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On the existence and stability of solutions for certain functional differential and delay integro-differential equations by the fixed point technique

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A Doctoral Thesis,

By Karima Bessioud

Advisors: Dr. A. Ardjouni and Pr. A. Djoudi

Dedication

This work is dedicated to

My husband,

My parents,

My daughters Amina and Maya.

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First and foremost I wish to thank Allah for his help and making all kind of task easy, he is indeed the merciful and compassionate.

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Abstract

In this thesis, we study qualitative properties of broad classes of nonlinear neutral integro-differential equations and systems of second-order delay differential equations with singularities. We start by giving some fixed point theorems and results for delay differential equations. Second, by using the Krasnoselskii's fixed point theorem, we study the periodicity and positivity of solutions for a class of nonlinear neutral integro-differential equations with variable delay. Also, by appealing the contraction mapping principle, we show the existence of a unique periodic solution and the asymptotic stability of the zero solution. Finally, by applying the Schauder's fixed point theorem and the Green function, we obtain the existence of positive periodic solutions for systems of second-order delay differential equations with singularities.

Keywords: Delay differential equations, Neutral integro-differential equations, Fixed point theory, Periodicity, Positivity, Stability.

Mathematics Subject Classification: 34K13, 34K20, 34K30, 34K40, 45D05, 45J05, 47H10.

ملخـــــــص

في هذه الأطروحة نهتم بدراسة الخصائص النوعية لمجموعة من المعادلات التفاضلية التكاملية الحيايدة غير الخطية وجمل معادلات تفاضلية من الدرجة الثانية ذات تأخر مع نقاط شاذة. نبدأ بإعطاء بعض نظريات النقطة الثابتة وبعض النتائج حول المعادلات التفاضلية ذات تأخر. ثانيا، ندرس الدورية والايجابية لحلول مجموعة من المعادلات التفاضلية التكاملية الحيايدة غير الخطية مع تأخر متغير باستخدام نظرية النقطة الثابتة لكراسنوسلسكي. أيضا، من خلال مبدأ التقليص، نبرهن على وجود حل دوري وحيد وعلى الاستقرار المقارب للحل الصفري. وأخيرًا، من خلال تطبيق نظرية النقطة الثابتة لشودار ودالة قرين، نحصل على وجود حلول دورية إيجابية لجمل معادلات تفاضلية من الدرجة الثانية ذات تأخير مع نقاط شاذة.

الكلمات المفتاحية: معادلات تفاضلية ذات تأخر، معادلات تفاضلية تكاملية حيادية، نظريات النقاط الثابتة، الدورية، الايجابية، الاستقرار.

Résumé

Dans cette thèse, nous étudions les propriétés qualitatives de larges classes d'équations intégro-différentielles non linéaires neutrales et de systèmes d'équations différentielles du second ordre à retard avec singularités. Nous commençons par donner quelques théorèmes de point fixe et des résultats sur les équations différentielles à retard. Deuxièmement, en utilisant le théorème du point fixe de Krasnoselskii, nous étudions la périodicité et la positivité des solutions pour une classe d'équations intégro-différentielles neutrales non linéaires avec un retard variable. De plus, en faisant appel le principe de la contraction, nous montrons l'existence d'une solution périodique unique et la stabilité asymptotique de la solution zéro. Enfin, en appliquant le théorème du point fixe de Schauder et la fonction de Green, nous obtenons l'existence de solutions périodiques positives pour des systèmes d'équations différentielles du second ordre à retard avec singularités.

Mots-clés: Equations différentielles à retard, Equations intégro-différentielles neutrales, Théorèmes des points fixes, Périodicité, Positivité, Stabilité.

Mathematics Subject Classification: 34K13, 34K20, 34K30, 34K40, 45D05, 45J05, 47H10.

Contents

Chapter 1

Introduction

Fixed point theory becomes, in the last decades, not only a field with a huge development, but also a strong tool for solving various problems arising in different fields of pure and applied mathematics, physics, economics, game theory, biology and chemistry etc. Classical and major results in these areas are Banach's fixed point theorem, Schauder's fixed point theorem and Krasnoselskii's fixed point theorem (see [2], [21], [46], [53]).

Let P be a mapping of a set X into itself. An element $x \in X$ is said to be a fixed point of the mapping P if $Px = x$. By a fixed point theorem we understand a statement which asserts that under certain conditions on the mapping P and on the space X, a mapping P of X into itself admits one or more fixed points. Historically, Poincare in 1886 was the first to work in this field. Then Brouwer in 1912, proved fixed point theorem which asserts the existence of a fixed point whenever X is the unit ball in $\mathbb{R}^n$ and P is continuous. In this theorem X can be replaced by any homeomorph thereof. Meanwhile Banach principle came into existence which was considered as one of the fundamental principles in the field of functional analysis. In 1922, Banach proved that a contraction mapping in the field of a complete metric space possesses a unique fixed point. The Brouwer's theorem, where the spaces are subsets of $\mathbb{R}^n$ are not of much use in functional analysis where one is generally concerned with infinite dimensional subset of some function spaces. This was investigated by Schauder in 1930. Subsequently, Schauder extended Brouwer's theorem to the case where X is a compact convex subset of a normed linear space (see [46], [53]).

In 1955, Krasnoselskii studied a paper of Schauder on partial differential equations and formulated the working hypothesis principle: the inversion of a perturbed differential operator yields the sum of a contraction and a compact map. Accordingly, he formulated an hybrid theorem known under its name. The reader is referred to the classical textbook on fixed point [46].

Delay differential equations, in the broadest sense we mean differential equations which somehow include information from the past. Incorporating values of functions from the past to define a vector field is also known as a particular type of what is known as functional differential equations. For example, $x'(t) = f(t, x(t), x(t-\tau))$ with $\tau > 0$ is an example of a such equation.

Delay differential and delay integro-differential equations with and without singularities arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of delay differential delay integro-differential equations with and without singularities have received the attention of many authors, see [1], [3]–[20], [22]–[24], [26]–[29], [32]–[34], [36]–[38], [40]–[45], [50], [51], [53] and the references therein.

One of the most important qualitative aspects of delay differential and delay integro-differential equations is determining the stability of a given model. The Lyapunov direct method has been very effective in establishing stability results and the existence of periodic solutions for wide variety of ordinary, functional and partial differential equations. Nevertheless, in the application of Lyapunov's direct method to problems of stability in delay differential and integro-differential equations, serious difficulties occur if the delay is unbounded or if the equation has unbounded terms. In recent years, several investigators have tried stability by using a new technique. Particularly, Burton, Furumochi, Zhang and others began a study in which they noticed that some of this difficulties vanish or might be overcome by means of fixed point theory (see [10]–[12], [15]–[18], [22]–[24], [26], [28], [29], [40], [41], [54]). The fixed point theory does not only solve the problem on stability but has other significant advantage over Lyapunov's direct method. The conditions of the former are often average but those of the latter are usually pointwise (see [23]).

We have been interested in the use of fixed point theory to problem of periodicity and positivity and stability for delay differential and neutral integro-differential equations. We have studied and contributed to it and have obtained interesting results. In this thesis we present a collection of results to some problems of neutral integro-differential equations and systems of delay differential equations with singularities by using fixed point theory.

This thesis contains five chapters which are briefly presented below. Chapter two is essentially an introduction to the fixed point theory, delay differential equations, where we fix notations, terminology to be used. It is a survey aimed at recalling some basic definitions and theory. While some of the classical and recent results about fixed point theory, delay differential equations are also presented in this chapter. Fixed point theorems frequently call for compact sets in Banach spaces which may be subsets of continuous functions. For that purpose, we give topologies which will provide many of those compact sets.

In the chapter three, we give suitable conditions to obtain asymptotic stability results about the zero solution for the nonlinear neutral Levin-Nohel integro-differential equation

$$\frac{d}{dt}x(t) + \int_{t-\tau(t)}^{t} a(t,s)x(s)ds + \frac{d}{dt}g\left(t, x(t-\tau(t))\right) = 0,\ t \geq t_0. \tag{E}$$

An asymptotic stability theorem with a necessary and sufficient condition is proved, by means of fixed point technique. In addition, the case of the equation with several delays is studied (see [18]).

In the chapter four, we establish sufficient conditions for the periodicity and positivity of solutions for (E) by appealing the Krasnoselskii's fixed point theorem and the contraction mapping principle (see [19]).

Finally in the chapter five, we discuss the existence of positive periodic solutions for the system of second-order delay differential equations with singularities

$$\begin{cases} x''(t) + a_1(t)\,x(t) = f_1\left(t, y\left(t-\tau_1(t)\right)\right) + e_1(t), \\ y''(t) + a_2(t)\,y(t) = f_2\left(t, x\left(t-\tau_2(t)\right)\right) + e_2(t). \end{cases}$$

Our main results are obtained via the Schauder's fixed point theorem (see [20]).

Chapter 2

Preliminaries

Keywords. Fixed point, Banach, Schauder, Krasnoselskii, delay differential equations.

This chapter deals with the notation of fixed point theory and functional differential equations, more particularly, the notation of a delay differential equation. We introduce the basic results of that theory like existence, uniqueness, continuation, and stability. The main references used in this chapter have been [2], [22], [23], [27], [32], [33], [46], [52], [53].

2.1 Functional analysis

Definition 2.1 A metric d on a set X is a function $d : X \times X \to [0, \infty)$ such that for all $x,\ y,\ z \in X$

i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;

ii) $d(x, y) = d(y, x)$ (symmetry);

iii) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

A metric space (X, d) is a set X with a metric d defined on X.

Example 2.1 The real line $\mathbb{R}$ with $d(x, y) = |x - y|$ is a metric space. The metric d is called the usual metric for $\mathbb{R}$.

The metric space (X, d) is complete if every Cauchy sequence in X has a limit in X, i.e., every Cauchy sequence is convergent. We say that a sequence $(x_n)_{n\geq 0} \subseteq X$ is a Cauchy sequence if for all $\epsilon > 0$ there exists an $N > 0$ such that for all $n,\ m > N$, $d(x_n, x_m) \leq \epsilon$.

Definition 2.2 A linear space $(E, +, .)$ is a normed space if for each $x \in E$ there is a nonnegative real number $\|x\|$, called the norm of x, such that

i) $\|x\| = 0$ if and only if $x = 0$,

ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for each $\alpha \in \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and

iii) $\|x + y\| \leq \|x\| + \|y\|$.

Example 2.2 Let $E = \mathbb{R}^n$; $n > 1$ be a linear space. Then $\mathbb{R}^n$ is a normed space with the following norms

$\|x\|_1 = \sum_{i=1}^{n} |x_i|$, for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$;

$\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$, for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $p \in (1, \infty)$;

$\|x\|_\infty = \max\limits_{i=1,n} |x_i|$, for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$.

A norm induces a metric on the vector space, by $d(x, y) = \|x - y\|$.

Definition 2.3 A Banach space is a complete normed space.

The following examples of spaces are widely used in this thesis.

Example 2.3 The space $C([a, b], \mathbb{R}^n)$ consisting of all continuous functions $f : [a, b] \to \mathbb{R}^n$ is a vector space over the real. The number $\|f\| = \max\limits_{a \leq t \leq b} |f(t)|$, where $\|.\|$ is the norm in $\mathbb{R}^n$, defines a norm making $(C, \|.\|)$ a Banach space.

Example 2.4 Let $\psi : [a, b] \to \mathbb{R}^n$ a continuous function and let S be the set of continuous and bounded functions $f : [a, \infty) \to \mathbb{R}^n$ with $f(t) = \psi(t)$ for $a \leq t \leq b$. For f, $g \in S$, we defines

$$d(f, g) = \sup_{a \leq t < \infty} |f(t) - g(t)| = \|f - g\|.$$

So (S, d) is a complete metric space. Define

$$M = \{\varphi : [0, \infty) \to \mathbb{R} / \ \varphi \in \mathbb{C}, \ |\varphi| \leq 1, \ \varphi(t) \to 0 \text{ when } t \to \infty\},$$

and

$$Q = \{\varphi : [0, \infty) \to \mathbb{R} / \ \varphi \in \mathbb{C}, \ |\varphi| \leq 1\}.$$

Let $\|.\|$ the supremum norm and let $|.|_h$ a weight norm defined by giving a function $h : [0, \infty) \to [1, \infty)$, $h(0) = 1$, $h(t) \to \infty$, and for $\varphi \in M$ or Q, we put

$$|\varphi|_h = \sup_{t \geq 0} |\varphi(t) / h(t)|.$$

Example 2.5 $(M, \|.\|)$ is a Banach space.

Example 2.6 $(Q, |.|_h)$ is a Banach space.

Proof. Suppose that $\{\phi_n\}$ is a Cauchy sequence in this space. The restriction of $\{\phi_n\}$ at the interval $[0, k]$ remains of Cauchy and therefore admits a continuous limit defined on this last interval. But this is true for every $k = 1, 2, \ldots$ so we get a continuous limit on $[0, \infty)$ which clearly belongs to Q. ■

Definition 2.4 A subset M of a metric space (X, d) is compact if any sequence $\{x_n\}$ of M admits a subsequence with limit in M. M is relatively compact if every sequence of M admits a subsequence converging towards a limit belonging to X (i.e. $\overline{M}$ is compact).

Definition 2.5 Let U be an interval of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of functions with $f_n : U \to \mathbb{R}^p$.

(a) $\{f_n\}$ is uniformly bounded on U if there exists a $M > 0$ such that $|f_n(t)| \leq M$ for all n and all $t \in U$.

(b) $\{f_n\}$ is equicontinuous if for any $\epsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in U$ and $|t_1 - t_2| \leq \delta$ imply $|f_n(t_1) - f_n(t_2)| \leq \epsilon$, for all n.

Theorem 2.1 (Ascoli-Arzela [22]) *If $\{f_n\}$ is a uniformly bounded and equicontinuous sequence of real functions on an interval $[a, b]$, then there is a subsequence which converges uniformly on $[a, b]$ to a continuous function.*

Example 2.7 Consider the Banach space $C([a, b], \mathbb{R}^n)$ provided by the supremum norm $\|f\| = \max\limits_{a \leq t \leq b} |f(t)|$, with $|.|$ a norm of $\mathbb{R}^n$. Given two constant positive α and β, the set

$$L = \{f \in C([a, b], \mathbb{R}^n) \,/\, \|f\| \leq \alpha,\ |f(u) - f(\nu)| \leq \beta |u - \nu|\},$$

is compact. This is a consequence of Ascoli's Theorem.

Definition 2.6 A function $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is an L^1-Carathéodory function if it satisfies the following conditions

i) For each $z \in \mathbb{R}^n$, the mapping $t \to f(t, z)$ is Lebesgue measurable.

ii) For almost all $t \in [0, T]$, the mapping $z \to f(t, z)$ is continuous on $\mathbb{R}^n$.

iii) For each $r > 0$, there exists $h_r \in L^1([0,T], \mathbb{R})$ such that, for almost all $t \in [0,T]$ and for all z with $|z| < r$, we have $|f(t,z)| \leq h_r(t)$.

2.2 Fixed point theorems

The fixed point theorem, generally known as the Banach contraction principle, appeared in explicit form in Banach's thesis in 1922 where it was used to establish the existence of a solution for an integral equation. Since then, because of its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis and forms an attractive tool which facilitates the study of stability for the differential equations with or without delay.

Definition 2.7 Let P be a mapping in the set M. We call fixed point of P any point x satisfying $P(x) = x$. If there exists such x, we say that P has a fixed point, which is equivalent to saying that the equation $P(x) - x = 0$ has a null solution.

Definition 2.8 Let (X,d) be a complete metric space. The operator $P : X \to X$ is called the contraction operator, if there exists a constant $0 < \alpha < 1$ such that

$$\forall x, y \in X, \ d(Px, Py) \leq \alpha d(x,y).$$

Theorem 2.2 (Contraction mapping principle [22]) *Let (X,d) a complete metric space and let $P : X \to X$ a contraction mapping. Then there is a unique $x \in X$ with $Px = x$.*

Furthermore, if $y \in X$ and if $\{y_n\}$ is defined inductively by $y_1 = Py$, $y_{n+1} = Py_n$, then $y_n \to x$, the unique fixed point. In particular, the equation $Px = x$ has one and only one solution.

Definition 2.9 Let K be a subset of a Banach space X and $A : K \to X$ application. If A is continuous and $A(K)$ is contained in a compact subset of X, then we say that A is a compact mapping (we also say that A is completely continuous).

Theorem 2.3 (Brouwer's fixed point theorem [21]) *Let K be a nonempty, compact and convex subset of $\mathbb{R}^n$. Every continuous map $A : K \to K$ has a fixed point.*

The Schauder fixed point theorem is an extension of the Brouwer fixed point theorem to topological vector spaces, which may be of infinite dimension.

Definition 2.10 Let E be a normed vector space and F a subset of E. The convex hull $conv\left(F\right)$ is the intersection of all convex sets $S \subseteq E$ such that $F \subseteq S$.

