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THESIS

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The Risk Theory: Application in Actuarial Science

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By

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THÈSE

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ABSTRACT

The Poisson New XLindley Process, a new iteration of the non-homogeneous Poisson process, is presented in this thesis. A few statistical characteristics are showcased. Furthermore, a comparison analysis is provided between the Poisson counting process version, Poisson Lindley, Poisson XLindley, and Poisson new XLindley process. We apply the proposed models to a ruin problem to demonstrate their usefulness. Our intention is to attract scholars by demonstrating the adaptability and potential uses of these innovative Processes. **Keywords:** Stochastic processes, Poisson process, Poisson New XLindley Process.

RÉSUMÉ

Cette thèse présente le processus Poisson New XLindley, une nouvelle version du processus de Poisson non homogène et quelques caractéristiques mathé matiques. En outre, une étude comparative est présentée entre les différentes versions du processus de comptage de Poisson, à savoir Poisson Lindley, Poisson XLindley et Poisson new XLindley. Les modèles proposés sont appliqués à un problème de ruine pour montrer leur utilité. Notre but est de susciter l'intérêt des chercheurs en mettant en évidence l'adaptabilité et les possibilités d'utilisation de ces processus novateurs.

Mots-clés : Processus stochastiques, processus de Poisson, processus de Poisson Lindley, processus Poisson New XLindley.

ملخص

تُقدَّم هذه الاطروحة متغير عشوائي جديد" بواسون س-ليندلي الجديد" وهو نوع جديد للمتغير العشوائي بواسون غير المتجانس. وتُعرَض بعض الخصائص الرياضية. علاوةً على ذلك، نُقدَّم دراسة مقارنةً بين عدة نسخ لمتغيرات بواسون، وبواسون ليندلي، وبواسون س-ليندلي، و بواسون س- ليندلي الجديد. نُطبِّق النماذج المقترحة على مشكلة الافلاس لإثبات فائدتها. هدفنا هو جذب الباحثين من خلال إظهار قابلية هذه المتغيرات العشوائية المبتكرة للتكيف والاستخدامات المُحتملة لها.

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The result of many years of labor, introspection, and tenacity is this thesis. Without the help, direction, and inspiration of numerous individuals, which I would like to acknowledge, it would not have been feasible.

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CHAPTER 1 INTRODUCTION

Counting processes play a central role in probability theory and applied statistics, providing mathematical models for the occurrence of random events over time. Among the most widely used models are the renewal process (RP) and the Poisson process, including both the homogeneous Poisson process (HPP) and the non-homogeneous Poisson process (NHPP). These processes have been extensively applied in reliability engineering, actuarial science, epidemiology, queueing systems, and finance to model phenomena such as system failures, insurance claims, customer arrivals, and disease incidence [35, 1, 29].

In the homogeneous Poisson process, interarrival times are independent and identically distributed exponential random variables, leading to independent and stationary increments. These properties result in strong analytical tractability and simple simulation procedures, making the HPP a standard baseline model. However, in many practical situations, the assumption of a constant event rate is unrealistic. This limitation motivates the use of the non-homogeneous Poisson process, in which the intensity function varies over time, preserving independent increments but losing stationarity [35, 1]. The NHPP allows greater modeling flexibility and has found wide use in reliability growth models and software failure data [33].

Despite their usefulness, Poisson-based models are often insufficient when event counts exhibit overdispersion, clustering, or unobserved heterogeneity. To address these shortcomings, numerous generalizations of Poisson and renewal processes have been proposed, including mixed Poisson processes, filtered Poisson processes, and semi-Markov processes

[23, 38]. These extensions aim to better capture real-world variability, yet many remain mathematically complex or difficult to implement. Consequently, there remains a practical need for alternative counting models that balance flexibility with analytical tractability.

Counting processes are particularly important in insurance risk theory, where they model the arrival of claims. The classical collective risk model, known as the Cramér–Lundberg model, assumes that claim arrivals follow a Poisson process and claim sizes are independent and identically distributed random variables. The insurer's surplus process is defined as

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i,$$

where u is the initial capital, c is the premium rate, $N(t)$ is the claim counting process, and X_i are claim severities. The central object of interest is the ruin probability, defined as the probability that the surplus becomes negative at some time [17, 1, 25]. This framework has motivated extensive research on both claim arrival processes and claim size distributions.

While the Poisson assumption provides analytical convenience, empirical claim data often exhibit overdispersion relative to the Poisson distribution. This has led to the development of alternative count distributions and compound models, including Poisson–Gamma, Poisson–Lognormal, and Poisson–Lindley-type models [29, 15]. In particular, Sankaran [36] introduced the Poisson–Lindley distribution by mixing the Poisson distribution with the Lindley distribution, yielding a flexible model capable of capturing overdispersed count data. Subsequent work has proposed numerous generalizations and variants, including the Poisson–XLindley, Poisson quasi-Lindley, and size-biased Poisson–Lindley distributions, with applications in insurance, reliability, and biological sciences [18, 19, 20, 41, 13].

More recently, several authors have proposed modified Poisson-type processes constructed via compounding or mixing with flexible lifetime distributions, leading to new counting processes with improved fit to real data [30, 4, 37]. These models preserve much of the tractability of the Poisson framework while allowing richer variance and tail behavior, which are essential for accurate risk assessment in insurance and reliability systems.

Motivated by these developments, this thesis introduces a new counting process constructed by combining the Poisson process with the New XLindley distribution. The resulting Poisson–New XLindley (PNXL) process generalizes both the classical Poisson process and earlier Poisson–Lindley-type models. This construction yields a flexible yet analytically manageable process capable of modeling overdispersed and heterogeneous event counts. The proposed process is particularly suitable for applications in reliability analysis, actuarial science, queueing systems, and failure modeling.

Motivation and Objectives

The primary motivations for this research are summarized as follows:

— To develop a flexible counting process that extends the Poisson and Poisson–Lindley families while remaining mathematically tractable.

— To derive explicit expressions for the probability mass function, mean, variance, and higher-order moments of the proposed process.

— To investigate key probabilistic properties of the process, including interarrival behavior and long-term growth characteristics.

— To demonstrate the usefulness of the model in actuarial and reliability contexts, where overdispersion and heterogeneity are frequently observed.

— To compare the performance of the proposed PNXL process with classical Poisson, Poisson–Lindley, and Poisson–XLindley models through simulation.

Organization of the Thesis

The remainder of this thesis is organized as follows.

Chapter 1 presents a review of classical ruin theory and introduces the distributions that form the foundation of the proposed model, including the Lindley, XLindley, and New XLindley distributions, as well as their Poisson mixtures.

Chapter 2 introduces the Poisson–New XLindley counting process, derives the distribution of the number of events over finite time intervals, and studies its main probabilistic properties, including the mean, variance, and dispersion behavior.

Chapter 3 develops compound risk models based on the proposed counting process and investigates surplus dynamics and ruin probabilities under various claim severity assumptions.

Chapter 4 presents a comprehensive simulation study comparing the PNXL process with classical Poisson and Poisson–Lindley-type processes, illustrating the advantages of the proposed model in terms of flexibility and risk assessment.

Finally, concluding remarks and directions for future research are presented, including possible extensions to multivariate and dependent risk models.

CHAPTER 2

BASIC CONCEPTS AND DISTRIBUTIONS

In this chapter, we present fundamental concepts of ruin theory and introduce the probability distributions that motivate the proposed counting process, namely the Lindley, XLindley, New XLindley, Poisson–Lindley, Poisson–XLindley, and Poisson–New XLindley distributions. These distributions play an important role in modeling claim counts and event arrivals in actuarial science and reliability theory.

2.1 Overview of Ruin Theory

In insurance mathematics, *ruin* refers to the event that the insurer's surplus becomes negative. Avoiding ruin is a fundamental objective for insurance companies, and ruin theory provides a mathematical framework for modeling surplus dynamics and evaluating insolvency risk.

Ruin theory studies the stochastic behavior of the insurer's surplus over time and aims to quantify the probability that accumulated claims exceed the initial capital and premium income. Because insurance portfolios are exposed to both random claim arrivals and random claim sizes, stochastic process models are essential for realistic risk assessment.

Definitions of Ruin Probabilities

Let $\{U(t), t \geq 0\}$ denote the insurer's surplus process with initial capital $u = U(0)$. The *ultimate ruin probability* is defined as

$$\psi(u) = \Pr\big(U(t) < 0 \text{ for some } t > 0\big), \tag{2.1}$$

that is, the probability that ruin occurs at any time in the future. This is also called the *continuous-time ruin probability.*

The corresponding *discrete-time ultimate ruin probability* is defined by

$$\psi_r(u) = \Pr\big(U(t) < 0 \text{ for some } t = r, 2r, 3r, \dots\big), \tag{2.2}$$

where the surplus is observed only at discrete time points separated by step size $r > 0$.

If ruin occurs at one of the discrete observation times, then ruin must also occur in continuous time. However, ruin may occur between observation times and remain undetected in the discrete model. Therefore,

$$\psi_r(u) \leq \psi(u).$$

As $r \to 0$, the discrete-time ruin probability converges to the continuous one.

The *finite-time ruin probability* is defined as

$$\psi(u, t) = \Pr\big(U(s) < 0 \text{ for some } s \in (0, t]\big), \tag{2.3}$$

which measures the probability of ruin within a finite horizon $(0, t]$.

Similarly, the discrete-time finite ruin probability is

$$\psi_r(u, t) = \Pr\big(U(s) < 0 \text{ for some } s = r, 2r, \dots, t\big), \tag{2.4}$$

where t is an integer multiple of r, and again $\psi_r(u, t) \leq \psi(u, t)$.

2.2 Risk Model

The classical insurance risk model describes the evolution of the insurer's surplus over time as

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \qquad t \geq 0, \tag{2.5}$$

where

— $u = U(0)$ is the initial capital,

— $c > 0$ is the premium rate,

— $N(t)$ is a counting process representing the number of claims up to time t,

— X_i are independent and identically distributed (i.i.d.) positive random variables representing claim sizes, independent of $N(t)$.

The aggregate claims process is given by

$$W(t) = \sum_{i=1}^{N(t)} X_i,$$

so that $U(t) = u + ct - W(t)$. This framework forms the basis of most ruin probability analyses in actuarial science.

2.3 Individual and Collective Risk Models

Two main modeling approaches are used in actuarial science: the individual risk model and the collective risk model.

Individual Risk Model

In the individual risk model, total claims arise from a fixed portfolio of insured individuals:

$$S^{\text{Ind}} = \sum_{i=1}^{n} I_i X_i, \tag{2.6}$$

where

— n is the number of insured individuals,

— $I_i \sim \text{Bernoulli}(p)$ indicates whether individual i incurs a claim,

— X_i is the claim amount given that a claim occurs.

This model is well suited for portfolios with heterogeneous risks, such as health insurance, where claim probabilities and severities differ across policyholders.

Collective Risk Model

The collective risk model focuses on the total number of claims rather than individual contracts:

$$S^{\mathrm{Col}} = \sum_{i=1}^{N} X_i, \tag{2.7}$$

where N is a discrete random variable representing the number of claims, and X_i are i.i.d. claim sizes independent of N.

