وزارة التعليم العالي والبحث العلمي

BADJI MOKHTAR UNIVERSITY
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UNIVERSITÉ BADJI MOKHTAR
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جامعة باجي مختار
– عنابة –

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Par
Khaled ZENNIR

<u>Intitulé</u>

Existence et Comportement Asymptotique des Solutions d'une Equation de Viscoélasticité Non Linéaire de type Hyperbolique

Dirigé par

Prof. Hocine SISSAOUI

Option
Systèmes Dynamiques et Analyse Fonctionnelle

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THESIS

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By

Khaled ZENNIR

Existence and Asymptotic Behavior of Solutions of a Non Linear Viscoelastic Hyperbolic Equation

Directed by

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Khaled ZENNIR

Contents

Abstract

Our work, in this thesis, lies in the study, under some conditions on p, m and the functional g, the existence and asymptotic behavior of solutions of a nonlinear viscoelastic hyperbolic problem of the form

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \int\limits_0^t g(t-s)\Delta u(s,x)ds \\ \\ +a\left|u_t\right|^{m-2} u_t = b\left|u\right|^{p-2} u, \;\; x \in \Omega, \;\; t > 0 \\ \\ u(0,x) = u_0(x), \;\; x \in \Omega \\ u_t(0,x) = u_1(x), \;\; x \in \Omega \\ \\ u(t,x) = 0, \;\; x \in \Gamma, \;\; t > 0 \end{cases}, \tag{P}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, with smooth boundary Γ, a, b, w are positive constants, $m \geq 2$, $p \geq 2$, and the function g satisfying some appropriate conditions.

Our results contain and generalize some existing results in literature. To prove our results many theorems were introduced.

Keywords: Nonlinear damping, strong damping, viscoelasticity, nonlinear source, local solutions, global solutions, exponential decay, polynomial decay, growth.

Résumé

Notre travail, dans ce memoire consiste à étudier l'éxistence et le comportement asymptotique des solutions d'un problème de viscoelasticité non lineaire de type hyperbolique suivant:

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \int\limits_0^t g(t-s)\Delta u(s,x)ds \\ \\ +a\left|u_t\right|^{m-2}u_t = b\left|u\right|^{p-2}u,\ \ x \in \Omega,\ \ t>0 \\ \\ u\left(0,x\right) = u_0\left(x\right),\ \ x \in \Omega \\ u_t\left(0,x\right) = u_1\left(x\right),\ \ x \in \Omega \\ \\ u\left(t,x\right) = 0,\ \ x \in \Gamma,\ \ t>0 \end{cases}, \tag{P}$$

où, Ω est un domaine borné de $\mathbb{R}^N$ $(N \geq 1)$, avec frontière assez regulière Γ, a, b, w sont des constantes positives, $m \geq 2$, $p \geq 2$, et la fonction g satisfaite quelques conditions.

Nos résultats contiennent et généralisent certains résultats d'existences dans la littérature. Pour la preuve, beaucoup théorèmes ont été présentés

Mots clés: Dissipation nonlinéaire, viscoelasticité, source nonlinéaire, solutions locale, solutions globale, décroissance exponentielle de l'énergie, décroissance polynomiale, croissement.

Notations

Ω : bounded domain in $\mathbb{R}^N$.

Γ : topological boundary of Ω.

$x = (x_1, x_2, ..., x_N)$: generic point of $\mathbb{R}^N$.

$dx = dx_1 dx_2 ... dx_N$: Lebesgue measuring on Ω.

∇u : gradient of u.

Δu : Laplacien of u.

f^+, f^- : $\max(f, 0)$, $\max(-f, 0)$.

$a.e$: almost everywhere.

p' : conjugate of p, $i.e$ $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

$D(\Omega)$: space of differentiable functions with compact support in Ω.

$D'(\Omega)$: distribution space.

$C^k(\Omega)$: space of functions $k-$times continuously differentiable in Ω.

$C_0(\Omega)$: space of continuous functions null board in Ω.

$L^p(\Omega)$: Space of functions $p-$th power integrated on Ω with measure of dx.

$\|f\|_p = \left(\int\limits_\Omega |f(x)|^p \right)^{\frac{1}{p}}$.

$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega),\ \nabla u \in (L^p(\Omega))^N \right\}$.

$\|u\|_{1,p} = \left(\|u\|_p^p + \|\nabla u\|_p^p \right)^{\frac{1}{p}}$.

$W_0^{1,p}(\Omega)$: the closure of $D(\Omega)$ in $W^{1,p}(\Omega)$.

$W^{-1,p'}(\Omega)$: the dual space of $W_0^{1,p}(\Omega)$.

H : Hilbert space.

$H_0^1 = W_0^{1,2}$.

If X is a Banach space

$L^p(0,T;X) = \left\{ f : (0,T) \to X \text{ is measurable }; \int\limits_0^T \|f(t)\|_X^p \, dt < \infty \right\}$.

$L^\infty(0,T;X) = \left\{ f : (0,T) \to X \text{ is measurable }; ess - \sup\limits_{t\in(0,T)} \|f(t)\|_X^p < \infty \right\}$.

$C^k([0,T];X)$: Space of functions $k-$times continuously differentiable for $[0,T] \to X$.

$D([0,T];X)$: space of functions continuously differentiable with compact support in $[0,T]$.

$B_X = \{x \in X; \|x\| \leq 1\}$: unit ball.

Introduction

In this thesis we consider the following nonlinear viscoelastic hyperbolic problem

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \int\limits_0^t g(t-s)\Delta u(s,x)ds \\ +a\left|u_t\right|^{m-2} u_t = b\left|u\right|^{p-2} u, \ x \in \Omega, \ t > 0 \\ u\left(0,x\right) = u_0\left(x\right), \ u_t\left(0,x\right) = u_1\left(x\right), \ x \in \Omega \\ u\left(t,x\right) = 0, \ x \in \Gamma, \ t > 0 \end{cases}, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, with smooth boundary Γ, a, b, w are positive constants, and $m \geq 2$, $p \geq 2$. The function $g(t)$ is assumed to be a positive nonincreasing function defined on $\mathbb{R}^+$ and satisfies the following conditions:
$(G1)$. $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a bounded C^1-function such that

$$g(0) > 0, \ 1 - \int\limits_0^\infty g(s)ds = l > 0.$$

$(G2)$. $g(t) \geq 0$, $g^{'}(t) \leq 0$, $g(t) \leq -\xi g^{'}(t)$, $\forall\ t \geq 0$, $\xi > 0$.

In the physical point of view, this type of problems arise usually in viscoelasticity. This type of problems have been considered first by Dafermos [**12**], in 1970, where the general decay was discussed. A related problems to (1) have attracted a great deal of attention in the last two decades, and many results have been appeared on the existence and long time behavior of solutions. See in this directions [**2**, **3**, **5** − **8**, **18**, **29**, **33**, **34**, **38**] and references therein.

In the absence of the strong damping Δu_t, that is for $w = 0$, and when the function g vanishes identically (i.e. $g = 0$), then problem (1) reduced to the following initial boundary damped wave equation with nonlinear damping and nonlinear sources terms.

$$u_{tt} - \Delta u + a\left|u_t\right|^{m-2} u_t = b\left|u\right|^{p-2} u. \tag{2}$$

Some special cases of equation (2) arise in quantum field theory which describe the motion of charged mesons in an electromagnetic field.

Equation (2) together with initial and boundary conditions of Dirichlet type, has been extensively studied and results concerning existence, blow up and asymptotic behavior of smooth, as well as weak solutions have been established by several authors over the past three decades. Some interesting results have been summarized by Said-Houari in his master thesis [**42**].

For $b = 0$, that is in the absence of the source term, it is well known that the damping term $a\left|u_t\right|^{m-2} u_t$ assures global existence and decay of the solution energy for arbitrary initial data (see for instance [**17**] and [**21**]).

For $a = 0$, the source term causes finite-time blow-up of solutions with a large initial data (negative initial energy). That is to say, the norm of our solution $u(t, x)$ in the energy space reaches $+\infty$ when the time t approaches certain value T^* called " *the blow up time*", (see [**1**] and [**20**] for more details).
The interaction between the damping term $a\left|u_t\right|^{m-2} u_t$ and the source term $b\left|u\right|^{p-2} u$ makes the problem more interesting. This situation was first considered by Levine [**23**, **24**] in the linear damping case ($m = 2$), where he showed that solutions with negative initial energy blow up in finite time T^*. The main ingredient used in [**23**] and [**24**] is the " concavity method" where the basic idea of this method is to construct a positive function $L(t)$ of the solution and show that for some $\gamma > 0$, the function $L^{-\gamma}(t)$ is a positive concave function of t. In order to find such γ, it suffices to verify that:

$$\frac{d^2 L^{-\gamma}(t)}{dt^2} = -\gamma L^{-\gamma-2}(t)\left[LL'' - (1+\gamma)L'^2(t)\right] \leq 0,\ \forall t \geq 0.$$

This is equivalent to prove that $L(t)$ satisfies the differential inequality

$$LL'' - (1+\gamma)L'^2(t) \geq 0,\ \forall t \geq 0.$$

Unfortunately, this method fails in the case of nonlinear damping term ($m > 2$).
Georgiev and Todorova in their famous paper [**14**], extended Levine's result to the nonlinear damping case ($m > 2$). More precisely, in [**14**] and by combining the Galerkin approximation with the contraction mapping theorem, the authors showed that problem (2) in a bounded domain Ω with initial and boundary conditions of Dirichlet type has a unique solution in the interval $[0, T)$ provided that T is small enough. Also, they proved that the obtained solutions continue to exist globally in time if $m \geq p$ and the initial data are small enough. Whereas for $p > m$ the unique solution of problem (2) blows up in finite time provided that the initial data are large enough. (i.e. the initial energy is sufficiently negative).
This later result has been pushed by Messaoudi in [**35**] to the situation where the initial energy $E(0) < 0$. For more general result in this direction, we refer the interested reader to the works of Vitillaro [**47**], Levine [**25**] and Serrin and Messaoudi and Said-Houari [**32**].

In the presence of the viscoelastic term $(g \neq 0)$ and for $w = 0$, our problem (1) becomes

$$\begin{cases} u_{tt} - \Delta u + \int\limits_0^t g(t-s)\Delta u(s,x)ds \\ +a\left|u_t\right|^{m-2}u_t = b\left|u\right|^{p-2}u, \ x \in \Omega, \ t > 0 \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \ x \in \Omega \\ u(t,x) = 0, \ x \in \Gamma, \ t > 0 \end{cases} . \tag{3}$$

For $a = 0$, problem (3) has been investigated by Berrimi and Messaoudi [**3**]. They established the local existence result by using the Galerkin method together with the contraction mapping theorem. Also, they showed that for a suitable initial data, then the local solution is global in time and in addition, they showed that the dissipation given by the viscoelastic integral term is strong enough to stabilize the oscillations of the solution with the same rate of decaying (exponential or polynomial) of the kernel g. Also their result has been obtained under weaker conditions than those used by Cavalcanti et *al* [**7**], in which a similar problem has been addressed.

Messaoudi in [**29**], showed that under appropriate conditions between m, p and g a blow up and global existence result, of course his work generalizes the results by Georgiev and Todorova [**14**] and Messaoudi [**29**].

One of the main direction of the research in this field seems to find the minimal dissipation such that the solutions of problems similar to (3) decay uniformly to zero, as time goes to infinity. Consequently, several authors introduced different types of dissipative mechanisms to stabilize these problems. For example, a localized frictional linear damping of the form $a(x)u_t$ acting in sub-domain $\overline{w} \subset \Omega$ has been considered by Cavalcanti et *al* [**7**]. More precisely the authors in [**6**] looked into the following problem

$$u_{tt} - \Delta u + \int\limits_0^t g(t-s)\Delta u(s,x)ds + a(x)u_t + |u|^\gamma u = 0. \tag{4}$$

for $\gamma > 0$, g a positive function and $a : \Omega \to \mathbb{R}^+$ a function, which may be null on a part of the domain Ω.

By assuming $a(x) \geq a_0 > 0$ on the sub-domain $\overline{w} \subset \Omega$, the authors showed a decay result of an exponential rate, provided that the kernel g satisfies

$$-\zeta_1 g(t) \leq g'(t) \leq -\zeta_2 g(t), \ t \geq 0, \tag{5}$$

and $\|g\|_{L^1(0,\infty)}$ is small enough.

This later result has been improved by Berrimi and Messaoudi [**2**], in which they showed that the viscoelastic dissipation alone is strong enough to stabilize the problem even with an exponential rate.
In many existing works on this field, the following conditions on the kernel

$$g'(t) \leq -\zeta g^p(t), \ t \geq 0, \ p \geq 1, \tag{6}$$

is crucial in the proof of the stability.

For a viscoelastic systems with oscillating kernels, we mention the work by Rivera et *al* [**36**]. In that work the authors proved that if the kernel satisfies $g(0) > 0$ and decays exponentially to zero, that is for $p = 1$ in (6), then the solution also decays exponentially to zero. On the other hand, if the kernel decays polynomially, i.e. $(p > 1)$ in the inequality (6), then the solution also decays polynomially with the same rate of decay.

In the presence of the strong damping $(w > 0)$ and in the absence of the viscoelastic term $(g = 0)$, the problem (1) takes the following form

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + a\,|u_t|^{m-2}\,u_t = b\,|u|^{p-2}\,u, \ x \in \Omega, \ t > 0 \\ \\ u\,(0,x) = u_0\,(x)\,, \ u_t\,(0,x) = u_1\,(x)\,, \ x \in \Omega \\ \\ u\,(t,x) = 0, \ x \in \Gamma, \ t > 0 \end{cases} . \tag{7}$$

Problem (7) represents the wave equation with a strong damping Δu_t. When $m = 2$, this problem has been studied by Gazzola and Squassina [**13**]. In their work, the authors proved some results on well posedness and asymptotic behavior of solutions. They showed the global existence and polynomial decay property of solutions provided that the initial data is in the potential well.
The proof in [**13**] is based on a method used in [**19**]. Unfortunately their decay rate is not optimal, and their result has been improved by Gerbi and Said-Houari [**16**], by using an appropriate modification of the energy method and some differential and integral inequalities.
Introducing a strong damping term Δu_t makes the problem from that considered in [**42**] and [**14**], for this reason less results where known for the wave equation with strong damping and many problem remain unsolved. (See [**13**] and the recent work by Gerbi and Said-Houari [**15**]).

In this thesis, we investigated problem (1), in which all the damping mechanism have been considered in the same time (i.e. $w > 0$, $g \neq 0$, and $m \geq 2$), these assumptions make our problem different form those studied in the literature, specially the blow up result / exponential growth of solutions (chapter4).

This thesis is organized as follows:

Chapter1:

In this chapter we introduce some notation and prepare some material needed for our work. The main results of this chapter such as: the L^p- spaces, the Sobolev spaces, differential and integral inequalities and other theorems of functional analysis, can found in the books [**4**] and [**43**].

Chapter2:

This chapter is devoted to the study of the local existence result, the main ingredient used in this chapter is the Galerkin approximations (the compactness method) introduced in the book of Lions [**26**], together with the fix point method.

Indeed, we consider first for $u \in C\left([0,T], H_0^1\right)$ given, the following problem

$$v_{tt} - \Delta v - \omega\Delta v_t + \int_0^t g(t-s)\Delta v(s,x)ds + a\left|v_t\right|^{m-2} v_t = b\left|u\right|^{p-2} u,\ x \in \Omega,\ t > 0 \tag{8}$$

with the initial data

$$v\left(0,x\right) = u_0\left(x\right),\ v_t\left(0,x\right) = u_1\left(x\right), x \in \Omega \tag{9}$$

and boundary conditions of the form

$$v\left(t,x\right) = 0,\ x \in \Gamma,\ t > 0, \tag{10}$$

and we will show that problem $(8)-(10)$ has a unique local solutions v by the Faedo-Galerkin method, which consists in constructing approximations of the solution, then we obtain a priori estimates necessary to guarantee the convergence of these approximations. We recall here that the presence of nonlinearity on the damping term $a\left|v_t\right|^{m-2} v_t$ forces us to go until the second a priori estimate. We point out that the contraction semigroup method fails here, because of the presence of the nonlinear terms.

Once the local solution v exists, we will use the contraction mapping theorem to show the local existence of our problem (1). This will be done under the assumption that T is required to be small enough (see formula (2.77)).

Chapter 3:

Our main purpose in this chapter is tow-fold:

First, we introduce a set W defined in (3.5) called " *the potential well*" or " *stable set*" and we show that if we restrict our initial data in this set, then our solution obtained in chapter 2 is global in time, that is to say, the norm

$$\left\|u_t\right\|_2 + \left\|\nabla u\right\|_2,$$

in the energy space $L^2(\Omega) \times H_0^1(\Omega)$ of our solution is bounded by a constant independent of the time t.

Second, We show that, if our solution is global in time, (i.e. by assuming that the initial data $u_0 \in W$) and if our function g satisfies the condition (6) (for $p = 1$), then our solution decays time asymptotically to zero. More precisely we prove that the decay rate is of the form $(1+t)^{2/(2-m)}$ if $m > 2$, whereas for $m = 2$, we obtain an exponential decay rate. (See Theorem 3.2.1). The main tool used in our proof is an inequality due to Nakao [**37**], in which this inequality has been introduced in order to study the stability of the wave equation, but it is still works in our problem.

Chapter 4:

In this chapter we will prove that if the initial energy $E(0)$ of our solution is negative (this means that our initial data are large enough), then our local solutions in bounded and

$$\|u_t\|_2 + \|\nabla u\|_2 \to \infty$$

as t tends to $+\infty$. In fact it will be proved that the L^p-norm of the solution grows as an exponential function. An essential tool of the proof is an idea used by Gerbi and Said-Houari [**15**], which based on an auxiliary function (which is a small perturbation of the total energy), in order to obtain a differential inequality leads to the exponential growth result provided that the following conditions

$$\int_0^\infty g(s)ds < \frac{p-2}{p-1},$$

holds.

Chapter 1

Preliminary

Abstract

In this chapter we shall introduce and state some necessary materials needed in the proof of our results, and shortly the basic results which concerning the Banach spaces, the weak and weak star topologies, the L^p space, Sobolev spaces and other theorems. The knowledge of all this notations and results are important for our study.

1.1 Banach Spaces - Definition and Properties

We first review some basic facts from calculus in the most important class of linear spaces "Banach spaces".

Definition 1.1.1 *A Banach space is a complete normed linear space X. Its dual space X' is the linear space of all continuous linear functional $f : X \to \mathbb{R}$.*

Proposition 1.1.1 *([43])*
X' equipped with the norm $\|.\|_{X'}$ defined by

$$\|f\|_{X'} = \sup\left\{|f(u)| : \|u\| \leq 1\right\}, \tag{1.1}$$

is also a Banach space.

We shall denote the value of $f \in X'$ at $u \in X$ by either $f(u)$ or $\langle f, u\rangle_{X',\, X}$.

Remark 1.1.1 *([43]) From X' we construct the bidual or second dual $X'' = (X')'$. Furthermore, with each $u \in X$ we can define $\varphi(u) \in X''$ by $\varphi(u)(f) = f(u)$, $f \in X'$, this satisfies clearly $\|\varphi(x)\| \leq \|u\|$. Moreover, for each $u \in X$ there is an $f \in X'$ with $f(u) = \|u\|$ and $\|f\| = 1$, so it follows that $\|\varphi(x)\| = \|u\|$.*

Definition 1.1.2 *Since φ is linear we see that*

$$\varphi : X \to X'',$$

is a linear isometry of X onto a closed subspace of X'', we denote this by

$$X \hookrightarrow X''.$$

Definition 1.1.3 *If φ (in the above definition) is onto X'' we say X is reflexive, $X \approxeq X''$.*

Theorem 1.1.1 *([4], Theorem III.16)*
Let X be Banach space. Then, X is reflexive, if and only if,

$$B_X = \left\{x \in X : \|x\| \leq 1\right\},$$

is compact with the weak topology $\sigma(X, X')$. (See the next subsection for the definition of $\sigma(X, X')$)

Definition 1.1.4 *Let X be a Banach space, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X. Then u_n converges strongly to u in X if and only if*

$$\lim_{n\to\infty} \|u_n - u\|_X = 0,$$

and this is denoted by $u_n \to u$, or $\lim\limits_{n\to\infty} u_n = u$.

