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Ministry of Higher Education and Scientific Research

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Controlled G-Neutral Stochastic Differential equations

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By
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Département de Mathématiques **Année : 2022/2023**

THÈSE

Présentée en vue de l'obtention du diplôme de Doctorat

G-EQUATIONS DIFFERENTIELLES STOCHASTIQUES NEUTRALES CONTROLEES

Filière
Mathématiques Appliquées

Spécialité
ACTUARIAT

Par
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Contents

Dedication

To my parents and my family, who instilled in me a love of learning and a sense of curiosity that has fueled my academic pursuits. Your unwavering support and encouragement have been invaluable to me throughout my journey.

To my supervisors, **Pr. Hacene BOUTABIA** and **Dr. Amel REDJIL** for your guidance, mentorship, and the countless hours you dedicated to shaping my research. Your brilliance, passion, and dedication to your work have been truly inspiring.

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List of Works

Publications

- N. Elgroud, H. Boutabia, A. Redjil and O Kebiri. *Existence of relaxed optimal control for G-neutral stochastic functional differential equations with uncontrolled diffusion.* Bulletin of the Institute of Mathematics, $17(2) : 143 - 172, 2022$. [21].

Submitted Papers

- O. Kebiri and N. Elgroud. Relaxed Optimal Control Problem for a Finite Horizon G-SDE with Delay and Its Application in Economics [45].

Communications

- Nabil Elgroud. International communication in 15/03/2022, presented orally "On Stochastic Optimal Control Problem for G-Stochastic Functional Differential Equations" in International Meeting on Backward Stochastic Differential Equations, Optimal Control and Stochastic Systems organized by Laboratory of Probabilities and Statistics (LaPS), Badji Mokhtar University Annaba, algeria, in partnership with BTU Cottbus, Germany (Project of Erasmus+).

- Nabil Elgroud. International communication in 20/09/2021, presented orally "Successive Approximation of G-Stochastic Functional Differential Equations with Finite Delay and Their Application" In The 1st online Yarmok Mathematics Conference on Differential Equations: Analysis, Modeling and Numerical Computations, (DEAMN-2021) organized by Faculty od Science, Yarmok University, Jordan.

- Nabil Elgroud. International communication in 16/03/2022, presented orally "Educational reform under the COVID-19 Pandemic: Problems and Solution by Integrate Smart Learning Systems" in SGVU-Global Science Summit 2020 on "RECENT ADVANCEMENTS AND INNOVATIONS IN SCIENCE and EDUCATION", organized by the Academic Staff College in Collaboration with, School of Applied Science and Department of International Relations.

- Nabil Elgroud. International communication in 21/08/2020, presented orally "Approximation and Stability Results In Relaxed Control problems to G-Stochastic Functional Differential Equations" in the first online international Day on Probabilities, IDP'22, organized by Laboratory of Stochastic Models, Statistics and Applications, University of Saida, algeria.

- Nabil Elgroud. National Communication in 02/06/2021, presented orally "Mean-field G-Stochastic differential equations and their applications", in Second National Mathematics Seminar organized by Applied mathematics and Modeling Laboratory, Freres Mentouri University Constantine, algeria.

- Nabil Elgroud. National Communication in 16/12/2021, presented orally "Acaratheodory Approximation of G-Neutral stochastic Differential Delay Equations", in National Day for Young Researchers organized by Department of Mathematics, Laboratory of Probabilities and Statictics (LaPS) Badji Mokhtar University Annaba, algeria.

- Nabil Elgroud. National Communication in 16/12/2021, presented orally "Stochastic Functional Differential Equations of Neutral Type", in National Meeting on Applied Mathematics (RNMA 2021), organized by Department of Mathematics, Laboratory of Numerical Analysis, and optimization (LANOS) Badji Mokhtar University Annaba, algeria.

- Nabil Elgroud. Talk in febuary 1st, 2023, Stochastic Optimal Control Problems under the G-expectation framework. Stochastic seminar, BTU Cottbus-Seftenberg- Germany.

Other training

- Participation in International CIMPA Researsh CIMPA'ASA 2019 school on "Stochastic Analysis and Applications" organized by : Laboratory of Stochastic Models, Statistics and Applications, University of Saida Dr Moulay Tahar, algeria.

- Participation in HOMECAMP LIMITLESS ALGERIA 2020 "AI, ML, DL and their applications " organized by ECE PARIS, Limitless Learning, and Yahoo! Silicon Valley.

Project

- Member of PRFU research Project 2020 - 2024, organized by: the Faculty of Science and Technology, El Tarf University. Stochastic analysis, statistics of processes and applications. Project title : "Stochastic Differential Equations, Stochastic Dynamic Systems and Statistics ".

Abstract

The purpose of this dissertation is to investigate the existence of a relaxed optimal control problem given by two stochastic delay equations within the framework of G-Brownian motion, where the neutral term in one of the equations and the diffusion term in both equations are independent of the control variable. We study the existence and uniqueness of solution for neutral stochastic functional differential equation driven by G-Brownian motion (G-NSFDE, in short) under refined Lipschitz hypothesis, in addition to certain numerical analysis simulations. We investigate the approximations and existence of a G-relaxed optimal control for G-NSFDE by defining the space of relaxed control. Next, we establish the existence of a G-relaxed optimal control of a stochastic differential delay equation driven by G-Brownian motion (G-SDDE in short) for the finite horizon case and propose an economic model represented by a G-SDDE, the optimization of which we studied. We related our controlled systems to Hamilton Jacobi Bellman's appropriate equation to a decoupled forward backward stochastic differential delay equations driven by G-Brownian motion (G-FBSDDEs in short). In a broader context, we conduct simulations on the decoupled G-FBSDDEs with the goal of identifying the optimal strategy.

Key words :

G-expectation, G-Brownian motion, G-neutral stochastic functional differential equations, stochastic differential delay equation, G-relaxed optimal control, economic model, forward backward stochastic differential delay equations.

Résumé

L'objectif de cette thèse est d'étudier l'existence d'un problème de contrôle optimal relaxé donné par deux équations stochastique avec retard dans le cadre du G-mouvement Brownien, où le terme neutre dans l'une des équations et le terme de diffusion dans les deux équations sont indépendants de la variable de contrôle. Nous étudions l'existence et l'unicité de la solution d'une équation différentielle stochastique fonctionnelle neutre dérivé par un G-mouvement Brownien (G–EDSFN, en abrégé) sous l'hypothèse raffinée de Lipschitz, en plus de certaines simulations d'analyse numérique. Nous étudions les approximations et l'existence d'un G-contrôle optimal relaxé pour G–EDSFN en définissant l'espace de contrôle détendu. Ensuite, nous établissons l'existence d'un G-contrôle optimal relaxé d'une équation différentielle stochastique avec retard dérivé par un G-mouvement Brownien (G-EDSR en abrégé) pour le cas d'un horizon fini et proposons un modèle économique représenté par un G-EDSR, le optimisation dont nous avons étudié. Nous avons associé nos systèmes contrôlés à l'équation appropriée de Hamilton Jacobi Bellman à une G-avant-arrière équations différentielles stochastiques avec retard (G-AAEDSR en abrégé) découplées. Dans un contexte plus large, nous simulons les G-AAEDSR pour trouver la strategie optimale et le cout.

Mots-clés :

G-espérance, G-mouvement Brownien, G-équations différentielles stochastiques fonctionnelles neutres, équations différentielles stochastiques avec retard, G-contrôle optimal relaxé, modèle économique, avant-arrière équations différentielles stochastiques avec retard.

ملخص

الغرض من هذه الأطروحة هو التحقق من وجود مشكلة تحكم مثالية مريحة تعطى من خلال معادلتين للتأخير العشوائي في إطار الحركة G-Brownian، حيث يكون الحد المحايد في إحدى المعادلات و حد الانتشار في كلتا المعادلتين مستقلين عن متغير التحكم . قمنا بدراسة وجود وتفرد حل للمعادلة التفاضلية الوظيفية العشوائية المحايدة المدفوعة بحركة G-Brownian (G-NSFDE، باختصار) في ظل فرضية Lipschitz المنقحة، بالإضافة إلى بعض عمليات محاكاة التحليل العددي. نحن نتحقق من التقريبات ووجود عنصر G-تحكم امثل مريح لـ G-NSFDE من خلال تحديد مساحة التحكم المريح. بعد ذلك، أثبتنا وجود G-تحكم امثل مريح لمعادلة تأخير تفاضلية عشوائية مدفوعة بحركة G-Brownian (G-SDDE باختصار) لحالة الأفق المحدود ونقترح نموذجًا اقتصاديًا يمثله G-SDDE، التحسين الذي درسنا. لقد ربطنا أنظمتنا الخاضعة للرقابة بمعادلة Hamilton Jacobi Bellman بمعادلات التأخير التفاضلي العشوائي المنفصلة إلى الأمام والخلف مدفوعة بحركة G-Brownian (G-FBSDDEs باختصار). في سياق أوسع، نقوم بمحاكاة G-FBSDDEs للعثور على الإستراتيجية المثلى.

الكلمات المفتاحية:

توقع G ، حركة G-Brownian، G-المعادلات التفاضلية الوظيفية العشوائية المحايدة، معادلة التأخير التفاضلي العشوائي، G-تحكم امثل مريح ، النموذج الاقتصادي، معادلات التأخير التفاضلي العشوائي إلى أمام خلف.

List of Figures

Notations

The objective of this part is to recall some basic notions that will be used throughout this work.

Notations Description

$\mathbb{A}$: is the compact polish space of $\mathbb{R}^n$.

$a^+ = \max\{0,a\},\ a^- = -\min\{0,a\},\ a \in \mathbb{R}$.

$(\mathbb{A}, d)$: is the separable metric space.

$B_t(w) = w_t,\ t \geq 0$: is the canonical process defined on Ω.

$\langle B \rangle$: is the universally defined quadratic variation of B.

$BC([-\tau, 0]; \mathbb{R})$: is the space of bounded and continuous functions with values in $\mathbb{R}$.

$\mathcal{B}(\mathbb{A})$: is the σ-algebra of Borel subsets of the set $\mathbb{A}$ of values taken by the strict control.

$C_{l,Lip}(\mathbb{R}^n)$: is the space of all linear and Lipschitz functions on $\mathbb{R}^n$.

$c(A) := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(A)$: is the natural Choquet capacity related to $\mathbb{P}$.

$C([-\tau, 0]; \mathbb{R}^n)$: is the space of $\mathbb{R}^n$-valued continuous functions.

$\delta_{u(t)}$: is the Dirac measure assigns a charge to the function $u(t)$ at every value of t.

$\delta_{(u(t),u(t-\tau))}$: is the Dirac measure assigns a charge to the functions $u(t)$ and $u(t-\tau)$.

$\widehat{\mathbb{E}}[.] = \sup\limits_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[.]$: is the G-expectation.

$\mathbb{E}^{\mathbb{P}}$: is the ordinary expectations for every $\mathbb{P} \in \mathcal{P}$.

$\|\eta\|_{p,T} = \widehat{\mathbb{E}}\left[\left(\int_0^T |\eta_t|^p\, dt\right)\right]^{\frac{1}{p}}$: is the norm on the space $\overline{M}_G^{p,m\times n}(0,T)$.

$\mathcal{F}_t^{\mathbb{P}} := \mathcal{F}_{t^+} \vee \mathcal{N}^{\mathbb{P}}(\mathcal{F}_{t^+})$: is the right-continuous $\mathbb{P}-$completed filtrations where $\mathbb{P} \in \mathcal{P}$.

$\mathbb{F}^{\mathcal{P}} = \left\{ \widehat{\mathcal{F}}_t^{\mathcal{P}} \right\}_{t \geq 0}$, $\widehat{\mathcal{F}}_t^{\mathcal{P}} := \bigcap_{\mathbb{P} \in \mathcal{P}} (\mathcal{F}_t^{\mathbb{P}} \vee \mathcal{N}_{\mathcal{P}})$: is the universal filtration.

$\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$: is the filtration generated by B, which is only left-continuous.

$\mathbb{F}^+ := (\mathcal{F}_{t^+})_{t \geq 0}$, $\mathcal{F}_{t^+} := \cap_{s>t} \mathcal{F}_s$: is the right-limit filtration.

$G(a) := \frac{1}{2} \widehat{\mathbb{E}} \left[aX^2 \right] = \frac{1}{2} \left(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^- \right)$: is the sublinear function.

Γ : is the non empty bounded and closed subset of $\mathbb{R}^{d \times d}$.

$\widetilde{\mathcal{H}}_T := \left\{ X = (X(t))_{t \in [0,T]} ,\ \mathbb{F}^{\mathcal{P}} - \text{adapted such that:} \int_0^T \widehat{\mathbb{E}} \left[|X(s)|^2 \right] ds < \infty \right\}$.

$\|K\|^p_{S^p([0,T];C)} = \widehat{\mathbb{E}}[\sup_{t \in [0,T]} |K_t|^p_C] = \widehat{\mathbb{E}}[\sup_{t \in [0,T]} \sup_{\theta \in [-\tau,0]} |K_t(\theta)|^p]$: is the norm defined in

$L^p_G(\Omega_T)$: is the completion of $Lip(\Omega_T)$.

$Lip(\Omega_t) := Lip(\Omega) \cap L^0(\Omega_t)$, $L^p_G(\Omega_t) := L^p_G(\Omega) \cap L^0(\Omega_t)$.

$Lip(\Omega_t) := \{ \varphi(B_{t_1}, ..., B_{t_n}) : \varphi \in C_{b,Lip}(\mathbb{R}^n) \text{ and } 0 \leq t_1, t_2, ..., t_n \leq t \}$.

$Lip(\Omega) := \underset{n \in \mathbb{N}}{\cup} Lip(\Omega_n)$.

$L^0(\Omega)$: is the space of all $\mathcal{B}(\Omega)$-measurable real functions.

$L^0(\Omega_t)$: is the space of all $\mathcal{B}(\Omega_t)$-measurable real functions.

$\mathbb{M} = \mathbb{M}([0,T];\mathbb{A})$: is the separable metrizable space.

$M_G^{0,p}(0,T)$: is the space of simple processes consisting of $\mathbb{F}$-progressively measurable with values in $\mathbb{R}^n$.

$M_G^p(0,T)$: is the completion of the set of elementary processes $M_G^{0,p}(0,T)$.

$M_G^{p,m \times n}(0,T)$: is the closure of $M_G^{p,m \times n}(0,T)$.

$\overline{M}_G^{p,m \times n}(0,T)$: is the closure of $M_G^{p,m \times n}(0,T)$.

$\widehat{\mu}_{\mathbb{P}} = \arg\min_{\mu \in \mathcal{R}} J^{\mathbb{P}}(\mu)$: is the optimal relaxed control for every $\mathbb{P} \in \mathcal{P}$.

$\overline{\mu} = \widehat{\mathbb{E}}[X]$: is the value of maximum mean uncertainty.

$\underline{\mu} = -\widehat{\mathbb{E}}[-X]$: is the value of minimum mean uncertainty.

$\left[\underline{\mu}, \overline{\mu} \right]$: is the interval of the mean uncertainty.

$\mu_t(d\xi)dt = \delta_{u(t)}(d\xi)dt$: is the relaxed control variable at time t.

$\mu_t(d\xi)dt = \delta_{(u(t),u(t-\tau))}(d\xi)\,dt$: is the relaxed control variable at time t and $t-\tau$.

$N_C(X) := \left(\int_0^T e^{-2Cs}\widehat{\mathbb{E}}\left(|X(s)|^2\right)ds\right)^{\frac{1}{2}}$: is the norms of $\widetilde{\mathcal{H}}_T$.

$\mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2])$ *: is the G-normal distribution.*

$\mathcal{N}^{\mathbb{P}}(\mathcal{G}) := \{D \subset \Omega$, there exists $\widetilde{D} \in \mathcal{G}$ such that $D \subset \widetilde{D}$ and $\mathbb{P}[\widetilde{D}] = 0\}$: is the $\mathbb{P}-$negligible set.

$\nabla_k V(t,k)$: is the gradient at $(t,k) \in [0,T] \times C([-\tau,0];\mathbb{R}^n)$ is the collection of n-tuple finite Borel measures defined on the interval $[-\tau,0]$.

$|\nabla_k V(t,k)|$: is the total variation norm.

$\nabla_0 V(t,k) = \nabla_k V(t,k)\{(0)\}$: is the masse located specifically at 0, corresponds to the component of $\nabla_k V(t,k)$.

$\langle \nabla_k V(t,k), g\rangle := \int_{[-\tau,0]} g(\theta)\cdot\nabla_k V(t,k)(d\theta)$: is the continuous map defined on $[0,T]\times C$, for $g \in C$.

$\Omega_T := \{w_{.\wedge T} : w \in \Omega\}$.

$\Omega := C_0^n(\mathbb{R}_+)$: is the space of real valued continuous functions on $\mathbb{R}_+$ such that $\omega(0) = 0$ measure on Ω.

$\left(\Omega, \mathcal{H}, \widehat{\mathbb{E}}\right)$: is the space comprising sublinear expectation.

$\mathbb{P}$: is the probability measure defined on the $\mathcal{B}(\Omega)$ of the space Ω.

$\mathcal{P}$: is the tight family of possibly mutually singular probability measures.

$\mathcal{P}(\mathbb{A})$: is the space of probability measures $\mathcal{B}(\mathbb{A})$.

$(.,.)$: is the inner product on $\mathbb{R}^d$.

$\Phi(u)(dt,d\xi) = \delta_{u(t)}(d\xi).dt$: is the mapping of relaxed control.

$q(\omega,dt,d\xi) = \mu_t(\omega,d\xi)dt$: is the $\mathbb{F}^{\mathcal{P}}$-progressively measurable random measure.

$\mathbb{Q}$: is the probability measure on $(\Omega,\mathcal{B}(\Omega))$ respect to $\mathcal{P}$.

$\mathcal{R}$: is the class of relaxed stochastic controls.

$\rho\left(w^1,w^2\right) := \sum_{i=1}^{\infty} 2^{-i}\left(\left(\max_{0\le t\le N}\left|w_t^1 - w_t^2\right|\right)\wedge 1\right)$: is the distance.

$\mathbb{S}_d$: is the space of all $d\times d$ symetric matrices.

$\mathbb{S}_d^+$: is the set of symetric positive semi definite matrices of $\mathbb{S}_d$.

$S^p([0,T];C)$: is the space of C-valued stochastic processes.

$\sup_{\gamma\in[\sigma_{min},\sigma_{max}]} \mathbb{E}^{\mathbb{P}_\gamma}[\phi(K)] = \widehat{\mathbb{E}}[\phi(\widehat{K})]$: is the G-expectation constracted by expected valued over $\mathbb{P}_\gamma$ due the worst case senario.

$\sum := \{\gamma\gamma^*,\ \gamma \in \Gamma\}$: is the matrix of covariance uncertainty of X.

$\overline{\sigma}^2 := \widehat{\mathbb{E}}\left[X^2\right]$: is the value of maximum variance uncertainty.

$\underline{\sigma}^2 := -\widehat{\mathbb{E}}\left[-X^2\right]$: is the value of minimum variance uncertainty.

$\left[\underline{\sigma}^2, \overline{\sigma}^2\right]$: is the interval of the variance uncertainty.

$\mathcal{U} = \mathcal{U}\left([0,T]\right)$: is the space of admisible controls.

$u\left(t\right)$: is the strict control variable at time t.

$u\left(t-\tau\right)$: is the strict control variable at time $t-\tau$.

$\|X\|^p_{L^p_G(\Omega_T)} := \widehat{\mathbb{E}}[|X|^p]$: is the norm of the space $L^p_G(\Omega_T)$.

Introduction

The objective of this thesis is to investigate delayed system issues and stochastic optimal control problems in the framework of G-Brownian motion [21, 45].

There has been a great deal of interest in the classical stochastic optimal control issue for delayed systems. Stochastic differential delay equations (SDDEs in short) describe systems whose dynamics are impacted by both the current and previous values of the state, Mao et al [53, 52] focused on this type of equations and also the dynamics can depend on the past evolution rate of the state value which are called neutral stochastic functional differential equations (NSFDEs in short) studied by different authors, for more details the reader can see [7, 8, 9, 35, 46]. This particular model finds utility across numerous situations in many systems with after-effects (memory), such as domains where this model is applicable include economics and finance, along with population dynamics in biology (see [48, 18, 33, 51, 73]). An introduction to the probabilistic methods in stochastic control is given by Mazliak [55] and Bismut [13]. In 1975, Fleming et al [30] interested in optimal deterministic and stochastic controls. Many authors [3, 37, 47, 14, 15, 78] have studied different problems in stochastic optimal control theory. Benssoussan [11] worked on the study of stochastic control by functional analysis methods.

A study of maximum principle of stochastic delay system and a verification theorem for a variational inequality was demonstrated by Elsanosi et al [25]. Ivanov [42] studied a controlled SDDEs and presented an economic application.

A necessary and sufficient optimality conditions of delayed stochastic maximum principle was studied by Menoukeu-Pamen [56] in problem of non-Markovian under uncertainty model with delay. In [57] Øksendal et al studied a sufficient and necessary stochastic maximum principle of time delayed SDEs with jumps.

A maximum principle for stochastic optimal control problem in the infinite horizon within the context of partial information proved by Agram et al [1] for a FBSDDEs. Theoretically, in the different problems in the theory of optimal control it is supposed the optimal control exists, which allow to study the existence of optimal control (see [22, 23, 38]).

Balachandran [6] studied the existence of an optimal control for non-linear systems with multiple-delay characterized by an implicit derivative by appropriately applying certain techniques, the system is subjected to criteria of quadratic performance.

The problem of optimal control for a systems presented by a delay differential equations with a quadratic functional cost was studied by Patel et al [58]. The relaxed procedures of the controls show that the state of controls are given by the relaxed control studied by [2, 3, 29]. Rosenblueth [70] proposed two different procedures for a problem of relaxed optimal control involving to the transformations of control functions state confirms that a solution exists for the resulting of relaxed problem, that ensure the existence of a minimizer. The cases of controlled of NSFDEs has been treated by different authors, for more details (see [2, 74]).

Inspired by the uncertainty idea, which arises in many fields of science, containing inaccurate parameters, generally, risks arising from dark fluctuations, market liquidity, and the influence on asset price movement and financial crises. Namely, optimal portfolio choice issues in which the volatility and the processes of premium risk are unknown.

The concept of uncertainty in fluctuations were studied by Peng [59, 64, 63, 65] established a specific kind of non-linear expectation theory that was done using capacity theory in [19, 20] within the framework of G-Brownian motion, and then related with the G-Brownian motion (see [62, 61, 60]) to use the G-stochastic calculus and existence and uniqueness of the stochastic differential equations driven by G-Brownian motion for more details see [5, 34, 49, 50, 80]. In addition, Faizullah [26],and Faizullah et al in 2017 [27] studied the existence and uniqueness of solution of G-neutral stochastic functional differential equations (G-NSFDEs). Moreover, the existence of solution and the stability result of highly nonlinear G-SDDEs was proved by Fei et al [28]. Yuan [79] studied some characteristics of the numerical results of solutions for a semilinear G-SDDEs. In [19, 20], the authors studied the G-expectation in a tight family of possibly mutually singular probability measures to used. Recently, Biagini et al and Hu et al in [12, 41] solved the optimal control problem with uncertainty of G-Brownian motion and its quadratic variation.

Different authors [12, 39, 41, 67, 68, 69, 77] have studied the problem of optimal control within the framework of G-Brownian motion.

The results of existence and uniqueness of the solution for G-BSDEs proved by Xu [75] to furnish the viscosity solution of a class of Hamilton-Jacobi-Bellman equations (HJB equations for short) by an interpretation of the probabilistic concept.

Additionally, it is demonstrated that the certain class of stochastic optimal control problems can be effectively characterized by G-BSDEs. In [36] the authors employed a class of HJB equations, which is acknowledged as the dynamic programming equation associated with the problem of optimal control.