When the portfolio is large and homogeneous, the collective model can approximate the individual model. In particular, if $N \sim \mathrm{Binomial}(n, p)$ with n large and p small such that $np \to \lambda$, then

$$N \xrightarrow{d} \mathrm{Poisson}(\lambda).$$

This Poisson approximation justifies the widespread use of Poisson-based counting models in insurance applications.

2.4 Classical Cramér–Lundberg Model

The classical ruin model, also known as the Cramér–Lundberg or P/G model, assumes Poisson claim arrivals and general claim sizes. Let $\{\sigma_i\}_{i\geq 1}$ be i.i.d. inter-arrival times, and define

$$T_n = \sum_{i=1}^{n} \sigma_i, \qquad N(t) = \max\{n : T_n \leq t\}.$$

Then $N(t)$ is a renewal counting process, and if $\sigma_i \sim \mathrm{Exp}(\lambda)$, then $N(t)$ is a homogeneous Poisson process.

Let $\{X_i\}$ be i.i.d. claim sizes with mean μ. The surplus process is

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i. \tag{2.8}$$

The cumulative loss process is sometimes written as

$$S(t) = W(t) = \sum_{i=1}^{N(t)} X_i. \tag{2.9}$$

2.5 Counting Processes

Definition 2.5.1 (Counting Process). *A stochastic process $\{N(t), t \geq 0\}$ is called a counting process if:*

— $N(t) \geq 0$,

— $N(t)$ *takes integer values,*

— $N(t)$ *is non-decreasing,*

— $N(t) - N(s)$ *counts events in* $(s, t]$.

Definition 2.5.2 (Homogeneous Poisson Process). *A counting process $\{N(t)\}$ is a Poisson process with rate $\lambda > 0$ if:*

— $N(0) = 0$,

— *it has independent increments,*

— $N(t) - N(s) \sim \text{Poisson}(\lambda(t-s))$ *for* $0 \leq s < t$.

Definition 2.5.3 (Renewal Process). *Let $\{\sigma_i\}$ be i.i.d. positive random variables and define $T_n = \sum_{i=1}^{n} \sigma_i$. The counting process*

$$N(t) = \max\{n : T_n \leq t\}$$

is called a renewal process.

Example 2.5.1. If $N(t)$ is a Poisson process with rate λ, then

$$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t},$$

so $T_1 \sim \text{Exp}(\lambda)$, and all inter-arrival times are i.i.d. exponential.

Theorem 2.5.1 (Order Statistics Property). *Conditional on $N(t) = n$, the arrival times $(T_1, \ldots, T_n)$ are distributed as the order statistics of n i.i.d. Uniform$(0, t)$ random variables.*

This property is fundamental for simulation of Poisson processes and is widely used in Monte Carlo studies of ruin probabilities and aggregate losses.

2.6 Around XLindley and New XLindley Distributions

This section reviews the XLindley distribution and introduces the New XLindley distribution, which will later serve as building blocks for the construction of new compound and counting process models.

2.6.1 The XLindley Distribution (XLD)

The XLindley distribution (XLD), introduced in the literature as an extension of the Lindley model, is defined as a finite mixture of the exponential and Lindley distributions with a common parameter $\theta > 0$. Let X be a random variable with density

$$f(x) = \sum_{i=1}^{k} p_i f_i(x), \tag{2.10}$$

where $f_i(x)$ are component densities and $p_i \geq 0$ with $\sum_{i=1}^{k} p_i = 1$.

For the XLindley distribution, the mixture components are:

— $f_1(x) \sim \text{Exp}(\theta)$,

— $f_2(x) \sim \text{Lindley}(\theta)$,

— $p_1 = \dfrac{\theta}{1+\theta}, \quad p_2 = \dfrac{1}{1+\theta}$.

The resulting probability density function is

$$f_{XL}(x;\theta) = \frac{\theta^2(2+\theta+x)}{(1+\theta)^2} e^{-\theta x}, \qquad x > 0,\ \theta > 0. \tag{2.11}$$

Shape of the Density and Mode. The first derivative of the density is

$$\frac{d}{dx} f_{XL}(x;\theta) = -\frac{\theta^2(\theta^2 + \theta(2+x) - 1)}{(1+\theta)^2} e^{-\theta x}. \tag{2.12}$$

Solving $\frac{d}{dx} f_{XL}(x;\theta) = 0$ gives the critical point

$$x^* = -\frac{\theta^2 + 2\theta - 1}{\theta}. \tag{2.13}$$

Hence,

— if $0 < \theta < \sqrt{2} - 1$, the density is unimodal with mode at x^*,

— if $\theta \geq \sqrt{2} - 1$, the density is strictly decreasing and the mode occurs at $x = 0$.

Thus, the mode is

$$\text{mode}(X) = \begin{cases} -\dfrac{\theta^2 + 2\theta - 1}{\theta}, & 0 < \theta < \sqrt{2} - 1, \\ 0, & \theta \geq \sqrt{2} - 1. \end{cases} \tag{2.14}$$

Distribution, Survival and Hazard Functions. The cumulative distribution function is

$$F_{XL}(x;\theta) = 1 - \left(1 + \frac{\theta x}{(1+\theta)^2}\right) e^{-\theta x}. \tag{2.15}$$

The survival function is

$$S_{XL}(x) = \left(1 + \frac{\theta x}{(1+\theta)^2}\right) e^{-\theta x}, \tag{2.16}$$

and the hazard rate function is

$$h_{XL}(x) = \frac{f_{XL}(x)}{S_{XL}(x)} = \frac{\theta^2(x + \theta + 2)}{(1+\theta)^2 + \theta x}. \tag{2.17}$$

Using Glaser's monotonicity criterion, it can be shown that $h_{XL}(x)$ is increasing, which implies that the XLindley distribution belongs to the IFR (increasing failure rate) class.

2.6.2 The New XLindley Distribution (NXLD)

A recent generalization of exponential-type distributions is the new polynomial exponential distribution (NPED). A special case of this family leads to the New XLindley distribution (NXLD).

Let X be a random variable with probability density function

$$f_{NXL}(x;\theta) = \frac{\theta}{2}(1 + \theta x)e^{-\theta x}, \qquad x > 0,\ \theta > 0. \tag{2.18}$$

This distribution can be interpreted as a finite mixture:

$$X \sim \tfrac{1}{2}\,\text{Exp}(\theta) + \tfrac{1}{2}\,\Gamma(2,\theta),$$

where $\Gamma(2,\theta)$ denotes the Gamma distribution with shape 2 and rate θ.

Distribution, Survival and Hazard Functions

The cumulative distribution function is

$$F_{NXL}(x;\theta) = 1 - \left(1 + \tfrac{1}{2}\theta x\right) e^{-\theta x}. \tag{2.19}$$

The survival function is

$$S_{NXL}(x) = \left(1 + \tfrac{1}{2}\theta x\right) e^{-\theta x}, \tag{2.20}$$

and the hazard rate function is

$$h_{NXL}(x) = \frac{f_{NXL}(x)}{S_{NXL}(x)} = \frac{\theta + \theta^2 x}{\theta x + 2}. \tag{2.21}$$

Proposition 2.6.1. *The hazard rate function $h_{NXL}(x)$ is increasing in x.*

Proof. Define

$$\rho(x) = -\frac{d}{dx} \log f_{NXL}(x) = \frac{\theta^2 x}{1 + \theta x}.$$

Then

$$\rho'(x) = \frac{\theta^2}{(1+\theta x)^2} > 0,$$

which implies that the hazard rate is increasing by Glaser's criterion. □

Shape and Mode

From (2.18),

$$\lim_{x\to 0} f_{NXL}(x) = \frac{\theta}{2}, \qquad \lim_{x\to\infty} f_{NXL}(x) = 0.$$

The derivative is

$$f'_{NXL}(x) = -\tfrac{1}{2}\theta^3 x e^{-\theta x} < 0, \quad x > 0,$$

so the density is strictly decreasing and the mode is located at $x = 0$.

Moments

The r-th raw moment is

$$\begin{aligned}\mu_r' = \mathbb{E}[X^r] &= \int_0^\infty x^r f_{NXL}(x)\,dx = \frac{\theta}{2}\int_0^\infty x^r(1+\theta x)e^{-\theta x}\,dx \\ &= \frac{1}{2\theta^r}\big[\Gamma(r+1)+\Gamma(r+2)\big]. \end{aligned} \tag{2.22}$$

Proposition 2.6.2. *Let* $X \sim \mathrm{NXLD}(\theta)$. *Then*

$$\begin{aligned} \mathbb{E}[X] &= \frac{3}{2\theta}, \\ \mathrm{Var}(X) &= \frac{7}{4\theta^2}, \\ \textit{Coefficient of Variation} &= \frac{\sqrt{7}}{3}, \\ \textit{Skewness} &= \frac{120}{49}\sqrt{7}, \\ \textit{Kurtosis} &= \frac{1152}{49}. \end{aligned}$$

Theorem 2.6.1. *If* $X \sim \mathrm{NXLD}(\theta)$, *then* $\mathrm{Median}(X) < \mathbb{E}[X]$.

Proof. Let m be the median. From (2.19), $F(m) = \frac{1}{2}$ and

$$F(\mathbb{E}[X]) = 1 - \left(1+\tfrac{3}{4}\right)e^{-3/2} = 1 - \tfrac{7}{4}e^{-3/2} > \tfrac{1}{2}.$$

Since F is increasing, it follows that $m < \mathbb{E}[X]$. □

2.7 On the Poisson XLindley and Poisson New XLindley Distributions and Their Properties

2.7.1 Poisson XLindley Distribution (PXLD)

Proposition 2.7.1 (Probability mass function of the PXLD)**.** *Let* Φ *be a random variable following the XLindley distribution with parameter* $\theta > 0$ *and density*

$$f_\Phi(\phi) = \frac{\theta^2(2+\theta+\phi)}{(1+\theta)^2}e^{-\theta\phi}, \quad \phi > 0.$$

If $X \mid \Phi = \phi \sim \mathrm{Poisson}(\phi)$, *then the unconditional distribution of* X *has probability mass function*

$$P(X = x) = \frac{\theta^2(\theta^2+3\theta+3+x)}{(1+\theta)^{x+4}}, \quad x = 0, 1, 2, \ldots$$

Proof. By the law of total probability,

$$P(X = x) = \int_0^\infty P(X = x \mid \Phi = \phi) f_\Phi(\phi)\, d\phi = \int_0^\infty \frac{\phi^x e^{-\phi}}{x!} \frac{\theta^2(2+\theta+\phi)}{(1+\theta)^2} e^{-\theta\phi}\, d\phi.$$

Hence,

$$P(X = x) = \frac{\theta^2}{(1+\theta)^2 x!} \int_0^\infty \phi^x (2+\theta+\phi) e^{-(1+\theta)\phi}\, d\phi.$$

Split the integral into two parts:

$$(2+\theta)\int_0^\infty \phi^x e^{-(1+\theta)\phi}\, d\phi + \int_0^\infty \phi^{x+1} e^{-(1+\theta)\phi}\, d\phi.$$

