. \end{cases}$$

We give a conditional stability in the following theorem.

Theorem 3.2.1. (Conditional stability) *If $f \in \mathbb{H}_{\frac{p}{2}}$, i.e., f satisfies the following a priori bound condition:*

$$\|f\|_{\frac{p}{2}} \leq E, \quad p > 0, \quad (3.2.33)$$

Then, we have

$$\|f\| \leq CE^{\frac{2}{p+2}} \|g\|^{\frac{p}{p+2}}, \quad p > 0, \quad (3.2.34)$$

Here, the constant C is given by $C = \sqrt{a_1^{\frac{-2p}{p+2}} + a_2^{\frac{-2p}{p+2}}}$ and depends on the values of p, γ, T and ε_0 .

Proof. Using (3.2.17) along with Holder's inequality, we obtain:

$$\begin{aligned} \|f\|^2 &= \sum_{k=0}^{\infty} f_{1k}^2 + \sum_{k=1}^{\infty} f_{2k}^2 \\ &= \sum_{k=0}^{\infty} \left(\frac{\lambda_{2k+1} g_{1k}}{z_{2k+1}(T)} \right)^2 + \sum_{k=1}^{\infty} \left(\frac{\lambda_{2k} g_{2k}}{z_{2k}(T)} \right)^2 \\ &\leq \left(\sum_{k=0}^{\infty} \left(\frac{\lambda_{2k+1}}{z_{2k+1}(T)} \right)^{p+2} g_{1k}^2 \right)^{\frac{2}{p+2}} \left(\sum_{k=0}^{\infty} g_{1k}^2 \right)^{\frac{p}{p+2}} \\ &\quad + \left(\sum_{k=1}^{\infty} \left(\frac{\lambda_{2k}}{z_{2k}(T)} \right)^{p+2} g_{2k}^2 \right)^{\frac{2}{p+2}} \left(\sum_{k=1}^{\infty} g_{2k}^2 \right)^{\frac{p}{p+2}}. \end{aligned} \quad (3.2.35)$$

By employing Lemma 3.2.1, we derive:

$$\begin{aligned} \sum_{k=0}^{\infty} \left(\frac{\lambda_{2k+1}}{z_{2k+1}(T)} \right)^{p+2} g_{1k}^2 &\leq \sum_{k=0}^{\infty} \left(\frac{\lambda_{2k+1} g_{1k}}{z_{2k+1}(T)} \right)^2 \left(\frac{\lambda_{2k+1}}{a_2} \right)^p \\ &= \sum_{k=0}^{\infty} f_{1k}^2 \lambda_{2k+1}^p \left(\frac{1}{a_2} \right)^p \leq \left(\frac{1}{a_2} \right)^p \|f\|_{\frac{p}{2}}^2, \end{aligned} \quad (3.2.36)$$

and

$$\begin{aligned} \sum_{k=1}^{\infty} \left(\frac{\lambda_{2k}}{z_{2k}(T)} \right)^{p+2} g_{2k}^2 &\leq \sum_{k=1}^{\infty} \left(\frac{\lambda_{2k} g_{2k}}{z_{2k}(T)} \right)^2 \left(\frac{\lambda_{2k}}{\alpha_1} \right)^p \\ &= \sum_{k=1}^{\infty} f_{2k}^2 \lambda_{2k}^p \left(\frac{1}{a_1} \right)^p \leq \left(\frac{1}{a_1} \right)^p \|f\|_{\frac{p}{2}}^2. \end{aligned} \quad (3.2.37)$$

By injecting (3.2.35) and (3.2.36) into (3.2.37), we obtain the desired result. \square

Remark 3.2.3. Drawing from the proof of Theorem 3.2.1, we can rephrase the conditional stability in the following manner:

$$\|f\| \leq C \|f\|_{\frac{p}{2}}^{\frac{2}{p+2}} \|Kf\|_{\frac{p}{2}}^{\frac{p}{p+2}}.$$

This indicates that we can establish an upper bound for the L^2 norm of f by estimating both $\|f\|_{\frac{p}{2}}$ and the L^2 norm of Kf .

3.3 Modified quasi-boundary value method and convergence rates

In this section, we adopt a variant of the modified quasi-boundary value method applied to our problem (3.1.1). We follow the same approach developed in the work [99] with an additional calculation generated by two sums of two Fourier series. In this context, We establish two convergence estimates: one based on an a priori regularization parameter selection criterion and the other on an a posteriori regularization parameter selection criterion.

Let $u_{\alpha}^{\delta}(x, t)$ denote the solution to the following regularized problem:

$$\begin{cases} D_t^{\gamma} \left[u_{\alpha}^{\delta}(x, t) + Lu_{\alpha}^{\delta}(x, t) \right] + Lu_{\alpha}^{\delta}(x, t) = f_{\alpha}^{\delta}(x), & (x, t) \in Q, \\ u_{\alpha}^{\delta}(0, t) = u_{\alpha}^{\delta}(\pi, t) = 0, & t \in (0, T), \\ u_{\alpha}^{\delta}(x, 0) = 0, & x \in (0, \pi), \\ u_{\alpha}^{\delta}(x, T) + \alpha L f_{\alpha}^{\delta}(x) = g^{\delta}(x), & x \in (0, \pi), \end{cases} \quad (3.3.1)$$

where $\alpha > 0$ is a regularization parameter.

Through the process of separating variables, it is evident that $u_{\alpha}^{\delta}(x, t)$ takes on the fol-

lowing form:

$$u_{\alpha}^{\delta}(x, t) = \sum_{k=0}^{\infty} \frac{f_{1k}}{\lambda_{2k+1}} z_{2k+1}(t) \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{f_{2k}}{\lambda_{2k}} z_{2k}(t) \varphi_{2k}(x). \quad (3.3.2)$$

Then, we get (in the case of inexact data g^{δ}):

$$f_{\alpha}^{\delta}(x) = \sum_{k=0}^{\infty} \frac{\lambda_{2k+1} g_{1k}^{\delta} \varphi_{2k+1}(x)}{\alpha \lambda_{2k+1}^2 + z_{2k+1}(T)} + \sum_{k=1}^{\infty} \frac{\lambda_{2k} g_{2k}^{\delta} \varphi_{2k}(x)}{\alpha \lambda_{2k}^2 + z_{2k}(T)}, \quad (3.3.3)$$

and (in the case of inexact data g):

$$f_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{\lambda_{2k+1} g_{1k} \varphi_{2k+1}(x)}{\alpha \lambda_{2k+1}^2 + z_{2k+1}(T)} + \sum_{k=1}^{\infty} \frac{\lambda_{2k} g_{2k} \varphi_{2k}(x)}{\alpha \lambda_{2k}^2 + z_{2k}(T)}. \quad (3.3.4)$$

In the subsequent analysis, we present two convergence estimates for $\|f_{\alpha}^{\delta} - f\|$ while employing both an a priori and a posteriori choice rule for the regularization parameter. These convergence estimates are instrumental in evaluating the accuracy and efficacy of the regularization process in approximating the true solution $f(x)$ as we progress with the solution method.

3.3.1 Convergence estimate under an a priori regularization parameter choice rule

Theorem 3.3.1. *Assuming that the a priori condition (3.2.33) and the noise Assumption (3.2.5) are satisfied, then:*

1. If $0 < p < 4$ and choose $\alpha = \left(\frac{\delta}{E}\right)^{\frac{4}{p+2}}$, we obtain a convergence estimate

$$\|f_{\alpha}^{\delta} - f\| \leq \left(\sqrt{\frac{1}{4a_1} + \frac{1}{4a_2}} + C_1 \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \quad (3.3.5)$$

2. If $p \geq 4$ and choose $\alpha = \left(\frac{\delta}{E}\right)^{\frac{2}{3}}$, we obtain a convergence estimate

$$\|f_\alpha^\delta - f\| \leq \left(\sqrt{\frac{1}{4a_1} + \frac{1}{4a_2}} + C_2\right) E^{\frac{1}{3}} \delta^{\frac{2}{3}}, \quad (3.3.6)$$

where C_1, C_2 are positive constants depending on p, γ, T and ε_0 .

Proof. By using the triangle inequality, we have

$$\|f_\alpha^\delta - f\| \leq \|f_\alpha^\delta - f_\alpha\| + \|f_\alpha - f\|. \quad (3.3.7)$$

Let us begin by providing an estimate for the first term. Using (3.3.3), (3.3.4) and (3.2.5), we arrive

$$\begin{aligned} \|f_\alpha^\delta - f_\alpha\|^2 &= \sum_{k=0}^{\infty} \left(\frac{\lambda_{2k+1} (g_{1k}^\delta - g_{1k})}{\alpha \lambda_{2k+1}^2 + z_{2k+1}(T)} \right)^2 + \left(\sum_{k=1}^{\infty} \frac{\lambda_{2k} (g_{2k}^\delta - g_{2k})}{\alpha \lambda_{2k}^2 + z_{2k}(T)} \right)^2 \\ &\leq \left(\left(\sup_{k \geq 0} A_{2k+1} \right)^2 + \left(\sup_{k \geq 1} A_{2k} \right)^2 \right) \delta^2. \end{aligned} \quad (3.3.8)$$

By using (3.2.21) and (3.2.22), we derive

$$\sup_{k \geq 0} A_{2k+1} \leq \frac{1}{2\sqrt{a_1}} \frac{1}{\sqrt{\alpha}}, \quad (3.3.9)$$

and

$$\sup_{k \geq 1} A_{2k} \leq \frac{1}{2\sqrt{a_2}} \frac{1}{\sqrt{\alpha}}. \quad (3.3.10)$$

Substituting (3.3.9) and (3.3.10) in (3.3.8), we obtain

$$\|f_\alpha^\delta - f_\alpha\| \leq \sqrt{\frac{1}{4a_1} + \frac{1}{4a_2}} \frac{\delta}{\sqrt{\alpha}}. \quad (3.3.11)$$