In fact, the estimation of the noise parameter σ precisely is not feasible, and all we have as information is a range interval $[\sigma_{min}, \sigma_{max}]$ where belongs, therefore the challenge is to examine the worst-case scenario, which is complicated to analyze directly. The worst-case scenarios may be changed to a G-NSFDE and G-SDDE respectively., and assuming that the objective is to govern the dynamics of the particle under certain restrictions, as a consequence, the G-NSFDE (respectively. G-SDDE) will be optimally controlled.

Motivation ([21])**:** To motivate our work let consider a Brownian particle moving in an unbounded medium. Let $X(t)$ be the position and $Y(t)$ the velocity of the particle at time t. So The dynamic is represented by

$$X'(t) = Y(t) \qquad\qquad mdY(t) = b(t)dt + \sigma d\xi_t, \tag{1}$$

where m is the mass of the particle and $\sigma d\xi_t$ is the noise part of the medium on the particle. According to Boussinesq representation in [16], $b(t) = -hY(t) - Y'(t)\sigma\sqrt{\frac{hm_1^3}{\pi}}\int_0^\infty Y'^2\,(s)\,ds$, which represents the systematic action of the medium on the particle, where $-hY(t)$ is the Stokes friction force at time t and m the apparent additional mass which is half the mass of the material of the medium ousted by the body. The $\int_0^\infty Y'^2\,(s)\,ds$ is the viscous hydrodynamic aftereffect. These models represent a NSFDE in the classical case.

In this thesis, we investigate the existence of a relaxed optimal control problem given by two stochastic delay equations within the framework of G-Brownian motion, where the neutral term in one of the equations and diffusion term in both equations are independent of the control variable, and we give the motivation and economic application. In chapter 1, we present some initial G-stochastic calculus notations that will be employed to validate our finndings. We target to introduce the definition of one-dimensional neutral stochastic functional differential equation driven by G-Brownian motion (G-NSFDE, in short), the G-Itô formula, and we study the existence and uniqueness of solution for G-NSFDE under refined Lipschitz hypothesis, in addition to certain numerical analysis simulations, in chapter 2. The third chapter is devoted to providing some concepts of stochastic control in the G-framework, along with two examples of control models. We investigate the approximations and existence of a relaxed optimal control for a one-dimesional G-NSFDE by defining the space of relaxed control.

The last chapter examines the existence of a relaxed optimal control of a n-dimensional stochastic differential delay equation driven by G-Brownian motion (G-SDDE in short) for the finite horizon case and proposes an economic model represented by a G-SDDE, the optimization of which we studied. We related our controlled systems to Hamilton Jacobi Bellman appropriate equation to a decoupled forward backward stochastic differential delay equations driven by G-Brownian motion (G-FBSDDEs in short). Overall, we simulate the G-FBSDDEs to find the optimal strategy and cost.

Thesis Outline

In the first chapter of this thesis, we present certain preliminaries, definitions, lemmas, and theorems of G-stochastic calculus that will be used to confirm our result.

The next three Chapters pertain to the articles that have been published during the thesis work ([21, 45]).

The second chapter will introduce the concept of G-NSFDEs, the G-Itô formula, and the existence and uniqueness of the one-dimensional G-NSFDE solution. Lastly, we present some numerical analysis simulations. The third chapter focuses on stochastic control concepts in the G-framework, with two example control models. We investigate the approximations and existence of a G-relaxed optimal control for G-NSFDE by defining the space of G-relaxed control. The last chapter, the presence of a G-relaxed optimal control for n-dimensional G-SDDE in the context of a finite horizon is investigated. The optimization of the economic model based on and the corresponding HJB equation and decoupled G-FBSDDEs is demonstrated. Finally, we present the simulations of G-FBSDDEs to determine the best optimal strategy and associated cost.

Chapter 1

Preliminaries

In this chapter, we begin with some initial content, specifically on sublinear expectation theory and the G-Brownian motion, and the aforementioned would be associated with the G-normal distribution. The main aim is to review certain ideas and initial outcomes from the G-framework that will be utilized in the upcoming sections, used by many authors, Denis et al [19, 20], Peng [62, 61, 64, 63, 60], Soner et al [71, 72].

1.1 Sublinear expectation

Let a given set Ω and let us consider the linear space $\mathcal{H}$ defined on Ω is space of the real functions meet the following requirements: if $X \in \mathcal{H}$, we have $|X| \in \mathcal{H}$ and $c \in \mathcal{H}$ for each constant c. $\mathcal{H}$ is called also the random variables space.

Definition 1 *Let* $\widehat{\mathbb{E}} : \mathcal{H} \rightarrow \mathbb{R}$ *is the function called sublinear expectation on* $\mathcal{H}$ *that holds these conditions:*

1. Monotonicity*:* $Y \leq X \Rightarrow \widehat{\mathbb{E}}[Y] \leq \widehat{\mathbb{E}}[X], \forall\ Y, X \in \mathcal{H}.$

2. Constants preservation *:* $\widehat{\mathbb{E}}[c] = c, \forall\ c \in \mathbb{R}.$

3. Subadditivity*:* $\widehat{\mathbb{E}}[X - Y] \geq \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y], \forall Y, X \in \mathcal{H}.$

4. Positive homogeneity*:* $\widehat{\mathbb{E}}[\beta X] = \beta \widehat{\mathbb{E}}[X], \forall\ \beta \in [0, \infty).$

The space of sublinear expectation is denoted by $\left(\Omega, \mathcal{H}, \widehat{\mathbb{E}}\right)$. If just condition 1 *and* 2 *are fulfilled,* $\widehat{\mathbb{E}}$ represents the nonlinear expectation, and if inequality 3 *is an equality, then* $\widehat{\mathbb{E}}$ *is a classical linear expectation, i.e.,* $\widehat{\mathbb{E}}$ *is a linear function tha meets the criteria of both* 1 *and* 2*.*

Remark 2 *The properties* 3 *and* 4 *confer upon us the characteristic of convexity for* $\alpha \in [0, 1]$ *:*

$$\widehat{\mathbb{E}}[(1 - \alpha) Y + \alpha X] \leq (1 - \alpha) \widehat{\mathbb{E}}[Y] + \alpha \widehat{\mathbb{E}}[X].$$

Mention that the following condition is identical to condtion 4:

$$\widehat{\mathbb{E}}[\beta X] = \beta^{-}\widehat{\mathbb{E}}[-X] + \beta^{+}\widehat{\mathbb{E}}[X], \text{ for } \beta \in \mathbb{R},$$

where $\beta^{+} = \max(\beta, 0)$ and $\beta^{-} = \max(-\beta, 0)$.

It should be pointed out that the space $C_{l,Lip}(\mathbb{R}^n)$ comprises a real-valued functions that map to real numbers such that for all $y, x \in \mathbb{R}^n$ for any given elements $y, x \in \mathbb{R}^n$, we have

$$|\phi(x) - \phi(y)| \leq C(1 + |x|^{\epsilon} + |y|^{\epsilon})|x - y|,$$

here, $C > 0$ is a constant depends only on ϕ and $\epsilon \in \mathbb{N}^*$.

Theorem 3 ([*17*], [*61*])**:** *Consider that the sublinear expectation $\widehat{\mathbb{E}}$ is defined on $\mathcal{H}$, then there exists $\mathcal{P}$ family of probability measures defined on $(\Omega, \mathcal{B}(\Omega))$ such that for any X belongs to $\mathcal{H}$.*

$$\widehat{\mathbb{E}}[X] = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[X],$$

where $\mathcal{B}(\Omega)$ denotes the Borellian tribe $\sigma-$algebra belonging Ω.

Now, let introduce on $\mathcal{H}$ the Choquet capacity as follow

$$c(D) := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}(D),$$

where $D \in \mathcal{B}(\Omega)$.

For additional information, the reader may refer to [20, 39, 81].

Definition 4 *Let D be a set belonging to $\mathcal{B}(\Omega)$ is considered polar if and only if $c(D) = 0$ i.e., $\mathbb{P}(D) = 0$, $\mathbb{P} \in \mathcal{P}$. If a property holds $\mathbb{P}$-almost-surely ($\mathbb{P} - a.s.$ in short) for all $\mathbb{P} \in \mathcal{P}$ means that holds everywhere except possibly on a polar set, then it is said to hold $\mathcal{P}$-quasi surely (q.s. in short).*

The theory of stochastic processes in continuous time within framework of a G-expectation has been established, Specifically, Itô's formula, some stochastic inequalities, and G-SDE. It is possible to establish these results in the sense of quasi-sure, according to the G-expectation representation's and the "quasi-sure" notions introduced.

1.1.1 Notions of distributions and independence

Besides the principles of the usual framework, [64] introduced the notions of distributions with random variables independence in the following new concepts.

However, these concepts are more functional and less probabilistic, and they can be described in the space $C_{l,Lip}(\mathbb{R}^n)$, $n \geq 1$ by the families of test functions.

Definition 5 *The distribution of random vector $X = (X_1, ..., X_n)$, $X \in \mathcal{H}^n$ is given by the following functional $\mathbb{F}_X[.]$*

$$\mathbb{F}_X[\phi] := \widehat{\mathbb{E}}[\phi(X)], \ \forall \phi \in C_{l,Lip}(\mathbb{R}^n).$$

A space of nonlinear expectation is formed by $(\mathbb{R}^n, C_{l,Lip}(\mathbb{R}^n), \mathbb{F}_X)$. Moreover, if X' is another n-dimensional random vector $\mathbb{F}_X[\phi] = \mathbb{F}_{X'}[\phi]$ for each ϕ in $C_{l,Lip}(\mathbb{R}^n)$, X and X' are said to be identically distributed, we write $X \overset{d}{=} X'$.

Definition 6 *(see [17],[55]) One can prove that the distribution family $\left\{\mathbb{F}_X^\theta(.)\right\}_{\theta\in\Theta}$ exists on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, then*

$$\mathbb{F}_X[\phi] = \sup_{\theta\in\Theta} \int_{\mathbb{R}^n} \phi(x) \mathbb{F}_X^\theta(dx),$$

where $\phi \in C_{l,Lip}(\mathbb{R}^n)$ and $\mathbb{F}_X[\phi]$ *describes the uncertainty distribution of X.*

To be mentioned that if $\mathbb{F}_X$ the distribution of $X \in \mathcal{H}$ does not correspond to a linear expectation, then X is subject to distribution uncertainty. The four typical parameters of the distribution of X are as follows:

$$\underline{\mu} = -\widehat{\mathbb{E}}[-X], \ \overline{\mu} = \widehat{\mathbb{E}}[X], \ \underline{\sigma}^2 = -\widehat{\mathbb{E}}\left[-X^2\right] \text{ and } \overline{\sigma}^2 = \widehat{\mathbb{E}}\left[X^2\right].$$

The intervals $\left[\underline{\mu}, \overline{\mu}\right]$ and $\left[\underline{\sigma}^2, \overline{\sigma}^2\right]$ characterize the mean and variance of X respectively. subject to the uncertainty.

Proposition 7 ([64]) *Let X and Y be two elements of $\mathcal{H}$ such that $-\widehat{\mathbb{E}}[-Y] = \widehat{\mathbb{E}}[Y]$, means that Y has no uncertainty in the mean. In this case, we can write :*

$$\widehat{\mathbb{E}}[\alpha Y + X] = \alpha \widehat{\mathbb{E}}[Y] + \widehat{\mathbb{E}}[X].$$

Particularly, if $-\widehat{\mathbb{E}}[-Y] = \widehat{\mathbb{E}}[Y] = 0$, then $\widehat{\mathbb{E}}[\alpha Y + X] = \widehat{\mathbb{E}}[X]$.

Proposition 8 ([64])*If $Y \in \mathcal{H}^n$ be a random vector defined on $\left(\Omega, \mathcal{H}, \widehat{\mathbb{E}}\right)$, and $\phi \in C_{l,Lip}(\mathbb{R}^{m+n})$. Then, we have:*

1) *The function $\phi(x, .) \in C_{l,Lip}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^m$,*

2) The expected values $\widehat{\mathbb{E}}[\phi(., Y)] \in C_{l,lip}(\mathbb{R}^m)$.

Definition 9 *Let $Y \in \mathcal{H}^n$ and $X \in \mathcal{H}^m$ defined on $\left(\Omega, \mathcal{H}, \widehat{\mathbb{E}}\right)$. we say that Y is independent of X, if*

$$\widehat{\mathbb{E}}[\phi(X, Y)] = \widehat{\mathbb{E}}\left[\widehat{\mathbb{E}}[\phi(x, Y)] \,|_{x=X}\right], \ \forall \phi \in C_{l,Lip}\left(\mathbb{R}^{m+n}\right).$$

Remark 10 *The independence property in a space of sublinear expectation is not symmetric (see [81]).*

1.1.2 G-normal distribution

After introducing the fundamental definition, we will now proceed to present the interpretation and concept of the G-normal distribution.

Definition 11 *Let $X = (X_1, ..., X_d)$ is a d-dimensional random vector defined on $\left(\Omega, \mathcal{H}, \widehat{\mathbb{E}}\right)$, if $\forall\ b, a \geq 0$:*

$$aX + b\overline{X} \overset{d}{=} \sqrt{a^2 + b^2}X,$$

for each copy $\overline{X}$ of X, then X is called G-normally distributed.

Let the generatrice functional defined on the space of symetric matrices $\mathbb{S}_d$ by:

$$G(A) := \frac{1}{2}\widehat{\mathbb{E}}\left[\langle AX, X\rangle\right].$$

In [62, 63], Peng shows that $X = (X_1, ..., X_d)$ is G-normally distributed if and only if

$$u(t,x) := \widehat{\mathbb{E}}\left[\phi\left(x + \sqrt{t}X\right)\right], (t,x) \in [0,\infty) \times \mathbb{R}^d, \phi \in C_{l,Lip}\left(\mathbb{R}^d\right),$$

is the unique viscosity solution of the following parabolic PDE, called the heat equation in the G-fromework, which is given for all $(t, x) \in [0, T] \times \mathbb{R}^d$ by

$$\begin{cases} \partial_t u(t,x) = G(Du(t,x)), \\ u(0,x) = \phi(x), \end{cases} \tag{1.1}$$

with the Hessian matrix $Du(t,x)$ of $u(t,x)$ i.e., $Du(t,x){=}\left(\partial^2_{x_i x_i} u(t,x)\right)^d_{i,j=1}$. G is a monotonic sublinear function on $\mathbb{S}_d$. It leads us to conclude that there is a subset $\Sigma \in \mathbb{S}_d^+$ which is bounded, convex, and closed such that

$$G(A) = \frac{1}{2}\sup_{B\in\Sigma} tr(AB),\ A \in \mathbb{S}_d,$$

with the set $\mathbb{S}_d^+$ is symetric semi-definite positive matrices set of $\mathbb{S}_d$. We say that X follow $\mathcal{N}(0; \Sigma)$.

Remark 12 *The instance where the dimension is equal to one $(d = 1)$ corresponds to $G = G_{\overline{\sigma},\underline{\sigma}}$ and $\Sigma = \left[\underline{\sigma}^2, \overline{\sigma}^2\right]$ referring to the sublinear function that has been parameterized as $\overline{\sigma}$ and $\underline{\sigma}$ for all $\alpha \in \mathbb{R}$:*

$$G(\alpha) = \frac{1}{2}\left(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-\right),$$

with $\overline{\sigma}^2 = \widehat{\mathbb{E}}\left[X^2\right]$ and $\underline{\sigma}^2 = -\widehat{\mathbb{E}}\left[-X^2\right]$.

Corollary 13 ([64])*For the case of $\mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2]) = \mathcal{N}(0, \underline{\sigma}^2)$ is the classical normal distribution in the case $\underline{\sigma}^2 = \overline{\sigma}^2 > 0$.*

1.2 G-expectation subject to canonical G-Brownian motion

Peng established the notion sublinear expectation subject to the canonical G-Brownian motion on the space of continuous functions in 2007. The target of this section is to review the definition of one and d-dimensional G-Brownian motion and the appropriate G-expectation.

1.2.1 G-Brownian motion

Definition 14 *We say that the process $(X_t)_{t\geq 0}$ defined on $\left(\Omega, \mathcal{H}, \widehat{\mathbb{E}}\right)$ is d-dimensional if the random variable $X_t \in \mathcal{H}$ for each $t \geq 0$.*

Now, we can use the following definition of the G-Brownian motion on $\left(\Omega, \mathcal{H}, \widehat{\mathbb{E}}\right)$.

Definition 15 *Let a process $(B_t)_{t\geq 0}$ be a G-Brownian motion in d-dimensional case defined on $\left(\Omega, \mathcal{H}, \widehat{\mathbb{E}}\right)$, if the subsequent conditions are met :*

i. $B_0 = 0$.

ii. $\forall\ s, t \geq 0$, *the increment $B_{t+s} - B_t$ conforms to the* $\mathcal{N}(0, s\Sigma)$ $-$distributed and is not dependent of $(B_{t_1}, ..., B_{t_n})$, $\forall\ n \in \mathbb{N}$ *and* $0 \leq t_1 \leq ... \leq t_n \leq t$.

The property shows in particular that the G-Brownian motion increments are independent and also B_s and $B_{t+s} - B_t$ are identically and follow $\mathcal{N}(0, \left[s\underline{\sigma}^2,\ s\overline{\sigma}^2\right])$-distributed.

Notation 16 *In the sequel, we set*

$$B_t^a := (a, B_t)\,,\ \forall a \in \mathbb{R}^d,$$

where $a = (a_1, ..., a_d)^T$, and $(.,.)$ is the inner product on $\mathbb{R}^d$.

According to the previous definition, we already have the next proposition which will be essential for improvements.

Proposition 17 *(see [34], [64]) Let a G-Brownian motion $(B_t)_{t\geq 0}$ is a d-dimensional process defined on $\left(\Omega, \mathcal{H}, \widehat{\mathbb{E}}\right)$. Then, a real G_a-Brownian motion $(B_t^a)_{t\geq 0}$ is a real G_a-Brownian motion, where the generated function $G := Ga$ defined by*

$$G_a(\alpha) = \frac{1}{2}\left(\overline{\sigma}^2_{aa^T}\alpha^+ - \underline{\sigma}^2_{aa^T}\alpha^-\right),$$

where $\underline{\sigma}^2_{aa^T} = -\widehat{\mathbb{E}}\left[-(a, B_1)^2\right]$ and $\overline{\sigma}^2_{aa^T} = \widehat{\mathbb{E}}\left[(a, B_1)^2\right]$. Particularly, $\forall s, t \geq 0, B^a_{t+s} - B^a_t \sim \mathcal{N}\left(0; \left[s\underline{\sigma}^2_{aa^T}; s\overline{\sigma}^2_{aa^T}\right]\right)$.

As a result of this concept, all of the components $\left(B_t^i\right)_{t\geq 0}$ of $(B_t)_{t\geq 0}$ are also considered as G-Brownian motions. In this case, The function that generates $\left(B_t^i\right)_{t\geq 0}$ is defined as follows:

$$G_i(\alpha) = \frac{1}{2}\left(\overline{\sigma}_i^2\alpha^+ - \underline{\sigma}_i^2\alpha^-\right),$$

where $\underline{\sigma}_i^2 = -\widehat{\mathbb{E}}\left[-\left(B_1^i\right)^2\right]$ and $\overline{\sigma}_i^2 = \widehat{\mathbb{E}}\left[\left(B_1^i\right)^2\right]$, $\forall\, i \in \overline{1,d}$. Relating to this case, we have for any function ϕ that is convex:

$$\widehat{\mathbb{E}}\left[\phi\left(B_{t+s}^i - B_t^i\right)\right] = \frac{1}{\sqrt{2\pi\overline{\sigma}_i^2 s}}\int_{-\infty}^{+\infty}\phi(y)\exp\left(-\frac{y^2}{2\overline{\sigma}_i^2 s}\right)dy,$$

for any function ϕ with $\underline{\sigma}_i^2 > 0$ that is concave, we have

$$\widehat{\mathbb{E}}\left[\phi\left(B_{t+s}^i - B_t^i\right)\right] = \frac{1}{\sqrt{2\pi\underline{\sigma}_i^2 s}}\int_{-\infty}^{+\infty}\phi(y)\exp\left(-\frac{y^2}{2\underline{\sigma}_i^2 s}\right)dy.$$

Specifically, we obtain

$$\widehat{\mathbb{E}}\left[\left(B_t^i - B_s^i\right)^2\right] = \overline{\sigma}_i^2(t-s),\ \widehat{\mathbb{E}}\left[\left(B_t^i - B_s^i\right)^4\right] = 3\overline{\sigma}_i^4(t-s)^2,$$

$$\widehat{\mathbb{E}}\left[-\left(B_t^i - B_s^i\right)^2\right] = -\underline{\sigma}_i^2(t-s),\ \widehat{\mathbb{E}}\left[-\left(B_t^i - B_s^i\right)^4\right] = -3\underline{\sigma}_i^4(t-s)^2,$$

for more details (see Peng [60, 63, 64])

1.2.2 Existence of the G-Brownian motion within the context of G-expectation.

Following that, we use the space $\Omega = C_0^n(\mathbb{R}_+)$ including $\omega : \mathbb{R}_+ \to \mathbb{R}^n$ continuous functions at 0, equipped with the metric:

$$\rho\left(\omega^1,\omega^2\right) = \sum_{i=1}^{\infty}2^{-i}\left[\left(\max_{t\in[0,i]}\left|\omega_t^1 - \omega_t^2\right|\right)\wedge 1\right].$$

We set $\Omega_T = \{\omega_{.\wedge T} : \omega \in \Omega\}$ for any $T \geq 0$ fixed, with $B_t(\omega) = \omega_t$, $t \geq 0$ if $\omega \in \Omega$, the canonical process. Consider the following space of random variables:

$$\begin{aligned} Lip(\Omega_T) \ &: = \left\{\phi\left(B_{t_1\wedge T},\cdots,B_{t_n\wedge T}\right) : t_i \geq 0, i = \overline{1,n},\ \phi \in C_{l.Lip}(\mathbb{R}^n)\right\}, \\ Lip(\Omega) \ &: = \bigcup_{n=1}^{\infty} Lip(\Omega_n). \end{aligned}$$

It is obvious that $Lip(\Omega_t) \subset Lip(\Omega_T)$, $\forall\, t \leq T$.

Remark 18 *It is obvious that $Lip\left(\Omega_T\right)$, $Lip\left(\Omega\right)$ and $C_{l.Lip}\left(\mathbb{R}^n\right)$ are a spaces of vector. It is also worth noting that $\phi, \psi \in C_{l.Lip}\left(\mathbb{R}^n\right)$ indicates that $\phi.\psi \in C_{l.Lip}\left(\mathbb{R}^n\right)$ is the product. Thus, if $X, Y \in Lip\left(\Omega_T\right)$ then $X.Y \in Lip\left(\Omega_T\right)$. Namely, $B_t \in Lip\left(\Omega\right)$ for any $t \in [0, \infty)$.*

A sublinear expectation defined on $(\Omega, Lip\left(\Omega\right))$ has constructed by Peng [61], such that the canonical process $(B_t)_{t\geq 0}$ is a G-Brownian motion defined as follows: consider the sequence $(\xi_i)_{i=1}^{\infty}$ is called a sequence of random vectors in d-dimensional case on $\left(\Omega, \widetilde{\mathcal{H}}, \widehat{\mathbb{E}}\right)$ such that ξ_i is G-normally distributed with ξ_{i+1} is not dependent of $(\xi_1, ..., \xi_i)$ $\forall$ $i = 1, 2, ...$

Then, he introduced a sublinear expectation $\widehat{\mathbb{E}}$ on $Lip\left(\Omega\right)$, according to the following technique: $\forall X \in Lip\left(\Omega\right)$ and $0 \leq t_0 < t_1 < ... < t_n < T$, written as

$$X = \phi\left(B_{t_1} - B_{t_0}, ..., B_{t_n} - B_{t_{n-1}}\right), \forall \phi \in C_{l.Lip}(\mathbb{R}^n),$$

and we give :

$$\widehat{\mathbb{E}}\left[X\right] = \widehat{\mathbb{E}}\left[\phi\left(\xi_1\sqrt{t_1 - t_0}, ..., \xi_n\sqrt{t_n - t_{n-1}}\right)\right].$$

Definition 19 *The sublinear expectation $\widehat{\mathbb{E}} : Lip\left(\Omega\right) \to \mathbb{R}$ given by the previous notions is known as G-expectation.*

Let $\mathbb{F} := \mathbb{F}^B = \{\mathcal{F}_t\}_{t\geq 0}$ be the be the right continious filtration generated by B, and let $\mathbb{F}^+ := \{\mathcal{F}_t^+, t \geq 0\}$ be the filtration, defined by $\mathcal{F}_t^+ := \cap_{s>t}\mathcal{F}_s$. We set $\mathcal{F}_t^{\mathbb{P}} := \mathcal{F}_t^+ \vee \mathcal{N}^{\mathbb{P}}(\mathcal{F}_t^+)$ and $\widehat{\mathcal{F}}_t^{\mathbb{P}} := \mathcal{F}_t^+ \vee \mathcal{N}^{\mathbb{P}}(\mathcal{F}_\infty)$ for every probability measure $\mathbb{P}$ belonging on $(\Omega, \mathcal{B}(\Omega))$ respectively. right continuous $\mathbb{P}$-completed filtration and $\mathcal{F}_t^+$(resp. $\mathcal{F}_\infty$), with $\mathcal{N}^{\mathbb{P}}(\mathcal{F}_t^+)$ (resp. $\mathcal{N}^{\mathbb{P}}\left(\mathcal{F}_\infty\right)$) are the sets of $\mathbb{P}$-negligible events of the σ-algebra $\mathcal{F}_t^+$ (resp. $\mathcal{F}_\infty$). The corresponding filtrations will be referred to as $\mathbb{F}^{\mathbb{P}}$ and $\widehat{\mathbb{F}}^{\mathbb{P}}$.

