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# THESIS

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**On the numerical approximation of a class of quasi-variational  
inequalities with term sources depending on the solution**

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Applied Mathematics

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Faculté des Sciences

Département de Mathématiques

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## THÈSE

Présentée en vue de l'obtention du diplôme de Doctorat

**Sur l'approximation numérique d'une classe d'inéquation quasi-variationnelles avec des termes sources dépendants de la solution**

**Filière**  
**Mathématiques Appliquées**

**Spécialité**  
**Contrôle optimal**

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## DEDICATION

*To my **father**, for his strength and guidance.*

*To my **mother**, for her endless love.*

*To my **brothers**, for their constant support.*

*To my **sister**, for her kindness and encouragement.*

# Acknowledgements

First for most I want to thank Allah for all that has been given me strength, courage and above all knowledge, for making it all possible for me to complete this dissertation, for showing me that there is a light to every tunnel.

A verse that I will forever live by in the holybook of Islam in the Quran that says :

وَيَرْزُقُهُ مِنْ حَيْثُ لَا يَحْتَسِبُ وَمَنْ يَتَوَكَّلْ عَلَى اللَّهِ فَهُوَ حَسْبُهُ إِنَّ اللَّهَ بَالِغُ أَمْرِهِ قَدْ جَعَلَ اللَّهُ لِكُلِّ شَيْءٍ قَدْرًا

And whoever puts their trust in Allah, then He alone is sufficient for them. Certainly, Allah achieves His Will. Allah has already set a destiny for everything.

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# ABSTRACT

This thesis focuses on the numerical study of non-coercive elliptic quasi-variational inequalities in which both the obstacle and the right-hand side depend on the solution. We propose two distinct iterative approaches to solve this problem. For the first, we prove a geometric convergence theorem; while the second establishes the uniform convergence of the solutions. In both cases, we obtain an optimal error estimate between the continuous solution  $m$  and the discrete solution  $m_h$ .

**KEYWORDS:** Algorithmic approach, elliptic variational inequalities, elliptic quasi-variational inequalities, finite element method, maximum norm analysis, error estimate.



# RÉSUMÉ

Cette thèse porte sur l'analyse numérique des inéquations quasi-variationnelles elliptiques non coercives, dans lesquelles l'obstacle et le second membre dépendent de la solution. Nous proposons deux approches itératives distinctes pour résoudre ce problème. Pour le premier, nous établissons un théorème de convergence géométrique; tandis que la seconde permet d'établir la convergence uniforme des solutions. Dans les deux cas, on obtient une estimation optimale de l'erreur entre la solution continue  $m$  et la solution discrète  $m_h$ .

**Mots clés:** Approche algorithmique, Inéquations variationnelles elliptiques, Inéquations quasi-variationnelles elliptiques, méthode d'élément finis, estimation de l'erreur.

# ملخص

تتناول هذه الأطروحة التحليل العددي للمراجحات الشبه متغايرة الغير قابلة للإجبار، حيث يعتمد الحاجز و الطرف الثاني على الحل نفسه. نقتراح منهجين تكرارين مختلفين لحل هذا النوع من المسائل. بالنسبة للمنهج الأول، نثبت مبرهنة تقارب هندسي؛ بينما يتيح المنهج الثاني إثبات تقارب منتظم للحلول. وفي كلا الحالتين، يتم الحصول على تقدير أمثل للخطأ بين الحل المستمر والحل المنفصل.

**الكلمات المفتاحية:** المقاربة الخوارزمية، المراجحات المتغايرة، المراجحات الشبه المتغايرة، طريقة العناصر المنتهية، تقدير الخطأ.





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# INTRODUCTION

Mathematical modeling consists in representing complex phenomena through mathematical equations, with the objective of analyzing and approximating their solutions in order to better understand and predict their behavior. However, for many real-world problems, obtaining exact solutions is often impossible due to the complexity or nonlinearity of the models involved.

With the rise of computational methods, numerical analysis has become an indispensable discipline in applied mathematics. Among its most effective tools, the finite element method stands out for its ability to solve a wide variety of problems arising in fields such as physics, economics, and biology.

In particular, many of these complex systems are naturally formulated as problems involving constraints and nonlinear interactions, which are effectively described by variational and quasi-variational inequalities.

Quasi-variational inequalities (QVIs) were introduced in the 1970s by A. Bensoussan and J. L. Lions in the context of impulse control problems, a class of optimal control problems where interventions occur at discrete times and involve sudden changes in the system. Impulse control type situations are common in economics or inventory management, thermal power plant start-up, statistics and financial mathematics. The problem of impulse control was initiated by A. Bensoussan and J. L. Lions in [4] [5] and [31] then, reworked and formalized by the same authors in [9] and other authors, including the work of [33]. A. Bensoussan and J. L. Lions considered a system whose state at time  $t$  is a vector  $x$  of  $\mathbb{R}^N$  given either by the solution of the ordinary differential

equation, in the deterministic case or dynamic system

$$\begin{aligned} y(0) &= x \in \mathbb{R}^N \\ \frac{dy}{dt} &= j(y), \end{aligned} \tag{1}$$

or through the solution of an Itô type stochastic differential equation .

$$\begin{aligned} y(0) &= x \in \mathbb{R}^N, \\ dy &= j(y) dt + \sigma(y) dw(t). \end{aligned} \tag{2}$$

$\sigma$  and  $j$  are Lipschitzian and bounded functions.

$$\begin{aligned} j: \overline{\Omega} \subset \mathbb{R}^N &\rightarrow \mathbb{R}^N \\ y \rightarrow j(y) &= \begin{pmatrix} j_1(y) \\ \dots \\ j_N(y) \end{pmatrix} \end{aligned}$$

The solution of (1) and of (2) is denoted by

$$y_x(s), \text{ where } 0 \leq s \leq \tau(x).$$

$\tau(x) = T$  is the exit time of  $\overline{\Omega}$ . A control is exercised over the evolution in the following manner. In a series of moments  $0 \leq t_1 \leq t_2 \leq \dots \leq t_N \leq T$ , the state of the system instantly changes from  $y(t_i)$  to  $y(t_i) + \xi_i$ . The instants  $t_i$  are called impulse instants and the values  $\xi^i$  are called impulse levels. Between two successive impulse instants, the evolution of the system is described either by the equation (1) or equation (2). The set

$$\nu = \{t_1, \xi_1; t_2, \xi_2; \dots; t_N, \xi_N\}$$

is called an impulse control. Let  $J_x(\nu)$  defined by

$$J_x(\nu) = E \left[ \int_t^T z(y_x(t, \nu)) dt + kN(\nu) \right] \tag{3}$$

a criterion or a cost function, where  $k > 0$  and  $z$  a continuous function

$$z : \overline{\Omega} \subset \mathbb{R}^N \rightarrow \mathbb{R}.$$

The problem then consists of solving

$$m(x) = \inf_{0 \leq v \leq \tau(x)} J_x(v) \quad (4)$$

The solution function  $m$  of (4) is the optimal impulse control.

The authors A. Bensoussan and J. L. Lions in [4] [5], [31], [9] and in the paper [24] demonstrated that the solution  $m$  of (4) is characterized by

$$\begin{aligned} -\frac{\partial m}{\partial t} + Am - z &\leq 0 \text{ in } \Omega \\ m - Dm &\leq 0 \text{ in } \Omega \\ \left(-\frac{\partial m}{\partial t} + Am - z\right)(m - Dm) &= 0 \text{ in } \Omega \end{aligned} \quad (5)$$

where  $A$  is a second-order elliptic operator

$$Av = - \sum_{1 \leq i, j \leq N} \frac{\partial}{\partial x_i} (c_{ij}(x)) \frac{\partial v}{\partial x_j} + \sum_{i=1}^N c_i(x) \frac{\partial v}{\partial x_i} + c_0(x)v$$

which operator reduces to a first-order operator in the deterministic case [33]. The obstacle function  $D$  is defined by

$$Dm = k + \inf_{\xi \geq 0} m(x + \xi).$$

The stationary problem corresponding to (5) is

$$\begin{aligned} Am - z &\leq 0 \text{ in } \Omega \\ m - Dm &\leq 0 \text{ in } \Omega \\ (Am - z)(m - Dm) &= 0 \text{ in } \Omega \end{aligned} \quad (6)$$

The problem (6) is a free boundary problem with operator  $D$  an implicit nonlinear and

solution-dependent obstacle attached to the cost structure  $k$ . The weak formulation of (6) yields the resolution of the quasi-variational inequality, The solution  $m \in P(m)$ ,

$$\begin{aligned} s(m, v - m) &\geq (z, v - m) \\ m &\leq Dm. \end{aligned} \tag{7}$$

The bilinear form  $s(.,.)$  in (7) being non coercive, the problem (7) is transformed into a new quasi-variational inequality whose bilinear form  $d$  is coercive; The continuous solution  $m \in P(m)$ , such that

$$\begin{aligned} d(m, v - m) &\geq (z + \lambda m, v - m) \\ m &\leq Dm \end{aligned} \tag{8}$$

with

$$d(v, v) = s(v, v) + \lambda v$$

and  $P(m)$  is the implicit closet convex set defined by

$$P(m) = \{v \in H^1(\Omega), v \leq Dm\}.$$

In [12] and by discretizing the domain  $\Omega$  by the finite elements method [21], the author has studied the numerical analysis of the problem (7) and obtained an error estimate by using the concept of subsolution.

The terminology of quasi-variational inequalities reflects the fact that both the source term and the obstacle depend on the solution itself. This dependence plays a crucial role in the mathematical analysis, particularly in the derivation of optimal error estimates for numerical approximations. To address these challenges, iterative methods play a fundamental role to solving these problems, aiming to iteratively improve an initial solution until an optimum is derived.

In this work, we focus on the finite element approximation of elliptic quasi-variational inequalities with a non-coercive operator, resulting from an impulse control problem, where both the right-hand side and the obstacle depend on the solution.

The main objective is to show that the problem can be effectively approximated using a finite element method, and to derive an optimal error estimate in  $L^\infty$  norm which coincides with that well known for the elliptic free boundary variational inequalities obtained in the work of de Ph. Cortey-Dumont ([18]). That is,  $m \in P(m)$ , such that

$$\begin{cases} d(m, v - m) \geq (z(m) + \lambda m, v - m) \\ m \leq Dm. \end{cases} \quad (9)$$

The discrete counterpart of (9) in the context of the finite elements method, is defined by;

$$\begin{cases} d(m_h, v - m_h) \geq (z(m_h) + \lambda m_h, v - m_h) \\ m \leq r_h Dm_h \end{cases} \quad (10)$$

and  $P(m_h)$  is the discrete implicit closet convex set defined by

$$P(m_h) = \{v \in V_h, v \leq r_h Dm_h\}.$$

Let  $m$  be the unique solution of (9) and  $m_h$  the unique discrete solution of (10). We get

$$\|m - m_h\|_{L^\infty(\Omega)} \leq Ch^2 |\log h|^2. \quad (11)$$

In order to get the optimal error estimate (11), the algorithmic approach is highlighted in this thesis where two different iterative Benssoussan-Lions schemes are explicitly studied both in the continuous and the discrete framework. The thesis is divided into three chapters and organized as follows

Chapter 1 is devoted to recalling the well-known classical properties and results related to existence and uniqueness of the solution of elliptic variational inequalities of free boundary type with Dirichlet boundary conditions and the obstacle function belongs to  $W^{2,\infty}(\Omega)$  and  $\mu \geq 0$  on  $\partial\Omega$ . The most interesting result to highlight is the error estimate in the maximum norm obtained by Ph. Cortey-Dumont in[18].

Chapter 2 is the subject of a synthesis in which we present the definition and a class of elliptic quasi-variational inequalities where the obstacle function is independent of the solution and the source terms are nonlinear functional depending on the solution.

The numerical analysis of this family of quasi-variational inequalities has been studied in several paper some of which are cited [13][15] where an optimal error estimate is obtained.

Chapter 3 is concerned with the main objective of this thesis, namely the numerical analysis of for a class of elliptic quasi-variational inequalities with noncoercive operator related to impulse control problems where the source term is a nonlinear functional depending on the solution defined in (9).

Knowing that the concept of quasi-variational inequalities generalizes that of variational inequalities, the algorithmic approach is recommended for the numerical analysis study where two different iterative schemes of Bensoussan-Lions type are explicitly detailed as well in the continuous framework as in the discrete framework by a confirming finite elements method is applied. The two iterative schemes introduced in this thesis differ in their nature: the first is nonlinear, whereas the second is linear. We show that the  $L^\infty(\Omega)$  - error estimate (11) holds for the both schemes presented in this thesis.

# PRELIMINARY

## Notations and assumptions

1. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ , whose boundary  $\partial\Omega$  is sufficiently smooth.
2. We consider the bilinear form

$$s(\xi, v) = \int_{\Omega} \left( \sum_{i,j=1}^N c_{ij}(x) \frac{\partial \xi}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^N c_i(x) \frac{\partial \xi}{\partial x_i} v + c_0 \xi v \right) dx. \quad (12)$$

3. Associated with the operator  $A$  defined by

$$A_{\xi} = - \sum_{i,j} \frac{\partial}{\partial x_i} \left( c_{ij}(x) \frac{\partial \xi}{\partial x_j} \right) + \sum_i c_i(x) \frac{\partial \xi}{\partial x_i} + c_0 \xi. \quad (13)$$

4. The coefficients  $c_{ij}(x)$ ,  $c_i(x)$  and  $c_0(x)$  are in  $C^2(\overline{\Omega})$ , such that  $x \in \overline{\Omega}$  for all  $1 \leq i, j \leq N$ , and sufficiently smooth, such that  $c_0(x)$  satisfies

$$c_0(x) \geq \beta > 0. \quad (14)$$

5. The bilinear form  $s$  is supposed to be continuous and strongly coercive

$$\exists \alpha > 0; \quad \alpha \|y\|_{H^1(\Omega)}^2 \leq s(y, y). \quad (15)$$

6. The  $H^1(\Omega)$  space is defined by

$$H^1(\Omega) = \left\{ v \in L^2(\Omega); \frac{\partial v}{\partial x_i} \in L^2(\Omega), \forall i = 1, \dots, n \right\}. \quad (16)$$

With the norm

$$\|\xi\|_{H^1(\Omega)} = \left( \|\xi\|_{L^2(\Omega)}^2 + \sum_{i=1}^n \left\| \frac{\partial \xi}{\partial x_i} \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}. \quad (17)$$

7. We are given the right-hand side  $z$ , a non-decreasing Lipschitz function

$$|z(x) - z(y)| \leq L|x - y|, \forall x, y \in \mathbb{R}, \quad (18)$$

with Lipschitz constant  $L$  such that

$$L < \beta. \quad (19)$$

8. The obstacle function  $\mu$  satisfies

$$\mu \in W^{2,\infty}(\Omega) \quad (20)$$

and the boundary condition  $j$  satisfies

$$j \in L^\infty(\partial\Omega). \quad (21)$$

9. We define the subsolution notion for variational inequality (1.1)

**definition** [28].

$w \in H^1(\Omega)$  said to be a subsolution for variational inequality (1.1) if

$$\begin{cases} s(w, v) \leq (z, v) & \forall v \geq 0 \\ w \leq \mu, w \leq j \text{ on } \partial\Omega. \end{cases} \quad (22)$$

Let  $X$  represents the set of such subsolutions. Then we have the following theorem.

