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THESIS

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Existence and stability for fractional differential equations.

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By
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"Mathematics, rightly viewed, possesses not only truth, but supreme beauty."
– Bertrand Russell

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Abstract

This thesis investigates the theoretical properties and applications of the Riesz-Caputo fractional derivative. Key contributions include the rigorous analysis of various boundary value problems (BVPs), such as multi-point BVPs with integral boundary conditions, pantograph-type delay differential equations, and a fractional thermostat model with nonlocal boundary conditions. For these problems, criteria for the existence, uniqueness, positivity, and Ulam-Hyers stability of solutions are established using advanced fixed-point theorems (Krasnoselskii, Leray-Schauder Alternative, Schaefer, Guo-Krasnoselskii) and detailed Green's function analysis. The work highlights the suitability of the symmetric Riesz-Caputo operator for modeling systems with bidirectional memory effects and non-local interactions, where its Caputo-type formulation facilitates the use of physically interpretable initial/boundary conditions. The thesis also explores the conceptual underpinnings of fractional calculus, including its historical development and potential links to fractal geometry, and surveys applications in fields like viscoelasticity and anomalous diffusion. Numerical examples complement the theoretical findings. The research aims to advance the understanding of this specific fractional operator and its role in mathematical modeling of complex phenomena.

Keywords: Fractional Calculus, Riesz-Caputo Derivative, Boundary Value Problems, Fixed Point Theorems, Stability Analysis, Memory Effects, Pantograph Equations, Thermostat Model.

Résumé

Cette thèse étudie les propriétés théoriques et les applications de la dérivée fractionnaire de Riesz-Caputo. Les contributions clés comprennent l'analyse rigoureuse de divers problèmes aux limites (BVP), tels que les BVP multipoints avec conditions aux limites intégrales, les équations différentielles à retard de type pantographe, et un modèle de thermostat fractionnaire avec conditions aux limites non locales. Pour ces problèmes, des critères d'existence, d'unicité, de positivité et de stabilité de Hyers-Ulam des solutions sont établis en utilisant des théorèmes de point fixe avancés (Krasnoselskii, Alternative de Leray-Schauder, Schaefer, Guo-Krasnoselskii) et une analyse détaillée de la fonction de Green. Le travail met en évidence l'adéquation de l'opérateur symétrique de Riesz-Caputo pour la modélisation de systèmes avec des effets de mémoire bidirectionnels et des interactions non locales, où sa formulation de type Caputo facilite l'utilisation de conditions initiales/aux limites physiquement interprétables. La thèse explore également les fondements conceptuels du calcul fractionnaire, y compris son développement historique et ses liens potentiels avec la géométrie fractale, et examine les applications dans des domaines tels que la viscoélasticité et la diffusion anormale. Des exemples numériques complètent les résultats théoriques. La recherche vise à faire progresser la compréhension de cet opérateur fractionnaire spécifique et de son rôle dans la modélisation mathématique de phénomènes complexes.

Mots-clés : Calcul Fractionnaire, Dérivée de Riesz-Caputo, Problèmes aux Limites, Théorèmes de Point Fixe, Analyse de Stabilité, Effets de Mémoire, Équations de Pantographe, Modèle de Thermostat.