Let us now estimate the second term in (3.3.7). By referring to (3.3.4), we can infer that:

$$f_\alpha(x) - f(x) = \sum_{k=0}^{\infty} \left(\frac{\lambda_{2k+1} g_{1k}}{\alpha \lambda_{2k+1}^2 + z_{2k+1}(T)} - \frac{\lambda_{2k+1} g_{1k}}{z_{2k+1}(T)} \right) \varphi_{2k+1}(x)$$

$$\begin{aligned}
& + \sum_{k=1}^{\infty} \left(\frac{\lambda_{2k} g_{2k}}{\alpha \lambda_{2k}^2 + z_{2k}(T)} - \frac{\lambda_{2k} g_{2k}}{z_{2k}(T)} \right) \varphi_{2k}(x) \\
& = \sum_{k=0}^{\infty} \left(\frac{\lambda_{2k+1} g_{1k}}{z_{2k+1}(T)} \frac{-\alpha \lambda_{2k+1}^2}{\alpha \lambda_{2k+1}^2 + z_{2k+1}(T)} \right) \varphi_{2k+1}(x) \\
& + \sum_{k=1}^{\infty} \left(\frac{\lambda_{2k} g_{2k}}{z_{2k}(T)} \frac{-\alpha \lambda_{2k}^2}{\alpha \lambda_{2k}^2 + z_{2k}(T)} \right) \varphi_{2k}(x). \tag{3.3.12}
\end{aligned}$$

Applying the a priori bound condition (3.2.33), we obtain

$$\begin{aligned}
& \|f_{\alpha} - f\|^2 \\
& = \sum_{k=0}^{\infty} \left(\frac{\lambda_{2k+1} g_{1k}}{z_{2k+1}(T)} \right)^2 \lambda_{2k+1}^p \left(\frac{\alpha \lambda_{2k+1}^2}{\alpha \lambda_{2k+1}^2 + z_{2k+1}(T)} \right)^2 \frac{1}{\lambda_{2k+1}^p} \\
& + \sum_{k=1}^{\infty} \left(\frac{\lambda_{2k} g_{2k}}{z_{2k}(T)} \right)^2 \lambda_{2k}^p \left(\frac{\alpha \lambda_{2k}^2}{\alpha \lambda_{2k}^2 + z_{2k}(T)} \right)^2 \frac{1}{\lambda_{2k}^p} \\
& \leq E^2 \left(\left(\sup_{k \geq 0} B_{2k+1} \right)^2 + \left(\sup_{k \geq 1} B_{2k} \right)^2 \right), \tag{3.3.13}
\end{aligned}$$

where

$$B_{2k+1} = \frac{\alpha \lambda_{2k+1}^{2-\frac{p}{2}}}{\alpha \lambda_{2k+1}^2 + z_{2k+1}(T)},$$

and

$$B_{2k} = \frac{\alpha \lambda_{2k}^{2-\frac{p}{2}}}{\alpha \lambda_{2k}^2 + z_{2k}(T)}.$$

Now, using (3.2.31) and (3.2.32) combined with (3.3.11), we obtain

$$\|f_{\alpha}^{\delta} - f\| \leq \sqrt{\frac{1}{4\alpha_1} + \frac{1}{4a_2} \frac{\delta}{\sqrt{\alpha}}} + \begin{cases} \sqrt{b_1^2 + c_1^2} E \alpha^{\frac{p}{4}}, & 0 < p < 4, \\ \sqrt{b_2^2 + c_2^2} E \alpha, & p \geq 4. \end{cases} \tag{3.3.14}$$

Choose the regularization parameter α by

$$\alpha = \begin{cases} \left(\frac{\delta}{E} \right)^{\frac{4}{p+2}}, & 0 < p < 4, \\ \left(\frac{\delta}{E} \right)^{\frac{2}{3}}, & p \geq 4. \end{cases} \tag{3.3.15}$$

Finally, we obtain

$$\|f_\alpha^\delta - f\| \leq \begin{cases} \left(\sqrt{\frac{1}{4a_1} + \frac{1}{4a_2} + C_1} \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, & 0 < p < 4, \\ \left(\sqrt{\frac{1}{4a_1} + \frac{1}{4a_2} + C_2} \right) E^{\frac{1}{3}} \delta^{\frac{2}{3}}, & p \geq 4. \end{cases} \quad (3.3.16)$$

The proof is completed. \square

3.3.2 Convergence estimate under an a posteriori regularization parameter choice rule

In this subsection, we adopt an a posteriori regularization parameter choice strategy, specifically Morozov's discrepancy principle, to determine the appropriate value of the regularization parameter α . By leveraging the conditional stability estimate outlined in Theorem 3.2.1, we derive a convergence rate for the regularized solution (3.3.3). This convergence rate serves as a valuable measure of the accuracy and reliability of the regularization process in approximating the desired solution.

Morozov's discrepancy principle in our context entails identifying α under the condition:

$$\|Kf_\alpha^\delta - g^\delta\| = \tau\delta. \quad (3.3.17)$$

Here, $\tau > 1$ is a constant. As per the subsequent lemma, it becomes evident that (3.3.17) possesses a unique solution if $\|g^\delta\| > \tau\delta > 0$.

Lemma 3.3.1. *Setting $\rho(\alpha) = \|Kf_\alpha^\delta - g^\delta\|$, we have the following properties:*

- (a) $\rho(\alpha)$ is a continuous function,
- (b) $\lim_{\alpha \rightarrow 0} \rho(\alpha) = 0$,
- (c) $\lim_{\alpha \rightarrow \infty} \rho(\alpha) = \|g^\delta\|$,
- (d) $\rho(\alpha)$ is a monotonically increasing function on $(0, \infty)$.

Proof. The proofs are direct consequences derived from the following expression:

$$(\rho(\alpha))^2 = \sum_{k=0}^{\infty} \left(\frac{\alpha \lambda_{2k+1}^2}{z_{2k+1}(T) + \alpha \lambda_{2k+1}^2} \right)^2 (g_{1k}^\delta)^2 + \sum_{k=1}^{\infty} \left(\frac{\alpha \lambda_{2k}^2}{z_{2k}(T) + \alpha \lambda_{2k}^2} \right)^2 (g_{2k}^\delta)^2.$$

□

Theorem 3.3.2. *Under the assumption that the a priori condition (3.2.33) and the noise assumption (3.2.5) are met, and if there exists a constant $\tau > 1$ such that $\|g^\delta\| > \tau\delta > 0$. and furthermore, the regularization parameter $\alpha > 0$ is chosen using Morozov's discrepancy principle (3.3.17). we can conclude the following:*

1. For $0 < p < 2$, we have a convergence estimate

$$\|f_\alpha^\delta - f\| \leq \left(C (\tau + 1)^{\frac{p}{p+2}} + \sqrt{\frac{1}{4a_1} + \frac{1}{4a_2}} \left(\frac{\sqrt{\tilde{b}_1^2 + \tilde{c}_1^2}}{\tau - 1} \right)^{\frac{2}{p+2}} \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.$$

2. For $p \geq 2$, we have a convergence estimate

$$\|f_\alpha^\delta - f\| \leq \left(C (\tau + 1)^{\frac{p}{p+2}} + \sqrt{\frac{1}{4a_1} + \frac{1}{4a_2}} \left(\frac{\sqrt{\tilde{b}_2^2 + \tilde{c}_2^2}}{\tau - 1} \right)^{\frac{1}{2}} \right) E^{\frac{1}{2}} \delta^{\frac{1}{2}}.$$

Proof. We have

$$\|f_\alpha^\delta - f\| \leq \|f_\alpha^\delta - f_\alpha\| + \|f_\alpha - f\|.$$

Let us start by providing an estimate for the second term. We have

$$\mathbf{K}f_\alpha^\delta(x) - g^\delta(x) = \sum_{k=0}^{\infty} \frac{-\alpha\lambda_{2k+1}^2 g_{2k+1}^\delta}{z_{2k+1}(T) + \alpha\lambda_{2k+1}^2} \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{-\alpha\lambda_{2k}^2 g_{2k}^\delta}{z_{2k}(T) + \alpha\lambda_{2k}^2} \varphi_{2k}(x).$$

$$\mathbf{K}(f_\alpha(x) - f(x)) = \Delta_1 + \Delta_2$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} \frac{-\alpha\lambda_{2k+1}^2 (g_{2k+1} - g_{2k+1}^\delta)}{z_{2k+1}(T) + \alpha\lambda_{2k+1}^2} \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{-\alpha\lambda_{2k}^2 (g_{2k} - g_{2k}^\delta)}{z_{2k}(T) + \alpha\lambda_{2k}^2} \varphi_{2k}(x) \\ &+ \sum_{k=0}^{\infty} \frac{-\alpha\lambda_{2k+1}^2 g_{2k+1}^\delta}{z_{2k+1}(T) + \alpha\lambda_{2k+1}^2} \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{-\alpha\lambda_{2k}^2 g_{2k}^\delta}{z_{2k}(T) + \alpha\lambda_{2k}^2} \varphi_{2k}(x). \end{aligned}$$

By using (3.2.5) and (3.3.17), we can write

$$\|\Delta_1\|^2 = \sum_{k=0}^{\infty} \left[\frac{-\alpha \lambda_{2k+1}^2 (g_{2k+1} - g_{2k+1}^\delta)}{z_{2k+1}(T) + \alpha \lambda_{2k+1}^2} \right]^2 + \sum_{k=1}^{\infty} \left[\frac{-\alpha \lambda_{2k}^2 (g_{2k} - g_{2k}^\delta)}{z_{2k}(T) + \alpha \lambda_{2k}^2} \right]^2 \leq \|g - g^\delta\|^2 \leq \delta^2,$$

and

$$\|\Delta_2\|^2 = \|\mathbf{K}f_\alpha^\delta - g^\delta\|^2 = (\tau\delta)^2.$$

By the triangle inequality $\|\mathbf{K}(f_\alpha - f)\| = \|\Delta_1 + \Delta_2\| \leq \|\Delta_1\| + \|\Delta_2\|$, we obtain the desired estimate

$$\|\mathbf{K}(f_\alpha - f)\| \leq \delta + \tau\delta = (\tau + 1)\delta. \quad (3.3.18)$$