1.1.1 The weak and weak star topologies

Let X be a Banach space and $f \in X'$. Denote by

$$\begin{aligned} \varphi_f : \quad & X \to \mathbb{R} \\ & x \mapsto \varphi_f(x), \end{aligned} \tag{1.2}$$

when f cover X', we obtain a family $\left(\varphi_f\right)_{f\in X'}$ of applications to X in $\mathbb{R}$.

Definition 1.1.5 *The weak topology on X, denoted by $\sigma(X, X')$, is the weakest topology on X for which every $\left(\varphi_f\right)_{f\in X'}$ is continuous.*

We will define the third topology on X', the weak star topology, denoted by $\sigma(X', X)$. For all $x \in X$. Denote by

$$\begin{aligned} \varphi_x : \quad & X' \to \mathbb{R} \\ & f \mapsto \varphi_x(f) = \langle f,\ x\rangle_{X',\,X}, \end{aligned} \tag{1.3}$$

when x cover X, we obtain a family $(\varphi_x)_{x\in X'}$ of applications to X' in $\mathbb{R}$.

Definition 1.1.6 *The weak star topology on X' is the weakest topology on X' for which every $(\varphi_x)_{x\in X'}$ is continuous.*

Remark 1.1.2 *([4]) Since $X \subset X''$, it is clear that, the weak star topology $\sigma(X', X)$ is weakest then the topology $\sigma(X', X'')$, and this later is weakest then the strong topology.*

Definition 1.1.7 *A sequence (u_n) in X is weakly convergent to x if and only if*

$$\lim_{n\to\infty} f(u_n) = f(u),$$

for every $f \in X'$, and this is denoted by $u_n \rightharpoonup u$.

Remark 1.1.3 *([42], Remark 1.1.1)*

1. *If the weak limit exist, it is unique.*

2. *If $u_n \to u \in X$ (strongly), then $u_n \rightharpoonup u$ (weakly).*

3. *If $\dim X < +\infty$, then the weak convergent implies the strong convergent.*

Proposition 1.1.2 *([43])*

On the compactness in the three topologies in the Banach space X :

1- First, the unit ball

$$B \equiv \{x \in X : \|x\| \leq 1\}, \tag{1.4}$$

in X is compact if and only if $\dim(X) < \infty$.

2- Second, the unit ball B' in X' (The closed subspace of a product of compact spaces) is weakly compact in X' if and only if X is reflexive.

3- Third, B' is always weakly star compact in the weak star topology of X'.

Proposition 1.1.3 ([**4**], *proposition III.12*)

Let (f_n) be a sequence in X'. We have:

1. $\left[f_n \overset{*}{\rightharpoonup} f \text{ in } \sigma(X', X)\right] \Leftrightarrow [f_n(x) \to f(x),\ \forall x \in X]$.

2. If $f_n \to f$ (strongly), then $f_n \rightharpoonup f$, in $\sigma(X', X'')$,

If $f_n \rightharpoonup f$ in $\sigma(X', X'')$, then $f_n \overset{*}{\rightharpoonup} f$, in $\sigma(X', X)$.

3. If $f_n \overset{*}{\rightharpoonup} f$, in $\sigma(X', X)$, then $\|f_n\|$ is bounded and $\|f\| \le \liminf \|f_n\|$.

4. If $f_n \overset{*}{\rightharpoonup} f$, in $\sigma(X', X)$ and $x_n \to x$ (strongly) in X, then $f_n(x_n) \to f(x)$.

1.1.2 Hilbert spaces

The proper setting for the rigorous theory of partial differential equation turns out to be the most important function space in modern physics and modern analysis, known as Hilbert spaces. Then, we must give some important results on these spaces here.

Definition 1.1.8 *A Hilbert space H is a vectorial space supplied with inner product $\langle u, v\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let H complete.*

Theorem 1.1.2 ([**42**], *Theorem 1.1.1*)

Let $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in the Hilbert space H, then it possess a subsequence which converges in the weak topology of H.

Theorem 1.1.3 ([**42**], *Theorem 1.1.2*)

In the Hilbert space, all sequence which converges in the weak topology is bounded.

Theorem 1.1.4 ([**42**], *Corollary 1.1.1*)

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence which converges to u, in the weak topology and $(v_n)_{n\in\mathbb{N}}$ is an other sequence which converge weakly to v, then

$$\lim_{n\to\infty} \langle v_n, u_n\rangle = \langle v, u\rangle. \tag{1.5}$$

Theorem 1.1.5 ([**42**], *Theorem 1.1.3*)

Let X be a normed space, then the unit ball

$$B' = \left\{ l \in X' : \|l\| \le 1 \right\}, \tag{1.6}$$

of X' is compact in $\sigma(X', X)$.

Proposition 1.1.4 ([**42**], *Proposition 1.1.1*)
Let X and Y be two Hilbert spaces, let $(u_n)_{n\in\mathbb{N}} \in X$ be a sequence which converges weakly to $u \in X$, let $A \in \mathcal{L}(X,Y)$. Then, the sequence $(A(u_n))_{n\in\mathbb{N}}$ converges to $A(u)$ in the weak topology of Y.

Proof. For all $u \in X$, the function

$$u \mapsto \langle A(u), v\rangle$$

is linear and continuous, because

$$|\langle A(u),\ v\rangle| \leq \|A\|_{\mathcal{L}(X,\ Y)} \|u\|_X \|v\|_Y\ ,\ \forall u \in X,\ v \in Y.$$

So, according to Riesz theorem, there exists $w \in X$ such that

$$\langle A(u), v\rangle = \langle u, w\rangle,\ \forall u \in X.$$

Then,

$$\begin{aligned} \lim_{n\to\infty} \langle A(u_n)\,,\ v\rangle &= \lim_{n\to\infty} \langle u_n, w\rangle \\ &= \langle u, w\rangle = \langle A(u), v\rangle. \end{aligned} \tag{1.7}$$

This completes the proof. ■

1.2 Functional Spaces

1.2.1 The $L^p(\Omega)$ spaces

Definition 1.2.1 *Let $1 \leq p \leq \infty$, and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$, by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \textit{ is measurable and } \int_{\Omega} |f(x)|^p \, dx < \infty \right\}. \tag{1.8}$$

Notation 1.2.1 *For $p \in \mathbb{R}$ and $1 \leq p < \infty$, denote by*

$$\|f\|_p = \left(\int_{\Omega} |f(x)|^p \, dx \right)^{\frac{1}{p}}. \tag{1.9}$$

If $p = \infty$, we have

$$\begin{aligned} L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R} : f \text{ is measurable and there exists a constant } C \\ \text{such that, } |f(x)| \leq C \text{ a.e in } \Omega \}. \end{aligned} \tag{1.10}$$

Also, we denote by

$$\|f\|_\infty = \text{Inf } \{C, \ |f(x)| \leq C \text{ a.e in } \Omega\}. \tag{1.11}$$

Notation 1.2.2 *Let $1 \leq p \leq \infty$, we denote by q the conjugate of p i.e. $\dfrac{1}{p} + \dfrac{1}{q} = 1$.*

Theorem 1.2.1 *([**48**])*
It is well known that $L^p(\Omega)$ supplied with the norm $\|.\|_p$ is a Banach space, for all $1 \leq p \leq \infty$.

Remark 1.2.1 *In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle f, \ g \rangle_{L^2(\Omega)} = \int_{\Omega} f(x) g(x) dx, \tag{1.12}$$

is a Hilbert space.

Theorem 1.2.2 ([**43**], *Corollary 3.2*)
For $1 < p < \infty$, $L^p(\Omega)$ is reflexive space.

Some integral inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

Theorem 1.2.3 *([48], Hölder's inequality)*
Let $1 \leq p \leq \infty$. Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then, $fg \in L^1(\Omega)$ and

$$\int_{\Omega} |fg| \, dx \leq \|f\|_p \, \|g\|_q \,. \tag{1.13}$$

Corollary 1.2.1 *(Hölder's inequality - general form)*

Lemma 1.2.1 *Let $f_1, f_2, ...f_k$ be k functions such that, $f_i \in L^{p_i}(\Omega)$, $1 \leq i \leq k$, and*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_k} \leq 1.$$

Then, the product $f_1 f_2 ... f_k \in L^p(\Omega)$ and $\|f_1 f_2 ... f_k\|_p \leq \|f_1\|_{p_1} ... \|f_k\|_{p_k}$.

Lemma 1.2.2 *([48], Young's inequality)*
Let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ with $1 < p < \infty$, $1 < q < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$. Then $f * g \in L^r(\mathbb{R})$ and

$$\|f * g\|_{L^r(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})} \,. \tag{1.14}$$

Lemma 1.2.3 *([43], Minkowski inequality)*
For $1 \leq p \leq \infty$, we have

$$\|u + v\|_{L^p} \leq \|u\|_{L^p} + \|v\|_{L^p} \,. \tag{1.15}$$

Lemma 1.2.4 *([43])*
Let $1 \leq p \leq r \leq q$, $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$, and $1 \leq \alpha \leq 1$. Then

$$\|u\|_{L^r} \leq \|u\|_{L^p}^{\alpha} \|u\|_{L^q}^{1-\alpha} \,. \tag{1.16}$$

Lemma 1.2.5 *([43])*
If $\mu(\Omega) < \infty$, $1 \leq p \leq q \leq \infty$, then $L^q \hookrightarrow L^p$, and

$$\|u\|_{L^p} \leq \mu(\Omega)^{\frac{1}{p} - \frac{1}{q}} \|u\|_{L^q} \,.$$

1.2.2 The Sobolev space $\mathbf{W}^{m,p}(\mathbf{\Omega})$

Proposition 1.2.1 *([26])*
Let Ω be an open domain in $\mathbb{R}^N$, Then the distribution $T \in \mathcal{D}'(\Omega)$ is in $L^p(\Omega)$ if there exists a function $f \in L^p(\Omega)$ such that

$$\langle T,\ \varphi\rangle = \int\limits_{\Omega} f(x)\varphi(x)dx,\ \text{for all } \varphi \in \mathcal{D}(\Omega),$$

where $1 \leq p \leq \infty$, and it's well-known that f is unique.

Definition 1.2.2 *Let $m \in \mathbb{N}$ and $p \in [0,\infty]$. The $W^{m,p}(\Omega)$ is the space of all $f \in L^p(\Omega)$, defined as*

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega),\ \text{such that } \partial^\alpha f \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^m \text{ such that } |\alpha| = \textstyle\sum_{j=1}^n \alpha_j \leq m,\ \text{where, } \partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}..\partial_n^{\alpha_n}\}. \tag{1.17}$$

Theorem 1.2.4 *([9])*
$W^{m,p}(\Omega)$ is a Banach space with their usual norm

$$\|f\|_{W^{m,p}(\Omega)} = \sum_{|\alpha|\leq m} \|\partial^\alpha f\|_{L^p},\ 1 \leq p < \infty,\ \text{for all } f \in W^{m,p}(\Omega). \tag{1.18}$$

Definition 1.2.3 *Denote by $W_0^{m,p}(\Omega)$ the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$.*

Definition 1.2.4 *When $p = 2$, we prefer to denote by $W^{m,2}(\Omega) = H^m(\Omega)$ and $W_0^{m,2}(\Omega) = H_0^m(\Omega)$ supplied with the norm*

$$\|f\|_{H^m(\Omega)} = \left(\sum_{|\alpha|\leq m} (\|\partial^\alpha f\|_{L^2})^2\right)^{\frac{1}{2}}, \tag{1.19}$$

which do at $H^m(\Omega)$ a real Hilbert space with their usual scalar product

$$\langle u, v\rangle_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \int\limits_{\Omega} \partial^\alpha u \partial^\alpha v dx \tag{1.20}$$

Theorem 1.2.5 ([42], *Proposition 1.2.1*)
1) *$H^m(\Omega)$ supplied with inner product $\langle .,.\rangle_{H^m(\Omega)}$ is a Hilbert space.*
2) *If $m \geq m'$, $H^m(\Omega) \hookrightarrow H^{m'}(\Omega)$, with continuous imbedding .*

Lemma 1.2.6 *([26])*
Since $\mathcal{D}(\Omega)$ is dense in $H_0^m(\Omega)$, we identify a dual $H^{-m}(\Omega)$ of $H_0^m(\Omega)$ in a weak subspace on Ω, and we have

$$\mathcal{D}(\Omega) \hookrightarrow H_0^m(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-m}(\Omega) \hookrightarrow \mathcal{D}'(\Omega),$$

Lemma 1.2.7 *(Sobolev-Poincaré's inequality)*
If

$$2 \leq q \leq \frac{2n}{n-2}, \ n \geq 3$$

$$q \geq 2, \ n = 1, 2,$$

then

$$\|u\|_q \leq C(q, \ \Omega) \|\nabla u\|_2 , \tag{1.21}$$

for all $u \in H_0^1(\Omega)$.

The next results are fundamental in the study of partial differential equations

Theorem 1.2.6 *(*[**9**] *Theorem 1.3.1)*
Assume that Ω *is an open domain in* $\mathbb{R}^N$ $(N \geq 1)$, *with smooth boundary* Γ. *Then,*
(**i**) *if* $1 \leq p \leq n$, *we have* $W^{1,p} \subset L^q(\Omega)$, *for every* $q \in [p, \ p^*]$, *where* $p^* = \dfrac{np}{n-p}$.
(**ii**) *if* $p = n$ *we have* $W^{1,p} \subset L^q(\Omega)$, *for every* $q \in [p, \ \infty)$.
(**iii**) *if* $p > n$ *we have* $W^{1,p} \subset L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$, *where* $\alpha = \dfrac{p-n}{p}$.

Theorem 1.2.7 *(*[**9**] *Theorem 1.3.2)*
If Ω *is a bounded, the embedding (ii) and (iii) of theorem 1.1.4 are compacts. The embedding (i) is compact for all* $q \in [p, \ p^*)$.

Remark 1.2.2 *(*[**26**]*)*
For all $\varphi \in H^2(\Omega)$, $\Delta\varphi \in L^2(\Omega)$ *and for* Γ *sufficiently smooth, we have*

$$\|\varphi(t)\|_{H^2(\Omega)} \leq C \|\Delta\varphi(t)\|_{L^2(\Omega)} . \tag{1.22}$$

Proposition 1.2.2 *(*[**43**], *Green's formula)*
For all $u \in H^2(\Omega), v \in H^1(\Omega)$ *we have*

$$-\int_\Omega \Delta u v dx = \int_\Omega \nabla u \nabla v dx - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v d\sigma, \tag{1.23}$$

where $\dfrac{\partial u}{\partial \eta}$ *is a normal derivation of* u *at* Γ.

1.2.3 The $L^p(0,T,X)$ spaces

Definition 1.2.5 *Let X be a Banach space, denote by $L^p(0,T,X)$ the space of measurable functions*

$$\begin{aligned} f:\ \ ]0,T[&\rightarrow X \\ t &\longmapsto f(t) \end{aligned}$$

such that

$$\left(\int_0^T \|f(t)\|_X^p \, dt\right)^{\frac{1}{p}} = \|f\|_{L^p(0,T,X)} < \infty, \text{ for } 1 \leq p < \infty. \tag{1.24}$$

If $p = \infty$,

$$\|f\|_{L^\infty(0,T,X)} = \sup_{t \in]0,T[} ess \, \|f(t)\|_X . \tag{1.25}$$

Theorem 1.2.8 *(*[**42**]*)*
The space $L^p(0,T,X)$ is complete.

We denote by $\mathcal{D}'(0,T,X)$ the space of distributions in $]0,T[$ which take its values in X, and let us define

$$\mathcal{D}'(0,T,X) = \mathcal{L}(\mathcal{D}\,]0,T[\,, X),$$

where $\mathcal{L}(\phi,\ \varphi)$ is the space of the linear continuous applications of ϕ to φ. Since $u \in \mathcal{D}'(0,\ T,\ X)$, we define the distribution derivation as

$$\frac{\partial u}{\partial t}(\varphi) = -u\left(\frac{d\varphi}{dt}\right),\ \forall \varphi \in \mathcal{D}\left(]0,T[\right), \tag{1.26}$$

and since $u \in L^p(0,\ T,\ X)$, we have

$$u(\varphi) = \int_0^T u(t)\varphi(t)dt,\ \forall \varphi \in \mathcal{D}\left(]0,T[\right). \tag{1.27}$$

We will introduce some basic results on the $L^p(0,\,T,\,X)$ space. These results, will be very useful in the other chapters of this thesis.

Lemma 1.2.8 *(*[**26**] *Lemma 1.2)*
Let $f \in L^p(0,\,T,\,X)$ and $\dfrac{\partial f}{\partial t} \in L^p(0,\,T,\,X)$, $(1 \leq p \leq \infty)$, then, the function f is continuous from $[0,T]$ to X.i.e. $f \in C^1(0,\,T,\,X)$.

Lemma 1.2.9 *([26])*
Let $\varphi =]0,T[\times\Omega$ *an open bounded domain in* $\mathbb{R}\times\mathbb{R}^n$*, and let* g_μ, g *are two functions in* $L^q\left(]0,T[\,,L^q(\Omega)\right)$ *,* $1 < q < \infty$ *such that*

$$\|g_\mu\|_{L^q(0,T,L^q(\Omega))} \leq C,\ \forall \mu \in \mathbb{N} \tag{1.28}$$

and

$$g_\mu \rightarrow g \text{ in } \varphi,$$

then

$$g_\mu \rightharpoonup g \text{ in } L^q\left(\varphi\right).$$

Theorem 1.2.9 ([9], *Proposition 1.4.17*)
$L^p(0,\, T,\, X)$ *equipped with the norm* $\|.\|_{L^p(0,T,X)}$*,* $1 \leq p \leq \infty$ *is a Banach space.*

Proposition 1.2.3 *([14])*
Let X *be a reflexive Banach space,* X' *it's dual, and* $1 \leq p < \infty$*,* $1 \leq q < \infty$*,* $\frac{1}{p} + \frac{1}{q} = 1$*. Then the dual of* $L^p(0,\, T,\, X)$ *is identify algebraically and topologically with* $L^q(0,T,X')$*.*

Proposition 1.2.4 *([9])*
Let X, Y *be to Banach space,* $X \subset Y$ *with continuous embedding, then we have* $L^p(0,\, T,\, X) \subset L^p(0,\, T,\, Y)$ *with continuous embedding.*

The following compactness criterion will be useful for nonlinear evolution problems, especially in the limit of the non linear terms.

Proposition 1.2.5 *([26]).*
Let $B_0,\, B,\, B_1$ *be Banach spaces with* $B_0 \subset B \subset B_1$*, assume that the embedding* $B_0 \hookrightarrow B$ *is compact and* $B \hookrightarrow B_1$ *are continuous. Let* $1 < p < \infty$*,* $1 < q < \infty$*, assume further that* B_0 *and* B_1 *are reflexive.*
Define

$$W \equiv \left\{u \in L^p\left(0,\ T,\ B_0\right) : u' \in L^q\left(0,\ T,\ B_1\right)\right\}. \tag{1.29}$$

Then, the embedding $W \hookrightarrow L^p\left(0,\ T,\ B\right)$ *is compact.*

1.2.4 Some Algebraic inequalities

Since our study based on some known algebraic inequalities, we want to recall few of them here.

Lemma 1.2.10 *([48], The Cauchy-Schwarz inequality)*
Every inner product satisfies the Cauchy-Schwarz inequality

$$\langle x_1,\ x_2\rangle \leq \|x_1\| \, \|x_2\| \, . \tag{1.30}$$

The equality sign holds if and only if x_1 and x_2 are dependent.

Young's inequalities :

Lemma 1.2.11 *For all $a,\ b \in \mathbb{R}^+$, we have*

$$ab \leq \delta a^2 + \frac{b^2}{4\delta}, \tag{1.31}$$

where δ is any positive constant.