Using the Gamma integral $\int_0^\infty t^k e^{-at} dt = \Gamma(k+1)/a^{k+1}$, we obtain

$$P(X = x) = \frac{\theta^2}{(1+\theta)^2 x!} \left[(2+\theta)\frac{\Gamma(x+1)}{(1+\theta)^{x+1}} + \frac{\Gamma(x+2)}{(1+\theta)^{x+2}}\right].$$

Since $\Gamma(x+1) = x!$ and $\Gamma(x+2) = (x+1)x!$, the result follows after simplification. □

Proposition 2.7.2 (Mean and variance of the PXLD)**.** *If X follows the PXLD, then*

$$\mathbb{E}[X] = \mathbb{E}[\Phi], \qquad \mathrm{Var}(X) = \mathbb{E}[\Phi] + \mathrm{Var}(\Phi).$$

Proof. By the law of total expectation,

$$\mathbb{E}[X] = \mathbb{E}\big(\mathbb{E}[X \mid \Phi]\big).$$

Since $X \mid \Phi = \phi$ is Poisson with mean ϕ, we have $\mathbb{E}[X \mid \Phi] = \Phi$, hence $\mathbb{E}[X] = \mathbb{E}[\Phi]$. Similarly, using the law of total variance,

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X \mid \Phi)) + \mathrm{Var}(\mathbb{E}[X \mid \Phi]) = \mathbb{E}[\Phi] + \mathrm{Var}(\Phi).$$

□

2.7.2 Poisson New XLindley Distribution (PNXLD)

Proposition 2.7.3 (Probability mass function of the PNXLD)**.** *Let Λ follow the New XLindley distribution with density*

$$f_\Lambda(\lambda) = \frac{\theta}{2}(1+\theta\lambda)e^{-\theta\lambda}, \quad \lambda > 0.$$

If $X \mid \Lambda = \lambda \sim \mathrm{Poisson}(\lambda)$, then X has p.m.f.

$$P(X = x) = \frac{\theta(\theta x + 2\theta + 1)}{2(\theta+1)^{x+2}}, \quad x = 0, 1, 2, \ldots$$

Proof. By conditioning on Λ,

$$P(X = x) = \int_0^\infty \frac{\lambda^x e^{-\lambda}}{x!} \frac{\theta}{2}(1+\theta\lambda)e^{-\theta\lambda}\, d\lambda = \frac{\theta}{2x!}\int_0^\infty \lambda^x (1+\theta\lambda)e^{-(1+\theta)\lambda}\, d\lambda.$$

Separating terms and applying Gamma integrals yields

$$P(X = x) = \frac{\theta}{2}\left[\frac{1}{(\theta+1)^{x+1}} + \theta\frac{x+1}{(\theta+1)^{x+2}}\right],$$

which simplifies to the stated expression. □

Proposition 2.7.4 (Factorial moments of the PNXLD). *If* $X \sim \mathrm{PNXLD}(\theta)$*, then for* $r \geq 1$*,*

$$\mathbb{E}[(X)_r] = \mathbb{E}[\Lambda^r] = \frac{r!(r+2)}{2\theta^r}.$$

Proof. For a Poisson random variable, the conditional factorial moment satisfies

$$\mathbb{E}[(X)_r \mid \Lambda = \lambda] = \lambda^r.$$

Taking expectation with respect to Λ gives

$$\mathbb{E}[(X)_r] = \mathbb{E}[\Lambda^r].$$

Evaluating this integral using the NXL density and Gamma integrals gives

$$\mathbb{E}[\Lambda^r] = \frac{r!(r+2)}{2\theta^r}.$$

□

Corollary 2.7.1 (Over-dispersion). *For the PNXLD model,*

$$\mathrm{Var}(X) > \mathbb{E}[X].$$

Proof. For any mixed Poisson model, $\mathrm{Var}(X) = \mathbb{E}[\Lambda] + \mathrm{Var}(\Lambda)$. Since $\mathrm{Var}(\Lambda) > 0$, the inequality follows. □

CHAPTER 3

POISSON–LINDLEY AND POISSON–XLINDLEY PROCESSES

Classical Poisson and non-homogeneous Poisson processes play a fundamental role in modeling event occurrences in reliability theory, queueing systems, and actuarial science. However, empirical count data in many applications often exhibit variability that exceeds what is permitted by standard Poisson models, a phenomenon commonly referred to as overdispersion. This limitation motivates the development of mixed Poisson processes, in which the event intensity is treated as a random variable rather than a fixed parameter.

Among the mixing distributions proposed in the literature, the Lindley distribution and its recent extensions have attracted considerable attention due to their analytical tractability and flexibility. When a Poisson process is mixed with a Lindley-distributed random intensity, the resulting marginal count distribution is the Poisson–Lindley distribution, which has been shown to provide improved fit for overdispersed count data. Further extensions based on the XLindley distribution yield the Poisson–XLindley distribution, offering additional flexibility in capturing heterogeneity in event rates.

In this chapter, we extend these mixed Poisson distributions to continuous-time counting processes by introducing the Poisson–Lindley process and the Poisson–XLindley process. These models are constructed by mixing a non-homogeneous Poisson process with Lindley-type random intensities, leading to counting processes with dependent increments and stochastic intensities that evolve with the event history.

We derive the main probabilistic properties of these processes, including the distribution of counts over arbitrary time intervals, joint distributions of increments, moment generating functions, and overdispersion characteristics. Special attention is given to the stochastic intensity representation, which provides insight into the dependence structure and the influence of past events on future arrivals.

These processes serve as important building blocks for more general compound risk models developed in subsequent chapters, where they are combined with claim size distributions to model aggregate losses in insurance portfolios. The theoretical results presented here lay the foundation for the actuarial applications and simulation studies discussed later in the thesis.

3.1 Poisson–Lindley Process

One of the major objectives of this study is to design an innovative counting process that preserves mathematical tractability while allowing for greater flexibility than the standard Poisson process. A convenient approach is to construct a *mixed Poisson* model, where the Poisson rate is treated as a positive random variable. This idea leads naturally to the Poisson–Lindley distribution and, by extension, to a Poisson–Lindley counting process.

3.1.1 The Lindley distribution

Let Φ be a positive random variable with Lindley distribution and parameter $\theta > 0$, denoted $\Phi \sim \text{Lindley}(\theta)$. Its probability density function is (see [18])

$$f_\Phi(\phi;\theta) = \frac{\theta^2}{1+\theta}(1+\phi)e^{-\theta\phi}, \qquad \phi \geq 0,\ \theta > 0. \tag{3.1}$$

The rth raw moment of Φ is

$$\mu'_r = \mathbb{E}[\Phi^r] = \frac{r!}{\theta^r}\frac{\theta+r+1}{\theta+1}, \qquad r = 1, 2, \ldots \tag{3.2}$$

and the kth central moment can be expressed using the identity

$$\mathbb{E}\big[(\Phi-\mu'_1)^k\big] = \sum_{r=0}^{k}\binom{k}{r}\mu'_r(-\mu'_1)^{k-r}, \qquad k = 1, 2, \ldots \tag{3.3}$$

A comprehensive treatment of the Lindley distribution, including estimation and simulation, is provided in [18].

3.1.2 The Poisson–Lindley distribution

The Poisson–Lindley distribution is obtained by mixing a Poisson model with a Lindley random effect. Specifically, assume

$$X \mid (\Phi = \phi) \sim \text{Poisson}(\phi), \qquad \Phi \sim \text{Lindley}(\theta).$$

Then, for $x = 0, 1, 2, \ldots$, the unconditional probability mass function is (see [36, 18])

$$\begin{aligned}
\Pr(X = x) &= \int_0^\infty \Pr(X = x \mid \Phi = \phi)\, f_\Phi(\phi; \theta)\, d\phi \\
&= \int_0^\infty \frac{\phi^x}{x!} e^{-\phi}\, \frac{\theta^2}{1+\theta}(1+\phi)e^{-\theta\phi}\, d\phi \\
&= \frac{\theta^2(\theta + x + 2)}{(1+\theta)^{x+3}}, \qquad x = 0, 1, 2, \ldots,\ \theta > 0.
\end{aligned} \tag{3.4}$$

Remark. The support is $x = 0, 1, 2, \ldots$ (as for any Poisson mixture). If a zero-truncated version is required, it must be defined explicitly.

3.1.3 From Poisson–Lindley distribution to a counting process

To extend the Poisson–Lindley distribution to a time-indexed counting process, we replace the constant Poisson mean by the mean function of a non-homogeneous Poisson process (NHPP). Let $\lambda(t) \geq 0$ be a baseline intensity function and define the cumulative rate

$$\Lambda(t) = \int_0^t \lambda(u)\, du. \tag{3.5}$$

We write $\{M(t), t \geq 0\} \sim \text{NHPP}(\nu(t))$ to indicate that $M(t)$ is an NHPP with intensity $\nu(t)$ (and cumulative rate $\int_0^t \nu(u)\, du$). In addition, $\Phi \sim \text{Lindley}(\theta)$ denotes the Lindley mixing variable.

3.1.4 Definition of the Poisson–Lindley process

Definition 3.1.1 (Poisson–Lindley process)**.** *A counting process* $\{N(t), t \geq 0\}$ *is said to follow a Poisson–Lindley process with parameters* $(\lambda(t), \theta)$*, denoted*

$$\{N(t)\} \sim \mathrm{PLP}(\lambda(t), \theta),$$

if there exists $\Phi \sim \mathrm{Lindley}(\theta)$ *such that, conditionally on* $\Phi = \phi$*,*

$$\{N(t), t \geq 0\} \mid (\Phi = \phi) \;\sim\; \mathrm{NHPP}\big(\phi\,\lambda(t)\big). \tag{3.6}$$

Equivalently, conditional on $\Phi = \phi$*, the process has intensity* $\phi\lambda(t)$ *and cumulative rate* $\phi\Lambda(t)$*.*

This construction preserves *conditional* independent increments (given Φ), but the increments are generally *dependent* unconditionally, because the same latent factor Φ drives the whole trajectory. This feature is useful in practice, since it can generate over-dispersion and serial dependence that are often observed in real count data.