By employing the a priori bound condition for f , we get

$$\begin{aligned} \|f_\alpha - f\|_{\frac{p}{2}}^2 &= \sum_{k=0}^{\infty} \left(\frac{\lambda_{2k+1} g_{2k+1}}{z_{2k+1}(T)} \frac{\alpha \lambda_{2k+1}^2}{\alpha \lambda_{2k+1}^2 + z_{2k+1}(T)} \right)^2 \lambda_{2k+1}^p + \sum_{k=1}^{\infty} \left(\frac{\lambda_{2k} g_{2k}}{z_{2k}(T)} \frac{\alpha \lambda_{2k}^2}{\alpha \lambda_{2k}^2 + z_{2k}(T)} \right)^2 \lambda_{2k}^p \\ &\leq \sum_{k=0}^{\infty} \left(\frac{\lambda_{2k+1} g_{2k+1}}{z_{2k+1}(T)} \right)^2 \lambda_{2k+1}^p + \sum_{k=1}^{\infty} \left(\frac{\lambda_{2k} g_{2k}}{z_{2k}(T)} \right)^2 \lambda_{2k}^p = \|f\|_{\frac{p}{2}}^2 \leq E^2. \end{aligned}$$

From (3.2.34) and (3.3.18), we deduce that

$$\|f_\alpha - f\| \leq C (\tau + 1)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \quad (3.3.19)$$

Now, let us establish the bound for the first term. Analogous to (3.3.11), we obtain

$$\|f_\alpha^\delta - f_\alpha\| \leq \sqrt{\frac{1}{4a_1} + \frac{1}{4a_2} \frac{\delta}{\sqrt{\alpha}}}. \quad (3.3.20)$$

From (3.3.17), there holds

$$\begin{aligned} \tau\delta &= \left\| \sum_{k=0}^{\infty} \frac{\alpha \lambda_{2k+1}^2 g_{2k+1}^\delta \varphi_{2k+1}}{z_{2k+1}(T) + \alpha \lambda_{2k+1}^2} + \sum_{k=1}^{\infty} \frac{\alpha \lambda_{2k}^2 g_{2k}^\delta \varphi_{2k}}{z_{2k}(T) + \alpha \lambda_{2k}^2} \right\| \\ &\leq \left\| \sum_{k=0}^{\infty} \frac{\alpha \lambda_{2k+1}^2 (g_{2k+1}^\delta - g_{2k+1}) \varphi_{2k+1}}{z_{2k+1}(T) + \alpha \lambda_{2k+1}^2} + \sum_{k=1}^{\infty} \frac{\alpha \lambda_{2k}^2 (g_{2k}^\delta - g_{2k}) \varphi_{2k}}{z_{2k}(T) + \alpha \lambda_{2k}^2} \right\| \\ &\quad + \left\| \sum_{k=0}^{\infty} \frac{\alpha \lambda_{2k+1}^2 g_{2k+1} \varphi_{2k+1}(x)}{z_{2k+1}(T) + \alpha \lambda_{2k+1}^2} + \sum_{k=1}^{\infty} \frac{\alpha \lambda_{2k}^2 g_{2k} \varphi_{2k}}{z_{2k+1}(T) + \alpha \lambda_{2k}^2} \right\| \end{aligned}$$

$$\leq \delta + J. \quad (3.3.21)$$

Once again, by employing the a priori bound condition for f , we get

$$\begin{aligned}
J^2 &= \left\| \sum_{k=0}^{\infty} \frac{\alpha \lambda_{2k+1}^2 g_{2k+1} \varphi_{2k+1}}{z_{2k+1}(T) + \alpha \lambda_{2k+1}^2} + \sum_{k=1}^{\infty} \frac{\alpha \lambda_{2k}^2 g_{2k} \varphi_{2k}}{z_{2k}(T) + \alpha \lambda_{2k}^2} \right\|^2 \\
&\leq \sum_{k=0}^{\infty} \left(\frac{\alpha \lambda_{2k+1}^2 g_{2k+1}}{z_{2k+1}(T) + \alpha \lambda_{2k+1}^2} \right)^2 + \sum_{k=1}^{\infty} \left(\frac{\alpha \lambda_{2k}^2 g_{2k}}{z_{2k}(T) + \alpha \lambda_{2k}^2} \right)^2 \\
&\leq \sum_{k=0}^{\infty} \left(\frac{\alpha \lambda_{2k+1}^2 g_{2k+1}}{z_{2k+1}(T) + \alpha \lambda_{2k+1}^2} \frac{z_{2k+1}(T) \lambda_{2k+1}^{\frac{p}{2}}}{z_{2k+1}(T) \lambda_{2k+1}^{\frac{p}{2}}} \right)^2 + \sum_{k=1}^{\infty} \left(\frac{\alpha \lambda_{2k}^2 g_{2k}}{z_{2k}(T) + \alpha \lambda_{2k}^2} \frac{z_{2k}(T) \lambda_{2k}^{\frac{p}{2}}}{z_{2k}(T) \lambda_{2k}^{\frac{p}{2}}} \right)^2 \\
&\leq E^2 \sup_{k \geq 0} \left(\frac{\alpha \lambda_{2k+1} z_{2k+1}(T)}{z_{2k+1}(T) + \alpha \lambda_{2k+1}^2} \right)^2 \frac{1}{\lambda_{2k+1}^p} + E^2 \sup_{k \geq 1} \left(\frac{\alpha \lambda_{2k} z_{2k}(T)}{z_{2k}(T) + \alpha \lambda_{2k}^2} \right)^2 \frac{1}{\lambda_{2k}^p} \\
&\leq E^2 \sup_{k \geq 0} \left(\frac{\alpha \lambda_{2k+1}^{1-\frac{p}{2}}}{a_1 + \alpha \lambda_{2k+1}^2} \right)^2 + E^2 \sup_{k \geq 1} \left(\frac{\alpha \lambda_{2k}^{1-\frac{p}{2}}}{a_2 + \alpha \lambda_{2k}^2} \right)^2 \\
&\leq \begin{cases} \left(\sqrt{\tilde{b}_1^2 + \tilde{c}_1^2} E \alpha^{\frac{p+2}{4}} \right)^2, & 0 < p < 2, \\ \left(\sqrt{\tilde{b}_2^2 + \tilde{c}_2^2} E \alpha \right)^2, & p \geq 2. \end{cases} \quad (3.3.22)
\end{aligned}$$

Combining (3.3.21)-(3.3.22), we obtain

$$(\tau - 1) \delta \leq \begin{cases} \sqrt{\tilde{b}_1^2 + \tilde{c}_1^2} E \alpha^{\frac{p+2}{4}}, & 0 < p < 2, \\ \sqrt{\tilde{b}_2^2 + \tilde{c}_2^2} E \alpha, & p \geq 2. \end{cases}$$

It is clear that

$$\frac{1}{\alpha} \leq \begin{cases} \left(\frac{\sqrt{\tilde{b}_1^2 + \tilde{c}_1^2}}{\tau - 1} \right)^{\frac{4}{p+2}} \left(\frac{E}{\delta} \right)^{\frac{4}{p+2}}, & 0 < p < 2, \\ \frac{\sqrt{\tilde{b}_2^2 + \tilde{c}_2^2} E}{\tau - 1} \frac{1}{\delta}, & p \geq 2. \end{cases} \quad (3.3.23)$$

Substituting (3.3.23) into (3.3.20), we obtain

$$\|f_\alpha^\delta - f_\alpha\| \leq \sqrt{\frac{1}{a_1} + \frac{1}{4a_2}} \begin{cases} \left(\frac{\sqrt{\widetilde{b}_1^2 + \widetilde{c}_1^2}}{\tau - 1} \right)^{\frac{2}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, & 0 < p < 2, \\ \left(\frac{\sqrt{\widetilde{b}_2^2 + \widetilde{c}_2^2}}{\tau - 1} \right)^{\frac{1}{2}} E^{\frac{1}{2}} \delta^{\frac{1}{2}}, & p \geq 2. \end{cases} \quad (3.3.24)$$

Finally, by combining (3.3.19) and (3.3.24), we derive

$$\|f_\alpha^\delta - f\| \leq C (\tau + 1)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}} + \sqrt{\frac{1}{4a_1} + \frac{1}{4a_2}} \begin{cases} \left(\frac{\sqrt{\widetilde{b}_1^2 + \widetilde{c}_1^2}}{\tau - 1} \right)^{\frac{2}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, & 0 < p < 2, \\ \left(\frac{\sqrt{\widetilde{b}_2^2 + \widetilde{c}_2^2}}{\tau - 1} \right)^{\frac{1}{2}} E^{\frac{1}{2}} \delta^{\frac{1}{2}}, & p \geq 2. \end{cases}$$

This completes the proof. \square

3.4 Quasi-reversibility method and convergence rates

In this section, motivated by the regularizing strategy developed in [31] and the recent work [32], we present a novel quasi-reversibility method for solving problem (3.1.1) with the given perturbation (3.2.5). We provide two convergence estimates, each of which is obtained through distinct regularization parameter choice rules. The first estimate is derived using an a priori regularization parameter choice rule, whereas the second estimate is based on a posteriori regularization parameter choice rule. These convergence estimates are essential for assessing the accuracy and effectiveness of the proposed quasi-reversibility method in approximating the solutions to the given problem with a specific perturbation.

Let $u_\alpha^\delta(x, t)$ be the solution of the following regularized problem

$$\begin{cases} D_t^\gamma [u_\alpha^\delta(x, t) + Lu_\alpha^\delta(x, t)] + Lu_\alpha^\delta(x, t) = f(x) + \alpha Lf(x), & (x, t) \in Q, \\ u_\alpha^\delta(0, t) = u_\alpha^\delta(\pi, t) = 0, & t \in (0, T), \\ u_\alpha^\delta(x, 0) = 0, & x \in (0, \pi), \\ u_\alpha^\delta(x, T) = g^\delta(x), & x \in (0, \pi), \end{cases} \quad (3.4.1)$$

where $\alpha > 0$ is a regularization parameter.

We apply the separation of variables method, it becomes clear that $u_\alpha^\delta(x, t)$ takes on the subsequent form:

$$u_\alpha^\delta(x, t) = \sum_{k=0}^{\infty} \frac{(1 + \alpha\lambda_{2k+1}) f_{2k+1}}{\lambda_{2k+1}} z_{2k+1}(t) \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{(1 + \alpha\lambda_{2k}) f_{2k}}{\lambda_{2k}} z_{2k}(t) \varphi_{2k}(x). \quad (3.4.2)$$

Then, we get (in the case of exact data)

$$f_\alpha(x) = \sum_{k=0}^{\infty} \frac{\lambda_{2k+1} g_{2k+1}}{(1 + \alpha\lambda_{2k+1}) z_{2k+1}(T)} \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{\lambda_{2k} g_{2k}}{(1 + \alpha\lambda_{2k}) z_{2k}(T)} \varphi_{2k}(x), \quad (3.4.3)$$

and (in the case of inexact data)

$$f_\alpha^\delta(x) = \sum_{k=0}^{\infty} \frac{\lambda_{2k+1} g_{2k+1}^\delta}{(1 + \alpha\lambda_{2k+1}) z_{2k+1}(T)} \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{\lambda_{2k} g_{2k}^\delta}{(1 + \alpha\lambda_{2k}) z_{2k}(T)} \varphi_{2k}(x). \quad (3.4.4)$$

In the following, we give two convergence estimates for $\|f_\alpha^\delta - f\|$ by using an a priori and an a posteriori choice rule for the regularization parameter.