**theoreme[8]**

The solution of the variational inequality (1.1) is the maximum element of the set  $X$ .

10. We define the comparison result by

**definition**

Let  $m = \partial(z, j, \mu)$  and  $\tilde{m} = \partial(\tilde{z}, \tilde{j}, \tilde{\mu})$  the solutions of the variational inequality (1.1), the comparison result is:

if  $z \leq \tilde{z}$ ,  $Dm \leq D\tilde{m}$ , and  $j \leq \tilde{j}$ . Then

$$m \leq \tilde{m}.$$

11. We provide the definition of the continuous maximum principle, which will be used later

**definition [22].**

Let  $m \in H^1(\Omega)$  the solution of the following equation

$$\begin{cases} s(m, v) = (z(x), v); \forall v \in H^1(\Omega) \\ v = j \text{ on } \partial\Omega \end{cases} \quad (23)$$

and  $\tilde{m} \in H^1(\Omega)$  solution of

$$\begin{cases} s(\tilde{m}, v) = (\tilde{z}(x), v) \forall v \in H^1(\Omega) \\ v = \tilde{j} \text{ on } \partial\Omega. \end{cases} \quad (24)$$

If

$$z \leq \tilde{z} \text{ in } \Omega$$

and

$$j \leq \tilde{j} \text{ on } \partial\Omega$$

then,

$$m \leq \tilde{m} \in \overline{\Omega}.$$

12. The  $L^\infty(\Omega)$  is the Banach space of measurable and bounded functions defined

almost everywhere on  $\Omega$  into  $\mathbb{R}$ , with the norm

$$\|\xi\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x \in \Omega} |\xi(x)|.$$

13.  $\tau_h$  is a triangulation of  $\Omega$  with meshsize  $h$  positive.
14. We denote by  $V_h$  the finite element space of continuous, piecewise linear functions, defined as

$$V_h = \left\{ v_h \in C(\overline{\Omega}) \cap H^1(\Omega) : v_h|_K \in P_1; \forall K \in \tau_h \right\}.$$

15.  $\varphi_i; i = 1, 2, \dots, t(h)$ , is the basis functions of the space  $V_h$ .
16.  $r_h$  is the usual finite element restriction operator on  $\Omega$ .
17.  $\pi_h$  is an interpolation operator on  $\partial\Omega$ .

**18. Discrete maximum principle**

Let  $A$  is a square matrix such that  $s_{ii} > 0$  and  $s_{ji} \leq 0$  with  $i \neq j$ . If  $A^{-1}$  exist and its elements are all non-negative, so we say that  $A$  is an M-matrix. The corresponding matrix  $A = s(\varphi_i, \varphi_j)$  that emerges from the discretization of problem (1.3) is assumed to be an M-matrix.

We introduce the following notations

19.  $d(m, v) = s(m, v) + \lambda \int_{\Omega} m v dx$ .
20.  $Z(m) = z(m) + \lambda m$ .
21.  $E^n = \|m - m^n\|_{L^\infty(\Omega)}$ .
22.  $e_h^n = \|m_h - m_h^n\|_{L^\infty(\Omega)}$ .
23.  $W^n = \|w_h^n - m^n\|_{L^\infty(\Omega)}$ .
24.  $W_h^n = \|w_h^n - m_h^n\|_{L^\infty(\Omega)}$ .
25.  $E_h^n = \|m^n - m_h^n\|_{L^\infty(\Omega)}$ .

# ELLIPTIC VARIATIONAL INEQUALITIES OF FREE BOUNDARY TYPE



## 1.1 Elliptic variational inequalities of free boundary type

In this section, we review some classical properties and results for elliptic variational inequalities. These properties can be found, for instance, in [2], [28], [29]...

Throughout this chapter, we use the assumptions and notations introduced before

### 1.1.1 Statement of the continuous problem

Under the same assumptions and notations defined earlier, we consider the following variational inequality problem

$$\begin{cases} s(\xi, v - \xi) \geq (z, v - \xi), \forall v \in P^j \\ \xi \leq \mu. \end{cases} \quad (1.1)$$

where  $P^j$  is the non empty convex set defined by

$$P^j = \{v \in H^1(\Omega); v = j \text{ on } \partial\Omega; v \leq \mu \text{ in } \Omega\}. \quad (1.2)$$

#### Continuous monotony property of the solution

As shown in [27], we shall present the continuous monotonous property of the solution based on the source term, boundary condition, and obstacle function. This is also known as the comparison result.

**Proposition 1.1.1** [27].

The solutions of the VI (1.1) corresponding to the data  $(z, j, \mu)$  and  $(\tilde{z}, \tilde{j}, \tilde{\mu})$  are denoted by  $\xi$  and  $\tilde{\xi}$ , respectively ; if  $z \leq \tilde{z}$  in  $\Omega$ ,  $j \leq \tilde{j}$  on  $\partial\Omega$  and  $\mu \leq \tilde{\mu}$ , then  $\xi \leq \tilde{\xi}$ .

**Proof:**

The proof of proposition (1.1.1) is detailed in [27].

### A Lipschitz continuous dependence

**Proposition 1.1.2** [27].

Under the conditions of proposition (1.1.1), we have

$$\|\xi - \tilde{\xi}\|_{L^\infty(\Omega)} \leq \max \left\{ \frac{1}{\beta} \|z - \tilde{z}\|_{L^\infty(\Omega)}, \|j - \tilde{j}\|_{L^\infty(\partial\Omega)}, \|\mu - \tilde{\mu}\|_{L^\infty(\Omega)} \right\}.$$

**Proof**

The proof of proposition (1.1.2) is detailed in [27].

### Existence and uniqueness of the continuous solution

**Theorem 1.1.1** [23].

Let  $V$  be a Hilbert space,  $s(.,.)$  be a bilinear continuous and coercive form, the linear term source  $z$  is a continuous function. Then, the problem (1.1) has a unique solution  $m$  in the closet convex  $P^j$ .

### 1.1.2 Statement of the discrete problem

The discrete variational inequality related to (1.1) is

$$\begin{cases} s(\xi_h, v_h - \xi_h) \geq (z, v_h - \xi_h), \forall v_h \in P_h^j \\ \xi_h \leq r_h \mu. \end{cases} \quad (1.3)$$

where  $P_h^j$  is the discrete closet convex given by

$$P_h^j = \{v_h \in V_h; v = \pi_h j \text{ on } \partial\Omega; v_h \leq r_h \mu \text{ on } \Omega\}. \quad (1.4)$$

### Discrete monotony property of the solution

**Proposition 1.1.3** [27].

The solutions of the variational inequality (1.3) corresponding to the data  $(z, j, r_h \mu)$  and  $(\tilde{z}, \tilde{j}, r_h \tilde{\mu})$  are denoted by  $\xi_h$  and  $\tilde{\xi}_h$  respectively; if  $z \leq \tilde{z}$  in  $\Omega$ ,  $j \leq \tilde{j}$  on  $\partial\Omega$  and  $r_h \mu \leq r_h \tilde{\mu}$ , then  $\xi \leq \tilde{\xi}$ .

**Proof.**

The proof is the same as in the continuous case and it is based on the discrete maximum principle.

### A Lipschitz discrete dependence property

**Proposition 1.1.4** [27].

Under the conditions of proposition (1.1.3), we have

$$\|\xi_h - \tilde{\xi}_h\|_{L^\infty(\Omega)} \leq \max \left\{ \frac{1}{\beta} \|z - \tilde{z}\|_{L^\infty(\Omega)}, \|j - \tilde{j}\|_{L^\infty(\partial\Omega)} \|r_h \mu - r_h \tilde{\mu}\|_{L^\infty(\Omega)} \right\}.$$

**Proof.**

The proof follows the one of Proposition (1.1.2) and it is based on the discrete maximum principle.

### Existence and uniqueness of the discrete solution

**Theorem 1.1.2** [23].

Let  $V$  a Hilbert space,  $s(.,.)$  a bilinear continuous and coercive form, the linear term source  $z$  is a continuous function. Then, the problem (1.3) has a unique solution.

### 1.1.3 Error estimate in the maximum norm

**Theorem 1.1.3** [19].

Let  $\xi$  and  $\xi_h$  be the solutions of (1.1) and (1.3), respectively. Then, there exist  $C$  independent of  $h$  such that

$$\|\xi - \xi_h\|_{L^\infty(\Omega)} \leq Ch^2 |\log h|^2.$$



# ELLIPTIC QUASI-VARIATIONAL INEQUALITIES WITH NONLINEAR SOURCE TERM DEPENDING ON THE SOLUTION

In this chapter, we use the same notations and assumptions as those used in the previous one.

## 2.1 Elliptic quasi-variational inequalities with nonlinear source term depending on the solution

In this section we focus on the analysis of the quasi-variational inequality in which the source term is dependent on the solution  $m$ .

### 2.1.1 Statement of the continuous problem

The quasi-variational inequalities are a generalization of variational inequalities. The main difference is that its terminology is based on the source term's dependence on the solution, which is the primary distinction.

## 2.1. Elliptic quasi-variational inequalities with nonlinear source term depending on the solution

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The numerical analysis of the quasi-variational inequalities with non linear source term depending on solution have been studied by many authors. We cite references ([14],[13],[15]) in which the author has made detailed studies and obtained optimal error estimate.

Then, we consider the following quasi-variational inequality problem

$$\begin{cases} s(\xi, v - \xi) \geq (z(\xi), v - \xi), \forall v \in P^j \\ \xi \leq \mu. \end{cases} \quad (2.1)$$

### 2.1.2 Existence and uniqueness

By relying on assumption (14), it will be demonstrated that the fixed point of the contraction mapping  $O$  is precisely the solution of (2.1), ensuring the problem's uniqueness. Indeed, we consider the following mapping

$$\begin{aligned} O : L^\infty(\Omega) &\rightarrow L^\infty(\Omega) \\ \omega &\rightarrow O\omega = \zeta, \end{aligned}$$

where  $\zeta \in P_h$  is a solution of

$$\begin{cases} s(\zeta, v - \zeta) \geq (z(\omega), v - \zeta) \\ \zeta \leq \mu. \end{cases} \quad (2.2)$$

**Theorem 2.1.1** [15].

*Under the assumption (14), the mapping  $O$  is a contraction in  $L^\infty(\Omega)$ . therefore, it admits a unique fixed point which corresponds to the solution of problem (2.1).*

### 2.1.3 Statement of discrete problem

The same finite element discretization are used in this part as in the chapter 1.

We define the discrete quasi-variational inequality wich is the discrete counterpart of

(2.1) by

$$\begin{cases} s(\xi_h, v_h - \xi_h) \geq (z(\xi_h), v_h - \xi_h) \\ \xi_h \leq r_h \mu. \end{cases} \quad (2.3)$$

With

$$P_h^j = \{v_h \in V_h; v_h \leq r_h \mu \text{ on } \Omega\}. \quad (2.4)$$

### 2.1.4 Existence and uniqueness

By relying on assumption (14), We shall demonstrate that the fixed point of a contraction mapping is equal to the solution of (2.3), ensuring the problem's uniqueness. Indeed, consider the following discrete mapping

$$\begin{aligned} O_h : L^\infty(\Omega) &\rightarrow V_h \\ \omega &\rightarrow O_h \omega = \zeta_h, \end{aligned}$$

where  $\zeta_h \in P_h$  is a solution of the following variational inequality

$$\begin{cases} s(\zeta_h, v_h - \zeta_h) \geq (z(\omega), v_h - \zeta_h) \\ \zeta_h \leq r_h \mu. \end{cases} \quad (2.5)$$

**Theorem 2.1.2** [15].

*Under the discrete maximum principle and the assumption (14), the mapping  $O_h$  is a contraction in  $L^\infty(\Omega)$ . Then, it admits a unique fixed point which corresponds with the solution of the problem (2.3).*

### 2.1.5 Error estimate in the maximum norm

**Theorem 2.1.3** [14],[13],[15].

*Let  $\xi$  and  $\xi_h$  be the solutions of (2.1) and (2.3), respectively. Then, there exist  $C$  independent of  $h$  such that*

$$\|\xi - \xi_h\|_{L^\infty(\Omega)} \leq Ch^2 |\log h|^2.$$



# THE NONCOERCIVE QUASI-VARIATIONAL INEQUALITY RELATED TO IMPULSE CONTROL PROBLEM WITH NONLINEAR SOURCE TERM

## 3.1 Some examples of application of quasi-variational inequalities related to impulse control problems

Impulse control it is a decision-making method where actions are taken suddenly and significantly to influence a system. These actions are not continuous but discrete, like pressing a button to initiate a software update, with the goal of causing an immediate and significant change in the controlled system. It involves determining the timing and magnitude of these control impulses to achieve a specific objective while minimizing a given cost function.

Here is an overview of some examples of impulsive control applications. Starting by giving an impulse control problem applied to the inventory management.

### 3.1.1 Example 1: Inventory management

The example addressed concerns a stock management problem which is explicitly detailed in [30] where the goal is to optimize the restocking process for several goods subject to random demand. The objective is to determine the optimal times  $t_i$  for placing an order and the quantities to be ordered  $\xi_i$ , taking into account three types of costs:

- ▶ A fixed cost for each order.
- ▶ A storage cost when the stock level is positive.
- ▶ A stockout cost when the stock is insufficient (negative).