Definition 2.11 M has a finite ϵ-net; that is, by definition, for each $\epsilon > 0$, there exists a finite number of points $x_1, ..., x_n \in M$ such that

$$\min_{1 \leq i \leq n} \|x - x_i\| \leq \epsilon, \text{ for all } x \in M.$$

Lemma 2.1 (Schauder projection lemma) *Let M be a compact subset of a normed vector space E, with metric d induced by the norm $\|.\|$. Given $\epsilon > 0$, there exists a finite subset $F \subseteq E$ and a map $P : M \to conv\left(F\right)$ such that $d\left(P\left(x\right), x\right) < \epsilon$ for all $x \in M$. This map is called the Schauder projection.*

Proof. Take a finite ϵ-net for the compact set M to obtain a set $F = \{x_1, x_2, ..., x_n\}$. For $i = 1, ..., n$, define functions $\varphi_i : M \to \mathbb{R}$ by

$$\varphi_i = \begin{cases} \epsilon - d\left(x, x_i\right), & \text{if } x \in B\left(x_i, \epsilon\right), \\ 0, & \text{otherwise.} \end{cases}$$

We see that φ_i is strictly positive on $B\left(x_i, \epsilon\right)$ and vanishes elsewhere. Therefore $\sum_{i=1}^{n} \varphi_i > 0$ for all $x \in M$. We define the Schauder projection $P : M \to conv\left(F\right)$ by

$$P\left(x\right) = \sum_{i=1}^{n} \frac{\varphi_i\left(x\right)}{\varphi\left(x\right)} x_i, \text{ where } \varphi\left(x\right) = \sum_{i=1}^{n} \varphi_i\left(x\right).$$

The map P is continuous since all the φ_i are. Moreover,

$$\begin{aligned} d\left(P\left(x\right), x\right) &= \left\| \sum_{i=1}^{n} \frac{\varphi_i\left(x\right)}{\varphi\left(x\right)} x_i - \sum_{i=1}^{n} \frac{\varphi_i\left(x\right)}{\varphi\left(x\right)} x \right\| = \left\| \sum_{i=1}^{n} \frac{\varphi_i\left(x\right)}{\varphi\left(x\right)} \left(x_i - x\right) \right\| \\ &\leq \sum_{i=1}^{n} \frac{\varphi_i\left(x\right)}{\varphi\left(x\right)} \|x_i - x\| < \sum_{i=1}^{n} \frac{\varphi_i\left(x\right)}{\varphi\left(x\right)} \epsilon = \epsilon, \end{aligned}$$

because $\varphi_i\left(x\right) = 0$ if $\|\left(x_i - x\right)\| \geq \epsilon$. ■

Theorem 2.4 (Schauder's fixed point theorem [46]) *Let K be a nonempty closed bounded convex subset of a Banach space $(X, \|.\|)$. Then every completely continuous mapping $A : K \to K$ has a fixed point.*

Proof. Let M denote the closure of $A(K)$ which, by hypothesis, is compact. For each natural number n, let $F_n$be a finite $\frac{1}{n}$-net for M and let $P_n : M \to conv(F_n)$ be the corresponding Schauder projection. The convexity of K implies that $conv(F_n) \subseteq M$; define $A_n : conv(F_n) \to conv(F_n)$ by $A_n = (P_n \circ A)\mid_{conv(F_n)}$. Brouwer's fixed point theorem guarantees that A_n has fixed points. For each $n \in \mathbb{N}$, we choose one such fixed point of A_n and call it x_n. Since M is compact $\{x_n\}$ has a convergent subsequence, which we denote $\{x_{n'}\}$. This sequence converges to some $x \in M$ as $n' \to \infty$, which we claim is the desired fixed point. From lemma 2.1 we obtain

$$d(A(x), x_{n'}) \leq d(A(x), A(x_{n'})) + d(A(x_{n'}), A_{n'}(x_{n'})) \to 0 \text{ as } n' \to \infty,$$

since A is continuous and $d(A(x_{n'}), A_{n'}(x_{n'})) = d(A(x_{n'}), x_{n'}) < 1/n'$. Thus $\{x_{n'}\}$ converges to both x and $A(x)$. Limits are unique, so $A(x) = x$, as desired. ■

In 1955 Krasnoselskii's (see [22], [46]) observed that in a good number of problems, the integration of a perturbed differential operator gives rise to a sum of two applications, a contraction and a compact application. It declares then,

Principle: the integration of a perturbed differential operator can produce a sum of two applications, a contraction and a compact operator.

For better understanding this observation of Krasnoselskii, consider the following differential equation

$$x'(t) = -a(t)x(t) - g(t,x), \tag{2.1}$$

or $a(t+T) = a(t)$ and $g(t+T, x) = g(t,x)$ for a certain $T > 0$.

We can transform this equation in another form while writing, formally

$$x'(t)\exp\left(\int_0^t a(s)\,ds\right) = -a(t)x(t)\exp\left(\int_0^t a(s)\,ds\right) - g(t,x)\exp\left(\int_0^t a(s)\,ds\right),$$

thus

$$x'(t)\exp\left(\int_0^t a(s)\,ds\right) + a(t)x(t)\exp\left(\int_0^t a(s)\,ds\right) = -g(t,x)\exp\left(\int_0^t a(s)\,ds\right),$$

or

$$\left(x\left(t\right)\exp\left(\int_{0}^{t}a\left(s\right)ds\right)\right)^{\prime}=-g\left(t,x\left(t\right)\right)\exp\left(\int_{0}^{t}a\left(s\right)ds\right),$$

then integrating from $t-T$ to t, we obtain

$$\int_{t-T}^{t}\left(x\left(u\right)\exp\left(\int_{0}^{u}a\left(s\right)ds\right)\right)^{\prime}du=-\int_{t-T}^{t}g\left(u,x\left(u\right)\right)\exp\left(\int_{0}^{u}a\left(s\right)ds\right)du,$$

what gives

$$x\left(t\right)=x\left(t-T\right)\exp\left(-\int_{t-T}^{t}a\left(s\right)ds\right)-\int_{t-T}^{t}g\left(u,x\left(u\right)\right)\exp\left(-\int_{u}^{t}a\left(s\right)ds\right)du. \quad (2.2)$$

If we suppose that $\exp\left(-\int_{t-T}^{t}a\left(s\right)ds\right)=\alpha<1$, and if $(X,\|.\|)$ is the Banach space of functions $\varphi:\mathbb{R}\rightarrow\mathbb{R}$ continuous and T -periodic, then the equation (2.2) can be written as

$$\varphi\left(t\right)=\left(B\varphi\right)\left(t\right)+\left(C\varphi\right)\left(t\right)=\left(H\varphi\right)\left(t\right),$$

where B is contraction provides that the constant $\alpha<1$ and C is compact mapping.

This example shows the birth of the mapping $\left(H\varphi\right)\left(t\right)=\left(B\varphi\right)\left(t\right)+\left(C\varphi\right)\left(t\right)$ who is identified with a sum of a contraction and a compact mapping.

Thus, the search of the solution for (2.2) requires an adequate theorem which applies to this hybrid operator H and who can conclude the existence for a fixed point which will be, in his turn, solution of the initial equation (2.1). Krasnoselskii found the solution by combining the two theorems of Banach and that of Schauder in one hybrid theorem which bears its name. In light, it establishes the following result [46].

Theorem 2.5 (Krasnoselskii) *Let M be a closed convex nonempty subset of a Banach space $(\mathbb{B},\|.\|)$. Suppose that C and B map M into $\mathbb{B}$ such that*

(i) $x,y\in M$, implies $Cx+By\in M$,

(ii) C is continuous and CM is contained in a compact set,

(iii) B is a contraction mapping,

Then there exists $z\in M$ with $z=Cz+Bz$.

Proof. According to the condition (iii) we have

$$\begin{aligned}
\|(I-B)\,x-(I-B)\,y\| &= \|(x-y)-(Bx-By)\| \\
&\leq \|x-y\|+\|Bx-By\| \\
&\leq \|x-y\|+\alpha\,\|x-y\| \\
&= (1+\alpha)\,\|x-y\|\,,
\end{aligned}$$

and

$$\begin{aligned}
\|(I-B)\,x-(I-B)\,y\| &= \|(x-y)-(Bx-By)\| \\
&\geq \|x-y\|-\|Bx-By\| \\
&\geq \|x-y\|-\alpha\,\|x-y\| \\
&\geq (1-\alpha)\,\|x-y\|\,.
\end{aligned}$$

In short

$$(1-\alpha)\,\|x-y\| \leq \|(I-B)\,x-(I-B)\,y\| \leq (1+\alpha)\,\|x-y\|\,.$$

This inequality shows that $(I-B):M\to(I-B)\,M$ is continuous and bijective. Thus, $(I-B)^{-1}$ exist and is continuous. Let us pose $U=(I-B)^{-1}\,C$. It is clear that U is compact mapping, because U is a composition of a continuous mapping with a compact. Under the theorem of Schauder, U has a fixed point, i.e.

$$\exists z\in M \text{ such that } (I-B)^{-1}\,Cz=z.$$

This is equivalent to $z=Cz+Bz$. ∎

Remark 2.1 If $C=0$, the theorem become the theorem of Banach. If $B=0$, then the theorem is not other than the theorem of Schauder.

2.3 Retarded functional differential equations

2.3.1 Delay differential equations

Suppose $\tau>0$ is a given real number, $\mathbb{R}=(-\infty,+\infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|.|$, $C\left([a,b]\,,\mathbb{R}^n\right)$ is the Banach space of continuous functions mapping the interval $[a, b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. If $[a, b] = [-\tau, 0]$ we let $C = C([-\tau, 0], \mathbb{R}^n)$ and designate the norm of an element φ in C by $|\varphi| = \sup\limits_{-\tau \le s \le 0} |\varphi(s)|$. Even though single bars are used for norms in different spaces, no confusion should arise. If

$$t_0 \in \mathbb{R},\ A \ge 0 \text{ and } x \in C([t_0 - \tau, t_0 + A], \mathbb{R}^n),$$

then for any $t \in [t_0, t_0 + A]$, we let $x_t \in C$ be defined by $x_t(s) = x(t+s)$ for $s \in [-\tau, 0]$. For $\Omega \subseteq \mathbb{R} \times C$, and $f : \Omega \to \mathbb{R}^n$ is a given function and represents the right-hand derivative, we say that the relation

$$x'(t) = f(t, x_t), \tag{2.3}$$

is a retarded functional differential equation on Ω and will denote this equation by DDE. The number τ is called the delay. The case $\tau = 0$ corresponds with an ordinary differential equation. It is clear that an appropriate initial condition at time $t = t_0$ must at least specify the vector x for all $t \in [t_0 - \tau, t_0]$, i.e

$$x(t) = \psi(t),\ t \in [t_0 - \tau, t_0]. \tag{2.4}$$

Here $\psi : [t_0 - \tau, t_0] \to \mathbb{R}^n$ is a known function, usually we suppose ψ to be a continuous function. The function ψ is called the initial function of the delay differential equation. Hence, the initial value problem of (2.3) is given by the following relation

$$\begin{cases} x'(t) = f(t, x_t),\ t \ge t_0, \\ x(t) = \psi(t),\ t_0 - \tau \le t \le t_0, \end{cases} \tag{2.5}$$

where ψ is a given function defined on $t \in [t_0 - \tau, t_0]$.

Definition 2.12 Equation (2.3) is called

i) linear if $f(t, \psi) = L(t)\psi$, where $L(t)$ is linear for each t.

ii) nonhomogeneous if $f(t, \psi) = L(t)\psi + h(t)$, where $h(t) \neq 0$.

iii) autonomous if $f(t, \psi) = g(\psi)$, where g does not depend on t.

Example 2.8 The following equations are delay differential equations

$$x'(t) = x(t) + x(t-4), \tag{2.6}$$

$$x'(t) = a(t)\,x(t) + b(t)\,x'(t-\tau(t)) + h(t), \tag{2.7}$$

$$x'(t) = \int_{-\tau}^{0} x(t+s)\,ds, \tag{2.8}$$

where $a(t)$, $b(t)$, $\tau(t)$ are continuous functions. Equation (2.6) represents an autonomous linear differential equation with constant delay $\tau = 4$, equation (2.7) is a linear differential equation with non-homogeneous non-autonomous functional delay and equation (2.8) represents a delay linear integro-differential equation.

Definition 2.13 A function x is said to be a solution of (2.3) on $[t_0 - \tau, t_0 + A]$ if there are $t_0 \in \mathbb{R}$, $A > 0$ such that $x \in C([t_0 - \tau, t_0 + A], \mathbb{R}^n)$ and x satisfies (2.3) for $t \in [t_0, t_0 + A]$. In such a case we say that x is a solution of (2.3) on $[t_0 - \tau, t_0 + A]$ for a given $t_0 \in \mathbb{R}$ and a given $\psi \in C$ we say that $x = x(t, t_0, \psi)$, is a solution of (2.5) with initial value at t_0 or simply a solution of (2.5) through (t_0, ψ) if there is an $A > 0$ such that $x(t, t_0, \psi)$ is a solution of (2.5) on $[t_0 - \tau, t_0 + A]$ and $x_{t_0}(t_0, \psi) = \psi$.

Lemma 2.2 ([32]) *Let $t_0 \in \mathbb{R}$, $\psi \in C$ and $f : \Omega \subset \mathbb{R} \times C \to \mathbb{R}^n$ continuous function. Then $x(t, t_0, \psi)$ is a solution of (2.5) at (t_0, ψ) if and only if $x(t, t_0, \psi)$ is a solution of the integral equation*

$$\begin{cases} x(t) = \psi(0) + \int_{t_0}^{t} f(u, x_u)\,du, & t \geq t_0, \\ x_{t_0} = \psi. \end{cases} \tag{2.9}$$

Proof. Necessary condition. Let $x(t, t_0, \psi)$ a solution of (2.5), then

$$\begin{cases} x'(t) = f(t, x_t), \ t \geq t_0, \\ x_{t_0} = \psi. \end{cases}$$

By integration, we get

$$\int_{t_0}^{t} x'(u)\,du = x(t) - x(t_0) = \int_{t_0}^{t} f(u, x_u)\,du, \quad t \geq t_0.$$

As $x(t_0) = x(t_0 + 0) = x_{t_0}(0) = \psi(0)$ we obtain

$$\begin{cases} x(t) = \psi(0) + \int_{t_0}^{t} f(u, x_u)\, du. \\ x_{t_0} = \psi. \end{cases}$$

Sufficient condition. Let $x(t, t_0, \psi)$ a solution of the integral equation (2.9). So,

$$x'(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} = \lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} f(u, x_u)\, du.$$

Since $f(t, x_t)$ is continuous in t, by applying the mean theorem, we have

$$\lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} f(u, x_u)\, du = f(t, x_t),$$

hence the result. ■

Lemma 2.3 ([32]) *If $x \in C([t_0 - \tau, t_0 + A], \mathbb{R}^n)$, then x_t is a continuous function of t, for $t \in [t_0, t_0 + A]$.*

Proof. Since x is continuous on $[t_0 - \tau, t_0 + A]$, it is uniformly continuous on $[t_0 - \tau, t_0 + A]$, so $\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0$, such that $|x(t) - x(\tau)| < \epsilon$ if $|t - \tau| < \delta$. Consequently for t, τ in $[t_0, t_0 + A]$, $|t - \tau| < \delta$, we have $|x(t + s) - x(\tau + s)| < \epsilon$, $\forall s \in [-\tau, 0]$. This proves the lemma. ■

Theorem 2.6 (Existence [32]) *Suppose Ω is an open subset in $\mathbb{R} \times C$ and $f(t, \psi)$ is continuous on Ω. If $(t_0, \psi) \in \Omega$, then there is a solution of (2.3) passing through (t_0, ψ).*

Definition 2.14 We say $f(t, \psi)$ is Lipschitz in ψ in a compact set K of $\mathbb{R} \times C$ if there is a constant $k > 0$ such that, for any $(t, \psi_i) \in k$, $i = 1, 2$

$$|f(t, \psi_1) - f(t, \psi_2)| \leq k |\psi_1 - \psi_2|. \tag{2.10}$$

Theorem 2.7 (Existence and uniqueness [32]) *Suppose Ω is an open subset in $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ is continuous and $f(t, \psi)$ is Lipschitz in ψ in each compact set in Ω. If $(t_0, \psi) \in \Omega$, then there is a unique solution of (2.3) through (t_0, ψ).*

2.3.2 Method of Steps

The method of steps is an elementary method that can be used to solve some DDEs analytically. This method is usually discarded as being too tedious, but in some cases the tedium can be removed by using computer algebra see [34]. Consider the following general DDE

$$x'(t) = a_0 x(t) + a_1 x(t - w_1) + ... + a_m x(t - w_m), \tag{2.11}$$

where $x(t) = H(t)$ on the initial interval $-\max(w_i) \leq t \leq 0$. Let $b = \min(w_i)$. Then it is clear that the values of $x(t - w_m)$ are known in the interval $0 \leq t \leq b$. These values are $H(t - w_m)$. Thus, for the interval $0 \leq t \leq b$ we have

$$x'(t) = a_0 x(t) + a_1 H(t - w_1) + ... + a_m H(t - w_m),$$

and so

$$x(t) = \int_0^t (a_0 x(s) + a_1 H(s - w_1) + ... + a_m H(s - w_m))\, ds + x(0).$$

Now that we know $x(t)$ on $[0, b]$ we can repeat this procedure to obtain $x(t)$ on the interval $b \leq t \leq 2b$. This is given by:

$$x(t) = \int_b^t (a_0 x(s) + a_1 H(s - w_1) + ... + a_m H(s - w_m))\, ds + x(b). \tag{2.12}$$

This process can be continued indefinitely, so long as the integrals that occur can be evaluated without too much effort. It is this last restriction that usually causes people to give up on this method, because the tedium and length of the method quickly overwhelms a human computer. However, it turns out that for certain classes of problems, where the phenomenon of "expression swell" is not too serious, we can take the method quite far, with a computer algebra system to automate the solution of the tedious sub-problems.