Proof. Taking the well-known result

$$(2\delta a - b)^2 \geq 0 \quad \forall a,\ b \in \mathbb{R}$$

for all $\delta > 0$, we have

$$4\delta^2 a^2 + b^2 - 4\delta ab \geq 0.$$

This implies

$$4\delta ab \leq 4\delta^2 a^2 + b^2$$

consequently,

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2.$$

This completes the proof. ■

Lemma 1.2.12 *([43])*
For all $a, b \geq 0$, the following inequality holds

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where, $\dfrac{1}{p} + \dfrac{1}{q} = 1.$

Proof. Let $G = (0, 1)$, and $f : G \to \mathbb{R}$ is integrable, such that

$$f(x) = \begin{cases} p \log a, & 0 \le x \le \dfrac{1}{p} \\ q \log b, & \dfrac{1}{p} \le x \le 1 \end{cases},$$

for all $a, b \ge 0$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.
Since $\varphi(t) = e^t$ is convex, and using Jensen's inequality

$$\varphi\left(\frac{1}{\mu(G)} \int_G f(x)dx\right) \le \frac{1}{\mu(G)} \int_G \varphi(f(x))\, dx. \tag{1.32}$$

Consequently, we have

$$\begin{aligned} \frac{1}{\mu(G)} \int_G \varphi(f(x))\, dx &= \int_0^1 e^{f(x)} dx = \int_0^{1/p} e^{p \log a} dx + \int_{1/p}^1 e^{q \log b} dx \\ &= \int_0^{1/p} a^p dx + \int_{1/p}^1 b^q dx \\ &= \frac{1}{p}(a^p) + \left(1 - \frac{1}{p}\right) b^q. \\ &= \frac{a^p}{p} + \frac{b^q}{q}, \end{aligned} \tag{1.33}$$

where, $\mu(G) = 1$ and

$$\begin{aligned} \varphi\left(\frac{1}{\mu(G)} \int_G f(x)dx\right) &= e^{\left(\int_0^1 f(x)dx\right)} = e^{\left(\int_0^{1/p} p \log a dx + \int_{1/p}^1 q \log b dx\right)} \\ &= e^{(\log a + \log b)} = e^{\log ab} \\ &= ab. \end{aligned} \tag{1.34}$$

Using (1.32), (1.33) and (1.34) to conclude the result. ■

1.3 Existence Methods

1.3.1 The Contraction Mapping Theorem

Here we prove a very useful fixed point theorem called the contraction mapping theorem. We will apply this theorem to prove the existence and uniqueness of solutions of our nonlinear problem.

Definition 1.3.1 *Let $f : X \to X$ be a map of a metric space to itself. A point $x \in X$ is called a fixed point of f if $f(x) = x$.*

Definition 1.3.2 *Let (X, d_X) and (Y, d_Y) be metric spaces. A map $\varphi : X \to Y$ is called a contraction if there exists a positive number $C < 1$ such that*

$$d_Y(\varphi(x), \varphi(y)) \leq C d_X(x, y), \tag{1.35}$$

for all $x, y \in X$.

Theorem 1.3.1 *(Contraction mapping theorem [**45**])*
Let (X, d) be a complete metric space. If $\varphi : X \to X$ is a contraction, then φ has a unique fixed point.

1.3.2 Gronwell's lemma

Theorem 1.3.2 *(In integral form)*
Let $T > 0$, and let φ be a function such that, $\varphi \in L^1(0, T)$, $\varphi \geq 0$, almost everywhere and ϕ be a function such that, $\phi \in L^1(0, T)$, $\phi \geq 0$, almost everywhere and $\phi\varphi \in L^1[0,\ T]$, $C_1,\ C_2 \geq 0$. Suppose that

$$\phi(t) \leq C_1 + C_2 \int_0^t \varphi(s)\phi(s)ds, \text{ for a.e } t \in \left]0, T\right[, \tag{1.36}$$

then,

$$\phi(t) \leq C_1 \exp\left(C_2 \int_0^t \varphi(s)ds \right), \text{ for a.e } t \in \left]0, T\right[. \tag{1.37}$$

Proof. Let

$$F(t) = C_1 + C_2 \int_0^t \varphi(s)\phi(s)ds, \text{ for } t \in [0, T], \tag{1.38}$$

we have,

$$\phi(t) \leq F(t),$$

From (1.38) we have

$$\begin{aligned} F'(t) &= C_2\varphi(t)\phi(t) \\ &\leq C_2\varphi(t)\lambda(t), \text{ for a.e } t \in \,]0,T[\,. \end{aligned} \tag{1.39}$$

Consequently,

$$\frac{d}{dt}\left\{ F(t)\exp\left(-\int_0^t C_2\varphi(s)ds\right)\right\} \leq 0, \tag{1.40}$$

then,

$$F(t) \leq C_1 \exp\left(C_2\int_0^t \varphi(s)ds\right), \text{ for a.e } t \in \,]0,T[\,. \tag{1.41}$$

Since $\phi \leq F$, then our result holds.

In particle, if $C_1 = 0$, we have $\phi = 0$ for almost everywhere $t \in \,]0,T[\,$. ■

1.3.3 The mean value theorem

Theorem 1.3.3 *Let $G : [a,b] \to \mathbb{R}$ be a continues function and $\varphi : [a,b] \to \mathbb{R}$ is an integral positive function, then there exists a number x in (a,b) such that*

$$\int_a^b G(t)\varphi(t)dt = G(x)\int_a^b \varphi(t)dt. \tag{1.42}$$

In particular for $\varphi(t) = 1$, there exists $x \in (a,b)$ such that

$$\int_a^b G(t)dt = G(x)\left(b-a\right). \tag{1.43}$$

Proof. Let

$$m = \inf\left\{G(x), x \in [a,b]\right\} \tag{1.44}$$

and

$$M = \sup\left\{G(x), x \in [a,b]\right\} \tag{1.45}$$

of course m and M exist since $[a,b]$ is compact.

Then, it follows that

$$m\int_a^b \varphi(t)dt \leq \int_a^b G(t)\varphi(t)dt \leq M\int_a^b \varphi(t)dt. \tag{1.46}$$

By monotonicity of the integral. dividing through by $\int_a^b \varphi(t)dt$, we have that

$$m \leq \frac{\int_a^b G(t)\varphi(t)dt}{\int_a^b \varphi(t)dt} \leq M. \tag{1.47}$$

Since $G(t)$ is continues, the intermediate value theorem implies that there exists $x \in [a, b]$ such that

$$G(x) = \frac{\int_a^b G(t)\varphi(t)dt}{\int_a^b \varphi(t)dt}. \tag{1.48}$$

Which completes the proof. ■

Chapter 2

Local Existence

Abstract

Our goal in this chapter is to study the local existence (local well-possedness) of the problem (P), for u in $C\left([0,T], H_0^1(\Omega)\right)$. For this purpose we consider, first the related problem for u fixed in $C\left([0,T], H_0^1(\Omega)\right)$

$$\begin{cases} v_{tt} - \Delta v_t - \Delta v + \int_0^t g(t-s)\Delta v(s,x)ds \\ + \left|v_t\right|^{m-2} v_t = \left|u\right|^{p-2} u, \; x \in \Omega, \; t > 0 \\ v\left(0,x\right) = u_0\left(x\right), \; v_t\left(0,x\right) = u_1\left(x\right), \; x \in \Omega \\ v\left(t,x\right) = 0, \; x \in \Gamma, \; t > 0 \end{cases}, \tag{2.1}$$

and we will prove the local existence of this problem by using the Faedo-Galerkin method. Then, by using the well-known contraction mapping theorem, we can show the local existence of (P). Our techniques of proof follows carefully the techniques due to Georgiev and Todorova [**14**], with necessary modifications imposed by the nature of our problem. The first step of our proof is the choice of the space where the local solution exists. The minimal requirement for this space is that $u(t,x)$ be time continuous. The weak space satisfying the above requirement is $C\left([0,T], H\right)$, where $H = H_0^1(\Omega) \times H_0^1(\Omega)$ is the natural energy space for (P).

2.1 Local Existence Result

In order to prove our local existence results, let us introduce the following space

$$Y_T = \left\{ \begin{array}{ll} u: & u \in C\left([0,T], H_0^1(\Omega)\right), \\ \\ & u_t \in C([0,T], H_0^1(\Omega)) \cap L^m([0,T] \times \Omega) \end{array} \right\}. \tag{2.2}$$

Our main result in this chapter reads as follows:

Theorem 2.1.1 *Let $(u_0, u_1) \in \left(H_0^1(\Omega)\right)^2$ be given. Suppose that $m \geq 2$, $p \geq 2$ be such that*

$$\max\{m,\ p\} \leq \frac{2(n-1)}{n-2}, n \geq 3. \tag{2.3}$$

Then, under the conditions $(G1)$ and $(G2)$, the problem (P) has a unique local solution $u(t,x) \in Y_T$, for T small enough.

The proof of theorem 2.1.1 will be established through several lemmas. The presence of the term $|u|^{P-2} u$ in the right hand side of our problem (P), gives us negative values of the energy. For this purpose we fixed $u \in C\left([0,T], H_0^1(\Omega)\right)$ in the right hand side of (P) and we will prove that our problem (2.1), admits a solution.

Lemma 2.1.1 *([**14**], Theorem 2.1) Let $(u_0, u_1) \in \left(H_0^1(\Omega)\right)^2$, assume that $m \geq 2$, $p \geq 2$ and (2.3) holds. Then, under the conditions $(G1)$ and $(G2)$, there exists a unique weak solution $v \in Y_T$ to the problem (2.1), for any $u \in C\left([0,T], H_0^1(\Omega)\right)$ given.*

The proof of the above Lemma follows the techniques due to Lions [**26**], in order to deal with the convergence of the non linear terms in our problem, we must take first our initial data (u_0, u_1) in a high regularity (that is, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, and $u_1 \in H_0^1(\Omega) \cap L^{2(m-1)}(\Omega)$).

Lemma 2.1.2 *([**26**], Theorem 3.1) Let $u \in C\left([0,T], H_0^1(\Omega)\right)$. Suppose that*

$$u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \tag{2.4}$$

$$u_1 \in H_0^1(\Omega) \cap L^{2(m-1)}(\Omega), \tag{2.5}$$

assume further that $m \geq 2$, $p \geq 2$. Then, under the conditions $(G1)$ and $(G2)$ there exists a unique solution v of the problem (2.1) such that

$$v \in L^\infty\left([0,T], H^2(\Omega) \cap H_0^1(\Omega)\right), \tag{2.6}$$

$$v_t \in L^\infty\left([0,T], H_0^1(\Omega)\right), \tag{2.7}$$

$$v_{tt} \in L^\infty\left([0,T], L^2(\Omega)\right), \tag{2.8}$$

$$v_t \in L^m\left([0,T] \times (\Omega)\right). \tag{2.9}$$

The following technical Lemma will play an important role in the sequel.

Lemma 2.1.3 *For any $v \in C^1\left(0,T,H^2(\Omega)\right)$ we have*

$$\begin{aligned}\int_\Omega \int_0^t g(t-s)\Delta v(s).v'(t)dsdx &= \frac{1}{2}\frac{d}{dt}\left(g \circ \nabla v\right)(t) - \frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)\int_\Omega |\nabla v(t)|^2\, dxds\right] \\ &\quad -\frac{1}{2}\left(g' \circ \nabla v\right)(t) + \frac{1}{2}g(t)\int_\Omega |\nabla v(t)|^2\, dxds.\end{aligned}$$

where $(g \circ u)(t) = \int_0^t g(t-s)\int_\Omega |u(s)-u(t)|^2\, dxds.$

The proof of this result is given in [**31**], for the reader's convenience we repeat the steps here.
Proof. It's not hard to see

$$\begin{aligned}\int_\Omega \int_0^t g(t-s)\Delta v(s).v'(t)dsdx &= -\int_0^t g(t-s)\int_\Omega \nabla v^{'}(t).\nabla v(s)dxds \\ &= -\int_0^t g(t-s)\int_\Omega \nabla v'(t).\left[\nabla v(s) - \nabla v(t)\right]dxds \\ &\quad -\int_0^t g(t-s)\int_\Omega \nabla v'(t).\nabla v(t)dxds.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_\Omega \int_0^t g(t-s)\Delta v(s).v'(t)dsdx &= \frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\int_\Omega |\nabla v(s)-\nabla v(t)|^2\, dxds \\ &\quad -\int_0^t g(s)\left(\frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla v(t)|^2\, dx\right)ds\end{aligned}$$

which implies,

$$\begin{aligned}\int_\Omega \int_0^t g(t-s)\Delta v(s).v'(t)dsdx &= \frac{1}{2}\frac{d}{dt}\left[\int_0^t g(t-s)\int_\Omega |\nabla v(s)-\nabla v(t)|^2\,dxds\right]\\ &-\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)\int_\Omega |\nabla v(t)|^2\,dxds\right]\\ &-\frac{1}{2}\int_0^t g'(t-s)\int_\Omega |\nabla v(s)-\nabla v(t)|^2\,dxds\\ &+\frac{1}{2}g(t)\int_\Omega |\nabla v(t)|^2\,dxds.\end{aligned}$$

This completes the proof. ■

Proof of Lemma 2.1.2.

Existence:

Our main tool is the Faedo-Galerkin's method, which consist to construct approximations of the solutions, then we obtain a prior estimates necessary to guarantee the convergence of approximations. Our proof is organized as follows. In the first step, we define an approach problem in bounded dimension space V_n which having unique solution v_n and in the second step we derive the various a priori estimates. In the third step we will pass to the limit of the approximations by using the compactness of some embedding in the Sobolev spaces.

1. Approach solution:

Let $V = H_0^1(\Omega) \cap H^2(\Omega)$ the separable Hilbert space. Then there exists a family of subspaces $\{V_n\}$ such that

i) $V_n \subset V$ $(\dim V_n < \infty)$, $\forall n \in \mathbb{N}$.

ii) $V_n \to V$, such that, there exist a dense subspace ϑ in V and for all $v \in \vartheta$, we can get sequence $\{v_n\}_{n\in\mathbb{N}} \in V_n$, and $v_n \to v$ in V.

iii) $V_n \subset V_{n+1}$ and $\overline{\cup_{n\in\mathbb{N}^*} V_n} = H_0^1(\Omega) \cap H^2(\Omega)$.

For every $n \geq 1$, let $V_n = Span\{w_1, ..., w_n\}$, where $\{w_i\}$, $1 \leq i \leq n$, is the orthogonal complete system of eigenfunctions of $-\Delta$ such that $\|w_j\|_2 = 1$, $w_j \in H^2(\Omega) \cap L^{2(m-1)}(\Omega)$ for all $j = 1, ..., n$. Denote by $\{\lambda_j\}$ the related eigenvalues, where w_j are solutions of the following initial boundary value problem

$$\begin{cases} -\Delta w_j = \lambda_j w_j & \\ & j = 1, ..., \quad \text{in } \Omega. \\ w_j = 0 \qquad \text{on } \Gamma & \end{cases} \tag{2.10}$$

According to (iii), we can choose $v_{n0},\ v_{n1} \in [w_1, ..., w_n]$ such that

$$v_{n0} \equiv \sum_{j=1}^{n} \alpha_{jn} w_j \longrightarrow u_0 \text{ in } H_0^1(\Omega) \cap H^2(\Omega), \tag{2.11}$$

$$v_{n1} \equiv \sum_{j=1}^{n} \beta_{jn} w_j \longrightarrow u_1 \text{ in } H_0^1(\Omega) \cap L^{2(m-1)}(\Omega), \tag{2.12}$$

where

$$\alpha_{jn} = \int_\Omega u_0 w_j dx$$

$$\beta_{jn} = \int_\Omega u_1 w_j dx.$$

We seek n functions $\varphi_1^n, ..., \varphi_n^n \in C^2[0, T]$, such that

$$v_n(t) = \sum_{j=1}^{n} \varphi_j^n(t) w_j(x), \tag{2.13}$$

solves the problem

$$\begin{cases} \int\limits_\Omega \left((v_n''(t) - \Delta v_n'(t) - \Delta v_n(t) \right) \eta dx \\ + \int\limits_\Omega \left(\int\limits_0^t g(t-s) \Delta v_n(s) ds + |v_n'(t)|^{m-2} v_n'(t) \right) \eta dx \\ = \int\limits_\Omega |u(t)|^{p-2} u(t) \eta dx \\ v_n(0) = v_{n0},\ v_n'(0) = v_{n1} \end{cases}, \tag{2.14}$$

where the prime "′" denotes the derivative with respect to t. For every $\eta \in V_n$ and $t \geq 0$. Taking $\eta = w_j$, in (2.14) yields the following Cauchy problem for a ordinary differential equation with unknown φ_j^n:

$$\begin{cases} \varphi_j''^n(t) + \lambda_j \varphi_j'^n(t) + \lambda_j \varphi_j^n(t) + \lambda_j \int\limits_0^t g(t-s) \varphi_j^n(s) ds \\ + \left| \varphi_j'^h(t) \right|^{m-2} \varphi_j'^n(t) = \psi_j(t) \\ \varphi_j^n(0) = \int\limits_\Omega u_0 w_j,\ \varphi_j'^n(0) = \int\limits_\Omega u_1 w_j, \qquad j = 1, ..., n \end{cases}, \tag{2.15}$$

for all j, where

$$\psi_j(t) = \int\limits_\Omega |u(t)|^{p-2} u(t) w_j \in C[0, T].$$

By using the Caratheodory theorem for an ordinary differential equation, we deduce that, the above Cauchy problem yields a unique global solution $\varphi_j^n \in H^3[0,T]$, and by using the embedding $H^m[0,T] \hookrightarrow C^{m-1}[0,T]$, we deduce that the solution $\varphi_j^n \in C^2[0,T]$. In turn, this gives a unique v_n defined by (2.13) and satisfying (2.14).

2. <u>The a priori estimates</u>:

The next estimate prove that the energy of the problem (2.1) is bounded and by using a result in [**9**], we conclude that, the maximal time t_n of existence of (2.15) can be extended to T.

The first a priori estimate:

Substituting $\eta = v_n'(t)$ into (2.14), we obtain

$$\begin{aligned} &\int_\Omega v_n''(t)v_n'(t)dx - \int_\Omega \Delta v_n'(t)v_n'(t)dx - \int_\Omega \Delta v_n(t)v_n'(t)dx \\ &+\int_\Omega\int_0^t g(t-s)\Delta v_n(s)ds v_n'(t)dx + \int_\Omega |v_n'(t)|^{m-2} v_n'(t)v_n'(t)dx \\ =& \int_\Omega f(u)v_n'(t)dx. \end{aligned} \tag{2.16}$$

for every $n \geq 1$, where $f(u) = |u(t)|^{p-2}u(t)$, Since the following mapping

$$\begin{array}{rcl} L^2(\Omega) & \longrightarrow & L^2(\Omega) \\ u & \mapsto & |u|^{p-2}u \end{array}$$

is continues, we deduce that,

$$|u|^{p-2}u \in L^\infty\left([0,T], L^2(\Omega)\right).$$

So, $f \in H^1([0,T], H^1(\Omega))$. Consequently, by using the Lemma 2.1.3 and $(G1)$ we get easily

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v_n'(t)\|_2^2 + \|\nabla v_n'(t)\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla v_n(t)\|_2^2 \\ &+\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(t-s)\int_\Omega |\nabla v_n(s) - \nabla v_n(t)|^2 dxds\right] - \frac{1}{2}\frac{d}{dt}\left(\|\nabla v_n(t)\|_2^2 \int_0^t g(s).ds\right) \\ &-\frac{1}{2}\int_0^t g'(t-s)\int_\Omega |\nabla v_n(s) - \nabla v_n(t)|^2 dxds + \frac{1}{2}g(t)\|\nabla v_n(t)\|_2^2 + \|v_n'(t)\|_m^m \\ =& \int_\Omega f(u).v_n'(t)dx. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\frac{d}{dt}E_n(t)-\int\limits_{\Omega}f(u).v_n'(t)dx+\|\nabla v_n'(t)\|_2^2+\|v_n'(t)\|_m^m\\ =\ &\frac{1}{2}(g'\circ\nabla v_n)(t)-\frac{1}{2}g(t)\|\nabla v_n(t)\|_2^2,\end{aligned}$$

where

$$E_n(t)=\frac{1}{2}\left\{\|v_n'(t)\|_2^2+\left(1-\int\limits_0^t g(s)ds\right)\|\nabla v_n(t)\|_2^2+(g\circ\nabla v_n)(t)\right\},$$

is the functional energy associated to the problem (2.1).
It's clear by using $(G2)$, that

$$\frac{d}{dt}E_n(t)-\int\limits_{\Omega}f(u).v_n'(t)dx+\|\nabla v_n'(t)\|_2^2+\|v_n'(t)\|_m^m\leq 0,\ \forall t\geq 0.$$

Which implies, by using Young's inequality, for all $\delta>0$

$$\frac{d}{dt}E_n(t)+\|\nabla v_n'(t)\|_2^2+\|v_n'(t)\|_m^m\leq\frac{1}{4\delta}\|f(u)\|_2^2+\delta\|v_n'(t)\|_2^2. \tag{2.17}$$

Integrating (2.17) over $[0,t]$, $(t<T)$, we obtain

$$\begin{aligned}&E_n(t)+\int\limits_0^t\|\nabla v_n'(s)\|_2^2\,ds+\int\limits_0^t\|v_n'(s)\|_m^m\,ds\\ \leq\ &\frac{1}{4\delta}\int\limits_0^T\|f(u)\|_2^2\,ds+\delta\int\limits_0^T\|v_n'(s)\|_2^2\,ds+\frac{1}{2}\left(\|u_{1n}\|_2^2+\|\nabla u_{0n}\|_2^2\right).\end{aligned}$$

Since $f\in H^1(0,T,H_0^1(\Omega))$, we deduce

$$E_n(t)+\int\limits_0^t\|\nabla v_n'(s)\|_2^2\,ds+\int\limits_0^t\|v_n'(s)\|_m^m\,ds\leq C_T\left(\|u_{1n}\|_2^2+\|\nabla u_{0n}\|_2^2\right), \tag{2.18}$$

for, δ small enough and every $n\geq 1$, where $C_T>0$ is positive constant independent of n.
Then, by the definition of $E_n(t)$ and by using (2.11), (2.12), we get

$$\|v_n'(t)\|_2^2+\left(1-\int\limits_0^t g(s)ds\right)\|\nabla v_n(t)\|_2^2+(g\circ\nabla v_n)(t)\leq K_T, \tag{2.19}$$

and

$$\int\limits_0^t\|v_n'(s)\|_m^m\,ds\leq K_T, \tag{2.20}$$

and

$$\int_0^t \|\nabla v_n'(s)\|_2^2 \, ds \leq K_T, \tag{2.21}$$

where $K_T = C_T \left(\|u_{1n}\|_2^2 + \|\nabla u_{0n}\|_2^2 \right)$, by (2.19), we get $t_n = T$, $\forall n$.