The restocking occurs at discrete moments  $t_1, t_2, \dots, t_N$  where  $t \leq t_1 < \dots < t_N < T$ . Employing impulse control in the modeling of the problem is warranted by the impulsive nature of the orders. The control is given by

$$\{t_1, \xi_1; t_2, \xi_2; \dots; t_N, \xi_N\}.$$

#### Associated quasi-variational inequality

This problem has been the subject of recent studies ([3],[7],[24],[30]), whereby it has been demonstrated that the solution is found by solving a specific system of partial differential inequalities known as a quasi-variational inequality.

The QVI has the form:

$$\begin{cases} s(m, v - m) \geq (z, v - m), \forall v \in P^j \\ m \leq Dm, m \in P^j \end{cases} \quad (3.1)$$

where

$$P^j(m) = \{v \in H^1(\Omega); v = j \text{ on } \partial\Omega \text{ and } v \leq Dm\}$$

such that the operator  $D$  is defined by

$$\begin{aligned} D &: L^\infty(\Omega) \rightarrow L^\infty(\Omega) \\ Dm &= k + \inf_{\xi \geq 0} m(x + \xi) \end{aligned}$$

where

- ▶  $\xi$ : corresponds to the quantity ordered, causing a discontinuity in the stock level. In other words, it is the decision variable that determines how much to order during each restocking event.
- ▶  $k$ : represent a fixed order cost.
- ▶  $x$ : denotes the stock level . It is the system's state, evolving based on orders and demand.  $x \geq 0$  indicates a sufficient stock, while  $x < 0$  signifies a stockout.
- ▶  $z(x)$ : is the cost function associated with the stock level  $x$ . It includes the storage cost when  $x \geq 0$  and the stockout cost when  $x < 0$ .
- ▶  $m(x)$ : is the optimization problem's value function or solution. It shows the minimal cost depending on the stock level  $x$ .  $m(x)$  determines the optimal decisions, such as how much to order.

Let us consider now a second example of an impulse control problem in the management of technological change in a company

### 3.1.2 Example 2: Change of technology in a company

The problem is about managing a system where a machine suffers wear over time. The continuous operation of the machine causes a wear cost  $z(x)$  depending on its state  $x$ . In parallel, the user has the possibility to replace the machine at selected times, which generates a replacement cost  $c(x, \xi)$  when the machine in state  $x$  is replaced by a new one in state  $\xi$ .

This situation is modeled by the following quasi-variational inequality introduced by A.Bensoussan and J.L Lions [6], [5]

$$\left\{ \begin{array}{l} -Am + m \geq z(x) \\ m \leq Dm \\ (-Am + m - z)(m - Dm) = 0. \end{array} \right.$$

The problem is to determine an optimal strategy that minimizes the total cost, taking into account

- ▶  $m(x)$ : the impulse control problem's optimal total cost.
- ▶  $z(x)$  : the cost of machine operation.
- ▶  $c(x, \xi)$ : the cost of buying a new machine.
- ▶  $x$ : the condition of the ancient machine.
- ▶  $\xi$ : the new machine's condition.
- ▶  $k$ : the fixed cost  $k$  that applies each time a replacement is made.
- ▶  $\inf m(x+\xi)$ : the minimum of the total cost function  $u$  among these possible future states.

These examples illustrate two instances of impulse control problems, but many other situations involve similar techniques as production, maintenance, finance, marketing, as well as fields such as mechanics, control systems, ecology, game theory,... In what follows, we are interested in the numerical analysis of the non-coercive quasi-variational inequality related to the impulse control problem with a non linear source term dependent of the unique solution.

The algorithmic approach is chosen to deal with the numerical study of the non coercive quasi-variational inequality related to impulse control with non linear source terms. Two different algorithms are presented, and their detailed analysis is provided.

## 3.2 Statement of the continuous problem

We use the same notations and assumptions as those used in chapter 1.

The coefficients  $c_i(x)$  of the operator  $A$  might be small, when modeling control problems, due to practical or design constraints, which can render the bilinear form non coercive. Therefore, it is relevant to relax the coercivity assumption and Suppose that there is a  $\lambda > 0$  such that

$$s(y, y) + \lambda(y, y) \geq \alpha \|y\|_{H^1(\Omega)}^2. \quad (3.2)$$

We are given the non linear continuous operator  $D$  defined from  $L^\infty(\Omega)$  into itself, by

$$Dv(x) = k + \inf_{\xi \geq 0} v(x + \xi), \quad x \in \Omega, \quad \xi \geq 0, \quad x + \xi \in \Omega, \quad k > 0 \quad (3.3)$$

satisfies the following three assumptions

- $D$  is increasing, such that

$$m \leq v \rightarrow Dm \leq Dv \quad (3.4)$$

- $D(m + c) \leq Dm + c$ ;  $c$  is constant.

- Under the two previous assumptions, we have

$$\|Dm - Dv\|_{L^\infty(\Omega)} \leq \|m - v\|_{L^\infty(\Omega)}. \quad (3.5)$$

Thus, we consider the following quasi-variational inequality problem

$$\begin{cases} s(m, v - m) + \lambda(m, v - m) \geq (z(m) + \lambda m, v - m), \quad \forall v \in P^j(m) \\ m \leq Dm, \end{cases} \quad (3.6)$$

equivalent to

$$\begin{cases} d(m, v - m) \geq (z(m) + \lambda m, v - m), \quad \forall v \in P^j(m) \\ m \leq Dm, \end{cases} \quad (3.7)$$

where  $P^j(m)$  is a non-empty convex set defined by

$$P^j(m) = \{v \in H^1(\Omega) / v = j \text{ in } \partial\Omega, v \leq Dm \text{ in } \Omega\}. \quad (3.8)$$

### 3.3 Statement of the discrete problem

We will now tackle the discrete problem and perform an analysis similar to that of the continuous case. Any proofs not explicitly provided are the same as in the continuous case. To emphasize the symmetry of our approach, we will follow the

same approach used for the continuous problem.

The same finite element discretization are used in this part as in chapter 1

The discrete quasi-variational inequality related to the continuous one (3.7) is

$$\begin{cases} d(m_h, v - m_h) \geq (z(m_h) + \lambda m_h, v - m_h), & \forall v \in P_j^h(m_h) \\ m_h \leq r_h Dm_h, \end{cases} \quad (3.9)$$

where

$$P_j^h(m_h) = \{v_h \in V_h \subset H^1(\Omega) \mid v_h = \pi_h j \text{ on } \partial\Omega, v_h \leq r_h Dm_h \text{ on } \tau_h\}.$$

Based on the discretization of problem (3.9), we assume that the corresponding matrix  $A = s(\varphi_i, \varphi_j)$  is an M-matrix.

## 3.4 A first approach to deal with the numerical analysis of the problem

In this section, we introduce the first iterative scheme and prove its geometric convergence to the unique solution  $m$  of the continuous problem. We begin by establishing the monotonicity property of the continuous solution, as well as its Lipschitz continuous dependence with respect to the source term  $z$ , the boundary condition  $j$ , and the obstacle function  $\mu$ .

### 3.4.1 Continuous monotony property of the solution of problem (3.7)

In the sequel we will demonstrate the continuous monotony property of the solution based on the obstacle function, boundary condition, and source term dependence of the solution. which will paly an important role in the rest of our work.

#### **Proposition 3.4.1 .**

*The solutions of the QVI (3.7) corresponding to the data  $(Z(m), Dm, j)$  and  $(\tilde{Z}(\tilde{m}), D\tilde{m}, \tilde{j})$  are denoted by  $m$  and  $\tilde{m}$  respectively; if  $Z \leq \tilde{Z}$  in  $\Omega$  and  $j \leq \tilde{j}$  on  $\partial\Omega$ , then  $m \leq \tilde{m}$  in  $\overline{\Omega}$ .*

**Proof**

► For  $n = 0$

Let  $m^0 \in H^1(\Omega)$  the solution of the following equation

$$\begin{cases} d(m^0, v) = (Z(m^0), v); \forall v \in H^1(\Omega) \\ v = j \text{ on } \partial\Omega \end{cases} \quad (3.10)$$

and  $\tilde{m}^0 \in H^1(\Omega)$  solution of

$$\begin{cases} d(\tilde{m}^0, v) = (\tilde{Z}(\tilde{m}^0), v) \\ v = \tilde{j} \text{ on } \partial\Omega. \end{cases} \quad (3.11)$$

Since  $Z \leq \tilde{Z}$  in  $\Omega$  and  $j \leq \tilde{j}$  in  $\partial\Omega$ , then for all positive  $v$

$$d(m^0, v) = (Z(m^0), v) \leq (\tilde{Z}(\tilde{m}^0), v) = d(\tilde{m}^0, v).$$

Thus, by using the continuous maximum principle defined in definition(11), we obtain

$$m^0 \leq \tilde{m}^0 \text{ in } \overline{\Omega}.$$

As the operator  $D$  is non-decreasing, we get

$$Dm^0 \leq D\tilde{m}^0.$$

► For  $n=1$ ;

let

$$m^1 = \partial(Z(m^0), Dm^0, j)$$

and

$$\tilde{m}^1 = \partial(\tilde{Z}(\tilde{m}^0), D\tilde{m}^0, \tilde{j}).$$

By applying the comparison result, we arrive at

$$m^1 \leq \tilde{m}^1 \text{ in } \overline{\Omega}.$$

► Now, let's suppose

$$m^{n-1} \leq \tilde{m}^{n-1} \text{ in } \overline{\Omega}$$

then, according to the monotony property of the operator  $D$  (3.4), we get

$$Dm^{n-1} \leq D\tilde{m}^{n-1}.$$

As

$$m^{n+1} = \partial(Z(m^n), Dm^n, j)$$

and

$$\tilde{m}^{n+1} = \partial(\tilde{Z}(\tilde{m}^n), D\tilde{m}^n, \tilde{j}).$$

Consequently

$$m^{n+1} \leq \tilde{m}^{n+1} \text{ in } \overline{\Omega}.$$

Now, let us take the limit as  $n \rightarrow \infty$  [10]; we obtain

$$m \leq \tilde{m} \text{ in } \overline{\Omega}.$$

This completes the proof.

### 3.4.2 A Lipschitz continuous dependence property of the solution of problem (3.7)

**Proposition 3.4.2 .**

*Under the conditions of proposition (3.4.1), we have*

$$\|m - \tilde{m}\|_{L^\infty(\Omega)} \leq \max \left\{ \frac{1}{\lambda + \beta} \|Z(m) - \tilde{Z}(\tilde{m})\|_{L^\infty(\Omega)}, \|j - \tilde{j}\|_{L^\infty(\partial\Omega)}, \|Dm - D\tilde{m}\|_{L^\infty(\Omega)} \right\}.$$

**Proof**

Let

$$\Phi = \max \left\{ \frac{1}{\lambda + \beta} \|Z(m) - \tilde{Z}(\tilde{m})\|_{L^\infty(\Omega)}, \|j - \tilde{j}\|_{L^\infty(\partial\Omega)}, \|Dm - D\tilde{m}\|_{L^\infty(\Omega)} \right\}.$$

Then

$$\begin{aligned} Z(m) &\leq \tilde{Z}(\tilde{m}) + \|Z(m) - \tilde{Z}(\tilde{m})\|_{L^\infty(\Omega)} \\ &\leq \tilde{Z}(\tilde{m}) + \left( \frac{c_0(x) + \lambda}{\lambda + \beta} \right) \|Z(m) - \tilde{Z}(\tilde{m})\|_{L^\infty(\Omega)} \end{aligned}$$

so

$$Z(m) \leq \tilde{Z}(\tilde{m}) + (c_0(x) + \lambda)\Phi$$

and

$$j \leq \tilde{j} + \Phi.$$

We introduce the bilinear form

$$\begin{aligned} d(m + \Phi, v - m) &= d(m, v - m) + d(\Phi, v - m) \\ &\geq (Z(m), v - m) + \Phi(c_0, v - m) \\ &= (F(u) + a_0 \phi, v - u). \end{aligned}$$

So,

$$d(m + \Phi, (v + \Phi) - (m + \Phi)) \geq (Z(m) + c_0\Phi, (v + \Phi) - (m + \Phi)) \quad (3.12)$$

and we denote

$$\bar{m} = m + \Phi \quad \text{and} \quad \bar{v} = v + \Phi$$

then (3.12) is reformulated as follow

$$d(\bar{m}, \bar{v} - \bar{m}) \geq (Z(m) + c_0\Phi, \bar{v} - \bar{m})$$

hence,

$$\bar{m} = m + \Phi = j + \Phi \text{ on } \partial\Omega$$

and,

$$\bar{m} = m + \Phi \leq Dm + \Phi.$$

Therefore  $\bar{m}$  is the solution of  $\partial(Z(m) + c_0\Phi, j + \Phi, Dm + \Phi)$ , thus

$$\begin{aligned}\partial(Z(m) + c_0\Phi, j + \Phi, Dm + \Phi) &= \bar{m} = m + \Phi \\ &= \partial(Z(m), j, Dm) + \Phi.\end{aligned}$$

So, due to proposition (3.4.1), we get

$$\begin{aligned}\partial(Z(m), j, Dm) &\leq \partial(\tilde{Z}(\tilde{m}) + c_0\Phi, \tilde{j} + \Phi, D\tilde{m} + \Phi) \\ &= \partial(\tilde{Z}(\tilde{m}), \tilde{j}, D\tilde{m}) + \Phi\end{aligned}$$

then

$$m \leq \tilde{m} + \Phi.$$

Similarly, interchanging the roles of the couples  $(Z(m), j, Dm)$  and  $(\tilde{Z}(\tilde{m}), \tilde{j}, D\tilde{m})$ , we obtain

$$\tilde{m} \leq m + \Phi$$

so

$$\tilde{m} - m \leq \Phi$$

consequently

$$\|m - \tilde{m}\|_{L^\infty(\Omega)} \leq \Phi.$$

### 3.4.3 The continuous iterative scheme related to the first approach

Let  $m_0$  be an initial value defined by the following equation

$$d(m^0, v) = (Z(m^0), v), \quad \forall v \in H^1(\Omega). \quad (3.13)$$

For  $0 < \omega < 1$ , we define the sequence  $(m^{n+1})_{n \geq 0}$  such that  $m^{n+1} \in H^1(\Omega)$  solves the following quasi-variational inequality

$$\begin{cases} d(m^{n+1}, v - m^{n+1}) \geq (\omega Z(m^{n+1}) + (1 - \omega)Z(m^n), v - m^{n+1}) \\ m^{n+1} \leq Dm^n, \\ m^{n+1} = j \text{ on } \partial\Omega. \end{cases} \quad (3.14)$$

### 3.4.4 Geometrical convergence of the continuous iterative scheme related of the first approach

**Theorem 3.4.1 .**

*The continuous sequence  $(m^n)_{n \geq 0}$  defined in (3.14) converges geometrically to the solution  $m$  of QVI (3.7), More specifically, for every  $n \geq 0$ , there exists  $0 < q < 1$ .*

$$E^{n+1} \leq q^{n+1} E^0.$$

Where

$$q = \frac{\eta(1 - \omega)}{1 - \eta \omega}.$$

and

$$0 < \eta = \frac{L + \lambda}{\lambda + \beta} < 1$$

This quantity will be used throughout the proof.