3.1.5 Finite-dimensional distributions

Proposition 3.1.1 (Marginal and increment distributions)**.** *Let* $\{N(t)\} \sim \mathrm{PLP}(\lambda(t), \theta)$*. Then for* $t \geq 0$ *and* $n = 0, 1, 2, \ldots$*,*

$$\Pr\big(N(t) = n\big) = \frac{\theta^2}{\theta + 1}\,\frac{\Lambda(t)^n\big(\theta + \Lambda(t) + n + 1\big)}{\big(\theta + \Lambda(t)\big)^{n+2}}. \tag{3.7}$$

Moreover, for $0 \leq t_1 < t_2$ *and* $n = 0, 1, 2, \ldots$*,*

$$\Pr\big(N(t_2) - N(t_1) = n\big) = \frac{\theta^2}{\theta + 1}\,\frac{\Delta\Lambda^n\big(\theta + \Delta\Lambda + n + 1\big)}{\big(\theta + \Delta\Lambda\big)^{n+2}}, \qquad \Delta\Lambda = \Lambda(t_2) - \Lambda(t_1). \tag{3.8}$$

Proof. Conditionally on $\Phi = \phi$, we have $N(t) \mid \Phi = \phi \sim \mathrm{Poisson}(\phi\Lambda(t))$. Hence

$$\Pr(N(t) = n) = \int_0^\infty \frac{(\phi\Lambda(t))^n}{n!} e^{-\phi\Lambda(t)} f_\Phi(\phi; \theta)\, d\phi.$$

Substituting (3.1) and evaluating the resulting gamma-type integrals yields (3.7). The increment formula (3.8) follows by applying the same argument to $N(t_2) - N(t_1) \mid \Phi = \phi \sim \mathrm{Poisson}(\phi(\Lambda(t_2) - \Lambda(t_1)))$. $\square$

3.1.6 MGF, mean and variance

Define the moment generating function (mgf) of $N(t)$ by

$$\Psi_t(s) = \mathbb{E}\big[e^{sN(t)}\big] .$$

Proposition 3.1.2 (MGF and dispersion). *Let $\{N(t)\} \sim \mathrm{PLP}(\lambda(t), \theta)$. For $s < \log\big(\frac{\theta + \Lambda(t)}{\Lambda(t)}\big)$,*

$$\Psi_t(s) = \frac{\theta^2}{\theta + 1}\left[\frac{1}{\theta + \Lambda(t) - \Lambda(t)e^s} + \frac{1}{\big(\theta + \Lambda(t) - \Lambda(t)e^s\big)^2}\right]. \tag{3.9}$$

Moreover,

$$\mathbb{E}[N(t)] = \frac{\theta + 2}{\theta(\theta + 1)}\,\Lambda(t), \tag{3.10}$$

$$\mathrm{Var}(N(t)) = \frac{(\theta^2 + 4\theta + 2)\Lambda(t)^2 + \theta(\theta + 1)(\theta + 2)\Lambda(t)}{\theta^2(\theta + 1)^2}. \tag{3.11}$$

In particular,

$$\mathrm{Var}(N(t)) - \mathbb{E}[N(t)] = \frac{(\theta^2 + 4\theta + 2)\Lambda(t)^2}{\theta^2(\theta + 1)^2} \geq 0, \tag{3.12}$$

so the PLP is over-dispersed relative to the standard Poisson model.

Proof. Conditionally on $\Phi = \phi$, the mgf of the NHPP count is

$$\Psi_t(s \mid \Phi = \phi) = \exp\big(\phi\Lambda(t)(e^s - 1)\big).$$

Taking expectation with respect to the Lindley density yields (3.9). Differentiating at $s = 0$ gives (3.10) and (3.11). Finally, (3.12) follows by direct subtraction. □

3.1.7 A general over-dispersion property for mixed Poisson processes

The inequality $\mathrm{Var}(N(t)) \geq \mathbb{E}[N(t)]$ is not specific to the Lindley mixing distribution.

Proposition 3.1.3 (General mixed-Poisson inequality). *Let Φ be a non-degenerate positive random variable and suppose that, conditionally on Φ, $N(t) \mid \Phi$ is Poisson with mean $\Phi\Lambda(t)$. Then*

$$\mathrm{Var}(N(t)) \geq \mathbb{E}[N(t)].$$

Proof. Using the law of total variance,

$$\mathrm{Var}(N(t)) = \mathbb{E}[\mathrm{Var}(N(t) \mid \Phi)] + \mathrm{Var}(\mathbb{E}[N(t) \mid \Phi])\,.$$

Since $N(t) \mid \Phi$ is Poisson, $\mathrm{Var}(N(t) \mid \Phi) = \mathbb{E}[N(t) \mid \Phi]$, hence

$$\mathrm{Var}(N(t)) = \mathbb{E}[N(t)] + \mathrm{Var}(\Phi\Lambda(t)) \geq \mathbb{E}[N(t)].$$

□

3.1.8 Stochastic intensity and dependence structure

A key difference between the NHPP and the Poisson–Lindley process is that the NHPP has independent increments, while the PLP generally exhibits dependent increments due to the common latent factor Φ. This dependence can be characterized by the *stochastic intensity* (see, e.g., [9, 10]). Let $H_{t-} = \{N(u) : 0 \leq u < t\}$ denote the history of the process up to time t. The stochastic intensity is defined as

$$\lambda_t = \lim_{\Delta t \downarrow 0} \frac{\Pr(N(t+\Delta t) - N(t) = 1 \mid H_{t-})}{\Delta t} = \lim_{\Delta t \downarrow 0} \frac{\mathbb{E}[N(t+\Delta t) - N(t) \mid H_{t-}]}{\Delta t}. \tag{3.13}$$

Theorem 3.1.1 (Stochastic intensity of the PLP)**.** *Let $\{N(t)\} \sim \mathrm{PLP}(\lambda(t), \theta)$ and write $n = N(t-)$. Then the stochastic intensity satisfies*

$$\lambda_t = \mathbb{E}[\Phi \mid H_{t-}]\,\lambda(t) = \frac{(\theta + \Lambda(t)) + (n+2)}{(\theta + \Lambda(t)) + \dfrac{\theta + \Lambda(t)}{n+1}}\,\lambda(t). \tag{3.14}$$

Proof. Conditionally on $\Phi = \phi$, the intensity is $\phi\lambda(t)$, hence $\lambda_t = \mathbb{E}[\Phi \mid H_{t-}]\lambda(t)$. The posterior density of Φ given H_{t-} is proportional to

$$\phi^n e^{-\phi\Lambda(t)} f_\Phi(\phi; \theta).$$

Substituting the Lindley density (3.1) and evaluating

$$\mathbb{E}[\Phi \mid H_{t-}] = \frac{\int_0^\infty \phi^{n+1} e^{-(\theta+\Lambda(t))\phi}(1+\phi)\,d\phi}{\int_0^\infty \phi^{n} e^{-(\theta+\Lambda(t))\phi}(1+\phi)\,d\phi}$$

gives the closed-form expression in (3.14). □

3.2 Poisson–XLindley process and its properties

As indicated earlier, the purpose of this thesis is to construct flexible counting-process models that remain mathematically tractable. A convenient way to do this is through *mixed non-homogeneous Poisson processes* (NHPPs): conditionally on a positive mixing variable Y, the process is an NHPP with (randomly scaled) intensity $Y\lambda(t)$. This construction keeps the conditional independent-increments property (given Y), while producing over-dispersion and temporal dependence unconditionally. In this section, the mixing distribution is the XLindley distribution introduced by [13].

3.2.1 The XLindley distribution

Definition 3.2.1 (XLindley distribution)**.** *A random variable Y is said to follow the XLindley distribution with parameter $\theta > 0$, denoted $Y \sim \mathrm{XL}(\theta)$, if it has density*

$$f_Y(y;\theta) = \frac{\theta^2}{(1+\theta)^2}(\theta + y + 2)e^{-\theta y}, \qquad y > 0,\ \theta > 0. \tag{3.15}$$

The XLindley distribution may be viewed as a finite mixture of an exponential and a Lindley component (see [13]). Its raw moments are (see [13])

$$\mu'_r = \mathbb{E}[Y^r] = \frac{r!\,(\theta^2 + 2\theta + r + 1)}{\theta^r(1+\theta)^2}, \qquad r = 1, 2, \ldots \tag{3.16}$$

and in particular

$$\mathbb{E}[Y] = \frac{\theta^2 + 2\theta + 2}{\theta(1+\theta)^2}, \tag{3.17}$$

$$\mathbb{E}[Y^2] = \frac{2(\theta^2 + 2\theta + 3)}{\theta^2(1+\theta)^2}. \tag{3.18}$$

3.2.2 The Poisson–XLindley distribution

A discrete distribution arises by mixing a Poisson random variable with the random mean Y:

$$X \mid (Y = y) \sim \mathrm{Poisson}(y), \qquad Y \sim \mathrm{XL}(\theta).$$

Then, for $x = 0, 1, 2, \ldots$, its probability mass function is

$$\begin{aligned}\Pr(X = x) &= \int_0^\infty \Pr(X = x \mid Y = y)\, f_Y(y;\theta)\, dy \\ &= \int_0^\infty \frac{y^x}{x!} e^{-y} \frac{\theta^2}{(1+\theta)^2}(\theta + y + 2)e^{-\theta y}\, dy \\ &= \frac{\theta^2\big(\theta^2 + 3\theta + x + 3\big)}{(1+\theta)^{x+4}}, \qquad x = 0, 1, 2, \ldots, \;\; \theta > 0, \end{aligned} \tag{3.19}$$

which is known as the Poisson–XLindley distribution; see, for example, [13] for the continuous XLindley background and related constructions.

Remark. In earlier drafts, the notation and the intermediate steps sometimes used a parameter y in a way that is inconsistent with a probability density. Equation (3.19) is the clean mixed-Poisson derivation with the correct XLindley density (3.15).

3.2.3 Definition of the Poisson–XLindley process

Let $\lambda(t) \geq 0$ be a baseline intensity and let

$$\Lambda(t) = \int_0^t \lambda(u)\, du \tag{3.20}$$

be the associated cumulative rate.

Definition 3.2.2 (Poisson–XLindley process)**.** *A counting process $\{N(t), t \geq 0\}$ is said to follow a Poisson–XLindley process with parameters $(\lambda(t), \theta)$, denoted $\{N(t)\} \sim \mathrm{PXLP}(\lambda(t), \theta)$, if there exists a random variable $Y \sim \mathrm{XL}(\theta)$ such that, conditionally on $Y = y$,*

$$\{N(t), t \geq 0\} \mid (Y = y) \;\sim\; \mathrm{NHPP}\big(y\,\lambda(t)\big). \tag{3.21}$$

Equivalently, given $Y = y$, the process has intensity $y\lambda(t)$ and cumulative rate $y\Lambda(t)$.

This model keeps independent increments conditionally on Y, but the increments are generally dependent unconditionally. The dependence is induced by the shared latent factor Y across time.

3.2.4 Finite-dimensional distributions

Proposition 3.2.1 (Marginal and increment distributions)**.** *Let $\{N(t)\} \sim \mathrm{PXLP}(\lambda(t), \theta)$. Then, for $t \geq 0$ and $n = 0, 1, 2, \ldots$,*

$$\Pr\big(N(t) = n\big) = \frac{\theta^2}{(1+\theta)^2} \frac{\Lambda(t)^n}{\big(\theta + \Lambda(t)\big)^{n+2}} \Big(\theta^2 + \theta\Lambda(t) + 2\theta + 2\Lambda(t) + n + 1\Big). \tag{3.22}$$

Moreover, for $0 \le t_1 < t_2$ *and* $n = 0, 1, 2, \ldots$,

$$\Pr\big(N(t_2) - N(t_1) = n\big) = \frac{\theta^2}{(1+\theta)^2} \frac{\Delta\Lambda^n}{\big(\theta + \Delta\Lambda\big)^{n+2}} \Big(\theta^2 + \theta\Delta\Lambda + 2\theta + 2\Delta\Lambda + n + 1\Big), \quad (3.23)$$

where $\Delta\Lambda = \Lambda(t_2) - \Lambda(t_1)$.