Lemma 20 *(Soner et al [72]): Let consider an arbitrary $\mathbb{P}$ which is a probability measure on $(\Omega, \mathcal{F}_\infty)$. Then, we have:*

1) *For every random variable $\widehat{\xi}$, which is an $\widehat{\mathcal{F}}_t^{\mathbb{P}}$-measurable,* an *$\mathcal{F}_t-$measurable random variable ξ exists and unique almost surely (a.s.) under $\mathbb{P}$ such that $\xi = \widehat{\xi}$, almost surely (a.s.) under $\mathbb{P}$.*

2) *For every process $\widehat{X}$, which is an $\widehat{\mathbb{F}}^{\mathbb{P}}$-progressively measurable, an $\mathbb{F}$-progressively measurable unique process X* exists *such that $\widehat{X} = X$, $dt \times \mathbb{P}$ almost surely (a.s.).* Further*more, in the case where $\widehat{X}$ is almost surely (a.s.) continuous under $\mathbb{P}$, then, it is possible to select X to be almost surely (a.s.) continuous under $\mathbb{P}$.*

1.2.3 The G-expectation and its dual representation.

Denis et al. [20] as well as Denis and Martini [19] has established the G-expectation representations of in restricted terms, family of weakly compact $\mathcal{P}$ of probability measures mutually singular on $(\Omega, \mathcal{B}(\Omega))$. This dual form establishes the computed G-expectation by means of a robust expectation based on $\mathcal{P}$. For the explicit construction of $\mathcal{P}$, we refer to [19] and [20].

Proposition 21 *(see [19], [20]) For all $\xi \in L_G^1(\Omega)$, a weakly compac of probability measures family $\mathcal{P}$ exists on $(\Omega, B(\Omega))$, such that*

$$\widehat{\mathbb{E}}[\xi] = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi].$$

The standard Choquet capacity stated in the first subsection is taken into consideration.

It should be noted that the set $\mathcal{N}_{\mathcal{P}} := \bigcap_{\mathbb{P}\in\mathcal{P}} \mathcal{N}^{\mathbb{P}}(\mathcal{F}_\infty)$ represents the $\mathcal{P}$-polar sets.

Definition 22 *Let $\mathbb{Q}$ is a probability measure defined on the measurable space $(\Omega, B(\Omega))$ is considered to be absolutely continuous with respect to $\mathcal{P}$ if $\mathbb{Q}(C) = 0$ for every C belonging to the set $\mathcal{N}_{\mathcal{P}}$.*

Soner et al [72] considered $\mathbb{F}^{\mathcal{P}}$ the well-known universal filtration, according to $\mathbb{P} \in \mathcal{P}$ the mutually singular measures probability, we have

$$\begin{aligned} \mathbb{F}^{\mathcal{P}} \quad &: \quad = \{\widehat{\mathcal{F}}_t^{\mathcal{P}}\}_{t\geq 0}, \qquad (1.2) \\ \widehat{\mathcal{F}}_t^{\mathcal{P}} \quad &: \quad = \bigcap_{\mathbb{P}\in\mathcal{P}} (\mathcal{F}_t^{\mathbb{P}} \vee \mathcal{N}_{\mathcal{P}}). \qquad (1.3) \end{aligned}$$

The knowledge related to continuous stochastic processes theory is studied through sublinear expectation theory has evolved as a consequence of the introduction of the sublinear expectation representation and the notion of "quasi sure", and the G-framework is the source of some stochastic inequalities, Itô formula, and SDE.

1.3 Formulation of G-stochastic integrals

In [64] Peng presented the Itô type stochastic integrals, with respect to the G-Brownian morion $\left(B_t^i\right)_{t\geq 0}$ similar to the classical case (for further details, can refer to [62, 60]).

1.3.1 Notion of G-Bochner Integral

The value of $T \in \mathbb{R}^+$ will be fixed in the rest. Let the subdivision $\pi_T = \{t_0, t_1, ..., t_N\}$ of $[0, T]$ and $p \in [1, \infty[$. By using the simple processes using values in $\mathbb{R}^{m\times n}, m,n\in\mathbb{N}^*$, such that for $\xi_\ell \in L_G^P(\Omega_{t_\ell}), \ \ell=\overline{0,N-1}$

$$\eta_t(\omega)=\sum_{\ell=0}^{N-1}\xi_\ell(\omega)\,\mathbf{I}_{[t_\ell,t_{\ell+1})}(t).$$

We use the notation $M_G^{p,m\times n}(0,T)$ to represent the set of all simple processes over the time interval $[0,T]$.

Definition 23 *Let η belongs to $M_G^{p,m\times n}(0,T)$, we have*

$$\eta_t(\omega)=\sum_{\ell=0}^{N-1}\xi_\ell(\omega)\,\mathbf{I}_{[t_\ell,t_{\ell+1})}(t),$$

then, can be expressed the Bochner integral as follows

$$\int_0^T\eta_t(\omega)\,dt:=\sum_{\ell=0}^{N-1}\xi_\ell(\omega)\,(t_{\ell+1}-t_\ell).$$

Definition 24 *For all $p\geq 1$, we denote $\overline{M}_G^{p,m\times n}(0,T)$ the closure of $M_G^{p,m\times n}(0,T)$ according to the following norm:*

$$\|\eta\|_{p,T}=\widehat{\mathbb{E}}\left[\left(\int_0^T|\eta_t|^p\,dt\right)\right]^{\frac{1}{p}}.$$

It is worth noting that $\overline{M}_G^{p,m\times n}(0,T)$ constitutes a Banach space. Subsequently, we use the notation $\overline{M}_G^{p,m}(0,T):=\overline{M}_G^{p,m\times m}(0,T)$.

1.3.2 Notion of G-Itô integral

The description of the notion of Itô integral with respect to the G-Brownian motion $\left(B_t^i\right)_{t\geq 0}$ is now provided.

Definition 25 *For all process η defined on $M^{2,1}(0,T)$, we have*

$$\eta_t(\omega)=\sum_{\ell=0}^{N-1}\xi_\ell(\omega)(t).$$

Definition 26 *The G-Itô integral is expressed for every $t\geq 0$ by*

$$I(\eta)=\int_0^T\eta_t dB_t^i:=\sum_{\ell=0}^{N-1}\xi_\ell\left(B_{t_{\ell+1}}^i-B_{t_\ell}^i\right).$$

Lemma 27 *For all $\eta \in M_G^{2,1}(0,T)$, we have*

$$\widehat{\mathbb{E}}\left(\int_0^T \eta_t dB_t^i\right) = 0, \tag{1.4}$$

and

$$\widehat{\mathbb{E}}\left[\left(\int_0^T \eta_t dB_t^i\right)^2\right] \leq \overline{\sigma}_i^2 \left(\int_0^T \widehat{\mathbb{E}}\left[\eta_t^2\right] dt\right), \tag{1.5}$$

which gives the following lemma

Lemma 28 *Let the continuous map $I : M_G^{2,1}(0,T) \longrightarrow L_G^2(\Omega_T)$ is linear, then it is possible to extend it continuously to $\overline{M}_G^{2,1}(0,T)$.*

Definition 29 *For all $\alpha \in \overline{M}_G^{2,1}(0,T)$ fixed, we define the following stochastic integral*

$$I(\alpha) = \int_0^T \alpha_t dB_t^i.$$

It is obvious that for $\eta \in \overline{M}_G^{2,1}(0,T)$ the notions (1.4) and (1.5) stay true.

We give an Itô integral properties with respect to the G-Brownian motion, This follows from the following definition.

Proposition 30 *Let $\eta, \theta \in \overline{M}_G^{2,1}(0,T)$ and $0 \leq s \leq v \leq t \leq T$. Following that, we have the following properties :*

(i) $\int_s^t \eta_u dB_u^i = \int_s^v \eta_u dB_u^i + \int_v^t \eta_u dB_u^i.$

(ii) $\int_s^t (\alpha \eta_u + \theta_u) dB_u^i = \alpha \int_s^t \eta_u dB_u^i + \int_s^t \theta_u dB_u^i,$

if α is bounded and belongs in $L_G^1(\Omega_s)$.

1.3.3 G-quadratic variation process

The quadratic variation of G-Brownian motion is a significant process. We say that the G-Brownian motions had a no mean uncertainty and a variance uncertainty. This uncertainty or ambiguity is focused on their quadratic variations $\langle B^i \rangle$.

Let the subdivision sequence $\left\{\pi_t^N, N \geq 1\right\}$ of $[0,t]$ such that the mesh $\mu\left(\pi_t^N\right) := \max\limits_{0\leq \ell\leq N-1}\left|t_{\ell+1}^N - t_\ell^N\right|$ tends to zero if N goes to ∞. We consider the double equality

$$\begin{aligned}\left(B_t^i\right)^2 &= \sum_{\ell=0}^{N-1}\left(\left(B^i_{t^N_{\ell+1}}\right)^2 - \left(B^i_{t^N_\ell}\right)^2\right)\\ &= \sum_{\ell=0}^{N-1}\left(B^i_{t^N_{\ell+1}} - B^i_{t^N_\ell}\right)^2 + \sum_{\ell=0}^{N-1} 2B^i_{t^N_\ell}\left(B^i_{t^N_{\ell+1}} - B^i_{t^N_\ell}\right)\\ &= I_1 + I_2.\end{aligned}$$

In second equality, the term I_2 tends to $2\int\limits_0^t B_s^i dB_s^i$ in $L_G^2(\Omega)$, the first term I_1 must be converge to a limit denoted $\left\langle B^i\right\rangle_t$, i.e.,

$$\left\langle B^i\right\rangle_t = \lim_{\mu\left(\pi_t^N\right)\to 0}\sum_{\ell=0}^{N-1}\left(B^i_{t^N_{\ell+1}} - B^i_{t^N_\ell}\right)^2 = \left(B_t^i\right)^2 - 2\int\limits_0^t B_s^i dB_s^i. \tag{1.6}$$

According to the preceding structure, $\left(\left\langle B^i\right\rangle_t\right)_{t\geq 0}$ is the increasing process, and $\left\langle B^i\right\rangle_0 = 0$. It will be known as the quadratic variation of the G-Brownian motion. It is crucial to remain mindful that a process $\left\langle B^i\right\rangle_t$ is not a deterministic, except in the case where $\underline{\sigma}_i^2 = \overline{\sigma}_i^2$, *i.e.*, when $\left(B_t^i\right)_{t\geq 0}$ represents a Brownian motion in classical case. The integral of a process $\eta \in \overline{M}_G^{2,1}(0,T)$ is defined according with $\langle B\rangle_t$ in the following. In first, we introduce the following mapping $Q_{0,T}(\eta) : M_G^{2,1}(0,T) \longrightarrow L_G^1(\Omega_T)$

$$Q_{0,T}(\eta) = \int\limits_0^T \eta_t d\left\langle B^i\right\rangle_t := \sum_{\ell=0}^{N-1}\xi_\ell\left(\left\langle B^i\right\rangle_{t_{\ell+1}} - \left\langle B^i\right\rangle_{t_\ell}\right).$$

Lemma 31 *For all* $\eta \in \overline{M}_G^{1,1}(0,T)$, *we have*

$$\overline{\sigma}_i^2\widehat{\mathbb{E}}\left[\int\limits_0^T |\eta_t|\, dt\right] \geq \widehat{\mathbb{E}}\left[\left|Q_{0,T}(\eta)\right|\right]. \tag{1.7}$$

As a result, the map $Q_{0,T}(\eta) : M_G^{1,1}(0,T) \longrightarrow L_G^1(\Omega_T)$ is linear and continuous. Therefore, by extending with continuity the application $Q_{0,T}$ on $\overline{M}_G^{1,1}(0,T)$, which is also denoted by $Q_{0,T}(\eta)$, that is

$$Q_{0,T}(\eta) = \int\limits_0^T \eta_t d\left\langle B^i\right\rangle_t,\ \forall \eta \in \overline{M}_G^{1,1}(0,T).$$

It should be noted that the inequality (1.7) holds true for $\eta \in \overline{M}_G^{1,1}(0,T)$.

Corollary 32 *For all $0 \leq s \leq t \leq T$, we have*

$$\overline{\sigma}_i^2 t \geq \left(\left\langle B^i\right\rangle_{t+s} - \left\langle B^i\right\rangle_s\right) \geq \underline{\sigma}_i^2 t.$$

Proposition 33 *For all fixed process η defined on $M_G^{2,1}(0,T)$, we have*

$$\overline{\sigma}_i^2 \widehat{\mathbb{E}}\left[\int_0^T \eta_t^2\, dt\right] \geq \widehat{\mathbb{E}}\left[\left(\int_0^T \eta_t\, dB_t^i\right)^2\right] \geq \underline{\sigma}_i^2 \widehat{\mathbb{E}}\left[\int_0^T \eta_t^2\, dt\right]. \tag{1.8}$$

Proof. First of all, we have

$$\begin{aligned}
\widehat{\mathbb{E}}\left[\left(\int_0^T \eta_t dB_t^i\right)^2\right] &= \widehat{\mathbb{E}}\left[\left(\int_0^{t_{N-1}} \eta_t dB_t^i + \xi_{N-1}\left(B_{t_N}^i - B_{t_{N-1}}^i\right)\right)^2\right] \\
&= \widehat{\mathbb{E}}\left[\left(\int_0^{t_{N-1}} \eta_t\, dB_t^i\right)^2 + \eta_{N-1}^2\left(B_{t_N}^i - B_{t_{N-1}}^i\right)^2\right. \\
&\quad \left. +2\left(\int_0^{t_{N-1}} \eta_t\, dB_t^i\right)\eta_{N-1}\left(B_{t_N}^i - B_{t_{N-1}}^i\right)\right] \\
&= \widehat{\mathbb{E}}\left[\left(\left(\int_0^{t_{N-1}} \eta_t dB_t^i\right)^2 + \xi_{N-1}^2\left(B_{t_N}^i - B_{t_{N-1}}^i\right)^2\right)\right] \\
&= \widehat{\mathbb{E}}\left[\sum_{i=0}^{N-1} \xi_i^2\left(B_{t_{i+1}}^i - B_{t_i}^i\right)^2\right] \\
&= \widehat{\mathbb{E}}\left[\sum_{i=0}^{N-1} \eta_i^2\left(\left\langle B^i\right\rangle_{t_{i+1}} - \left\langle B^i\right\rangle_{t_i}\right)\right].
\end{aligned}$$

By Corollary 32, we have

$$\overline{\sigma}_i^2\left(t_{i+1} - t_i\right) \geq \left(\left\langle B^i\right\rangle_{t_{i+1}} - \left\langle B^i\right\rangle_{t_i}\right) \geq \underline{\sigma}_i^2\left(t_{i+1} - t_i\right),$$

multiplying by η_i^2 term to term, we obtain

$$\overline{\sigma}_i^2 \eta_i^2\left(t_{i+1} - t_i\right) \geq \eta_i^2\left(\left\langle B^i\right\rangle_{t_{i+1}} - \left\langle B^i\right\rangle_{t_i}\right) \geq \underline{\sigma}_i^2 \eta_i^2\left(t_{i+1} - t_i\right).$$

As a result of summing with according to i with taking $\widehat{\mathbb{E}}$, we arrive at

$$\begin{aligned}
\widehat{\mathbb{E}}\left[\sum_{i=0}^{N-1} \underline{\sigma}_i^2 \eta_i^2\left(t_{i+1} - t_i\right)\right] &\leq \widehat{\mathbb{E}}\left[\sum_{i=0}^{N-1} \eta_i^2\left(\left\langle B^i\right\rangle_{t_{i+1}} - \left\langle B^i\right\rangle_{t_i}\right)\right] \\
&\leq \widehat{\mathbb{E}}\left[\sum_{i=0}^{N-1} \overline{\sigma}_i^2 \eta_i^2\left(t_{i+1} - t_i\right)\right].
\end{aligned}$$

Furthermore, the inequality mentioned above can be expressed by

$$\overline{\sigma}^2 \widehat{\mathbb{E}} \left[\int_0^T \eta_t^2 \, dt \right] \geq \widehat{\mathbb{E}} \left[\left(\int_0^T \eta_t \, dB_t \right)^2 \right] \geq \underline{\sigma}^2 \widehat{\mathbb{E}} \left[\int_0^T \eta_t^2 \, dt \right],$$

which complete the proof. ■

Definition 34 *Let the quadratic co-variation process of G-Brownian motions B^i and B^j given as follow*

$$\left\langle B^i, B^j \right\rangle = \frac{1}{4} \left[\left\langle B^i + B^j \right\rangle - \left\langle B^i - B^j \right\rangle \right], \text{ for all } i, j \in \overline{1, d} \, .$$

It is worth noting that now the processes $\left\langle B^i - B^j \right\rangle_{t \geq 0}$ and $\left(B^i + B^j \right)_{t \geq 0}$ are G-Brownian motions (see [62]).

Remark 35 *Since* $\left\langle B^{i-j} \right\rangle = \left\langle B^{j-i} \right\rangle = \left\langle -B^{i-j} \right\rangle$, *then* $\left\langle B^i, B^j \right\rangle_t = \left\langle B^j, B^i \right\rangle_t$.

We reference the following as one of the stability results from [20], which is crucial to our research.

Lemma 36 (Lemma 29, [20]) *If* $\{\mathbb{P}_n\}_{n=1}^{\infty} \subset \mathcal{P}$ *converges weakly to* $\mathbb{P} \in \mathcal{P}$, *then*

$$E^{\mathbb{P}_n}[\xi] \to E^{\mathbb{P}}[\xi], \text{ for each } \xi \in \mathbb{L}_G^1(\Omega_T).$$

Chapter 2

G-Neutral Stochastic Functional Differential Equations

This chapter is divided into three sections. The definition of G-NSFDEs is given in the first section. After that, we consider giving various inequalities and the G-Itô formula, and in the second section, we prove the existence and uniqueness of the solution under the Lipshitz condition by constructing a mapping for which the fixed point is the G-NSFDE solution. In the final part, we illustrate the numerical analysis for G-NSFDE using the Euler Maruyama method. the author in [21] illustrated the existence and uniqueness of solution for G-NSFDE via simulation findings. Initially, we will address the G-NSFDE, which is going to be covered in the third chapter.

2.1 G-Neutral stohastic functional differential equations

We define a one-dimensional G-NSFDE in which the variable's evolution rate is affected by its current state, previous state, and past evolution rate of the state variable.

Definition 37 *Let* $(B_t)_{t\geq 0}$ *the process called* G*-Brownian motion in one-dimensional case defined on* $\left(\Omega, \mathcal{H}, \widehat{\mathbb{E}}\right)$ *related to the universal filtraton* $\mathbb{F}^{\mathcal{P}} = \left\{\widehat{\mathcal{F}}_t^{\mathcal{P}}\right\}_{t\geq 0}$ *and its quadratic variation* $\{\langle B\rangle_t\ ,\ t \geq 0\}$*. One called one-dimensional neutral stochastic functional differential equation driven by a* G*-Brownian motion*

(G-NSFDE) which is written by the following form :

$$\begin{cases} d\left[X(t) - Q(t, X_t)\right] = b(t, X_t)\, dt + \gamma(t, X_t)\, d\langle B\rangle_t + \sigma(t, X_t)\, dB_t, \\ X_0 = \eta, t \in [0, T], \end{cases} \tag{2.1}$$

where η is the initial data on $[0,T]$, and $X_t = \{X(t+\theta) : -\tau \leq \theta \leq 0\}$, $\tau > 0$, the functions b, γ, σ and Q defined on $[0,T] \times BC([-\tau, 0]; \mathbb{R})$ and deterministic, with $BC([-\tau, 0]; \mathbb{R})$ is the family of $\mathbb{R}$-valued bounded continuous functions defined on $[-\tau, 0]$.

2.1.1 Important inequalities

Using the dual formulation of the G-expectation, we present the G-Burkholder-Davis-Gundy-type estimates (G-BDG in short) in one dimension.

Lemma 38 *(see [34])For each $p \geq 2$ and $\eta \in M_G^p(0,T)$, there exists some constant C_p depending only on p and T such that*

$$\widehat{\mathbb{E}}\left[\sup_{s \leq u \leq t} \left| \int_s^u \eta_r dB_r \right|^p \right] \leq C_p |t-s|^{\frac{p}{2}-1} \int_s^t \widehat{\mathbb{E}}[|\eta_r|^p|] dr.$$

Lemma 39 *(see [34])For $p \geq 1$ and η belonging to $M_G^p(0,T)$ there exists a constant $\bar{\sigma} > 0$ given that $\frac{d\langle B\rangle_t}{dt} \leq \bar{\sigma}$ holds q.s., follows that*

$$\widehat{\mathbb{E}}\left[\sup_{s \leq u \leq t} \left| \int_s^u \eta_r d\langle B\rangle_r \right|^p \right] \leq \bar{\sigma}^p |t-s|^{p-1} \int_s^t \widehat{\mathbb{E}}[|\eta_r|^p] dr.$$

We have also the following 'isometry'.

Lemma 40 *([60]) For each $p \geq 1$, for each $\eta \in M_G^1(0,T)$, we have*

$$\widehat{\mathbb{E}}\left[\int_0^T \eta(t) d\langle B\rangle_t\right] \leq \bar{\sigma}^2 \widehat{\mathbb{E}}\left[\int_0^T |\eta(t)|\, dt\right],$$

$$\widehat{\mathbb{E}}\left[\left(\int_0^T \eta(t) dB_t\right)^2\right] = \widehat{\mathbb{E}}\left[\int_0^T \eta^2(t) d\langle B\rangle_t\right], \quad \textit{for each } \eta \in M_G^2(0,T), (\textit{isometry}),$$

and

$$\widehat{\mathbb{E}}\left[\int_0^T |\eta(t)|^p\, dt\right] \leq \int_0^T \widehat{\mathbb{E}}\left[|\eta(t)|^p\right] dt, \quad \textit{for each } \eta \in M_G^p(0,T).$$

2.2 $G-$Itô formula

Finally, we provide a G-Itô formula in a d-dimensional case (for details, see [62, 80]).

The notion "one-dimensional G-Itô process" is used to refer to a specific type of stochastic process.