**Proof**

We proceed by induction.

- ▶ We initiate the induction with  $n = 0$ ,

From proposition (3.4.2), we have

$$E^1 \leq \max \left\{ \frac{1}{\beta + \lambda} \|Z(m) - (\omega Z(m^1) + (1 - \omega)Z(m^0))\|_{L^\infty(\Omega)}, \|Dm - Dm^0\|_{L^\infty(\Omega)} \right\}.$$

This yields,

$$\begin{aligned} E^1 &\leq \max \left\{ \frac{1}{\beta + \lambda} (\omega \|Z(m) - Z(m^1)\|_{L^\infty(\Omega)} + (1 - \omega) \|Z(m) - Z(m^0)\|_{L^\infty(\Omega)}), \|m - m^0\|_{L^\infty(\Omega)} \right\} \\ &\leq \max \left\{ \frac{\lambda + L}{\beta + \lambda} (\omega E^1 + (1 - \omega) E^0), E^0 \right\}. \end{aligned}$$

We define

$$\eta = \frac{\lambda + L}{\beta + \lambda}.$$

Since  $L < \beta$ , we have  $\eta < 1$ , and clearly  $\eta > 0$ .

Then,

$$E^1 \leq \max \left\{ \eta \omega E^1 + \eta(1 - \omega) E^0, E^0 \right\}. \quad (3.15)$$

We now consider two cases

- **case 1**

$$\begin{aligned} \max \left\{ \eta \omega E^1 + \eta(1 - \omega) E^0, E^0 \right\} = \\ \eta \omega E^1 + \eta(1 - \omega) E^0 \end{aligned} \quad (3.16)$$

Then, from (3.15) and (3.16), we get:

$$E^1 \leq \eta \omega E^1 + \eta(1 - \omega) E^0 \quad (3.17)$$

and also,

$$E^0 \leq \eta \omega E^1 + \eta(1 - \omega) E^0. \quad (3.18)$$

Solving (3.17), we get

$$E^1 \leq \frac{\eta(1 - \omega)}{1 - \eta \omega} E^0. \quad (3.19)$$

Plugging (3.19) into (3.18) gives

$$E^0 \leq \frac{\eta(1 - \omega)}{1 - \eta \omega} E^0,$$

which is a contradiction (since  $0 < \eta < 1$ ). Therefore, this case cannot occur.

• case 2

$$\max\{\eta\omega E^1 + \eta(1-\omega)E^0, E^0\} = E^0. \quad (3.20)$$

Then,

$$E^1 \leq E^0 \quad (3.21)$$

and also,

$$\eta\omega E^1 + \eta(1-\omega)E^0 \leq E^0. \quad (3.22)$$

From (3.22) and (3.21), we get

$$E^1 \leq \eta\omega E^1 + \eta(1-\omega)E^0 \leq E^0$$

or

$$\eta\omega E^1 + \eta(1-\omega)E^0 \leq E^1 \leq E^0.$$

Then, we have

$$E^1 \leq \frac{\eta(1-\omega)}{1-\eta\omega} E^0 \leq E^0 \quad (3.23)$$

or

$$\frac{\eta(1-\omega)}{1-\eta\omega} E^0 \leq E^1 \leq E^0 \quad (3.24)$$

that implies

$$E^1 = \frac{\eta(1-\omega)}{1-\eta\omega} E^0.$$

Hence,

$$E^1 \leq \frac{\eta(1-\omega)}{1-\eta\omega} E^0.$$

Let

$$q = \frac{\eta(1-\omega)}{1-\eta\omega}.$$

Under the assumptions  $(0 < \omega < 1)$  and  $(0 < \eta < 1)$ , we have  $(1 - \omega > 0)$  and  $(1 - \eta\omega > 0)$  then

$$q > 0.$$

Moreover,

$$q < 1 \iff \frac{\eta(1-\omega)}{1-\eta\omega} < 1.$$

Since  $(1 - \eta\omega > 0)$ , this inequality is equivalent to

$$\eta(1 - \omega) < 1 - \eta\omega.$$

Expanding both sides, we obtain

$$\eta - \eta\omega < 1 - \eta\omega,$$

which is equivalent to  $(\eta < 1)$ , a condition that is satisfied.

Therefore, we conclude that

$$0 < q < 1.$$

Thus,

$$E^1 \leq qE^0. \tag{3.25}$$

► We suppose that for  $n \geq 0$ , we have

$$E^n \leq q^n E^0, \tag{3.26}$$

► And we establish that

$$E^{n+1} \leq q^{n+1} E^0. \tag{3.27}$$

As before, we obtain

$$E^{n+1} \leq \max\{\eta\omega E^{n+1} + \eta(1-\omega)E^n, E^n\} \tag{3.28}$$

Now we consider two cases

• **case 1**

$$\begin{aligned} \max\{\eta\omega E^{n+1} + \eta(1-\omega)E^n, E^n\} = \\ \eta\omega E^{n+1} + \eta(1-\omega)E^n \end{aligned} \tag{3.29}$$

Then,

$$E^{n+1} \leq \eta\omega E^{n+1} + \eta(1-\omega)E^n \quad (3.30)$$

and also,

$$E^n \leq \eta\omega E^{n+1} + \eta(1-\omega)E^n. \quad (3.31)$$

Which implies

$$E^{n+1} \leq \frac{\eta(1-\omega)}{1-\eta\omega} E^n. \quad (3.32)$$

Plugging (3.32) into (3.31), yields

$$E^n \leq \frac{\eta(1-\omega)}{1-\eta\omega} E^n,$$

which is a contradiction( since  $0 < \eta < 1$  ), and the first case (3.29) is impossible.

• case 2

$$\max\{\eta\omega E^{n+1} + \eta(1-\omega)E^n, E^n\} = E^n. \quad (3.33)$$

Then,

$$E^{n+1} \leq E^n$$

and also,

$$\eta\omega E^{n+1} + \eta(1-\omega)E^n \leq E^n.$$

That is,

$$E^{n+1} \leq \eta\omega E^{n+1} + \eta(1-\omega)E^n \leq E^n$$

or

$$\eta\omega E^{n+1} + \eta(1-\omega)E^n \leq E^{n+1} \leq E^n.$$

Thus,

$$E^{n+1} \leq \frac{\eta(1-\omega)}{1-\eta\omega} E^n \leq E^n \quad (3.34)$$

or

$$\frac{\eta(1-\omega)}{1-\eta\omega} E^n \leq E^{n+1} \leq E^n. \quad (3.35)$$

Hence,

$$E^{n+1} = \frac{\eta(1-\omega)}{1-\eta\omega} E^n.$$

Consequently

$$E^{n+1} \leq qE^n.$$

Furthermore, by applying the induction assumption (3.26), we obtain

$$E^{n+1} \leq q^{n+1} E^0.$$

which completes the induction, and shows geometric convergence of the sequence  $(m^n)_{n \geq 0}$  to  $m$  in the  $L^\infty$ -norm.

### 3.4.5 Discrete monotony property of the solution of problem (3.9)

In this part, we present the discrete monotony property of the solution with respect to the source term dependent of the unique discrete solution  $m_h$  of (3.9), the boundary conditions, and the obstacle function.

**Proposition 3.4.3 .**

Let  $m_h = \partial_h(Z(m_h), r_h Dm_h, \pi_h j)$  and  $\tilde{m}_h = \partial_h(\tilde{Z}(\tilde{m}_h), r_h D\tilde{m}_h, \pi_h \tilde{j})$  be solutions of (3.9) associated with the right hand side  $Z$  (respectively  $\tilde{Z}$ ) and the boundary condition  $j$  (respectively  $\tilde{j}$ ). then, according to the discrete maximum principle, we have if  $Z \leq \tilde{Z}$  and  $j \leq \tilde{j}$ , we obtain

$$m_h \leq \tilde{m}_h \text{ in } \overline{\Omega}.$$

### 3.4.6 A Lipschitz discrete dependence property of the solution of problem (3.9)

Maintaining the previous notations, we state the following proposition

**Proposition 3.4.4 .**

$$\|m_h - \tilde{m}_h\|_{L^\infty(\Omega)} \leq \max \left\{ \frac{1}{\lambda + \beta} \|Z(m_h) - \tilde{Z}(m_h)\|_{L^\infty(\Omega)}, \|j - \tilde{j}\|_{L^\infty(\partial\Omega)}, \|Dm_h - D\tilde{m}_h\|_{L^\infty(\Omega)} \right\}.$$

**Remark**

The proofs of the two preceding propositions are similar to those provided for the propositions in the continuous problem, and both proof rest on the discrete maximum principle, so they will be omitted.

**3.4.7 The discrete iterative scheme related of the first approach**

Let  $m_h^0$ ; be an initial value defined by the following equation

$$d(m_h^0, v) = (Z(m_h^0), v).$$

For  $0 < \omega < 1$ , we define the discrete sequence  $(m_h^n)_{n \geq 0}$  which solves the following quasi-variational inequality, for all  $n \geq 0$ ,  $m_h^{n+1} \in P_h^j(r_h m_h^n)$

$$\left\{ \begin{array}{l} d(m_h^{n+1}, v_h - m_h^{n+1}) \geq (\omega Z(m_h^{n+1}) + (1 - \omega)Z(m_h^n), v_h - m_h^{n+1}) \\ m_h^{n+1} \leq r_h D m_h^n, \\ m_h^{n+1} = j \text{ on } \partial\Omega. \end{array} \right. \quad (3.36)$$

**3.4.8 Geometrical convergence of the discrete iterative scheme related of the first approach**

In this subsection, we will present the geometrical convergence of the discrete iterative scheme associated with the first approach in this theorem

**Theorem 3.4.2** *The discrete sequence  $(m_h^n)_{n \geq 0}$  defined in (3.36) converges geometrically to the discrete solution  $m_h$  of (3.9) , more precisely there exists  $0 < q < 1$ , such that for all  $n \geq 0$*

$$e_h^{n+1} \leq q^{n+1} e_h^0.$$

**Proof.** The continuous case's evidence is the same as this one.

### 3.4.9 Error estimate in the maximum norm

This part focuses on proving the main result, which is the error estimate between the discrete and continuous solutions.

#### Auxiliary iteratif scheme related to the first approach

First, starting from  $w_h^0 = m_h^0$  we introduce the following auxiliary iterative scheme  $w_h^{n+1} \in V_h$  solution of

$$\begin{cases} d(w_h^{n+1}, v - w_h^{n+1}) \geq (\omega Z(m^{n+1}) + (1 - \omega)Z(m^n), v - w_h^{n+1}) \\ w_h^{n+1} \leq r_h D m^n = r_h(k + \inf_{n \geq 0}(m^n)). \end{cases} \quad (3.37)$$

We give the following lemma which will play an important role in the rest of our work

**Lemma 3.4.1** [27].

Let  $(m^{n+1})$  defined in (3.14), and  $(w_h^{n+1})$  defined in (3.37), then, for all  $n \geq 0$

$$W^n \leq Ch^2 |\log h|^2. \quad (3.38)$$

The following key lemma provides the error estimate between  $m^{n+1}$  defined in (3.14) and its discrete counterparts  $m_h^{n+1}$  defined in (3.36).