3.4.1 Convergence estimate under an a priori regularization parameter choice rule

Theorem 3.4.1. *Assuming that the a priori condition (3.2.33) and the noise Assumption (3.2.5) are satisfied, then:*

1. For $0 < p < 2$ If we choose $\alpha = \left(\frac{\delta}{E}\right)^{\frac{2}{p+2}}$, we obtain the following convergence

$$\|f_\alpha^\delta - f\| \leq \left(\sqrt{\frac{1}{a_1^2} + \frac{1}{a_2^2}} + C_1\right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \quad (3.4.5)$$

2. For $p \geq 2$ If we choose $\alpha = \left(\frac{\delta}{E}\right)^{\frac{1}{2}}$, we obtain

$$\|f_\alpha^\delta - f\| \leq \left(\sqrt{\frac{1}{a_1^2} + \frac{1}{a_2^2}} + C_2\right) E^{\frac{1}{2}} \delta^{\frac{1}{2}}, \quad (3.4.6)$$

where C_1, C_2 are positive constants depending on p, γ, T and ε_0 .

Proof. Applying the triangle inequality, it is evident that:

$$\|f_\alpha^\delta - f\| \leq \|f_\alpha^\delta - f_\alpha\| + \|f_\alpha - f\|. \quad (3.4.7)$$

Let us begin by providing an estimate for the first term. By using equations (3.4.3), (3.4.4) and (3.2.5), we arrive at:

$$\begin{aligned} \|f_\alpha^\delta - f_\alpha\|^2 &= \sum_{k=0}^{\infty} \left(\frac{g_{1k}^\delta - g_{1k}}{z_{2k+1}(T)} \frac{\lambda_{2k+1}}{(1 + \alpha\lambda_{2k+1})} \right)^2 \\ &\quad + \left(\sum_{k=1}^{\infty} \frac{g_{2k}^\delta - g_{2k}}{z_{2k}(T)} \frac{\lambda_{2k}}{(1 + \alpha\lambda_{2k})} \right)^2 \\ &\leq \left(\left(\sup_{k \geq 0} A_{2k+1} \right)^2 + \left(\sup_{k \geq 1} A_{2k} \right)^2 \right) \delta^2, \end{aligned} \quad (3.4.8)$$

where

$$A_{2k+1} = \frac{1}{z_{2k+1}(T)} \frac{\lambda_{2k+1}}{(1 + \alpha\lambda_{2k+1})},$$

and

$$A_{2k} = \frac{1}{z_{2k}(T)} \frac{\lambda_{2k}}{(1 + \alpha\lambda_{2k})}.$$

Using Lemma 3.2.1, we obtain

$$A_{2k+1} \leq \frac{\lambda_{2k+1}}{a_2(1 + \alpha\lambda_{2k+1})} \leq \frac{1}{a_2} \frac{1}{\alpha},$$

and

$$A_{2k} \leq \frac{\lambda_{2k}}{a_1 (1 + \alpha \lambda_{2k})} \leq \frac{1}{a_1} \frac{1}{\alpha}.$$

Thus, inequality (3.4.8), gives

$$\|f_\alpha^\delta - f_\alpha\| \leq \sqrt{\left(\frac{1}{a_1}\right)^2 + \left(\frac{1}{a_2}\right)^2} \frac{\delta}{\alpha}. \quad (3.4.9)$$

Let us now estimate the second term in (3.4.7). Using (3.4.4), we can infer that

$$\begin{aligned} & f_\alpha(x) - f(x) \\ &= \sum_{k=0}^{\infty} \left(\frac{g_{1k}}{z_{2k+1}(T)} \frac{\lambda_{2k+1}}{(1 + \alpha \lambda_{2k+1})} - \frac{\lambda_{2k+1} g_{1k}}{z_{2k+1}(T)} \right) \varphi_{2k+1}(x) \\ & \quad + \sum_{k=1}^{\infty} \left(\frac{g_{2k}}{z_{2k}(T)} \frac{\lambda_{2k}}{(1 + \alpha \lambda_{2k})} - \frac{\lambda_{2k} g_{2k}}{z_{2k}(T)} \right) \varphi_{2k}(x) \\ &= \sum_{k=0}^{\infty} \left(\frac{\lambda_{2k+1} g_{1k}}{z_{2k+1}(T)} \frac{-\alpha \lambda_{2k+1}}{(1 + \alpha \lambda_{2k+1})} \right) \varphi_{2k+1}(x) \\ & \quad + \sum_{k=1}^{\infty} \left(\frac{\lambda_{2k} g_{2k}}{z_{2k}(T)} \frac{-\alpha \lambda_{2k}}{(1 + \alpha \lambda_{2k})} \right) \varphi_{2k}(x). \end{aligned} \quad (3.4.10)$$

Applying the a priori bound condition given by equation (3.2.33), we acquire:

$$\begin{aligned} & \|f_\alpha - f\|^2 \\ &= \sum_{k=0}^{\infty} \left(\frac{\lambda_{2k+1} g_{1k}}{z_{2k+1}(T)} \right)^2 \lambda_{2k+1}^p \left(\frac{\alpha \lambda_{2k+1}}{(1 + \alpha \lambda_{2k+1})} \right)^2 \frac{1}{\lambda_{2k+1}^p} \\ & \quad + \sum_{k=1}^{\infty} \left(\frac{\lambda_{2k} g_{2k}}{z_{2k}(T)} \right)^2 \lambda_{2k}^p \left(\frac{\alpha \lambda_{2k}}{(1 + \alpha \lambda_{2k})} \right)^2 \frac{1}{\lambda_{2k}^p} \\ & \leq E^2 \left(\left(\sup_{k \geq 0} \widehat{F}_{2k+1} \right)^2 + \left(\sup_{k \geq 1} \widehat{F}_{2k} \right)^2 \right), \end{aligned} \quad (3.4.11)$$

where

$$\widehat{F}_{2k+1} = \frac{\alpha \lambda_{2k+1}^{1-\frac{p}{2}}}{1 + \alpha \lambda_{2k+1}},$$

and

$$\widehat{F}_{2k} = \frac{\alpha \lambda_{2k}^{1-\frac{p}{2}}}{1 + \alpha \lambda_{2k+1}}.$$

From Lemma 3.2.3, we have

$$\widehat{F}_{2k+1} \leq \begin{cases} \widehat{b}_1 \alpha^{\frac{p}{2}}, & 0 < p < 2, \\ \widehat{b}_2 \alpha, & p \geq 2, \end{cases}$$

and

$$\widehat{F}_{2k} \leq \begin{cases} \widehat{c}_1 \alpha^{\frac{p}{2}}, & 0 < p < 2, \\ \widehat{c}_2 \alpha, & p \geq 2. \end{cases}$$

Substituting the above inequality into (3.4.11) and using (3.4.9), we obtain

$$\begin{aligned} \|f_\alpha^\delta - f\| &\leq \sqrt{\left(\frac{1}{a_1}\right)^2 + \left(\frac{1}{a_2}\right)^2} \frac{\delta}{\alpha} \\ &\quad + \begin{cases} \sqrt{\widehat{b}_1^2 + \widehat{c}_1^2} E \alpha^{\frac{p}{2}}, & 0 < p < 2, \\ \sqrt{\widehat{b}_2^2 + \widehat{c}_2^2} E \alpha, & p \geq 2. \end{cases} \end{aligned} \quad (3.4.12)$$

By choosing the regularization parameter α as:

$$\alpha = \begin{cases} \left(\frac{\delta}{E}\right)^{\frac{2}{p+2}}, & 0 < p < 2, \\ \left(\frac{\delta}{E}\right)^{\frac{1}{2}}, & p \geq 2, \end{cases} \quad (3.4.13)$$

then, we get

$$\|f_\alpha^\delta - f\| \leq \begin{cases} \left(\sqrt{\frac{1}{a_1^2} + \frac{1}{a_2^2}} + C_1\right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, & 0 < p < 2, \\ \left(\sqrt{\frac{1}{a_1^2} + \frac{1}{a_2^2}} + C_2\right) E^{\frac{1}{2}} \delta^{\frac{1}{2}}, & p \geq 2. \end{cases} \quad (3.4.14)$$

The proof is completed. \square

3.4.2 Convergence estimate under an a posteriori regularization parameter choice rule

In this subsection, we employ a posteriori regularization parameter selection approach to determine the value of the regularization parameter α in equation (3.4.3). We can further derive a convergence rate for the regularized solution (3.4.3) using this parameter selection

rule.

We adopt the discrepancy principle in the following manner:

$$\left\| \alpha L (1 + \alpha L)^{-1} (Kf_\alpha^\delta - g^\delta) \right\| = \tau \delta. \quad (3.4.15)$$

Here, $\tau > 1$ is a constant. As per the lemma below, a unique solution exists for (3.4.15) when $\|g^\delta\| > \tau \delta > 0$.

Lemma 3.4.1. *Set $\rho(\alpha) = \left\| \alpha L (1 + \alpha L)^{-1} (Kf_\alpha^\delta - g^\delta) \right\|$. If $\|g^\delta\| > \tau \delta > 0$, then the following holds:*

- (a) $\rho(\alpha)$ is a continuous function,
- (b) $\lim_{\alpha \rightarrow 0} \rho(\alpha) = 0$,
- (c) $\lim_{\alpha \rightarrow \infty} \rho(\alpha) = \|g^\delta\|$,
- (d) $\rho(\alpha)$ is a monotonically increasing function on $(0, \infty)$.