Definition 41 *We mean that the one-dimensional G-Itô process is specific type of stochastic process for all $t \in [0, T]$, $\nu = \overline{1, n}$, defined by*

$$X_t^\nu = X_0^\nu + \int_0^t \sigma_s^\nu dB_s + \int_0^t b_s^\nu ds + \int_0^t \gamma_s^\nu d\langle B\rangle_s,$$

with the unitial condition $X_0 \in L_G^2(\Omega_T)$, $\sigma^\nu \in \overline{M}_G^{2,1}(0, T)$ and $\gamma^\nu, b^\nu \in \overline{M}_G^{1,1}(0, T)$.

Theorem 42 *Consider $X_t = \left(X_t^1, ..., X_t^n\right)$ be a vector process of G-Itô of the form*

$$X_t^\nu = X_0^\nu + \int_0^t \sigma_s^\nu dB_s + \int_0^t b_s^\nu ds + \int_0^t \gamma_s^\nu d\langle B\rangle_s,\ t \in [0, T],$$

with the bounded processes $\sigma^\nu \in \overline{M}_G^{2,1}(0, T)$ and $\gamma^\nu, b^\nu \in \overline{M}_G^{1,1}(0, T)$, where $\nu = \overline{1, n}$.

Assuming that *the function* ψ is an element of the space $C^2\left(\mathbb{R}^d\right)$ such that $\left\{\partial^2_{x^\mu x^\nu}\psi\right\}_{\mu,\nu=1}^n$ are uniformly Lipschitz continuous. From $L_G^2\left(\Omega_t\right)$ and for all $t \geq 0$, it can be shown that

$$\begin{aligned}\psi\left(X_t\right) &= \psi\left(X_0\right) + \int_0^t \partial_{x^\nu}\psi\left(X_s\right)\sigma_s^\nu dB_s + \int_0^t \partial_{x^\nu}\psi\left(X_s\right)\beta_s^\nu ds \\ &\quad + \int_0^t \left[\partial_{x^\nu}\psi\left(X_s\right)\gamma_s^\nu + \frac{1}{2}\partial^2_{x^\mu x^\nu}\psi\left(X_s\right)\sigma_s^\mu\sigma_s^\nu\right] d\langle B\rangle_s.\end{aligned}$$

2.2.1 Problem and assumptions

With the definition of G-Itô's integral, for all $0 \leq t \leq T$, the one-dimensional G-NSFDE (2.1) written in the following integral form ([21]),

$$\begin{aligned}X(t) &= \eta(0) + Q(t, X_t) - Q(0, \eta) + \int_0^t b(s, X_s)\, ds \\ &\quad + \int_0^t \gamma(s, X_s)\, d\langle B\rangle_s + \int_0^t \sigma(s, X_s)\, dB_s, \qquad (2.2)\end{aligned}$$

and

$$Q, b, \gamma, \sigma : [0, T] \times BC([-\tau, T]; \mathbb{R}) \times \Omega \longrightarrow \mathbb{R},$$

are measurable Borel functions, as well as $Q, b, \gamma, \sigma \in \mathbb{M}_{\mathbb{G}}^{2}([0, T], \mathbb{R})$, with a random initial condition such that $\eta \in BC([-\tau, 0]; \mathbb{R})$,

$$X_0 = \eta = \{\eta(\theta) : -\tau \le \theta \le 0\},$$

The following assumptions will be in effect throughout this work.

(A_1) There exists $K_1 > 0$ such that

$$|H(t, x, u) - H(t, y, u)| \le K_1 |x(0) - y(0)|,$$

uniformly with respect to (t, ω) for each $x, y \in BC([-\tau, 0]; \mathbb{R})$, where $H = b, \gamma, \sigma$.

(A_2) There exists $0 < k_0 < \frac{1}{4}$ such that

$$|Q(t, x) - Q(t, y)| \le k_0 |x(0) - y(0)|, \tag{2.3}$$

uniformly with respect to (t, ω) for each $x, y \in BC([-\tau, 0]; \mathbb{R})$.

Note that, since $|Q(0, \eta)| \le k_0 |\eta(0)| + |Q(0, 0)|$, then $Q(0, \eta) \in L_G^2(\Omega_T)$ for all $\eta \in BC([-\tau, 0]; \mathbb{R})$.

Remark 43 *Indeed, the functions Q, b, γ, and σ defined by*

$$\begin{aligned} b(t, x, u) &: = c(t, x(0), u), \\ \gamma(t, x, u) &: = \alpha(t, x(0), u), \\ \sigma(t, x) &: = \beta(t, x(0)), \end{aligned}$$

and

$$Q(t, x) := \lambda(t, x(0)),$$

with the functions c, α and β are K_1- *Lipschitz, and λ is k_0-Lipschitz* uniformly with respect to (t, ω) for each $x \in BC([-\tau, 0]; \mathbb{R})$*, satisfies the assumptions (A_1) and (A_2).*

2.3 Existence and uniqueness of the solution for G-NSFDE

Given the following integral equation, we investigate the existence and uniqueness of the solution for G-NSFDE ([21]),

$$\begin{aligned} X(t) &= \eta(0) + Q(t, X_t) - Q(0, \eta) + \int_0^t b(s, X_s)\, ds \\ &\quad + \int_0^t \gamma(s, X_s)\, d\langle B\rangle_s + \int_0^t \sigma(s, X_s)\, dB_s, \ t \in [0, T], \end{aligned} \tag{2.4}$$

with random initial data

$$\eta=\{\eta(\theta)\}_{-\tau\leq\theta\leq 0}\in BC([-\tau,0];\mathbb{R}),$$

Let the space

$$\widetilde{\mathcal{H}}_T:=\left\{X=(X(t))_{t\in[0,T]},\ \mathbb{F}^{\mathcal{P}}\text{-adapted such that: }\int_0^T\widehat{\mathbb{E}}\left[|X(s)|^2\right]ds<\infty\right\},$$

equipped with the norms $N_C(X):=\left(\int_0^T\exp(-2Cs)\,\widehat{\mathbb{E}}\left(|X(s)|^2\right)ds\right)^{\frac{1}{2}}$, where $C\geq 0$. Since

$$\exp(-2CT)\,N_0(X)\leq N_C(X)\leq N_0(X),$$

then these norms are equivalent. Moreover, the functions

$$\sigma,Q:[0,T]\times BC([-\tau,0];\mathbb{R})\times\Omega\to\mathbb{R},$$

$$b,\gamma:[0,T]\times BC([-\tau,0];\mathbb{R})\times\Omega\to\mathbb{R},$$

are measurable, the random variable $Q(0,0)\in L_G^2(\Omega_T)$ as well as $Q(.,x)$, $\sigma(.,x)$, $b(.,x)$, $\gamma(.,x)\in\widetilde{\mathcal{H}}_T$ for each $x\in BC([-\tau,0];\mathbb{R})$.

In order to consider G-NSFDE (2.4), we first study the question of existence and uniqueness of solution. To this end, we need the following assumptions:

Theorem 44 *Let the assumptions (A_1) and (A_2) are satisfied. Then, the integral equation (2.4) has an unique solution $X\in\widetilde{\mathcal{H}}_T$.*

Proof. Let the mapping $\Theta:\widetilde{\mathcal{H}}_T\to\widetilde{\mathcal{H}}_T$ defined by: for each $t\in[0,T]$,

$$\begin{aligned}\Theta(X)(t) &= \eta(0)+Q(t,X_t)-Q(0,\eta)+\int_0^t b(s,X_s)\,ds\\ &\quad+\int_0^t\gamma(s,X_s)\,d\langle B\rangle_s+\int_0^t\sigma(s,X_s)\,dB_s. \end{aligned}\tag{2.5}$$

We have for all $X,\overline{X}\in\widetilde{\mathcal{H}}_T$

$$\begin{aligned}&\left|\Theta(X)(t)-\Theta\left(\overline{X}\right)(t)\right|\\ \leq\ &\left|Q(t,X_t)-Q\left(t,\overline{X}_t\right)\right|+\left|\int_0^t\left[b(s,X_s)-b\left(s,\overline{X}_s\right)\right]ds\right|\\ &+\left|\int_0^t\left[\gamma(s,X_s)-\gamma\left(s,\overline{X}_s\right)\right]d\langle B\rangle_s\right|\\ &+\left|\int_0^t\left[\sigma(s,X_s)-\sigma\left(s,\overline{X}_s\right)\right]dB_s\right|. \end{aligned}\tag{2.6}$$

Taking G-expectation on both sides, and using the following inequality

$$\left(\sum_{i=1}^k d_i\right)^2\leq 2^{k-1}\sum_{i=1}^k d_i^2,\ \text{for each } d_1...d_k>0,\tag{2.7}$$

we have

$$\begin{aligned}
& \widehat{\mathbb{E}}\left[\left|\Theta\left(X\right)\left(0t\right)-\Theta\left(\overline{X}\right)\left(t\right)\right|^{2}\right] \\
\leq\ & 8\widehat{\mathbb{E}}\left[\left|Q\left(t,X_{t}\right)-Q\left(t,\overline{X}_{t}\right)\right|^{2}\right] \\
& +8\widehat{\mathbb{E}}\left[\left|\int_{0}^{t}\left[b\left(s,X_{s}\right)-b\left(s,\overline{X}_{s}\right)\right]ds\right|^{2}\right] \\
& +8\widehat{\mathbb{E}}\left[\left|\int_{0}^{t}\left[\gamma\left(s,X_{s}\right)-\gamma\left(s,\overline{X}_{s}\right)\right]d\left\langle B\right\rangle_{s}\right|^{2}\right] \\
& +8\widehat{\mathbb{E}}\left[\left|\int_{0}^{t}\left[\sigma\left(s,X_{s}\right)-\sigma\left(s,\overline{X}_{s}\right)\right]dB_{s}\right|^{2}\right] \\
:\ & =8\sum_{i=1}^{4}U_{i}. \qquad (2.8)
\end{aligned}$$

Now, we have by assumption (A_2),

$$U_{1}\leq k_{0}^{2}\widehat{\mathbb{E}}\left[\left|X\left(t\right)-\overline{X}\left(t\right)\right|^{2}\right]. \qquad (2.9)$$

By applying Hölder inequality and (A_1), we have

$$\begin{aligned}
U_{2} & \leq T\int_{0}^{T}\widehat{\mathbb{E}}\left[\left|\left[b\left(s,X_{s}\right)-b\left(s,\overline{X}_{s}\right)\right]\right|^{2}\right]ds \\
& \leq TK_{1}^{2}\int_{0}^{T}\widehat{\mathbb{E}}\left[\left|X\left(s\right)-\overline{X}\left(s\right)\right|^{2}\right]ds. \qquad (2.10)
\end{aligned}$$

In similar, by employing the G-BDG inequalities we acquire the following results

$$\begin{aligned}
U_{3}+U_{4} & \leq T\bar{\sigma}^{2}\int_{0}^{T}\widehat{\mathbb{E}}\left[\left|\gamma\left(s,X_{s}\right)-\gamma\left(s,\overline{X}_{s}\right)\right|^{2}\right]ds \\
& +C_{2}\int_{0}^{T}\widehat{\mathbb{E}}\left[\left|\sigma\left(s,X_{s}\right)-\sigma\left(s,\overline{X}_{s}\right)\right|^{2}\right]ds, \\
& =K_{1}^{2}\left[T\bar{\sigma}^{2}+C_{2}\right]\int_{0}^{T}\widehat{\mathbb{E}}\left[\left|X\left(s\right)-\overline{X}\left(s\right)\right|^{2}\right]ds. \qquad (2.11)
\end{aligned}$$

Combining (2.9),(2.10), and (2.11), we get

$$\begin{aligned}
& \widehat{\mathbb{E}}\left[\left|\Theta\left(X\right)\left(t\right)-\Theta\left(\overline{X}\right)\left(t\right)\right|^{2}\right] \\
\leq\ & 8k_{0}^{2}\widehat{\mathbb{E}}\left[\left|X\left(t\right)-\overline{X}\left(t\right)\right|^{2}\right] \\
& +C\int_{0}^{T}\widehat{\mathbb{E}}\left[\left|X\left(s\right)-\overline{X}\left(s\right)\right|^{2}\right]ds, \qquad (2.12)
\end{aligned}$$

where $C = 8K_1^2\left(T + T\bar{\sigma}^2 + C_2\right)$.

Multiplying by $\exp(-2Ct)$ both sides of inequality (2.12) and integrating on $[0,T]$, we obtain

$$\begin{aligned}
N_C^2\left[\Theta(X) - \Theta\left(\overline{X}\right)\right] &\leq 8k_0^2 N_C^2\left[X - \overline{X}\right] \\
&\quad + C\int_0^T \exp(-2Ct)\left(\int_0^t \widehat{\mathbb{E}}\left[\left|X(s) - \overline{X}(s)\right|^2\right] ds\right) dt \\
&\leq 8k_0^2 N_C^2\left[X - \overline{X}\right] \\
&\quad + C\int_0^T \left(\widehat{\mathbb{E}}\left[\left|X(s) - \overline{X}(s)\right|^2\right] \int_s^T \exp(-2Ct)\, dt\right) ds \\
&\leq 8k_0^2 N_C^2\left[X - \overline{X}\right] \\
&\quad + \int_0^T \widehat{\mathbb{E}}\left[\left|X(s) - \overline{X}(s)\right|^2\right]\left(\frac{e^{-2Cs} - e^{-2CT}}{2}\right) ds \\
&\leq 8k_0^2 N_C^2\left[X - \overline{X}\right] + \frac{1}{2} N_C^2\left[X - \overline{X}\right]. \qquad (2.13)
\end{aligned}$$

Thus, we obtain the following estimation

$$N_C\left[\Theta(X) - \Theta\left(\overline{X}\right)\right] \leq \sqrt{8k_0^2 + \frac{1}{2}}\, N_C\left[X - \overline{X}\right]. \qquad (2.14)$$

We have, by using Hölder inequality,

$$\begin{aligned}
N_0^2\left(\int_0^{\cdot} b(s,0)\, ds\right) &= \int_0^T \widehat{\mathbb{E}}\left[\left|\int_0^t b(s,0)\, ds\right|^2\right] dt \\
&\leq T\int_0^T \int_0^t \widehat{\mathbb{E}}\left[\left|b(s,0)\right|^2\right] ds dt \\
&\leq T^2 N_0^2\left(b(.,0)\right).
\end{aligned}$$

Similarly, it easy to check, by G-BDG inequalities, that

$$N_0^2\left(\int_0^{\cdot} \gamma(s,0)\, d\langle B\rangle_s\right) \leq \overline{\sigma}^2 T^2 N_0^2\left(\gamma(.,0)\right),$$

and

$$N_0^2\left(\int_0^{\cdot} \sigma(s,0)\, dB_s\right) \leq C_2 T N_0^2\left(\sigma(.,0)\right).$$

Now observe that,

$$\begin{aligned}\Theta(0)(t) &= \eta(0)+Q(t,0)-Q(0,\eta)+\int_0^t b(s,0)\,ds \\ &\quad +\int_0^t \gamma(s,0)\,d\langle B\rangle_s+\int_0^t \sigma(s,0)\,dB_s.\end{aligned}$$

It follows that

$$\begin{aligned}N_0(\Theta(0)) &\leq \sqrt{T}\left(\|Q(0,\eta)\|_{L_G^2(\Omega_T)}+|\eta(0)|\right)+N_0(Q(.,0)) \\ &\quad +TN_0(b(.,0))+\overline{\sigma}TN_0(\gamma(.,0)) \\ &\quad +\sqrt{C_2T}N_0(\sigma(.,0)),\end{aligned} \tag{2.15}$$

then the process $\Theta(0)\in\widetilde{\mathcal{H}}_T$, so that if $X\in\widetilde{\mathcal{H}}_T$ then

$$N_C(\Theta(X))\leq N_C(\Theta(X)-\Theta(0))+N_C(\Theta(0))\leq N_0(X)+N_0(\Theta(0))<\infty.$$

This means that $\Theta(X)\in\widetilde{\mathcal{H}}_T$, which implies that Θ is well defined.

Finally, taking into account the fact that $\sqrt{8k_0^2+\frac{1}{2}}<1$ and assumption (A_2), we deduce by the formula (2.14) that $\Theta(X)$ is a contraction on $\widetilde{\mathcal{H}}_T$, then the fixed point $X\in\widetilde{\mathcal{H}}_T$ is the unique solution of (2.4). The proof is completed. ■

2.4 Numerical analysis: Euler-Maruyama method for G-NSFDE

In Section 2.4 ([21]), we present a numerical analysis of a G-NSFDE. The idea is to use the Euler-Maruyama scheme to solve the G-NSFDE (2.1). Let $\tau>0, T>\tau, N\in\mathbb{N}, h=\frac{T+\tau}{N}$ and $t_0=-\tau, t_1=-\tau+h,\cdots,t_{N_0}=0,\cdots,t_N=T$ be a discretization of the interval $[-\tau,T]$. Consider the following Euler-Maruyama scheme:

$$\begin{cases}\text{Given an initial data } \eta:[-\tau,0]\to\mathbb{R}^n, \text{ and put } X(t)=\eta(t) \text{ for } t=t_0,t_1,\cdots t_{N_0} \\ \text{Now for } i=N_0,N_0+1\cdots N \\ X(t_{i+1})=X(t_i)+Q\left(t_{i+1},X_{t_{i+1}}\right)-Q(t_i,X_{t_i}) \\ \qquad +b(t_i,X_{t_i})h+\gamma(t_i,X_{t_i})\left(\langle B\rangle_{t_{i+1}}-\langle B\rangle_{t_i}\right) \\ \qquad +\sigma(t_i,X_{t_i})(B_{t_{i+1}}-B_{t_i}),\end{cases} \tag{2.16}$$

where, $X_{t_i}:=\{X_{t_i}(\lambda):-\tau\leq\lambda\leq 0\}$, $X_{t_i}(\lambda):=X(t_{i+k})+\frac{\lambda-t_k}{h}[X(t_{i+k+1})-X(t_{i+k})], k$ is such that $t_k\leq\lambda<t_{k+1}$. In order that our algorithm works we have to give a value for X_{t_0-h}, we can set it equal $X_{t_0-h}=X_{t_0}=\eta(-\tau)$.

We employ the same strategy as [76] to simulate the increments of the G-Brownian motion and its quadratic variation by simulating its corresponding G-PDE using finite difference.

In Figure 2.1 (resp. Figure 2.2) we represent the simulation of the density (resp. distribution) of the G-Normal BM for $\sigma_{\max} = 1$ and different values of $\sigma_{\min}$.

Simulation of the $\mathbf{G-Normal}$ density and distribution for $\boldsymbol{\sigma}_{\max} = 1$ and different $\boldsymbol{\sigma}_{\min}$.

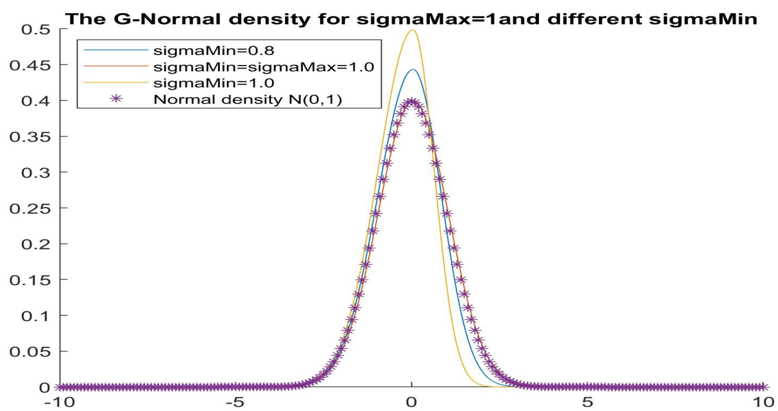

Figure 2.1: G-Normal density

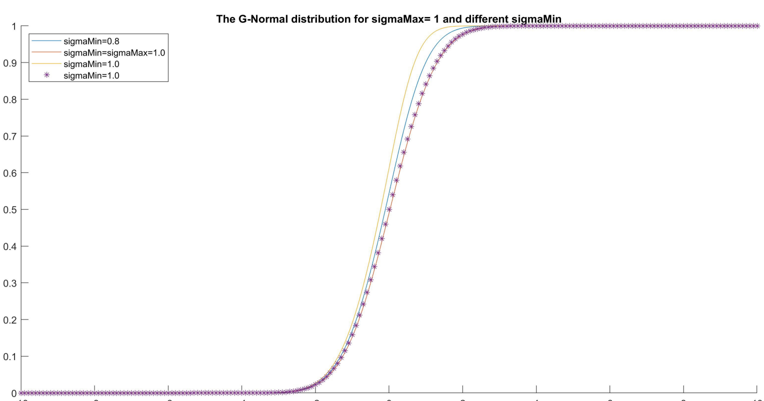

Figure 2.2: G-Normal distribution

Also, in Figure 2.3 (resp. Figure 2.4) we represent the simulation of the density (resp. distribution) of the G-Normal BM for $\sigma_{\min} = 1$ and different values of $\sigma_{\max}$.

Simulation of the G − Normal density and distribution for $\boldsymbol{\sigma}_{\min} = 1$ and different $\boldsymbol{\sigma}_{\max}$.

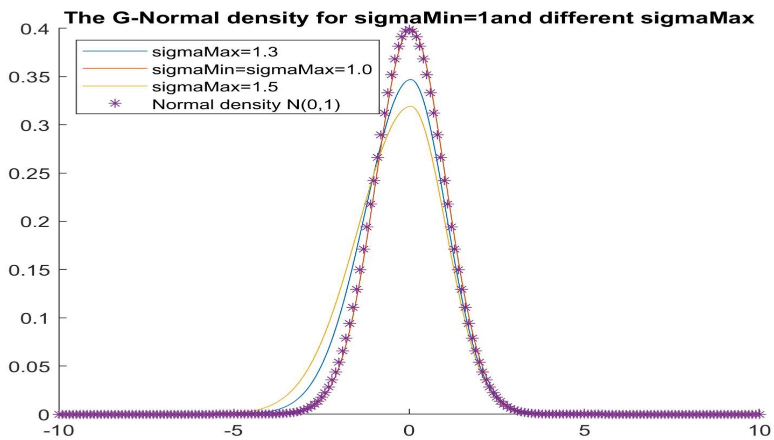

Figure 2.3: G-Normal density

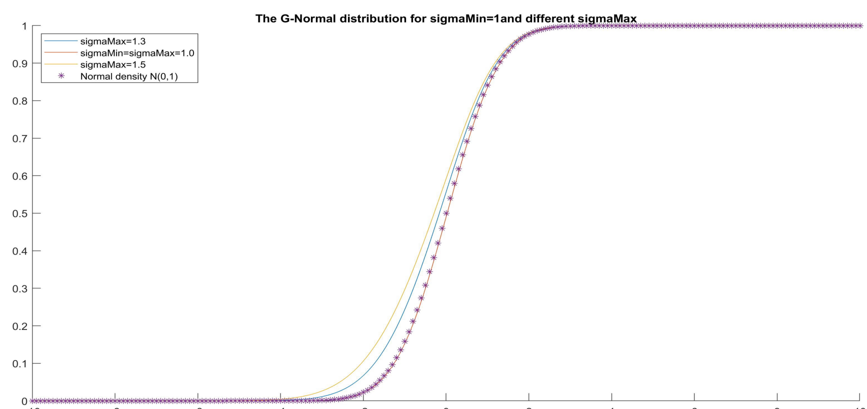

Figure 2.4: G-Normal distribution

Now, let take $T = 1, \tau = 0.1$ and the coefficients of the G-NSFDE (2.16) given by:

$$Q(t, X_t) := 0.3 \int_{t-\tau}^{t} X(s)ds$$

$$b(t, X_t) := 10 \int_{t-\tau}^{t} X(s)ds$$

$$\gamma(t, X_t) := 0.4 \int_{t-\tau}^{t} X(s)ds$$

$$\sigma(t, X_t) := 5 \int_{t-\tau}^{t} X(s)ds.$$

For these given data and coefficients we get the following results:

In Figure 2.5 (resp. Figure 2.6) we represent the trajectories of the solution of the $G-$NSFDE where the G-Brownian motion is with $\sigma_{max} = 1$ (resp. $\sigma_{max} = 3$), $\sigma_{min} = 0.65$ and the initial condition $(X_0(t))_{-\tau \le t \le 0}$ solution of $dX_0(t) = dW(t)$ with $X_0(-\tau) = 0$ where W is the standard Brownian motion.