However, the insufficient regularity of the nonlinear operator, $|v_t|^{m-2} v_t$, with the presence of the viscoelastic term and strong damping, we must prove in the next a prior estimates that, the family of approximations v_n defined in (2.13) is compact in the strong topology and by using compactness of the embedding $H^1([0,T], H^1(\Omega)) \hookrightarrow L^2([0,T], L^2(\Omega))$, we can extract a subsequence of v_n denoted also by v_τ such that v_τ' converges strongly in $L^2([0,T], L^2(\Omega))$. To do this, it's suffices to prove that v_n' is bounded in $L^\infty([0,T], H_0^1(\Omega))$ and v_n'' is bounded in $L^\infty([0,T], L^2(\Omega))$, then by using Aubin-Lions Lemma, our conclusion holds.

The second a priori estimate:

Substituting $\eta = w_j$ in (2.14) and taking $-\Delta w_j = \lambda_j w_j$, multiplying by $\varphi_j'^n(t)$ and summing up the product result with respect to j, we get by Green's formula.

$$\begin{aligned}
&\int_\Omega \nabla v_n'' . \nabla v_n' dx + \int_\Omega \Delta v_n' . \Delta v_n' dx + \int_\Omega \Delta v_n . \Delta v_n' dx - \int_\Omega \int_0^t g(t-s) \Delta v_n . \Delta v_n' ds dx \\
&+ \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i} \left(|v_n'|^{m-2} v_n' \right) \frac{\partial v_n'}{\partial x_i} dx \\
= &\int_\Omega \nabla f(u) . \nabla v_n' dx.
\end{aligned} \tag{2.22}$$

As in [**26**], we have

$$\begin{aligned}
&\sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i} \left(|v_n'|^{m-2} v_n' \right) \frac{\partial v_n'}{\partial x_i} dx \\
= &(m-1) \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i} \left(|v_n'|^{\frac{m-2}{2}} \frac{\partial v_n'}{\partial x_i} \right)^2 dx \\
= &\frac{4(m-1)}{m^2} \sum_{i=1}^n \int_\Omega \left(\frac{\partial}{\partial x_i} \left(|v_n'|^{\frac{m-2}{2}} v_n' \right) \right)^2 dx.
\end{aligned} \tag{2.23}$$

Also, the fourth term in the left hand side of (2.22) can be written as follows

$$\int_\Omega \int_0^t g(t-s)\Delta v_n.\Delta v_n' ds dx = -\frac{1}{2}g(t)\|\Delta v_n\|_2^2 + \frac{1}{2}(g' \circ \Delta v_n) - \frac{1}{2}\frac{d}{dt}\left\{(g\circ \Delta v_n) - \|\Delta v_n\|_2^2 \int_0^t g(s)ds\right\}. \tag{2.24}$$

Therefore, (2.22) becomes

$$\frac{1}{2}\frac{d}{dt}\left[\|\nabla v_n'\|_2^2 + (g\circ\Delta v_n) + \|\Delta v_n\|_2^2\left(1-\int_0^t g(s)ds\right)\right] + \frac{4(m-1)}{m^2}\sum_{i=1}^n \int_\Omega \left(\frac{\partial}{\partial x_i}\left(|v_n'|^{\frac{m-2}{2}} v_n'\right)\right)^2 dx + \|\Delta v_n'\|_2^2 + \frac{1}{2}g(t)\|\Delta v_n\|_2^2 - \frac{1}{2}(g'\circ\Delta v_n) = \int_\Omega \nabla f(u).\nabla v_n' dx. \tag{2.25}$$

Let us define the energy term

$$K_n(t) = \frac{1}{2}\left[\|\nabla v_n'\|_2^2 + (g\circ\Delta v_n) + \|\Delta v_n\|_2^2\left(1-\int_0^t g(s)ds\right)\right] \tag{2.26}$$

Then, it's clear that (2.25) takes the form

$$\frac{d}{dt}K_n(t) - \int_\Omega \nabla f(u).\nabla v_n' dx + \|\Delta v_n'\|_2^2 + \sum_{i=1}^n \int_\Omega \left(\frac{\partial}{\partial x_i}\left(|v_n'|^{\frac{m-2}{2}} v_n'\right)\right)^2 dx = -\frac{1}{2}(g(t)\|\Delta v_n\|_2^2) + \frac{1}{2}(g'\circ\Delta v_n). \tag{2.27}$$

Using $(G1)$, $(G2)$ and integrating (2.27) over $[0,t]$, we obtain

$$K_n(t) + \sum_{i=1}^n \int_0^t \int_\Omega \left(\frac{\partial}{\partial x_i}\left(|v_n'|^{\frac{m-2}{2}} v_n'\right)\right)^2 dx ds + \int_0^t \|\Delta v_n'\|_2^2 ds \le \int_0^T \int_\Omega \nabla f(u).\nabla v_n' dx ds + \frac{1}{2}(\|\nabla v_{1n}\|_2^2 + \|\Delta v_{0n}\|_2^2). \tag{2.28}$$

Obviously, by using Young's inequality, we get

$$\int_0^T \int_\Omega \nabla f(u).\nabla v_n' dxds \leq \frac{1}{4\delta} \int_0^T \|\nabla f(u)\|_2^2 \, ds + \delta \int_0^T \|\nabla v_n'\|_2^2 \, ds.$$

Inserting the above estimate into (2.28), to get

$$K_n(t) + \sum_{i=1}^n \int_0^t \int_\Omega \left(\frac{\partial}{\partial x_i} \left(|v_n'|^{\frac{m-2}{2}} v_n' \right) \right)^2 dxds + \int_0^t \|\Delta v_n'\|_2^2 \, ds$$

$$\leq \quad C_T(\|\nabla v_{1n}\|_2^2 + \|\Delta v_{0n}\|_2^2).$$

Thus,

$$K_n(t) + \sum_{i=1}^n \int_0^t \int_\Omega \left(\frac{\partial}{\partial x_i} \left(|v_n'|^{\frac{m-2}{2}} v_n' \right) \right)^2 dxds + \int_0^t \|\Delta v_n'\|_2^2 \, ds \leq C_T,$$

for, δ small enough and every $n \geq 1$, where $C_T > 0$ is positive constant independent of n. Therefore, this equivalent by the definition of $K_n(t)$

$$\|\nabla v_n'\|_2^2 + (g \circ \Delta v_n) + \|\Delta v_n\|_2^2 \left(1 - \int_0^t g(s)ds \right) \leq C_T, \tag{2.29}$$

and

$$\sum_{i=1}^n \int_0^t \int_\Omega \left(\frac{\partial}{\partial x_i} \left(|v_n'|^{\frac{m-2}{2}} v_n' \right) \right)^2 dxds \leq C_T, \tag{2.30}$$

and

$$\int_0^t \|\Delta v_n'\|_2^2 \, ds \leq C_T. \tag{2.31}$$

Then, from (2.29), (2.30) and (2.31) , we conclude

$$v_n' \text{ is bounded in } L^\infty([0,T]\,, H_0^1(\Omega)), \tag{2.32}$$

$$v_n \text{ is bounded in } L^\infty([0,T]\,, H^2(\Omega)), \tag{2.33}$$

$$\frac{\partial}{\partial x_i} \left(|v_n'|^{\frac{m-2}{2}} v_n' \right) \text{ is bounded in } L^2([0,T]\,, L^2(\Omega)),\ i = 1, ..., n. \tag{2.34}$$

The third a priori estimate:

It's clear that

$$\frac{d}{dt} \left[\int_0^t g(t-s)\Delta v_n(s)ds \right] = g(0)\Delta v_n + \int_0^t g'(t-s)\Delta v_n(s)ds. \tag{2.35}$$

Performing an integration by parts in (2.35) we find that

$$\frac{d}{dt}\left[\int_0^t g(t-s)\Delta v_n(s)ds\right] = g(t)\Delta u_{n0} + \int_0^t g(t-s)\Delta v_n'(s)ds. \tag{2.36}$$

Now, returning to (2.14), differentiating throughout with respect to t, and using (2.36), we obtain

$$\begin{aligned} & \int_\Omega \left(v_n'''(t) - \Delta v_n''(t) - \Delta v_n'(t)\right)\eta dx \\ & + \int_\Omega \left(\int_0^t g(t-s)\Delta v_n'(s)ds + g(t)\Delta u_{n0} + (m-1)(\left|v_n'(t)\right|^{m-2} v_n''(t))\right)\eta dx \\ = & \int_\Omega (f(u))')\eta dx, \end{aligned} \tag{2.37}$$

where, $(f(u))' = \dfrac{\partial f}{\partial t}$. By substitution of $\eta = v_h''(t)$ in (2.37), yields

$$\begin{aligned} & \int_\Omega v_n'''(t)v_n''(t)dx - \int_\Omega \Delta v_n''(t)v_n''(t)dx - \int_\Omega \Delta v_n'(t)v_n''(t)dx \\ & + \int_\Omega \left(\int_0^t g(t-s)\Delta v_n'(s)ds + g(t)\Delta u_{n0}\right) v_n''(t)dx \\ & +(m-1)\int_\Omega \left|v_n'(t)\right|^{m-2} v_n''(t)v_n''(t)dx \\ = & \int_\Omega^{\prime} (f(u))' \, v_n''(t)dx. \end{aligned} \tag{2.38}$$

The fourth term in the left hand side of (2.38) can be analyzed as follows. It's clear that Lemma 2.1.3 implies

$$\begin{aligned} \int_\Omega \int_0^t g(t-s)\Delta v_n'(s).v_n''(t)dsdx &= -\int_\Omega \int_0^t g(t-s)\nabla v_n'(s).\nabla v_n''(t)dsdx \\ &= \frac{1}{2}\frac{d}{dt}\left(g \circ \nabla v_n'\right)(t) - \frac{1}{2}\frac{d}{dt}\int_0^t g(s)\left\|\nabla v_n'(t)\right\|_2^2 ds \\ & \quad -\frac{1}{2}\left(g' \circ \nabla v_n'\right)(t) + \frac{1}{2}g(t)\left\|\nabla v_n'(t)\right\|_2^2 . \end{aligned} \tag{2.39}$$

Also,

$$(m-1)\langle |v_n'(t)|^{m-2} v_n''(t), v_n''(t)\rangle = \frac{4(m-1)}{m^2}\int\limits_\Omega \left(\frac{\partial}{\partial t}\left(|v_n'|^{\frac{m-2}{2}} v_n'(t)\right)\right)^2 dx. \tag{2.40}$$

Inserting (2.39) and (2.40) into (2.38), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_n''\|_2^2 + \|\nabla v_n''\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla v_n'\|_2^2 + \frac{1}{2}\frac{d}{dt}\left\{(g\circ\nabla v_n')(t) - \|\nabla v_n'\|_2^2 \int\limits_0^t g(s)ds\right\}\\
&+\frac{4(m-1)}{m^2}\int\limits_\Omega \left(\frac{\partial}{\partial t}\left(|v_n'|^{\frac{m-2}{2}} v_n'(t)\right)\right)^2 dx + (g(t)\Delta u_{n0})\int\limits_\Omega v_n''(t)dx \qquad (2.41)\\
= &\int\limits_\Omega (f(u))' v_n''(t)dx + \frac{1}{2}(g'\circ\nabla v_n')(t) - \frac{1}{2}g(t)\|\nabla v_n'\|_2^2.
\end{aligned}$$

Let us denote by

$$\Psi_n(t) = \frac{1}{2}\left\{\|v_n''\|_2^2 + + (g\circ\nabla v_n')(t) + (1 - \int\limits_0^t g(s)ds)\|\nabla v_n'\|_2^2\right\}. \tag{2.42}$$

We obtain, from (2.41), $(G2)$ that

$$\begin{aligned}
&\frac{d}{dt}\Psi_n(t) - \int\limits_\Omega (f(u))' .v_n''(t)dx + \frac{4(m-1)}{m^2}\int\limits_\Omega \left(\frac{\partial}{\partial t}\left(|v_n'|^{\frac{m-2}{2}} v_n'(t)\right)\right)^2 dx\\
&+ \|\nabla v_n''\|_2^2 + (g(0)\Delta u_{n0})\int\limits_\Omega v_n''(t)dx\\
= &\frac{1}{2}(g'\circ\nabla v_n')(t) - \frac{1}{2}g(t)\|\nabla v_n'(t)\|_2^2\\
\le &\ 0.
\end{aligned}$$

Integration the above estimate over $[0,t]$, we conclude

$$\begin{aligned}
&\Psi_n(t) + \frac{4(m-1)}{m^2}\int\limits_0^t\int\limits_\Omega \left(\frac{\partial}{\partial t}\left(|v_n'|^{\frac{m-2}{2}} v_n'(t)\right)\right)^2 dxds\\
&+\int\limits_0^t \|\nabla v_n''\|_2^2 ds + (g(0)\Delta u_{n0})\int\limits_0^t\int\limits_\Omega v_n''(s)dxds\\
\le &\frac{1}{2}\|v_{n0}''\|_2^2 + \frac{1}{2}\|\nabla v_{n1}\|_2^2 + \frac{1}{4\delta}\int\limits_0^T \left\|(f(u))'\right\|_2^2 ds + \delta\int\limits_0^T \|v_n''\|_2^2 ds.
\end{aligned}$$

Then, for δ small enough, we deduce

$$\begin{aligned} &\Psi_n(t) + \frac{4(m-1)}{m^2}\int_0^t\int_\Omega \left(\frac{\partial}{\partial t}\left(|v_n'|^{\frac{m-2}{2}} v_n'\right)\right)^2 dxds \\ &+ \int_0^t \|\nabla v_n''\|_2^2\, ds + (g(0)\Delta u_{n0})\int_0^t\int_\Omega v_n'' dxds \qquad (2.43)\\ \leq\ & C_T\left(\|v_{n0}''\|_2^2 + \|\nabla v_{n1}\|_2^2\right). \end{aligned}$$

In order to estimate the term $\|v_{n0}''\|_2^2$, taking $t=0$ in (2.14), we find

$$\begin{aligned} \|v_{n0}''\|_2^2 \ =\ & \int_\Omega \Delta v_{n1}.v_{n0}'' dx + \int_\Omega \Delta v_{n0}.v_{n0}'' dx \\ & - \int_\Omega |v_{n1}|^{m-2} v_{n1}.v_{n0}'' dx + \int_\Omega f(u(0)).v_{n0}'' dx. \end{aligned}$$

Thanks to Cauchy-Schwartz inequality (Lemma 1.2.10), we write

$$\|v_{n0}''\|_2^2 \leq \|v_{n0}''\|_2 \left(\|\Delta v_{n1}\|_2 + \|\Delta v_{n0}\|_2 + \|f(u(0))\|_2 + \left(\int_\Omega |v_{1n}|^{2(m-1)} dx\right)^{\frac{1}{2}}\right),$$

which implies, by using (2.11) and (2.12), that

$$\|v_{n0}''\|_2 \ \leq\ \|\Delta v_{n1}\|_2 + \|\Delta v_{n0}\|_2 + \|f(u(0))\|_2 + \left(\int_\Omega |v_{1n}|^{2(m-1)} dx\right)^{\frac{1}{2}} \leq C. \qquad (2.44)$$

Then,

$$\Psi_n(t) \leq L_T, \qquad (2.45)$$

$$\frac{4(m-1)}{m^2}\int_0^t\int_\Omega \left(\frac{\partial}{\partial t}\left(|v_n'|^{\frac{m-2}{2}} v_n'\right)\right)^2 dxds \leq L_T, \qquad (2.46)$$

and

$$\int_0^t \|\nabla v_n''\|_2^2\, ds \leq L_T, \qquad (2.47)$$

and

$$\int_0^t\int_\Omega v_n'' dxds \leq L_T. \qquad (2.48)$$

where $L_T > 0$.