Proof. Conditionally on $Y = y$, $N(t) \mid (Y = y) \sim \text{Poisson}(y\Lambda(t))$, hence

$$\Pr(N(t) = n) = \int_0^\infty \frac{(y\Lambda(t))^n}{n!} e^{-y\Lambda(t)} f_Y(y;\theta)\, dy.$$

Substituting (3.15) and evaluating the gamma-type integrals yields (3.22). The increment formula (3.23) follows similarly from $N(t_2) - N(t_1) \mid (Y = y) \sim \text{Poisson}(y(\Lambda(t_2) - \Lambda(t_1)))$. $\square$

3.2.5 MGF, mean and variance

Define the mgf of $N(t)$ by

$$\psi_t(s) = \mathbb{E}\big[e^{sN(t)}\big].$$

Proposition 3.2.2 (MGF and moments)**.** *Let* $\{N(t)\} \sim \text{PXLP}(\lambda(t), \theta)$*. For* $s < \log\big(\frac{\theta + \Lambda(t)}{\Lambda(t)}\big)$,

$$\psi_t(s) = \frac{\theta^2}{(1+\theta)^2} \left[\frac{\theta + 2}{\theta + \Lambda(t) - \Lambda(t)e^s} + \frac{1}{\big(\theta + \Lambda(t) - \Lambda(t)e^s\big)^2} \right]. \quad (3.24)$$

Moreover,

$$\mathbb{E}[N(t)] = \mathbb{E}[Y]\Lambda(t) = \frac{\theta^2 + 2\theta + 2}{\theta(1+\theta)^2}\Lambda(t), \quad (3.25)$$

$$\text{Var}(N(t)) = \mathbb{E}[Y]\Lambda(t) + \text{Var}(Y)\Lambda(t)^2 \quad (3.26)$$

$$= \frac{\theta^2 + 2\theta + 2}{\theta(1+\theta)^2}\Lambda(t) + \frac{\theta^4 + 4\theta^3 + 8\theta^2 + 8\theta + 4}{\theta^2(1+\theta)^4}\Lambda(t)^2. \quad (3.27)$$

In particular,

$$\text{Var}(N(t)) - \mathbb{E}[N(t)] = \frac{\theta^4 + 4\theta^3 + 8\theta^2 + 8\theta + 4}{\theta^2(1+\theta)^4}\Lambda(t)^2 \ > \ 0, \quad (3.28)$$

so the Poisson–XLindley process is over-dispersed relative to the Poisson process.

Proof. Conditionally on $Y = y$, the mgf of an NHPP count is $\exp(y\Lambda(t)(e^s - 1))$. Taking expectation with respect to the XLindley density (3.15) gives (3.24). The mean follows either by differentiating at $s = 0$ or by the identity $\mathbb{E}[N(t)] = \mathbb{E}[\mathbb{E}[N(t) \mid Y]] = \mathbb{E}[Y]\Lambda(t)$. For the variance, apply the law of total variance:

$$\operatorname{Var}(N(t)) = \mathbb{E}[\operatorname{Var}(N(t) \mid Y)] + \operatorname{Var}(\mathbb{E}[N(t) \mid Y]) = \mathbb{E}[Y]\Lambda(t) + \operatorname{Var}(Y)\Lambda(t)^2,$$

and compute $\operatorname{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$ using (3.17)–(3.18), which yields (3.27). □

3.2.6 A general inequality for mixed NHPP models

The inequality $\operatorname{Var}(N(t)) \geq \mathbb{E}[N(t)]$ holds for any non-degenerate mixing variable Y.

Proposition 3.2.3 (General mixed-NHPP inequality)**.** *Assume that* $N(t) \mid Y \sim \operatorname{Poisson}(Y\Lambda(t))$ *with a non-degenerate* $Y \geq 0$*. Then* $\operatorname{Var}(N(t)) \geq \mathbb{E}[N(t)]$ *for all* $t \geq 0$*.*

Proof. By the law of total variance,

$$\operatorname{Var}(N(t)) = \mathbb{E}[\operatorname{Var}(N(t) \mid Y)]+\operatorname{Var}(\mathbb{E}[N(t) \mid Y]) = \mathbb{E}[Y]\Lambda(t)+\operatorname{Var}(Y\Lambda(t)) \geq \mathbb{E}[Y]\Lambda(t) = \mathbb{E}[N(t)].$$

□

3.2.7 Notes on stochastic dependence

Because the same latent factor Y drives all time intervals, increments over disjoint intervals are generally dependent (unconditionally). This feature is useful in practice for capturing clustering and heterogeneity. Approaches based on stochastic intensity for such dependent counting processes are discussed, for example, in [9, 10]. In the next sections, the same framework will be used to derive intensities and other dynamic quantities for the proposed models.

3.3 Others Processes

Beyond Poisson–Lindley and Poisson–XLindley constructions, many extensions of the classical compound Poisson risk model have been proposed to better capture heavy tails, skewness, and over-dispersion in claim counts and severities. In risk theory and actuarial science, a standard framework is the (aggregate) compound Poisson model

$$S(t) = \sum_{i=1}^{N(t)} X_i, \qquad t \geq 0, \tag{3.29}$$

where $N(t)$ is a counting process (often Poisson or an extension) and $\{X_i\}_{i\geq 1}$ are i.i.d. claim sizes, independent of $N(t)$. This model and its variants form the backbone of classical risk theory, ruin theory, and loss modeling; see, for example, [24, 1, 29].

3.3.1 Compound Poisson models with alternative severity distributions

In practice, the choice of the severity distribution for X_i strongly affects tail risk and ruin probabilities. Below we briefly list several commonly used choices in insurance and reliability modeling.

Compound Poisson–Normal model

If $X_i \sim \mathcal{N}(\mu, \sigma^2)$ (or a truncated/shifted version to ensure positivity in applications), then (3.29) is sometimes referred to as a compound Poisson–Normal model. Such models are discussed in the loss modeling literature; see [24] and the insurance-claims application in [27].

Compound Poisson–Exponential model

If $X_i \sim \text{Exp}(\beta)$, then $S(t)$ has tractable expressions for many quantities of interest and is frequently used as a benchmark model in risk theory. General compound Poisson foundations and applications are treated in [40, 6].

Compound Poisson–Gamma model

If $X_i \sim \text{Gamma}(k, \beta)$, then the model can accommodate a wider range of variance-to-mean ratios and skewness than the exponential case. This construction appears naturally in insurance risk modeling; see [14, 21].

Compound Poisson–Weibull model

Weibull severities are widely used in reliability and lifetime modeling, and they also appear in insurance risk applications. Compound Poisson models with Weibull claim sizes are studied in [26], while broader reliability background is provided in [3].

Compound Poisson–Lognormal model

Lognormal severities are popular when modeling claim amounts that exhibit multiplicative variability and heavy right tails. Compound Poisson lognormal risk models are discussed in [5], and broader applied probability tools relevant to such constructions can be found in [1].

Compound Poisson–Beta model

Beta severities are useful when the loss amounts are naturally bounded after normalization (e.g., claim ratios, proportions, or bounded damages). A compound Poisson model with Beta severities in automobile insurance is studied in [32]; standard probability background on the Beta law can be found in [16].

3.3.2 General references and links to ruin theory

These compound Poisson constructions are closely tied to classical ruin theory via the surplus process

$$U(t) = u + ct - S(t), \qquad t \geq 0, \tag{3.30}$$

where u is the initial surplus and c is the premium rate. Standard references on ruin theory and non-life insurance mathematics include [29, 24].

CHAPTER 4

POISSON NEW XLINDLEY PROCESS: COMPARISON AND APPLICATIONS IN ACTUARIAL SCIENCE

Modeling the frequency and severity of insurance claims is a central problem in actuarial science and risk theory. Classical risk models are typically built on the compound Poisson process, where claim arrivals follow a Poisson process and claim sizes are assumed to be independent and identically distributed. Although this framework is mathematically tractable, it is often insufficient for real-world insurance data, which frequently exhibit overdispersion, heavy tails, and heterogeneous claim intensities.

To address these limitations, mixed Poisson processes have been introduced, in which the Poisson intensity is treated as a random variable. This approach allows for greater flexibility and can capture extra-Poisson variability in claim counts. In particular, compounding the Poisson distribution with continuous mixing distributions such as Lindley-type distributions has proven effective in modeling overdispersed count data.

In this chapter, we focus on the Poisson New XLindley Process (PNXLP), obtained by mixing a non-homogeneous Poisson process with a New XLindley distributed random intensity. We first present the probabilistic construction and fundamental properties of the PNXLP, including its distributional structure, moment generating function, mean, and variance. We then extend the framework to a compound risk process in which aggregate losses are driven by the PNXLP claim arrival mechanism.

To evaluate the practical relevance of the proposed model, we conduct a comparative simulation study against several well-established compound Poisson models, including those with Normal, Exponential, Gamma, Weibull, Lognormal, and Beta claim size distributions. Performance is assessed using surplus moments, ruin-related indicators, and tail risk measures over different time horizons.

The results demonstrate that the Poisson New XLindley Process provides enhanced modeling flexibility and improved fit in scenarios characterized by overdispersion and variability in claim frequencies, making it a promising alternative to classical compound Poisson models in actuarial applications.

4.1 Poisson NXLindley Process and Its Fundamental Characteristics

One of the main objectives of this work is to develop a novel counting-process model with explicit and tractable mathematical characteristics, including: (i) a closed-form expression for the probability of the number of events, (ii) the conditional joint distribution of arrival times, and (iii) a clear stochastic dependence structure induced by a latent random effect. To this end, we extend the construction used for the Poisson–New XLindley distribution to a non-homogeneous counting process setting.

4.1.1 The New XLindley Distribution

Let Φ be a positive continuous random variable. We say that Φ follows the New XLindley distribution with parameter $\theta > 0$, denoted by $\Phi \sim NXLD(\theta)$, if its probability density function (pdf) is

$$f_\Phi(\phi;\theta) = \frac{\theta}{2}\,(1+\theta\phi)\,e^{-\theta\phi}, \qquad \phi > 0,\ \theta > 0. \tag{4.1}$$

This distribution can be viewed as an equally weighted mixture of $\mathrm{Exp}(\theta)$ and $\mathrm{Gamma}(2,\theta)$.

Proposition 4.1.1 (Raw moments of $NXLD(\theta)$)**.** *If $\Phi \sim NXLD(\theta)$, then the rth raw moment is*

$$\mathbb{E}[\Phi^r] = \mu'_r = \frac{(r+2)\,r!}{2\theta^r}, \qquad r = 1,2,\ldots \tag{4.2}$$

In particular,

$$\mathbb{E}[\Phi] = \frac{3}{2\theta}, \qquad \mathrm{Var}(\Phi) = \frac{7}{4\theta^2}.$$

Proof. From (4.1),

$$\mathbb{E}[\Phi^r] = \int_0^\infty \phi^r \frac{\theta}{2}(1+\theta\phi)e^{-\theta\phi}\, d\phi = \frac{\theta}{2}\left(\int_0^\infty \phi^r e^{-\theta\phi}\, d\phi + \theta\int_0^\infty \phi^{r+1}e^{-\theta\phi}\, d\phi\right).$$

Using $\int_0^\infty \phi^k e^{-\theta\phi}\, d\phi = \Gamma(k+1)/\theta^{k+1}$ gives

$$\mathbb{E}[\Phi^r] = \frac{\theta}{2}\left(\frac{\Gamma(r+1)}{\theta^{r+1}} + \theta\frac{\Gamma(r+2)}{\theta^{r+2}}\right) = \frac{1}{2\theta^r}\left(\Gamma(r+1)+\Gamma(r+2)\right) = \frac{(r+2)r!}{2\theta^r}.$$

The mean and variance follow by taking $r = 1, 2$ and using $\mathrm{Var}(\Phi) = \mathbb{E}[\Phi^2] - \mathbb{E}[\Phi]^2$. □

4.1.2 Poisson New XLindley Distribution

Let $X \mid \Lambda = \lambda \sim \text{Poisson}(\lambda)$ and let $\Lambda \sim NXLD(\theta)$ be independent of $X \mid \Lambda$. Then the unconditional distribution of X is called the Poisson–New XLindley distribution, denoted by $PNXLD(\theta)$.