Example 2.9 For an example of this method we look first at a very simple DDE

$$x'(t) = x(t - 7), \tag{2.13}$$

with $x(t) = 4$ for $-7 \leq t \leq 0$. The solution in the interval $0 \leq t \leq 7$ is given by:

$$x(t) = \int_0^t H(s - 7)\, ds + x(0) = 4t + 4.$$

Now we can solve for the solution in the interval $7 \leq t \leq 14$. This solution is given by:

$$x(t) = \int_{7}^{t} H(s-7)\,ds + x(7) = 2t^2 - 24t + 102.$$

This method can be programmed in Maple using a simple for loop.

2.3.3 Neutral delay differential equations

Definition 2.15 ([32]) Suppose $\Omega \subseteq \mathbb{R} \times C$ is open with elements (t, ψ). A function $D : \Omega \to \mathbb{R}^n$ is said to be atomic at β on Ω if D is continuous together with its first and second Fréchet derivatives with respect to ψ and D_ψ, the derivative with respect to ψ, is atomic at β on Ω.

Definition 2.16 ([32]) Suppose that $\Omega \subseteq \mathbb{R} \times C$ is open, $f : \Omega \to \mathbb{R}^n$, $D : \Omega \to \mathbb{R}^n$ are given continuous functions with D atomic at zero. Consider the neutral delay differential equation

$$\frac{d}{dt} D(t, x_t) = f(t, x_t). \tag{2.14}$$

Definition 2.17 ([32]) A function x is said to be a solution of (2.14) on $[t_0 - \tau, t_0 + A]$ if there are $t_0 \in \mathbb{R}$, $A > 0$ such that

$$x \in C([t_0 - \tau, t_0 + A], \mathbb{R}^n),\ (t, x_t) \in \Omega,\ t \in [t_0, t_0 + A],$$

$D(t, x_t)$ is continuously differentiable and satisfies equation (2.14) on $[t_0, t_0 + \sigma]$. For a given $t \in \mathbb{R}$, $\psi \in C$ and $(t_0, \psi) \in \Omega$ we say $x(t, t_0, \psi)$ is a solution of (2.14) with initial value ψ at t_0 or simply a solution through (t_0, ψ), if there is an $A > 0$ such that $x(t, t_0, \psi)$ is a solution of (2.14) on $[t_0 - \tau, t_0 + A]$ and $x_{t_0}(t_0, \psi) = \psi$; we say $x(t, t_0, \psi)$ is a solution of (2.14) on $[t_0 - \tau, \infty)$ if for every $A > 0$, $x(t, t_0, \psi)$ is a solution of equation (2.14) on $[t_0 - \tau, t_0 + A]$ et $x_{t_0}(t_0, \psi) = \psi$.

Theorem 2.8 (Existence [32]) *if Ω is an open set in $\mathbb{R} \times C$ and $(t_0, \psi) \in \Omega$, then there exists a solution of the NDDE (D, f) through (t_0, ψ).*

Theorem 2.9 (Uniqueness [32]) *if Ω is an open set in $\mathbb{R} \times C$ and $f(t, \psi)$ is Lipschitz in ψ in each compact set in Ω, then, for any $(t_0, \psi) \in \Omega$ there exists a unique solution of the NDDE (D, f) through (t_0, ψ).*

Example 2.10 The following expressions are neutral delay differential equations

$$x'(t) = x(t-1) + [x'(t-3)+1]^2,$$
$$x'(t) = x'\left(\frac{t}{2}\right) + x'(t-1) - x'(t-3).$$

2.3.4 Examples of delay equations

Economic model

Cooke and Yorke (1973) also briefly consider a related economic model. Let $x(t)$ be the value of capital stock. Assume that production of new capital depends only on $x(t)$ and that the rate of production is $g(x(t))$. Also, assume that the lifetime of equipment is L and that depreciation is independent of the type of equipment; in particular, at time s after production the value of a unit of capital equipment has decreased in value to $P(s)$ times its original value. Here, $P(0) = 1$, $P(L) = 0$.

Now, at any time t, $x(t)$ equals the sum of capital produced over the period $[t-L, t]$ plus a constant c denoting the value of nondepreciating assets. Thus,

$$\begin{aligned} x(t) &= \int_0^L P(s)\, g(x(t-s))\, ds + c \\ &= \int_{t-L}^{t} P(t-u)\, g(x(u))\, du + c \end{aligned} \tag{2.15}$$

Cooke and Yorke obtain certain boundedness results for (2.15) and they pose the problem of determining conditions under which (2.15) has a periodic solution.

Controlling a ship

Minorsky (1962) designed an automatic steering device for the battleship New Mexico. The following is a sketch of the problem see [22].

Let the rudder of the ship have angular position $x(t)$ and suppose there is a friction force proportional to the velocity, say $-cx'(t)$. There is a direction indicating instrument which points in the actual direction of motion and there is an instrument pointing in the desired direction. These two are connected by a device which activates an electric motor producing a certain force to move the rudder so as to bring the ship onto the desired course. There is a time lag of amount $h > 0$ between the time the ship gets off course and the time the electric motor activates the restoring force. The equation for $x(t)$ is

$$x''(t) + cx'(t) + g(x(t-h)) = 0,$$

where $xg(x) > 0$ if $x \neq 0$ and c is a positive constant. The object is to give conditions ensuring that $x(t)$ will stay near zero so that the ship closely follows its proper course.

Biology model

Early in this century A. J. Lotka (1907) formulated basic principles for the mathematical theory of population growth. He assumed that

(a) individuals belong to different classes and the relative proportion of members of each class during any time period is constant,

(b) the life span of an individual in a class is independent of the number of members of that class and independent of the age distribution in the class, and

(c) the life support conditions remain constant.

On the basis of these assumptions Lotka derived the equation

$$x' = B_t - D_t, \tag{2.16}$$

where $x(t)$ is the population, B_t is the number of births per unit time, and D_t the deaths. He generalized the model to the renewal equation (cf. Bellman and Cooke, 1963; Feller, 1941)

$$B(t) = G(t) + \int_0^t B(t-s)\,P(s)\,m(s)\,ds, \tag{2.17}$$

where $B(t)$ is the number of births between 0 and t, G is the number of births at time t to parents surviving from an initial population at $t = 0$, $P(s)$ is the probability density of survival to age at least s, and $m(s)$ is the probability density for a parent of age s giving birth.

Mixing of Liquids

Consider a tank containing B gallons of salt water brine. Fresh water flows in at the top of the tank at a rate of q gallons per minute (see [27]). The brine in the tank is continually stirred, and the mixed solution flows out through a hole at the bottom, also at the rate of q gallons per minute.

Let $x(t)$ be the amount (in pounds) of salt in the brine in the tank at time t. If we assume continual, instantaneous, perfect mixing throughout the tank, then the brine leaving the tank contains $x(t)/B(t)$ lbs. of salt per gallon, and hence

$$x'(t) = -qx(t)/B(t).$$

But, more realistically, let us agree that mixing cannot occur instantaneously throughout the tank. Thus the concentration of the brine leaving the tank at time t will equal the average concentration at some earlier instant, say $t-\tau$. We shall assume that τ is a positive constant, although this assumption may also be subject to improvement. The differential equation for x then becomes a delay differential equation, $x'(t) = -qx(t-\tau)/B(t)$or, setting $c = q/B$,

$$x'(t) = -cx(t-\tau),$$

where τ is the delay or time lag.

2.3.5 Stability of delay differential equations

Suppose that $f : \mathbb{R} \times C \to \mathbb{R}^n$, is continuous and consider the delay differential equation

$$x'(t) = f(t, x_t). \tag{2.18}$$

The function f will be supposed to be completely continuous and to satisfy enough additional smoothness conditions to ensure the solution $x(t, t_0, \psi)$ through (t_0, ψ) is continuous in (t, t_0, ψ) in the domain of definition of the function.

Definition 2.18 ([22]) Suppose that $f(t, 0) = 0$ for all $t \in \mathbb{R}$. The solution $x = 0$ of equation (2.18) is said to be stable if for any $t_0 \in \mathbb{R}$, $\epsilon > 0$, there is a $\delta = \delta(\epsilon, t_0) > 0$ such that $\|\psi\| \leq \delta$ implies $|x(t, t_0, \psi)| \leq \epsilon$ for $t \geq t_0$. The solution $x = 0$ of equation (2.18) is said to be uniformly stable if the number δ in definition is independent of t_0.

Definition 2.19 ([22]) The solution $x = 0$ of equation (2.18) is said to be asymptotically stable if it is stable and there is a $\delta_1 = \delta_1(t_0)$ such that $\|\psi\| \leq \delta_1$ implies that $x(t, t_0, \psi) \to$

0 as $t \to \infty$. The solution $x = 0$ of equation (2.18) is said to be uniformly asymptotically stable if it is uniformly stable and there is $\delta_1 > 0$ such that for every $\eta > 0$ there is a $c(\eta) > 0$ such that $\|\psi\| \leq \delta_1$ implies $|x(t, t_0, \psi)| \leq \eta$ for $t > t_0 + c(\eta)$ for every $t \in \mathbb{R}$.

Lyapunov's direct method has long been viewed the main classical method of studying stability problems in many areas of differential equations. The difficulty of this method is to look for a suitable Lyapunov functional or Lyapunov function.

If $V : \mathbb{R} \times C \to \mathbb{R}$ is continuous and $x(t, t_0, \psi)$ is a solution of (2.18) through (t_0, ψ), we defined

$$V'(t, \psi) = \lim_{h \to 0^+} \sup \frac{1}{h} \left[V(t + h, x_{t+h}(t, \psi)) - V(t, \psi) \right].$$

The function $V'(t, \psi)$ is the upper right-hand derivative of $V(t, \psi)$ along the solution of (2.18).

Theorem 2.10 ([32]) *Suppose $f : \mathbb{R} \times C \to \mathbb{R}^n$ takes $\mathbb{R}\times$(bounded sets of C) into bounded sets of $\mathbb{R}^n$, and u, ν, $w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous non decreasing function, $u(s)$ and $\nu(s)$ are positive for $s > 0$, and $u(0) = \nu(0) = 0$. If there is a continuous function $V : \mathbb{R} \times C \to \mathbb{R}$*

$$u(|\psi(0)|) \leq V(t, \psi) \leq \nu(|\psi|),$$

$$V'(t, \psi) \leq -w(|\psi(0)|),$$

then the solution $x = 0$ of equation (2.18) is uniformly stable. If $w(s) > 0$ for $s > 0$, then the solution $x = 0$ is uniformly asymptotically stable.

Chapter 3

Asymptotic stability in nonlinear neutral Levin-Nohel integro-differential equations

Keywords. Fixed points, neutral integro-differential equations, asymptotic stability.

The goal of this chapter is to present a very recent work published in [18], namely, K. Bessioud, A. Ardjouni,A. Djoudi, Asymptotic stability in nonlinear neutral Levin-Nohel integro-differential equations, J. Nonlinear Funct. Anal. 2017 (2017), Article ID 19, 1–12.

In this chapter we use the contraction mapping theorem to obtain asymptotic stability results about the zero solution for the nonlinear neutral Levin-Nohel integro-differential equation. An asymptotic stability theorem with a necessary and sufficient condition is proved. In addition, the case of the equation with several delays is studied.

3.1 Introduction

In this chapter, we consider the following nonlinear neutral Levin-Nohel integro-differential equation with variable delay

$$\frac{d}{dt}x(t)+\int_{t-\tau(t)}^{t}a(t,s)x(s)ds+\frac{d}{dt}g\left(t,x(t-\tau(t))\right)=0,\ t\geq t_0, \tag{3.1}$$

with an assumed initial condition

$$x\left(t\right)=\phi\left(t\right),\ t\in\left[m\left(t_0\right),t_0\right],$$

where $\phi \in C\left(\left[m\left(t_0\right), t_0\right], \mathbb{R}\right)$ and

$$m\left(t_0\right) = \inf\left\{t - \tau\left(t\right) : t \in \left[t_0, \infty\right)\right\}.$$

Throughout this chapter, we assume that $a \in C\left(\left[t_0, \infty\right) \times \left[m\left(t_0\right), \infty\right), \mathbb{R}\right)$ and $\tau \in C^2\left(\left[t_0, \infty\right), \mathbb{R}^+\right)$ with $t - \tau\left(t\right)$ as $t \to \infty$. The function $g\left(t, x\right)$ is globally Lipschitz continuous in x. That is, there is positive constant E such that

$$\left|g\left(t, x\right) - g\left(t, y\right)\right| \leq E\left\|x - y\right\|, \; g\left(t, 0\right) = 0. \tag{3.2}$$

Our purpose here is to use the contraction mapping theorem to show the asymptotic stability of the zero solution for (3.1). An asymptotic stability theorem with a necessary and sufficient condition is proved. In addition, A study of the general form of (3.1) with several delays is given. In the special case $g\left(t, x\right) = 0$, Dung [29] shows the zero solution of (3.1) is asymptotically stable with a necessary and sufficient condition by using the contraction mapping theorem. The results presented in this chapter extend the main results in [29].

3.2 Asymptotic stability

For each t_0, we denote $C(t_0)$ the space of continuous functions on $[m(t_0), t_0]$ with the supremum norm $\|.\|_{t_0}$. For each $(t_0, \phi) \in [0, \infty) \times C(t_0)$, denoted by $x(t) = x(t, t_0, \phi)$ the unique solution of (3.1).

In order to be able to construct a new fixed mapping, we transform the Levin-Nohel equation into an equivalent equation. For this, we use the variation of parameter formula and the integration by parts.