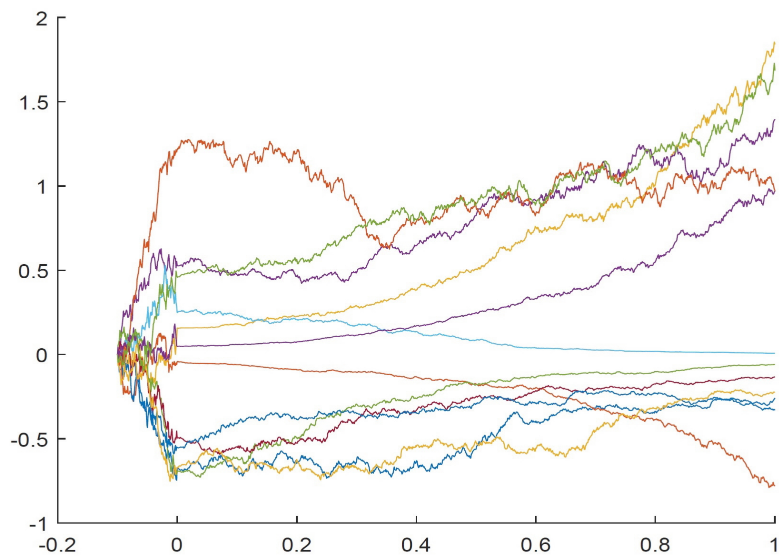

Figure 2.5: Solution of G-NSFDE with random initial condition.

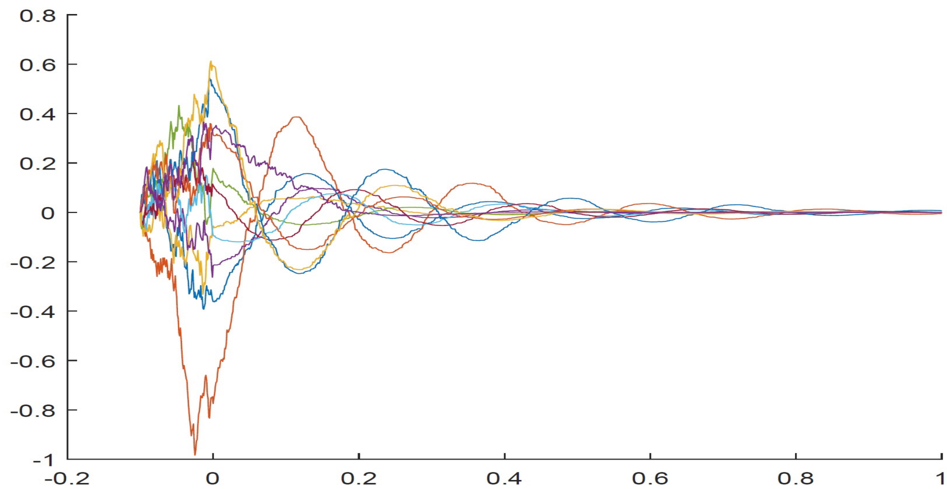

Figure 2.6: Solution of G-NSFDE with random initial condition BM.

In Figure 2.7 we represent the trajectories of the solution of the G-NSFDE where the G-Brownian motion is with $\sigma_{max} = 1, \sigma_{min} = 0.65$ and deterministic initial condition $(X_0(t))_{-\tau \leq t \leq 0}$ given by $X_0(t) = \exp(t)$.

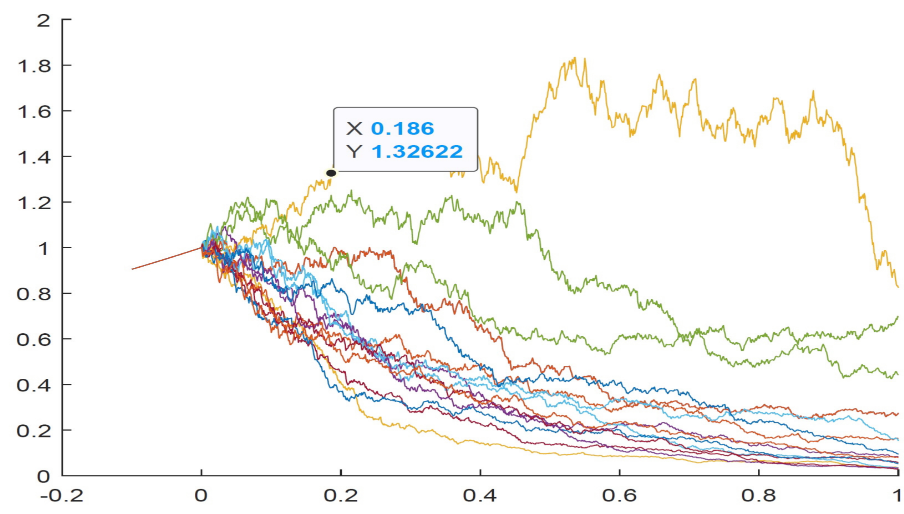

Figure 2.7: Solution of G-NSFDE with deterministic initial condition $\exp(t)$.

In Figure 2.8 we represent the trajectories of the solution of the $G-$NSFDE where the G-Brownian motion is with $\sigma_{max} = 1, \sigma_{min} = 0.65$ and deterministic initial condition $(X_0(t))_{-\tau \leq t \leq 0}$ given by: $\forall t \in [-\tau, 0]$, $X_0(t, \omega)$ a fixed value between $[-0.2, 0.2]$.

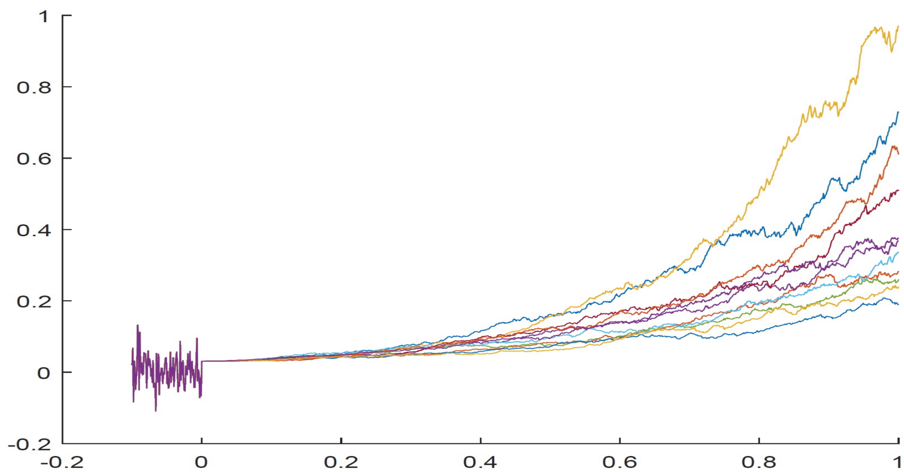

Figure 2.8: Solution of G-NSFDE with deterministic initial condition $X_0(t)$.

Chapter 3

Optimal Control Problem Of G-Neutral Stochastic Functional Differential Equation

This Chapter, devided in two sections, in section 3.1, related to the G-framework, we give some notions of stochastic control, and we illustrate the position of controlled problem for a one-dimensional G-NSFDE with two cases where the diffusion term σ controlled (respectively., uncontrolled). Following that, we will illustrate some examples of stochastic control models. In section 3.3 ([21]), we define G-relaxed control space. Then we study the approximation and we prove that the G-relaxed optimal control for G-NSFDE exists in which the diffusion and neutral terms are independent of the control variable. The result is achieved by employing tightness and the methods of weak convergence are applied to every probability measure belonging to the set of all possible probabilities in our system dynamics. In the finite horizon case, the minimizer of the cost functional of our controlled system is presented.

3.1 The problem of G-optimal control

We investugate a stochastic differential equation (SDE) subjected to control and the presence of an optimal control $\widehat{u} \in \mathcal{U}$ is demonstrated with values in the action space $\mathbb{A}$. This problem has been studied by Hu et al [39], Biagini et al [12] and Matoussi et al [54], where the authors proposed the necessary and sufficient conditions of optimality. To clarify, both dynamic programming and Pontryagin's principle are discussed. In the possibility that assumptions convexity are not present. The problem of optimal control may not have a viable solution when $\mathcal{U}$ is insufficiently large to fulfill certain conditions to possess a minimizer.

We are searching for a space $\mathcal{R}$ that contains $\mathcal{U}$ the set of strict controls. To achieve this fundamental finding, we construct the set $\mathcal{R}$ of relaxed controls as a subset of the space of probability measures on the action space $\mathbb{A}$, as in the classical case. We shall point out that the G-relaxed optimal control exists for a G-NSFDE.

3.2 G-Stochastic control notions

The admissible controls are the basic tools in our control problem.

Definition 45 *We call "admissible control" any process with values in $\mathbb{A}$, measurable and $\mathbb{F}^{\mathcal{P}}-$adapted.*

Recall that a process $u : \Omega \times [0,T] \longrightarrow \mathbb{A}$ is $\mathbb{F}^{\mathcal{P}}-$adapted if $u(t)$ is an $\widehat{\mathcal{F}}_t^{\mathbb{P}}-$measurable application for each $\mathbb{P} \in \mathcal{P}$.

Remark 46 *The process u must be $\mathbb{F}^{\mathcal{P}}-$adapted to ensure knowledge of the history of the process up to time t.*

Definition 47 *Let $\mathbb{A}$ a compact metric space of $\mathbb{R}$. The strict control u is an $\mathbb{F}^{\mathcal{P}}$-adapted process with values in $\mathbb{A}$, with the expected cost functional is given by*

$$J(u) = \widehat{\mathbb{E}}\big[\int_0^T \mathcal{L}(t, X_t, u(t))dt + \Psi(X_T)\big], \tag{3.1}$$

where

$$\begin{aligned} \mathcal{L} &: [0,T] \times BC([-\tau,0];\mathbb{R}) \times \mathbb{A} \longrightarrow \mathbb{R}, \\ \Psi &: BC([-\tau,0];\mathbb{R}) \longrightarrow \mathbb{R}. \end{aligned}$$

Our goal is to minimize the cost functional (3.1) over a set of admisible controls, which processes are $\widehat{\mathcal{F}}_t^{\mathbb{P}}$-measurable and valued in $\mathbb{A}$. The set of all admissible controls is denoted by $\mathcal{U}$ instead of $\mathcal{U}([0,T])$.

We say that the admissible control $\widehat{u}$ if it satisfies $J(\widehat{u}) = \inf\limits_{u\in\mathcal{U}} J(u)$, it is considered an optimal control.

3.2.1 Position of controlled problem for G-NSFDE

Case where the diffusion term σ is controlled

Given a dynamic system where the state is modeled by the G-NSFDE

$$\begin{cases} d[X(t) - Q(t,X_t)] = b(t,X_t,u(t))dt + \gamma(t,X_t,u(t))\, d\langle B\rangle_t + \sigma(t,X_t,u(t))dB_t, \\ X_0 = \eta,\ \eta \in BC([-\tau,0];\mathbb{R}). \end{cases} \tag{3.2}$$

The functions

$$\begin{aligned} b, \gamma, \sigma \quad &: \quad [0,T] \times BC\left([-\tau,0];\mathbb{R}\right) \times \mathbb{A} \times \Omega \longrightarrow \mathbb{R}, \\ Q \quad &: \quad [0,T] \times BC\left([-\tau,0];\mathbb{R}\right) \times \Omega \longrightarrow \mathbb{R}, \end{aligned}$$

are a Borellian and satisfying the assumptions (A_1), (A_2), and respect to the space of variables uniformly in (t,u) for each $u \in \mathbb{A}$.

It is clear that the random variable X_T is associated with an admissible control u for which we define the cost $J(u)$. The objective of the control problem is to determine the infimum for J and subsequently find a control, if it exists, that achieves it to obtain the expected optimality. Therefore, we must prove the existence of an admissible optimal control $\widehat{u}$ verifying for all $\eta \in BC\left([-\tau,0];\mathbb{R}\right)$:

$$V(\eta) := \inf_{u \in \mathcal{U}} J^u(\eta) = J^{\widehat{u}}(\eta). \tag{3.3}$$

The function V is called the optimal cost or value function.

Case where the diffusion term σ is uncontrolled

In the case where the diffusion term is uncontrolled, the state of our dynamic system is described by the following G-NSFDE:

$$\begin{cases} d\left[X(t) - Q(t, X_t)\right] = b(t, X_t, u(t))dt + \gamma\left(t, X_t, u(t)\right) d\left\langle B\right\rangle_t \ + \sigma(t, X_t)dB_t, \\ X_0 = \eta, \ \eta \in BC\left([-\tau,0];\mathbb{R}\right). \end{cases} \tag{3.4}$$

where b, γ, σ are a Borellian functions defined above. By the previous result of existence and uniqueness of solution, under our assumptions (A_1) and (A_2) the G-NSFDE (3.4) has a unique solution X^u for each fixed control $u \in \mathbb{A}$. The cost functional is defined, in this case, in the same way as in (3.1).

3.2.2 Example of stochastic control models

Optimal stopping ([78])

Consider the situation where the stopping time is the part of the control. In the formulation of such models, an admissible control/stopping time is a pair (u,τ) defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$ in addition with a Brownian motion B in m-dimensional case, where τ is an $\{\mathcal{F}_t\}_{t \geq 0}$-stopping time and u is the control that meets the usual conditions.

To minimize the following cost functional we use the stopping/optimal control probelm

$$J(u,\tau) = E\left\{ \int_0^{\tau} \mathcal{L}\left(t, X(t), u(t)\right) dt + \Psi\left(X(\tau)\right) \right\}, \tag{3.5}$$

in accordance with (3.4) across the stopping times and the admissible controls set.

Risk-sensitive control ([78])

Let the function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a monotonic increasing, we define the cost by

$$J(u) = E\left[\phi\left(\int_0^T \mathcal{L}(t, X(t), u(t))\, dt + \Psi(X(T))\right)\right]. \tag{3.6}$$

It should be highlighted that $X(t)$ is the solution of the SDE depicting the dynamic system to be investigated at time t. In economic theory, this sort of issue is significant. Often, cost functionals have the form

$$\phi^{\lambda}(x) = \lambda \exp(\lambda x),$$

where $\lambda \in \mathbb{R}$. represents a parameter defining the degree of risk sensitivity of the criterion. Particularly, ϕ^{λ} represents a concave (respectively, convex function) when $\lambda < 0$ (respectively. $\lambda > 0$). When λ is sufficiently small.

Another frequently used disutility function is the so-called **HARA** utility

The '**HARA** utility' is another widely utilized disutility function.

$$\phi^{\delta}(x) = \begin{cases} \frac{1}{\delta} x^{\delta}, & \text{if } \delta \neq 0, \\ \ln\ x, & \text{if } \delta = 0, \end{cases} \qquad x > 0,$$

in which the following situations hold

$$\begin{cases} \delta < 1 \text{ is a risk-seeking}, \\ \delta > 1 \text{ is a risk-averse}, \\ \delta = 1 \text{ is a risk-neutral}. \end{cases}$$

3.3 Formulation of G-relaxed control problem

In Section 3.3 ([21]), we initiate the problem of G-relaxed optimal control on the relaxed space $\mathcal{R}$, indicating that $\mathcal{R}$ seems to have a stable convergence topology (weak topology). The existence of an optimal control for the problems modeled by the G-NSFDE in which the diffusion does not depend on the control, was shown in an explicit way by Fleming [31]. Later, these results were generalized in the case where the duffusion is controlled by El Karoui [24].

3.3.1 The space of G-relaxed controls

Let $(\mathbb{A}, d)$ be a separable metric space and $\mathcal{P}(\mathbb{A})$ the space of probability measures on $\mathbb{A}$ endowed with its Borel σ-algebra $\mathcal{B}(\mathbb{A})$. The class $\mathbb{M}([0,T] \times \mathbb{A})$ of G-relaxed controls considered in this work is a subset of $\mathbb{M}([0,T] \times \mathbb{A})$, the set of Radon measures $\nu(dt, d\xi)$ on $[0,T] \times \mathbb{A}$ having for projection on $[0,T]$ the Lebesgue measure dt for projection on $\mathbb{A}$ a probability measure $\mu_t(d\xi) \in \mathcal{P}(\mathbb{A})$ (i.e., $\nu(dt, d\xi) := \mu_t(d\xi)dt$). equipped this space with the topology of the stable convergence of the measures. The topology of the stable convergence of the measures is the thinnest topology that makes the application

$$q \mapsto \int_0^T \int_{\mathbb{A}} \varphi(t,\xi) q(dt, d\xi),$$

continuous, for any bounded measurable function $\varphi(t,\xi)$ continuous in ξ, for any fixed t, $\mathbb{M} := \mathbb{M}([0,T] \times \mathbb{A})$ is a separable metrizable space. Moreover, $\mathbb{M}$ is compact if $\mathbb{A}$ is. It should be noted that the topology of stable convergence of the measures leads to the topology of weak convergence (See El karoui [24, 23] for more details). Then, we introduce the class of relaxed stochastic controls on $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$.

Definition 48 *A relaxed stochastic control on $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is an $\mathbb{F}^{\mathcal{P}}$-progressively measurable random measure of the form $q(\omega, dt, d\xi) = \mu_t(\omega, d\xi)dt$ such that*

$$\begin{aligned} X(t) &= \eta(0) + Q(t, X_t) - Q(0, \eta) + \int_0^t \int_{\mathbb{A}} b(s, X_s, \xi) \mu_s(d\xi) ds \\ &\quad + \int_0^t \int_{\mathbb{A}} \gamma(s, X_s, \xi) \mu_s(d\xi) d\langle B \rangle_s + \int_0^t \sigma(s, X_s) dB_s, \end{aligned} \tag{3.7}$$

Note that each strict control can be considered as a relaxed control via the mapping

$$\Phi(u)(dt, d\xi) = \delta_{u(t)}(d\xi)\, dt, \tag{3.8}$$

where $\delta_{u(t)}$ is a Dirac measure charging $u(t)$ for each t.

Remark 49 *We mean by "the process $q(\omega, dt, d\xi)$ is $\mathbb{F}^{\mathcal{P}}$- progressively measurable" that for every $C \in \mathcal{B}(\mathbb{A})$ and for every $t \in [0,T]$, the mapping $(s,\omega) \mapsto \mu_s(\omega, C)$ is $\mathcal{B}([0,t]) \otimes \widehat{\mathcal{F}}_t^{\mathcal{P}}$-measurable. In particular, the process $(\mu_t(C))_{t\in[0,T]}$ is adapted to $\mathbb{F}^{\mathcal{P}}$.*

We denote by $\mathcal{R}$ the class of relaxed stochastic controls. The set of relaxed controls admits a very useful compactness property.

Proposition 50 *(see El Karoui [24]) If we assume that the space $\mathbb{A}$ is compact, then $\mathcal{R}$ is too.*

3.3.2 Problem of G-NSFDE relaxed control

In this section, we consider a relaxed control problem (3.7). Let X^μ denotes the solution of equation (3.7) associated with the G-relaxed control. We establish the existence of a minimizer of the cost corresponding to μ,

$$J(\mu) = \widehat{\mathbb{E}} \left[\int_0^T \int_{\mathbb{A}} \mathcal{L}(t, X_t^\mu, \xi) \mu_t(d\xi)\ dt + \Psi(X_T^\mu) \right], \tag{3.9}$$

the functions

$$
\begin{aligned}
\mathcal{L} &: [0,T] \times BC\left([-\tau,0];\mathbb{R}\right) \times \mathbb{A} \longrightarrow \mathbb{R}, \\
\Psi &: [0,T] \times BC\left([-\tau,0];\mathbb{R}\right) \longrightarrow \mathbb{R},
\end{aligned}
$$

satisfy the following assumption:

(A_3) $\mathcal{L}$, Ψ are bounded and for each $t \in [0,T]$ and $x \in BC\left([-\tau,0];\mathbb{R}\right)$ the functions $\mathcal{L}(t,x,\cdot)$, $\Psi(x)$ are continuous. Additionally, we suppose that:

$$
\left|\mathcal{L}\left(t,x,u\right) - \mathcal{L}\left(t,y,u\right)\right| + \left|\Psi\left(x\right) - \Psi\left(y\right)\right| \leq K_1 \left|x\left(0\right) - y\left(0\right)\right|.
$$

We recall that in the strict control problem

$$
J(u) = \widehat{\mathbb{E}}\left[\int_0^T \mathcal{L}(t, X_t^u, u\left(t\right))\, dt + \Psi(X_T^u)\right], \tag{3.10}
$$

over the set $\mathcal{U}$,

$$
\begin{aligned}
X^u\left(t\right) &= \eta\left(0\right) + Q\left(t, X_t^u\right) - Q\left(0,\eta\right) + \int_0^t b(s, X_s^u, u\left(s\right))ds \\
&\quad + \int_0^t \gamma(s, X_s^u, u\left(s\right))d\left\langle B\right\rangle_s + \int_0^t \sigma(s, X_s^u)dB_s.
\end{aligned} \tag{3.11}
$$

Then, we have

$$
\begin{aligned}
X^\mu(t) &= \eta\left(0\right) + Q\left(t, X_t^\mu\right) - Q\left(0,\eta\right) + \int_0^t \int_{\mathbb{A}} b(s, X_s^\mu, \xi)\mu_s(d\xi)ds \\
&\quad + \int_0^t \int_{\mathbb{A}} \gamma(s, X_s^\mu, \xi)\mu_s(d\xi)d\left\langle B\right\rangle_s + \int_0^t \sigma(s, X_s^\mu)dB_s.
\end{aligned} \tag{3.12}
$$

We suppose as well that the coefficients of the G-NSFDE verify the following condition

(A_4) The coefficients b, γ, σ are bounded and for every fixed $t \in [0,T]$ and $x \in BC\left([-\tau,0];\mathbb{R}\right)$ the functions $b(t,x,\cdot), \gamma(t,x,\cdot)$ and $\sigma(t,x)$ are continuous $q.s.$

3.3.3 Approximations and existence of G-relaxed optimal control

By initiating the relaxed control issue, the following lemma, which expands the celebrated Chattering Lemma, claims that each G-relaxed control in $\mathcal{R}$ can be approximated by strict controls.

Taking use of the fact that under $\mathbb{P} \in \mathcal{P}$, B is a continuous martingale with a quadratic variation process $\langle B\rangle$ such that $c_t := \frac{d\langle B\rangle_t}{dt}$ is bounded. Let X^μ and

X^n the corresponding solutions satisfy the following integral equations type of G-NSFDEs:

$$\begin{aligned} X^{\mu}(t) &= \eta(0) + Q(t, X_t^{\mu}) - Q(0, \eta) \\ &\quad + \int_0^t \int_{\mathbb{A}} (b(s, X_s^{\mu}, \xi) + c_s \gamma(s, X_s^{\mu}, \xi)) \mu_s(d\xi) ds + \int_0^t \sigma(s, X_s^{\mu}) dB_s, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} X^{n}(t) &= \eta(0) + Q(t, X_t^{n}) - Q(0, \eta) \\ &\quad + \int_0^t \int_{\mathbb{A}} (b(s, X_s^{n}, \xi) + c_s \gamma(s, X_s^{n}, \xi)) \delta_{u^n(s)}(d\xi)\, ds + \int_0^t \sigma(s, X_s^{n}) dB_s, \end{aligned} \tag{3.14}$$

with random initial data

$$X_0^{\mu} = X_0^n = \eta \in BC([-\tau, 0]; \mathbb{R}). \tag{3.15}$$

Definition 51 ***(stable convergence)*** *Let $\mu^n, \mu \in \mathcal{R}, n \in \mathbb{N}^*$. We say that, we have a stable convergence, if for any continuous function $f : [0, T] \times \mathbb{A} \to \mathbb{R}$, we have*

$$\lim_{n \to \infty} \int_{[0,T] \times \mathbb{A}} f(t, \xi)\, \mu^n(dt, d\xi) = \int_{[0,T] \times \mathbb{A}} f(t, \xi)\, \mu(dt, d\xi).$$

Lemma 52 *([67])* ***(G-Chattering Lemma)*** *Let $(\mathbb{A}, d)$ be a separable compact metric space. Let $(\mu_t)_{t \geq 0}$ be an $\mathbb{F}^{\mathcal{P}}$-progressively measurable process taking values in $\mathcal{P}(\mathbb{A})$. Then there exists a sequence $(u^n(t))_{n \geq 0}$ of $\mathbb{F}^{\mathcal{P}}$-progressively measurable processes taking values in $\mathbb{A}$, such that the sequence of random measures $\delta_{u^n(t)}(d\xi)dt$ converges in the sense of stable convergence (thus weakly) to $\mu_t(d\xi)dt$ q.s.*

We have the following Lemma of stability results for G-NSFDE with a performance $J^{\mathbb{P}}$, for any $\mathbb{P} \in \mathcal{P}$.