Proof. The proof is a straightforward result based on the fact

$$\rho(\alpha) = \left(\sum_{k=0}^{\infty} \left(\frac{\alpha \lambda_{2k+1}}{1 + \alpha \lambda_{2k+1}} \right)^4 (g_{1k}^\delta)^2 + \sum_{k=1}^{\infty} \left(\frac{\alpha \lambda_{2k}}{1 + \alpha \lambda_{2k}} \right)^4 (g_{2k}^\delta)^2 \right)^{\frac{1}{2}}.$$

□

Theorem 3.4.2. *Assuming that the a priori condition (3.2.33) and the noise assumption (3.2.5) are satisfied and that, there exists a constant $\tau > 1$ such that $\|g^\delta\| > \tau \delta > 0$. The regularization parameter $\alpha > 0$ is selected using the discrepancy principle (3.4.15). Under these conditions, we obtain*

1. If $0 < p < 2$, we obtain a convergence estimate

$$\left\| f_\alpha^\delta - f \right\| \leq \left(C (\tau + 1)^{\frac{p}{p+2}} + \sqrt{\frac{1}{a_1^2} + \frac{1}{\alpha_2^2}} \left(\frac{\widehat{b}_1^2 + \widehat{c}_1^2}{\tau - 1} \right)^{\frac{2}{p+2}} \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.$$

2. If $p \geq 2$, we obtain a convergence estimate

$$\|f_\alpha^\delta - f\| \leq \left(C(\tau + 1)^{\frac{1}{2}} + \sqrt{\frac{1}{a_1^2} + \frac{1}{\alpha_2^2}} \left(\frac{(\widehat{b}_2^2 + \widehat{c}_2^2)}{\tau - 1} \right)^{\frac{1}{2}} \right) E^{\frac{1}{2}} \delta^{\frac{1}{2}}.$$

Proof. 1. For $0 < p < 2$, similar to (3.4.7), we have

$$\|f_\alpha^\delta - f\| \leq \|f_\alpha^\delta - f_\alpha\| + \|f_\alpha - f\|.$$

Let us begin by providing an estimate for the second term. We have

$$\|f_\alpha - f\|^2 = \left\| \sum_{k=0}^{\infty} \frac{\alpha \lambda_{2k+1}}{1 + \alpha \lambda_{2k+1}} f_{1k} \varphi_{2k+1} \right\|^2 + \left\| \sum_{k=1}^{\infty} \frac{\alpha \lambda_{2k}}{1 + \alpha \lambda_{2k}} f_{2k} \varphi_{2k} \right\|^2. \quad (3.4.16)$$

By using equation (3.4.3), Lemma 3.2.4, and the a priori bound condition (3.2.33), we derive:

$$\begin{aligned} & \left\| \sum_{k=0}^{\infty} \frac{\alpha \lambda_{2k+1}}{1 + \alpha \lambda_{2k+1}} f_{1k} \varphi_{2k+1} \right\| \\ &= \left\| \sum_{k=0}^{\infty} \left(\frac{\alpha \lambda_{2k+1} \theta_{2k+1}}{1 + \alpha \lambda_{2k+1}} \right)^{\frac{p}{2}} \left(\frac{\alpha \lambda_{2k+1}}{1 + \alpha \lambda_{2k+1}} \right)^{1 - \frac{p}{2}} \frac{f_{1k}}{(\theta_{2k+1})^{\frac{p}{2}}} \varphi_{2k+1} \right\| \\ &\leq \left\| \sum_{k=0}^{\infty} \left(\frac{\alpha \lambda_{2k+1} \theta_{2k+1}}{1 + \alpha \lambda_{2k+1}} \right)^{\frac{p+2}{2}} \left(\frac{\alpha \lambda_{2k+1}}{1 + \alpha \lambda_{2k+1}} \right)^{1 - \frac{p}{2}} \frac{f_{1k}}{(\theta_{2k+1})^{\frac{p}{2}}} \varphi_{2k+1} \right\|^{\frac{p}{p+2}} \\ &\quad \times \left\| \sum_{k=0}^{\infty} \left(\frac{\alpha \lambda_{2k+1}}{1 + \alpha \lambda_{2k+1}} \right)^{1 - \frac{p}{2}} \frac{f_{1k}}{(\theta_{2k+1})^{\frac{p}{2}}} \varphi_{2k+1} \right\|^{\frac{2}{p+2}} \\ &\leq \left\| \sum_{k=0}^{\infty} \left(\frac{\alpha \lambda_{2k+1}}{1 + \alpha \lambda_{2k+1}} \right)^2 \theta_{2k+1} f_{1k} \varphi_{2k+1} \right\|^{\frac{p}{p+2}} \left\| \sum_{k=0}^{\infty} \frac{f_{1k}}{(\theta_{2k+1})^{\frac{p}{2}}} \varphi_{2k+1} \right\|^{\frac{2}{p+2}} \\ &\leq \left\| \sum_{k=0}^{\infty} \left(\frac{\alpha \lambda_{2k+1}}{1 + \alpha \lambda_{2k+1}} \right)^2 g_{1k} \varphi_{2k+1} \right\|^{\frac{p}{p+2}} \left\| \sum_{k=0}^{\infty} \lambda_{2k+1}^{\frac{p}{2}} f_{1k} \varphi_{2k+1} \right\|^{\frac{2}{p+2}} a_2^{-\frac{p}{p+2}} \\ &\leq \left(\left\| \sum_{k=0}^{\infty} \left(\frac{\alpha \lambda_{2k+1}}{1 + \alpha \lambda_{2k+1}} \right)^2 (g_{1k} - g_{1k}^\delta) \varphi_{2k+1} \right\| \right. \\ &\quad \left. + \left\| \sum_{k=0}^{\infty} \left(\frac{\alpha \lambda_{2k+1}}{1 + \alpha \lambda_{2k+1}} \right)^2 g_{1k}^\delta \varphi_{2k+1} \right\| \right)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} a_2^{-\frac{p}{p+2}} \end{aligned}$$

$$\leq a_2^{-\frac{p}{p+2}} (\tau + 1)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, \quad (3.4.17)$$

and

$$\begin{aligned} & \left\| \sum_{k=1}^{\infty} \frac{\alpha \lambda_{2k}}{1 + \alpha \lambda_{2k}} f_{2k} \varphi_{2k} \right\| \\ &= \left\| \sum_{k=1}^{\infty} \left(\frac{\alpha \lambda_{2k} \theta_{2k}}{1 + \alpha \lambda_{2k}} \right)^{\frac{p}{2}} \left(\frac{\alpha \lambda_{2k}}{1 + \alpha \lambda_{2k}} \right)^{1 - \frac{p}{2}} \frac{f_{2k}}{(\theta_{2k})^{\frac{p}{2}}} \varphi_{2k} \right\| \\ &\leq \left\| \sum_{k=1}^{\infty} \left(\frac{\alpha \lambda_{2k} \theta_{2k}}{1 + \alpha \lambda_{2k}} \right)^{\frac{p+2}{2}} \left(\frac{\alpha \lambda_{2k}}{1 + \alpha \lambda_{2k}} \right)^{1 - \frac{p}{2}} \frac{f_{2k}}{(\theta_{2k})^{\frac{p}{2}}} \varphi_{2k} \right\|^{\frac{p}{p+2}} \\ &\quad \times \left\| \left(\frac{\alpha \lambda_{2k}}{1 + \alpha \lambda_{2k}} \right)^{1 - \frac{p}{2}} \frac{f_{2k}}{(\theta_{2k})^{\frac{p}{2}}} \varphi_{2k} \right\|^{\frac{2}{p+2}} \\ &\leq \left\| \sum_{k=1}^{\infty} \left(\frac{\alpha \lambda_{2k}}{1 + \alpha \lambda_{2k}} \right)^2 \theta_{2k} f_{2k} \varphi_{2k} \right\|^{\frac{p}{p+2}} \left\| \sum_{k=1}^{\infty} \frac{f_{2k}}{(\theta_{2k})^{\frac{p}{2}}} \varphi_{2k} \right\|^{\frac{2}{p+2}} \\ &\leq \left\| \sum_{k=1}^{\infty} \left(\frac{\alpha \lambda_{2k}}{1 + \alpha \lambda_{2k}} \right)^2 g_{2k} \varphi_{2k} \right\|^{\frac{p}{p+2}} \left\| \sum_{k=1}^{\infty} \lambda_{2k}^{\frac{p}{2}} f_{2k} \varphi_{2k} \right\|^{\frac{2}{p+2}} a_1^{-\frac{p}{p+2}} \\ &\leq \left\| \sum_{k=1}^{\infty} \left(\frac{\alpha \lambda_{2k}}{1 + \alpha \lambda_{2k}} \right)^2 (g_{2k} - g_{2k}^{\delta}) \varphi_{2k} \right\| \\ &\quad + \left\| \sum_{k=1}^{\infty} \left(\frac{\alpha \lambda_{2k}}{1 + \alpha \lambda_{2k}} \right)^2 g_{2k}^{\delta} \varphi_{2k} \right\|^{\frac{p}{p+2}} E^{\frac{2}{p+2}} a_1^{-\frac{p}{p+2}} \\ &\leq a_1^{-\frac{p}{p+2}} (\tau + 1)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \end{aligned} \quad (3.4.18)$$

Combining (3.4.16)-(3.4.18), we obtain

$$\|f_{\alpha} - f\| \leq C (\tau + 1)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}} \quad (3.4.19)$$

Now we give the bound for the first term. Similar to (3.4.9), we have

$$\|f_{\alpha}^{\delta} - f_{\mu}\| \leq \sqrt{\frac{1}{\alpha_1^2} + \frac{1}{a_2^2} \delta}. \quad (3.4.20)$$