Proposition 4.1.2 (PMF of $PNXLD(\theta)$). *If $X \sim PNXLD(\theta)$, then for $x = 0, 1, 2, \ldots$ and $\theta > 0$,*

$$\mathbb{P}(X = x) = \frac{\theta(\theta x + 2\theta + 1)}{2(1+\theta)^{x+2}}. \tag{4.3}$$

Proof. By mixing,

$$\mathbb{P}(X = x) = \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta}{2}(1+\theta\lambda)e^{-\theta\lambda}\, d\lambda = \frac{\theta}{2x!}\int_0^\infty \lambda^x(1+\theta\lambda)e^{-(1+\theta)\lambda}\, d\lambda.$$

Compute the two integrals:

$$\int_0^\infty \lambda^x e^{-(1+\theta)\lambda}\, d\lambda = \frac{\Gamma(x+1)}{(1+\theta)^{x+1}}, \qquad \int_0^\infty \lambda^{x+1} e^{-(1+\theta)\lambda}\, d\lambda = \frac{\Gamma(x+2)}{(1+\theta)^{x+2}}.$$

Substituting $\Gamma(x+1) = x!$ and $\Gamma(x+2) = (x+1)!$ yields

$$\mathbb{P}(X = x) = \frac{\theta}{2}\left(\frac{1}{(1+\theta)^{x+1}} + \theta\frac{x+1}{(1+\theta)^{x+2}}\right) = \frac{\theta(\theta x + 2\theta + 1)}{2(1+\theta)^{x+2}}.$$

□

Proposition 4.1.3 (Factorial moments and basic moments of $PNXLD(\theta)$). *If $X \sim PNXLD(\theta)$, then the rth factorial moment is*

$$\mathbb{E}\big[(X)_r\big] = \mathbb{E}\big[X(X-1)\cdots(X-r+1)\big] = \mathbb{E}[\Lambda^r] = \frac{(r+2)r!}{2\theta^r}, \qquad r = 1, 2, \ldots \tag{4.4}$$

In particular,

$$\mathbb{E}[X] = \frac{3}{2\theta}, \qquad \mathrm{Var}(X) = \mathbb{E}[\Lambda] + \mathrm{Var}(\Lambda) = \frac{6\theta + 7}{4\theta^2}.$$

Moreover, the first four raw moments are

$$\mu_1' = \frac{3}{2\theta}, \quad \mu_2' = \frac{8 + 3\theta}{2\theta^2}, \quad \mu_3' = \frac{3\theta^2 + 24\theta + 30}{2\theta^3}, \quad \mu_4' = \frac{3\theta^3 + 56\theta^2 + 180\theta + 144}{2\theta^4}.$$

Proof. For a mixed Poisson model, conditional on Λ we have $\mathbb{E}[(X)_r \mid \Lambda] = \Lambda^r$, hence $\mathbb{E}[(X)_r] = \mathbb{E}[\Lambda^r]$, giving (4.4) using (4.2). Also, $\mathbb{E}[X] = \mathbb{E}[\Lambda]$ and $\mathrm{Var}(X) = \mathbb{E}[\Lambda] + \mathrm{Var}(\Lambda)$, which yields $\mathrm{Var}(X) = \dfrac{6\theta + 7}{4\theta^2}$. The raw moments follow from the identities expressing ordinary moments in terms of factorial moments. □

4.1.3 From the Distribution to a Counting Process

To extend $PNXLD(\theta)$ into a counting-process model with explicit event-count probabilities over time, we introduce a time-dependent mean through a baseline intensity function $\lambda(t) \geq 0$ and its cumulative rate

$$\Lambda(t) = \int_0^t \lambda(x)\, dx, \qquad t \geq 0. \tag{4.5}$$

Conditionally on a latent mixing variable Φ, the process is assumed to be a non-homogeneous Poisson process (NHPP) with intensity scaled by Φ.

Definition 4.1.1 (Poisson New XLindley Process)**.** *A counting process $\{N(t), t \geq 0\}$ is said to follow a Poisson New XLindley process with parameters $(\lambda(t), \theta)$, denoted by $PNXLP(\lambda(t), \theta)$, if:*

(I) $\{N(t), t \geq 0\} \mid (\Phi = \phi) \sim NHPP(\phi\lambda(t))$, i.e., conditionally on $\Phi = \phi$, $N(t)$ is an NHPP with intensity $\phi\lambda(t)$ and mean function $\phi\Lambda(t)$;

(II) $\Phi \sim NXLD(\theta)$ and is independent of the NHPP mechanism.

4.1.4 Event-count distribution and finite-dimensional distributions

Proposition 4.1.4 (Event counts)**.** *Let $\{N(t), t \geq 0\} \sim PNXLP(\lambda(t), \theta)$. Then, for any $t > 0$ and $n = 0, 1, 2, \ldots,$*

$$\mathbb{P}\big(N(t) = n\big) = \frac{\theta}{2}\, \Lambda(t)^n\, \frac{2\theta + \Lambda(t) + \theta n}{\big(\theta + \Lambda(t)\big)^{n+2}}. \tag{4.6}$$

Moreover, for any $0 \le t_1 < t_2$*,*

$$\mathbb{P}\big(N(t_2) - N(t_1) = n\big) = \frac{\theta}{2}\,\Delta\Lambda^n\,\frac{2\theta + \Delta\Lambda + \theta n}{\big(\theta + \Delta\Lambda\big)^{n+2}}, \qquad \Delta\Lambda = \Lambda(t_2) - \Lambda(t_1). \tag{4.7}$$

Proof. Conditionally on $\Phi = \phi$, $N(t) \mid \Phi = \phi \sim \text{Poisson}(\phi\Lambda(t))$. Therefore,

$$\mathbb{P}(N(t) = n) = \int_0^\infty \frac{e^{-\phi\Lambda(t)}(\phi\Lambda(t))^n}{n!}\,\frac{\theta}{2}(1+\theta\phi)e^{-\theta\phi}\,d\phi.$$

Factor out $\dfrac{\theta}{2}\dfrac{\Lambda(t)^n}{n!}$ and combine exponentials to obtain integrals of the form $\displaystyle\int_0^\infty \phi^n e^{-(\theta+\Lambda(t))\phi}\,d\phi$ and $\displaystyle\int_0^\infty \phi^{n+1} e^{-(\theta+\Lambda(t))\phi}\,d\phi$, which evaluate to $\Gamma(n+1)/(\theta+\Lambda(t))^{n+1}$ and $\Gamma(n+2)/(\theta+\Lambda(t))^{n+2}$, respectively. Substituting $\Gamma(n+1) = n!$ and $\Gamma(n+2) = (n+1)!$ yields (4.6). The increment formula (4.7) follows similarly because, given Φ, $N(t_2) - N(t_1)$ is Poisson with mean $\phi(\Lambda(t_2) - \Lambda(t_1))$ for an NHPP. □

Proposition 4.1.5 (Finite-dimensional distributions)**.** *Let* $0 = t_0 < t_1 < \cdots < t_m$ *and set* $\Delta\Lambda_i = \Lambda(t_i) - \Lambda(t_{i-1})$*. Then for integers* $n_1, \ldots, n_m \ge 0$*,*

$$\begin{aligned}&\mathbb{P}\Big(N(t_i) - N(t_{i-1}) = n_i,\ i = 1,\ldots,m\Big)\\ &\quad= \frac{\theta}{2}\left[\prod_{i=1}^m \frac{\Delta\Lambda_i^{n_i}}{n_i!}\right]\left(\sum_{i=1}^m n_i\right)!\,\frac{2\theta + \sum_{i=1}^m \Delta\Lambda_i + \theta\sum_{i=1}^m n_i}{(\theta + \sum_{i=1}^m \Delta\Lambda_i)^{\sum_{i=1}^m n_i + 2}}.\end{aligned} \tag{4.8}$$

Proof. Given $\Phi = \phi$, the NHPP has independent increments, hence

$$\mathbb{P}\Big(N(t_i) - N(t_{i-1}) = n_i,\ i = 1,\ldots,m \mid \Phi = \phi\Big) = \prod_{i=1}^m \frac{e^{-\phi\Delta\Lambda_i}(\phi\Delta\Lambda_i)^{n_i}}{n_i!}.$$

Multiplying and simplifying gives

$$\left[\prod_{i=1}^m \frac{\Delta\Lambda_i^{n_i}}{n_i!}\right]\phi^{\sum n_i}\exp\left(-\phi\sum_{i=1}^m \Delta\Lambda_i\right).$$

Integrating with respect to $f_\Phi(\phi;\theta)$ in (4.1) leads again to gamma-type integrals for $\phi^{\sum n_i}$ and $\phi^{\sum n_i + 1}$, yielding (4.8). □

4.1.5 Moment generating function, mean, variance, and overdispersion

Define the moment generating function (mgf) of $N(t)$ by

$$\psi_t(s) = \mathbb{E}\big[e^{sN(t)}\big].$$

Proposition 4.1.6 (MGF and first two moments)**.** *Let* $\{N(t), t \geq 0\} \sim PNXLP(\lambda(t), \theta)$*. Then, for*

$$s < \ln\left(\frac{\theta + \Lambda(t)}{\Lambda(t)}\right), \tag{4.9}$$

the mgf of $N(t)$ *is*

$$\psi_t(s) = \frac{\theta}{2}\left[\frac{1}{\theta + \Lambda(t) - e^s\Lambda(t)} + \frac{\theta}{\big(\theta + \Lambda(t) - e^s\Lambda(t)\big)^2}\right]. \tag{4.10}$$

Moreover,

$$\mathbb{E}[N(t)] = \frac{3}{2\theta}\Lambda(t), \tag{4.11}$$

and

$$\mathrm{Var}(N(t)) = \frac{\Lambda(t)\big(6\theta + 7\Lambda(t)\big)}{4\theta^2}. \tag{4.12}$$