**Lemma 3.4.2** .

Let  $0 < \eta < 1$  defined in theorem (3.4.2) and for all  $0 < \omega < 1$ , we obtain

$$E_h^{n+1} \leq (\gamma(1 + q + \dots + q^n) + q^{n+1}) Ch^2 |\log h|^2. \quad (3.39)$$

Where

$$\gamma = \frac{1}{1 - \eta\omega} \quad (3.40)$$

and

$$q = \frac{\eta(1 - \omega)}{1 - \eta\omega}. \quad (3.41)$$

**Proof**

The proof will be by induction

- We initiate the induction with  $n = 0$ ,

$$E_h^1 \leq W^1 + W_h^1.$$

Then, from proposition(3.4.4)

$$E_h^1 \leq W^1 + \max\left\{\frac{1}{\lambda + \beta} \left[ \left( \omega Z(m^1) + (1 - \omega)Z(m^0) \right) - \left( \omega Z(m_h^1) + (1 - \omega)Z(m_h^0) \right) \right], \right. \\ \left. \|m_h^0 - m^0\|_{L^\infty(\Omega)} \cdot \right\}$$

So,

$$E_h^1 \leq W^1 + \max\left\{ \eta\omega E_h^1 + \eta(1 - \omega)E_h^0, E_h^0 \right\}. \quad (3.42)$$

We consider two cases

- **case 1**

$$\max\left\{ \eta\omega E_h^1 + \eta(1 - \omega)E_h^0, E_h^0 \right\} = \\ \eta\omega E_h^1 + \eta(1 - \omega)E_h^0, \quad (3.43)$$

Then, from (3.42)and (3.43), we find

$$E_h^1 \leq W^1 + \eta\omega E_h^1 + \eta(1 - \omega)E_h^0 \quad (3.44)$$

and also,

$$E_h^0 \leq \eta\omega E_h^1 + \eta(1 - \omega)E_h^0. \quad (3.45)$$

Thus,

$$E_h^1 \leq \frac{1}{1 - \eta\omega} W^1 + \frac{\eta(1 - \omega)}{1 - \eta\omega} E_h^0.$$

By lemma (3.4.1), we have

$$E_h^1 \leq \frac{1 + \eta(1 - \omega)}{1 - \eta\omega} Ch^2 |\log h|^2. \quad (3.46)$$

That is,

$$E_h^1 \leq (\gamma + q)Ch^2 |\log h|^2.$$

We plugging (3.46) into (3.45), we find

$$E_h^0 \leq \frac{\eta}{1-\eta\omega} Ch^2 |\log h|^2 < Ch^2 |\log h|^2$$

• case 2

$$\max\{\eta\omega E_h^1 + \eta(1-\omega)E_h^0, E_h^0\} = E_h^0. \quad (3.47)$$

Then,

$$E_h^1 \leq W^1 + E_h^0 \quad (3.48)$$

and also,

$$\eta\omega E_h^1 + \eta(1-\omega)E_h^0 \leq E_h^0. \quad (3.49)$$

Then,

$$E_h^1 - W^1 \leq E_h^0$$

and

$$\eta\omega E_h^1 + \eta(1-\omega)E_h^0 \leq E_h^0.$$

Therefore, we must distinguish the following cases

$$E_h^1 - W^1 \leq \eta\omega E_h^1 + \eta(1-\omega)E_h^0 \quad (3.50)$$

or

$$\eta\omega E_h^1 + \eta(1-\omega)E_h^0 \leq E_h^1 - W^1. \quad (3.51)$$

So,

$$E_h^1 \leq \frac{1}{1-\eta\omega} W^1 + \frac{\eta(1-\omega)}{1-\eta\omega} E_h^0$$

which implies

$$E_h^1 \leq \frac{1+\eta(1-\omega)}{1-\eta\omega} Ch^2 |\log h|^2. \quad (3.52)$$

Or

$$\frac{1+\eta(1-\omega)}{1-\eta\omega} Ch^2 |\log h|^2 \leq E_h^1 \leq W^1 + E_h^0$$

by lemma(3.4.1), we obtain

$$\frac{1 + \eta(1 - \omega)}{1 - \eta\omega} Ch^2 |\log h|^2 \leq E_h^1 \leq Ch^2 |\log h|^2$$

which is impossible because

$$\frac{1 + \eta(1 - \omega)}{1 - \eta\omega} > 1.$$

Thus, equations (3.43) and (3.47) imply

$$E_h^1 \leq (\gamma + q) Ch^2 |\log h|^2. \quad (3.53)$$

► For  $n \geq 0$ , we suppose that we have

$$E_h^n \leq (\gamma(1 + q + \dots + q^{n-1}) + q^n) Ch^2 |\log h|^2. \quad (3.54)$$

► And we establish that,

$$E_h^{n+1} \leq (\gamma(1 + q + \dots + q^n) + q^{n+1}) Ch^2 |\log h|^2. \quad (3.55)$$

We have

$$E_h^{n+1} \leq E_h^{n+1} + W^{n+1}.$$

Then, by applying Proposition (3.4.4), we obtain

$$E_h^{n+1} \leq W^{n+1} + \max\{\eta\omega E_h^{n+1} + \eta(1 - \omega)E_h^n, E_h^n\}. \quad (3.56)$$

We consider two cases

• **case 1**

$$\max\{\eta\omega E_h^{n+1} + \eta(1 - \omega)E_h^n, E_h^n\} = \eta\omega E_h^{n+1} + \eta(1 - \omega)E_h^n \quad (3.57)$$

Then, from (3.56) and (3.57), we get

$$E_h^{n+1} \leq W^{n+1} + \eta\omega E_h^{n+1} + \eta(1-\omega)E_h^n$$

and also

$$E_h^n \leq \eta\omega E_h^{n+1} + \eta(1-\omega)E_h^n. \quad (3.58)$$

So,

$$E_h^{n+1} \leq \frac{1}{1-\eta\omega} W^{n+1} + \frac{\eta(1-\omega)}{1-\eta\omega} E_h^n. \quad (3.59)$$

Plugging (3.59) into (3.58), gives

$$E_h^n \leq \eta\omega \left( \frac{1}{1-\eta\omega} W^{n+1} + \frac{\eta(1-\omega)}{1-\eta\omega} E_h^n + \eta(1-\omega)E_h^n \right) \leq \frac{\eta\omega}{1-\eta\omega} W^{n+1} + \frac{\eta(1-\omega)}{1-\eta\omega} E_h^n.$$

(3.56) in conjunction with (3.60) implies

$$E_h^{n+1} \leq \gamma ch^2 |\log h|^2 + q\{(\gamma(1+q+\dots+q^{n-1})+q^n)Ch^2|\log h|^2\}.$$

Since,  $\eta\omega < 1$ , we have

$$E_h^n \leq \gamma ch^2 |\log h|^2 + q\{(\gamma(1+q+\dots+q^{n-1})+q^n)Ch^2|\log h|^2\}.$$

Thus,

$$E_h^{n+1} \leq (\gamma(1+q+\dots+q^n)+q^{n+1})Ch^2|\log h|^2$$

and

$$E_h^n \leq (\gamma(1+q+\dots+q^n)+q^{n+1})Ch^2|\log h|^2$$

which coincides with (3.54).

• case 2

$$\max\{\eta\omega E_h^{n+1} + \eta(1-\omega)E_h^n, E_h^n\} = E_h^n. \quad (3.60)$$

Then,

$$E_h^{n+1} \leq W^{n+1} + E_h^n$$

and also

$$\eta\omega E_h^{n+1} + \eta(1-\omega)E_h^n \leq E_h^n.$$

Then,

$$E_h^{n+1} - W^{n+1} \leq E_h^n$$

and

$$\eta\omega E_h^{n+1} + \eta(1-\omega)E_h^n \leq E_h^n.$$

Therefore, we must distinguish the following cases

$$E_h^{n+1} - W^{n+1} \leq \eta\omega E_h^{n+1} + \eta(1-\omega)E_h^n \quad (3.61)$$

or

$$\eta\omega E_h^{n+1} + \eta(1-\omega)E_h^n \leq E_h^{n+1} - W^{n+1}. \quad (3.62)$$

This implies that

$$E_h^{n+1} \leq \frac{1}{1-\eta\omega} W^{n+1} + \frac{\eta(1-\omega)}{1-\eta\omega} E_h^n$$

or

$$\frac{1}{1-\eta\omega} W^{n+1} + \frac{\eta(1-\omega)}{1-\eta\omega} E_h^n \leq E_h^{n+1}.$$

So,

$$E_h^{n+1} \leq \gamma Ch^2 |\log h|^2 + q((\gamma(1+q+\dots+q^{n-1})+q^n)Ch^2 |\log h|^2)$$

or

$$\begin{aligned} \gamma Ch^2 |\log h|^2 + q((\gamma(1+q+\dots+q^{n-1})+q^n)Ch^2 |\log h|^2) &\leq E_h^{n+1} \\ &\leq Ch^2 |\log h|^2 + \gamma(1+q+\dots+q^{n-1})+q^n)Ch^2 |\log h|^2. \end{aligned}$$

which is impossible because

$$(\gamma(1+q+\dots+q^n)+q^{n+1})Ch^2 |\log h|^2 > Ch^2 |\log h|^2 + \gamma(1+q+\dots+q^{n-1})+q^n)Ch^2 |\log h|^2$$

Thus, only equation (3.61) is true. Consequently, we have

$$E_h^{n+1} \leq (\gamma(1 + q + \dots + q^n) + q^{n+1})Ch^2|\log h|^2.$$

which concludes the proof.

In this sequel, we will present the main result in this work and it characterizes the error between the continuous and the discrete solutions.

### The main result of the first approach

The following theorem characterizes the first main result of this thesis, which is the optimal error estimate between the continuous solution  $m$  and the discrete solution  $m_h$ .

#### Theorem 3.4.3 .

Let  $m$  be the solution of (3.7) and  $m_h$  the solution of (3.9). Then, there exist  $C$  independent of  $h$  such that

$$E_h \leq \frac{\gamma}{1-q} Ch^2|\log h|^2.$$

Where  $\gamma$  is defined in ( 3.40), and  $q$  is defined in (3.41).

#### Proof .

We use the triangular inequality and the results of geometrical convergence of both continuous and discrete schemes

$$\begin{aligned} E_h &\leq E^{n+1} + E_h^{n+1} + e_h^n \\ &\leq q^{n+1}E^0 + \eta \left( \frac{1-q^{n+1}}{1-q} \right) Ch^2|\log h|^2 + q^{n+1}e_h^0. \end{aligned}$$

As  $n \rightarrow \infty$ , we obtain

$$E_h \leq \frac{\gamma}{1-q} Ch^2|\log h|^2.$$

Which completes the proof.

## 3.5 A second approach to deal with the numerical analysis of the problem

As, we will use a linear iterative scheme in this second approach, we just recall the following two propositions which we have given a detailed proof in the chapter 1.

### 3.5.1 Continuous monotony property of the solution of Q.V.I with linear source term

**Proposition 3.5.1 .**

*The solutions of the QVI (3.7) corresponding to the data  $(Z, Dm, j)$  and  $(\tilde{Z}, D\tilde{m}, \tilde{j})$  denoted by  $m$  and  $\tilde{m}$  respectively; if  $Z \leq \tilde{Z}$  in  $\Omega$  and  $j \leq \tilde{j}$  in  $\partial\Omega$ , then  $m \leq \tilde{m}$ .*

**proof.**

The proof is similar to proposition (3.4.1).

### 3.5.2 A Lipschitz continuous property of the solution of Q.V.I with linear source term

**Proposition 3.5.2 .**

*Under the conditions of proposition (3.5.1), we have*

$$\|m - \tilde{m}\|_{L^\infty(\Omega)} \leq \max \left\{ \frac{1}{\lambda + \beta} \|Z - \tilde{Z}\|_{L^\infty(\Omega)}, \|j - \tilde{j}\|_{L^\infty(\partial\Omega)}, \|Dm - D\tilde{m}\|_{L^\infty(\Omega)} \right\}.$$

**proof.**

The proof is similar to proposition (3.4.2).

### 3.5.3 The continuous iterative scheme related to the second approach

Let  $m_0$  be an initial value defined in (3.13). And  $m^1 \in H^1(\Omega)$  be the solution of the following quasi-variational inequality

$$\begin{cases} d(m^1, v - m^1) \geq (Z(m^0), v - m^1), \forall v \in H^1(\Omega) \\ m^1 \leq Dm^0 \end{cases} \quad (3.63)$$

where

$$Dm^0 = k + \inf_{\xi \geq 0} m^0(x + \xi).$$

For  $0 < \omega < 1$ , we define the sequence  $(m^n)_{n \geq 0}$  which  $m^n \in H^1(\Omega)$  solves the following elliptic variational inequality

$$\begin{cases} d(m^{n+1}, v - m^{n+1}) \geq (\omega Z(m^n) + (1 - \omega)Z(m^{n-1}), v - m^{n+1}) \\ m^{n+1} \leq Dm^n \\ m^{n+1} = j \text{ on } \partial\Omega. \end{cases} \quad (3.64)$$

### 3.5.4 Uniforme convergence of the continuous iterative scheme related to the second approach

The following theorem describes the uniform convergence of the continuous sequence (3.64) to the unique solution of the continuous coercive problem (3.7).

**Theorem 3.5.1** *The continuous sequence  $(m^{n+1})_{n \geq 0}$  defined in (3.63)-(3.64) converges uniformly to the unique solution  $m$  of the continuous problem (3.7), more specifically, for all  $n \geq 0$ , there exists  $0 < \eta < 1$  such that,*

$$E^{n+1} \leq \eta^{\lfloor \frac{n+1}{2} \rfloor} E^0 \quad (3.65)$$

where

$$\eta = \frac{L + \lambda}{\beta + \lambda} \quad (3.66)$$

and  $\lfloor \frac{n+1}{2} \rfloor$  represents the integer part of the real number  $\frac{n+1}{2}$ .

This quantity will be used throughout the proof.

**Proof**

We proceed the theorem (3.5.1) by induction.

- ▶ We initiate the induction with  $n = 0$ ,

From proposition (3.5.2), and according to the definition of  $m$  and  $m^1$ , we have

$$E^1 \leq \max\left\{\left(\frac{1}{\beta + \lambda}\right)\|Z(m) - Z(m^0)\|_{L^\infty(\Omega)}; \|Dm - Dm^0\|_{L^\infty(\Omega)}\right\}. \quad (3.67)$$

According to (3.5), since the functional  $Z$  is Lipschitz, we obtain

$$E^1 \leq \max\left\{\left(\frac{L + \lambda}{\beta + \lambda}\right)E^0; E^0\right\}. \quad (3.68)$$

Then,

$$E^1 \leq \max\{\eta E^0; E^0\} = E^0. \quad (3.69)$$

Thus,

$$E^1 \leq \eta^{[\frac{1}{2}]} E^0. \quad (3.70)$$

- ▶ For  $n = 1$ , we get

$$E^2 \leq \max\left\{\left(\frac{1}{\beta + \lambda}\right)\|Z(m) - (\omega Z(m^1) + (1 - \omega)Z(m^0))\|_{L^\infty(\Omega)}; \|Dm - Dm^1\|_{L^\infty(\Omega)}\right\}. \quad (3.71)$$

Then,

$$E^2 \leq \max\left\{\left(\frac{1}{\beta + \lambda}\right)\|(\omega(Z(m) - Z(m^1)) + (1 - \omega)(Z(m) - Z(m^0)))\|_{L^\infty(\Omega)}; \|Dm - Dm^1\|_{L^\infty(\Omega)}\right\}. \quad (3.72)$$

So,

$$E^2 \leq \max\{\omega\eta E^1 + (1 - \omega)\eta E^0; E^1\}. \quad (3.73)$$

As, (3.73) may attain one of two distinct values, it is necessary to analyze the following two scenarios separately, We begin by analyzing the first case.