From (3.4.15), there holds

$$\begin{aligned}
\tau\delta &= \left\| \sum_{k=0}^{\infty} \left(\frac{\alpha\lambda_{2k+1}}{1+\alpha\lambda_{2k+1}} \right)^2 g_{1k}^\delta \varphi_{2k+1} + \sum_{k=1}^{\infty} \left(\frac{\alpha\lambda_{2k}}{1+\alpha\lambda_{2k}} \right)^2 g_{2k}^\delta \varphi_{2k} \right\| \\
&\leq \left\| \sum_{k=0}^{\infty} \left(\frac{\alpha\lambda_{2k+1}}{1+\alpha\lambda_{2k+1}} \right)^2 (g_{1k}^\delta - g_{1k}) \varphi_{2k+1} + \sum_{k=1}^{\infty} \left(\frac{\alpha\lambda_{2k}}{1+\alpha\lambda_{2k}} \right)^2 (g_{2k}^\delta - g_{2k}) \varphi_{2k} \right\| \\
&\quad + \left\| \sum_{k=0}^{\infty} \left(\frac{\alpha\lambda_{2k+1}}{1+\alpha\lambda_{2k+1}} \right)^2 g_{1k} \varphi_{2k+1} + \sum_{k=1}^{\infty} \left(\frac{\alpha\lambda_{2k}}{1+\alpha\lambda_{2k}} \right)^2 g_{2k} \varphi_{2k} \right\| \\
&\leq \delta + \left\| \sum_{k=0}^{\infty} \left(\frac{\alpha\lambda_{2k+1}}{1+\alpha\lambda_{2k+1}} \right)^2 g_{1k} \varphi_{2k+1} + \sum_{k=1}^{\infty} \left(\frac{\alpha\lambda_{2k}}{1+\alpha\lambda_{2k}} \right)^2 g_{2k} \varphi_{2k} \right\| \\
&\leq \delta + J.
\end{aligned} \tag{3.4.21}$$

By Lemma 3.2.4, we have

$$\begin{aligned}
J^2 &= \left\| \sum_{k=0}^{\infty} \left(\frac{\alpha\lambda_{2k+1}}{1+\alpha\lambda_{2k+1}} \right)^2 g_{1k} \varphi_{2k+1} + \sum_{k=1}^{\infty} \left(\frac{\alpha\lambda_{2k}}{1+\alpha\lambda_{2k}} \right)^2 g_{2k} \varphi_{2k} \right\|^2 \\
&\leq \sum_{k=0}^{\infty} \left(\frac{\alpha\lambda_{2k+1}}{1+\alpha\lambda_{2k+1}} \right)^4 (g_{1k})^2 + \sum_{k=1}^{\infty} \left(\frac{\alpha\lambda_{2k}}{1+\alpha\lambda_{2k}} \right)^4 (g_{2k})^2 \\
&\leq E^2 \sup_{k \geq 0} \left(\frac{\alpha\lambda_{2k+1}}{1+\alpha\lambda_{2k+1}} \right)^4 \left(\frac{z_{2k+1}(T)}{\lambda_{2k+1}} \right)^2 \frac{1}{\lambda_{2k+1}^p} \\
&\quad + E^2 \sup_{k \geq 1} \left(\frac{\alpha\lambda_{2k}}{1+\alpha\lambda_{2k}} \right)^4 \left(\frac{z_{2k}(T)}{\lambda_{2k}} \right)^2 \frac{1}{\lambda_{2k}^p} \\
&\leq E^2 \sup_{k \geq 0} \left(\frac{\alpha\lambda_{2k+1}^{1-\frac{p+2}{4}}}{1+\alpha\lambda_{2k+1}} \right)^4 + E^2 \sup_{k \geq 1} \left(\frac{\alpha\lambda_{2k}^{1-\frac{p+2}{4}}}{1+\alpha\lambda_{2k}} \right)^4 \\
&\leq \left((\widehat{b}_1^2 + \widehat{c}_1^2) E \alpha^{\frac{p+2}{2}} \right)^2.
\end{aligned} \tag{3.4.22}$$

Combining (3.4.21)-(3.4.22), we obtain

$$(\tau - 1)\delta \leq \left(\widehat{b}_1^2 + \widehat{c}_1^2 \right) E \alpha^{\frac{p+2}{2}}.$$

This yields

$$\frac{1}{\alpha} \leq \left(\frac{\widehat{b}_1^2 + \widehat{c}_1^2}{\tau - 1} \right)^{\frac{2}{p+2}} \left(\frac{E}{\delta} \right)^{\frac{2}{p+2}}. \tag{3.4.23}$$

Substituting (3.4.23) into (3.4.20), we obtain

$$\|f_\alpha^\delta - f_\alpha\| \leq \sqrt{\frac{1}{a_1^2} + \frac{1}{a_2^2}} \left(\frac{(\widehat{b}_1^2 + \widehat{c}_1^2)}{\tau - 1} \right)^{\frac{2}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \quad (3.4.24)$$

Combining (3.4.24) and (3.4.19), we obtain

$$\|f_\alpha^\delta - f\| \leq \left(C (\tau + 1)^{\frac{p}{p+2}} + \sqrt{\frac{1}{a_1^2} + \frac{1}{a_2^2}} \left(\frac{(\widehat{b}_1^2 + \widehat{c}_1^2)}{\tau - 1} \right)^{\frac{2}{p+2}} \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \quad (3.4.25)$$

2. For $p \geq 2$, we use the same methodology, we can draw the aforementioned conclusion. This completes the proof. □

3.5 Iterative regularization method and convergence rates

In this section, we propose an iterative technique to address the inverse source problem. In addition, we provide two convergence assessments based on distinct approaches for selecting the regularization parameter: one employing an a priori rule and the other using an a posteriori rule.

Let $v^m(x, t)$ denote the solution to the following problem:

$$\begin{cases} D_t^\gamma [v^m(x, t) + Lv^m(x, t)] + Lv^m(x, t) = f^{m,\delta}(x), & (x, t) \in Q, \\ v^m(0, t) = v^m(\pi, t) = 0, & t \in (0, T), \\ v^m(x, 0) = 0, & x \in (0, \pi), \\ f^{m,\delta}(x) = f^{m-1,\delta}(x) - s \left(v^{m-1}(x, T) - g^\delta(x) \right), & x \in (0, \pi), \end{cases} \quad (3.5.1)$$

where m stands for the regularization parameter and s serves as an accelerating factor that meets the condition:

$$0 < \frac{s}{\lambda_k} z_k(T) < 1 \quad \text{for all } k \in \mathbb{N}^*.$$

In the subsequent subsections, we will demonstrate that $f^{m,\delta}(x)$ constitutes a regularized solution to the inverse source problem. For the sake of simplicity, we will use the notation

$\sigma_{2k+1} = \frac{s}{\lambda_{2k+1}} z_{2k+1}(T)$ and $\sigma_{2k} = \frac{s}{\lambda_{2k}} z_{2k}(T)$ and take $f^{0,\delta}(x) = 0$.

Denote $f_{1k}^{m,\delta} = (f^{m,\delta}, \varphi_{2k+1})$, and $f_{2k}^{m,\delta} = (f^{m,\delta}, \varphi_{2k})$, then we have

$$\begin{aligned} v^m(x, t) &= \sum_{k=0}^{\infty} \frac{f_{1k}^{m-1,\delta}}{\lambda_{2k+1}} z_{2k+1}(t) \varphi_{2k+1}(x) \\ &\quad + \sum_{k=1}^{\infty} \frac{f_{2k}^{m-1,\delta}}{\lambda_{2k}} z_{2k}(t) \varphi_{2k}(x). \end{aligned}$$

From (3.5.1)), we obtain

$$\begin{aligned} f_{1k}^{m,\delta} &= f_{1k}^{m-1,\delta} - s \left(\frac{f_{1k}^{m-1,\delta}}{\lambda_{2k+1}} z_{2k+1}(T) - g_{1k}^{\delta} \right) \\ &= (1 - \sigma_{2k+1}) f_{1k}^{m-1,\delta} + s g_{1k}^{\delta} \\ &= (1 - \sigma_{2k+1})^m f_{1k}^{0,\delta} + \sum_{i=0}^{m-1} (1 - \sigma_{2k+1})^i s g_{1k}^{\delta}, \end{aligned} \quad (3.5.2)$$

and

$$\begin{aligned} f_{2k}^{m,\delta} &= f_{2k}^{m-1,\delta} - s \left(\frac{f_{2k}^{m-1,\delta}}{\lambda_{2k}} z_{2k}(T) - g_{2k}^{\delta} \right) \\ &= (1 - \sigma_{2k}) f_{2k}^{m-1,\delta} + s g_{2k}^{\delta} \\ &= (1 - \sigma_{2k})^m f_{2k}^{0,\delta} + \sum_{i=0}^{m-1} (1 - \sigma_{2k})^i s g_{2k}^{\delta}. \end{aligned} \quad (3.5.3)$$

If we take $f_{1k}^{0,\delta} = 0$, and $f_{2k}^{0,\delta} = 0$, we have $f_{1k}^{m,\delta} = \sum_{i=0}^{m-1} (1 - \sigma_{2k+1})^i s g_{1k}^{\delta}$, and $f_{2k}^{m,\delta} =$

$\sum_{i=0}^{m-1} (1 - \sigma_{2k})^i s g_{2k}^{\delta}$, we can write

$$\begin{aligned} f^{m,\delta}(x) &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{m-1} (1 - \sigma_{2n+1})^i s g_{1n}^{\delta} \right) \varphi_{2n+1}(x) \\ &\quad + \sum_{n=1}^{\infty} \left(\sum_{i=0}^{m-1} (1 - \sigma_{2n})^i s g_{2n}^{\delta} \right) \varphi_{2n}(x) \\ &= \sum_{n=0}^{\infty} p_m(\sigma_{2n+1}) s g_{1n}^{\delta} \varphi_{2n+1}(x) + \sum_{n=1}^{\infty} p_m(\sigma_{2n}) s g_{2n}^{\delta} \varphi_{2n}(x). \end{aligned} \quad (3.5.4)$$

Denote $f^{m,\delta}(x) = \sum_{k=0}^{\infty} p_m(\sigma_{2k+1}) s g_{1k}^{\delta} \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} p_m(\sigma_{2k}) s g_{2k}^{\delta} \varphi_{2k}(x)$, we have

$$\begin{aligned}
f^{m,\delta}(x) - f(x) &= f^{m,\delta}(x) - f^m(x) + f^m(x) - f(x) \\
&= \sum_{k=0}^{\infty} p_m(\sigma_{2k+1}) s (g_{1k}^{\delta} - g_{1k}) \varphi_{2k+1}(x) \\
&\quad + \sum_{k=1}^{\infty} p_m(\sigma_{2k}) s (g_{2k}^{\delta} - g_{2k}) \varphi_{2k}(x) \\
&\quad + \sum_{k=0}^{\infty} (-r_m(\sigma_{2k+1})) \frac{\lambda_{2k+1} g_{1k} \varphi_{2k+1}(x)}{z_{2k+1}(T)} \\
&\quad + \sum_{k=1}^{\infty} (-r_m(\sigma_{2k})) \frac{\lambda_{2k} g_{2k} \varphi_{2k}(x)}{z_{2k}(T)}. \tag{3.5.5}
\end{aligned}$$

Below, we provide two convergence assessments for $\|f^{m,\delta} - f\|$ employing both an a priori and a posteriori selection approach for the regularization parameter.