Lemma 53 ***(stability results)*** *Let μ be a relaxed control, and let (u^n) be a sequence defined as in (G-Chattering Lemma). Then we have*

(i) For every $\mathbb{P} \in \mathcal{P}$, it holds that

$$\lim_{n \to \infty} \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |X^n(t) - X^{\mu}(t)|^2 \right] = 0, \tag{3.16}$$

and

$$\lim_{n \to \infty} \widehat{\mathbb{E}} \left[\sup_{0 \leq t \leq T} |X^n(t) - X^{\mu}(t)|^2 \right] = 0. \tag{3.17}$$

(ii) *Let* $J(u^n)$ *and* $J(\mu)$ *be the corresponding cost functionals to* u^n *and* μ *respectively. Then, there exists a subsequence* (u^{n_k}) *of* (u^n) *such that for every* $\mathbb{P} \in \mathcal{P}$

$$\lim_{k\to\infty} J^{\mathbb{P}}(u^{n_k}) = J^{\mathbb{P}}(\mu), \tag{3.18}$$

and

$$\lim_{k\to\infty} J(u^{n_k}) = J(\mu). \tag{3.19}$$

Moreover,

$$\inf_{u\in\mathcal{U}} J^{\mathbb{P}}(u) = \inf_{\mu\in\mathcal{R}} J^{\mathbb{P}}(\mu), \tag{3.20}$$

and there exists a G-relaxed optimal control $\widehat{\mu}_{\mathbb{P}} \in \mathcal{R}$ *such that*

$$J^{\mathbb{P}}(\widehat{\mu}_{\mathbb{P}}) = \inf_{\mu\in\mathcal{R}} J^{\mathbb{P}}(\mu). \tag{3.21}$$

Proof.

(i) The proof of this result is inspired by [67]. Subtracting (3.13) from (3.14) term by term, we have

$$\begin{aligned}
X^n(t) - X^\mu(t) &= \left[Q\left(t, X_t^n\right) - Q\left(t, X_t^\mu\right)\right] \\
&\quad + \int_0^t \int_{\mathbb{A}} (b(s, X_s^n, \xi) + c_s \gamma(s, X_s^n, \xi)) \delta_{u^n(s)}\left(d\xi\right) ds \\
&\quad - \int_0^t \int_{\mathbb{A}} (b(s, X_s^\mu, \xi) + c_s \gamma(s, X_s^\mu, \xi)) \mu_s(d\xi) ds \\
&\quad + \int_0^t \left[\sigma(s, X_s^n) - \sigma(s, X_s^\mu)\right] dB_s \\
&= \left[Q\left(t, X_t^n\right) - Q\left(t, X_t^\mu\right)\right] + \mathcal{I}_n\left(t\right).
\end{aligned} \tag{3.22}$$

Taking G-expectation on both sides and using the assumptions (A_1) and (A_2), it follows that

$$\begin{aligned}
\widehat{\mathbb{E}}\left[\sup_{0\le t\le T} \left|X^n\left(t\right) - X^\mu\left(t\right)\right|^2\right] &\le 2k_0^2 \widehat{\mathbb{E}}\left[\sup_{0\le t\le T} \left|X^n\left(t\right) - X^\mu\left(t\right)\right|^2\right] \\
&+ 2\widehat{\mathbb{E}}\left[\sup_{0\le t\le T} \left|\mathcal{I}_n\left(s\right)\right|^2\right],
\end{aligned}$$

then

$$\widehat{\mathbb{E}}\left[\sup_{0\le t\le T} \left|X^n\left(t\right) - X^\mu\left(t\right)\right|^2\right] \le \frac{2}{(1-2k_0^2)} \widehat{\mathbb{E}}\left[\sup_{0\le t\le T} \left|\mathcal{I}_n\left(s\right)\right|^2\right]. \tag{3.23}$$

We have

$$
\begin{aligned}
& \widehat{\mathbb{E}}\left[\sup_{0\leq t\leq T}\left|\mathcal{I}_n\left(s\right)\right|^2\right] \\
\leq\ & \widehat{\mathbb{E}}\left(\sup_{0\leq t\leq T}\left|\int_0^t\int_{\mathbb{A}}(b(s,X_s^n,\xi)+c_s\gamma(s,X_s^n,\xi))\delta_{u^n(s)}\left(d\xi\right)ds\right.\right. \\
& \left.\left.-\int_0^t\int_{\mathbb{A}}(b(s,X_s^\mu,\xi)+c_s\gamma(s,X_s^\mu,\xi))\mu_s(d\xi)ds\right|^2\right) \\
& +\widehat{\mathbb{E}}\left(\sup_{0\leq t\leq T}\left|\int_0^t\left[\sigma(s,X_s^n)-\sigma(s,X_s^\mu)\right]dB_s\ \right|^2\right) \\
\leq\ & \widehat{\mathbb{E}}\left(\sup_{0\leq t\leq T}\left|\int_0^t\int_{\mathbb{A}}(b(s,X_s^n,\xi)+c_s\gamma(s,X_s^n,\xi))\delta_{u^n(s)}\left(d\xi\right)ds\right.\right. \\
& \left.\left.-\int_0^t\int_{\mathbb{A}}(b(s,X_s,\xi)+c_s\gamma(s,X_s,\xi))\delta_{u^n(s)}(d\xi)ds\right|^2\right) \\
& +\widehat{\mathbb{E}}\left(\sup_{0\leq t\leq T}\left|\int_0^t\int_{\mathbb{A}}(b(s,X_s,\xi)+c_s\gamma(s,X_s,\xi))\delta_{u^n(s)}\left(d\xi\right)ds\right.\right. \\
& \left.\left.-\int_0^t\int_{\mathbb{A}}(b(s,X_s^\mu,\xi)+c_s\gamma(s,X_s^\mu,\xi))\mu_s(d\xi)ds\right|^2\right) \\
& +\widehat{\mathbb{E}}\left(\sup_{0\leq t\leq T}\left|\int_0^t\left[\sigma(s;X_s^n)-\sigma(s,X_s^\mu)\right]dB_s\ \right|^2\right).
\end{aligned}
\tag{3.24}
$$

Let $\varepsilon>0$. Then, there exists $\mathbb{P}^\varepsilon\in\mathcal{P}$ such that

$$
\begin{aligned}
& \widehat{\mathbb{E}}\left[\sup_{0\leq t\leq T}\left|\mathcal{I}_n\left(s\right)\right|^2\right] \\
\leq\ & \mathbb{E}^{\mathbb{P}^\varepsilon}\left(\sup_{0\leq t\leq T}\left|\int_0^t\int_{\mathbb{A}}(b(s,X_s^n,\xi)+c_s\gamma(s,X_s^n,\xi))\delta_{u^n(s)}\left(d\xi\right)ds\right.\right. \\
& \left.\left.-\int_0^t\int_{\mathbb{A}}(b(s,X_s,\xi)+c_s\gamma(s,X_s,\xi))\delta_{u^n(s)}(d\xi)ds\right|^2\right) \\
& +\mathbb{E}^{\mathbb{P}^\varepsilon}\left(\sup_{0\leq t\leq T}\left|\int_0^t\int_{\mathbb{A}}b(s,X_s,\xi)\delta_{u^n(s)}\left(d\xi\right)ds-\int_0^t\int_{\mathbb{A}}b(s,X_s^\mu,\xi)\mu_s(d\xi)ds\right|^2\right) \\
& +\mathbb{E}^{\mathbb{P}^\varepsilon}\left(\sup_{0\leq t\leq T}\left|\int_0^t\int_{\mathbb{A}}c_s\gamma(s,X_s,\xi)\delta_{u^n(s)}\left(d\xi\right)ds\right.\right. \\
& \left.\left.-\int_0^t\int_{\mathbb{A}}c_s\gamma(s,X_s^\mu,\xi))\mu_s(d\xi)ds\right|^2\right) \\
& +\mathbb{E}^{\mathbb{P}^\varepsilon}\left(\sup_{0\leq t\leq T}\left|\int_0^t\left[\sigma(s,X_s^n)-\sigma(s,X_s^\mu)\right]dB_s\ \right|^2\right)+\varepsilon.
\end{aligned}
\tag{3.25}
$$

Then, we have

$$
\begin{aligned}
& \widehat{\mathbb{E}}\left[\sup_{0\leq t\leq T}\left|\mathcal{I}_{n}\left(s\right)\right|^{2}\right] \\
\leq\ & 16\mathbb{E}^{\mathbb{P}^{\varepsilon}}\left(\sup_{0\leq t\leq T}\left|\int_{0}^{t}\int_{\mathbb{A}}(b(s,X_{s}^{n},\xi)+c_{s}\gamma(s,X_{s}^{n},\xi))\delta_{u^{n}(s)}\left(d\xi\right)ds\right.\right. \\
& \left.\left.-\int_{0}^{t}\int_{\mathbb{A}}(b(s,X_{s},\xi)+c_{s}\gamma(s,X_{s},\xi))\delta_{u^{n}(s)}(d\xi)ds\right|^{2}\right) \\
& +16\mathbb{E}^{\mathbb{P}^{\varepsilon}}\left(\sup_{0\leq t\leq T}\left|\left[\int_{0}^{t}\int_{\mathbb{A}}(b(s,X_{s},\xi)\delta_{u^{n}(s)}\left(d\xi\right)ds\right.\right.\right. \\
& \left.\left.\left.-\int_{0}^{t}\int_{\mathbb{A}}b(s,X_{s}^{\mu},\xi)\mu_{s}(d\xi)ds\right]\ \right|^{2}\right) \\
& +16\mathbb{E}^{\mathbb{P}^{\varepsilon}}\left(\sup_{0\leq t\leq T}\left|\left[\int_{0}^{t}\int_{\mathbb{A}}c_{s}\gamma(s,X_{s},\xi)\delta_{u^{n}(s)}\left(d\xi\right)ds\right.\right.\right. \\
& \left.\left.\left.-\int_{0}^{t}\int_{\mathbb{A}}c_{s}\gamma(s,X_{s}^{\mu},\xi))\mu_{s}(d\xi)ds\right]\right|^{2}\right) \\
& +16\mathbb{E}^{\mathbb{P}^{\varepsilon}}\left(\sup_{0\leq t\leq T}\left|\int_{0}^{t}\left[\sigma(s,X_{s}^{n})-\sigma(s,X_{s}^{\mu})\right]dB_{s}\ \right|^{2}\right)+16\varepsilon^{2} \\
=\ & 16\left\{\left(\mathcal{I}_{(n,1)}+\mathcal{I}_{(n,2)}+\mathcal{I}_{(n,3)}+\mathcal{I}_{(n,4)}\right)+\varepsilon^{2}\ \right\}. \qquad (3.26)
\end{aligned}
$$

Since b, γ are bounded and continuous in the control variable ξ, then, by using the dominated convergence theorem, and the stable convergence of $\delta_{u^{n}(t)}\left(d\xi\right)dt$ to $\mu_{t}(d\xi)dt$, we have

$$
\lim_{n\to\infty}\mathcal{I}_{(n,2)}=\lim_{n\to\infty}\mathcal{I}_{(n,3)}=0. \qquad (3.27)
$$

Similarly, we use the assumption (A_{1}), then

$$
\lim_{n\to\infty}\left(\mathcal{I}_{(n,1)}+\mathcal{I}_{(n,4)}\right)\leq K_{1}^{2}\lim_{n\to\infty}\left[E^{\mathbb{P}^{\varepsilon}}\left(\int_{0}^{T}\left|X^{n}\left(s\right)-X^{\mu}\left(s\right)\right|^{2}ds\right)+\varepsilon^{2}\right]. \qquad (3.28)
$$

Consequently, from theorem of dominated convergence, it can be obatin that

$$
\begin{aligned}
& \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}^{\varepsilon}} \left[\int_0^T \left| X^n(s) - X^{\mu}(s) \right|^2 ds \right] \\
\leq & \int_0^T \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}^{\varepsilon}} \left[\left| X^n(s) - X^{\mu}(s) \right|^2 \right] ds \\
\leq & \int_0^T \lim_{n\to\infty} \widehat{\mathbb{E}} \left[\sup_{0\leq \upsilon \leq s} \left| X^n(\nu) - X^{\mu}(\nu) \right|^2 \right] ds. \qquad (3.29)
\end{aligned}
$$

Using $Z(\delta) = \lim_{n\to\infty} \widehat{\mathbb{E}} \left[\sup_{0\leq t\leq \delta} \left| X^n(t) - X^{\mu}(t) \right|^2 \right]$, for each $\delta > 0$, then that gives from (3.28) and (3.29),

$$
Z(T) \leq \frac{32K_1^2}{1-2k_0^2} \left(\int_0^T Z(s)\, ds + \varepsilon^2 \right). \qquad (3.30)
$$

By using Gronwall's lemma, we conclude that

$$
\lim_{n\to\infty} \widehat{\mathbb{E}} \left[\sup_{0\leq t\leq T} \left| X^n(t) - X^{\mu}(t) \right|^2 \right] = 0. \qquad (3.31)
$$

(ii) According to property (i), it can be deduced that a subsequence $(X^{n_k}(t))_{n_k}$ exists and converges to $X^{\mu}(t)$ $q.s.$ uniformly in t. For each $\mathbb{P} \in \mathcal{P}$, we have

$$
\begin{aligned}
\left| J^{\mathbb{P}}(u^{n_k}) - J^{\mathbb{P}}(\mu) \right| \leq\ & \mathbb{E}^{\mathbb{P}} \left[\int_0^T \int_{\mathbb{A}} \left| \mathcal{L}(t, X_t^{n_k}, \xi) - \mathcal{L}(t, X_t, \xi) \right| \delta_{u^{n_k}(t)}(d\xi) dt \right] \\
& + \mathbb{E}^{\mathbb{P}} \left[\left| \int_0^T \int_{\mathbb{A}} \mathcal{L}(t, X_t, \xi) \delta_{u^{n_k}(t)}(d\xi) dt \right.\right. \\
& \left.\left. - \int_0^T \int_{\mathbb{A}} \mathcal{L}(t, X_t^{\mu}, \xi) \mu_t(d\xi) dt \right| \right] \\
& + \mathbb{E}^{\mathbb{P}} \left[\left| \Psi(X_T^{n_k}) - \Psi(X_T^{\mu}) \right| \right]. \qquad (3.32)
\end{aligned}
$$

The first and third terms in the right-hand side converge to 0 as a result of the continuity and boundness assumptions on $\mathcal{L}$ and Ψ with respect to X that, and the second term on the right-hand side tends to 0, due to the continuity and the boundness of $\mathcal{L}$ in the variable ξ, and by the weak convergence of $\delta_{u^{n_k}(t)}(d\xi)dt$ to $\mu_t(d\xi)dt$, we use the dominated convergence theorem to conclude. Using Lemma 60 (stability results), we obtain for each $\mathbb{P} \in \mathcal{P}$,

$$
\lim_{k\to\infty} J^{\mathbb{P}}(u^{n_k}) = J^{\mathbb{P}}(\mu), \qquad (3.33)
$$

then

$$\lim_{k\to\infty} J(u^{n_k}) = J(\mu), \tag{3.34}$$

we have $J^{\mathbb{P}}(u) = J^{\mathbb{P}}(\delta_u)$, this yields $\inf_{u\in\mathcal{U}} J^{\mathbb{P}}(u) \geq \inf_{\mu\in R} J^{\mathbb{P}}(\mu)$. Given an arbitrary $\mu \in \mathcal{R}$. From Lemma 52 (G-Chattering Lemma), to obtain a sequence of strict controls $(u^{n_k}) \subset \mathcal{U}$ such that $\delta_{u^{n_k}(t)}(d\xi)dt$ converges weakly to $\mu_t(d\xi)dt$, we obtain

$$J^{\mathbb{P}}(\mu) = \lim_{n\to\infty} J^{\mathbb{P}}(u^n) \geq \inf_{u\in\mathcal{U}} J^{\mathbb{P}}(u), \tag{3.35}$$

since μ is arbitrary, we have:

$$\inf_{\mu\in R} J^{\mathbb{P}}(\mu) \geq \inf_{u\in\mathcal{U}} J^{\mathbb{P}}(u). \tag{3.36}$$

■

The ultimate effect is that the following theorem is proved. It should be noted that this conclusion applies to G-NSFDEs with an uncontrolled diffusion coefficient. We demonstrate the existence of an optimal solution for the relaxed control issue via the presence of G-relaxed optimal control for each $\mathbb{P} \in \mathcal{P}$ and a tightness argument.

Theorem 54 *For every $u \in \mathcal{U}$ and $\mu \in \mathcal{R}$, we have*

$$\inf_{u\in\mathcal{U}} J(u) = \inf_{\mu\in\mathcal{R}} J(\mu). \tag{3.37}$$

Moreover, there exists a G-relaxed optimal control $\widehat{\mu} \in \mathcal{R}$ such that

$$J(\widehat{\mu}) = \min_{\mu\in\mathcal{R}} J(\mu), \tag{3.38}$$

recall that

$$J(\mu) = \sup_{\mathbb{P}\in\mathcal{P}} J^{\mathbb{P}}(\mu), \tag{3.39}$$

where for each $\mathbb{P} \in \mathcal{P}$, the relaxed cost functional is given as follow

$$J^{\mathbb{P}}(\mu) = \mathbb{E}^{\mathbb{P}}\left[\int_0^T \int_{\mathbb{A}} \mathcal{L}(t, X_t^{\mu}, \xi)\mu_t(d\xi)\, dt + \Psi(X_T^{\mu})\right]. \tag{3.40}$$

Let $(\mu^n,\, X^{\mu^n})_{n\geq 0}$ be a minimizing sequence of $\inf_{\mu\in\mathcal{R}} J^{\mathbb{P}}(\mu)$ such that

$$\lim_{n\to\infty} J^{\mathbb{P}}(\mu^n) = \inf_{\mu\in\mathcal{R}} J^{\mathbb{P}}(\mu), \tag{3.41}$$

where X^{μ^n} is the unique solution of (3.7), corresponding to the random variables μ^n which belongs to the compact set $\mathbb{M}$.

The existence of a G-relaxed optimal control is proved that the sequence of of the processes distributions $(\mu^n,\, X^{\mu^n})_{n\geq 0}$ is tight for a given topology on the state space, and then demonstrating that a subsequence can be extracted that conforms in law to a process $(\mu,\, X^{\mu})$ that satisfies (3.12). To achieve the proof, we show that under some regularity conditions of $\left(J^{\mathbb{P}}(\mu^n)\right)_n$ converges to $J^{\mathbb{P}}(\widehat{\mu})$ which is equal to $\inf\limits_{\mu\in\mathcal{R}} J^{\mathbb{P}}(\mu)$ and then $(\widehat{\mu},\, X^{\widehat{\mu}})$ is optimal.

Lemma 55 *([4]) The sequence of distributions of the relaxed controls* $(\mu^n)_{n\geq 0}$ *is relatively compact in* $\mathbb{M}$*.*

Proof of Theorem 61. The relaxed controls μ^n are random variables in the compact set $\mathbb{M}$. Then by Prohorov's theorem the associated family of distribution $(\mu^n)_{n\geq 0}$ is tight on the space $\mathbb{M}$, then it is relatively compact in $\mathbb{M}$. Thus, there exists a subsequence $(\mu^{n_k}, X^{\mu^{n_k}})_{k\geq 0}$ of $(\mu^n, X^{\mu^n})_{n\geq 0}$ that weakly converges to $(\widehat{\mu}, X^{\widehat{\mu}})$ which solves (3.12). Using Skorohod's embedding theorem, the continuity and boundness assumptions of the functions $\mathcal{L}$ and Ψ, and Lebesgue Dominated Convergence Theorem, we finally obtain :

$$\inf_{\mu\in\mathcal{R}} J^{\mathbb{P}}(\mu) = \lim_{k\to\infty} J^{\mathbb{P}}(\mu^{n_k}) = J^{\mathbb{P}}(\widehat{\mu}).$$

Then, from Lemma 60 (stability results), for every $\mathbb{P}\in\mathcal{P}$ there exists a G-relaxed optimal control $\widehat{\mu}\in\mathcal{R}$ such that

$$\widehat{\mu}_{\mathbb{P}} = \arg\min_{\mu\in\mathcal{R}} J^{\mathbb{P}}(\mu).$$

Then, we conclude that

$$J(\widehat{\mu}) = \min_{\mu\in\mathcal{R}} J(\mu).$$

■

Remark 56 *The relaxed model is a genuine extension of the strict model since the infimum of the two cost functions is identical, and the G-relaxed model has an optimal solution, as indicated by previous research.*

Chapter 4

Optimal Control Problem for a Finite Horizon of G-SDDE and its Application in Economics

In this Chapter ([45]), the presence of a G-relaxed optimal control of n-dimensional G-SDDE is examined in this chapter. First, we demonstrate that optimal control exist for the finite horizon situation, and we provide a stochastic model in economics and its optimization with delay and randomness in the production cycle. In the last section, where the system's noise is high, we are unable to estimate the noise parameter. This will end in a G-SDDE. To minimize investment capital under labor assumptions, we build a probabilistic model of the HJB solution, stated as a system of decoupled G-FBSDDE.

4.1 Optimal control problem for a finite horizon of G-SDDE

We study the optimal control problem for a finite horizon $(T < \infty)$ of n-dimensional G-SDDE that corresponds to a particular case of G-NSFDE with neutral term $Q = 0$ and present state value at time t.

We are interested in the controlled system denoted by a G-SDDE and given by

$$\begin{cases} dX(t) = b(t, X(t), X(t-\tau), u(t), u(t-\tau))\, dt \\ \qquad\quad + \gamma(t, X(t), X(t-\tau), u(t), u(t-\tau))\, d\langle B\rangle_t \\ \qquad\quad + \sigma(t, X(t), X(t-\tau))\, dB_t, \quad t \in [0, T], \\ X(t) = \eta(t), \quad t \in [-\tau, 0], \tau > 0, \end{cases} \tag{4.1}$$

where $\eta=\{\eta(\theta)\}_{-\tau\leq\theta\leq 0}$ is $\widehat{\mathcal{F}}_0^{\mathcal{P}}-$adapted $C([-\tau,0];\mathbb{R}^n)$-value random variable, where $C=C([-\tau,0];\mathbb{R}^n)$ is a space of continuous functions equipped with the norm $|\phi|_C=\sup_{\theta\in[-\tau,0]}|\phi(\theta)|_C$, as well as $\eta\in M_G^2([-\tau,0];\mathbb{R}^n)$. Then, the integral equation is given by

$$\begin{aligned} X(t) &= \eta(0)+\int_0^t b(s,X(s),X(s-\tau),u(s),u(s-\tau))\,ds \\ &\quad+\int_0^t \gamma(s,X(s),X(s-\tau),u(s),u(s-\tau))\,d\langle B\rangle_s \\ &\quad+\int_0^t \sigma(s,X(s),X(s-\tau))\,dB_s, \end{aligned} \tag{4.2}$$

with $(\langle B\rangle_t)_{t\geq 0}$ denotes a process of quadratic variation for m-dimensional G-Brownian motion defined on $\left(\Omega,\mathcal{H},\widehat{\mathbb{E}}\right)$, and $u(t),u(t-\tau)\in\mathbb{A}$, are called the strict control variables for each $t\in[0,T]$. Moreovere, the functions

$$\begin{aligned} b &: [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{A}\times\mathbb{A}\to\mathbb{R}^n, \\ \gamma &: [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{A}\times\mathbb{A}\to\mathbb{R}^{m\times n}, \\ \sigma &: [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{m\times n}, \end{aligned}$$

as well as $\sigma(.,x,y)$, $b(.,x,y,u,u_\tau)$, $\gamma(.,x,y,u,u_\tau)\in M_G^2([0,T];\mathbb{R}^n)$ for each $x,y\in\mathbb{R}^n$ and for each strict control u.