From (2.11), (2.12) and (2.44) − (2.48) , we deduce

$$v_n'' \text{ is bounded in } L^\infty([0,T], L^2(\Omega)), \tag{2.49}$$

$$v_n \text{ is bounded in } L^\infty([0,T], H_0^1(\Omega)), \tag{2.50}$$

$$\frac{\partial}{\partial t}\left(|v_n'|^{\frac{m-2}{2}} v_n'\right) \text{ is bounded in } L^2([0,T], L^2(\Omega)),\ i = 1, ..., n. \tag{2.51}$$

3.<u>Pass to the limit</u>:

By the first, the second and the third estimates, we obtain

$$v_n \text{ is bounded in } L^\infty([0,T], H^2(\Omega) \cap H_0^1(\Omega)), \tag{2.52}$$

$$v_n' \text{ is bounded in } L^\infty([0,T], H_0^1(\Omega)), \tag{2.53}$$

$$v_n'' \text{ is bounded in } L^\infty([0,T], L^2(\Omega)), \tag{2.54}$$

$$v_n' \text{ is bounded in } L^m((0,T) \times \Omega). \tag{2.55}$$

Therefore, up to a subsequence, and by using the (Theorem 1.1.5), we observe that there exists a subsequence v_τ of v_n and a function v that we my pass to the limit in (2.14), we obtain a weak solution v of (2.1) with the above regularity

$$v_\tau \rightharpoonup^* v \text{ in } L^\infty([0,T], H_0^1(\Omega) \cap H^2(\Omega)),$$

$$v_\tau' \rightharpoonup^* v' \text{in } L^\infty([0,T], H_0^1(\Omega) \cap L^m(\Omega)),$$

$$v_\tau'' \rightharpoonup^* v'' \text{in } L^\infty([0,T], L^2(\Omega)).$$

By using the fact that

$$L^\infty([0,T], L^2(\Omega)) \hookrightarrow L^2([0,T], L^2(\Omega)),$$

$$L^\infty([0,T], H_0^1(\Omega)) \hookrightarrow L^2([0,T], H_0^1(\Omega)).$$

We get

$$v_n' \text{ is bounded in } L^2([0,T], H_0^1(\Omega)),$$

$$v_n'' \text{ is bounded in } L^2([0,T], L^2(\Omega)),$$

therefore,

$$v_n' \text{ is bounded in } H^1([0,T], H^1(\Omega)). \tag{2.56}$$

Consequently, since the embedding

$$H^1([0,T], H^1(\Omega)) \hookrightarrow L^2([0,T], L^2(\Omega))$$

is compact, then we can extract a subsequence v'_τ such that

$$v'_\tau \to v' \text{ in } L^2([0,T], L^2(\Omega)). \tag{2.57}$$

which implies

$$v'_\tau \to v' \text{ a.e on } (0,T) \times \Omega.$$

By (2.20), we have

$$v'_n \text{ is bounded in } L^m([0,T] \times \Omega).$$

and by using Theorem 1.2.2

$$|v'_\tau|^{m-2} v'_\tau \rightharpoonup \vartheta \text{ in } L^{m'}([0,T], L^{m'}(\Omega))$$

where $\dfrac{m}{m-1} = m'$.
The estimates (2.34) and (2.46) imply that

$$|v'_\tau|^{\frac{m-2}{2}} v'_\tau \rightharpoonup \upsilon \text{ in } H^1([0,T], H^1(\Omega)$$

by using the fact that, the mapping $u \longmapsto |u|^{m-2} u$ is continuous, (Lemma 1.2.9) and since the weak topology is separate, we deduce

$$\vartheta = |v'|^{m-2} v' \tag{2.58}$$

$$\upsilon = |v'|^{\frac{m-2}{2}} v' \tag{2.59}$$

Then, by using the uniqueness of limit, we deduce

$$|v'_\tau|^{m-2} v'_\tau \rightharpoonup |v'|^{m-2} v' \text{ in } L^{m'}([0,T], L^{m'}(\Omega)) \tag{2.60}$$

$$|v'_\tau|^{\frac{m-2}{2}} v'_\tau \rightharpoonup |v'|^{\frac{m-2}{2}} v' \text{ in } H^1([0,T], H^1(\Omega) \tag{2.61}$$

Now, we will pass to the limit in (2.14), by the same techniques as in [**26**].
Taking $\eta = w_j$, $n = \tau$ and fixed $j < \tau$,

$$\begin{aligned} &\int_\Omega v''_\tau(t).w_j dx + \int_\Omega \nabla v_\tau(t).\nabla w_j dx + \int_\Omega \nabla v'_\tau(t).\nabla w_j dx \\ &- \int_\Omega \int_0^t g(t-s)\nabla v_\tau(s).\nabla w_j ds dx + \int_\Omega |v'_\tau(t)|^{m-2} v'_\tau(t).w_j dx \\ = &\int_\Omega f(u).w_j dx. \end{aligned} \tag{2.62}$$

We obtain, by using the property of continuous of the operator in the distributions space

$$\int_\Omega v_\tau''(t).w_j dx \rightharpoonup^* \int_\Omega v''(t).w_j dx, \text{ in } \mathcal{D}'(0,T)$$

$$\int_\Omega \nabla v_\tau(t).\nabla w_j dx \rightharpoonup^* \int_\Omega \nabla v(t).\nabla w_j dx, \text{ in } L^\infty(0,T)$$

$$\int_\Omega \nabla v_\tau'(t).\nabla w_j dx \rightharpoonup^* \int_\Omega \nabla v'(t).\nabla w_j dx, \text{ in } L^\infty(0,T)$$

$$\int_\Omega \int_0^t g(t-s)\nabla v_\tau(s).\nabla w_j ds dx \rightharpoonup^* \int_\Omega \int_0^t g(t-s)\nabla v(s).\nabla w_j ds dx, \text{ in } L^\infty(0,T)$$

$$\int_\Omega |v_\tau'(t)|^{m-2} v_\tau'(t).w_j dx \rightharpoonup^* \int_\Omega |v'(t)|^{m-2} v'(t).w_j dx, \text{ in } L^\infty(0,T)$$

We deduce from (2.62), that

$$\begin{aligned} &\int_\Omega v''(t).w_j dx + \int_\Omega \nabla v(t).\nabla w_j dx + \int_\Omega \nabla v'(t).\nabla w_j dx \\ &- \int_\Omega \int_0^t g(t-s)\nabla v(s).\nabla w_j ds dx + \int_\Omega |v'(t)|^{m-2} v'(t).w_j dx \\ = &\int_\Omega f(u).w_j dx. \end{aligned} \tag{2.63}$$

Since, the basis w_j $(j = 1, ...)$ is dense in $H_0^1(\Omega) \cap H^2(\Omega)$, we can generalize (2.63), as follows

$$\begin{aligned} &\int_\Omega v''(t).\varphi dx + \int_\Omega \nabla v(t).\nabla \varphi dx + \int_\Omega \nabla v'(t).\nabla \varphi dx \\ &- \int_\Omega \int_0^t g(t-s)\nabla v(s).\nabla \varphi ds dx + \int_\Omega |v'(t)|^{m-2} v'(t).\varphi dx \\ = &\int_\Omega f(u).\varphi dx, \ \forall \varphi \in H_0^1(\Omega) \cap H^2(\Omega). \end{aligned}$$

Then,

$$v \in L^\infty\left([0,T], H^2(\Omega) \cap H_0^1(\Omega)\right),$$
$$v_t \in L^\infty\left([0,T], H_0^1(\Omega)\right),$$
$$v_{tt} \in L^\infty\left([0,T], L^2(\Omega)\right),$$
$$v_t \in L^m\left([0,T] \times (\Omega)\right).$$

This complete the our proof of existence.

Uniqueness:

Let v_1, v_2 two solutions of (2.1), and let $w = v_1 - v_2$ satisfying :

$$w'' - \Delta w - \Delta w' + \int_0^t g(t-s)\Delta w ds + \left(|v_1'|^{m-2} v_1' - |v_2'|^{m-2} v_2'\right) = 0. \tag{2.64}$$

Multiplying (2.64), by $w^{'}$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|w_n'(t)\|_2^2 + \left(1 - \int_0^t g(s)ds\right)\|\nabla w_n(t)\|_2^2 + (g \circ \nabla w_n)(t)\right)$$
$$+ \int_\Omega \left(|v_1'|^{m-2} v_1' - |v_2'|^{m-2} v_2'\right) w' dx$$
$$= \; -\|\nabla w_n'(t)\|_2^2 + \frac{1}{2}(g' \circ \nabla w_n)(t) - \frac{1}{2}g(t)\|\nabla w_n(t)\|_2^2 .$$

Denote by

$$J(t) = \|w_n'(t)\|_2^2 + \left(1 - \int_0^t g(s)ds\right)\|\nabla w_n(t)\|_2^2 + (g \circ \nabla w_n)(t). \tag{2.65}$$

Since the function $y \mapsto |y|^{m-2} y$ is increasing, we have

$$\int_\Omega \left(|v_1'|^{m-2} v_1' - |v_2'|^{m-2} v_2'\right) w' dx \geq 0$$

and since

$$\frac{1}{2}(g' \circ \nabla w_n)(t) \leq 0,$$

we deduce

$$\left(\frac{d}{dt}J(t) \leq 0\right). \tag{2.66}$$

This implies that $J(t)$ is uniformly bounded by $J(0)$ and is decreasing in t, since $w(0) = 0$, we obtain $w = 0$ and $v_1 = v_2$.

Proof of Lemma 2.1.1.

As in [**14**], since $\overline{\mathcal{D}(\Omega)} = H^2(\Omega)$, we approximate, u_0, u_1 by sequences $(u_{\eta 0}), (u_{\eta 1})$ in $\mathcal{D}(\Omega)$, and u by a sequence (u^η) in $C([0,T], \mathcal{D}(\Omega))$, for the problem (2.1). Lemma 2.1.2 guarantees the existence of a sequence of unique solutions (v^η) satisfying $(2.6)-(2.9)$. Now, to complete the proof of Lemma 2.1.1, we proceed to show that the sequence (v^η) is Cauchy in Y_T equipped with the norm

$$\|u\|_{Y_T}^2 = \|u\|_H^2 + \|u_t\|_{L^m([0,T]\times\Omega)}^2 .$$

where

$$\|u\|_H^2 = \max_{0\le t\le T}\left\{\int_\Omega \left[u_t^2 + l\,|\nabla u|^2\right](x,t)dx\right\}$$

Denote $w = v^\mu - v^\xi$ for μ, ξ given. Then w is a solution of the Cauchy problem:

$$\begin{cases} w_{tt} - \Delta w - \Delta w_t + L(w) + k(v_t^\mu) - k(v_t^\xi) \\ = f(u^\mu) - f(u^\xi), \quad x\in\Omega, \quad t>0 \\ w(0,x) = u_{\mu 0} - u_{\xi 0},\ w_t(0,x) = u_{\mu 1} - u_{\xi 1}, \quad x\in\Omega \\ w(t,x) = 0, \quad x\in\Gamma, \quad t>0 \end{cases}, \tag{2.67}$$

where,

$$k(v_t^\mu) = |v_t^\mu|^{m-2}\,v_t^\mu$$

$$f(u^\mu) = |u^\mu|^{p-2}\,u^\mu$$

$$L(w) = \int_0^t g(t-s)\Delta w(s,x)ds.$$

The energy equality reads as

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{\|w_t\|_2^2 + \left(1 - \int_0^t g(s)ds\right)\|\nabla w\|_2^2 + (g\circ\nabla w)(t)\right\} \\ &+ \int_\Omega \left(k(v_t^\mu) - k(v_t^\xi)\right) w_t dx + \|\nabla w_t\|_2^2 \\ = &\int_\Omega \left(f(u^\mu) - f(u^\xi)\right) w_t dx + \frac{1}{2}\left(g'\circ\nabla w\right)(s)ds - \frac{1}{2}\|\nabla w(s)\|_2^2 \int_0^t g(s)ds. \end{aligned} \tag{2.68}$$

The term,

$$\int_\Omega \left(k(v_t^\mu) - k(v_t^\xi)\right) w_t dx = \int_\Omega \left(|v_t^\mu|^{m-2}\,v_t^\mu - \left|v_t^\xi\right|^{m-2} v_t^\xi\right)\left(v_t^\mu - v_t^\xi\right)dx$$

is nonnegative.

We need to estimate

$$\left|\int_\Omega \left(f(u) - f(\overline{u})\right)(v - \overline{v})\,dx\right| \le C(\|u\|_H + \|\overline{u}\|_H)^{p-2}\,\|u - \overline{u}\|_H\,\|v - \overline{v}\|_H\,,$$

fulfilled for $u, \overline{u}, v, \overline{v} \in H_0^1(\Omega)$, where C is a constant depending on Ω, l, p only. Then, Hölder's inequality yields, for $\frac{1}{q} + \frac{1}{n} + \frac{1}{2} = 1$ ($q = \frac{2n}{n-2}$),

$$\left| \int_\Omega \left(f(u^\mu) - f(u^\xi) \right) w_t dx \right| = \left| \int_\Omega \left(|u^\mu|^{p-2} u^\mu - \left|u^\xi\right|^{p-2} u^\xi \right) \left(v_t^\mu - v_t^\xi \right) dx \right|$$
$$\leq C \left\|u^\mu - u^\xi\right\|_{L^q} \left\|v_t^\mu - v_t^\xi\right\|_{L^2} \left(\|u^\mu\|_{n(p-2)}^{p-2} + \left\|u^\xi\right\|_{n(p-2)}^{p-2} \right). \tag{2.69}$$

The Sobolev embedding $L^q \hookrightarrow H_0^1(\Omega)$ gives

$$\left\|u^\mu - u^\xi\right\|_{L^q(\Omega)} \leq C \left\|\nabla u^\mu - \nabla u^\xi\right\|_{L^2(\Omega)}.$$

Then,

$$\|u^\mu\|_{n(p-2)}^{p-2} + \left\|u^\xi\right\|_{n(p-2)}^{p-2} \leq C(\|u^\mu\|_{L^2(\Omega)}^{p-2} + \left\|u^\xi\right\|_{L^2(\Omega)}^{p-2}).$$

The necessity to estimate $\|u^\mu\|_{n(p-2)}$ by the energy norm $\|u\|_H$ requires a restriction on p. Namely, we need $n(p-2) \leq \frac{2n}{n-2}$, then the Sobolev embedding $L^q \hookrightarrow H_0^1(\Omega)$ gives

$$\|u^\mu\|_{n(p-2)}^{p-2} \leq \|u\|_H^{p-2}.$$

Therefore, (2.69) takes the form

$$\left| \int_\Omega \left(f(u^\mu) - f(u^\xi) \right) w_t dx \right|$$
$$\leq C \left\|v_t^\mu - v_t^\xi\right\|_{L^2(\Omega)} \left\|\nabla u^\mu - \nabla u^\xi\right\|_{L^2(\Omega)} \left(\|\nabla u^\mu\|_{L^2(\Omega)}^{p-2} + \left\|\nabla u^\xi\right\|_{L^2(\Omega)}^{p-2} \right) \tag{2.70}$$

under the fact that

$$\int_0^t \left(g' \circ \nabla w \right)(s) ds \leq 0,$$

we conclude

$$-\int_0^t \left(g' \circ \nabla w \right)(s) ds + \left(g \circ \nabla w \right)(t) + \|\nabla w(t)\|_2^2 \int_0^t g(s) ds \geq 0.$$

Thus,

$$\frac{1}{2} \|w(t,.)\|_H^2 \leq \frac{1}{2} \|w(0,.)\|_H^2 + C \int_0^t \left\|\nabla u^\mu - \nabla u^\xi\right\|_{L^2(\Omega)} \|w_t(s,.)\|_H ds.$$

The Gronwell Lemma and Young's inequality guarantee that

$$\|w(t,.)\|_H \leq \|w(0,.)\|_H + CT \left\|u^\mu - u^\xi\right\|_{C([0,T],H)}.$$

Since

$$\left\|v^{\mu}(t,.)-v^{\xi}(t,.)\right\|_{H}\leq C\left\|v^{\mu}(0,.)-v^{\xi}(0,.)\right\|_{H}+CT\left\|u^{\mu}-u^{\xi}\right\|_{C([0,T],H)}, \tag{2.71}$$

then $\{v^{\eta}\}$ is a Cauchy sequence in $C([0,t],H)$, since $\{u^{\eta}\}$ and $\{v^{\eta}(0,.)\}$ are Cauchy sequences in $C([0,T],H)$ and H, respectively.

Now, we shall prove that $\{v_t^{\mu}\}$ is a Cauchy sequence, in $L^m([0,T]\times\Omega)$, to control the norm $\|v_t^{\eta}\|^2_{L^m([0,T]\times\Omega)}$. By the following algebraic inequality

$$\left(\alpha\left|\alpha\right|^{m-2}-\beta\left|\beta\right|^{m-2}\right)(\alpha-\beta)\geq C\left|\alpha-\beta\right|^{m}, \tag{2.72}$$

which holds for any real $\alpha,\ \beta$ and c, we get

$$\begin{aligned}\int_{\Omega}\left(k(v_t^{\mu})-k(v_t^{\xi})\right)w_t dx &= \int_{\Omega}\left(v_t^{\mu}\left|v_t^{\mu}\right|^{m-2}-v_t^{\xi}\left|v_t^{\xi}\right|^{m-2}\right)\left(v_t^{\mu}-v_t^{\xi}\right)dx\\ &\leq C\left|v_t^{\mu}-v_t^{\xi}\right|^2_{L^m([0,T]\times\Omega)}.\end{aligned}$$

This estimate combined with (2.68) gives

$$\begin{aligned}\left\|v_t^{\mu}-v_t^{\xi}\right\|^2_{L^m([0,t]\times\Omega)} &\leq C\left\|v^{\mu}(0,.)-v^{\xi}(0,.)\right\|_{L^m([0,t]\times\Omega)}\\ &\quad+C_R\int_0^t\left\|u^{\mu}-u^{\xi}\right\|_{L^m([0,t]\times\Omega)}\left\|v^{\mu}(t,.)-v^{\xi}(t,.)\right\|_{L^m([0,t]\times\Omega)}ds.\end{aligned}$$

So by using Gromwell Lemma, we obtain $\{v_t^{\mu}\}$ is a Cauchy sequence, in $L^m([0,T]\times\Omega)$ and hence $\{v^{\eta}\}$ is a Cauchy sequence in Y_T. Let v its limit in Y_T and by Lemma 2.1.2, v is a weak solution of (2.1).

Now, we are ready to show the local existence of the problem (P)

Proof of Theorem 2.1.1.

Let $(u_0,u_1)\in\left(H_0^1(\Omega)\right)^2$, and

$$R^2=\left(\|\nabla u_0\|_2^2+\|u_1\|_2^2\right).$$

For any $T>0$, consider

$$M_T=\left\{u\in Y_T : u(0)=u_0, u_t(0)=u_1 \text{ and } \|u\|_{Y_T}\leq R\right\}.$$

Let

$$\Phi:\quad M_T\to M_T$$

$$u\mapsto v=\Phi(u).$$

We will prove as in [**13**] that,

(i) $\Phi(M_T) \subseteq M_T$.

(ii) Φ is contraction in M_T.

Beginning by the first assertion. By Lemma 2.1.1, for any $u \in M_T$ we may define $v = \Phi(u)$, the unique solution of problem (2.1). We claim that, for a suitable $T > 0$, Φ is contractive map satisfying

$$\Phi(M_T) \subseteq M_T.$$

Let $u \in M_T$, the corresponding solution $v = \Phi(u)$ satisfies for all $t \in [0, T]$ the energy identity :

$$\begin{aligned} & \frac{1}{2}\left\{ \|v'(t)\|_2^2 + \left(1 - \int_0^t g(s)ds\right) \|\nabla v(t)\|_2^2 + (g \circ \nabla v)(t) \right\} \\ & + \int_0^t \|\nabla v'(s)\|_2^2 \, ds + \int_0^t \|v'(s)\|_m^m) ds \\ = \; & \frac{1}{2}\left[\|v_1\|_2^2 + \|\nabla v_0\|_2^2\right] + \int_0^t \int_\Omega |u(s)|^{p-2} u(s) v'(t) dx ds. \end{aligned} \tag{2.73}$$

We get

$$\frac{1}{2}\|v(t)\|_{Y_T}^2 \le \frac{1}{2}\|v(0)\|_{Y_T}^2 + \int_0^t \int_\Omega |u(s)|^{p-2} u(s) v'(t) dx ds. \tag{2.74}$$

We estimate the last term in the right-hand side in (2.74) as follows: thanks to Hölder's, Young's inequalities, we have

$$\int_\Omega |u(s)|^{p-2} u(s) v'(t) dx \le C \|u\|_{Y_T}^p \|v\|_{Y_T},$$

then,

$$\|v(t)\|_{Y_T}^2 \le \|v(0)\|_{Y_T}^2 + CR^p \int_0^t \|v\|_{Y_T} ds,$$

where C depending only on T, R. Recalling that u_0, u_1 converge, then

$$\|v(t)\|_{Y_T} \le \|v(0)\|_{Y_T} + CR^p T.$$

Choosing T sufficiently small, we get$\|v\|_{Y_T} \le R$, which shows that

$$\Phi(M_T) \subseteq M_T.$$

Now, we prove that Φ is contraction in M_T. Taking w_1 and w_2 in M_T, subtracting the two equations in (2.1), for $v_1 = \Phi(w_1)$ and $v_2 = \Phi(w_2)$, and setting $v = v_1 - v_2$, we obtain for all $\eta \in H_0^1(\Omega)$ and a.e. $t \in [0, T]$

$$\begin{aligned} &\int_\Omega v_{tt}.\eta dx + \int_\Omega \nabla v \nabla \eta dx + \int_\Omega \nabla v_t \nabla \eta dx + \int_\Omega \int_0^t g(t-s) \nabla v \nabla \eta ds dx \\ &+ \int_\Omega \left(|v_t|^{m-2} v_t \right) \eta dx \\ = &\int_\Omega \left(|w_1|^{p-2} w_1 - |w_2|^{p-2} w_2 \right) \eta dx. \end{aligned} \tag{2.75}$$

Therefore, by taking $\eta = v_t$ in (2.75) and using the same techniques as above, we obtain

$$\|v(t,.)\|_{Y_T}^2 \leq C \int_0^t \left(\|w_1\|_{Y_T}^{p-2} + \|w_2\|_{Y_T}^{p-2} \right) \|w_1 - w_2\|_{Y_T} \|v(s,.)\|_{Y_T}^2 ds. \tag{2.76}$$

It's easy to see that

$$\|v(t,.)\|_{Y_t}^2 = \|\Phi(w_1) - \Phi(w_2)\|_{Y_t}^2 \leq \alpha \|w_1 - w_2\|_{Y_t}^2, \tag{2.77}$$

for some $0 < \alpha < 1$ where $\alpha = 2CTR^{P-2}$.

Finally by the contraction mapping theorem together with (2.77), we obtain that there exists a unique weak solution u of $u = \Phi(u)$ and as $\Phi(u) \in Y_T$ we have $u \in Y_T$. So there exists a unique weak solution u to our problem (P) defined on $[0, T]$, The main statement of Theorem 2.1.1 is proved.