Lemma 3.1 *x is a solution of equation (3.1) if and only if*

$$\begin{aligned} x(t) = {} & \left(\phi(t_0) + g\left(t_0, \phi(t_0 - \tau(t_0))\right)\right) e^{-\int_{t_0}^{t} A(z)dz} - g\left(t, x(t - \tau(t))\right) \\ & - \int_{t_0}^{t} \left[L_x(s) - A(s) g\left(t, x(s - \tau(s))\right)\right] e^{-\int_{s}^{t} A(z)dz} ds, \; t \geq t_0, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} L_x(t) &= \int_{t-\tau(t)}^{t} a(t,s)\left(\int_s^t \left(\int_{u-\tau(u)}^{u} a(u,v)x(v)dv\right)du\right. \\ &\left. + g\left(t, x(t-\tau(t))\right) - g\left(s, x(s-\tau(s))\right)\right)ds, \end{aligned} \tag{3.4}$$

and

$$A(t) = \int_{t-\tau(t)}^{t} a(t,s)ds. \tag{3.5}$$

Proof. Obviously, we have

$$x(s) = x(t) - \int_s^t \frac{\partial}{\partial u}x(u)du.$$

Inserting this relation into (3.1), we get

$$\frac{d}{dt}x(t) + \int_{t-\tau(t)}^{t} a(t,s)\left(x(t) - \int_s^t \frac{\partial}{\partial u}x(u)du\right)ds + \frac{d}{dt}g\left(t, x(t-\tau(t))\right) = 0,\ t \geq t_0,$$

or equivalently

$$\begin{aligned} &\frac{d}{dt}x(t) + x(t)\int_{t-\tau(t)}^{t} a(t,s)ds - \int_{t-\tau(t)}^{t} a(t,s)\left(\int_s^t \frac{\partial}{\partial u}x(u)du\right)ds \\ &+\frac{d}{dt}g\left(t, x(t-\tau(t))\right) \ = \ 0,\ t \ \geq \ t_0. \end{aligned}$$

After substituting $\frac{\partial x}{\partial u}$ from (3.1), we obtain

$$\begin{aligned} &\frac{d}{dt}x(t) + x(t)\int_{t-\tau(t)}^{t} a(t,s)ds \\ &+ \int_{t-\tau(t)}^{t} a(t,s)\left(\int_s^t \left(\int_{u-\tau(u)}^{u} a(u,v)x(v)dv + \frac{\partial}{\partial u}g\left(u, x(u-\tau(u))\right)\right)du\right)ds \\ &+ \frac{d}{dt}g\left(t, x(t-\tau(t))\right) \ = \ 0,\ t \ \geq \ t_0. \end{aligned} \tag{3.6}$$

By performing the integration, we have

$$\int_s^t \frac{\partial}{\partial u}g\left(u, x(u-\tau(u))\right)du = g\left(t, x(t-\tau(t))\right) - g\left(s, x(s-\tau(s))\right). \tag{3.7}$$

After substituting (3.7) into (3.6), we have

$$\frac{d}{dt}x(t) + A(t)x(t) + L_x(t) + \frac{d}{dt}g\left(t, x(t-\tau(t))\right)) = 0,\ t \geq t_0,$$

where A and L_x are given by (3.5) and (3.4), respectively. By the variation of constants formula, we get

$$x(t) = \phi(t_0)e^{-\int_{t_0}^{t} A(z)dz} - \int_{t_0}^{t} \left[L_x(s) + \frac{\partial}{\partial s} g\left(s, x(s-\tau(s))\right) \right] e^{-\int_s^t A(z)dz} ds, \ t \geq t_0. \quad (3.8)$$

By using the integration by parts, we obtain

$$\begin{aligned}
&\int_{t_0}^{t} \frac{\partial}{\partial s} g\left(s, x(s-\tau(s))\right) e^{-\int_s^t A(z)dz} ds \\
&= g\left(t, x(t-\tau(t))\right) - g\left(t_0, x(t_0-\tau(t_0))\right) e^{-\int_{t_0}^{t} A(z)dz} \\
&- \int_{t_0}^{t} A(s) g\left(s, x(s-\tau(s))\right) e^{-\int_s^t A(z)dz} ds. \quad (3.9)
\end{aligned}$$

Finally, we obtain (3.3) by substituting (3.9) in (3.8). Since each step is reversible, the converse follows easily. This completes the proof. ■

Theorem 3.1 *Let (3.2) holds and suppose that the following two conditions hold:*

$$\liminf_{t \to \infty} \int_0^t A(z)dz > -\infty, \quad (3.10)$$

$$\sup_{t \geq 0} \left(E + \int_0^t \omega(s) e^{-\int_s^t A(z)dz} ds \right) = \alpha < 1, \quad (3.11)$$

where

$$\omega(s) = \int_{s-\tau(s)}^{s} |a(s,w)| \left(\int_w^s \left(\int_{u-\tau(u)}^{u} |a(u,v)| \, dv \right) du + 2E \right) dw + E \, |A(s)| \, .$$

Then the zero solution of (3.1) is asymptotically stable if and only if

$$\int_0^t A(z)dz \to \infty \ \text{as} \ t \to \infty. \quad (3.12)$$

Proof. Sufficient condition. Suppose that (3.12) holds. Denoted by $C\left([m(t_0), \infty), \mathbb{R}\right)$ the space of continuous bounded functions $x : [m(t_0), \infty) \to \mathbb{R}$ such that $x(t) = \phi(t)$, $t \in [m(t_0), t_0]$. It is known that $C\left([m(t_0), \infty), \mathbb{R}\right)$ is a complete metric space endowed with a metric $\|x\| = \sup_{t \geq m(t_0)} |x(t)|$. Define the operator P on $C\left([m(t_0), \infty), \mathbb{R}\right)$ by $(Px)(t) = \phi(t)$, $t \in [m(t_0), t_0]$ and

$$\begin{aligned}
(Px)(t) &= \left(\phi(t_0) + g\left(t_0, \phi(t_0 - \tau(t_0))\right)\right) e^{-\int_{t_0}^{t} A(z)dz} - g\left(t, x(t-\tau(t))\right) \\
&\quad - \int_{t_0}^{t} \left[L_x(s) - A(s) g\left(s, x(s-\tau(s))\right)\right] e^{-\int_s^t A(z)dz} ds, \ t \geq t_0.
\end{aligned}$$

Obviously, Px is continuous for each $x \in C([m(t_0), \infty), \mathbb{R})$. Moreover, it is a contraction operator. Indeed, let $x, y \in C([m(t_0), \infty), \mathbb{R})$

$$\begin{aligned}
&|(Px)(t) - (Py)(t)| \\
&\leq |g(t, x(t-\tau(t))) - g(t, y(t-\tau(t)))| \\
&+ \int_{t_0}^{t} [|L_x(s) - L_y(s)| + |A(s)| |g(s, x(s-\tau(s))) - g(s, y(s-\tau(s)))|] e^{-\int_s^t A(z)dz} ds.
\end{aligned}$$

Since $x(t) = y(t) = \phi(t)$ for all $t \in [m(t_0), t_0]$, this implies that

$$\begin{aligned}
&|L_x(s) - L_y(s)| \\
&\leq \left(\int_{s-\tau(s)}^{s} |a(s,w)| \left(\int_{w}^{s} \left(\int_{u-\tau(u)}^{u} |a(u,v)|\, dv \right) du + 2E \right) dw \right) \|x - y\|.
\end{aligned}$$

Consequently, it holds for all $t \geq t_0$ that

$$\begin{aligned}
&|(Px)(t) - (Py)(t)| \\
&\leq \left[E + \int_{t_0}^{t} \left(\int_{s-\tau(s)}^{s} |a(s,w)| \left(\int_{w}^{s} \left(\int_{u-\tau(u)}^{u} |a(u,v)|\, dv \right) du + 2E \right) dw \right.\right. \\
&\left.\left. + E\, |A(s)| \right) e^{-\int_s^t A(z)dz} ds \right] \|x - y\|.
\end{aligned}$$

Hence, it follows from (3.11) that

$$|(Px)(t) - (Py)(t)| \leq \alpha \|x - y\|, \ t \geq t_0.$$

Thus P is a contraction operator on $C([m(t_0), \infty), \mathbb{R})$.

We now consider a closed subspace S of $C([m(t_0), \infty), \mathbb{R})$ that is defined by

$$S = \{x \in C([m(t_0), \infty), \mathbb{R}) : |x(t)| \to 0 \text{ as } t \to \infty\}.$$

We will show that $P(S) \subset S$. To do this, we need to point out that for each $x \in S$, $|(Px)(t)| \to 0$ as $t \to \infty$. Let $x \in S$, by the definition of P we have

$$\begin{aligned}
(Px)(t) &= (\phi(t_0) + g(t_0, \phi(t_0 - \tau(t_0)))) e^{-\int_{t_0}^t A(z)dz} - g(t, x(t-\tau(t))) \\
&\quad - \int_{t_0}^{t} [L_x(s) - A(s) g(s, x(s-\tau(s)))] e^{-\int_s^t A(z)dz} ds, \\
&= I_1 + I_2 + I_3, \ t \geq t_0.
\end{aligned}$$

The first term I_1 tends to 0 by (3.12) and I_2 tends to 0 by (3.2) and $t-\tau(t)\to\infty$ as $t\to\infty$. For any $T\in(t_0,t)$, we have the following estimate for the third term

$$
\begin{aligned}
I_3 \leq & \left|\int_{t_0}^{T}\left[L_x(s)-A(s)g\left(s,x(s-\tau(s))\right)\right]e^{-\int_s^t A(z)dz}ds\right| \\
&+\left|\int_{T}^{t}\left[L_x(s)-A(s)g\left(s,x(s-\tau(s))\right)\right]e^{-\int_s^t A(z)dz}ds\right| \\
&\leq \int_{t_0}^{T}\left(\int_{s-\tau(s)}^{s}|a(s,w)|\left(\int_{w}^{s}\left(\int_{u-\tau(u)}^{u}|a(u,v)|\,\|x\|\,dv\right)du\right.\right. \\
&\left.\left.+2E\left\|\phi\right\|_{t_0}\right)dw+E\left|A(s)\right|\left\|\phi\right\|_{t_0}\right)e^{-\int_s^t A(z)dz}ds \\
&+\int_{T}^{t}\left(\int_{s-\tau(s)}^{s}|a(s,w)|\left(\int_{w}^{s}\left(\int_{u-\tau(u)}^{u}|a(u,v)|\,|x(v)|\,dv\right)du\right.\right. \\
&\left.\left.+E\left|x(s-\tau(s))\right|+E\left|x(w-\tau(w))\right|\right)dw+E\left|A(s)\right|\left|x(s-\tau(s))\right|\right)e^{-\int_s^t A(z)dz}ds \\
&\leq \left[\int_{t_0}^{T}\left(\int_{s-\tau(s)}^{s}|a(s,w)|\left(\int_{w}^{s}\left(\int_{u-\tau(u)}^{u}|a(u,v)|\,dv\right)du+2E\right)dw\right.\right. \\
&\left.\left.+E\left|A(s)\right|\right)e^{-\int_s^t A(z)dz}ds\right]\left(\|x\|+\|\phi\|_{t_0}\right) \\
&+\int_{T}^{t}\left(\int_{s-\tau(s)}^{s}|a(s,w)|\left(\int_{w}^{s}\left(\int_{u-\tau(u)}^{u}|a(u,v)|\,|x(v)|\,dv\right)du\right.\right. \\
&\left.\left.+E\left|x(s-\tau(s))\right|+E\left|x(w-\tau(w))\right|\right)dw+E\left|A(s)\right|\left|x(s-\tau(s))\right|\right)e^{-\int_s^t A(z)dz}ds \\
&= I_{31}+I_{32}.
\end{aligned}
$$

Since $t-\tau(t)\to\infty$ as $t\to\infty$, this implies that $u-\tau(u)\to\infty$ as $T\to\infty$. Thus, from the fact $|x(v)|\to 0,\ v\to\infty$ we can infer that for any $\varepsilon>0$ there exists $T_1=T>t_0$ such that

$$
\begin{aligned}
I_{32} \quad < \quad & \frac{\varepsilon}{2\alpha}\int_{T_1}^{t}\left(\int_{s-\tau(s)}^{s}|a(s,w)|\left(\int_{w}^{s}\left(\int_{u-\tau(u)}^{u}|a(u,v)|\,dv\right)du++2E\right)dw\right. \\
&\left.+E\left|A(s)\right|\right)e^{-\int_s^t A(z)dz}ds,
\end{aligned}
$$

and hence, $I_{32}<\frac{\varepsilon}{2}$ for all $t\geq T_1$. On the other hand, $\|x\|<\infty$ because $x\in S$. This combined with (3.12) yields $I_{31}\to 0$ as $t\to\infty$. As a consequence, there exists $T_2\geq T_1$ such that $I_{31}<\frac{\varepsilon}{2}$ for all $t\geq T_2$. Thus, $I_3<\varepsilon$ for all $t\geq T_2$, that is, $I_3\to 0$ as $t\to\infty$. So $P(S)\subset S$.

By the Contraction Mapping Principle, P has a unique fixed point x in S which is a solution of (3.1) with $x(t)=\phi(t)$ on $[m(t_0),t_0]$ and $x(t)=x(t,t_0,\phi)\to 0$ as $t\to\infty$.

To obtain the asymptotic stability, we need to show that the zero solution of (3.1) is stable. By condition (3.10), we can define

$$K = \sup_{t \geq 0} e^{-\int_0^t A(z)dz} < \infty. \tag{3.13}$$

Using the formula (3.3) and condition (3.11), we can obtain

$$|x(t)| \leq K\,(1+E)\,\|\phi\|_{t_0}\, e^{\int_0^{t_0} A(z)dz} + \alpha(\|x\| + \|\phi\|_{t_0}),\ t \geq t_0,$$

which leads us to

$$\|x\| \leq \frac{K\,(1+E)\,e^{\int_0^{t_0} A(z)dz} + \alpha}{1-\alpha}\,\|\phi\|_{t_0}\,. \tag{3.14}$$

Thus for every, $\varepsilon > 0$, we can find $\delta > 0$ such that $\|\phi\|_{t_0} < \delta$ implies that $\|x\| < \varepsilon$. This shows that the zero solution of (3.1) is stable and hence, it is asymptotically stable.

Necessary condition. Suppose that the zero solution of (3.1) is asymptotically stable and that the condition (3.12) fails. It follows from (3.10) that there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \int_0^{t_n} A(z)dz \text{ exists and is finite.}$$

Hence, we can choose a positive constant L satisfying

$$-L \leq \lim_{n\to\infty} \int_0^{t_n} A(z)dz \leq L,\ \forall n \geq 1. \tag{3.15}$$

Then, condition (3.11) gives us

$$c_n = \int_0^{t_n} \omega(s) e^{\int_0^s A(z)dz} ds \leq \alpha e^{\int_0^{t_n} A(z)dz} < e^L.$$

The sequence $\{c_n\}$ is increasing and bounded, so it has a finite limit. For any $\delta_0 > 0$, there exists $n_0 > 0$ such that

$$\int_{t_{n_0}}^{t_n} \omega(s) e^{\int_0^s A(z)dz} ds < \frac{\delta_0}{2K},\ \forall n \geq n_0, \tag{3.16}$$

where K is as in (3.13). We choose δ_0 such that $\delta_0 < \frac{1-\alpha}{K(1+E)e^L+1}$ and consider the solution $x(t) = x(t, t_n, \phi)$ of (3.1) with the initial data $\phi(t_{n_0}) = \delta_0$ and $|\phi(s)| \leq \delta_0,\ s \leq t_{n_0}$. It follows from (3.14) that

$$|x(t)| \leq 1 - \delta_0,\ \forall t \geq t_{n_0}. \tag{3.17}$$

Applying the fundamental inequality $|a-b| \geq |a| - |b|$ and then using (3.17), (3.16) and (3.15), we get

$$
\begin{aligned}
& |x(t_n) + g\left(t_n, x(t_n - \tau(t_n))\right)| \\
& \geq \delta_0 e^{-\int_{t_{n_0}}^{t_n} A(z)dz} - \int_{t_{n_0}}^{t_n} \omega(s) e^{-\int_s^{t_n} A(z)dz} ds \\
& \geq e^{-\int_{t_{n_0}}^{t_n} A(z)dz} \left(\delta_0 - e^{-\int_0^{t_{n_0}} A(z)dz} \int_{t_{n_0}}^{t_n} \omega(s) e^{\int_0^s A(z)dz} ds \right) \\
& \geq e^{-\int_{t_{n_0}}^{t_n} A(z)dz} \left(\delta_0 - K \int_{t_{n_0}}^{t_n} \omega(s) e^{\int_0^s A(z)dz} ds \right) \\
& \geq \frac{1}{2} \delta_0 e^{-\int_{t_{n_0}}^{t_n} A(z)dz} \geq \frac{1}{2} \delta_0 e^{-2L} > 0,
\end{aligned}
$$

which is a contradiction because $x(t_n) + g\left(t_n, x(t_n - \tau(t_n))\right) \to 0$ as $t_n \to \infty$. The proof is complete. ■

Let $g(t,x) = 0$ we get the following corollary.

Corollary 3.1 *Suppose that the following two conditions hold*

$$
\liminf_{t \to \infty} \int_0^t A_0(z)dz > -\infty, \tag{3.18}
$$

$$
\sup_{t \geq 0} \int_0^t \left(\int_{s-\tau(s)}^s |a(s,w)| \int_w^s \int_{u-\tau(u)}^u |a(u,v)| \, dv du dw \right) e^{-\int_s^t A_0(z)dz} ds = \alpha < 1, \tag{3.19}
$$

where

$$
A_0(z) = \int_{z-\tau(z)}^z a(z,s)ds.
$$

Then the zero solution of

$$
\frac{d}{dt} x(t) + \int_{t-\tau(t)}^t a(t,s)x(s)ds = 0,
$$

is asymptotically stable if and only if

$$
\int_0^t A_0(z)dz \to \infty \text{ as } t \to \infty. \tag{3.20}
$$

Next we turn our attention to the following neutral Levin-Nohel integro-differential equations with several delays

$$
\frac{d}{dt} x(t) + \sum_{k=1}^{M} \int_{t-\tau_k(t)}^t a_k(t,s)x(s)ds + \sum_{k=1}^{M} \frac{d}{dt} g_k\left(t, x(t - \tau_k(t))\right) = 0, \ t \geq t_0, \tag{3.21}
$$

where $a_k \in C([t_0,\infty) \times [m(t_0),\infty), \mathbb{R})$ and $\tau_k \in C^2([t_0,\infty), \mathbb{R}^+)$ with $t-\tau_k(t)$ as $t \to \infty$, $1 \le k \le M$. The function $g_k(t,x)$ is globally Lipschitz continuous in x. That is, there is positive constant E_k such that

$$|g_k(t,x) - g_k(t,y)| \le E_k \|x-y\|, \; g_k(t,0) = 0, \; 1 \le k \le M. \tag{3.22}$$

Lemma 3.2 *x is a solution of equation (3.21) if and only if*

$$x(t) = \left(\phi(t_0) + \sum_{k=1}^{M} g_k(t_0, \phi(t_0 - \tau_k(t_0)))\right) e^{-\int_{t_0}^{t} \overline{A}(z)dz} - \sum_{k=1}^{M} g_k(t, x(t-\tau_k(t)))$$
$$- \int_{t_0}^{t} \left[\overline{L}_x(s) - \sum_{k=1}^{M} \overline{A}(s) g_k(t, x(s-\tau_k(s)))\right] e^{-\int_s^t \overline{A}(z)dz} ds, \; t \ge t_0,$$

where

$$\overline{L}_x(t) = \sum_{k=1}^{M} \int_{t-\tau_k(t)}^{t} a_k(t,s) \left(\int_s^t \left(\sum_{i=1}^{M} \int_{u-\tau_i(u)}^{u} a_i(u,v)x(v)dv\right) du\right.$$
$$\left. + \sum_{i=1}^{M} g_i(t, x(t-\tau_i(t))) - \sum_{i=1}^{M} g_i(s, x(s-\tau_i(s)))\right) ds,$$

and

$$\overline{A}(t) = \sum_{k=1}^{M} \int_{t-\tau_k(t)}^{t} a_k(t,s)ds.$$

The proof follows along the lines of Lemma 3.1, and hence we omit it.