4.2 Relaxed control of G-SDDE.

For that we consider the class of G-relaxed stochastic controls on $(\Omega,\mathcal{H},\widehat{\mathbb{E}})$.

Definition 57 (G-Relaxed stochastic control) *An $\mathbb{F}^{\mathcal{P}}$-progressively measurable random measure of the form $q(\omega,dt,d\xi)=\mu_t(\omega,d\xi)dt$ is a G-relaxed stochastic control on $(\Omega,\mathcal{H},\widehat{\mathbb{E}})$, such that*

$$\begin{aligned} X(t)=\eta(0)&+\int_0^t\int_{\mathbb{A}} b(s,X(s),X(s-\tau),\xi_1,\xi_2)\,\mu_s(d\xi)ds \\ &+\int_0^t\int_{\mathbb{A}} \gamma(s,X(s),X(s-\tau),\xi_1,\xi_2)\,\mu_s(d\xi)d\langle B\rangle_s \\ &+\int_0^t \sigma(s,X(s),X(s-\tau))\,dB_s, \end{aligned} \tag{4.3}$$

where $(\xi_1,\xi_2)=\xi\in\mathbb{A}$.

It is important to note that, the set $\mathcal{U}([0,T])=\mathcal{U}$ of 'strict' controls constituted of $\mathbb{F}^{\mathcal{P}}$-adapted processes each strict control u taking values in the set $\mathbb{A}$, can be considered as a G-relaxed control into the set $\mathcal{R}$ of G-relaxed controls via the mapping

$$\Phi:\quad \mathcal{U}\ni u\mapsto\Phi(u)(dt,d\xi)=\delta_{(u(t),u(t-\tau))}(d\xi)\,.dt\in\mathcal{R}, \tag{4.4}$$

where $\delta_{(u(t),u(t-\tau))}$ is a Dirac measure charging $u(t)$, $u(t-\tau)$ for each t.

To consider the control problem (4.2), we must first study the existence and uniqueness of solution to the following equation

$$
\begin{aligned}
X(t) &= \eta(0) + \int_0^t \int_{\mathbb{A}} b(s, X(s), X(s-\tau), \xi_1, \xi_2)\, \mu_s(d\xi) ds \\
&+ \int_0^t \int_{\mathbb{A}} \gamma(s, X(s), X(s-\tau), \xi_1, \xi_2)\, \mu_s(d\xi) d\langle B \rangle_s \\
&+ \int_0^t \sigma(s, X(s), X(s-\tau))\, dB_s, \quad t \in [0, T], \qquad (4.5)
\end{aligned}
$$

where $\mu_t(d\xi) = \delta_{(u(t),u(t-\tau))}(d\xi)$. To confirm the existence and uniqueness of the equation's solution, the following assumptions must be met (4.2).

(A_1) The functions b, γ and σ are bounded and Lipschitz continuous with respect to the space variables x, y uniformly in (t, u, u_τ). There exist $K_1, K_2 > 0$ such that

$$
\begin{aligned}
&|b(t,x,y,u,u_\tau) - b(t,x',y',u,u_\tau)|^2 + |\gamma(t,x,y,u,u_\tau) - \gamma(t,x',y',u,u_\tau)|^2 \\
&\quad + |\sigma(t,x,y) - \sigma(t,x',y')|^2 \leq K_1 \left(|x-x'|^2 + |y-y'|^2 \right),
\end{aligned}
$$

and

$$
|b(t,x,y,u,u_\tau)|^2 + |\gamma(t,x,y,u,u_\tau)|^2 + |\sigma(t,x,y)|^2 \leq K_2 \left(1 + |x|^2 + |y|^2 \right),
$$

for each $x, y, x', y' \in \mathbb{R}^n$ and $u, u_\tau \in \mathbb{A}$.

By the result of [28], under our assumptions (A_1) and (A_2) the G-SDDE (4.2) has a unique solution $(X_t^\mu)_{t\geq 0}$, for each fixed control.

4.3 Trajectory approximations and existence of G-relaxed optimal control

We consider a relaxed control problem (4.3). Let X^μ denotes the solution of equation (4.3) related to the G-relaxed control. Let establish the existence of a minimizer of the cost for finite horizon ($T < \infty$) corresponding to μ.

$$
J(\mu) = \widehat{\mathbb{E}} \left[\int_0^T \int_{\mathbb{A}} \mathcal{L}(t, X^\mu(t), X^\mu(t-\tau), \xi_1, \xi_2) \mu_t(d\xi) dt + \Psi(X^\mu(T)) \right],
$$

where the functions,

$$\begin{aligned} \mathcal{L} &: [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{A} \times \mathbb{A} \to \mathbb{R}, \\ \Psi &: \mathbb{R}^n \longrightarrow \mathbb{R}, \end{aligned}$$

fulfill the following important assumption

(A_2) The functions $\mathcal{L}$, Ψ are bounded, and the coefficient Ψ is Lipschitz continuous, and $\mathcal{L}$ is Lipschitz continuous with respect to the space variables x, y uniformly in time and control (t, u, u_τ).

We recall that in the strict control problem

$$J(u) = \widehat{\mathbb{E}}\left[\int_0^T \mathcal{L}(t, X^u(t), X^u(t-\tau), u(t), u(t-\tau))\, dt + \Psi(X^u(T))\right]. \quad (4.6)$$

From the set $\mathcal{U}$,

$$\begin{aligned} X^u(t) &= \eta(0) + \int_0^t b(s, X^u(s), X^u(s-\tau), u(s), u(s-\tau)) ds \\ &\quad + \int_0^t \gamma(s, X^u(s), X^u(s-\tau), u(s), u(s-\tau)) d\langle B\rangle_s \\ &\quad + \int_0^t \sigma(s, X^u(s), X^u(s-\tau))\, dB_s, \end{aligned} \quad (4.7)$$

then, we have

$$\begin{aligned} X^\mu(t) &= \eta(0) + \int_0^t \int_{\mathbb{A}} b(s, X^\mu(s), X^\mu(s-\tau), \xi_1, \xi_2)\mu_s(d\xi) ds \\ &\quad + \int_0^t \int_{\mathbb{A}} \gamma(s, X^\mu(s), X^\mu(s-\tau), \xi_1, \xi_2)\mu_s(d\xi) d\langle B\rangle_s \\ &\quad + \int_0^t \sigma(s, X^\mu(s), X^\mu(s-\tau))\, dB_s. \end{aligned} \quad (4.8)$$

Moreover, for each $\mu \in \mathcal{R}$,

$$\chi^\mu := \int_0^T \int_{\mathbb{A}} \mathcal{L}(t, X^\mu(t), X^\mu(t-\tau), \xi_1, \xi_2)\mu_t(d\xi) dt + \Psi(X^\mu(T)) \in \mathbb{L}_G^1(\Omega_T). \quad (4.9)$$

We employ the relaxed control issue to introduce the definition of stable convergence in order to validate the G-Chattering lemma, which asserts that each G-relaxed control in $\mathcal{R}$ may be approximated by a sequence of strict controls from $\mathcal{U}$.

Definition 58 (stable convergence) *([21]) Let* $\mu^n, \mu \in \mathcal{R}, n \in \mathbb{N}^*$*, If for any* $f : [0,T] \times \mathbb{A} \to \mathbb{R}^n$ *continuous function, We have a stable convergence, then*

$$\lim_{n\to\infty} \int_{[0,T]\times\mathbb{A}} f\left(t, \xi_1, \xi_2\right) \mu^n\left(dt, d\xi\right) = \int_{[0,T]\times\mathbb{A}} f\left(t, \xi_1, \xi_2\right) \mu\left(dt, d\xi\right). \tag{4.10}$$

Lemma 59 (G-Chattering lemma) *Let the process* $(\mu_t)_{t\geq 0}$ *is an* $\mathbb{F}^{\mathcal{P}}$*-progressively measurable with values in* $\mathcal{P}(\mathbb{A})$*. Then there exists a sequences* $(u^n(t), u^n\left(t-\tau\right))_{n\geq 0}$ *of* $\mathbb{F}^{\mathcal{P}}$*-progressively measurable processes with values in* $\mathbb{A}$*, such that*

$$\mu_t^n(d\xi)dt = \delta_{(u^n(t),u^n(t-\tau))}\left(d\xi\right)dt \to \mu_t(d\xi)dt,$$

converges in terms of stable convergence (thus weakly).

Proof. Given the G-relaxed control μ which is $\mathbb{F}^{\mathcal{P}}$-progressively measurable, the precise pathwise development of the approximating sequence $(\delta_{(u^n(t),u^n(t-\tau))}\left(d\xi\right)dt)_{n\geq 0}$ of G-relaxed control $\mu_t(d\xi)dt$ in $\mathcal{R}$ (see lemma after theorem 3, [29]), which easily needs extend to consider the strict controls $(u^n)_n$ $\mathbb{F}^{\mathcal{P}}$-progressively measurable. ■

Let X^n the corresponding solution of G-SDDE (4.2), associated with u^n $\left(\text{or } \delta_{(u^n(t),u^n(t-\tau))}(d\xi)\right)$, and satisfy that

$$\begin{aligned} X^n\left(t\right) &= \eta(0) + \int_0^t b(s, X^n\left(s\right), X^n\left(s-\tau\right), u^n\left(s\right), u^n\left(s-\tau\right))ds \\ &\quad + \int_0^t \gamma(s, X^n\left(s\right), X^n\left(s-\tau\right), u^n\left(s\right), u^n\left(s-\tau\right))d\left\langle B\right\rangle_s \\ &\quad + \int_0^t \sigma\left(s, X^n\left(s\right), X^n\left(s-\tau\right)\right)dB_s. \end{aligned} \tag{4.11}$$

The following important lemma prove the stability results for the G-SDDE (4.8),

and gives $J(\mu) = \sup\limits_{\mathbb{P}\in\mathcal{P}} J^{\mathbb{P}}(\mu)$ for every $\mathbb{P} \in \mathcal{P}$.

Lemma 60 (stability results) *Suppose that* b, γ *and* σ *satisfy assumption* (A_1)*. Let* μ *denotes a* G*-relaxed control, and let* (u^n) *represents a sequence defined in previous lemma. Then, it holds that*

(i) *For each* $\mathbb{P} \in \mathcal{P}$*, it follows that*

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T} \left|X^n(t) - X^\mu(t)\right|^2\right] = 0, \tag{4.12}$$

and

$$\lim_{n\to\infty} \widehat{\mathbb{E}}\left[\sup_{0\leq t\leq T} \left|X^n(t) - X^\mu(t)\right|^2\right] = 0. \tag{4.13}$$

(ii) Let $J(u^n)$ and $J(\mu)$ are the corresponding cost functionals to u^n and μ respectively. Then, there exists a subsequence (u^{n_k}) of (u^n) such that

$$\lim_{k\to\infty} J(u^{n_k}) = J(\mu), \tag{4.14}$$

and, for every $\mathbb{P} \in \mathcal{P}$

$$\lim_{k\to\infty} J^{\mathbb{P}}(u^{n_k}) = J^{\ \mathbb{P}}(\mu). \tag{4.15}$$

Furthermore,

$$\inf_{u\in\mathcal{U}} J^{\mathbb{P}}(u) = \inf_{\mu\in\mathcal{R}} J^{\mathbb{P}}(\mu), \tag{4.16}$$

then, there exists a G-relaxed optimal control $\widehat{\mu}_{\mathbb{P}} \in \mathcal{R}$ such that

$$J^{\mathbb{P}}(\widehat{\mu}_{\mathbb{P}}) = \inf_{\mu\in\mathcal{R}} J^{\mathbb{P}}(\mu). \tag{4.17}$$

Since, by using the singularity for each $\mathbb{P} \in \mathcal{P}$, the G-SDDE (4.8) becomes a standard SDDE, this result was confirmed by the technique in [3].

Proof.

(i) We set

$$\zeta_n := \sup_{0\le t\le T} |X^n(t) - X^{\mu}(t)|^2,$$

and note that $\zeta_n \in \mathbb{L}^1_G(\Omega_T)$ for each $n \geq 1$. If there is indeed a $\eta > 0$ such that

$$\widehat{\mathbb{E}}[\zeta_n] \geq \eta, \quad n \in \mathbb{N}^*,$$

we can find a probability $\mathbb{P}_n \in \mathcal{P}$ such that

$$\mathbb{E}^{\mathbb{P}_n}[\zeta_n] \geq \eta - \frac{1}{n}, \quad n \in \mathbb{N}^*.$$

There exists a subsequence $\{\mathbb{P}_{n_k}\}_{k=1}^{\infty}$ that converges weakly to some $\mathbb{P} \in \mathcal{P}$, according to $\mathcal{P}$ is weakly compact. Then, we have

$$\lim_{j\to\infty} \mathbb{E}^{\mathbb{P}}[\zeta_{n_j}] = \lim_{j\to\infty}\lim_{k\to\infty} \mathbb{E}^{\mathbb{P}_{n_k}}[\zeta_{n_j}] \geq \liminf_{k\to\infty} \mathbb{E}^{\mathbb{P}_{n_k}}[\zeta_{n_k}] \geq \eta. \tag{4.18}$$

This is a contradiction to the fact $\lim_{j\to\infty} \mathbb{E}^{\mathbb{P}}[\zeta_{n_j}] = 0$ from (4.12). The proof of (4.12) is based on G-BDG inequalities and the standard Gronwall inequality as well as the Dominated Convergence theorem, according to stable convergence in Lemma 58 of $\delta_{(u^n(t),u^n(t-\tau))}(d\xi)\,dt$ converges to $\mu_t(d\xi)\,dt$, and the proof method does not extend to proving (4.13) because the Dominated Convergence theorem (and even the celebrated Fatou's lemma) is no longer valid under sublinear expectation, but the G-BDG inequalities hold true for G-stochastic integrals and G-SDDEs.

(*ii*) Assume that $\mathcal{L}$ and Ψ satisfy assumption (A_2). We have $dt\delta_{(u^n(t),u^n(t-\tau))}(d\xi)$ converges weakly to $dt\mu_t(d\xi)$ quasi-surely. Then, there exists a subsequence (u^{n_k}) of (u^n), we obtain

$$\lim_{k\to\infty} J(u^{n_k}) = J(\mu). \tag{4.19}$$

By using Proposition 17 in [20] and (4.13) it follows that, there exists a subsequence $(X^{n_k}(t))_{n_k}$ that converges quasi-surely to $X^\mu(t)$ *i.e.* $\mathbb{P}$-*a.s.*, for all $\mathbb{P}\in\mathcal{P}$, uniformly in t. We can use (4.13) for every $\mathbb{P}\in\mathcal{P}$, to obtain that

$$\lim_{k\to\infty} J^{\mathbb{P}}(u^{n_k}) = J^{\mathbb{P}}(\mu).$$

From the notation (4.9), we can note that

$$J(u^{n_k}) = \widehat{\mathbb{E}}[\chi^{u^{n_k}}],$$

and

$$J(\mu) = \widehat{\mathbb{E}}[\chi^\mu],$$

where both $\chi^{u^{n_k}}$, $\chi^\mu \in \mathbb{L}^1_G(\Omega_T)$. If there is some $\eta > 0$ then

$$\widehat{\mathbb{E}}[\chi^{u^{n_k}}] \geq \widehat{\mathbb{E}}[\chi^\mu] + \eta, \quad n_k \geq l, l+1, ...,$$

thus, we can find a probability measure $\mathbb{P}_m \in \mathcal{P}$ such that

$$\mathbb{E}^{\mathbb{P}_m}[\chi^{u^{n_k}}] \geq \widehat{\mathbb{E}}[\chi^\mu] + \eta - \frac{1}{m}.$$

Then we can find a subsequence $\{\mathbb{P}_{m_k}\}_{k=1}^\infty$ under a weakly compact $\mathcal{P}$, that converges to $\mathbb{P}\in\mathcal{P}$. We have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[\chi^\mu] &= \lim_{k\to\infty} \mathbb{E}^{\mathbb{P}_{m_k}}[\chi^\mu] = \lim_{k\to\infty}\lim_{j\to\infty} \mathbb{E}^{\mathbb{P}_{m_k}}[\chi^{u^{n_j}}] \geq \liminf_{j\to\infty} \mathbb{E}^{\mathbb{P}_{m_j}}[\chi^{u^{n_j}}] \\
&\geq \liminf_{j\to\infty}\left(\widehat{\mathbb{E}}[\chi^\mu] + \eta - \frac{1}{m_j}\right) \\
&= \widehat{\mathbb{E}}[\chi^\mu] + \eta.
\end{aligned}$$

Thus,

$$\mathbb{E}^{\mathbb{P}}[\chi^\mu] \geq \widehat{\mathbb{E}}[\chi^\mu] + \eta,$$

We get quite a contradiction from the definition of the sublinear expectation. Therefore,

$$\lim_{k\to\infty} J(u^{n_k}) \leq J(\mu).$$

By using (4.15) to prove that

$$\lim_{k\to\infty} J(u^{n_k}) \geq J(\mu).$$

We have

$$\begin{aligned}
\lim_{k\to\infty} J(u^{n_k}) &\geq \lim_{k\to\infty} J^{\mathbb{P}}(u^{n_k}), \quad \mathbb{P}\in\mathcal{P} \\
&= J^{\mathbb{P}}(\mu), \quad \mathbb{P}\in\mathcal{P}.
\end{aligned}$$

Therefore, $\lim_{k\to\infty} J(u^{n_k}) \geq J(\mu)$. ■

The fundamental finding of our problem is the following theorem, which asserts that the two issues have anidentical infimum values for expected costs.

Theorem 61 *For every $u \in \mathcal{U}$ and $\mu \in \mathcal{R}$, we have*

$$\inf_{u\in\mathcal{U}} J(u) = \inf_{\mu\in\mathcal{R}} J(\mu). \tag{4.20}$$

Furthermore, there is $\widehat{\mu} \in \mathcal{R}$ a G-relaxed optimal control such that

$$J(\widehat{\mu}) = \min_{\mu\in\mathcal{R}} J(\mu), \tag{4.21}$$

recall that

$$J(\mu) = \sup_{\mathbb{P}\in\mathcal{P}} J^{\mathbb{P}}(\mu), \tag{4.22}$$

where the relaxed cost functional is given for each $\mathbb{P} \in \mathcal{P}$ as

$$J^{\mathbb{P}}(\mu) = \mathbb{E}^{\mathbb{P}}\left[\int_0^T \int_{\mathbb{A}} \mathcal{L}(t, X^{\mu}(t), X^{\mu}(t-\tau), \xi_1, \xi_2)\mu_t(d\xi)dt + \Psi(X^{\mu}(T))\right]. \tag{4.23}$$

To prove (4.20), using Lemma **59** of G-Chattering lemma in and Lemma **60** of stability results for the G-SDDE (4.8). According to Lemma **59**, given a G-relaxed control $\mu \in \mathcal{R}$, and there is a sequence $(u^n)_n \in \mathcal{U}$ of strict controls such that $\delta_{(u^n(t),u^n(t-\tau))}(d\xi)dt$ converges weakly to $\mu_t(d\xi)dt$ quasi-surely i.e. $\mathbb{P}$-*a.s.*, for all $\mathbb{P} \in \mathcal{P}$. The existence of a G-relaxed optimal control for each $\mathbb{P} \in \mathcal{P}$ and a tightness argument is used to prove (4.21).

Proof. From (4.4) and (4.19) we can easily obtain that

$$\inf_{u\in\mathcal{U}} J(u) \leq \inf_{\mu\in\mathcal{R}} J(\mu).$$

Therefore, for every $u \in \mathcal{U}$, $\delta_u \in \mathcal{R}$, we have

$$J(u) = J(\delta_u) \geq \inf_{\mu\in\mathcal{R}} J(\mu).$$

Hence,

$$\inf_{u\in\mathcal{U}} J(u) \geq \inf_{\mu\in\mathcal{R}} J(\mu),$$

which confirms (4.20). As $\mathcal{L}$ and Ψ are continuous and bounded, we now go back to the evidence of the enssured of existence of G-relaxed optimal control for each $\upsilon \in \mathcal{R}$.

$$\chi^{\mu} := \int_0^T \int_{\mathbb{A}} \mathcal{L}(t, X^{\upsilon}(t), X^{\upsilon}(t-\tau), \xi_1, \xi_2)\mu_t(d\xi)dt + \Psi(X^{\upsilon}(T)) \in \mathbb{L}^1_G(\Omega_T).$$

By using Lemma 36, we can deduce that for each $\upsilon \in \mathcal{R}$,

$$\lim_{n\to\infty} J^{\mathbb{P}_n}(\upsilon) = J^{\mathbb{S}}(\upsilon). \tag{4.24}$$

Then, the sequence $\{\mathbb{P}_n\}_{n=1}^{\infty} \in \mathcal{P}$ converges weakly to $\mathbb{S} \in \mathcal{P}$. Assume there is an $\varsigma > 0$ such that, for each $\upsilon \in \mathcal{R}$,

$$J(\upsilon) \geq \inf_{\mu \in \mathcal{R}} J(\mu) + \varsigma.$$

Next, by according to Lemma **60**, for each $\mathbb{P} \in \mathcal{P}$ there exists a G-relaxed optimal control $\widehat{\mu} \in \mathcal{R}$ such that

$$\widehat{\mu}_{\mathbb{P}} = \arg\left(\min_{\mu \in \mathcal{R}} J^{\mathbb{P}}(\mu)\right),$$

we get that

$$\begin{aligned} J(\upsilon) &\geq \sup_{\mathbb{P} \in \mathcal{P}} \left(\inf_{\mu \in \mathcal{R}} J^{\mathbb{P}}(\mu)\right) + \varsigma \\ &= \sup_{\mathbb{P} \in \mathcal{P}} J^{\mathbb{P}}(\widehat{\mu}_{\mathbb{P}}) + \varsigma. \end{aligned}$$

However, there exists $\mathbb{P}_n \in \mathcal{P}$ for every $n \geq 1$, such that

$$J^{\mathbb{P}_n}(\upsilon) \geq J(\upsilon) + \frac{1}{n}.$$

We can extract a subsequence $\{\mathbb{P}_{n_j}\}_{j=1}^{\infty} \in \mathcal{P}$, from the sequence $\{\mathbb{P}_n\}_{n=1}^{\infty} \in \mathcal{P}$ being weakly compact, which converges weakly to some $\mathbb{S} \in \mathcal{P}$. Thus, As a result of (4.24) for each $\upsilon \in \mathcal{R}$, it follows that

$$J^{\mathbb{S}}(\upsilon) = \lim_{j \to \infty} J^{\mathbb{P}_{n_j}}(\upsilon) \geq \sup_{\mathbb{P} \in \mathcal{P}} J^{\mathbb{P}}(\widehat{\mu}_{\mathbb{P}}) + \varsigma.$$

Particularly, we obtain for a given $\upsilon^{\mathbb{S}} \in \mathcal{R}$,

$$J^{\mathbb{S}}(\upsilon^{\mathbb{S}}) \geq \sup_{\mathbb{P} \in \mathcal{P}} J^{\mathbb{S}}(\upsilon^{\mathbb{S}}) + \varsigma,$$

which is a contradiction with the fact that $\varsigma > 0$. ■

4.4 Economics model

As an implementation of our theoretical finding, defining the rate of change of capital K and labor L in a market adopting Ramsey's in 1928 system of ODEs (for more details, see [66]), given by

$$\begin{cases} \frac{dK_s}{ds} = P_s - C_s, \\ \frac{dL_s}{ds} = a_s L_s, \end{cases} \tag{4.25}$$

where

P : production,

C : consumption rates,

a_s : the rate of growth of labor (population).