ملخص

تهدف هذه الأطروحة إلى دراسة الخصائص النظرية وتطبيقات مشتق ريس-كابوتو الكسري تشمل المساهمات الرئيسية التحليل الدقيق لمختلف مسائل القيم الحدودية، مثل مسائل القيم الحدودية متعددة النقاط مع شروط حدودية تكاملية، والمعادلات التفاضلية ذات التأخير من نوع البانتوغراف ونموذج منظم حرارة كسري مع شروط حدودية غير محلية . بالنسبة لهذه المسائل، تم تأسيس معايير لوجود الحلول وتفردها وإيجابيتها واستقرارها من نوع هايرز-أولام باستخدام نظريات النقطة الثابتة المتقدمة (كروسنوسيلسكي، بديل لراي-شودر، شيفر، غو-كروسنوسيلسكي) وتحليل مفصل لدالة غرين يسلط العمل الضوء على مدى ملاءمة مؤثر ريس-كابوتو المتماثل لنمذجة الأنظمة ذات تأثيرات الذاكرة ثنائية الاتجاه والتفاعلات غير المحلية، حيث تسهل صياغته من نوع كابوتو استخدام شروط ابتدائية/حدودية قابلة للتفسير الفيزيائي تستكشف الأطروحة أيضًا الأسس المفاهيمية للحساب الكسري، بما في ذلك تطوره التاريخي وروابطه المحتملة بالهندسة الكسورية ، وتستعرض تطبيقات في مجالات مثل اللزوجة المرنة والانتشار الشاذ. أمثلة عددية تكمل النتائج النظرية .يهدف البحث إلى تعزيز فهم هذا المؤثر الكسري المحدد ودوره في النمذجة الرياضية للظواهر المعقدة.

الكلمات المفتاحية: حساب كسري، مشتق ريس-كابوتو، مسائل قيم حدودية، نظريات النقطة الثابتة، تحليل الاستقرار، تأثيرات الذاكرة، معادلات البانتوغراف، نموذج منظم الحرارة.

Contents

Chapter 1

Introduction

1.1 Motivation and Background

The twenty-first century has witnessed a surge of interest in fractional calculus, a branch of mathematical analysis that extends the concept of differentiation and integration to arbitrary non-integer orders. This generalization, with historical roots tracing back to a query from L'Hopital to Leibniz in 1695 [67], has transitioned from a mathematical curiosity to an indispensable tool in modeling complex systems across diverse scientific and engineering disciplines. Unlike classical integer-order models, which often rely on assumptions of locality and instantaneous interactions, fractional-order models inherently possess non-local characteristics and memory [70, 28]. This intrinsic ability to capture hereditary and long-range dependence phenomena makes them particularly adept at describing systems where past states significantly influence current and future behavior.

Early intuitions about fractional operators emerged naturally in physical problems, such as Abel's 1823 solution to the Tautochrone problem [1]. The rigorous mathematical foundations were subsequently developed by luminaries including Liouville, Riemann, and Fourier [30, 52]. Among the various definitions proposed, the Caputo derivative [23] gained prominence due to its compatibility with physically interpretable initial conditions. More recently, symmetric fractional operators like the Riesz derivative, and its regularized counterpart, the Riesz-Caputo derivative, have garnered attention for modeling phenomena with bidirectional spatial interactions or symmetric memory kernels [80, 91]. These operators are proving crucial in fields such as anomalous diffusion in complex media, viscoelasticity theory [60], control systems with distributed memory, signal processing, and beyond [4, 14, 89].

The primary impetus for this thesis lies in addressing specific theoretical questions concerning boundary value problems formulated with the Riesz-Caputo fractional derivative. While the utility of one-sided fractional operators is well-documented, the analytical intricacies of the symmetric Riesz-Caputo operator, especially when coupled with complex boundary conditions or inherent system delays, present ongoing research challenges. This work aims to contribute to this evolving area by providing rigorous mathematical analysis for selected classes of such problems.

1.2 Objectives and Scope of the Thesis

The overarching goal of this thesis is to investigate the existence, uniqueness, positivity, and stability of solutions for nonlinear fractional differential equations involving the Riesz-Caputo derivative, under various boundary and system configurations. The specific objectives are:

- To establish rigorous conditions for the existence of solutions to selected classes of boundary value problems for Riesz-Caputo fractional differential equations, employing advanced fixed-point theorems.
- To analyze the conditions under which these solutions exhibit positivity, a crucial property in many physical applications, utilizing cone-theoretic fixed-point methods.
- To investigate the Ulam-Hyers stability of the derived solutions, thereby assessing the robustness of the considered fractional models to small perturbations.