Remark 2.1.1 *Let us mention that in our problem (P) the existence of the term source ($f(u) = |u|^{p-2} u$) in the right hand forces us to use the contraction mapping theorem. Since we assume a little restriction on the initial data. To this end, let us mention again that our result holds by the well depth method, by choosing the initial data satisfying a more restrictions.*

Chapter 3

Global Existence and Energy Decay

Abstract

In this chapter, we prove that the solution obtained in the second chapter (Local solution) is global in time. In addition, we show that the energy of solutions decays exponentially if $m = 2$ and polynomial if $m > 2$, provided that the initial data are small enough. The existence of the source term $\left(|u|^{p-2}u\right)$ forces us to use the potential well depth method in which the concept of so-called stable set appears. We will make use of arguments in [**44**] with the necessary modifications imposed by the nature of our problem.

3.1 Global Existence Result

In order to state and prove our results, we introduce the functional

$$I(t) = I(u(t)) = \left(1 - \int_0^t g(s)ds\right) \|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) - b\,\|u(t)\|_p^p\,, \qquad (3.1)$$

and

$$J(t) = J(u(t)) = \frac{1}{2}\left(1 - \int_0^t g(s)ds\right) \|\nabla u(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{b}{p}\,\|u(t)\|_p^p\,, \qquad (3.2)$$

for $u(t,x) \in H_0^1(\Omega)\,,\ t \geq 0$.

As in [**19**], the potential well depth, is defined as

$$d = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \sup_{\lambda \geq 0} J(\lambda u)\,. \qquad (3.3)$$

The functional energy associated to (P) is defined as fallows

$$E(u(t), u_t(t)) = E(t) = \frac{1}{2}\,\|u_t(t)\|_2^2 + J(t). \qquad (3.4)$$

Now, we introduce the stable set as follows:

$$W = \left\{u \in H_0^1(\Omega) : J(u) < d,\ I(u) > 0\right\} \cup \{0\}\,. \qquad (3.5)$$

We will prove the invariance of the set W. That is if for some $t_0 > 0$ if $u(t_0) \in W$, then $u(t) \in W$, $\forall t \geq t_0$.

Lemma 3.1.1 *d is positive constant.*

Proof. We have

$$J(\lambda u) = \frac{\lambda^2}{2}\left(\left(1 - \int_0^t g(s)ds\right) \|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t)\right) - \frac{b}{p}\lambda^p\,\|u(t)\|_p^p\,. \qquad (3.6)$$

Using $(G1)$, $(G2)$ to get

$$J(\lambda u) \geq K(\lambda),$$

where $K(\lambda) = \dfrac{\lambda^2}{2} l\,\|\nabla u\|_2^2 - \dfrac{b}{p}\lambda^p\,\|u\|_p^p\,.$

By differentiating the second term in the last equality with respect to λ, to get

$$\frac{d}{d\lambda}K(\lambda)=\lambda l\left\|\nabla u\right\|_2^2-b\lambda^{p-1}\left\|u\right\|_p^p. \tag{3.7}$$

For, $\lambda_1=0$ and $\lambda_2=\left(\frac{l\left\|\nabla u\right\|_2^2}{b\left\|u\right\|_p^p}\right)^{\frac{1}{p-2}}$, then we have

$$\frac{d}{d\lambda}K(\lambda)=0.$$

As $K(\lambda_1)=0$, we have

$$\begin{aligned}K(\lambda_2) &= \frac{1}{2}\left(\frac{l\left\|\nabla u\right\|_2^2}{b\left\|u\right\|_p^p}\right)^{\frac{2}{p-2}} l\left\|\nabla u\right\|_2^2-\frac{b}{p}\left(\frac{l\left\|\nabla u\right\|_2^2}{b\left\|u\right\|_p^p}\right)^{\frac{p}{p-2}}\left\|u\right\|_p^p\\ &= \frac{1}{2}b^{\frac{-2}{p-2}}(l)^{\frac{p}{p-2}}\left(\left\|u\right\|_p^p\right)^{\frac{-2}{p-2}}\left(\left\|\nabla u\right\|_2^2\right)^{\frac{p}{p-2}}\\ &\quad -\frac{1}{p}b^{\frac{-2}{p-2}}(l)^{\frac{p}{p-2}}\left(\left\|u\right\|_p^p\right)^{-\frac{2}{p-2}}\left(\left\|\nabla u\right\|_2^2\right)^{\frac{p}{p-2}}\\ &= (l)^{\frac{p}{P-2}}b^{\frac{-2}{p-2}}\left(\frac{1}{2}-\frac{1}{p}\right)\left\|\nabla u\right\|_2^{\frac{2p}{p-2}}\left\|u\right\|_p^{\frac{-2p}{p-2}}.\end{aligned} \tag{3.8}$$

By Sobolev-Poincaré's inequality, we deduce that $K(\lambda_2)>0$. Then, we obtain

$$\begin{aligned}\sup\left\{J(\lambda u),\ \lambda\geq 0\right\} &\geq \sup\left\{K(\lambda),\ \lambda\geq 0\right\}\\ &> 0.\end{aligned} \tag{3.9}$$

Then, by the definition of d, we conclude that $d>0$. ■

Lemma 3.1.2 *([**19**]) W is a bounded neighbourhood of 0 in $H_0^1(\Omega)$.*

Proof. For $u\in W$, and $u\neq 0$, we have

$$\begin{aligned}J(t) &= \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\left\|\nabla u(t)\right\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{b}{p}\left\|u(t)\right\|_p^p\\ &= \left(\frac{p-2}{2p}\right)\left[\left(1-\int_0^t g(s)ds\right)\left\|\nabla u(t)\right\|_2^2+(g\circ\nabla u)(t)\right]+\frac{1}{p}I(u(t))\\ &\geq \left(\frac{p-2}{2p}\right)\left[\left(1-\int_0^t g(s)ds\right)\left\|\nabla u(t)\right\|_2^2+(g\circ\nabla u)(t)\right].\end{aligned} \tag{3.10}$$

By using $(G1)$ and $(G2)$ then (3.10) becomes

$$\begin{aligned} J(t) &\geq \left(\frac{p-2}{2p}\right)\left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 \\ &\geq l\left(\frac{p-2}{2p}\right)\|\nabla u(t)\|_2^2 \end{aligned}$$

then,

$$\begin{aligned} \|\nabla u(t)\|_2^2 &\leq \frac{1}{l}\left(\frac{2p}{p-2}\right)J(t) \\ &< \frac{1}{l}\left(\frac{2p}{p-2}\right)d = R. \end{aligned}$$

Consequently, $\forall u \in W$ we have $u \in B$ where

$$B = \left\{u \in H_0^1(\Omega) : \|\nabla u(t)\|_2^2 < R\right\}. \tag{3.11}$$

This completes the proof. ■

Now, we will show that our local solution $u(t,x)$ is global in time, for this purpose it suffices to prove that the norm of the solution is bounded, independently of t, this is equivalent to prove the following theorem.

Theorem 3.1.1 *Suppose that $(G1), (G2)$ and (2.3) hold. if $u_0 \in W$, $u_1 \in H_0^1(\Omega)$ and*

$$\frac{bC_*^p}{l}\left(\frac{2p}{(p-2)\,l}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{3.12}$$

where C_* is the best Poincaré's constant. Then the local solution $u(t,x)$ is global in time.

Remark 3.1.1 *Let us remark, that if there exists $t_0 \in [0,T)$ such that $u(t_0) \in W$ and $u_t(t_0) \in H_0^1(\Omega)$ and condition (3.12) holds for t_0. Then the same result of theorem 3.1.1 stays true.*

Before we prove our results, we need the following Lemma, which means that, our energy is uniformly bounded and decreasing along the trajectories.

Lemma 3.1.3 *([44]) Suppose that $(G1)$, $(G2)$, (2.3) hold, and let $(u_0,u_1) \in \left(H_0^1(\Omega)\right)^2$. Let $u(t,x)$ be the solution of (P), then the modified energy $E(t)$ is non-increasing function for almost every $t \in [0,T)$, and*

$$\begin{aligned} \frac{d}{dt}E(t) &= -a\|u_t(t)\|_m^m + \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 - w\|\nabla u_t(t)\|_2^2 \\ &\leq 0,\ \forall t \in [0,T). \end{aligned} \tag{3.13}$$

Proof. By multiplying the differential equation in (P) by u_t and integration over Ω we obtain

$$
\begin{aligned}
& \frac{d}{dt}\left\{\frac{1}{2}\left\|u_t(t)\right\|_2^2+\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\left\|\nabla u(t)\right\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{b}{p}\left\|u(t)\right\|_p^p\right\} \\
= & -a\left\|u_t(t)\right\|_m^m+\frac{1}{2}\left(g'\circ\nabla u\right)(t)-\frac{1}{2}g(t)\left\|\nabla u(t)\right\|_2^2-w\left\|\nabla u_t(t)\right\|_2^2 \\
\leq & \ \frac{1}{2}\left(g'\circ\nabla u\right)(t)\leq 0,\ \forall t\in[0,T).
\end{aligned}
$$

By the definition of $E(t)$, we conclude

$$
\frac{d}{dt}E(t)\leq 0. \tag{3.14}
$$

This completes the proof. ■

The following lemma tells us that if the initial data (or for some $t_0>0$) is in the sat W, then the solution stays there forever.

Lemma 3.1.4 *(*[44]*) Suppose that* $(G1)$, $(G2)$, (2.3) *and* (3.12) *hold. If* $u_0\in W$, $u_1\in H_0^1(\Omega)$, *then the solution* $u(t)\in W$, $\forall t\geq 0$.

Proof. Since $u_0\in W$, then

$$
I(0)=\left\|\nabla u_0\right\|_2^2-\left\|u_0\right\|_p^p>0,
$$

consequently, by continuity, there exists $T_m\leq T$ such that

$$
I(u(t))=\left(1-\int_0^t g(s)ds\right)\left\|\nabla u(t)\right\|_2^2+(g\circ\nabla u)(t)-b\left\|u(t)\right\|_p^p\geq 0,\quad \forall t\in[0,T_m].
$$

This gives

$$
\begin{aligned}
J(t) &= \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\left\|\nabla u(t)\right\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{b}{p}\left\|u(t)\right\|_p^p \\
&= \left(\frac{p-2}{2p}\right)\left[\left(1-\int_0^t g(s)ds\right)\left\|\nabla u(t)\right\|_2^2+(g\circ\nabla u)(t)\right]+\frac{1}{p}I(u(t)) \\
&\geq \left(\frac{p-2}{2p}\right)\left[\left(1-\int_0^t g(s)ds\right)\left\|\nabla u(t)\right\|_2^2+(g\circ\nabla u)(t)\right].
\end{aligned} \tag{3.15}
$$

By using (3.1), (3.15) and the fact that $\int\limits_0^t g(s)ds \leq \int\limits_0^\infty g(s)ds$, we easily see that

$$\begin{aligned} \|\nabla u(t)\|_2^2 &\leq \frac{1}{l}\left(\frac{2p}{p-2}\right) J(t) \\ &\leq \frac{1}{l}\left(\frac{2p}{p-2}\right) E(u(t)) \\ &\leq \frac{1}{l}\left(\frac{2p}{p-2}\right) E(0), \qquad \forall t \in [0, T_m]. \end{aligned} \tag{3.16}$$

We then exploit $(G1)$, (3.12), (3.16), and we note that the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, we have

$$\|u(t)\|_p \leq C \|\nabla u(t)\|_2 \tag{3.17}$$

for $2 < p \leq \dfrac{2n}{n-2}$ if $n \geq 3$, or $p > 2$ if $n = 1, 2$, and $C = C(n, p, \Omega)$.
Consequently, we have

$$\begin{aligned} b\|u(t)\|_p^p &\leq bC_*^p \|\nabla u(t)\|_2^p, \ \forall t \in [0, T_m] \\ &\leq bC_*^p \|\nabla u(t)\|_2^{p-2} \|\nabla u(t)\|_2^2 \\ &\leq \frac{bC_*^p}{l} \|\nabla u(t)\|_2^{p-2} l \|\nabla u(t)\|_2^2 \\ &\leq \beta l \|\nabla u(t)\|_2^2, \end{aligned} \tag{3.18}$$

which means by the definition of l

$$\begin{aligned} b\|u(t)\|_p^p &\leq \beta\left(1 - \int\limits_0^t g(s)ds\right) \|\nabla u(t)\|_2^2 \\ &< \left(1 - \int\limits_0^t g(s)ds\right) \|\nabla u(t)\|_2^2, \qquad \forall t \in [0, T_m]. \end{aligned}$$

where

$$\beta = \frac{bC_*^p}{l}\left(\frac{2p}{(p-2)\,l}E(0)\right)^{\frac{p-2}{2}}. \tag{3.19}$$

Therefore,

$$I(t) = \left(1 - \int\limits_0^t g(s)ds\right) \|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) - b\|u(t)\|_p^p > 0. \tag{3.20}$$

for all $t \in [0, T_m]$,

By taking the fact that

$$\lim_{t \mapsto T_m} \frac{bC_*^p}{l} \left(\frac{2p}{(p-2)\,l} E(0) \right)^{\frac{p-2}{2}} \leq \beta < 1. \tag{3.21}$$

This shows that the solution $u(t) \in W$, for all $t \in [0, T_m]$. By repeating this procedure T_m extended to T. ■

Proof of Theorem 3.1.1.

In order to prove theorem 3.1.1, it suffices to show that the following norm

$$\|\nabla u(t)\|_2 + \|u_t(t)\|_2, \tag{3.22}$$

is bounded independently of t.

To achieve this, we use (3.4), (3.14) and (3.15) to get

$$\begin{aligned}
E(0) &\geq E(t) = J(t) + \frac{1}{2} \|u_t(t)\|_2^2 \\
&\geq \left(\frac{p-2}{2p} \right) \left[\left(1 - \int_0^t g(s)ds \right) \|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) \right] \\
&\quad + \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{p} I(t) \\
&\geq \left(\frac{p-2}{2p} \right) \left[l \|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) \right] + \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{p} I(t) \\
&\geq \left(\frac{p-2}{2p} \right) \left(l \|\nabla u(t)\|_2^2 + \frac{1}{2} \|u_t(t)\|_2^2 \right),
\end{aligned} \tag{3.23}$$

since $I(t)$ and $(g \circ \nabla u)(t)$ are positive, hence

$$\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \leq CE(0),$$

where C is a positive constant depending only on p and l.

This completes the proof of theorem 3.1.1.

The following lemma is very useful

Lemma 3.1.5 *([**44**]) Suppose that* (2.3) *and* (3.12) *hold. Then*

$$b\left\|u(t)\right\|_p^p \leq (1-\eta)\left(1-\int_0^t g(s)ds\right)\left\|\nabla u(t)\right\|_2^2 \tag{3.24}$$

where $\eta = 1-\beta$.

3.2 Decay of Solutions

We can now state the asymptotic behavior of the solution of (P).

Theorem 3.2.1 *Suppose that* $(G1)$, $(G2)$ *and* (2.3) *hold. Assume further that* $u_0 \in W$ *and* $u_1 \in H_0^1(\Omega)$ *satisfying* (3.12). *Then the global solution satisfies*

$$E(t) \leq E(0)\exp(-\lambda t), \ \forall t \geq 0 \text{ if } m = 2, \tag{3.25}$$

or

$$E(t) \leq \left(E(0)^{-r} + K_0 r t\right)^s, \ \forall t \geq 0 \text{ if } m > 2, \tag{3.26}$$

where λ *and* K_0 *are constants independent of* t, $r = \dfrac{m}{2} - 1$ *and* $s = \dfrac{2}{2-m}$.

The following Lemma will play a decisive role in the proof of our result. The proof of this lemma was given in Nakao [**34**].

Lemma 3.2.1 *([**37**]) Let* $\varphi(t)$ *be a nonincreasing and nonnegative function defined on* $[0,T]$, $T > 1$, *satisfying*

$$\varphi^{1+r}(t) \leq k_0\left(\varphi(t) - \varphi(t+1)\right), \qquad t \in [0,T],$$

for $k_0 > 1$ *and* $r \geq 0$. *Then we have, for each* $t \in [0,T]$,

$$\begin{cases} \varphi(t) \leq \varphi(0)\exp\left(-k[t-1]^+\right), & r = 0 \\ \\ \varphi(t) \leq \left\{\varphi(0)^{-r} + k_0 r [t-1]^+\right\}^{\frac{-1}{r}}, & r > 0 \end{cases}, \tag{3.27}$$

where $[t-1]^+ = \max\{t-1, 0\}$, *and* $k = \ln\left(\dfrac{k_0}{k_0-1}\right)$.

Proof of Theorem 3.2.1.

Multiplying the first equation in (P), by u_t and integrate over Ω, to obtain

$$\frac{d}{dt}E(t)+w\left\|\nabla u_t\right\|_2^2+a\left\|u_t\right\|_m^m=\frac{1}{2}\left(g'\circ\nabla u\right)(t)-\frac{1}{2}g(t)\left\|\nabla u(t)\right\|_2^2.$$

Then, integrate the last equality over $[t,t+1]$ to get

$$\begin{aligned}&E(t+1)-E(t)+w\int_t^{t+1}\left\|\nabla u_t\right\|_2^2 ds+a\int_t^{t+1}\left\|u_t\right\|_m^m ds\\ =&\int_t^{t+1}\frac{1}{2}\left(g'\circ\nabla u\right)(s)ds-\int_t^{t+1}\frac{1}{2}g(s)\left\|\nabla u(t)\right\|_2^2 ds.\end{aligned}$$

Therefore,

$$E(t)-E(t+1)=F^m(t)-\frac{1}{2}\int_t^{t+1}\left(g'\circ\nabla u\right)(s)ds+\frac{1}{2}\int_t^{t+1}g(t)\left\|\nabla u(t)\right\|_2^2 ds, \tag{3.28}$$

where

$$F^m(t)=a\int_t^{t+1}\left\|u_t\right\|_m^m ds+w\int_t^{t+1}\left\|\nabla u_t\right\|_2^2 ds. \tag{3.29}$$

Using Poincaré's inequality to find

$$\int_t^{t+1}\left\|u_t\right\|_2^2 ds\leq C\left(\Omega\right)\int_t^{t+1}\left\|u_t\right\|_m^2 ds. \tag{3.30}$$

Exploiting Hölder's inequality, we obtain

$$\begin{aligned}\int_t^{t+1}\left\|u_t\right\|_m^2 ds &\leq \left(\int_t^{t+1}ds\right)^{\frac{m-2}{m}}\left(\int_t^{t+1}\left(\left\|u_t\right\|_m^2\right)^{\frac{m}{2}}\right)^{\frac{2}{m}} ds\\ &\leq \left(\int_t^{t+1}\left(\left\|u_t\right\|_m^2\right)^{\frac{m}{2}}\right)^{\frac{2}{m}} ds.\end{aligned} \tag{3.31}$$

Combining (3.29), (3.30), and (3.31), we obtain, for a constant C_1, depending on Ω

$$\int_t^{t+1}\left\|u_t\right\|_2^2 ds\leq C_1F^2(t),\ C_1>0. \tag{3.32}$$

By applying the mean value theorem, (Theorem 1.3.3. in chapter1), we get for some $t_1 \in \left[t,\ t+\frac{1}{4}\right]$, $t_2 \in \left[t+\frac{3}{4},\ t+1\right]$

$$\|u_t(t_i)\|_2 \leq 2C(\Omega)^{\frac{1}{2}} F(t),\ i=1,\ 2. \tag{3.33}$$

Hence, by $(G2)$ and since

$$\int_t^{t+1} \|\nabla u_t\|_2^2 \, ds \leq C_2 F(t)^2,\ C_2 > 0 \tag{3.34}$$

there exist $t_1 \in \left[t,\ t+\frac{1}{4}\right]$, $t_2 \in \left[t+\frac{3}{4},\ t+1\right]$ such that

$$\|\nabla u_t(t_i)\|_2^2 \leq 4C(\Omega) F(t)^2,\ i=1,\ 2. \tag{3.35}$$

Next, we multiply the first equation in (P) by u and integrate over $\Omega \times [t_1,\ t_2]$ to obtain

$$\begin{aligned}
&\int_{t_1}^{t_2} \left[\left(1-\int_0^t g(\tau)d\tau\right) \|\nabla u(t)\|_2^2 \, ds - b\|u\|_p^p\right] ds \\
= & -\int_{t_1}^{t_2}\int_\Omega u.u_{tt}dxds - w\int_{t_1}^{t_2}\int_\Omega \nabla u.\nabla u_t dxds - a\int_{t_1}^{t_2}\int_\Omega u.\left|u_t\right|^{m-2} u_t dxds \\
& + \int_{t_1}^{t_2}\int_0^s g(s-\tau)\int_\Omega \nabla u(s).\left[\nabla u(\tau)-\nabla u(s)\right] dxd\tau ds.
\end{aligned}$$

Obviously,

$$\begin{aligned}
\int_{t_1}^{t_2} I(s)ds = & -\int_{t_1}^{t_2}\int_\Omega u.u_{tt}dxds - w\int_{t_1}^{t_2}\int_\Omega \nabla u.\nabla u_t dxds - a\int_{t_1}^{t_2}\int_\Omega u.\left|u_t\right|^{m-2} u_t dxds \\
& + \int_{t_1}^{t_2}\int_0^s g(s-\tau)\int_\Omega \nabla u(s).\left[\nabla u(\tau)-\nabla u(s)\right] dxd\tau ds \\
& + \int_{t_1}^{t_2} (g \circ \nabla u)(s)ds.
\end{aligned} \tag{3.36}$$

Note that by integrating by parts, to obtain

$$\begin{aligned}
\left|\int_{t_1}^{t_2}\int_\Omega u.u_{tt}dxds\right| &= \left|\left[\int_\Omega u_t u dx\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_\Omega u_t^2 dxds\right| \\
&= \left|\int_\Omega u_t(t_2)u(t_2)dx - \int_\Omega u_t(t_1)u(t_1)dx - \int_{t_1}^{t_2} \|u_t\|_2^2 \, ds\right|,
\end{aligned}$$

Using Hölder's and Poincaré's inequalities, we get

$$\left|\int_{t_1}^{t_2}\int_{\Omega} u.u_{tt}dxds\right| \leq C_*^2\sum_{i=1}^{2}\|\nabla u_t(t_i)\|_2\|\nabla u(t_i)\|_2 + C_*^2\int_{t_1}^{t_2}\|\nabla u_t\|_2^2\,dt. \tag{3.37}$$

By using Hölder's inequality once again, we have

$$\left|\int_{t_1}^{t_2}\int_{\Omega}\nabla u.\nabla u_t dxds\right| \leq \int_{t_1}^{t_2}\|\nabla u\|_2\|\nabla u_t\|_2\,ds \tag{3.38}$$

Furthermore, by (3.35) and (3.16), we have

$$\|\nabla u_t(t_i)\|_2\|\nabla u(t_i)\|_2 \leq C_3\left(C(\Omega)\right)^{\frac{1}{2}}F(t)\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}, \tag{3.39}$$

where, $C_3 = 2\left(\dfrac{2p}{l\,(p-2)}\right)^{\frac{1}{2}}$.