In [33] the production, capital and labor related by Cobb-Douglas formula as follow

$$P_s = DK_s^{\alpha} L_s^{\beta}, \tag{4.26}$$

where $D, \alpha, \beta > 0$ are a constants. If $\alpha = \beta = 1$, we get the linearity of (4.26) under certain scenario, which we will assume throughout our problem. The labor is constant given by $L_s = L_0$, which verify a specific markets, or we take in many years a relatively short time intervals. As a result, the production rate and capital have a dependent formela $P_s = MK_s$, where $M = DL_0$.

The major causes for affecting economic development are massive random perturbations (big noise) caused by political decisions, wars, and atmospheric conditions, they are the real reasons for influencing the change of the economy. The production rate may therefore be calculated as follows:

$$P_s = MK_s + \text{"}\textit{big noise}\text{"}. \tag{4.27}$$

By using (4.27) in (4.25), we obtain

$$\frac{dK_s}{ds} = MK_s - C_s + \text{"}\textit{big noise}\text{"}.$$

Then, we obtain the following system

$$dK_s = [MK_s - C_s]ds + \gamma\sigma(K_s)dW_s, \tag{4.28}$$

where W_s is a typical Brownian motion with γ is unknown parameter and the coefficient σ varies on K as it is greatly influenced by variables that change the economy because of the significant amount of noise. In many economic conditions during the war, such as the rise in oil and the impact on wheat production, the essential period is usually required for transition owing to the influence and change of the economy, such as the length of the production process.

As a consequence, given the change in the rate of capital K, the most reliable assumption is that the investment was made at time $s - \tau$, where s is the current time and is the length of the cycle needed to generate working funds during wartime delays and political decision delays, and as time passes, the noise increases.

$$dK_s = [MK_{s-\tau} - C_s]ds + \gamma\sigma(K_{s-\tau})dW_s, \tag{4.29}$$

which means that we can not estimate the value of γ, the only infromation that we can know is that gamma in $[\sigma_{min}, \sigma_{max}]$, for some $\sigma_{min}, \sigma_{max} \in \mathbb{R}_+^*$, the idea is to consider the worst case senario; i.e.,

$$\sup_{\gamma \in [\sigma_{min}, \sigma_{max}]} \mathbb{E}^{\mathbb{P}_\gamma}[\phi(K)] = \widehat{\mathbb{E}}[\phi(\widehat{K})],$$

where $\mathbb{E}^{\mathbb{P}_\gamma}$ is an expected valued over $\mathbb{P}_\gamma$, with $\mathbb{P}_\gamma$ is the probability indexed induced by (4.29), and $\widehat{\mathbb{E}}$ is the non-linear expectation, with ϕ is a given function, and $\widehat{K}$ is the solution of the following equation

$$d\widehat{K}_s = [M\widehat{K}_{s-\tau} - C_s]ds + \sigma(\widehat{K}_{s-\tau})dB_s, \tag{4.30}$$

where B_s is the G-Brownian motion follow $\mathcal{N}(0,\ s[\sigma^2_{min}, \sigma^2_{max}])$. It is necessary to study how we make decisions when we are often faced with probabilistic uncertainty. The main question is to control the consumption rate C by a control $u_\cdot$, i.e., it is of the form u_sC_s, then

$$d\widehat{K}_s = [M\widehat{K}_{s-\tau} - u_sC_s]ds + \sigma\left(\widehat{K}_{s-\tau}\right)dB_s, \tag{4.31}$$

with

$$\widehat{K}_\theta = k\left(\theta\right), \quad t-\tau \le \theta \le t. \tag{4.32}$$

We recommend investigating the modified Ramsey model uncertainty with delay introduced by (4.31) and (4.32). We aim to minimize the investment capital under the above mentioned assumptions by employing the cost function below.

$$J\left(u; t, k\right) = \widehat{\mathbb{E}}\left[\left[\widehat{K}_T - \alpha\right]^2 + \frac{1}{2}\int_t^T u_r^2 dr\right],$$

The objective is to bring our capital $\widehat{K}$ in the future at time T closer to α with minimal energy, where α is a random variable.

4.4.1 Optimization problem

To solve our optimal control, we join its HJB with a numerically solved G-FBSDDE. In fact, for random variables and processes in [64], the condition of quasi-continuity is not required. As a consequence, see [64] may be utilized in practical conditions that do not take place in quasi-continuous space. In the context of super-linear expectation, we assume some independence among components of G-Brownian motion and construct an obvious kind of HJB equation with a vector-valued control variable. It's obvious that the following conclusion employs stochastic control and the uncertainty volatility model.

To solve our optimal control, we connect its HJB with a G-FBSDDE that we solve numerically. In fact, the condition of quasi-continuity is not required for random variables and processes in [64].

Let define $(K_t)_{t\in[0,T]}$, as a C-valued stochastic processes from the space $S^p([0,T];C)(1 \le p < \infty)$, and the norm defined by

$$\|K\|^p_{S^p([0,T];C)} = \widehat{\mathbb{E}}[\sup_{t\in[0,T]} |K_t|^p_C] = \widehat{\mathbb{E}}[\sup_{t\in[0,T]}\sup_{\theta\in[-\tau,0]} |K_t(\theta)|^p] < \infty, \tag{4.33}$$

with C-continuous paths.

The gradient $\nabla_k V(t,k)$ at $(t,k) \in [0,T] \times C([-\tau,0];\mathbb{R}^n)$ is an n-tuple of finite Borel measures on $[-\tau,0]$. Let define

$$\nabla_0 V(t,k) = \nabla_k V(t,k)\{(0)\}, \tag{4.34}$$

with $|\nabla_k V(t,k)|$ is a total variation norm and define where $\nabla_0 V(t,k)$ is a vector in $\mathbb{R}^n$ which represent the masse at point 0 of the component of $\nabla_0 V(t,k)$, such that we have the following continuous map on $[0,T] \times C$ for every $g \in C$

$$(t,k) \longrightarrow \langle \nabla_k V(t,k), g\rangle = \int_{[-\tau,0]} g(\theta) \cdot \nabla_k V(t,k)(d\theta), \tag{4.35}$$

for more details see [32].

Then the value function define by

$$V(t,k) = \inf_u \widehat{\mathbb{E}} \left[[K_T - \alpha]^2 + \frac{1}{2} \int_t^T u_r^2 dr \mid_{K_t^u = k} \right]. \tag{4.36}$$

Then, the HJB equation given by

$$\begin{cases} \frac{\partial V(t,k)}{\partial t} + \inf_u \left\{ LV(t,k) + \frac{1}{2} u^2 \right\} = 0, \\ V(T,k) = [k-\alpha]^2, \end{cases} \tag{4.37}$$

where u is a control variable is selected dynamically within $\mathbb{R}^n$.

The infinitesimal generator L given by

$$L = \frac{1}{2} \sigma^2(k) \nabla_0^2 + Mk\nabla_0 - uC_t \nabla_0, \tag{4.38}$$

then, we get

$$\begin{aligned} &\frac{\partial V(t,k)}{\partial t} + \frac{1}{2}\sigma^2(k) \nabla_0^2 V(t,k) + Mk\nabla_0 V(t,k) \\ &+ \inf_u \left\{ -u\, C_t \nabla_0 V(t,k) + \frac{1}{2} u_t^2 \right\} = 0. \end{aligned} \tag{4.39}$$

Hence, the infinum is achieved when

$$u^* - C_t \nabla_0 V(t,k) = 0.$$

Therefore

$$u^* = C_t \nabla_0 V(t,k).$$

As a converse of Eq. (4.39), we have

$$\begin{cases} \frac{\partial V(t,k)}{\partial t} + \frac{1}{2}\sigma^2(k) \nabla_0^2 V(t,k) + Mk\nabla_0 V(t,k) - \frac{1}{2} C_t^2 (\nabla_0 V(t,k))^2 = 0, \\ V(T,k) = [k-\alpha]^2, \end{cases} \tag{4.40}$$

where $V \in C_b^{1,2}$ define the processes $Y_s^{t,k} = V(s,k)$ and

$$Z_s^{t,k} = \sigma(k)\nabla_0 V(s,k),$$

where σ is non-degenerate diffusion.

$$Y_s^{t,k} = \left[\widehat{K}_T^{t,k} - \alpha\right]^2 - \frac{1}{2}\int_s^T \left[C_r Z_r^{t,k}\sigma^{-1}\left(\widehat{K}_{r-\tau}^{t,k}\right)\right]^2 dr + \int_s^T Z_r^{t,k} dB_r,$$

and the G-FBSDDEs defined on $[t,T] \subseteq [0,T]$: for $s \in [t,T]$,

$$\begin{cases} \widehat{K}_s^{t,k} = k(t) + \int_t^s \left(M\widehat{K}_{r-\tau}^{t,k}\right) dr + \int_t^s \sigma\left(\widehat{K}_{r-\tau}^{t,k}\right) dB_r, \\ Y_s^{t,k} = \left[\widehat{K}_T^{t,k} - \alpha\right]^2 - \frac{1}{2}\int_s^T \left[C_r Z_r^{t,k}\sigma^{-1}\left(\widehat{K}_{r-\tau}^{t,k}\right)\right]^2 dr + \int_s^T Z_r^{t,k} dB_r. \end{cases} \tag{4.41}$$

4.4.2 Numerical study

In this section, we report numerical findings for G-FBSDE with delay (4.41), based on a variation of the least-squares Monte Carlo algorithm approach suggested in [10]. It has two phases, as do most numerical methods for G-FBSDDE:

- G-FBSDDE explicit time discretization with delay.
- The approach for approximation of conditional expectations.

In first, for the time discretization we consider a partition of the interval $[-\tau, T]$ as $\pi = \{t_0, t_1, ..., t_N\}$, *i.e.*, $-\tau = t_0 < t_1 < t_2 < t_{N_\tau} = 0 < t_{N_\tau+1} <\cdots< t_N = T$. For the pair $(\widehat{K}_s, Y_s)$. We motivate a natural time discretization that goes backwards in time. Defining $\Delta t_n = t_{n+1} - t_n$, $\Delta B_{t_n} = B_{t_{n+1}} - B_{t_n}$ for $t_n \in \{t_{N_\tau}, t_{N_\tau+1}, \cdots, t_N\}$, we have

$$\begin{cases} \widehat{K}_{t_n} = \widehat{K}_{t_{n+1}} + M\widehat{K}_{t_n-\tau}\Delta t_n + \sigma\left(\widehat{K}_{t_n-\tau}\right)\Delta B_{t_n}, \\ Y_{t_n} = Y_{t_{n+1}} + \frac{1}{2}\left[C_{t_n} Z_{t_{n+1}}\sigma^{-1}\left(\widehat{K}_{t_n-\tau}\right)\right]^2 \Delta t_n - Z_{t_{n+1}}\Delta B_{t_n}, \end{cases} \tag{4.42}$$

where $Y_{t_N} = \left[\widehat{K}_{t_N} - \alpha\right]^2$.

By taking a conditional expectation on both sides of G-BSDDE in (4.42) to get an approximation of Y_{t_n}, given $Y_{t_{n+1}}$ from the joint process $(\widehat{K}_{t_n}, Y_{t_n})$ which is adapted to the filtration generated by $(B_r)_{0\le r\le s}$, and by using the property of symmetric martingale, we obtain that

$$\begin{aligned} Y_{t_n} = \widehat{\mathbb{E}}\left[Y_{t_n}|\mathcal{F}_{t_n}\right] &= \widehat{\mathbb{E}}\left[(Y_{t_{n+1}} + \frac{1}{2}\left[C_{t_n} Z_{t_n}\sigma^{-1}\left(\widehat{K}_{t_n-\tau}\right)\right]^2 \Delta t - Z_{t_n}\Delta B_{t_n})|\mathcal{F}_{t_n}\right] \\ &= \widehat{\mathbb{E}}\left[(Y_{t_{n+1}} + \frac{1}{2}\left[C_{t_n} Z_{t_n}\sigma^{-1}\left(\widehat{K}_{t_n-\tau}\right)\right]^2 \Delta t|\mathcal{F}_{t_n}\right] \\ &+ \widehat{\mathbb{E}}\left[-Z_{t_n}\Delta B_{t_n}|\mathcal{F}_{t_n}\right] \\ &= \widehat{\mathbb{E}}\left[(Y_{t_{n+1}} + \frac{1}{2}\left[C_{t_n} Z_{t_n}\sigma^{-1}\left(\widehat{K}_{t_n-\tau}\right)\right]^2 \Delta t)|\mathcal{F}_{t_n}\right], \end{aligned} \tag{4.43}$$

from (4.43) we have the time discretization $(Y_{t_n}^{\pi}, Z_{t_n}^{\pi})$ for (Y, Z) by

$$Y_{t_n}^{\pi} = \widehat{\mathbb{E}}\left[Y_{t_n}^{\pi}|\mathcal{F}_{t_n}\right] = \widehat{\mathbb{E}}\left[(Y_{t_{n+1}}^{\pi} + \frac{1}{2}\left[C_{t_n}^{\pi} Z_{t_n}^{\pi} \sigma^{-1}\left(\widehat{K}_{t_n-\tau}^{\pi}\right)\right]^2 \Delta t)|\mathcal{F}_{t_n}\right].$$

Let use the value of $Z_s = \sigma(k)\nabla_0 V(s,k)$ $\left(\text{resp. } Z_{t_{n+1}} = \sigma(k_{t_n})\nabla_0 V(t_n, k_{t_n})\right)$ to get Y_{t_n}.

By least squar method [44],

$$Y_{t_n} = \underset{Y\in L^2}{\arg\min}\widehat{\mathbb{E}}\left[\left|Y\left(\widehat{K}_{t_n}\right) - Y_{t_{n+1}}^{\pi} - \frac{1}{2}\left[C_{t_n}^{\pi} Z_{t_{n+1}}^{\pi}\sigma^{-1}\left(\widehat{K}_{t_n-\tau}^{\pi}\right)\right]^2 \Delta t\right|^2\right], \tag{4.44}$$

Y iterates through all measurable functions with $\widehat{\mathbb{E}}\left[\left|Y\left(\widehat{K}_{t_n}\right)\right|^2\right] < \infty$.

Finally, we get that

$$Y_{t_n} = \underset{Y=Y\left(\widehat{K}_{t_n}\right)}{\arg\min}\ \widehat{\mathbb{E}}\left[\left|Y\left(\widehat{K}_{t_n}\right) - Y_{t_{n+1}}^{\pi} - \frac{1}{2}\left[C_{t_n}^{\pi} Z_{t_{n+1}}^{\pi}\sigma^{-1}\left(\widehat{K}_{t_n-\tau}^{\pi}\right)\right]^2 \Delta t\right|^2\right]. \tag{4.45}$$

By using $Y\left(\widehat{K}_{t_n}\right) \approx \sum_{i=1}^{L_1}\varpi_i(t_n)\chi_i\left(\widehat{K}_{t_n}\right)$, for some $L_1 \in \mathbb{N}$, where $\varpi_1(.), \ldots$

$, \varpi_{L_1}(.) \in \mathbb{R}$ is the coefficients of the basis $\chi_1, \ldots, \chi_{L_1} : \mathbb{R}^n \to \mathbb{R}$. By least-squar method

$$\varpi^{(L_1,L_2)}(t_n) = \underset{\alpha\in\mathbb{R}^{L_1}}{\arg\min}\left\|W_{t_n}^{(L_1,L_2)}\varpi - \mathcal{A}_{t_n}\right\|^2, \tag{4.46}$$

where $W_{t_n}^{(L_1,L_2)} = \left(\chi_i\left(\widehat{K}_{t_n}^{\pi}\right)\right)_{i=1,\ldots,L_2, i=1,\ldots,L1}$. Then, we get that

$$\varpi^{(L_1,L_2)}(t_n) = \left(\left(W_{t_n}^{(L_1,L_2)}\right)^T\left(W_{t_n}^{(L_1,L_2)}\right)\right)^{-1}\left(W_{t_n}^{(L_1,L_2)}\right)^T\mathcal{A}_{t_n},$$

is the solution to the least-squares problem (4.46), and for further information (*see* [43]).

To solve the forward backward G-SDDE, we must first simulate the increment of the G-Brownian motion, and then adopt the same approach as described in [76] to simulate the density (resp. distribution) of the G-Normal BM by using the finite difference method to simulate its equivalent G-PDE.

The simulated result in figure 4.1 (resp. Figure 4.2) illustrate the G-Normal density (resp. distribution) with $\sigma_{min} = 0.75$ and $\sigma_{\max} = 1.25$.

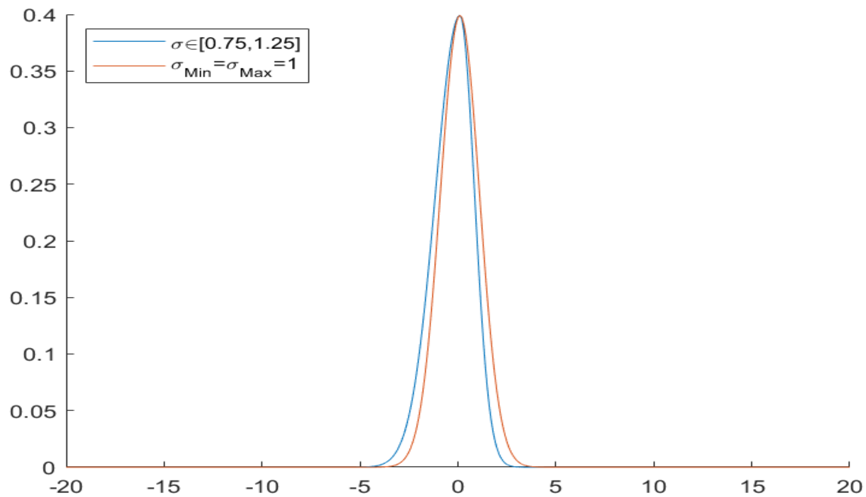

Figure 4.1: The G-Normal density and the standard normal density.

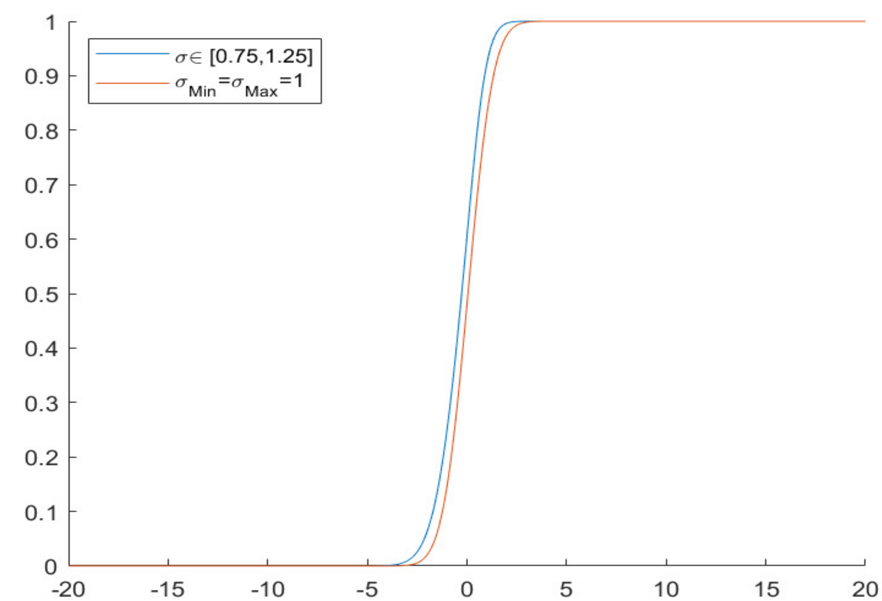

Figure 4.2: The G-Normal distribution and the standard normal distribution.

Concerning the choice of the parameters, we take the value of $\sigma_{min} = 0.75, \sigma_{max} = 1.25, T = 1, M = 1, \tau = 0.1, \alpha = 0$, and $\sigma(\widehat{K}) = 2 * \widehat{K}$. We take the value of $\widehat{K}$ in the interval $[-\tau, 0]$ are uniform random values in the interval $[1, 2]$. For this choice of coefficients and data, we obtain the following simulation result: Figure 4.3, shows that the trajectories of the solution of the G-FSDDE corresponding to $\sigma_{max} = 1.25, \sigma_{min} = 0.75$.

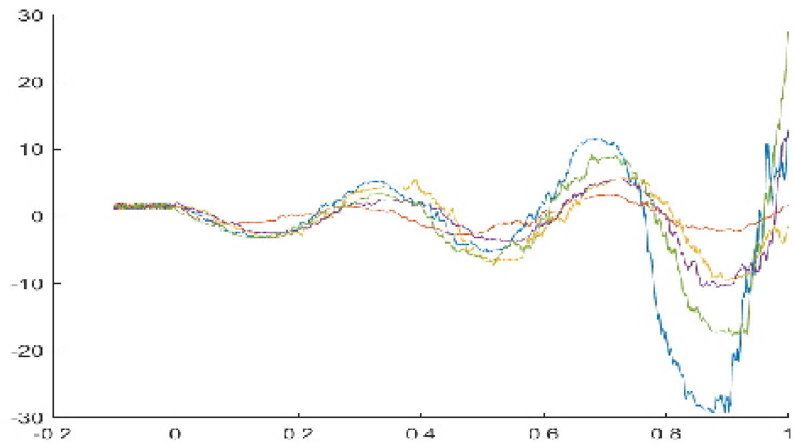

Figure 4.3: The solution of the G-FSDDE.

In Figure 4.4, we represent the trajectories of the solution of the G-BSDDE. According to a least-squar Monte Carlo scheme and based on the Euler discretisation.

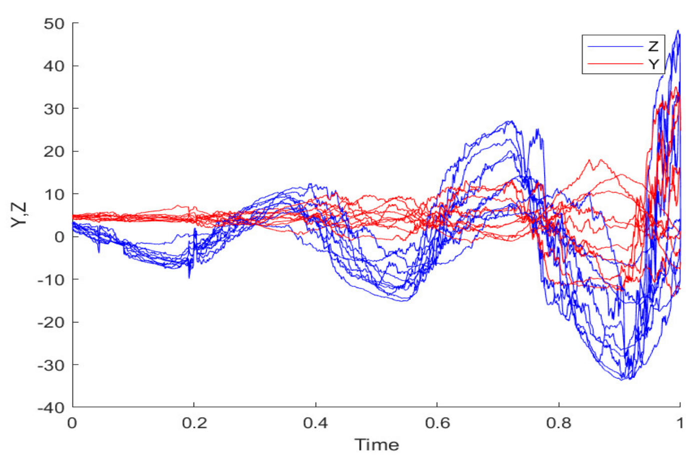

Figure 4.4: The solution of the G-BSDDE.

The simulation of G-FBSDDE give us the value of Y at unitial time $Y_0 = 3.9281$, which represent our optimal cost of our optimal control.

Conclusion and perspectives

Within this thesis, we have tackled compelling and complex problems, yielding results within the framework of sub-linear expectation spaces. These systems are characterized by the utilization of a recently introduced process known as G-Brownian motion. By examining existing outcomes in traditional scenarios and harnessing stochastic calculus principles associated with G-Brownian motion and its quadratic variation, we have achieved th following:

- We have provided the existence and uniqueness of the solution for a G-NSFDE and presented a numerical analysis using EMM for the G-NSFDE. We have proved the existence of a G-relaxed optimal control for G-NSFDE that verifies the regularity conditions using the results of Redjil et al. (2018). The main result is to prove that the infimum of the cost function in the space of strict controls is the same.

- We have also studied the existence of G-relaxed optimal control for G-SDDE and considered an economic uncertainty model with a delay connected with the worst-case scenario. Our finding is to minimize the investment capital under some assumptions, we connected its HJB with a decoupled G-FBSDDE and presented simulation results to get the optimal strategy.

We will continue our research about the existence of G-relaxed optimal control and their necessary conditions of optimality for other delay types of stochastic differential systems with a controlled diffusion coefficient that models phenomena in finance, biology, physics, etc.

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