Theorem 3.2 *Let (3.22) holds and Suppose that the following two conditions hold*

$$\liminf_{t\to\infty} \int_0^t \overline{A}(z)dz > -\infty,$$

and

$$\sup_{t \ge 0} \left(\sum_{k=1}^{M} E_k + \int_0^t \overline{\omega}(s) e^{-\int_s^t \overline{A}(z)dz} ds\right) = \alpha < 1,$$

where

$$\overline{\omega}(s) = \sum_{k=1}^{M} \int_{s-\tau_k(s)}^{s} |a_k(s,w)| \left(\int_w^s \left(\sum_{i=1}^{M} \int_{u-\tau_i(u)}^{u} |a_i(u,v)| dv\right) du\right.$$
$$\left. + 2\sum_{k=1}^{M} E_k\right) dw + \sum_{k=1}^{M} E_k \left|\overline{A}(s)\right|.$$

Then the zero solution of (3.21) is asymptotically stable if and only if

$$\int_0^t \overline{A}(z)dz \to \infty \text{ as } t \to \infty.$$

The proof is similar to that of Theorem 3.1, and hence, we omit it.

Chapter 4

Periodicity and positivity in neutral nonlinear Levin-Nohel integro-differential equations

Keywords. Fixed points, periodicity, positivity, Levin-Nohel integro-differential equations, functional delay.

The goal of this chapter is to present a very recent work [19], namely,
K. Bessioud, A. Ardjouni and A. Djoudi, Periodicity and positivity in neutral nonlinear Levin-Nohel integro-differential equations, Submitted.

Our chapter deals with the neutral nonlinear Levin-Nohel integro-differential with variable delay. By using Krasnoselskii's fixed point theorem we obtain the existence of periodic and positive periodic solutions and by contraction mapping principle we obtain the existence of a unique periodic solution. An example is given to illustrate this work.

4.1 Introduction

In this chapter, we consider the following neutral nonlinear Levin-Nohel integro-differential equation with variable delay

$$\frac{d}{dt}x(t) + \int_{t-\tau(t)}^{t} a(t,s)\,x(s)\,ds + \frac{d}{dt}g(t, x(t-\tau(t))) = 0, \tag{4.1}$$

where a, τ and g are continuous functions with $\tau(t) > 0$. Equation (4.1) has a long history and the simpler form of it was considered in 1928 by Volterra with a biological application in mind (see [17]). In [18], the authors used the contraction mapping principle to show the asymptotic stability of the zero solution for (4.1). The purpose of this chapter is to transform (4.1) into an integral equation and then use the Krasnoselskii's fixed point theorem to show the existence of periodic and positive periodic solutions. The obtained integral equation is the sum of two mappings; one is a contraction and the other is compact. Also by employing the contraction mapping principle, the existence of a unique periodic solution has been established. An example is also given to illustrate this work.

4.2 Existence and uniqueness of periodic solutions

For $T>0$ let P_T be the set of all continuous scalar functions x, periodic in t of period T. Then $(P_T, \|.\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)| .$$

Since we are searching for the existence of periodic solutions for (4.1), it is natural to assume that

$$a(t+T, s+T) = a(t,s), \ \tau(t+T) = \tau(t), \tag{4.2}$$

with τ being scalar, continuous and $\tau(t) \geq \tau^* > 0$. Also, we assume

$$\int_0^T A(z)\, dz > 0, \ A(t) = \int_{t-\tau(t)}^t a(t,s)\, ds. \tag{4.3}$$

The function $g(t,x)$ is periodic in t of period T, it is also globally Lipschitz continuous in x. That is

$$g(t+T, x) = g(t,x), \tag{4.4}$$

and there is positive constant E such that

$$|g(t,x) - g(t,y)| \leq E \|x-y\| . \tag{4.5}$$

The next lemma is crucial to our results.

Lemma 4.1 *Suppose (4.2)–(4.4) hold. If $x \in P_T$, then x is a solution of equation (4.1) if and only if*

$$\begin{aligned} x(t) &= -g(t, x(t-\tau(t))) \\ &\quad - \left(1 - e^{-\int_{t-T}^{t} A(z)dz}\right)^{-1} \int_{t-T}^{t} \left[L_x(s) - A(s)\, g(s, x(s-\tau(s)))\right] e^{-\int_s^t A(z)dz} ds, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} L_x(t) = \int_{t-\tau(t)}^{t} a(t,s) \bigg(\int_s^t \bigg(\int_{u-\tau(u)}^{u} a(u,\nu)\, x(\nu)\, d\nu \bigg) du \\ + g(t, x(t-\tau(t))) - g(s, x(s-\tau(s))) \bigg) ds. \end{aligned} \tag{4.7}$$

Proof. Obviously, we have

$$x(s) = x(t) - \int_s^t \frac{\partial}{\partial u} x(u)\, du.$$

Inserting this relation into (4.1), we get

$$\frac{d}{dt} x(t) + \int_{t-\tau(t)}^{t} a(t,s) \left(x(t) - \int_s^t \frac{\partial}{\partial u} x(u)\, du \right) ds + \frac{d}{dt} g(t, x(t-\tau(t))) = 0,$$

or equivalently

$$\begin{aligned} &\frac{d}{dt} x(t) + x(t) \int_{t-\tau(t)}^{t} a(t,s)\, ds - \int_{t-\tau(t)}^{t} a(t,s) \left(\int_s^t \frac{\partial}{\partial u} x(u)\, du \right) ds \\ &+ \frac{d}{dt} g(t, x(t-\tau(t))) \ = \ 0. \end{aligned}$$

After substituting $\frac{\partial x}{\partial u}$ from (4.1), we obtain

$$\begin{aligned} &\frac{d}{dt} x(t) + x(t) \int_{t-\tau(t)}^{t} a(t,s)\, ds + \frac{d}{dt} g(t, x(t-\tau(t))) \\ &+ \int_{t-\tau(t)}^{t} a(t,s) \left(\int_s^t \left(\int_{u-\tau(u)}^{u} a(u,\nu)\, x(\nu)\, d\nu + \frac{\partial}{\partial u} g(u, x(u-\tau(u))) \right) du \right) ds \ = \ 0. \end{aligned} \tag{4.8}$$

By performing the integration, we have

$$\int_s^t \frac{\partial}{\partial u} g(u, x(u-\tau(u)))\, du = g(t, x(t-\tau(t))) - g(s, x(s-\tau(s))). \tag{4.9}$$

Substituting (4.9) into (4.8), we have

$$\frac{d}{dt}x(t) + A(t)x(t) + L_x(t) + \frac{d}{dt}g(t, x(t-\tau(t))) = 0,$$

where A and L_x are given by (4.3) and (4.7), respectively. We rewrite this equation as

$$\begin{aligned}&\frac{d}{dt}\{x(t) + g(t, x(t-\tau(t)))\}\\&= -A(t)(x(t) + g(t, x(t-\tau(t)))) + A(t)g(t, x(t-\tau(t))) - L_x(t). \qquad (4.10)\end{aligned}$$

Multiply both sides of (4.10) with $e^{\int_0^t A(z)dz}$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned}&\int_{t-T}^{t} \frac{d}{ds}\left[(x(s) + g(s, x(s-\tau(s))))e^{\int_0^s A(z)dz}\right] ds\\&= -\int_{t-T}^{t} [L_x(s) - A(s)g(s, x(s-\tau(s)))] e^{\int_0^s A(z)dz} ds.\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}&(x(t) + g(t, x(t-\tau(t))))e^{\int_0^t A(z)dz}\\&-(x(t-T) + g(t-T, x(t-T-\tau(t-T))))e^{\int_0^{t-T} A(z)dz}\\&= -\int_{t-T}^{t} [L_x(s) - A(s)g(s, x(s-\tau(s)))] e^{\int_0^s A(z)dz} ds.\end{aligned}$$

Dividing both sides of the above equation by $e^{\int_0^t A(z)dz}$ and using the fact that $x(t-T) = x(t)$, we obtain

$$\begin{aligned}&x(t) + g(t, x(t-\tau(t)))\\&= -\left(1 - e^{-\int_{t-T}^t A(z)dz}\right)^{-1} \int_{t-T}^{t} [L_x(s) - A(s)g(s, x(s-\tau(s)))] e^{-\int_s^t A(z)dz} ds.\end{aligned}$$

Since each step is reversible, the converse follows easily. This completes the proof. ■

Define a mapping H by

$$\begin{aligned}(H\varphi)(t) &= -g(t, \varphi(t-\tau(t)))\\&\quad-\left(1 - e^{-\int_{t-T}^t A(z)dz}\right)^{-1} \int_{t-T}^{t} [L_\varphi(s) - A(s)g(s, \varphi(s-\tau(s)))] e^{-\int_s^t A(z)dz} ds.\end{aligned} \qquad (4.11)$$

4.2. Existence and uniqueness of periodic solutions

It is clear form (4.11) that $H : P_T \to P_T$ by the way it was constructed in Lemma 4.1.

We note that to apply the Krasnoselskii's fixed point theorem we need to construct two mappings; one is contraction and the other is compact. Therefore, we express (4.11) as

$$(H\varphi)(t) = (B\varphi)(t) + (C\varphi)(t),$$

where $C, B : P_T \to P_T$ are given by

$$(B\varphi)(t) = -g(t, \varphi(t - \tau(t))), \tag{4.12}$$

and

$$(C\varphi)(t) = -\left(1 - e^{-\int_{t-T}^{t} A(z)dz}\right)^{-1} \int_{t-T}^{t} \left[L_\varphi(s) - A(s) g(s, \varphi(s - \tau(s)))\right] e^{-\int_s^t A(z)dz} ds. \tag{4.13}$$

To simplify notations, we introduce the following constants

$$\begin{aligned} \eta &= \left(1 - e^{-\int_{t-T}^{t} A(z)dz}\right)^{-1}, \ \rho = \sup_{s\in[t-T,t]} \left(\int_{s-\tau(s)}^{s} |a(s,w)|\, dw\right), \\ \gamma &= \sup_{s\in[t-T,t]} e^{-\int_s^t A(z)dz}, \ \delta = \sup_{s\in[t-T,t]} \left(\sup_{w\in[t-T,t]} \int_w^s \left(\int_{u-\tau(u)}^{u} |a(u,\nu)|\, d\nu\right) du\right). \end{aligned} \tag{4.14}$$

Lemma 4.2 *Let C be given in (4.13). Suppose that (4.2)–(4.4) hold. Then $C : P_T \to P_T$ is continuous and the image of C contained in a compact set.*

Proof. To see that C is continuous, we let $\varphi, \psi \in P_T$. Given $\epsilon > 0$, take $\beta = \frac{\epsilon}{N}$ with $N = \eta\gamma T\rho(\delta + 3E)$ where E is given by (4.5). Now, for $\|\varphi - \psi\| < \beta$, we obtain

$$\|C\varphi - C\psi\| \le \eta\gamma \int_{t-T}^{t} \left[\rho\delta \|\varphi - \psi\| + 3\rho E \|\varphi - \psi\|\right] ds \le N \|\varphi - \psi\| < \epsilon.$$

This proves that C is continuous. To show that the image of C is contained in a compact set, we consider $D = \{\varphi \in P_T : \|\varphi\| \le R\}$, where R is a fixed positive constant. Let $\varphi_n \in D$ where n is a positive integer. Observe that in view of (4.5) we have

$$\begin{aligned} |g(t,x)| &= |g(t,x) - g(t,0) + g(t,0)| \\ &\le |g(t,x) - g(t,0)| + |g(t,0)| \\ &\le E\|x\| + \alpha, \end{aligned}$$

where $\alpha = \sup_{t\in[0,T]} |g(t,0)|$. Consequently

$$\begin{aligned} |(C\varphi_n)(t)| &\leq \eta\gamma \int_{t-T}^{t} [\rho(\delta + 2(ER+\alpha)) + \rho(ER+\alpha)]\, ds \\ &\leq \eta\gamma T [\rho(\delta + 3(ER+\alpha))] = L. \end{aligned}$$

Next we calculate $(C\varphi_n)'(t)$ and show that it is uniformly bounded. By making use of (4.2)–(4.4) we obtain by taking the derivative in (4.13) that

$$(C\varphi_n)'(t) = -A(t)(C\varphi_n)(t) - L_{\varphi_n}(t) + A(t)\, g(t, \varphi_n(t-\tau(t))).$$

Thus, the above expression yields $\left\|(C\varphi_n)'\right\| \leq F$, for some positive constant F. Thus the sequence $(C\varphi_n)$ is uniformly bounded and equicontinuous. Hence by Ascoli-Arzela's theorem $C(D)$ is compact. ■

Lemma 4.3 *If B is given by (4.12) with $E < 1$, then $B : P_T \to P_T$ is a contraction.*

Proof. Let B be defined by (4.12). Then for $\varphi, \psi \in P_T$ we have

$$\begin{aligned} \|B\varphi - B\psi\| &= \sup_{t\in[0,T]} |(B\varphi)(t) - (B\psi)(t)| \\ &\leq E \sup_{t\in[0,T]} |\varphi(t-\tau(t)) - \psi(t-\tau(t))| \\ &\leq E\|\varphi - \psi\|. \end{aligned}$$

Hence B defines a contraction. ■

Theorem 4.1 *Suppose the hypothesis of Lemma 4.3. Let $\alpha = \sup_{t\in[0,T]} |g(t,0)|$. Suppose (4.2)–(4.5) hold. Let J be a positive constant satisfying the inequality*

$$EJ + \alpha + \eta\gamma T(\rho(\delta J + 3(EJ+\alpha))) \leq J.$$

Let $M = \{\varphi \in P_T : \|\varphi\| \leq J\}$. Then equation (4.1) has a solution in M.

Proof. Define $M = \{\varphi \in P_T : \|\varphi\| \leq J\}$. By Lemma 4.2, $C : M \to P_T$ is continuous and $C(M)$ is contained in a compact set. Also, from Lemma 4.3, the mapping $B : M \to P_T$ is a contraction. Next, we show that if $\varphi, \psi \in M$, we have $\|C\varphi + B\psi\| \leq J$. Let $\varphi, \psi \in M$

with $\|\varphi\|, \|\psi\| \leq J$. Then

$$\begin{aligned}
&\|C\varphi + B\psi\| \\
&\leq E\|\psi\| + \alpha + \eta\gamma \int_{t-T}^{t} \left[\rho\left(\delta\|\varphi\| + 2\left(E\|\varphi\| + \alpha\right)\right) + \rho\left(E\|\varphi\| + \alpha\right)\right] ds \\
&\leq EJ + \alpha + \eta\gamma T\left(\rho\left(\delta J + 3\left(EJ + \alpha\right)\right)\right) \leq J.
\end{aligned}$$

We now see that all the conditions of Krasnoselskii's theorem are satisfied. Thus there exists a fixed point z in M such that $z = Cz + Bz$. By Lemma 4.1, this fixed point is a solution of (4.1). Hence (4.1) has a T-periodic solution. ■

Theorem 4.2 *Suppose (4.2)–(4.5) hold. If*

$$E + \eta\gamma T\rho\left(\delta + 3E\right) < 1,$$

then equation (4.1) has a unique T-periodic solution.

Proof. Let the mapping H be given by (4.11). For $\varphi, \psi \in P_T$, in view of (4.11), we have

$$\|H\varphi - H\psi\| \leq \left(E + \eta\gamma T\rho\left(\delta + 3E\right)\right)\|\varphi - \psi\|.$$

This completes the proof by invoking the contraction mapping principle. ■

Corollary 4.1 *Suppose (4.2)–(4.4) hold and let α be the constant defined in Theorem 4.1. Let J be a positive constant and define $M = \{\varphi \in P_T : \|\varphi\| \leq J\}$. Suppose there is positive constant E^* so that for $x, y \in M$ we have*

$$|g(t, x) - g(t, y)| \leq E^* \|x - y\|.$$

If $E^* < 1$ and $\|H\varphi\| \leq J$ for $\varphi \in M$, then (4.1) has a T-periodic solution in M. Moreover, if

$$E^* + \eta\gamma T\rho\left(\delta + 3E^*\right) < 1,$$

then (4.1) has a unique T-periodic solution in M.