• **case 1**

$$\max\{\omega\eta E^1 + (1 - \omega)\eta E^0; E^1\} = \omega\eta E^1 + (1 - \omega)\eta E^0 \quad (3.74)$$

Then,

$$E^2 \leq \omega\eta E^1 + (1 - \omega)\eta E^0, \quad (3.75)$$

and

$$E^1 \leq \omega\eta E^1 + (1 - \omega)\eta E^0. \quad (3.76)$$

So,

$$E^1 \leq \frac{(1 - \omega)\eta}{1 - \omega\eta} E^0. \quad (3.77)$$

We replace (3.77) in (3.75), we obtain

$$E^2 \leq qE^0. \quad (3.78)$$

Because  $q < \eta$ , we obtain

$$E^2 \leq \eta E^0. \quad (3.79)$$

Now, we pass to the second case

• **case 2**

$$\max\{\omega\eta E^1 + (1 - \omega)\eta E^0; E^1\} = E^1. \quad (3.80)$$

Then,

$$E^2 \leq E^1 \quad (3.81)$$

and also,

$$\omega\eta E^1 + (1 - \omega)\eta E^0 \leq E^1. \quad (3.82)$$

Thus,

$$qE^0 \leq E^1. \quad (3.83)$$

From(3.81) and (3.83) we establish the following two cases

$$E^2 \leq qE^0 \leq E^1 \quad (3.84)$$

or

$$qE^0 \leq E^2 \leq E^1, \quad (3.85)$$

according to (3.70), we get

$$E^2 \leq qE^0 \leq E^0 \quad (3.86)$$

or

$$qE^0 \leq E^2 \leq E^0. \quad (3.87)$$

Hence, (3.86) and (3.87) hold simultaneously, as they coincide respectively with (3.81) and (3.70). Therefore, either a contradiction arises—making case (3.80) impossible—or this case is possible only if

$$E^2 = qE^0 \quad (3.88)$$

thus, case 2 implies

$$E^2 \leq \eta E^0. \quad (3.89)$$

Therefore, in both cases (3.74) and (3.80), we have

$$E^2 \leq \eta^{\lfloor \frac{n+1}{2} \rfloor} E^0. \quad (3.90)$$

► Let us suppose that for all  $n \geq 0$ , we have

$$E^n \leq \eta^{\lfloor \frac{n}{2} \rfloor} E^0. \quad (3.91)$$

► And prove that

$$E^{n+1} \leq \eta^{\lfloor \frac{n+1}{2} \rfloor} E^0. \quad (3.92)$$

As in the previous iterates, we get

$$E^{n+1} \leq \max\{\omega\eta E^n + (1-\omega)\eta E^{n-1}; E^n\}. \quad (3.93)$$

We distinguish two cases, let's start by the first one

• **case 1**

$$\max\{\omega\eta E^n + (1-\omega)\eta E^{n-1}; E^n\} = \omega\eta E^n + (1-\omega)\eta E^{n-1} \quad (3.94)$$

then, (3.93) implies

$$E^{n+1} \leq \omega\eta E^n + (1-\omega)\eta E^{n-1} \quad (3.95)$$

with (3.94), we have

$$E^n \leq \omega\eta E^n + (1-\omega)\eta E^{n-1}. \quad (3.96)$$

Thus,

$$E^n \leq qE^{n-1}. \quad (3.97)$$

We replace (3.97) in (3.95), we obtain

$$E^{n+1} \leq qE^{n-1}. \quad (3.98)$$

According to (3.92), we get

$$E^{n+1} \leq \eta \eta^{\lfloor \frac{n-1}{2} \rfloor} E^0. \quad (3.99)$$

Then, the first case (3.94) causes us to have

$$E^{n+1} \leq \eta^{\lfloor \frac{n+1}{2} \rfloor} E^0. \quad (3.100)$$

Passing now to the second case,

• **case 2**

$$\max\{\omega \eta E^n + (1 - \omega) \eta E^{n-1}; E^n\} = E^n \quad (3.101)$$

then,

$$E^{n+1} \leq E^n \quad (3.102)$$

and also

$$\omega \eta E^n + (1 - \omega) \eta E^{n-1} \leq E^n. \quad (3.103)$$

Thus, (3.103) gives us

$$qE^{n-1} \leq E^n. \quad (3.104)$$

So, we distinguish the following two cases

$$E^{n+1} \leq qE^{n-1} \leq E^n \leq \eta^{\lfloor \frac{n}{2} \rfloor} E^0 \quad (3.105)$$

or

$$qE^{n-1} \leq E^{n+1} \leq E^n \leq \eta^{\lfloor \frac{n}{2} \rfloor} E^0. \quad (3.106)$$

Thus,(3.105) is always hold, because we have,

$$qE^{n-1} \leq \eta\eta^{\lfloor \frac{n-1}{2} \rfloor} E^0 \leq \eta^{\lfloor \frac{n}{2} \rfloor} E^0. \quad (3.107)$$

The second case (3.106) implies

$$E^{n-1} \leq \frac{1}{q} E^{n+1} \leq \frac{1}{q} \eta^{\lfloor \frac{n}{2} \rfloor} E^0. \quad (3.108)$$

And using (3.92) and (3.108), we get

$$E^{n-1} \leq \eta^{\lfloor \frac{n-1}{2} \rfloor} E^0 \leq \frac{1}{q} \eta^{\lfloor \frac{n}{2} \rfloor} E^0. \quad (3.109)$$

which corresponds to (3.92). As a result, both cases (3.105) and (3.106) hold simultaneously. This leads to two possibilities: either a contradiction occurs, rendering case (3.101) impossible, or case (3.101) remains possible provided that

$$E^{n+1} \doteq qE^{n-1} \quad (3.110)$$

consequently,

$$E^{n+1} \leq \eta^{\lfloor \frac{n+1}{2} \rfloor} E^0. \quad (3.111)$$

which completes the proof of the theorem .

### 3.5.5 Discrete monotony property of the solution of Q.V.I with linear source term

#### Proposition 3.5.3 .

Let  $m_h = \partial_h(Z, r_h D m_h, \pi_h j)$  and  $\tilde{m}_h = \partial_h(\tilde{Z}, r_h D \tilde{m}_h, \pi_h \tilde{j})$  be solutions of (3.9) associated with the right hand side  $Z$ (respectively  $\tilde{Z}$ ) and the boundary condition  $j$ (respectively  $\tilde{j}$ ). then, according to the discrete maximum principle, we have if  $Z \leq \tilde{Z}$  and  $j \leq \tilde{j}$ , we obtain

$$\partial_h(Z, r_h D m_h, \pi_h j) \leq \partial_h(\tilde{Z}, r_h D \tilde{m}_h, \pi_h \tilde{j}).$$

### 3.5.6 A Lipschitz discrete property of the solution of Q.V.I with linear source term

**Proposition 3.5.4 .**

$$\|m_h - \tilde{m}_h\|_{L^\infty(\Omega)} \leq \max \left\{ \frac{1}{\lambda + \beta} \|Z - \tilde{Z}\|_{L^\infty(\Omega)}, \|j - \tilde{j}\|_{L^\infty(\partial\Omega)}, \|Dm_h - D\tilde{m}_h\|_{L^\infty(\Omega)} \right\}.$$

### 3.5.7 The discrete iterative scheme related to the second approach

The discrete problem will be treated in this part with perfect symmetry, just as the continuous problem is. In fact, we define the discrete iterative sequence  $(m_h^{n+1})_{n \geq 0}$  and prove that it uniformly converges to unique discrete solution  $m_h$  of the discrete problem (3.9).

Let  $m_h^0 = r_h m^0$  denote the discrete counterpart of  $m^0$ , the solution of (3.13). Then,  $m_h^0 \geq 0$  and the standard Lagrange interpolation error estimate yields

$$E_h^0 \leq Ch^2 |\log h|^2. \quad (3.112)$$

Next, we define  $m_h^1 \in V_h$  as the solution of the following discrete VI

$$\begin{aligned} d(m_h^1, v - m_h^1) &\geq (Z(m_h^0), v - m_h^1) \\ m_h^1 &\leq r_h(Dm_h^0). \end{aligned} \quad (3.113)$$

For  $0 < \omega < 1$ , we introduce the following discrete iterative sequence  $(m_h^{n+1})_{n \geq 1}$  the solution of the following discrete elliptic variational inequality

$$\begin{aligned} d(m_h^{n+1}, v - m_h^{n+1}) &\geq (\omega Z(m_h^n) + (1 - \omega)Z(m_h^{n-1}), v - m_h^{n+1}) \\ m_h^{n+1} &\leq r_h(Dm_h^n), \\ m_h^{n+1} &= \pi_h j \end{aligned} \quad (3.114)$$

### 3.5.8 Uniforme convergence of the discrete iterative scheme related to the second approach

In this subsection, we establish the theorem that describes the uniform convergence of the discrete sequence (3.114) to the unique solution of the discrete coercive problem (3.9).

**Theorem 3.5.2** *The discrete sequence  $(m_h^{n+1})_{n \geq 0}$  defined in (3.113)-(3.114) converges uniformly to the unique discrete solution  $m_h$  of the discrete problem (3.9), that is there exists  $\eta$ ;  $0 < \eta < 1$  such that, for all  $n \geq 0$*

$$E_h^{n+1} \leq \eta^{\lfloor \frac{n+1}{2} \rfloor} E_h^0. \quad (3.115)$$

#### Proof

The proof of Theorem (3.5.2) follows a similar structure to that of Theorem (3.4.1).

### 3.5.9 Error estimate in the maximum norm

This part focuses on proving the main result, which is the error estimate between the continuous solution  $m$  of the continuous problem (3.7) and the discrete solution  $m_h$  of the discrete problem (3.9). We introduce the following lemma which will play a key role in our work [1].

**Lemma 3.5.1** *Let  $(m^{n+1})_{n \geq 0}$  be the continuous sequence defined in (3.64) and  $(m_h^{n+1})_{n \geq 0}$  be the discrete sequence defined in (3.114). Then, for all  $n \geq 0$*

$$E_h^{n+1} \leq 2 \left( \frac{1 - \eta^{n+1}}{1 - \eta} \right) Ch^2 |\log h|^2. \quad (3.116)$$

#### Proof

We proceed the proof by induction.

- We initiate the induction with  $n = 0$

Since,  $m^1 \in H^1(\Omega) \cap L^\infty(\Omega)$  is the unique solution of (3.63), it follows that  $m^1$  is a sub-solution; then,

$$\begin{cases} d(m^1, v) \leq (Z(m^0), v) \\ m^1(x) \leq Dm^0(x) = k + \inf_{\xi \geq 0} m^0(x + \xi). \end{cases} \quad (3.117)$$

We introduce now  $\bar{m}^1 \in H^1(\Omega) \cap L^\infty(\Omega)$  solution of the following continuous elliptic VI

$$\begin{aligned} d(\bar{m}^1, v - \bar{m}^1) &\geq (Z(m^0), v - \bar{m}^1) \\ \bar{m}^1 &\leq k. \end{aligned}$$

Since,  $\bar{m}^1$  is also a subsolution, we have the following inequality

$$\begin{aligned} d(\bar{m}^1, v) &\leq (Z(m^0), v) \\ \bar{m}^1 &\leq k. \end{aligned}$$

Given that,  $k \leq Dm^0 = k + \inf_{\xi \geq 0} m^0(x + \xi)$  it follows directly that

$$\begin{aligned} d(\bar{m}^1, v) &\leq (Z(m^0), v) \\ \bar{m}^1 &\leq k \leq Dm^0 \end{aligned}$$

thus,  $\bar{m}^1$  is a subsolution of  $m^1$ , and we have

$$\bar{m}^1 \leq m^1 \quad \text{in } \Omega. \quad (3.118)$$

We introduce  $\bar{m}_h^1$  the discrete counterpart of  $\bar{m}^1$ , which is solution of the following

discrete elliptic VI

$$d(\bar{m}_h^1, v - \bar{m}_h^1) \geq (Z(m^0), v - \bar{m}_h^1) \quad (3.119)$$

$$\bar{m}_h^1 \leq r_h(k).$$

Consequently, based on the Lipschitz continuity of  $Z$ , the following error estimate holds:

$$\|\bar{m}^1 - \bar{m}_h^1\|_{L^\infty(\Omega)} \leq Ch^2 |\log h|^2. \quad (3.120)$$

So,

$$\bar{m}_h^1 - Ch^2 |\log h|^2 \leq \bar{m}^1. \quad (3.121)$$

According to proposition (3.5.4), we get

$$\|\bar{m}_h^1 - m_h^1\|_{L^\infty(\Omega)} \leq \max\left\{\left(\frac{1}{\beta + \lambda}\right) \|Z(m^0) - Z(m_h^0)\|_{L^\infty(\Omega)}, \|r_h(k) - r_h(Dm_h^0)\|_{L^\infty(\Omega)}\right\}. \quad (3.122)$$

Then,

$$\|\bar{m}_h^1 - m_h^1\|_{L^\infty(\Omega)} \leq \max\{\eta \|m^0 - m_h^0\|_{L^\infty(\Omega)}, \|k - Dm_h^0\|_{L^\infty(\Omega)}\}. \quad (3.123)$$

Since

$$k - Dm_h^0(x) = k - \left(k + \inf_{\xi \geq 0} m_h^0(x + \xi)\right) = -\inf_{\xi \geq 0} m_h^0(x + \xi) \quad (3.124)$$

$$\leq -\inf_{\xi \geq 0} m_h^0(x + \xi) + \inf_{\xi \geq 0} m^0(x + \xi) \quad (3.125)$$

$$\leq \inf_{\xi \geq 0} \left(m^0(x + \xi) - m_h^0(x + \xi)\right) \quad (3.126)$$

meaning that

$$k - Dm_h^0(x) \leq E_h^0 \quad (3.127)$$

Also, for all positive  $\epsilon$ , we possess

$$\inf_{\xi \geq 0} m_h^0(x + \xi) \leq m_h^0(x + \xi) \leq \inf_{\xi \geq 0} m_h^0(x + \xi) + \epsilon \quad (3.128)$$

thus,

$$k - Dm_h^0(x) = -\inf_{\xi \geq 0} m_h^0(x + \xi) \geq -\inf_{\xi \geq 0} m_h^0(x + \xi) - \epsilon. \quad (3.129)$$