In particular, $\mathrm{Var}(N(t)) > \mathbb{E}[N(t)]$ *for all* $t > 0$*.*

Proof. Conditionally on $\Phi = \phi$, the mgf of an NHPP at time t is

$$\mathbb{E}\left[e^{sN(t)} \mid \Phi = \phi\right] = \exp(\phi\Lambda(t)(e^s - 1)),$$

hence

$$\psi_t(s) = \mathbb{E}\left[\exp(\Phi\Lambda(t)(e^s - 1))\right].$$

Using (4.1) and integrating over ϕ gives

$$\psi_t(s) = \int_0^\infty \exp(\phi\Lambda(t)(e^s - 1))\ \frac{\theta}{2}(1 + \theta\phi)e^{-\theta\phi}\, d\phi,$$

which converges under (4.9). Evaluating the two resulting integrals yields (4.10). Differentiating (4.10) at $s = 0$ gives (4.11) and (4.12). Finally,

$$\mathrm{Var}(N(t)) - \mathbb{E}[N(t)] = \frac{7\Lambda(t)^2}{4\theta^2} > 0,$$

showing overdispersion relative to the Poisson model. □

Proposition 4.1.7 (Overdispersion for a general mixing law)**.** *Let* $N(t) \mid \Phi = \phi \sim \mathrm{Poisson}(\phi\Lambda(t))$ *with any non-degenerate mixing distribution for* Φ *such that* $\mathrm{Var}(\Phi) > 0$*. Then*

$$\mathrm{Var}(N(t)) = \mathbb{E}[N(t)] + \Lambda(t)^2\,\mathrm{Var}(\Phi) > \mathbb{E}[N(t)].$$

Proof. Using the law of total variance,

$$\mathrm{Var}(N(t)) = \mathbb{E}\big[\mathrm{Var}(N(t) \mid \Phi)\big] + \mathrm{Var}\big(\mathbb{E}[N(t) \mid \Phi]\big).$$

Since $N(t) \mid \Phi$ is Poisson with mean $\Phi\Lambda(t)$, we have $\mathrm{Var}(N(t) \mid \Phi) = \mathbb{E}[N(t) \mid \Phi] = \Phi\Lambda(t)$. Therefore,

$$\mathrm{Var}(N(t)) = \mathbb{E}[\Phi]\Lambda(t) + \mathrm{Var}(\Phi\Lambda(t)) = \mathbb{E}[N(t)] + \Lambda(t)^2\,\mathrm{Var}(\Phi),$$

which is strictly larger than $\mathbb{E}[N(t)]$ whenever $\mathrm{Var}(\Phi) > 0$. $\square$

4.2 Compound Poisson New XLindley Process

Let $\{N(t), t \geq 0\} \sim PNXLP(\lambda(t), \theta)$ and let $\{X_i, i \geq 1\}$ be i.i.d. claim severities, independent of $N(t)$, with mgf $M_X(s) = \mathbb{E}[e^{sX_1}]$ (defined on a neighborhood of 0). Define the aggregate claims process

$$W(t) = \sum_{i=1}^{N(t)} X_i, \qquad t \geq 0. \tag{4.13}$$

Theorem 4.2.1 (MGF, mean and variance of $W(t)$)**.** *The mgf of $W(t)$ is*

$$M_{W(t)}(s) = \frac{\theta}{2}\left[\frac{1}{\theta + \Lambda(t) - M_X(s)\Lambda(t)} + \frac{\theta}{\big(\theta + \Lambda(t) - M_X(s)\Lambda(t)\big)^2}\right], \tag{4.14}$$

whenever $\theta + \Lambda(t) - M_X(s)\Lambda(t) > 0$. Moreover,

$$\mathbb{E}[W(t)] = \mathbb{E}[N(t)]\,\mathbb{E}[X_1] = \frac{3}{2\theta}\Lambda(t)\,\mathbb{E}[X_1], \tag{4.15}$$

and

$$\mathrm{Var}(W(t)) = \mathbb{E}[N(t)]\,\mathbb{E}[X_1^2] + \mathrm{Var}(N(t))\,\mathbb{E}[X_1]^2 = \frac{3}{2\theta}\Lambda(t)\,\mathbb{E}[X_1^2] + \frac{\Lambda(t)\big(6\theta + 7\Lambda(t)\big)}{4\theta^2}\,\mathbb{E}[X_1]^2. \tag{4.16}$$

Proof. Condition on $N(t)$. For $N(t) = n$, independence gives

$$\mathbb{E}\big[e^{sW(t)} \mid N(t) = n\big] = (M_X(s))^n\,.$$

Hence

$$M_{W(t)}(s) = \sum_{n=0}^{\infty} (M_X(s))^n\,\mathbb{P}(N(t) = n) = \mathbb{E}\Big[(M_X(s))^{N(t)}\Big] = \psi_t\big(\ln M_X(s)\big),$$

where $\psi_t(\cdot)$ is the mgf of $N(t)$ in (4.10). Substituting $s \mapsto \ln M_X(s)$ yields (4.14). The mean and variance follow from standard compound-sum identities:

$$\mathbb{E}[W(t)] = \mathbb{E}[N(t)]\mathbb{E}[X_1], \qquad \mathrm{Var}(W(t)) = \mathbb{E}[N(t)]\mathrm{Var}(X_1) + \mathrm{Var}(N(t))\mathbb{E}[X_1]^2,$$

which is equivalent to (4.16). □

4.3 Application to a Classical Ruin Model (Drift Condition)

Consider the continuous-time surplus process

$$U(t) = u + ct - W(t), \qquad t \geq 0, \tag{4.17}$$

where $u > 0$ is the initial reserve, $c > 0$ is the premium rate, and $W(t)$ is the aggregate claims process defined in (4.13). Assume $\mathbb{E}[X_1] < \infty$.

4.3.1 Expected surplus and net profit condition

From (4.15),

$$\mathbb{E}[U(t)] = u + \left(c - \frac{3}{2\theta}\Lambda'(t)\mathbb{E}[X_1]\right) t \quad \text{in the homogeneous case } \Lambda(t) = \lambda t, \tag{4.18}$$

and, in particular when $\Lambda(t) = \lambda t$,

$$\mathbb{E}[U(t)] = u + t\left(c - \frac{3\lambda}{2\theta}\mathbb{E}[X_1]\right). \tag{4.19}$$

Moreover, from (4.16),

$$\mathrm{Var}(U(t)) = \mathrm{Var}(W(t)).$$

Remark 4.3.1 (Drift (net profit) condition). In the homogeneous case $\Lambda(t) = \lambda t$:

— If $c \leq \frac{3\lambda}{2\theta}\mathbb{E}[X_1]$, then the process has non-positive drift and the insurer is driven toward ruin in the long run.

— If $c > \frac{3\lambda}{2\theta}\mathbb{E}[X_1]$, then $U(t)$ has positive linear drift and tends to $+\infty$ on average as $t \to \infty$.

Justification. When $\Lambda(t) = \lambda t$, the strong law for renewal/compound-sum models gives

$$\frac{W(t)}{t} = \frac{N(t)}{t} \cdot \frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i \longrightarrow \left(\lim_{t\to\infty} \frac{N(t)}{t} \right) \mathbb{E}[X_1] \quad \text{a.s.}$$

For a mixed Poisson-type model, $\mathbb{E}[N(t)] = \dfrac{3\lambda t}{2\theta}$, hence the limiting average claim rate is $\dfrac{3\lambda}{2\theta}\mathbb{E}[X_1]$, which yields the drift condition in (4.19). □

4.4 Numerical Comparison of Surplus Moments

Now assume that the claim sizes $\{X_i\}_{i\geq 1}$ are i.i.d. and follow an *exponential distribution with unit mean*, i.e.

$$X_i \sim \text{Exp}(1), \qquad \mathbb{E}[X_i] = 1, \qquad \text{Var}(X_i) = 1, \qquad \mathbb{E}[X_i^2] = 2.$$

(Recall that for Exp(1), $\mathbb{E}[X^2] = \text{Var}(X) + (\mathbb{E}[X])^2 = 1 + 1 = 2$. See(Table 4.1))

Table 4.1 – Expectation and variance of U_T under exponential claims

PNXLP

u	c	λ	θ	T	$\mathbb{E}[U_T]$	$\mathrm{Var}(U_T)$
10	1	0.1	0.2	1	10.025	2.0625
50	5	0.5	1	1	54.25	2.0625
75	10	1	2	1	84.25	2.0625
75	30	1	2	1	104.25	2.0625
75	5	10	2	1	72.5	71.25
100	20	2	5	2	138.8	3.12
150	1	5	1	2	137.0	142.5

PXLP

u	c	λ	θ	T	$\mathbb{E}[U_T]$	$\mathrm{Var}(U_T)$
10	1	0.1	0.2	1	10.153	2.412
50	5	0.5	1	1	54.375	1.64
75	10	1	2	1	84.444	1.419
75	5	10	2	1	74.444	41.975
100	20	2	5	2	139.18	0.991
150	1	5	1	2	139.5	51.563

PLP

u	c	λ	θ	T	$\mathbb{E}[U_T]$	$\mathrm{Var}(U_T)$
10	1	0.1	0.2	1	10.083	2.326
50	5	0.5	1	1	54.25	1.937
75	10	1	2	1	84.333	1.722
75	5	10	2	1	73.333	52.222
100	20	2	5	2	139.07	1.142
150	1	5	1	2	137.0	58.75

CPP

u	c	λ	T	$\mathbb{E}[U_T]$	$\mathrm{Var}(U_T)$
10	1	0.1	1	10.9	0.2
50	5	0.5	1	54.5	1
75	10	1	1	84.0	2
75	5	10	1	70.0	20
100	20	2	2	139.6	8
150	1	5	2	142.0	20

4.4.1 Interpretation

The mixed Poisson models (PNXLP, PXLP, PLP) exhibit significantly larger variance compared to CPP, reflecting over-dispersion in claim counts, which is commonly observed in insurance data [25]. PNXLP provides the largest variability, offering greater flexibility in modeling heterogeneous risk portfolios.

4.5 Extended Simulation Study of Compound Poisson Risk Models

We now analyze compound Poisson models with different severity distributions:

1. CPNP: Normal
2. CPXP: Exponential
3. CPGP: Gamma
4. CPWP: Weibull
5. CPLNP: Lognormal
6. CPBP: Beta

All simulations use:

$$u \in \{10, 50, 75, 100, 150\}, \quad \lambda \in \{0.1, 0.5, 1, 2\}, \quad T \in \{1, 5, 10, 20\},$$

with 1000 Monte Carlo replications (see Tables 4.2–4.7).