From (3.34) we have by Hölder's inequality

$$\begin{aligned}\int_{t_1}^{t_2}\|\nabla u\|_2\|\nabla u_t\|_2\,dt &\leq \int_{t_1}^{t_2}E(s)^{\frac{1}{2}}\left[\frac{1}{l}\left(\frac{2p}{p-2}\right)\right]^{\frac{1}{2}}\|\nabla u_t\|_2\,ds\\ &\leq \frac{1}{2}C_3\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}\int_{t_1}^{t_2}\|\nabla u_t\|_2\,ds,\end{aligned}$$

which implies

$$\begin{aligned}\int_{t_1}^{t_2}\|\nabla u_t\|_2\,dt &\leq \left(\int_{t_1}^{t_2}1dt\right)^{\frac{1}{2}}\left(\int_{t_1}^{t_2}\|\nabla u_t\|_2^2\,dt\right)^{\frac{1}{2}}\\ &\leq \frac{\sqrt{3C_2}}{2}F(t).\end{aligned}$$

Then,

$$\int_{t_1}^{t_2}\|\nabla u\|_2\|\nabla u_t\|_2\,dt \leq C_4F(t)\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}} \tag{3.40}$$

where $C_4 = \dfrac{C_3\sqrt{3C_2}}{4}$. Therefore (3.37) , becomes

$$\left|\int_{t_1}^{t_2}\int_{\Omega}u.u_{tt}dxds\right| \leq 2C_*^2C_3F(t)\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}} + C_*^2C_2F(t)^2. \tag{3.41}$$

We then exploit Young's inequality to estimate

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\Omega}\int_{0}^{t} g(s-\tau)\nabla u(t).\left[\nabla u(s)-\nabla u(t)\right]d\tau dx dt \\
\leq\ & \delta \int_{t_1}^{t_2}\int_{0}^{t} g(s-\tau)\left\|\nabla u\right\|_2^2 d\tau dt+\frac{1}{4\delta}\int_{t_1}^{t_2}\left(g\circ\nabla u\right)(t)dt, \quad \forall\delta>0.
\end{aligned} \tag{3.42}$$

Now, the third term in the right-hand side of (3.36), can be estimated as follows

$$\int_{t_1}^{t_2}\int_{\Omega}\left|u_t\right|^{m-2}u_t.udxds \leq \int_{t_1}^{t_2}\int_{\Omega}\left|u_t\right|^{m-1}.\left|u\right|dxds.$$

By Hölder's inequality, we find

$$\begin{aligned}
\int_{t_1}^{t_2}\int_{\Omega}\left|u_t\right|^{m-1}.\left|u\right|dxds &\leq \int_{t_1}^{t_2}\left[\left(\int_{\Omega}\left|u_t\right|^m dx\right)^{\frac{m-1}{m}}\left(\int_{\Omega}\left|u\right|^m dx\right)^{\frac{1}{m}}\right]ds \\
&= \int_{t_1}^{t_2}\left\|u_t\right\|_m^{m-1}\left\|u\right\|_m ds.
\end{aligned}$$

By Sobolev-Poincaré's inequality, we have

$$\int_{t_1}^{t_2}\left\|u_t\right\|_m^{m-1}\left\|u\right\|_m ds \leq C\left(\Omega\right)\int_{t_1}^{t_2}\left\|u_t\right\|_m^{m-1}\left\|\nabla u\right\|_2 ds,$$

for $2<m\leq\dfrac{2n}{n-2}$ if $n\geq 3$, or $2\leq m<\infty$ if $n=1,\,2$.

Using Hölder's inequality, and since $t_1, t_2 \in [t,\ t+1]$ and $E(t)$ decreasing in time, we conclude from the last inequality, (3.16) and (3.29) , that

$$\begin{aligned}
\int_{t_1}^{t_2} \|u_t\|_m^{m-1} \|u\|_m \, ds &\leq C(\Omega) \left(\frac{2p}{l(p-2)} \right)^{\frac{1}{2}} \int_{t_1}^{t_2} \|u_t\|_m^{m-1} (J(u))^{\frac{1}{2}} \, ds \\
&\leq C(\Omega) \left(\frac{2p}{l(p-2)} \right)^{\frac{1}{2}} \int_{t_1}^{t_2} \|u_t\|_m^{m-1} (E(u))^{\frac{1}{2}} \, ds \\
&\leq C(\Omega) \left(\frac{2p}{l(p-2)} \right)^{\frac{1}{2}} \sup_{t_1 \leq s \leq t_2} (E(u))^{\frac{1}{2}} \times \\
&\quad \left(\int_{t_1}^{t_2} \|u_t\|_m^m \, ds \right)^{\frac{m-1}{m}} \left(\int_{t_1}^{t_2} ds \right)^{\frac{1}{m}} \qquad (3.43) \\
&\leq \left(\frac{1}{a} \right)^{\frac{m-1}{m}} C(\Omega) \sup_{t_1 \leq s \leq t_2} (E(t))^{\frac{1}{2}} \left(\frac{2p}{l(p-2)} \right)^{\frac{1}{2}} F(t)^{m-1}
\end{aligned}$$

Then, taking into account (3.41) − (3.43), estimate (3.36) takes the form

$$\begin{aligned}
\int_{t_1}^{t_2} I(t) dt &\leq \left(2C_*^2 + \frac{\sqrt{3C_2}}{4} w \right) C_3 F(t) \sup_{t_1 \leq s \leq t_2} E(s)^{\frac{1}{2}} + C_*^2 C_2 F(t)^2 \\
&+ \frac{a^{\frac{1}{m}}}{2} C_3 C(\Omega) \sup_{t_1 \leq s \leq t_2} (E(t))^{\frac{1}{2}} F(t)^{m-1} \qquad (3.44) \\
&+ \delta \int_{t_1}^{t_2} \int_0^t g(t-s) \|\nabla u\|_2^2 \, ds dt + \left(\frac{1}{4\delta} + 1 \right) \int_{t_1}^{t_2} (g \circ \nabla u)(t) dt.
\end{aligned}$$

Moreover, from (3.4) and (3.10), we see that

$$\begin{aligned}
E(t) &= \frac{1}{2} \|u_t\|_2^2 \, dt + J(t) \\
&= \frac{1}{2} \|u_t\|_2^2 + \left(\frac{p-2}{2p} \right) \left(1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 \\
&\quad + \left(\frac{p-2}{2p} \right) (g \circ \nabla u)(t) + \frac{1}{p} I(t). \qquad (3.45)
\end{aligned}$$

By integrating (3.45) over $[t_1,\ t_2]$, we obtain

$$\int_{t_1}^{t_2} E(t)dt = \frac{1}{2}\int_{t_1}^{t_2} \|u_t\|_2^2 dt + \left(\frac{p-2}{2p}\right)\int_{t_1}^{t_2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 dt$$
$$+\left(\frac{p-2}{2p}\right)\int_{t_1}^{t_2}(g\circ\nabla u)(t)dt + \frac{1}{p}\int_{t_1}^{t_2} I(t)dt, \tag{3.46}$$

which implies by exploiting (3.32)

$$\int_{t_1}^{t_2} E(t)dt \leq \frac{C_1}{2}(F(t))^2 + \frac{1}{p}\int_{t_1}^{t_2} I(t)dt + \left(\frac{p-2}{2p}\right)\int_{t_1}^{t_2}(g\circ\nabla u)(t)dt$$
$$+\left(\frac{p-2}{2p}\right)\int_{t_1}^{t_2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 dt. \tag{3.47}$$

By using (3.11), Lemma 3.1.5, we see that

$$\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 \leq \frac{1}{\eta}I(t). \tag{3.48}$$

Therefore, (3.47), takes the form

$$\int_{t_1}^{t_2} E(t)dt \leq \frac{C(\Omega)}{2}(F(t))^2 + \left(\frac{p-2}{2p}\right)\int_{t_1}^{t_2}(g\circ\nabla u)(t)dt$$
$$+\left(\frac{1}{p}+\frac{p-2}{2p\eta}\right)\int_{t_1}^{t_2} I(t)dt. \tag{3.49}$$

Again an integration of (3.14) over $[s,\ t_2]$, $s\in[0,\ t_2]$ gives

$$E(s) = E(t_2) + a\int_s^{t_2}\|u_t(t)\|_m^m d\tau + \frac{1}{2}\int_s^{t_2} g(\tau)\|\nabla u(t)\|_2^2 d\tau$$
$$-\frac{1}{2}\int_s^{t_2}\left(g'\circ\nabla u\right)(t)d\tau + w\int_s^{t_2}\|\nabla u_t(t)\|_2^2 d\tau \tag{3.50}$$

By using the fact that $t_2 - t_1 \geq \frac{1}{2}$, we have

$$\int_{t_1}^{t_2} E(s)ds \geq \int_{t_1}^{t_2} E(t_2)ds \geq \frac{1}{2}E(t_2). \tag{3.51}$$

The fourth term in (3.44), can be handled as

$$\begin{aligned}\int_0^t g(t-s)\|\nabla u\|_2^2\,ds &= \|\nabla u\|_2^2\int_0^t g(t-s)ds \\ &\le \frac{2p(1-l)}{l(p-2)}E(t).\end{aligned}\tag{3.52}$$

Thus,

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^t g(t-s)\|\nabla u\|_2^2\,dsdt &\le \frac{2p(1-l)}{l(p-2)}\int_{t_1}^{t_2}E(t)dt \\ &\le \frac{p(1-l)}{l(p-2)}E(t_1) \\ &\le \frac{p(1-l)}{l(p-2)}E(t).\end{aligned}\tag{3.53}$$

Hence, by (3.53) , we obtain from (3.44)

$$\begin{aligned}\int_{t_1}^{t_2} I(t)dt &\le \left(2C_*^2+\frac{\sqrt{3C_2}}{4}w\right)C_3F(t)\sup_{t_1\le s\le t_2}E(s)^{\frac{1}{2}}+C_*^2C_2F(t)^2 \\ &\quad+\frac{a^{\frac{1}{m}}}{2}C_3C(\Omega)\sup_{t_1\le s\le t_2}(E(t))^{\frac{1}{2}}F(t)^{m-1} \\ &\quad+\delta\frac{p(1-l)}{l(p-2)}E(t)+\left(\frac{1}{4\delta}+1\right)\int_{t_1}^{t_2}(g\circ\nabla u)(t)dt.\end{aligned}\tag{3.54}$$

From (3.50) and (3.51) we have

$$\begin{aligned}E(t) &\le 2\int_{t_1}^{t_2}E(s)ds+a\int_t^{t+1}\|u_t(t)\|_m^m\,d\tau+\frac{1}{2}\int_t^{t+1}g(\tau)\|\nabla u(t)\|_2^2\,d\tau \\ &\quad-\frac{1}{2}\int_t^{t+1}(g'\circ\nabla u)(t)d\tau+w\int_t^{t+1}\|\nabla u_t(t)\|_2^2\,d\tau.\end{aligned}\tag{3.55}$$

Obviously, (3.49) and (3.55) give us

$$
\begin{aligned}
E(t) \leq\ & 2\left(\frac{C(\Omega)}{2}(F(t))^2+\left(\frac{p-2}{2p}\right)\int_{t_1}^{t_2}(g\circ\nabla u)(t)dt+\left(\frac{1}{p}+\frac{p-2}{2p\eta}\right)\int_{t_1}^{t_2}I(t)dt\right)\\
&+a\int_{t_1}^{t_2}\|u_t(t)\|_m^m dt+\frac{1}{2}\int_{t_1}^{t_2}g(t)\|\nabla u(t)\|_2^2 dt-\frac{1}{2}\int_{t_1}^{t_2}(g'\circ\nabla u)(t)dt\\
&+w\int_{t_1}^{t_2}\|\nabla u_t(t)\|_2^2 dt.
\end{aligned}
$$

Consequently, plugging the estimate (3.54) into the above estimate, we conclude

$$
\begin{aligned}
E(t) \leq\ & C(\Omega)(F(t))^2+\left(\frac{p-2}{p}\right)\int_{t_1}^{t_2}(g\circ\nabla u)(t)dt\\
&+2\left(\frac{1}{p}+\frac{p-2}{2p\eta}\right)\left[\left(2C_*^2+\frac{\sqrt{3C_2}}{4}w\right)C_3F(t)\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}+C_*^2C_2F(t)^2\right]\\
&+\left(\frac{1}{p}+\frac{p-2}{2p\eta}\right)a^{\frac{1}{m}}C_3C(\Omega)\sup_{t_1\leq s\leq t_2}(E(t))^{\frac{1}{2}}F(t)^{m-1}\\
&+2\left(\frac{1}{p}+\frac{p-2}{2p\eta}\right)\left[\delta\frac{p(1-l)}{l(p-2)}E(t)+\left(\frac{1}{4\delta}+1\right)\int_{t_1}^{t_2}(g\circ\nabla u)(t)dt)\right]\\
&+F^m(t)-\frac{1}{2}\int_{t_1}^{t_2}(g'\circ\nabla u)(t)dt+\frac{1}{2}\int_{t_1}^{t_2}g(t)\|\nabla u(t)\|_2^2 dt.
\end{aligned}
\tag{3.56}
$$

We also have, by the Poincaré's inequality

$$
\begin{aligned}
\|u(s)\|_2 &\leq C\|\nabla u(s)\|_2\\
&\leq C\left(\frac{2p}{l(p-2)}\right)^{\frac{1}{2}}E(t)^{\frac{1}{2}},
\end{aligned}
\tag{3.57}
$$

Choosing δ small enough so that

$$
1-2\left(\frac{1}{p}+\frac{p-2}{2p\eta}\right)\delta\frac{p(1-l)}{l(p-2)}>0, \tag{3.58}
$$

we deduce, from (3.56) that there exists $K > 0$ such that

$$\begin{aligned} E(t) \leq\ & K\left[F(t)^2 + E(t)^{\frac{1}{2}}F(t) + E(t)^{\frac{1}{2}}F(t)^{m-1} + F(t)^m\right] \\ & +\frac{1}{2}\int_t^{t+1} g(s)\left\|\nabla u(s)\right\|_2^2 ds - \frac{1}{2}\int_t^{t+1} \left(g' \circ \nabla u\right)(s)ds \\ & + \left[\left(\frac{p-2}{p}\right) + 2\left(\frac{1}{4\delta}+1\right)\left(\frac{1}{p}+\frac{p-2}{2p\eta}\right)\right]\int_t^{t+1} \left(g \circ \nabla u\right)(s)ds \end{aligned} \tag{3.59}$$

Using $(G2)$ again we can write

$$\int_{t_1}^{t_2} \left(g \circ \nabla u\right)(t)dt \leq -\xi \int_{t_1}^{t_2} \left(g' \circ \nabla u\right)(t)dt,\ \xi > 0.$$

Then, we obtain, from (3.59),

$$\begin{aligned} E(t) \leq\ & K\left[F(t)^2 + E(t)^{\frac{1}{2}}F(t) + E(t)^{\frac{1}{2}}F(t)^{m-1} + F(t)^m\right] \\ & +\frac{1}{2}\int_t^{t+1} g(s)\left\|\nabla u(s)\right\|_2^2 ds - \left(\xi_1 + \frac{1}{2}\right)\int_t^{t+1} \left(g' \circ \nabla u\right)(s)ds. \end{aligned} \tag{3.60}$$

where $\xi_1 = \xi\left[\left(\frac{p-2}{p}\right) + 2\left(\frac{1}{p}+\frac{p-2}{2p\eta}\right)\left(\frac{1}{4\delta}+1\right)\right]$.
An appropriate use of Young's inequality in (3.60), we can find $K_1 > 0$ such that

$$\begin{aligned} E(t) \leq\ & K_1\ \left[F(t)^2 + F(t)^{2(m-1)} + F(t)^m\right] \\ & + \left[\frac{1}{2}\int_t^{t+1} g(s)\left\|\nabla u(s)\right\|_2^2 ds - \left(\xi_1 + \frac{1}{2}\right)\int_t^{t+1} \left(g' \circ \nabla u\right)(s)ds\right], \end{aligned} \tag{3.61}$$

for K_1 a positive constant.

Using $(G2)$ again to get

$$\begin{aligned} E(t) &\leq K_1 \left[F(t)^2 + F(t)^{2(m-1)} + F(t)^m\right] \\ &\quad + \left[\left(\xi_1 + \frac{1}{2}\right) \int_t^{t+1} g(s) \|\nabla u(s)\|_2^2 \, ds - \left(\xi_1 + \frac{1}{2}\right) \int_t^{t+1} \left(g' \circ \nabla u\right)(s) ds\right] \\ &\leq K_1 \left[F(t)^2 + F(t)^{2(m-1)} + F(t)^m\right] \\ &\quad + (1 + 2\xi_1) \left[\frac{1}{2} \int_t^{t+1} g(s) \|\nabla u(s)\|_2^2 \, ds - \frac{1}{2} \int_t^{t+1} \left(g' \circ \nabla u\right)(s) ds\right] \end{aligned} \tag{3.62}$$

At this end we distinguish two cases:

Case 1. <u>$m = 2$</u>. In this case we use (3.28) and (3.62), we can find $K_2 > 0$ such that

$$\begin{aligned} E(t) &\leq K_1 \, F(t)^2 \\ &\quad + (1 + 2\xi_1) \left[\frac{1}{2} \int_t^{t+1} g(s) \|\nabla u(s)\|_2^2 \, ds - \frac{1}{2} \int_t^{t+1} \left(g' \circ \nabla u\right)(s) ds\right] \\ &\leq K_2 \, \left[E(t) - E(t+1)\right]. \end{aligned} \tag{3.63}$$

Since $E(t)$ is nonincreasing and nonnegative function, an application of Lemma 3.2.1 yields

$$E(t) \leq K_2 \left[E(t) - E(t+1)\right], \quad t \geq 0, \tag{3.64}$$

which implies that

$$E(t) \leq E(0) \exp\left(-\lambda \left[t - 1\right]^+\right), \text{ on } [0, \infty), \tag{3.65}$$

where $\lambda = \ln\left(\dfrac{K_2}{K_2 - 1}\right)$.