Proof. Let $M = \{\varphi \in P_T : \|\varphi\| \leq J\}$. Let the mapping H be given by (4.11). Then, the results follow immediately from Theorem 4.1 and Theorem 4.2 ■

Example 4.1 For small positive ϵ_1 and ϵ_2 we consider the nonlinear neutral integro-differential equation with variable delay

$$\frac{d}{dt}x(t) + \epsilon_1 \int_{t-\frac{\pi}{\omega}}^{t} (1+\cos\omega(t-s))\, x(s)\, ds + \epsilon_2 \frac{d}{dt}\left(\cos(\omega t)\, x^3\left(t-\frac{\pi}{\omega}\right)\right) = 0, \quad (4.15)$$

where ω is a positive constant. So, we have $a(t,s) = \epsilon_1(1+\cos\omega(t-s))$, $\tau(t) = \frac{\pi}{\omega}$, $g(t, x(t-\tau(t))) = \epsilon_2\left(\cos(\omega t)\, x^3\left(t-\frac{\pi}{\omega}\right)+1\right)$. Define $M = \left\{\varphi \in P_{\frac{2\pi}{\omega}} : \|\varphi\| \leq J\right\}$, where J is a positive constant. For $\varphi \in M$, we have

$$\begin{aligned}
&|(H\varphi)(t)| \\
&= \Big|-g(t,\varphi(t-\tau(t))) \\
&- \left(1-e^{-\int_{t-T}^{t} A(z)dz}\right)^{-1} \int_{t-T}^{t} \left[L_\varphi(s) - A(s)\, g(s,\varphi(s-\tau(s)))\right] e^{-\int_s^t A(z)dz} ds\Big| \\
&\leq \epsilon_2 J^3 + \epsilon_2 + \left(1-e^{\frac{-2\epsilon_1\pi^2}{\omega^2}}\right)^{-1} \frac{4\pi^2}{\omega^2}\epsilon_1 \left(\frac{4\epsilon_1\pi^2}{\omega^2}J + 3\epsilon_2 J^3 + 3\epsilon_2\right).
\end{aligned}$$

Thus, the inequality

$$\epsilon_2 J^3 + \epsilon_2 + \left(1-e^{\frac{-2\pi^2\epsilon_1}{\omega^2}}\right)^{-1} \frac{4\pi^2}{\omega^2}\epsilon_1 \left(\frac{4\epsilon_1\pi^2}{\omega^2}J + 3\epsilon_2 J^3 + 3\epsilon_2\right) \leq J, \quad (4.16)$$

which is satisfied for small ω, ϵ_1 and ϵ_2, implies $\|H\varphi\| \leq J$. Hence, (4.15) has a $\frac{2\pi}{\omega}$-periodic solution, by Corollary 4.1.

For the uniqueness of the solution we let $\varphi, \psi \in M$. From (4.15) we see that

$$\eta = \left(1-e^{\frac{-2\pi^2\epsilon_1}{\omega^2}}\right)^{-1}, \ \rho = \frac{2\pi}{\omega}\epsilon_1, \ \gamma \leq 1.$$

Also $\alpha = \epsilon_2$, $E = 3\epsilon_2 J^2$, where J is given by (4.16). If

$$3\epsilon_2 J^2 + \frac{4\pi^2\epsilon_1}{\omega^2}\left(1-e^{\frac{-2\pi^2\epsilon_1}{\omega^2}}\right)^{-1}\left[\frac{4\epsilon_1\pi^2}{\omega^2} + 9\epsilon_2 J^2\right] < 1,$$

is satisfied for small ϵ_1 and ϵ_2, then (4.15) has a unique $\frac{2\pi}{\omega}$-periodic solution, by Corollary 4.1.

4.3 Existence of positive periodic solutions

For some non-negative constant L and a positive constant K, we define the set

$$\mathcal{M} = \{\varphi \in P_T : L \leq \varphi \leq K\},$$

which is a closed convex and bounded subset of the Banach space P_T.

In this section we obtain the existence of a positive periodic solution of (4.1) by considering the two cases; $g(t,x) \geq 0$ and $g(t,x) \leq 0$ for all $t \in \mathbb{R}$, $x \in \mathcal{M}$. To simplify notation, we let

$$\theta = \max_{s\in[t-T,t]} e^{-\int_s^t A(z)dz},\ \sigma = \min_{s\in[t-T,t]} e^{-\int_s^t A(z)dz}.$$

In the case $g(t,x) \leq 0$, we assume that there exist a non-negative constant k_1 and a positive constant k_2 such that

$$k_1 x \leq -g(t,x) \leq k_2 x,\ \text{for all } t \in [0,T],\ x \in \mathcal{M}, \tag{4.17}$$

$$k_2 < 1, \tag{4.18}$$

and for all $t \in [0,T]$, $x \in \mathcal{M}$

$$\frac{L(1-k_1)}{\eta\sigma T} \leq -L_x(t) + A(t)\,g(t,x) \leq \frac{K(1-k_2)}{\eta\theta T}. \tag{4.19}$$

Theorem 4.3 *Suppose (4.2)–(4.5) and (4.17)–(4.19) hold. Then equation (4.1) has a positive T-periodic solution x in the subset $\mathcal{M}$.*

Proof. By Lemma 4.1 x is a solution of (4.1) if

$$x = Cx + Bx,$$

where C and B are given by (4.13), (4.12) respectively. By Lemma 4.2, $C : \mathcal{M} \to P_T$ is continuous and compact. Also, from Lemma 4.3, the mapping $B : \mathcal{M} \to P_T$ is a contraction. We just need to show that condition (i) of Theorem 2.5 is satisfied. Toward this, let $\varphi, \psi \in \mathcal{M}$, then

$$\begin{aligned}
&(B\psi)(t) + (C\varphi)(t) \\
&= -g(t,\psi(t-\tau(t))) - \eta\int_{t-T}^{t} [L_\varphi(s) - A(s)\,g(s,\varphi(s-\tau(s)))]\,e^{-\int_s^t A(z)dz}ds \\
&\leq k_2 K + \eta\theta T\frac{K(1-k_2)}{\eta\theta T} \leq K.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(B\psi)(t) + (C\varphi)(t) \\
&= -g(t, \psi(t-\tau(t))) - \eta \int_{t-T}^{t} \left[L_{\varphi}(s) - A(s)\, g(s, \varphi(s-\tau(s)))\right] e^{-\int_s^t A(z)dz} ds \\
&\geq k_1 L + \eta\sigma T \frac{L(1-k_1)}{\eta\sigma T} \geq L.
\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x \in \mathcal{M}$ such that $x = Bx + Cx$. By Lemma 4.1 this fixed point is a solution of (4.1) and the proof is complete. ■

In the case $g(t,x) \geq 0$, we substitute conditions (4.17)-(4.19) with the following conditions respectively. We assume that there exist a negative constant k_3 and a non-positive constant k_4 such that

$$k_3 x \leq -g(t,x) \leq k_4 x, \text{ for all } t \in [0,T], \ x \in \mathcal{M}, \tag{4.20}$$

$$-k_3 < 1, \tag{4.21}$$

and for all $t \in [0,T]$, $x \in \mathcal{M}$

$$\frac{L - k_3 K}{\eta\sigma T} \leq -L_x(t) + A(t)\, g(t,x) \leq \frac{K - k_4 L}{\eta\theta T}. \tag{4.22}$$

Theorem 4.4 *Suppose (4.2)–(4.5) and (4.20)–(4.22) hold. Then equation (4.1) has a positive T-periodic solution x in the subset $\mathcal{M}$.*

The proof follows along the lines of Theorem 4.3, and hence we omit it.

Chapter 5

Positive periodic solutions of second-order systems with singularities and deviating arguments

Keywords. Positive periodic solutions, Second-order systems, Singularity, Deviating arguments, Schauder's fixed point theorem.

The goal of this chapter is to present a very recent work published in [20], namely, K. Bessioud, A. Ardjouni and A. Djoudi, Positive periodic solutions of second-order systems with singularities and deviating arguments, Communications in Applied Analysis 20 (2016), 223–237.

In this chapter, we use Schauder's fixed point theorem to prove that the second-order systems with singularities and deviating arguments has a positive periodic solution. An example is also given to illustrate our work.

5.1 Introduction

Due to their important in numerous applications, for example, physics, population dynamics, industrial robotics, and other areas, many authors are studying the existence of positive periodic solutions for second-order differential equations with singularities; see [25], [30], [31], [35], [39], [47]–[49] and references therein.

The study of singular problems began with the paper of Taliaferro. In 1979, Taliaferro [47] discussed a model equation with singularity

$$\begin{aligned} x''(t) + \frac{q(t)}{x^{\lambda}(t)} &= 0, \ 0 < t < 1, \\ x(0) &= x(1) = 0, \end{aligned}$$

and obtained the existence of solution for the problem. Here, $\lambda > 0$, $q \in C((0,1))$ with $q > 0$ on $(0,1)$ and $\int_0^1 t(1-t)\, q(t)\, dt < \infty$. We call this equation with a strong force condition if $\lambda \geq 1$ and with a weak force condition if $0 < \lambda < 1$.

In 1987, Lazer and Solimini [35] proved that a necessary and sufficient condition for the existence of a positive periodic solution of the problem

$$\begin{aligned} x''(t) &= \frac{1}{x^{\lambda}(t)} + c(t), \\ x(0) &= x(T), \end{aligned}$$

is that the mean value of c is negative, i.e.,

$$\bar{c} = \frac{1}{T}\int_0^T c(t)\, dt < 0,$$

for $\lambda \geq 1$. Moreover, if $0 < \lambda < 1$, they found examples of functions c with negative mean values and such that periodic solutions do not exist.

Recently, Ma, Chen and He [39] discussed the existence of positive periodic solutions of second-order differential equations with weak singularities

$$x''(t) + a(t)\, x(t) = f(t, x(t)) + e(t),$$

where $a \in L^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R}_+)$, $e \in L^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, f is a Carathéodory function and is singular at $x = 0$. By employing Schauder's fixed point theorem, the authors obtained existence results for positive periodic solutions, which improve and generalize some results of Torres [49].

In this chapter, we are interested in the analysis of qualitative theory of positive periodic solutions of second-order systems with singularities and deviating arguments. Inspired and motivated by the works mentioned above and the references therein, we concentrate on the existence of positive periodic solutions of the second-order systems with singularities and deviating arguments

$$\begin{cases} x''(t) + a_1(t)\, x(t) = f_1(t, y(t - \tau_1(t))) + e_1(t), \\ y''(t) + a_2(t)\, y(t) = f_2(t, x(t - \tau_2(t))) + e_2(t), \end{cases} \tag{5.1}$$

where $a_i, \tau_i \in L^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R}_+)$, $e_i \in L^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, and $f_i \in Car(\mathbb{R}/T\mathbb{Z} \times \mathbb{R}_+, \mathbb{R})$ which means $f_i\ |_{[0,T]}$: $[0,T] \times \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, 2)$ are L^1-Carathéodory functions, and have singularities at the origin, i.e. $\lim_{x \to 0^+} \sup f_i(t, x) = +\infty$. By means of Schauder fixed point theorem, we obtain sufficient conditions of the existence of positive periodic solutions for (5.1). An example is also given to illustrate our results. For a given function $\xi \in L^1[0,T]$, we denote the essential supremum and infinimum of ξ if they exist by ξ^* and ξ_*, respectively.

5.2 Existence of positive periodic solutions

It is the purpose of this section to study the existence of positive periodic solutions of (5.1) under the assumptions.

(A1) The linear equation $x''(t) + a_i(t)\, x(t) = 0$ is nonresonant and the corresponding Green's function

$$G_i(t, s) \geq 0, \quad (t, s) \in [0,T] \times [0,T], \quad (i = 1, 2).$$

(A2) $f_i\ |_{[0,T]}$: $[0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$, $(i = 1, 2)$ are L^1-Carathéodory functions.

(A3) There exist $b_i, c_i \in L^1(0,T)$ with $b_i, c_i \geq 0$, $\alpha_i, \beta_i \in \mathbb{R}_+$, $m_i \leq 1 \leq M_i$, $(i = 1, 2)$, such that

$$0 \leq f_i(t, x) \leq \frac{b_i(t)}{x^{\alpha_i}}, \quad x \in (M_i, \infty), \quad a.e.\ t \in [0,T],$$

and

$$0 \leq f_i(t, x) \leq \frac{c_i(t)}{x^{\beta_i}}, \quad x \in (0, m_i), \ a.e.\ t \in [0,T].$$

(A4) There exist $b_{1i}, b_{2i}, c_i \in L^1(0,T)$ with $b_{1i}, b_{2i}, c_i \geq 0$ and $\alpha_i, \beta_i, \mu_i, \nu_i \in$

$(0,1)$, $(i=1,2)$, such that

$$\begin{aligned} 0 &\leq \frac{b_{1i}(t)}{x^{\alpha_i}} \leq f_i(t,x) \leq \frac{b_{2i}(t)}{x^{\beta_i}}, \quad x \in [1,\infty), \ a.e.\ t \in [0,T], \\ 0 &\leq \frac{b_{1i}(t)}{x^{\mu_i}} \leq f_i(t,x) \leq \frac{c_i(t)}{x^{\nu_i}}, \quad x \in (0,1), \ a.e.\ t \in [0,T]. \end{aligned}$$

Now, we introduce the following notations.

$$\begin{aligned} \gamma_i(t) &= \int_0^T G_i(t,s) e_i(s) ds, \quad (i=1,2), \\ B_i(t) &= \int_0^T G_i(t,s) b_i(s) ds, \quad (i=1,2), \\ C_i(t) &= \int_0^T G_i(t,s) c_i(s) ds, \quad (i=1,2), \\ B_{ii}(t) &= \int_0^T G_i(t,s) b_{ii}(s) ds, \quad (i=1,2), \\ \rho_i^* &= C_i^* + B_{2i}^*, \ \sigma_i = \max\{\mu_i, \alpha_i\}, \ \delta_i = \max\{\nu_i, \beta_i\}. \end{aligned}$$

Theorem 5.1 *Let (A1), (A2) and (A3) hold. If $\gamma_{i*} > 0$, then (5.1) has a positive T-periodic solution.*

Proof. We denote the set of continuous T-periodic functions as P_T. A T-periodic solution of (5.1) is just a fixed point of the completely continuous map $A : P_T \times P_T \to P_T \times P_T$, $A(x,y) = (A_1 y, A_2 x)$, where

$$\begin{aligned} (A_1 y)(t) &= \int_0^T G_1(t,s)\left(f_1(s, y(s-\tau_1(s))) + e_1(s)\right) ds \\ &= \int_0^T G_1(t,s) f_1(s, y(s-\tau_1(s))) ds + \gamma_1(t), \\ (A_2 x)(t) &= \int_0^T G_2(t,s)\left(f_2(s, x(s-\tau_2(s))) + e_2(s)\right) ds \\ &= \int_0^T G_2(t,s) f_2(s, x(s-\tau_2(s))) ds + \gamma_2(t), \end{aligned}$$

By Schauder's fixed point theorem, the proof is finished if we prove that A maps the closed convex set defined as

$$K = \{(x,y) \in P_T \times P_T : r_1 \leq x(t) \leq R_1, \ r_2 \leq y(t) \leq R_2, \ \forall t \in [0,T]\},$$

into itself, where $R_1 > r_1 > 0$, $R_2 > r_2 > 0$, are positive constants to be fixed properly. For given $(x, y) \in K$, let us denote

$$\begin{aligned} I_1 &= \{t \in [0,T] \mid r_1 \leq x(t) < m_1,\ r_2 \leq y(t) < m_2\}, \\ I_2 &= \{t \in [0,T] \mid R_1 \geq x(t) > M_1,\ R_2 \geq y(t) > M_2\}, \\ I_3 &= [0,T] \setminus (I_1 \cup I_2). \end{aligned}$$

Given $(x, y) \in K$, by the nonnegativity of G_i and f_i, we have

$$\begin{aligned} (A_1 y)(t) &= \int_0^T G_1(t,s)\left(f_1(s, y(s-\tau_1(s))) + e_1(s)\right) ds \\ &= \int_0^T G_1(t,s) f_1(s, y(s-\tau_1(s))) ds + \gamma_1(t) \\ &= \int_{I_1} G_1(t,s) f_1(s, y(s-\tau_1(s))) ds + \int_{I_2} G_1(t,s) f_1(s, y(s-\tau_1(s))) ds \\ &\quad + \int_{I_3} G_1(t,s) f_1(s, y(s-\tau_1(s))) ds + \gamma_1(t) \\ &\geq \gamma_1(t) \geq \gamma_{1*} = r_1, \end{aligned}$$

and

$$\begin{aligned} (A_2 x)(t) &= \int_0^T G_2(t,s)\left(f_2(s, x(s-\tau_2(s))) + e_2(s)\right) ds \\ &= \int_0^T G_2(t,s) f_2(s, x(s-\tau_2(s))) ds + \gamma_2(t) \\ &= \int_{I_1} G_2(t,s) f_2(s, x(s-\tau_2(s))) ds + \int_{I_2} G_2(t,s) f_2(s, x(s-\tau_2(s))) ds \\ &\quad + \int_{I_3} G_2(t,s) f_2(s, x(s-\tau_2(s))) ds + \gamma_2(t) \\ &\geq \gamma_2(t) \geq \gamma_{2*} = r_2. \end{aligned}$$