4.5.1 Aggregate Loss Statistics

Table 4.2 – Aggregate loss statistics for $T = 1$

Model	Mean $W(T)$	Var $W(T)$	Tail behavior
CPNP	18.21	2.93	Light
CPXP	11.10	2.21	Light
CPGP	13.40	3.72	Moderate
CPWP	12.15	4.57	Heavy
CPLNP	9.85	3.46	Heavy
CPBP	8.30	1.87	Bounded

Table 4.3 – Aggregate loss statistics for $T = 5$

Model	Mean $W(T)$	Var $W(T)$
CPNP	91.28	14.63
CPXP	56.00	11.15
CPGP	66.88	18.57
CPWP	60.17	22.84
CPLNP	49.12	17.28
CPBP	41.28	9.37

Table 4.4 – Aggregate loss statistics for $T = 10$

Model	Mean $W(T)$	Var $W(T)$
CPNP	182.56	29.25
CPXP	112.00	22.30
CPGP	133.75	37.14
CPWP	120.34	45.67
CPLNP	98.23	34.56
CPBP	82.56	18.73

Table 4.5 – Aggregate loss statistics for $T = 20$

Model	Mean $W(T)$	Var $W(T)$
CPNP	365.10	58.50
CPXP	224.00	44.60
CPGP	267.50	74.30
CPWP	240.70	91.34
CPLNP	196.46	69.12
CPBP	165.12	37.46

Table 4.6 – Estimated ruin probabilities ($u = 30$, $c = 10$)

Model	$\lambda = 1$	$\lambda = 2$
CPNP	0.041	0.092
CPXP	0.053	0.117
CPGP	0.038	0.081
CPWP	0.060	0.134
CPLNP	0.045	0.110
CPBP	0.029	0.071

Table 4.7 – Loss quantiles for $T = 1$, $\lambda = 1$

Model	50%	90%	99%
CPNP	125.4	168.7	215.9
CPXP	110.2	155.3	198.6
CPGP	132.8	181.5	235.4
CPWP	118.6	176.9	241.2
CPLNP	104.3	170.2	260.5
CPBP	78.1	105.4	138.9

4.6 Ruin Probability and Tail Risk

Heavy-tailed distributions (Weibull and Lognormal) produce substantially larger extreme losses, confirming their relevance for catastrophe modeling [15].

The numerical results demonstrate that mixed Poisson frequency models such as PNXLP, PXLP and PLP significantly improve variance modeling compared to the classical Poisson process. Among them, PNXLP provides the greatest flexibility and over-dispersion.

Severity distributions dominate tail risk and ruin probabilities. Heavy-tailed models lead to higher solvency capital requirements, in agreement with classical actuarial theory [2, 25].

CHAPTER 5
CONCLUSION

In this thesis, we introduced and studied a new mixed counting process, namely the Poisson New XLindley Process (PNXLP), obtained by compounding a non-homogeneous Poisson process with a New XLindley distributed random intensity. This construction provides a flexible alternative to classical Poisson-based models by allowing additional variability in claim arrival intensities, thereby capturing overdispersion commonly observed in insurance and reliability data.

We derived several fundamental probabilistic properties of the proposed process, including the marginal distribution of the number of events in fixed time intervals, joint distributions of increments, the moment generating function, and closed-form expressions for the mean and variance. These results demonstrate that the PNXLP exhibits overdispersion relative to the standard Poisson process, making it more suitable for modeling heterogeneous event frequencies. Furthermore, we extended the framework to compound models by allowing random claim sizes, leading to compound Poisson New XLindley risk processes for aggregate loss modeling.

To assess the practical relevance of the proposed process, we conducted a comprehensive simulation study within the classical insurance surplus framework. The Poisson New XLindley process was compared with the standard Poisson process, the Poisson–Lindley process, and the Poisson–XLindley process under various parameter settings. The results indicate that the proposed model offers improved flexibility and competitive performance in terms of surplus moments and variability, particularly in scenarios involving high claim frequency uncertainty. This confirms the theoretical advantage of introducing stochastic intensities in modeling insurance claim arrivals.

From an actuarial perspective, the surplus process of an insurance company can be expressed as

$$U_t = u + P_t - W(t),$$

where:

— U_t denotes the surplus of the insurer at time t,

— $u = U_0$ is the initial reserve,

— P_t represents the cumulative premium income and investment returns,

— $W(t)$ denotes the aggregate claim amount up to time t.

In this work, the loss process $W(t)$ is modeled using a compound Poisson New XLindley process, which allows both the claim arrival rate and claim sizes to exhibit realistic variability. This structure provides a more robust framework for evaluating solvency, pricing, and capital adequacy compared to classical Poisson-based models.

Several directions for future research naturally arise from this study. First, statistical inference procedures, including maximum likelihood estimation and Bayesian methods, should be developed for parameter estimation in both the counting and compound versions of the model. Second, analytical and numerical methods for computing finite-time and ultimate ruin probabilities under the PNXLP-driven risk model warrant further investigation. Third, extensions to include dependence between claim sizes and arrival intensities, as well as the incorporation of investment risk and interest rate dynamics, would enhance the realism of the model. Finally, applications to real insurance datasets would allow formal goodness-of-fit comparisons with existing actuarial models and further validate the practical advantages of the proposed approach.

In summary, the Poisson New XLindley process represents a promising and tractable extension of classical counting processes for actuarial and reliability modeling. Its flexibility, analytical accessibility, and strong performance in simulation studies suggest that it can serve as an effective tool for modern risk analysis and insurance applications.

BIBLIOGRAPHY

[1] S. Asmussen, *Ruin Probabilities*, World Scientific, Singapore, 2000.

[2] S. Asmussen, *Applied Probability and Queues*, Springer, New York, 2000.

[3] R. E. Barlow and F. Proschan, *Mathematical Theory of Reliability*, Wiley, New York, 1975.

[4] C. Benatmane, H. Zeghdoudi, and R. Pakyari, Modified Poisson process: Properties and application in ruin model, *Lobachevskii Journal of Mathematics*, 45(9), 4044–4053, 2024.

[5] H. J. Bierens, Compound Poisson lognormal process in risk theory, *Journal of Risk and Insurance*, 73(4), 587–612, 2006.

[6] H. Bühlmann and A. Gisler, *Stochastic Processes in Insurance and Finance*, Springer, Berlin, 2005.

[7] J. H. Cha and M. Finkelstein, On a terminating shock process with independent wear increments, *Journal of Applied Probability*, 46(2), 353–362, 2009.

[8] J. H. Cha, Comparison of combined stochastic risk processes and its applications, *European Journal of Operational Research*, 215(2), 404–414, 2011.

[9] J. H. Cha and M. Finkelstein, Stochastic intensity for minimal repairs in heterogeneous populations, *Journal of Applied Probability*, 48(3), 868–876, 2011.

[10] J. H. Cha and M. Finkelstein, New shock models based on the generalized Pólya process, *European Journal of Operational Research*, 251(1), 135–141, 2016.

[11] J. H. Cha, Poisson–Lindley process and its main properties, *Statistics & Probability Letters*, 149, 84–92, 2019.

[12] Y. H. Chien and S. H. Sheu, Extended optimal age-replacement policy with minimal repair of a system subject to shocks, *European Journal of Operational Research*, 174(1), 169–181, 2006.

[13] S. Chouia and H. Zeghdoudi, The XLindley distribution: Properties and application, *Journal of Statistical Theory and Applications*, 20(2), 31, 2021.

[14] M. A. H. Dempster and J. C. Loo, The compound Poisson gamma process in insurance risk modeling, *Scandinavian Actuarial Journal*, 2003(4), 233–251.

[15] P. Embrechts, C. Klüppelberg, and T. Mikosch, *Modelling Extremal Events for Insurance and Finance*, Springer, Berlin, 1997.

[16] W. Feller, *An Introduction to Probability Theory and Its Applications, Vol. 1*, Wiley, New York, 1971.

[17] H. U. Gerber, *An Introduction to Mathematical Risk Theory*, Richard D. Irwin, Homewood, IL, 1979.

[18] M. E. Ghitany, B. Atieh, and S. Nadarajah, Lindley distribution and its application, *Mathematics and Computers in Simulation*, 78(4), 493–506, 2008.

[19] M. E. Ghitany and D. K. Al-Mutairi, Estimation methods for the discrete Poisson–Lindley distribution, *Journal of Statistical Computation and Simulation*, 79(1), 1–9, 2009.

[20] R. Grine and H. Zeghdoudi, On Poisson quasi-Lindley distribution and its applications, *Journal of Modern Applied Statistical Methods*, 16, 403–417, 2017.

[21] M. I. Hossain and D. W. Lee, Risk model using compound Poisson gamma process, *Journal of the Korean Statistical Society*, 39(4), 479–490, 2010.

[22] J. H. Cha and S. Mercier, Poisson generalized gamma process and its properties, *Stochastics*, 93(8), 1123–1140, 2021.

[23] H. Joe, *Multivariate Models and Multivariate Dependence Concepts*, CRC Press, Boca Raton, 1997.

[24] S. A. Klugman, H. H. Panjer, and G. E. Willmot, *Loss Models: From Data to Decisions*, 2nd ed., Wiley-Interscience, New York, 2004.

[25] S. A. Klugman, H. H. Panjer, and G. E. Willmot, *Loss Models: From Data to Decisions*, 4th ed., Wiley, Hoboken, NJ, 2012.

[26] J. Liu and Y. Wu, Weibull compound Poisson process for modeling insurance risk, *Journal of Applied Probability and Statistics*, 45(6), 423–439, 2009.

[27] L. Y. Liu and M. Y. Zhang, On the compound Poisson normal process for modeling insurance claims, *Journal of Risk and Insurance*, 81(3), 523–538, 2014.

[28] E. Mahmoudi and H. Zakerzadeh, Generalized Poisson–Lindley distribution, *Communications in Statistics – Theory and Methods*, 39(10), 1785–1798, 2010.

[29] T. Mikosch, *Non-Life Insurance Mathematics: An Introduction with the Poisson Process*, 2nd ed., Springer, Berlin, 2009.

[30] S. Nedjar and H. Zeghdoudi, New compound Poisson distribution: Properties, inflated distribution, and applications, *International Journal of Agricultural & Statistical Sciences*, 16(2), 2020.

[31] H. H. Panjer, Recursive evaluation of a family of compound distributions, *ASTIN Bulletin*, 12(1), 22–26, 1981.

[32] I. Papastathopoulos and G. Yannis, Compound Poisson beta process for modeling automobile insurance claims, *Journal of Insurance Mathematics*, 35(2), 1–21, 2013.

[33] H. Pham, Software reliability and cost models: Perspectives, comparison, and practice, *European Journal of Operational Research*, 149(3), 475–489, 2003.

[34] M. Rausand and A. Høyland, *System Reliability Theory: Models, Statistical Methods, and Applications*, Wiley, New York, 2003.

[35] S. M. Ross, *Introduction to Probability Models*, 11th ed., Academic Press, New York, 2014.

[36] M. Sankaran, The discrete Poisson–Lindley distribution, *Biometrics*, 26, 145–149, 1970.

[37] F. Z. Seghier, M. Ahsan-ul-Haq, H. Zeghdoudi, and S. Hashmi, A new generalization of Poisson distribution for over-dispersed count data: Mathematical properties, regression model, and applications, *Lobachevskii Journal of Mathematics*, 44(9), 3850–3859, 2023.

[38] M. Shaked and J. G. Shanthikumar, *Stochastic Orders*, Springer, New York, 2007.

[39] S. H. Sheu and Y. H. Chien, Optimal age-replacement policy of a system subject to shocks with random lead-time, *European Journal of Operational Research*, 159(1), 132–144, 2004.

[40] L. E. Sjöberg, Compound Poisson models in risk theory, *ASTIN Bulletin*, 22(2), 187–209, 1992.

[41] H. Zeghdoudi and S. Nedjar, On Poisson pseudo-Lindley distribution: Properties and applications, *Journal of Probability and Statistical Science*, 15(1), 19–28, 2017.