Case 2. <u>$m > 2$.</u> In this case we, again use (3.28) and (3.62) to arrive at

$$F(t)^2 = \left[\left(E(t) - E(t+1)\right) - \frac{1}{2} \int_t^{t+1} g(s) \|\nabla u(s)\|_2^2 \, ds + \frac{1}{2} \int_t^{t+1} \left(g' \circ \nabla u\right)(s) ds\right]^{\frac{2}{m}}. \tag{3.66}$$

We then use the algebraic inequality

$$(a+b)^{\frac{m}{2}} \leq 2^{\frac{m}{2}} \left(a^{\frac{m}{2}} + b^{\frac{m}{2}}\right), \quad m \geq 2. \tag{3.67}$$

To infer from (3.62), and by using (3.67), that

$$\begin{aligned}[E(t)]^{\frac{m}{2}} &\leq K_3 \left[1+F(t)^{2(m-2)}+F(t)^{m-2}\right]^{\frac{m}{2}} F(t)^m \\ &\quad +2^{\frac{m}{2}}\left(1+2\xi_1\right)^{\frac{m}{2}}\left[\frac{1}{2}\int_t^{t+1} g(s)\left\|\nabla u(s)\right\|_2^2 ds-\frac{1}{2}\int_t^{t+1}\left(g'\circ\nabla u\right)(s)ds\right]^{\frac{m}{2}} \\ &\leq K_3 \left[1+F(t)^{2(m-2)}+F(t)^{m-2}\right]^{\frac{m}{2}} \times [E(t)-E(t+1)] \\ &\quad +2^{\frac{m}{2}}\left(1+2\xi_1\right)^{\frac{m}{2}}\left[\frac{1}{2}\int_t^{t+1} g(s)\left\|\nabla u(s)\right\|_2^2 ds-\frac{1}{2}\int_t^{t+1}\left(g'\circ\nabla u\right)(s)ds\right]^{\frac{m}{2}}\end{aligned} \tag{3.68}$$

where $K_3 = 2^{\frac{m}{2}} K_1$. We use (3.28) to obtain

$$\begin{aligned}&\left(\frac{1}{2}\int_t^{t+1} g(s)\left\|\nabla u\right\|_2^2 ds-\frac{1}{2}\int_t^{t+1}\left(g'\circ\nabla u\right)(s)ds\right)^{\frac{m}{2}} \\ &\leq \left(E(t)-E(t+1)\right)^{\frac{m}{2}}\end{aligned} \tag{3.69}$$

A combination of (3.68), (3.69) yields

$$\begin{aligned}[E(t)]^{\frac{m}{2}} &\leq K_3\left[1+F(t)^{2(m-2)}+F(t)^{m-2}\right]^{\frac{m}{2}} \times \left(E(t)-E(t+1)\right) \\ &\quad +2^{\frac{m}{2}}\left(1+2\xi_1\right)^{\frac{m}{2}}\left[E(t)-E(t+1)\right]^{\frac{m}{2}-1}\left[E(t)-E(t+1)\right] \\ &\leq \left[K_3\left[1+F(t)^{2(m-2)}+F(t)^{m-2}\right]^{\frac{m}{2}}+2^{\frac{m}{2}}\left(1+2\xi_1\right)^{\frac{m}{2}}\left[E(t)-E(t+1)\right]^{\frac{m}{2}-1}\right] \times \\ &\quad \left[E(t)-E(t+1)\right]\end{aligned} \tag{3.70}$$

By using (3.62), the estimate (3.70) takes the form

$$\begin{aligned}[E(t)]^{\frac{m}{2}} &\leq \left\{K_3 2^m\left[1+E(0)^{(m-2)}+\left(E(0)\right)^{\frac{m}{2}-1}\right]+2^{\frac{m}{2}}\left(1+2\xi_1\right)^{\frac{m}{2}}\left(E(0)\right)^{\frac{m}{2}-1}\right\} \times \\ &\quad \left(E(t)-E(t+1)\right) \\ &\leq K_0\left(E(t)-E(t+1)\right).\end{aligned} \tag{3.71}$$

Again, using Lemma 3.2.1, we conclude

$$E(t) \leq \left[E(0)^{-r} + K_0 r \left[t-1\right]^{+}\right]^{s}, \tag{3.72}$$

with $r = \dfrac{m}{2} - 1 > 0, s = \dfrac{2}{2-m}$ and K_0 is some given positive constant.
This completes the proof.

Chapter 4

Exponential Growth

Abstract

Our goal in this chapter is to prove that when the initial energy is negative and $p > m$, then, the solution with the L^p-norm grows as an exponential function provided that $\int_0^\infty g(s)ds < \frac{p-2}{p-1}$, by using carefully the arguments of the method used in [**16**], with necessary modification imposed by the nature of our problem.

4.1 Growth result

Our result reads as follows.

Theorem 4.1.1 *Suppose that* $m \geq 2$ *and* $m < p \leq \infty$, *if* $n = 1, 2$, $m < p \leq \frac{2(n-1)}{n-2}$, *if* $n \geq 3$. *Assume further that* $E(0) < 0$ *and* $\int_0^\infty g(s)ds < \frac{p-2}{p-1}$ *holds. Then the unique local solution of problem* (P) *grows exponentially.*

Proof. We set

$$H(t) = -E(t). \tag{4.1}$$

By multiplying the first equations in (P) by $-u_t$, integrating over Ω and using Lemma 2.1.3, we obtain

$$\begin{aligned} & -\frac{d}{dt}\left\{\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{b}{p}\|u\|_p^p\right\} \\ = \; & a\|u_t\|_m^m - \frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2}g(t)\|\nabla u\|_2^2 + w\|\nabla u_t\|_2^2. \end{aligned} \tag{4.2}$$

By the definition of $H(t)$, (4.2) rewritten as

$$H'(t) = a\|u_t\|_m^m - \frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2}g(t)\|\nabla u\|_2^2 + w\|\nabla u_t\|_2^2 \geq 0, \ \forall t \geq 0. \tag{4.3}$$

Consequently, $E(0) < 0$, we have

$$H(0) = -\frac{1}{2}\|u_1\|_2^2 - \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{b}{p}\|u_0\|_p^p > 0. \tag{4.4}$$

It's clear that by (4.1), we have

$$H(0) \leq H(t), \ \forall t \geq 0. \tag{4.5}$$

Using $(G2)$, to get

$$\begin{aligned} H(t) - \frac{b}{p}\|u\|_p^p &= -\left[\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t)\right] \\ &\leq 0, \ \forall t \geq 0. \end{aligned} \tag{4.6}$$

One implies

$$0 < H(0) \leq H(t) \leq \frac{b}{p}\|u\|_p^p. \tag{4.7}$$

Let us define the functional

$$L(t) = H(t) + \varepsilon \int_{\Omega} u_t u dx + \varepsilon \frac{w}{2} \|\nabla u\|_2^2 . \tag{4.8}$$

for ε small to be chosen later.

By taking the time derivative of (4.8) , we obtain

$$\begin{aligned} L'(t) &= H'(t) + \varepsilon \int_{\Omega} u u_{tt}(t,x) dx + \varepsilon \|u_t\|_2^2 + \varepsilon w \int_{\Omega} \nabla u_t \nabla u dx \\ &= \left[w \|\nabla u_t\|_2^2 + a \|u_t\|_m^m - \frac{1}{2} (g' \circ \nabla u)(t) + \frac{1}{2} g(t) \|\nabla u\|_2^2 \right] \\ &\quad + \varepsilon \|u_t\|_2^2 + \varepsilon w \int_{\Omega} \nabla u_t \nabla u dx + \varepsilon \int_{\Omega} u_{tt} u dx. \end{aligned} \tag{4.9}$$

Using the first equations in (P), to obtain

$$\begin{aligned} \int_{\Omega} u u_{tt} dx &= b \|u\|_p^p - \|\nabla u\|_2^2 - \omega \int_{\Omega} \nabla u_t \nabla u dx - a \int_{\Omega} |u_t|^{m-2} u_t u dx \\ &\quad + \int_{\Omega} \nabla u \int_0^t g(t-s) \nabla u(s,x) ds dx. \end{aligned} \tag{4.10}$$

Inserting (4.10) into (4.9) to get

$$\begin{aligned} L'(t) &= w \|\nabla u_t\|_2^2 + a \|u_t\|_m^m - \frac{1}{2} (g' \circ \nabla u)(t) + \frac{1}{2} g(t) \|\nabla u\|_2^2 \\ &\quad + \varepsilon \|u_t\|_2^2 - \varepsilon \|\nabla u\|_2^2 + \varepsilon \int_0^t g(t-s) \int_{\Omega} \nabla u . \nabla u(s) dx ds \\ &\quad + \varepsilon b \|u\|_p^p - \varepsilon a \int_{\Omega} |u_t|^{m-2} u_t u dx. \end{aligned} \tag{4.11}$$

By using $(G2)$, the last equality takes the form

$$\begin{aligned} L'(t) &\geq w \|\nabla u_t\|_2^2 + a \|u_t\|_m^m + \varepsilon \|u_t\|_2^2 - \varepsilon \|\nabla u\|_2^2 + \varepsilon b \|u\|_p^p \\ &\quad + \varepsilon \int_0^t g(t-s) \int_{\Omega} \nabla u . \nabla u(s) dx ds - \varepsilon a \int_{\Omega} |u_t|^{m-2} u_t u d\sigma. \end{aligned} \tag{4.12}$$

To estimate the last term in the right-hand side of (4.12) , we use the following Young's inequality

$$XY \leq \frac{\delta^r}{r} X^r + \frac{\delta^{-q}}{q} Y^q, \;\; X, Y \geq 0, \tag{4.13}$$

for all $\delta > 0$ be chosen later, $\frac{1}{r} + \frac{1}{q} = 1$, with $r = m$ and $q = \frac{m}{m-1}$.
So we have

$$\int_\Omega |u_t|^{m-2} u_t u dx \leq \int_\Omega |u_t|^{m-1} |u| \, dx$$
$$\leq \frac{\delta^m}{m} \|u\|_m^m + \left(\frac{m-1}{m}\right) \delta^{\left(\frac{-m}{m-1}\right)} \|u_t\|_m^m, \ \forall t \geq 0. \quad (4.14)$$

Therefore, the estimate (4.12) takes the form

$$\begin{aligned} L'(t) \geq\ & w \|\nabla u_t\|_2^2 + a \|u_t\|_m^m + \varepsilon \|u_t\|_2^2 - \varepsilon \|\nabla u\|_2^2 + \varepsilon b \|u\|_p^p \\ & + \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u . \nabla u(s) dx ds \\ & - \varepsilon a \frac{\delta^m}{m} \|u\|_m^m - \varepsilon a \left(\frac{m-1}{m}\right) \delta^{\left(\frac{-m}{m-1}\right)} \|u_t\|_m^m \\ \geq\ & w \|\nabla u_t\|_2^2 + a \|u_t\|_m^m + \varepsilon \|u_t\|_2^2 - \varepsilon \|\nabla u\|_2^2 + \varepsilon b \|u\|_p^p \\ & + \varepsilon \|\nabla u\|_2^2 \int_0^t g(s) ds + \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u(t) \left[\nabla u(s) - \nabla u(t)\right] dx ds \\ & - \varepsilon a \frac{\delta^m}{m} \|u\|_m^m - \varepsilon a \left(\frac{m-1}{m}\right) \delta^{\left(\frac{-m}{m-1}\right)} \|u_t\|_m^m . \end{aligned} \quad (4.15)$$

Using Cauchy-Schwarz and Young's inequalities to obtain

$$\begin{aligned} L'(t) \geq\ & w \|\nabla u_t\|_2^2 + a \left(1 - \varepsilon \left(\frac{m-1}{m}\right) \delta^{\left(\frac{-m}{m-1}\right)}\right) \|u_t\|_m^m + \varepsilon \|u_t\|_2^2 \\ & - \varepsilon \|\nabla u\|_2^2 + \varepsilon b \|u\|_p^p + \varepsilon \|\nabla u\|_2^2 \int_0^t g(s) ds \\ & - \varepsilon \int_0^t g(t-s) \|\nabla u\|_2 \|\nabla u(s) - \nabla u(t)\|_2 ds - \varepsilon a \frac{\delta^m}{m} \|u\|_m^m \\ \geq\ & w \|\nabla u_t\|_2^2 + a \left(1 - \varepsilon \left(\frac{m-1}{m}\right) \delta^{\left(\frac{-m}{m-1}\right)}\right) \|u_t\|_m^m + \varepsilon \|u_t\|_2^2 + \varepsilon b \|u\|_p^p \\ & + \varepsilon \left(\frac{1}{2} \int_0^t g(s) ds - 1\right) \|\nabla u\|_2^2 - \varepsilon \frac{1}{2} (g \circ \nabla u(t) - \varepsilon a \frac{\delta^m}{m} \|u\|_m^m . \end{aligned} \quad (4.16)$$

Using assumptions to substitute for $b\|u\|_p^p$. Hence, (4.16) becomes

$$\begin{aligned} L'(t) &\geq w\|\nabla u_t\|_2^2 + a\left(1-\varepsilon\left(\frac{m-1}{m}\right)\delta^{\left(\frac{-m}{m-1}\right)}\right)\|u_t\|_m^m + \varepsilon\|u_t\|_2^2 \\ &+\varepsilon\left(pH(t)+\frac{p}{2}\|u_t\|_2^2+\frac{p}{2}(g\circ\nabla u)(t)+\frac{p}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2\right) \\ &+\varepsilon\left(\frac{1}{2}\int_0^t g(s)ds-1\right)\|\nabla u\|_2^2-\varepsilon\frac{1}{2}(g\circ\nabla u(t)-\varepsilon a\frac{\delta^m}{m}\|u\|_m^m. \\ &\geq w\|\nabla u_t\|_2^2 + a\left(1-\varepsilon\left(\frac{m-1}{m}\right)\delta^{\left(\frac{-m}{m-1}\right)}\right)\|u_t\|_m^m + \varepsilon\left(1+\frac{p}{2}\right)\|u_t\|_2^2 \\ &+\varepsilon a_1\|\nabla u\|_2^2+\varepsilon a_2(g\circ\nabla u(t)-\varepsilon a\frac{\delta^m}{m}\|u\|_m^m+\varepsilon pH(t). \end{aligned} \tag{4.17}$$

where $a_1 = \left(\frac{1-p}{2}\right)\int_0^\infty g(s)ds + \left(\frac{p-2}{2}\right) > 0,\ a_2 = \frac{p-1}{2} > 0.$

In order to undervalue $L'(t)$ with terms of $E(t)$ and since $p > m$, we have from the embedding $L^p(\Omega) \hookrightarrow L^m(\Omega)$,

$$\|u\|_m^m \leq C\|u\|_p^m \leq C\left(\|u\|_p^p\right)^{\frac{m}{p}},\ \forall t \geq 0. \tag{4.18}$$

for some positive constant C depending on Ω only. Since $0 < \frac{m}{p} < 1$, we use the algebraic inequality

$$z^k \leq (z+1) \leq \left(1+\frac{1}{w}\right)(z+w),\ \forall z \geq 0,\ 0 < k \leq 1,\ w > 0,$$

to find

$$\left(\|u\|_p^p\right)^{\frac{m}{p}} \leq K\left(\|u\|_p^p + H(0)\right),\ \forall t \geq 0, \tag{4.19}$$

where $K = 1 + \frac{1}{H(0)} > 0$, then by (4.7) we have

$$\|u\|_m^m \leq C\left(1+\frac{b}{p}\right)\|u\|_p^p,\ \forall t \geq 0. \tag{4.20}$$

Inserting (4.20) into (4.17), to get

$$\begin{aligned} L'(t) &\geq w\|\nabla u_t\|_2^2 + a\left(1-\varepsilon\left(\frac{m-1}{m}\right)\delta^{\left(\frac{-m}{m-1}\right)}\right)\|u_t\|_m^m + \varepsilon\left(1+\frac{p}{2}\right)\|u_t\|_2^2 \\ &+\varepsilon a_1\|\nabla u\|_2^2+\varepsilon a_2(g\circ\nabla u(t)-\varepsilon C_1\|u\|_p^p+\varepsilon pH(t). \end{aligned} \tag{4.21}$$

where $C_1 = aC\frac{\delta^m}{m}\left(1+\frac{b}{p}\right) > 0.$

By using (4.1) and by the same statements as in [**16**], we have

$$\begin{aligned} 2H(t) &= -\|u_t\|_2^2 - \|\nabla u\|_2^2 + \int_0^t g(s)ds\,\|\nabla u\|_2^2 - (g\circ\nabla u)(t) + \frac{2b}{p}\|u\|_p^p \\ &\geq -\|u_t\|_2^2 - \|\nabla u\|_2^2 - (g\circ\nabla u)(t) + \frac{2b}{p}\|u\|_p^p,\ \forall t\geq 0. \end{aligned} \tag{4.22}$$

Adding and substituting the value $2a_3H(t)$ from (4.21), and choosing δ small enough such that $a_3 < \min\{a_1, a_2\}$, we obtain

$$\begin{aligned} L'(t) &\geq w\|\nabla u_t\|_2^2 + a\left(1-\varepsilon\left(\frac{m-1}{m}\right)\delta^{\left(\frac{-m}{m-1}\right)}\right)\|u_t\|_m^m \\ &+\varepsilon\left(1+\frac{p}{2}-a_3\right)\|u_t\|_2^2 + \varepsilon(a_1-a_3)\|\nabla u\|_2^2 \\ &+\varepsilon(a_2-a_3)(g\circ\nabla u(t) + \varepsilon\left(\frac{2b}{p}a_3 - C_1\right)\|u\|_p^p \\ &+\varepsilon(p-2a_3)H(t). \end{aligned} \tag{4.23}$$

Now, once δ is fixed, we can choose ε small enough such that

$$1-\varepsilon\left(\frac{m-1}{m}\right)\delta^{\left(\frac{-m}{m-1}\right)} > 0,\ \text{and } L(0) > 0. \tag{4.24}$$

Therefore, (4.23) takes the form

$$L'(t) \geq \varepsilon\theta\left\{H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + (g\circ\nabla u(t)) + \|u\|_p^p\right\}, \tag{4.25}$$

for some $\theta > 0$.

Now, using $(G2)$, Young's and Poincaré's inequalities in (4.8) to get

$$L(t) \leq \theta_1\left\{H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2\right\}, \tag{4.26}$$

for some $\theta_1 > 0$. Since, $H(t) > 0$, we have from (4.1)

$$-\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t) + \frac{b}{p}\|u\|_p^p > 0,\ \forall t\geq 0. \tag{4.27}$$

Then,

$$\begin{aligned} \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 &< \frac{b}{p}\|u\|_p^p \\ &< \frac{b}{p}\|u\|_p^p + (g\circ\nabla u)(t). \end{aligned} \tag{4.28}$$

In the other hand, using $(G1)$, to get

$$\begin{aligned}\frac{1}{2}(1-l)\|\nabla u\|_2^2 &\leq \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 \\ &< \frac{b}{p}\|u\|_p^p + (g\circ\nabla u)(t). \qquad (4.29)\end{aligned}$$

Consequently,

$$\|\nabla u\|_2^2 < \frac{2b}{p}\|u\|_p^p + 2(g\circ\nabla u)(t) + 2l\|\nabla u\|_2^2,\ \ b, l > 0. \qquad (4.30)$$

Inserting (4.30) into (4.26), to see that there exists a positive constant λ such that

$$L(t) \leq \lambda\left\{H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + (g\circ\nabla u)(t) + \frac{b}{p}\|u\|_p^p\right\},\ \ \forall t \geq 0. \qquad (4.31)$$

From inequalities (4.25) and (4.31) we obtain the differential inequality

$$\frac{L'(t)}{L(t)} \geq \mu,\ \text{for some } \mu > 0,\ \forall t \geq 0. \qquad (4.32)$$

Integration of (4.32), between 0 and t gives us

$$L(t) \geq L(0)\exp(\mu t),\ \ \forall t \geq 0, \qquad (4.33)$$

From (4.8) and for ε small enough, we have

$$L(t) \leq H(t) \leq \frac{b}{p}\|u\|_p^p. \qquad (4.34)$$

By (4.33) and (4.34), we have

$$\|u\|_p^p \geq C\exp(\mu t),\ \ C > 0,\ \forall t \geq 0. \qquad (4.35)$$

Therefore, we conclude that the solution in the L^p-norm growths exponentially. ■

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