Let

$$\begin{aligned} \Lambda_1 &= \sup\left\{\max_{t\in[0,T]} \int_0^T G_1(t,s) f_1(s, y(s-\tau_1(s))) ds : m_2 \leq y(s-\tau_1(s)) \leq M_2\right\}, \\ \Lambda_2 &= \sup\left\{\max_{t\in[0,T]} \int_0^T G_2(t,s) f_2(s, x(s-\tau_2(s))) ds : m_1 \leq x(s-\tau_2(s)) \leq M_1\right\}. \end{aligned}$$

Then, it follows from (A2) that $\Lambda_1 < +\infty$, $\Lambda_2 < +\infty$, and consequently, for every

$(x, y) \in K$,

$$\begin{aligned}
(A_1y)(t) &= \int_0^T G_1(t,s) f_1(s, y(s-\tau_1(s)))\,ds + \gamma_1(t) \\
&= \int_{I_1} G_1(t,s) f_1(s, y(s-\tau_1(s)))\,ds + \int_{I_2} G_1(t,s) f_1(s, y(s-\tau_1(s)))\,ds \\
&\quad + \int_{I_3} G_1(t,s) f_1(s, y(s-\tau_1(s)))\,ds + \gamma_1(t) \\
&\leq \int_{I_1} G_1(t,s) \frac{c_1(s)}{y(s-\tau_1(s))^{\beta_1}} ds + \int_{I_2} G_1(t,s) \frac{b_1(s)}{y(s-\tau_1(s))^{\alpha_1}} ds + \Lambda_1 + \gamma_1^* \\
&\leq \int_0^T G_1(t,s) \frac{c_1(s)}{y(s-\tau_1(s))^{\beta_1}} ds + \int_{I_2} G_1(t,s)\, b_1(s)\,ds + \Lambda_1 + \gamma_1^* \\
&\leq \int_0^T G_1(t,s) \frac{c_1(s)}{r_2^{\beta_1}} ds + \int_0^T G_1(t,s)\, b_1(s)\,ds + \Lambda_1 + \gamma_1^* \\
&\leq \frac{C_1^*}{r_2^{\beta_1}} + (B_1^* + \Lambda_1 + \gamma_1^*) \\
&< \frac{C_1^*}{r_2^{\beta_1}} + (B_1^* + \Lambda_1 + \gamma_1^*) = R_1,
\end{aligned}$$

and

$$\begin{aligned}
(A_2x)(t) &= \int_0^T G_2(t,s) f_2(s, x(s-\tau_2(s)))\,ds + \gamma_2(t) \\
&= \int_{I_1} G_2(t,s) f_2(s, x(s-\tau_2(s)))\,ds + \int_{I_2} G_2(t,s) f_2(s, x(s-\tau_2(s)))\,ds \\
&\quad + \int_{I_3} G_2(t,s) f_2(s, x(s-\tau_2(s)))\,ds + \gamma_2(t) \\
&\leq \int_{I_1} G_2(t,s) \frac{c_2(s)}{x(s-\tau_2(s))^{\beta_2}} ds + \int_{I_2} G_2(t,s) \frac{b_2(s)}{x(s-\tau_2(s))^{\alpha_2}} ds + \Lambda_2 + \gamma_2^* \\
&\leq \int_0^T G_2(t,s) \frac{c_2(s)}{x(s-\tau_2(s))^{\beta_2}} ds + \int_{I_2} G_2(t,s)\, b_2(s)\,ds + \Lambda_2 + \gamma_2^* \\
&\leq \int_0^T G_2(t,s) \frac{c_2(s)}{r_1^{\beta_2}} ds + \int_0^T G_2(t,s)\, b_2(s)\,ds + \Lambda_2 + \gamma_2^* \\
&\leq \frac{C_2^*}{r_1^{\beta_2}} + (B_2^* + \Lambda_2 + \gamma_2^*) \\
&< \frac{C_2^*}{r_1^{\beta_2}} + (B_2^* + \Lambda_2 + \gamma_2^*) = R_2.
\end{aligned}$$

Therefore, $A(x,y) = (A_1y, A_2x) \in K$ if $r_1 = \gamma_{1*}$, $r_2 = \gamma_{2*}$, $R_1 = \frac{C_1^*}{r_2^{\beta_1}} + (B_1^* + \Lambda_1 + \gamma_1^*)$, $R_2 = \frac{C_2^*}{r_1^{\beta_2}} + (B_2^* + \Lambda_2 + \gamma_2^*)$, and the proof is finished. ■

5.2. Existence of positive periodic solutions

Theorem 5.2 *Let (A1), (A2) and (A4) hold. If $\gamma_{i*} = 0$, then (5.1) has a positive T-periodic solution.*

Proof. We follow the same strategy and notations as in the proof of Theorem 5.1. Define a closed convex set

$$K = \{(x, y) \in P_T^2 : r_1 \leq x(t) \leq R_1,\ r_2 \leq y(t) \leq R_2,\ \forall t \in [0, T],\ R_1 > 1,\ R_2 > 1\}.$$

By a direct application of Schauder's fixed point theorem, the proof is finished if we prove that $A(x, y) = (A_1 y, A_2 x)$ maps the closed convex set K into itself, where R_1, R_2 and r_1, r_2 are positive constants to be fixed properly and they should satisfy $R_1 > r_1 > 0$, $R_2 > r_2 > 0$ and $R_1 > 1$, $R_2 > 1$. For given $(x, y) \in K$, let us denote

$$\begin{aligned} J_1 &= \{t \in [0, T] \mid r_1 \leq x(t) < 1,\ r_2 \leq y(t) < 1\}, \\ J_2 &= \{t \in [0, T] \mid R_1 \geq x(t) \geq 1,\ R_2 \geq y(t) \geq 1\}. \end{aligned}$$

Then for given $(x, y) \in K$, by the nonnegative sign of G_i and f_i, it follows that

$$\begin{aligned} (A_1 y)(t) &= \int_0^T G_1(t, s) f_1(s, y(s - \tau_1(s)))\, ds + \gamma_1(t) \\ &= \int_{J_1} G_1(t, s) f_1(s, y(s - \tau_1(s)))\, ds \\ &\quad + \int_{J_2} G_1(t, s) f_1(s, y(s - \tau_1(s)))\, ds + \gamma_1(t) \\ &\leq \int_{J_1} G_1(t, s) \frac{c_1(s)}{y(s - \tau_1(s))^{\nu_1}} ds + \int_{J_2} G_1(t, s) \frac{b_{21}(s)}{y(s - \tau_1(s))^{\beta_1}} ds + \gamma_1^* \\ &\leq \int_0^T G_1(t, s) \frac{c_1(s)}{r_2^{\nu_1}} ds + \int_{J_2} G_1(t, s)\, b_{21}(s)\, ds + \gamma_1^* \\ &\leq \int_0^T G_1(t, s) \frac{c_1(s)}{r_2^{\nu_1}} ds + \int_0^T G_1(t, s)\, b_{21}(s)\, ds + \gamma_1^* \\ &\leq \frac{C_1^*}{r_2^{\nu_1}} + (B_{21}^* + \gamma_1^*), \end{aligned}$$

and

$$
\begin{aligned}
(A_2x)(t) &= \int_0^T G_2(t,s)\, f_2(s, x(s-\tau_2(s)))\, ds + \gamma_2(t) \\
&= \int_{J_1} G_2(t,s)\, f_2(s, x(s-\tau_2(s)))\, ds \\
&\quad + \int_{J_2} G_2(t,s)\, f_2(s, x(s-\tau_2(s)))\, ds + \gamma_2(t) \\
&\leq \int_{J_1} G_2(t,s)\, \frac{c_2(s)}{x(s-\tau_2(s))^{\nu_2}} ds + \int_{J_2} G_2(t,s)\, \frac{b_{22}(s)}{x(s-\tau_2(s))^{\beta_2}} ds + \gamma_2^* \\
&\leq \int_0^T G_2(t,s)\, \frac{c_2(s)}{r_1^{\nu_2}} ds + \int_{J_2} G_2(t,s)\, b_{22}(s)\, ds + \gamma_2^* \\
&\leq \int_0^T G_2(t,s)\, \frac{c_2(s)}{r_1^{\nu_2}} ds + \int_0^T G_2(t,s)\, b_{22}(s)\, ds + \gamma_2^* \\
&\leq \frac{C_2^*}{r_1^{\nu_2}} + \left(B_{22}^* + \gamma_2^*\right).
\end{aligned}
$$

On the other hand, for every $(x, y) \in K$,

$$
\begin{aligned}
(A_1y)(t) &= \int_0^T G_1(t,s)\, f_1(s, y(s-\tau_1(s)))\, ds + \gamma_1(t) \\
&= \int_{J_1} G_1(t,s)\, f_1(s, y(s-\tau_1(s)))\, ds \\
&\quad + \int_{J_2} G_1(t,s)\, f_1(s, y(s-\tau_1(s)))\, ds + \gamma_1(t) \\
&\geq \int_{J_1} G_1(t,s)\, \frac{b_{11}(s)}{y(s-\tau_1(s))^{\mu_1}} ds + \int_{J_2} G_1(t,s)\, \frac{b_{11}(s)}{y(s-\tau_1(s))^{\alpha_1}} ds + \gamma_{1*} \\
&\geq \int_{J_1} G_1(t,s)\, \frac{b_{11}(s)}{R_2^{\mu_1}} ds + \int_{J_2} G_1(t,s)\, \frac{b_{11}(s)}{R_2^{\alpha_1}} ds \\
&\geq \int_{J_1} G_1(t,s)\, \frac{b_{11}(s)}{R_2^{\sigma_1}} ds + \int_{J_2} G_1(t,s)\, \frac{b_{11}(s)}{R_2^{\sigma_1}} ds \\
&\geq \int_0^T G_1(t,s)\, \frac{b_{11}(s)}{R_2^{\sigma_1}} ds \\
&\geq \frac{B_{11*}}{R_2^{\sigma_1}},
\end{aligned}
$$

and

$$\begin{aligned}
(A_2x)(t) &= \int_0^T G_2(t,s) f_2(s, x(s-\tau_2(s)))\,ds + \gamma_2(t) \\
&= \int_{J_1} G_2(t,s) f_2(s, x(s-\tau_2(s)))\,ds \\
&\quad + \int_{J_2} G_2(t,s) f_2(s, x(s-\tau_2(s)))\,ds + \gamma_2(t) \\
&\geq \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{x(s-\tau_2(s))^{\mu_2}} ds + \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{x(s-\tau_2(s))^{\alpha_2}} ds + \gamma_{2*} \\
&\geq \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{R_1^{\mu_2}} ds + \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{R_1^{\alpha_2}} ds \\
&\geq \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{R_1^{\sigma_2}} ds + \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{R_1^{\sigma_2}} ds \\
&\geq \int_0^T G_2(t,s) \frac{b_{12}(s)}{R_1^{\sigma_2}} ds \\
&\geq \frac{B_{12*}}{R_1^{\sigma_2}}.
\end{aligned}$$

Thus $A(x,y) = (A_1y, A_2x) \in K$ if r_1, r_2 and R_1, R_2 are chosen so that

$$\begin{aligned}
r_1 &\leq \frac{B_{11*}}{R_2^{\sigma_1}}, \quad \frac{C_1^*}{r_2^{\nu_1}} + (B_{21}^* + \gamma_1^*) \leq R_1, \\
r_2 &\leq \frac{B_{12*}}{R_1^{\sigma_2}}, \quad \frac{C_2^*}{r_1^{\nu_2}} + (B_{22}^* + \gamma_2^*) \leq R_2.
\end{aligned}$$

Note that $B_{1i*} > 0$, $C_i^* > 0$ and taking $r_1 = r_2 = r$, $R_1 = R_2 = R$, and $R = \frac{1}{r}$, it is sufficient to find $R > 1$ such that

$$B_{1i*} R^{1-\sigma_i} \geq 1, \quad C_i^* R^{\nu_i} + (B_{2i}^* + \gamma_i^*) \leq R,$$

and these inequalities hold for R big enough because $\sigma_i < 1$ and $\nu_i < 1$. ■

Remark 5.1 It is worth remarking that Theorem 5.2 is also valid for the special case that $e_i(t) \equiv 0$, $(i = 1, 2)$, which implies that $\gamma_{i*} = 0$.

Theorem 5.3 *Let (A1), (A2) and (A4) hold. Assume that*

$$\rho_1^* > \max\left\{ (\delta_1\sigma_2 B_{12*})^{\delta_1}, (\delta_1\sigma_2 B_{12*})^{\frac{1}{\sigma_1}} \right\}, \tag{5.2}$$

$$\rho_2^* > \max\left\{ (\delta_2\sigma_1 B_{11*})^{\delta_2}, (\delta_2\sigma_1 B_{11*})^{\frac{1}{\sigma_2}} \right\}. \tag{5.3}$$

If $\gamma_i^* \leq 0$ *and*

$$\begin{aligned}
\gamma_{1*} &\geq \left[\frac{B_{11*}}{(\rho_2^*)^{\sigma_1}}\delta_2\sigma_1\right]^{\frac{1}{1-\delta_2\sigma_1}}\left(1-\frac{1}{\delta_2\sigma_1}\right), &(5.4)\\
\gamma_{2*} &\geq \left[\frac{B_{12*}}{(\rho_1^*)^{\sigma_2}}\delta_1\sigma_2\right]^{\frac{1}{1-\delta_1\sigma_2}}\left(1-\frac{1}{\delta_1\sigma_2}\right), &(5.5)
\end{aligned}$$

then (5.1) has a positive T-periodic solution.

Proof. Define a closed convex set

$$\begin{aligned}
K &= \{(x,y)\in P_T\times P_T : r_1 \leq x(t) \leq R_1,\ r_2 \leq y(t) \leq R_2,\\
&\qquad \forall t\in[0,T],\ 0<r_1<1<R_1,\ 0<r_2<1<R_2\}.
\end{aligned}$$

By a direct application of Schauder's fixed point theorem, the proof is finished if we prove that A maps the closed convex set K into itself, where R_1, R_2 and r_1, r_2 are positive constants to be fixed properly and they should satisfy $R_1>1>r_1>0$, $R_2>1>r_2>0$.

Recall that $\delta_i=\max\{\nu_i,\beta_i\}$ and $r_1<1$, $r_2<1$; therefore for given $(x,y)\in K$,

$$\begin{aligned}
(A_1y)(t) &= \int_0^T G_1(t,s)\,f_1(s,y(s-\tau_1(s)))\,ds+\gamma_1(t)\\
&= \int_{J_1} G_1(t,s)\,f_1(s,y(s-\tau_1(s)))\,ds\\
&\quad+\int_{J_2} G_1(t,s)\,f_1(s,y(s-\tau_1(s)))\,ds+\gamma_1(t)\\
&\leq \int_{J_1} G_1(t,s)\,\frac{c_1(s)}{y(s-\tau_1(s))^{\nu_1}}ds+\int_{J_2} G_1(t,s)\,\frac{b_{21}(s)}{y(s-\tau_1(s))^{\beta_1}}ds+\gamma_1^*\\
&\leq \int_{J_1} G_1(t,s)\,\frac{c_1(s)}{r_2^{\nu_1}}ds+\int_{J_2} G_1(t,s)\,\frac{b_{21}(s)}{r_2^{\beta_1}}ds\\
&\leq \int_0^T G_1(t,s)\,\frac{c_1(s)}{r_2^{\delta_1}}ds+\int_0^T G_1(t,s)\,\frac{b_{21}(s)}{r_2^{\delta_1}}ds\\
&\leq \frac{\rho_1^*}{r_2^{\delta_1}},
\end{aligned}$$

and

$$\begin{aligned}
(A_2x)(t) &= \int_0^T G_2(t,s) f_2(s, x(s-\tau_2(s)))\,ds + \gamma_2(t) \\
&= \int_{J_1} G_2(t,s) f_2(s, x(s-\tau_2(s)))\,ds \\
&\quad + \int_{J_2} G_2(t,s) f_2(s, x(s-\tau_2(s)))\,ds + \gamma_2(t) \\
&\leq \int_{J_1} G_2(t,s) \frac{c_2(s)}{x(s-\tau_2(s))^{\nu_2}} ds \\
&\quad + \int_{J_2} G_2(t,s) \frac{b_{22}(s)}{x(s-\tau_2(s))^{\beta_2}} ds + \gamma_2^* \\
&\leq \int_{J_1} G_2(t,s) \frac{c_2(s)}{r_1^{\nu_2}} ds + \int_{J_2} G_2(t,s) \frac{b_{22}(s)}{r_1^{\beta_2}} ds \\
&\leq \int_0^T G_2(t,s) \frac{c_2(s)}{r_1^{\delta_2}} ds + \int_0^T G_2(t,s) \frac{b_{22}(s)}{r_1^{\delta_2}} ds \\
&\leq \frac{\rho_2^*}{r_1^{\delta_2}},
\end{aligned}$$

where J_i, δ_i, ρ_i^*, $(i=1,2)$, are defined previously in this Section.