And for  $\epsilon = \inf_{\xi \geq 0} m_h^0(x + \xi) + E_h^0$ , we get

$$k - Dm_h^0(x) \geq -E_h^0. \quad (3.130)$$

So,

$$\|k - Dm_h^0\|_{L^\infty(\Omega)} \leq E_h^0. \quad (3.131)$$

We note that the estimate (3.131) holds for every iteration; in other words, for all integers  $n$ , the following inequality remains true

$$\|k - Dm_h^n\|_{L^\infty(\Omega)} \leq E_h^n. \quad (3.132)$$

Plugging (3.131) in (3.123), we obtain

$$\|\overline{m}_h^1 - m_h^1\|_{L^\infty(\Omega)} \leq \max\{\eta E_h^0, E_h^0\} = E_h^0 \quad (3.133)$$

so,

$$m_h^1 - E_h^0 \leq \overline{m}_h^1. \quad (3.134)$$

Upon combining inequalities (3.118), (3.121), and (3.134), we arrive at the following result

$$-E_h^0 - Ch^2 |\log h|^2 \leq m^1 - m_h^1. \quad (3.135)$$

Similarly, since  $m_h^1 \in V_h$  is the unique discrete solution of (3.113) it follows that  $m_h^1$  is a discrete sub-solution that is, for all basis functions  $\phi_s$ , with  $s = 1, \dots, t_h$ , the following inequality holds

$$\begin{aligned} d(m_h^1, \phi_s) &\leq (Z(m_h^0), \phi_s) \\ m_h^1(x) &\leq r_h(Dm_h^0)(x) = r_h(k + \inf_{\xi \geq 0} m_h^0(x + \xi)). \end{aligned}$$

With  $\underline{m}_h^1 \in V_h$ , we present the discrete solution to the following discrete elliptic VI,

$$\begin{aligned} d(\underline{m}_h^1, v - \underline{m}_h^1) &\geq (Z(m_h^0), v - \underline{m}_h^1) \\ \underline{m}_h^1 &\leq r_h(k). \end{aligned}$$

Also  $\underline{m}_h^1$  is a discrete sub-solution, then,

$$\begin{aligned} d(\underline{m}_h^1, \phi_s) &\leq (Z(m_h^0), \phi_s) \\ \underline{m}_h^1 &\leq r_h(k) \leq r_h(k + \inf_{\xi \geq 0} m_h^0(x + \xi)). \end{aligned}$$

That is

$$\underline{m}_h^1 \leq m_h^1 \text{ in } \Omega. \quad (3.136)$$

Let  $\underline{m}^1$  be the continuous counterpart of  $\underline{m}_h^1$ , the solution of the following continuous elliptic VI

$$d(\underline{m}^1, v - \underline{m}^1) \geq (Z(m_h^0), v - \underline{m}^1) \quad (3.137)$$

$$\underline{m}^1 \leq k. \quad (3.138)$$

Therefore, due to the Lipschitz continuity of the function  $Z$ , we have

$$\|\underline{m}^1 - \underline{m}_h^1\|_{L^\infty(\Omega)} \leq Ch^2 |\log h|^2, \quad (3.139)$$

so,

$$\underline{m}^1 - Ch^2 |\log h|^2 \leq \underline{m}_h^1. \quad (3.140)$$

According to proposition (3.5.2), we obtain

$$\|\underline{m}^1 - m^1\|_{L^\infty(\Omega)} \leq \max\left\{\left(\frac{1}{\beta + \lambda}\right) \|Z(m_h^0) - Z(m^0)\|_{L^\infty(\Omega)}, \|k - Dm^0\|_{L^\infty(\Omega)}\right\}. \quad (3.141)$$

Thus

$$\|\underline{m}^1 - m^1\|_{L^\infty(\Omega)} \leq \max\{\eta E_h^0, \|k - Dm^0\|_{L^\infty(\Omega)}\}. \quad (3.142)$$

Then, as before, we get

$$\|k - Dm^0\|_{L^\infty(\Omega)} \leq E_h^0. \quad (3.143)$$

Consequently

$$\|k - Dm^n\|_{L^\infty(\Omega)} \leq E_h^n. \quad (3.144)$$

We replace (3.143) in (3.142), we get

$$\|\underline{m}^1 - m^1\|_{L^\infty(\Omega)} \leq \max\{\eta E_h^0, E_h^0\} = E_h^0 \quad (3.145)$$

hence

$$m^1 - E_h^0 \leq \underline{m}^1. \quad (3.146)$$

By using (3.136), (3.140), and (3.146), we get

$$-\|m^0 - m_h^0\|_{L^\infty(\Omega)} - Ch^2|\log h|^2 \leq m_h^1 - m^1. \quad (3.147)$$

Thus, (3.135) and (3.147) gives

$$E_h^1 \leq E_h^0 + Ch^2|\log h|^2. \quad (3.148)$$

Then,

$$E_h^1 \leq 2Ch^2|\log h|^2. \quad (3.149)$$

► For  $n = 1$ ,

Since  $m^2 \in H^1(\Omega) \cap L^\infty(\Omega)$  is the unique solution of (3.64) it follows that,  $m^2$  is a sub-solution, then

$$\begin{aligned} d(m^2, v) &\leq (\omega Z(m^1) + (1 - \omega)Z(m^0), v) \\ m^2(x) &\leq Dm^1(x) = k + \inf_{\xi \geq 0} m^1(x + \xi). \end{aligned}$$

Now, let's present the unique solution  $\bar{m}^2 \in H^1(\Omega) \cap L^\infty(\Omega)$  of the following continuous elliptic VI

$$\begin{aligned} d(\bar{m}^2, v - \bar{m}^2) &\geq (\omega Z(m^1) + (1 - \omega)Z(m^0), v - \bar{m}^2) \\ \bar{m}^2 &\leq k. \end{aligned}$$

Since,  $\bar{m}^2$  is also a sub-solution, we have the following inequality

$$d(\bar{m}^2, v) \leq (\omega Z(m^1) + (1 - \omega)Z(m^0), v)$$

$$\bar{m}^1 \leq k \leq Dm^1.$$

So,

$$\bar{m}^2 \leq m^2 \text{ in } \Omega. \quad (3.150)$$

We introduce  $\bar{m}_h^2$  the discrete counterpart of  $\bar{m}^2$ , the solution of the following discrete elliptic VI

$$d(\bar{m}_h^2, v - \bar{m}_h^2) \geq (\omega Z(m^1) + (1 - \omega)Z(m^0), v - \bar{m}_h^2) \quad (3.151)$$

$$\bar{m}_h^2 \leq r_h(k).$$

Then,

$$\|\bar{m}^2 - \bar{m}_h^2\|_{L^\infty(\Omega)} \leq Ch^2 |\log h|^2 \quad (3.152)$$

hence,

$$\bar{m}_h^2 - Ch^2 |\log h|^2 \leq \bar{m}^2. \quad (3.153)$$

Using proposition (3.5.4), we get

$$\|\bar{m}_h^2 - m_h^2\|_{L^\infty(\Omega)} \leq \max\left\{\left(\frac{1}{\beta+\lambda}\right)\|\omega(Z(m^1) - Z(m_h^1)) + (1 - \omega)(Z(m^0) - Z(m_h^0))\|_{L^\infty(\Omega)},\right.$$

$$\left.\|r_h(k) - r_h(Dm_h^1)\|_{L^\infty(\Omega)}\right\}. \quad (3.154)$$

And due to (3.132), we find

$$\|\bar{m}_h^2 - m_h^2\|_{L^\infty(\Omega)} \leq \max\{\eta\omega E_h^1 + \eta(1 - \omega)E_h^0, E_h^1\}. \quad (3.155)$$

Then,

$$m_h^2 - \|\overline{m}_h^2 - m_h^2\|_{L^\infty(\Omega)} \leq \overline{m}_h^2. \quad (3.156)$$

And using inequalities (3.150), (3.153), and (3.156), we have

$$-\|\overline{m}_h^2 - m_h^2\|_{L^\infty(\Omega)} - Ch^2|\log h|^2 \leq m^2 - m_h^2. \quad (3.157)$$

Then,

$$-\max\{\eta\omega E_h^1 + \eta(1-\omega)E_h^0, E_h^1\} - Ch^2|\log h|^2 \leq m^2 - m_h^2. \quad (3.158)$$

Similarly, since  $m_h^2 \in V_h$  is the unique discrete solution of (3.114) it follows that,  $m_h^2$  is a discrete sub-solution that is for all basis function  $\phi_s$ , with  $s = 1, \dots, t_h$  the following inequality holds

$$\begin{aligned} d(m_h^2, \phi_s) &\leq (\omega Z(m_h^1) + (1-\omega)Z(m_h^0), \phi_s) \\ m_h^2(x) &\leq r_h(Dm_h^1)(x) = r_h(k + \inf_{\xi \geq 0} m_h^1(x + \xi)). \end{aligned}$$

With  $\underline{m}_h^2 \in V_h$ , we present the unique discrete solution of the following discrete elliptic VI

$$\begin{aligned} d(\underline{m}_h^2, v - \underline{m}_h^2) &\geq (\omega Z(m_h^1) + (1-\omega)Z(m_h^0), v - \underline{m}_h^2) \\ \underline{m}_h^2 &\leq r_h(k). \end{aligned}$$

Also  $\underline{m}_h^2$  is a discrete sub-solution, then,

$$\begin{aligned} d(\underline{m}_h^2, \phi_s) &\leq (\omega Z(m_h^1) + (1-\omega)Z(m_h^0), \phi_s) \\ \underline{m}_h^2 &\leq r_h(k) \leq r_h(k + \inf_{\xi \geq 0} m_h^1(x + \xi)). \end{aligned}$$

Thus,

$$\underline{m}_h^2 \leq m_h^2 \text{ in } \Omega. \quad (3.159)$$

Let  $\underline{m}^2$  be the continuous counterpart of  $\underline{m}_h^2$  the solution of the following continuous elliptic VI

$$d(\underline{m}^2, v - \underline{m}^2) \geq (\omega Z(m_h^1) + (1 - \omega)Z(m_h^0), v - \underline{m}^2) \quad (3.160)$$

$$\underline{m}^2 \leq k. \quad (3.161)$$

So,

$$\|\underline{m}^2 - \underline{m}_h^2\|_{L^\infty(\Omega)} \leq Ch^2 |\log h|^2, \quad (3.162)$$

then,

$$\underline{m}^2 - Ch^2 |\log h|^2 \leq \underline{m}_h^2. \quad (3.163)$$

Due to proposition (3.5.2), we get

$$\|\underline{m}^2 - m^2\|_{L^\infty(\Omega)} \leq \max\left\{\left(\frac{1}{\beta+\lambda}\right)\|\omega(Z(m_h^1) - Z(m^1))\|_{L^\infty(\Omega)} + (1 - \omega)\|Z(m_h^0) - Z(m^0)\|_{L^\infty(\Omega)}, \|k - Dm^1\|_{L^\infty(\Omega)}\right\}. \quad (3.164)$$

Thus

$$\|\underline{m}^2 - u^m\|_{L^\infty(\Omega)} \leq \max\{\eta\omega E_h^1 + \eta(1 - \omega)E_h^0, \|k - Dm^1\|_{L^\infty(\Omega)}\}. \quad (3.165)$$

According to (3.144)

$$\|\underline{m}^2 - m^2\|_{L^\infty(\Omega)} \leq \max\{\eta\omega E_h^1 + \eta(1 - \omega)E_h^0, E_h^1\}, \quad (3.166)$$

and

$$m^2 - \|\underline{m}^2 - m^2\|_{L^\infty(\Omega)} \leq \underline{m}^2. \quad (3.167)$$

Using inequalities (3.159), (3.163), and (3.167) respectively, we get

$$-\|m^2 - m^2\|_{L^\infty(\Omega)} - Ch^2|\log h|^2 \leq m_h^2 - m^2. \quad (3.168)$$

Thus,

$$-\max\{\eta\omega E_h^1 + \eta(1-\omega)E_h^0, E_h^1\} - Ch^2|\log h|^2 \leq m_h^2 - m^2. \quad (3.169)$$

Then, according to (3.158) and (3.169), we get

$$E_h^2 \leq Ch^2|\log h|^2 + \max\{\eta\omega E_h^1 + \eta(1-\omega)E_h^0, E_h^1\}. \quad (3.170)$$

Then , we establish two cases, starting by the first one

• **case 1**

$$\max\{\eta\omega E_h^1 + \eta(1-\omega)E_h^0, E_h^1\} = \eta\omega E_h^1 + \eta(1-\omega)E_h^0. \quad (3.171)$$

then

$$E_h^2 \leq Ch^2|\log h|^2 + \eta\omega E_h^1 + \eta(1-\omega)E_h^0. \quad (3.172)$$

By using results of the iterates (3.112) and (3.149), (3.172)

$$E_h^2 \leq (1 + 2\eta\omega + \eta(1-\omega))Ch^2|\log h|^2. \quad (3.173)$$

As  $0 < \eta\omega < 1$ , case (3.171) gives

$$E_h^2 \leq 2(1 + \eta)Ch^2|\log h|^2. \quad (3.174)$$

Passing now to the second case

• **case 2**

$$\max\{\eta\omega E_h^1 + \eta(1-\omega)E_h^0, \|m^1 - m_h^1\|_{L^\infty(\Omega)}\} = E_h^1. \quad (3.175)$$

then

$$E_h^2 \leq Ch^2 |\log h|^2 + E_h^1, \quad (3.176)$$

and also

$$\eta\omega E_h^1 + \eta(1-\omega)E_h^0 \leq E_h^1. \quad (3.177)$$

So

$$qE_h^0 \leq E_h^1 \quad (3.178)$$

due to (3.112), we have

$$qE_h^0 \leq qCh^2 |\log h|^2. \quad (3.179)$$

Which brings us to consider the following two distinct possibilities.

$$E_h^1 \leq qCh^2 |\log h|^2. \quad (3.180)$$

Or

$$qCh^2 |\log h|^2 \leq E_h^1. \quad (3.181)$$

So,

$$E_h^1 \leq qCh^2 |\log h|^2 \leq 2Ch^2 |\log h|^2. \quad (3.182)$$

Or

$$qCh^2 |\log h|^2 \leq E_h^1 \leq 2Ch^2 |\log h|^2. \quad (3.183)$$

Then, both cases (3.182) and (3.183) hold simultaneously, as they both coincide with (3.149). Therefore, either a contradiction arises and case (3.175) is impossible, or case (3.175) is possible only if

$$E_h^1 = qCh^2 |\log h|^2. \quad (3.184)$$

We replace (3.184) in (3.176), we get

$$E_h^2 \leq Ch^2 |\log h|^2 + qCh^2 |\log h|^2 = (1+q)Ch^2 |\log h|^2. \quad (3.185)$$