3.5.1 Convergence analysis with a priori regularization parameter selection.

Theorem 3.5.1. *Assuming that the a priori condition (3.2.33) and the noise Assumption (3.2.5) are satisfied, if we opt for $m = \left\lceil \frac{1}{s} \left(\frac{E}{\delta} \right)^{\frac{2}{p+2}} \right\rceil$, then the ensuing estimate holds:*

$$\|f^{m,\delta} - f\| \leq \left(\sqrt{2} + 2\sqrt{\alpha_1^{-p} + a_2^{-p} \theta_{\frac{p}{2}}} \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, \tag{3.5.6}$$

where $[m]$ denotes the largest integer not exceeding m .

Proof. Using triangular inequality, we get

$$\|f^{m,\delta} - f\| \leq \|f^{m,\delta} - f^m\| + \|f^m - f\|. \tag{3.5.7}$$

We begin by estimating the first term. From (3.5.5) and (3.2.5), we obtain

$$\|f^{m,\delta} - f^m\|^2 = \sum_{k=0}^{\infty} \left(p_m(\sigma_{2k+1}) s (g_{1k}^{\delta} - g_{1k}) \right)^2$$

$$\begin{aligned}
& + \sum_{k=1}^{\infty} \left(p_m(\sigma_{2k}) s (g_{2k}^{\delta} - g_{2k}) \right)^2 \\
& \leq \left(\left(\sup_{k \geq 0} p_m(\sigma_{2k+1}) \right)^2 + \left(\sup_{k \geq 1} p_m(\sigma_{2k}) \right)^2 \right) (s\delta)^2, \tag{3.5.8}
\end{aligned}$$

where

$$p_m(\sigma_{2k+1}) = \sum_{i=0}^{m-1} (1 - \sigma_{2k+1})^i,$$

and

$$p_m(\sigma_{2k}) = \sum_{i=0}^{m-1} (1 - \sigma_{2k})^i.$$

By Lemma 3.2.5 with $\mu = 0$, we get

$$p_m(\sigma_{2k+1}) \leq m.$$

and

$$p_m(\sigma_{2k}) \leq m.$$

Then, using Equation (3.5.8), we obtain

$$\|f^{m,\delta} - f^m\| \leq \sqrt{2} sm\delta. \tag{3.5.9}$$

On the other hand, we estimate the second term in (3.5.7). By (3.5.5), we can deduce that

$$\begin{aligned}
f^m(x) - f(x) & = \sum_{k=0}^{\infty} (-r_m(\sigma_{2k+1})) \frac{\lambda_{2k+1} g_{1k} \varphi_{2k+1}(x)}{z_{2k+1}(T)} \\
& \quad + \sum_{k=1}^{\infty} (-r_m(\sigma_{2k})) \frac{\lambda_{2k} g_{2k} \varphi_{2k}(x)}{z_{2k}(T)}. \tag{3.5.10}
\end{aligned}$$

Applying the a priori bound condition (3.2.33), we obtain

$$\begin{aligned}
& \|f^m - f\|^2 \\
& = \sum_{k=0}^{\infty} \frac{r_m^2(\sigma_{2k+1})}{\lambda_{2k+1}^p} \left(\frac{\lambda_{2k+1} g_{1k}}{z_{2k+1}(T)} \right)^2 \lambda_{2k+1}^p \\
& \quad + \sum_{k=1}^{\infty} \frac{r_m^2(\sigma_{2k})}{\lambda_{2k}^p} \left(\frac{\lambda_{2k} g_{2k}}{z_{2k}(T)} \right)^2 \lambda_{2k}^p
\end{aligned}$$

$$\leq E^2 \left(\left(\sup_{k \geq 0} D_{2k+1} \right)^2 + \left(\sup_{k \geq 1} D_{2k} \right)^2 \right), \quad (3.5.11)$$

where

$$D_{2k+1} = \frac{r_m(\sigma_{2k+1})}{\lambda_{2k+1}^{\frac{p}{2}}},$$

and

$$D_{2k} = \frac{r_m(\sigma_{2k})}{\lambda_{2k}^{\frac{p}{2}}}.$$

From Lemmas 3.2.1 and 3.2.4 with $\nu = \frac{p}{2}$, we have

$$\begin{aligned} D_{2k+1} &\leq (sa_2)^{-\frac{p}{2}} r_m(\sigma_{2k+1}) \sigma_{2k+1}^{\frac{p}{2}} \\ &\leq \theta_{\frac{p}{2}} (sa_2)^{-\frac{p}{2}} \frac{1}{(m+1)^{\frac{p}{2}}}, \end{aligned}$$

and

$$\begin{aligned} D_{2k} &\leq (sa_1)^{-\frac{p}{2}} r_m(\sigma_{2k}) \sigma_{2k}^{\frac{p}{2}} \\ &\leq \theta_{\frac{p}{2}} (sa_1)^{-\frac{p}{2}} \frac{1}{(m+1)^{\frac{p}{2}}}. \end{aligned}$$

Using (3.5.11) and (3.5.9), we obtain

$$\begin{aligned} \|f^{m,\delta} - f\| &\leq \sqrt{2}sm\delta \\ &\quad + \sqrt{a_1^{-p} + a_2^{-p}} \theta_{\frac{p}{2}} s^{-\frac{p}{2}} \frac{E}{(m+1)^{\frac{p}{2}}}. \end{aligned} \quad (3.5.12)$$

By choosing the regularization parameter m as follows

$$m = \left\lceil \frac{1}{s} \left(\frac{E}{\delta} \right)^{\frac{2}{p+2}} \right\rceil, \quad (3.5.13)$$

we get

$$\|f^{m,\delta} - f\| \leq \left(\sqrt{2} + 2\sqrt{a_1^{-p} + a_2^{-p}} \theta_{\frac{p}{2}} \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \quad (3.5.14)$$

The proof is completed. \square

3.5.2 Convergence analysis with a posteriori regularization parameter selection

In this subsection, we employ an a posteriori regularization parameter selection rule that is not dependent on the a priori bound E .

We apply the discrepancy principle in the following manner:

$$\|Kf^{m,\delta} - g^\delta\| \leq \tau\delta < \|Kf^{m-1,\delta} - g^\delta\|, \quad (3.5.15)$$

where $\tau > 1$ is a constant.

Theorem 3.5.2. *Assuming that the a priori condition (3.2.33) and the noise assumption (3.2.5) are satisfied, and the regularization parameter m is determined using the discrepancy principle (3.5.15), we can establish a convergence estimate as follows:*

$$\|f^{m,\delta} - f\| \leq \left(C(\tau + 1)^{\frac{p}{p+2}} + \sqrt{2} \left(\frac{\sqrt{a_1^{-p} + a_2^{-p}} \theta^{\frac{p+2}{2}}}{\tau - 1} \right)^{\frac{2}{p+2}} \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.$$

Proof. Similar to (3.5.7), we have

$$\|f^{m,\delta} - f\| \leq \|f^{m,\delta} - f^m\| + \|f^m - f\|.$$

Let us begin by estimating the second term. In accordance with Remark 3.2.3, we can derive the upper bound of $\|f^m - f\|$ by estimating $\|K(f^m - f)\|$ and $\|f^m - f\|_{\frac{1}{2}}$. Referring to (3.5.5), we obtain:

$$\begin{aligned} & \|K(f^m - f)\| \\ &= \left\| \sum_{k=0}^{\infty} r_m(\sigma_{2k+1}) g_{1k} \varphi_{2k+1} + \sum_{k=1}^{\infty} r_m(\sigma_{2k}) g_{2k} \varphi_{2k} \right\| \\ &\leq \left\| \sum_{k=0}^{\infty} r_m(\sigma_{2k+1}) (g_{1k} - g_{1k}^\delta) \varphi_{2k+1} + \sum_{k=1}^{\infty} r_m(\sigma_{2k}) (g_{2k} - g_{2k}^\delta) \varphi_{2k} \right\| \\ &\quad + \left\| \sum_{k=0}^{\infty} r_m(\sigma_{2k+1}) g_{1k}^\delta \varphi_{2k+1} + \sum_{k=1}^{\infty} r_m(\sigma_{2k}) g_{2k}^\delta \varphi_{2k} \right\| \\ &\leq \delta + \tau\delta = (1 + \tau)\delta. \end{aligned}$$

Using the a priori bound condition for $f(x)$, we derive:

$$\begin{aligned} \|f^m - f\|_{D(L^{\frac{p}{2}})}^2 &= \sum_{k=0}^{\infty} r_m^2 (\sigma_{2k+1}) \left(\frac{\lambda_{2k+1} g_{1k}}{z_{2k+1}(T)} \right)^2 \lambda_{2k+1}^p \\ &\quad + \sum_{k=1}^{\infty} r_m^2 (\sigma_{2k}) \left(\frac{\lambda_{2k} g_{2k}}{z_{2k}(T)} \right)^2 \lambda_{2k}^p \\ &\leq E^2. \end{aligned}$$

On the basis of Theorem 3.2.1, we deduce

$$\|f^m - f\| \leq C (\tau + 1)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \quad (3.5.16)$$

On the other hand, we establish an upper bound for the first term. Similar to (3.5.9), we find:

$$\|f^{m,\delta} - f^m\| \leq \sqrt{2} s m \delta. \quad (3.5.17)$$

From (3.5.15), there holds

$$\begin{aligned} \tau \delta &\leq \left\| \sum_{k=0}^{\infty} r_{m-1} (\sigma_{2k+1}) g_{1k}^{\delta} \varphi_{2k+1} + \sum_{k=1}^{\infty} r_{m-1} (\sigma_{2k}) g_{2k}^{\delta} \varphi_{2k} \right\| \\ &\leq \left\| \sum_{k=0}^{\infty} r_{m-1} (\sigma_{2k+1}) (g_{1k}^{\delta} - g_{1k}) \varphi_{2k+1} + \sum_{k=1}^{\infty} r_{m-1} (\sigma_{2k}) (g_{2k}^{\delta} - g_{2k}) \varphi_{2k} \right\| \\ &\quad + \left\| \sum_{k=0}^{\infty} r_{m-1} (\sigma_{2k+1}) g_{1k} \varphi_{2k+1} + \sum_{k=1}^{\infty} r_{m-1} (\sigma_{2k}) g_{2k} \varphi_{2k} \right\| \\ &\leq \delta + \left\| \sum_{k=0}^{\infty} r_{m-1} (\sigma_{2k+1}) g_{1k} \varphi_{2k+1} + \sum_{k=1}^{\infty} r_{m-1} (\sigma_{2k}) g_{2k} \varphi_{2k} \right\| \\ &\leq \delta + J. \end{aligned} \quad (3.5.18)$$