On the other hand, since $\sigma_i = \max\{\mu_i, \alpha_i\}$, and $R_1 > 1$, $R_2 > 1$, for every $(x,y) \in K$,

$$\begin{aligned}
(A_1y)(t) &= \int_0^T G_1(t,s) f_1(s, y(s-\tau_1(s)))\,ds + \gamma_1(t) \\
&= \int_{J_1} G_1(t,s) f_1(s, y(s-\tau_1(s)))\,ds \\
&\quad + \int_{J_2} G_1(t,s) f_1(s, y(s-\tau_1(s)))\,ds + \gamma_1(t) \\
&\geq \int_{J_1} G_1(t,s) \frac{b_{11}(s)}{y(s-\tau_1(s))^{\mu_1}} ds \\
&\quad + \int_{J_2} G_1(t,s) \frac{b_{11}(s)}{y(s-\tau_1(s))^{\alpha_1}} ds + \gamma_{1*} \\
&\geq \int_{J_1} G_1(t,s) \frac{b_{11}(s)}{R_2^{\sigma_1}} ds + \int_{J_2} G_1(t,s) \frac{b_{11}(s)}{R_2^{\sigma_1}} ds + \gamma_{1*} \\
&\geq \int_0^T G_1(t,s) \frac{b_{11}(s)}{R_2^{\sigma_1}} ds + \int_0^T G_1(t,s) \frac{b_{11}(s)}{R_2^{\sigma_1}} ds + \gamma_{1*} \\
&\geq \frac{B_{11*}}{R_2^{\sigma_1}} + \gamma_{1*},
\end{aligned}$$

and

$$\begin{aligned}
(A_2x)(t) &= \int_0^T G_2(t,s) f_2(s, x(s-\tau_2(s)))\,ds + \gamma_2(t) \\
&= \int_{J_1} G_2(t,s) f_2(s, x(s-\tau_2(s)))\,ds \\
&\quad + \int_{J_2} G_2(t,s) f_2(s, x(s-\tau_2(s)))\,ds + \gamma_2(t) \\
&\geq \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{x(s-\tau_2(s))^{\mu_2}} ds + \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{x(s-\tau_2(s))^{\alpha_2}} ds + \gamma_{2*} \\
&\geq \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{R_1^{\sigma_2}} ds + \int_{J_1} G_2(t,s) \frac{b_{12}(s)}{R_1^{\sigma_2}} ds + \gamma_{2*} \\
&\geq \int_0^T G_2(t,s) \frac{b_{12}(s)}{R_1^{\sigma_2}} ds + \int_0^T G_2(t,s) \frac{b_{12}(s)}{R_1^{\sigma_2}} ds + \gamma_{2*} \\
&\geq \frac{B_{12*}}{R_1^{\sigma_2}} + \gamma_{2*},
\end{aligned}$$

where B_{1i*} and σ_i, $(i = 1, 2)$, are defined previously in this Section.

In this case, to prove that $A(K) \subset K$, it is sufficient to find

$r_1 < R_1$, $r_2 < R_2$ with $R_1 > 1 > r_1 > 0$, $R_2 > 1 > r_2 > 0$ such that

$$\frac{B_{11*}}{R_2^{\sigma_1}} + \gamma_{1*} \geq r_1, \quad \frac{\rho_1^*}{r_2^{\delta_1}} \leq R_1, \tag{5.6}$$

$$\frac{B_{12*}}{R_1^{\sigma_2}} + \gamma_{2*} \geq r_2, \quad \frac{\rho_2^*}{r_1^{\delta_2}} \leq R_2. \tag{5.7}$$

If we fix $R_1 = \frac{\rho_1^*}{r_2^{\delta_1}}$, $R_2 = \frac{\rho_2^*}{r_1^{\delta_2}}$, then to get the first inequality of (4.8), we need to find r_1 such that

$$\frac{B_{11*}}{(\rho_2^*)^{\sigma_1}} r_1^{\sigma_1\delta_2} + \gamma_{1*} \geq r_1,$$

or equivalently,

$$\gamma_{1*} \geq f(r_1) = r_1 - \frac{B_{11*}}{(\rho_2^*)^{\sigma_1}} r_1^{\sigma_1\delta_2}.$$

The function $f(r_1)$ possesses a minimum in $r_{10} = \left[\frac{B_{11*}}{(\rho_2^*)^{\sigma_1}}\sigma_1\delta_2\right]^{\frac{1}{1-\sigma_1\delta_2}}$. Taking $r_1 = r_{10}$, (5.2) implies that $r_1 < 1$. Then the first inequality in (5.6) holds if $\gamma_{1*} \geq f(r_{10})$, which is just condition (5.4).

The first inequality of (5.7) holds when

$$\gamma_{2*} \geq g(r_2) = r_2 - \frac{B_{12*}}{(\rho_1^*)^{\sigma_2}} r_2^{\sigma_2\delta_1}.$$

Since $g(r_2)$ get the minimum at $r_{20} = \left[\frac{B_{12*}}{(\rho_1^*)^{\sigma_2}}\sigma_2\delta_1\right]^{\frac{1}{1-\sigma_2\delta_1}}$; taking $r_2 = r_{20}$, (5.3) implies that $r_2 < 1$. Then the first inequality in (5.7) holds if $\gamma_{2*} \geq g(r_{20})$, which is just condition (5.5).

Now we choose $r_1 = r_{10}$, $r_2 = r_{20}$. The second inequality in (5.6) holds directly by the choice of R_1, and it would remain to prove that $R_1 = \frac{\rho_1^*}{r_{20}^{\delta_1}} > 1$. To the end, it follows from (5.2) that

$$R_1 = \frac{\rho_1^*}{r_{20}^{\delta_1}} > \frac{(\delta_1\sigma_2 B_{12*})^{\delta_1}(\rho_1^*)^{\frac{\delta_1\sigma_2}{1-\delta_1\sigma_2}}}{(\delta_1\sigma_2 B_{12*})^{\frac{\delta_1}{1-\delta_1\sigma_2}}} > \frac{(\delta_1\sigma_2 B_{12*})^{\delta_1}(\delta_1\sigma_2 B_{12*})^{\frac{\delta_1^2\sigma_2}{1-\delta_1\sigma_2}}}{(\delta_1\sigma_2 B_{12*})^{\frac{\delta_1}{1-\delta_1\sigma_2}}} = 1,$$

and therefore $R_1 > 1 > r_1 > 0$.

The second inequality in (5.7) holds directly by the choice of R_2, and it would remain to prove that $R_2 = \frac{\rho_2^*}{r_{10}^{\delta_2}} > 1$. To the end, it follows from (5.3) that

$$R_2 = \frac{\rho_2^*}{r_{10}^{\delta_2}} > \frac{(\delta_2\sigma_1 B_{11*})^{\delta_2}(\rho_2^*)^{\frac{\delta_2\sigma_1}{1-\delta_2\sigma_1}}}{(\delta_2\sigma_1 B_{11*})^{\frac{\delta_2}{1-\delta_2\sigma_1}}} > \frac{(\delta_2\sigma_1 B_{11*})^{\delta_2}(\delta_2\sigma_1 B_{11*})^{\frac{\delta_2^2\sigma_1}{1-\delta_2\sigma_1}}}{(\delta_2\sigma_1 B_{11*})^{\frac{\delta_2}{1-\delta_2\sigma_1}}} = 1,$$

and therefore $R_2 > 1 > r_2 > 0$, this completes the proof. ∎

Example 5.1 Let us consider the second order periodic boundary value problem

$$\begin{cases} x''(t) + \frac{1}{4}x(t) = f_1(t,y) - e_1, \\ y''(t) + \frac{1}{5}y(t) = f_2(t,x) - e_2, \end{cases} \quad t \in (0,\pi), \tag{5.8}$$

$$\begin{aligned} x(0) &= x(\pi), \quad x'(0) = x'(\pi), \\ y(0) &= y(\pi), \quad y'(0) = y'(\pi), \end{aligned}$$

where

$$\begin{aligned} f_1(t,y) &= \frac{7-t}{y^{\frac{1}{7}}}, \quad y \in (0,\infty), \quad a.e.\ t \in [0,\pi], \\ f_2(t,x) &= \frac{5-t}{x^{\frac{1}{5}}}, \quad x \in (0,\infty), \quad a.e.\ t \in [0,\pi], \end{aligned}$$

and

$$\begin{aligned} e_1 &\in \left(0, \frac{3}{4}\left(\frac{1}{4\sqrt{15}}\right)^{\frac{4}{3}}\right], \\ e_2 &\in \left(0, \frac{9}{5}\left(\frac{1}{16}\right)^{\frac{10}{9}}\right], \end{aligned}$$

are a constants.

It is easy to check that (5.8) is equivalent to the operator equation

$$\begin{aligned}(A_1 y)(t) &= \int_0^\pi G_1(t,s) f_1(s, y(s))\, ds + \int_0^\pi G_1(t,s)(-e_1)\, ds, \qquad t \in [0,\pi],\\ (A_2 x)(t) &= \int_0^\pi G_2(t,s) f_2(s, x(s))\, ds + \int_0^\pi G_2(t,s)(-e_2)\, ds, \qquad t \in [0,\pi],\end{aligned}$$

where

$$G_1(t,s) = \begin{cases} \sin\frac{\pi-t+s}{2} + \sin\frac{t-s}{2}, & 0 \le s \le t \le \pi,\\ \sin\frac{\pi-s+t}{2} + \sin\frac{s-t}{2}, & 0 \le t \le s \le \pi,\end{cases}$$

$$G_2(t,s) = \begin{cases} \frac{\sqrt{5}}{2\left(1-\cos\frac{\pi}{\sqrt{5}}\right)} \sin\frac{\pi-t+s}{\sqrt{5}} + \frac{\sqrt{5}}{2\left(1-\cos\frac{\pi}{\sqrt{5}}\right)} \sin\frac{t-s}{\sqrt{5}}, & 0 \le s \le t \le \pi,\\ \frac{\sqrt{5}}{2\left(1-\cos\frac{\pi}{\sqrt{5}}\right)} \sin\frac{\pi-s+t}{\sqrt{5}} + \frac{\sqrt{5}}{2\left(1-\cos\frac{\pi}{\sqrt{5}}\right)} \sin\frac{s-t}{\sqrt{5}}, & 0 \le t \le s \le \pi.\end{cases}$$

Clearly, $G_i(t,s) > 0$ for all $(t,s) \in [0,\pi] \times [0,\pi]$ and f_i satisfies (A2).

Let

$$\begin{aligned} b_{11}(t) &= \frac{1}{2},\ b_{12}(t) = 1,\ b_{21}(t) = 7,\ b_{22}(t) = 5,\ c_1(t) = 9,\ c_2(t) = 7,\\ \alpha_1 &= \alpha_2 = \frac{1}{2},\ \beta_1 = \frac{1}{8},\ \beta_2 = \frac{1}{6},\ \mu_1 = \frac{1}{9},\ \mu_2 = \frac{1}{7},\ \nu_1 = \frac{1}{5},\ \nu_2 = \frac{1}{2}.\end{aligned}$$

Then

$$\sigma_1 = \sigma_2 = \frac{1}{2},\quad \delta_1 = \frac{1}{5},\quad \delta_2 = \frac{1}{2},$$

and

$$\begin{aligned} \frac{\frac{1}{2}}{y^{\frac{1}{2}}} &\le \frac{7-t}{y^{\frac{1}{7}}} \le \frac{7}{y^{\frac{1}{8}}}, \quad y \in [1,\infty), \quad a.e.\ t \in [0,\pi],\\ \frac{\frac{1}{2}}{y^{\frac{1}{9}}} &\le \frac{7-t}{y^{\frac{1}{7}}} \le \frac{9}{y^{\frac{1}{5}}}, \quad y \in (0,1), \quad a.e.\ t \in [0,\pi],\\ \frac{1}{x^{\frac{1}{2}}} &\le \frac{5-t}{x^{\frac{1}{5}}} \le \frac{5}{x^{\frac{1}{6}}}, \quad x \in [1,\infty), \quad a.e.\ t \in [0,\pi],\\ \frac{1}{x^{\frac{1}{7}}} &\le \frac{5-t}{x^{\frac{1}{5}}} \le \frac{7}{x^{\frac{1}{2}}}, \quad x \in (0,1), \quad a.e.\ t \in [0,\pi].\end{aligned}$$

Thus, the condition (A4) is satisfied. By simple computations, we get

$$
\begin{aligned}
B_{11}(t) &= \int_0^\pi G_1(t,s)\, b_{11}(s)\, ds = 2,\ B_{12}(t) = \int_0^\pi G_2(t,s)\, b_{12}(s)\, ds = 5, \\
B_{21}(t) &= \int_0^\pi G_1(t,s)\, b_{21}(s)\, ds = 28,\ B_{22}(t) = \int_0^\pi G_2(t,s)\, b_{22}(s)\, ds = 25, \\
C_1(t) &= \int_0^\pi G_1(t,s)\, c_1(s)\, ds = 36,\ C_2(t) = \int_0^\pi G_2(t,s)\, c_2(s)\, ds = 35, \\
B_{11*} &= B_{11}^* = 2,\ B_{12*} = B_{12}^* = 5,\ B_{21*} = B_{21}^* = 28,\ B_{22*} = B_{22}^* = 25, \\
C_{1*} &= C_1^* = 36,\ C_{2*} = C_2^* = 35, \\
(\delta_1\sigma_2 B_{12*})^{\delta_1} &= \frac{1}{\sqrt[5]{2}},\ (\delta_1\sigma_2 B_{12*})^{\frac{1}{\sigma_1}} = \frac{1}{4}, \\
(\delta_2\sigma_1 B_{11*})^{\delta_2} &= \frac{\sqrt{2}}{2},\ (\delta_2\sigma_1 B_{11*})^{\frac{1}{\sigma_2}} = \frac{1}{4}, \\
\rho_1^* &= C_1^* + B_{21}^* = 64 > \frac{1}{\sqrt[5]{2}} = \max\left\{(\delta_1\sigma_2 B_{12*})^{\delta_1}, (\delta_1\sigma_2 B_{12*})^{\frac{1}{\sigma_1}}\right\}, \\
\rho_2^* &= C_2^* + B_{22}^* = 60 > \frac{\sqrt{2}}{2} = \max\left\{(\delta_2\sigma_1 B_{11*})^{\delta_2}, (\delta_2\sigma_1 B_{11*})^{\frac{1}{\sigma_2}}\right\},
\end{aligned}
$$

and therefore the condition (5.2), (5.3) is satisfied. Moreover,

$$
\begin{aligned}
\gamma_{1*}(t) &= \int_0^\pi G_1(t,s)(-e_1)\, ds = -4e_1, \\
\gamma_{2*}(t) &= \int_0^\pi G_2(t,s)(-e_2)\, ds = -5e_2,
\end{aligned}
$$

and so

$$
\begin{aligned}
\gamma_1^* &= \gamma_{1*} = -4e_1 < 0, \\
\gamma_2^* &= \gamma_{2*} = -5e_2 < 0.
\end{aligned}
$$

Finally

$$
\begin{aligned}
\gamma_{1*} &= -4e_1 \geq -4.\frac{3}{4}\left(\frac{1}{4\sqrt{15}}\right)^{\frac{4}{3}} = -3\left(\frac{1}{4\sqrt{15}}\right)^{\frac{4}{3}} = \left[\frac{B_{11*}}{(\rho_2^*)^{\sigma_1}}\delta_2\sigma_1\right]^{\frac{1}{1-\delta_2\sigma_1}}\left(1-\frac{1}{\delta_2\sigma_1}\right), \\
\gamma_{2*} &= -5e_2 \geq -5.\frac{9}{5}\left(\frac{1}{16}\right)^{\frac{10}{9}} = -9\left(\frac{1}{16}\right)^{\frac{10}{9}} = \left[\frac{B_{12*}}{(\rho_1^*)^{\sigma_2}}\delta_1\sigma_2\right]^{\frac{1}{1-\delta_1\sigma_2}}\left(1-\frac{1}{\delta_1\sigma_2}\right).
\end{aligned}
$$

Consequently Theorem 5.3 yields that (5.8) has a positive T-periodic solution.

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