So, as  $q < \eta$ , we have

$$E_h^2 \leq 2(1 + \eta)Ch^2|\log h|^2. \quad (3.186)$$

Thus, in either case—(3.171) or (3.175)—the following error estimate holds

$$E_h^2 \leq 2(1 + \eta)Ch^2|\log h|^2. \quad (3.187)$$

► Let's suppose that  $n \geq 0$ , we have

$$E_h^n \leq 2(1 + \eta + \eta^2 + \dots + \eta^{n-1})Ch^2|\log h|^2, \quad (3.188)$$

► and prove

$$E_h^{n+1} \leq 2(1 + \eta + \eta^2 + \dots + \eta^{n-1} + \eta^n)Ch^2|\log h|^2. \quad (3.189)$$

We introduce  $m^{n+1} \geq 0$  and  $m^{n+1} \in H^1(\Omega) \cap L^\infty(\Omega)$  the unique solution of (3.64)

then,  $m^{n+1}$  is a sub-solution and for all positive  $v$

$$\begin{aligned} d(m^{n+1}, v) &\leq (\omega Z(m^n) + (1 - \omega)Z(m^{n-1}), v) \\ m^{n+1}(x) &\leq Dm^n(x) = k + \inf_{\xi \geq 0} m^n(x + \xi). \end{aligned}$$

We introduce also a positive  $\bar{m}^{n+1}$  with  $\bar{m}^{n+1} \in H^1(\Omega) \cap L^\infty(\Omega)$  the unique solution of the following continuous elliptic VI

$$\begin{aligned} d(\bar{m}^{n+1}, v - \bar{m}^{n+1}) &\geq (\omega Z(m^n) + (1 - \omega)Z(m^{n-1}), v - \bar{m}^{n+1}) \\ \bar{m}^{n+1} &\leq k. \end{aligned}$$

Since  $\bar{m}^{n+1}$  is also a sub-solution, then

$$\begin{aligned} d(\bar{m}^{n+1}, v) &\leq (\omega Z(m^n) + (1 - \omega)Z(m^{n-1}), v) \\ \bar{m}^{n+1} &\leq k \leq Dm^n. \end{aligned}$$

Thus,  $\bar{m}^{n+1}$  is a subsolution of  $m^{n+1}$ , and we have

$$\bar{m}^{n+1} \leq m^{n+1} \quad \text{in } \Omega. \quad (3.190)$$

Let  $\bar{m}_h^{n+1}$  be the discrete counterpart to  $\bar{m}^{n+1}$  so,

$$\begin{aligned} d(\bar{m}_h^{n+1}, v - \bar{m}_h^{n+1}) &\geq (\omega Z(m^n) + (1 - \omega)Z(m^{n-1}), v - \bar{m}_h^{n+1}) \\ \bar{m}_h^{n+1} &\leq r_h(k). \end{aligned} \quad (3.191)$$

So,

$$\|\bar{m}^{n+1} - \bar{m}_h^{n+1}\|_{L^\infty(\Omega)} \leq Ch^2 |\log h|^2 \quad (3.192)$$

hence

$$\bar{m}_h^{n+1} - Ch^2 |\log h|^2 \leq \bar{m}^{n+1}. \quad (3.193)$$

As before, we have

$$\|\bar{m}_h^{n+1} - m_h^{n+1}\|_{L^\infty(\Omega)} \leq \max\{\eta\omega E_h^n + \eta(1 - \omega)E_h^{n-1}, \|k - Dm_h^n\|_{L^\infty(\Omega)}\}, \quad (3.194)$$

Using (3.132), we get

$$\|\bar{m}_h^{n+1} - m_h^{n+1}\|_{L^\infty(\Omega)} \leq \max\{\eta\omega E_h^n + \eta(1 - \omega)E_h^{n-1}, E_h^n\}. \quad (3.195)$$

Then, according to The definition of the maximum norm,

$$m_h^{n+1} - \|\bar{m}_h^{n+1} - m_h^{n+1}\|_{L^\infty(\Omega)} \leq \bar{m}_h^{n+1}. \quad (3.196)$$

Now, by applying (3.190), (3.193), and (3.196) respectively, we get

$$-\|\bar{m}_h^{n+1} - m_h^{n+1}\|_{L^\infty(\Omega)} - Ch^2|\log h|^2 \leq m^{n+1} - m_h^{n+1}. \quad (3.197)$$

Then, (3.195) with (3.197) gives

$$-\max\{\eta\omega E_h^n + \eta(1-\omega)E_h^{n-1}, E_h^n\} - Ch^2|\log h|^2 \leq m^{n+1} - m_h^{n+1}. \quad (3.198)$$

Similarly, as  $m_h^{n+1} \geq 0$  and  $m_h^{n+1} \in V_h$  is the unique discrete solution of (3.114)

then,  $m_h^{n+1}$  is a discrete sub-solution that is for all basis function  $\phi_s, s = 1, \dots, t_h$

$$\begin{aligned} d(m_h^{n+1}, \phi_s) &\leq (\omega Z(m_h^n) + (1-\omega)Z(m_h^{n-1}), \phi_s) \\ m_h^{n+1}(x) &\leq r_h(Dm_h^n)(x) = r_h(k + \inf_{\xi \geq 0} m_h^n(x + \xi)). \end{aligned}$$

Let  $\underline{m}_h^{n+1} \in V_h$  the unique discrete solution of the following discrete elliptic VI

$$\begin{aligned} d(\underline{m}_h^{n+1}, v - \underline{m}_h^{n+1}) &\geq (\omega Z(m_h^n) + (1-\omega)Z(m_h^{n-1}), v - \underline{m}_h^{n+1}) \\ \underline{m}_h^{n+1} &\leq r_h(k). \end{aligned}$$

Since  $\underline{m}_h^{n+1}$  is also a discrete sub-solution, then

$$\begin{aligned} d(\underline{m}_h^{n+1}, \phi_s) &\leq (\omega Z(m_h^n) + (1-\omega)Z(m_h^{n-1}), \phi_s) \\ \underline{m}_h^{n+1} &\leq r_h(k) \leq r_h(k + \inf_{\xi \geq 0} m_h^n(x + \xi)). \end{aligned}$$

gives that  $\underline{m}_h^{n+1}$  is a discrete subsolution of  $m_h^{n+1}$ , so

$$\underline{m}_h^{n+1} \leq m_h^{n+1} \quad \mathbf{in} \quad \Omega. \quad (3.199)$$

Let  $\underline{m}^{n+1}$  be the continuous counterpart of  $\underline{m}_h^{n+1}$  then,  $\underline{m}^{n+1} \in H^1(\Omega) \cap L^\infty(\Omega)$  is solution of the following continuous elliptic VI

$$d(\underline{m}^{n+1}, v - \underline{m}^{n+1}) \geq (\omega Z(m_h^n) + (1 - \omega)Z(m_h^n), v - \underline{m}^{n+1}) \quad (3.200)$$

$$\underline{m}^{n+1} \leq k.$$

So,

$$\|\underline{m}^{n+1} - \underline{m}_h^{n+1}\|_{L^\infty(\Omega)} \leq Ch^2 |\log h|^2, \quad (3.201)$$

thus,

$$\underline{m}^{n+1} - Ch^2 |\log h|^2 \leq \underline{m}_h^{n+1}. \quad (3.202)$$

As before we will use proposition (3.5.2) and we get

$$\|\underline{m}^{n+1} - m^{n+1}\|_{L^\infty(\Omega)} \leq \max\{\eta\omega E_h^n + \eta(1 - \omega)E_h^{n-1}, E_h^n\}, \quad (3.203)$$

and

$$m^{n+1} - \|\underline{m}^{n+1} - m^{n+1}\|_{L^\infty(\Omega)} \leq \underline{m}^{n+1}. \quad (3.204)$$

Using inequalities (3.199), (3.202), and (3.204) respectively, we get

$$-\|\underline{m}^{n+1} - m^{n+1}\|_{L^\infty(\Omega)} - Ch^2 |\log h|^2 \leq m_h^{n+1} - m^{n+1}. \quad (3.205)$$

Thus, (3.203) with (3.205) gives

$$-\max\{\eta\omega E_h^n + \eta(1 - \omega)E_h^{n-1}, E_h^n\} - Ch^2 |\log h|^2 \leq m_h^{n+1} - m^{n+1}. \quad (3.206)$$

Then, according to (3.198) and (3.206), we get

$$E_h^{n+1} \leq Ch^2 |\log h|^2 + \max\{\eta\omega E_h^n + \eta(1 - \omega)E_h^{n-1}, E_h^n\}. \quad (3.207)$$

So, we establish two possibilities

• case 1

$$\begin{aligned} & \max\{\eta\omega E_h^n + \eta(1-\omega)E_h^{n-1}, E_h^n\} \\ & = \eta\omega E_h^n + \eta(1-\omega)E_h^{n-1}. \end{aligned} \quad (3.208)$$

Then

$$E_h^{n+1} \leq Ch^2|\log h|^2 + \eta\omega E_h^n + \eta(1-\omega)E_h^{n-1}. \quad (3.209)$$

By using (3.188) and (3.209), we give

$$E_h^{n+1} \leq (1+2\eta\omega(1+\eta+\eta^2+\dots+\eta^{n-2}+\eta^{n-1})+2\eta(1-\omega)(1+\eta+\eta^2+\dots+\eta^{n-2}))Ch^2|\log h|^2. \quad (3.210)$$

Thus,

$$E_h^{n+1} \leq (1+2\eta^n\omega+2\eta(1+\eta+\eta^2+\dots+\eta^{n-2}))Ch^2|\log h|^2. \quad (3.211)$$

As  $0 < \omega < 1$ , we obtain

$$E_h^{n+1} \leq 2(1+\eta+\eta^2+\dots+\eta^n)Ch^2|\log h|^2. \quad (3.212)$$

• case2

$$\max\{\eta\omega E_h^n + \eta(1-\omega)E_h^{n-1}, E_h^n\} = E_h^n. \quad (3.213)$$

Then

$$E_h^{n+1} \leq Ch^2|\log h|^2 + E_h^n, \quad (3.214)$$

and

$$\eta\omega E_h^n + \eta(1-\omega)E_h^{n-1} \leq E_h^n. \quad (3.215)$$

Then

$$qE_h^{n-1} \leq E_h^n \quad (3.216)$$

Due to (3.188), we have

$$qE_h^{n-1} \leq 2q \left( \sum_{k=0}^{n-2} \eta^k \right) Ch^2 |\log h|^2. \quad (3.217)$$

This leads us to consider the following two possibilities

$$E_h^n \leq 2q \left( \sum_{k=0}^{n-2} \eta^k \right) Ch^2 |\log h|^2. \quad (3.218)$$

Or

$$2q \left( \sum_{k=0}^{n-2} \eta^k \right) Ch^2 |\log h|^2 \leq E_h^n. \quad (3.219)$$

So, according to (3.188), we have

$$E_h^n \leq 2q \left( \sum_{k=0}^{n-2} \eta^k \right) Ch^2 |\log h|^2 \leq 2 \left( \sum_{k=0}^{n-1} \eta^k \right) Ch^2 |\log h|^2. \quad (3.220)$$

Or

$$2q \left( \sum_{k=0}^{n-2} \eta^k \right) Ch^2 |\log h|^2 \leq E_h^n \leq 2 \left( \sum_{k=0}^{n-1} \eta^k \right) Ch^2 |\log h|^2. \quad (3.221)$$

Then, both cases (3.220) and (3.221) hold simultaneously, as they both coincide with (3.188). Therefore, either a contradiction arises and case (3.213) is impossible, or case (3.213) is possible only if the following condition is satisfied

$$E_h^n = 2q(1 + \eta + \eta^2 + \dots + \eta^{n-2})Ch^2 |\log h|^2. \quad (3.222)$$

We replace (3.222) in (3.214), we get

$$E_h^{n+1} \leq Ch^2 |\log h|^2 + 2q(1 + \eta + \eta^2 + \dots + \eta^{n-2})Ch^2 |\log h|^2. \quad (3.223)$$

Because  $q < \eta$ , we get

$$E_h^{n+1} \leq 2(1 + \eta + \eta^2 + \dots + \eta^n)Ch^2|\log h|^2. \quad (3.224)$$

Consequently, in both cases (3.208) and (3.213) we have the following error estimate

$$E_h^{n+1} \leq 2(1 + \eta + \eta^2 + \dots + \eta^n)Ch^2|\log h|^2. \quad (3.225)$$

### 3.5.10 The main result

The next theorem establishes the principal result of this work.

#### Theorem 3.5.3 .

Let  $m$  be the solution of the continuous problem (3.7) and  $m_h$  be the discrete problem (3.9). Then, there exists a constant  $C$  independent of  $h$  such that

$$E_h \leq \frac{2}{1-\eta}Ch^2|\log h|^2. \quad (3.226)$$

#### Proof

Let's use the following triangular inequality

$$E_h \leq E^{n+1} + E_h^{n+1} + \|m_h^{n+1} - m_h\|_{L^\infty(\Omega)}. \quad (3.227)$$

Then

$$E_h \leq \eta^{\lfloor \frac{n+1}{2} \rfloor} E^0 + 2\left(\frac{1-\eta^{n+1}}{1-\eta}\right)Ch^2|\log h|^2 + \eta^{\lfloor \frac{n+1}{2} \rfloor} \|m_h^0 - m_h\|_{L^\infty(\Omega)}. \quad (3.228)$$

By passing to the limit, as  $n \rightarrow +\infty$ , we get

$$E_h \leq \frac{2}{1-\eta}Ch^2|\log h|^2.$$



# CONCLUSION

In this work, we have characterized the unique solution of a noncoercive quasi-variational inequality arising from an impulse control problem with a nonlinear source term. For the numerical analysis, we adopted an algorithmic approach, presenting two Bensoussan–Lions type iterative schemes in both the continuous and discrete framework, the latter treated using the finite element method. It has been established that both iterative schemes yield the same optimal error estimate in the  $L^\infty(\Omega)$  norm,

$$\|m - m_h\|_{L^\infty(\Omega)} \leq Ch^2 |\log h|^2.$$

Finally, we outline some open problems for future research:

- Numerical experimentation for systems of quasi-variational inequalities with noncoercive operators established in this thesis.
- The application of the domain decomposition method.



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