From Lemmas 3.2.1 and 3.2.5, we have

$$\begin{aligned} J^2 &= \left\| \sum_{k=0}^{\infty} r_{m-1} (\sigma_{2k+1}) g_{1k} \varphi_{2k+1} + \sum_{k=1}^{\infty} r_{m-1} (\sigma_{2k}) g_{2k} \varphi_{2k} \right\|^2 \\ &\leq E^2 \sup_{k \geq 0} \frac{r_{m-1}^2 (\sigma_{2k+1})}{\lambda_{2k+1}^p} \left(\frac{z_{2k+1}(T)}{\lambda_{2k+1}} \right)^2 \end{aligned}$$

$$\begin{aligned}
& + E^2 \sup_{k \geq 1} \frac{r_{m-1}^2(\sigma_{2k})}{\lambda_{2k}^p} \left(\frac{z_{2k}(T)}{\lambda_{2k}} \right)^2 \\
& \leq a_2^{-p} E^2 \sup_{k \geq 0} r_{m-1}^2(\sigma_{2k+1}) \left(\frac{z_{2k+1}(T)}{\lambda_{2k+1}} \right)^{2+p} \\
& \quad + a_1^{-p} E^2 \sup_{k \geq 1} r_{m-1}^2(\sigma_{2k}) \left(\frac{z_{2k}(T)}{\lambda_{2k}} \right)^{2+p} \\
& \leq a_2^{-p} \theta_{1+\frac{p}{2}}^2 E^2 \frac{1}{(sk)^{p+2}} + a_1^{-p} \theta_{1+\frac{p}{2}}^2 E^2 \frac{1}{(sm)^{p+2}} \\
& \leq \left(\sqrt{a_1^{-p} + a_2^{-p} \theta_{1+\frac{p}{2}}^2} E \frac{1}{(sm)^{\frac{p+2}{2}}} \right)^2. \tag{3.5.19}
\end{aligned}$$

Combining (3.5.18) and (3.5.19), we obtain

$$(\tau - 1) \delta \leq \sqrt{a_1^{-p} + a_2^{-p} \theta_{1+\frac{p}{2}}^2} E \frac{1}{(sm)^{\frac{p+2}{2}}}.$$

This yields

$$sm \leq \left(\frac{\sqrt{a_1^{-p} + a_2^{-p} \theta_{1+\frac{p}{2}}^2}}{\tau - 1} \right)^{\frac{2}{p+2}} \left(\frac{E}{\delta} \right)^{\frac{2}{p+2}}. \tag{3.5.20}$$

By using (3.5.20) into (3.5.17), we get

$$\|f^{m,\delta} - f^m\| \leq \sqrt{2} \left(\frac{\sqrt{a_1^{-p} + a_2^{-p} \theta_{1+\frac{p}{2}}^2}}{\tau - 1} \right)^{\frac{2}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \tag{3.5.21}$$

Combining (3.5.16) and (3.5.21), we obtain the desired result. \square

3.6 Numerical experiments

In this section, we give a numerical example to validate the theoretical results obtained and consequently the feasibility and efficiency of the proposed approaches. We consider the following example:

$$\left\{ \begin{array}{l} D_t^\gamma [u(x, t) - u_{xx}(x, t) + \varepsilon u_{xx}(\pi - x, t)] - u_{xx}(x, t) + \varepsilon u_{xx}(\pi - x, t) = x(\pi - x), \\ u(0, t) = u(\pi, t) = 0, \\ u(x, 0) = 0, \\ u(x, 1) = g_N(x), \end{array} \right. \quad (3.6.1)$$

where $(x, t) \in (0, \pi) \times (0, T)$, $\gamma = 0.5$, $\varepsilon = 0.25$ and $T = 1$.

We compute a finite approximation of the final condition g by taking:

$$\begin{aligned} g_N(x) = u(x, 1) &= \sum_{k=0}^{N=20} \frac{f_{2k+1}}{\lambda_{2k+1}} \left(1 - E_{\gamma,1} \left(-\frac{\lambda_{2k+1}}{1 + \lambda_{2k+1}} \right) \right) \varphi_{2k+1}(x) \\ &+ \sum_{k=1}^{N=20} \frac{f_{2k}}{\lambda_{2k}} \left(1 - E_{\gamma,1} \left(-\frac{\lambda_{2k}}{1 + \lambda_{2k}} \right) \right) \varphi_{2k}(x), \end{aligned}$$

where the coefficients f_{2k} and f_{2k+1} have been numerically evaluated by the trapezoidal rule to approximate the integral using an equidistant grid $0 = x_0 < x_1 < \dots < x_{M=499} = \pi$, $h = \frac{\pi}{M}$.

Now, we add noise to the data g using a random perturbation (obtained by the MATLAB command `randn`), we obtain the vector g_δ :

$$\begin{aligned} noise &= randn(size(g_N)); \\ noise &= \delta \times noise \times norm(g_N)/norm(noise); \\ g_\delta &= g_N + noise; \end{aligned}$$

where $\delta = 0.1$ denotes the noise level in the information provided by the measurement. The function "`randn(.)`" generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$, and standard deviation $\sigma = 1$. "`randn(size(ψ))`" returns an array of random entries that is the same size as g .

Our inverse problem is to reconstruct the source term f_δ from the final data g_δ , and to give an estimate of the difference $\|f - f_\delta\|$, where f is the exact source function.

Our inverse problem is to reconstruct the unknown source $f(x)$ from the final condition $g_N(x) = u(x, 1)$.

The results of the numerical program are given in Figures 3.1-3.3

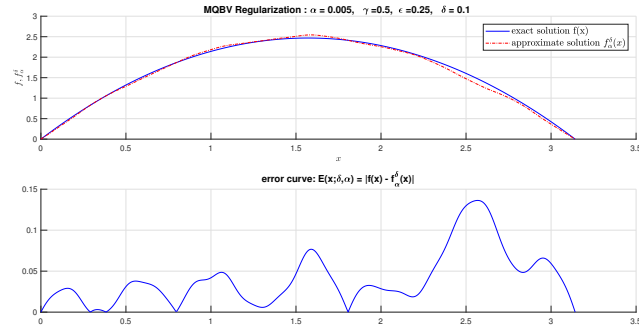


Figure 3.1: The exact and regularized source terms given by the a priori parameter choice: MQBV-method

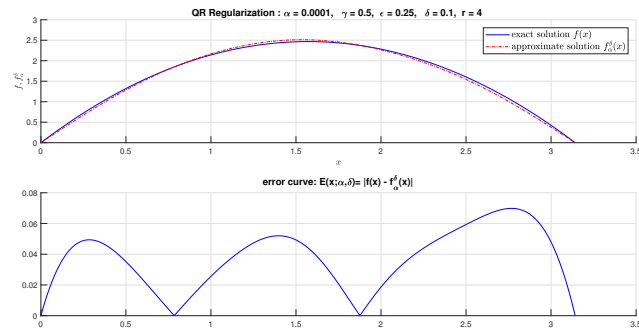


Figure 3.2: The exact and regularized source terms given by the a priori parameter choice: MQR-method

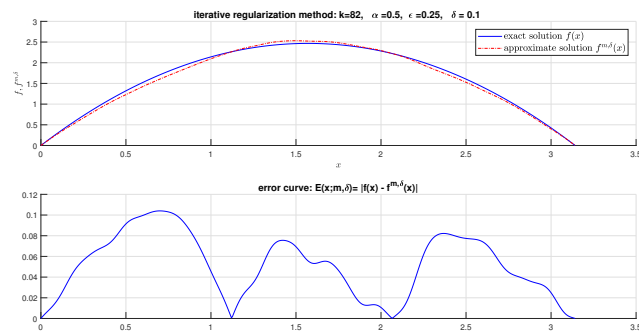


Figure 3.3: The exact and regularized source terms given by the a priori iteration parameter choice: Iterative-method

Conclusion

The considered example shows that the three methods are stable with respect to the strong noise level and the truncation error in the Fourier series. We can confirm that the three methods are close from a numerical point of view, except that the iterative method is simple in implementation.

This study set out to determine the regularization techniques for fractional pseudo-parabolic differential equations with involution terms. The comprehensive exploration covered theoretical advancements and practical methodologies. The primary findings suggest that the modified pseudo-parabolic regularization method effectively approximates the original ill-posed problems. Furthermore, the iterative methods for inverse problems demonstrated significant improvements in accuracy and stability. These findings contribute substantially to the field, offering valuable insights and practical solutions for researchers and practitioners dealing with complex inverse problems.

As perspectives, future research should continue to refine these methods and explore their applications in broader contexts. Specifically, there are several promising directions for further investigation:

- **Extension to Multi-Dimensional Problems:** While this thesis focused on one-dimensional cases, extending the regularization techniques to multi-dimensional problems could provide deeper insights and wider applicability.
- **Real-World Applications:** Implementing these methodologies in practical scenarios, such as geophysical exploration, medical imaging, and financial mathematics, can validate their effectiveness and reveal new challenges and opportunities.
- **Algorithm Optimization:** Further refinement of the computational algorithms to enhance efficiency and reduce computational costs would be beneficial, especially for large-scale problems.
- **Integration with Machine Learning:** Combining these regularization techniques with machine learning models could improve the predictive accuracy and provide more robust solutions to inverse problems.
- **Experimental Validation:** Conducting experimental studies to validate the theoretical

results and computational methods in laboratory settings or controlled environments would help bridge the gap between theory and practice.

- **Interdisciplinary Collaboration:** Encouraging collaboration between mathematicians, engineers, and scientists from various fields can lead to innovative approaches and new perspectives on tackling inverse problems.

These perspectives not only aim to enhance the current methodologies but also to open new avenues for research and practical applications, thereby broadening the impact and utility of the findings presented in this thesis.

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