The scope of this study is primarily theoretical, focusing on mathematical analysis and the development of proofs. However, the derived results are anticipated to have significant implications for applied fields where phenomena with symmetric memory and non-local interactions are prevalent, such as in the modeling of certain viscoelastic materials, anomalous transport processes, and specific control system designs. Numerical illustrations will be provided to complement the theoretical findings.

1.3 Outline of the Thesis

This thesis is organized as follows:

Chapter 2 provides a comprehensive overview of the mathematical preliminaries required throughout this work. This includes fundamental concepts from functional analysis, definitions and properties of essential special functions (Gamma, Beta, Mittag-Leffler, Lerch Transcendent), precise definitions of various fractional integrals and derivatives (Riemann-Liouville, Caputo, Riesz-Caputo), and a compendium of key fixed-point theorems (Banach, Krasnoselskii, Guo-Krasnoselskii, Schaefer, Leray-Schauder Alternative) and stability notions. It also delves into the conceptual underpinnings and diverse manifestations of fractional calculus, exploring its historical account, the "ontological question" concerning its fundamental meaning and links with fractal geometry, and surveying its role in both mathematics and applied sciences.

Chapter 3 focuses on establishing the existence of solutions for a class of nonlinear multi-point boundary value problems involving the Riesz-Caputo fractional derivative, coupled with integral boundary conditions. Krasnoselskii's fixed-point Theorem and the Leray-Schauder nonlinear alternative are the primary analytical tools employed.

Chapter 4 investigates a Riesz-Caputo fractional pantograph-type delay differential equation. This chapter establishes criteria for the existence, uniqueness, and positivity of solutions, alongside an analysis of Ulam-Hyers stability, leveraging its equivalent Volterra integral equation form.

Chapter 5 addresses a nonlinear Riesz-Caputo fractional thermostat model characterized by a specific nonlocal boundary condition. The analysis includes the derivation and study of the associated Green's function, proofs of solution existence (using Schaefer's theorem) and positivity (using cone-theoretic methods), and a demonstration of Hyers-Ulam stability.

Finally, **Chapter 6** summarizes the main contributions of the thesis, discusses limitations of the current study, and outlines promising directions for future research, particularly concerning the further analytical development and application of the Riesz-Caputo operator.

Chapter 2

Preliminaries

This chapter provides the fundamental mathematical framework necessary for understanding the subsequent analysis of fractional differential equations. We begin by establishing a foundation in functional analysis, reviewing essential function spaces crucial for defining and analyzing solutions. Following this, we introduce a suite of special functions that are ubiquitous in fractional calculus. With these prerequisites in place, we formally define the various fractional integral and differential operators, which constitute the core mathematical tools of this thesis. Building upon these definitions, we then delve into the rich historical development of fractional calculus, exploring its conceptual underpinnings and philosophical implications, particularly in relation to fractal geometry. The chapter further surveys the diverse applications of fractional analysis, both within pure mathematics and across various scientific and engineering disciplines, illustrating its capacity to model complex systems with memory. Finally, we present the key fixed-point theorems that underpin our existence and uniqueness proofs, and introduce fundamental concepts of stability relevant to dynamical systems.

2.1 Functional Analysis Preliminaries

We recall standard definitions of function spaces relevant to the analysis of differential equations.

2.1.1 Lebesgue Spaces

Let $\Omega = [a, b]$ be a finite interval on the real line $\mathbb{R}$.

Definition 2.1 (Lebesgue Space $L^p(\Omega)$ [76])**.** *For $1 \leq p < \infty$, the Lebesgue space $L^p(\Omega)$ consists of all (equivalence classes of) measurable functions $f : \Omega \to \mathbb{R}$ such that $|f|^p$ is Lebesgue integrable on Ω. The norm is defined as:*

$$\|f\|_{L^p} = \left(\int_a^b |f(x)|^p dx \right)^{1/p}.$$

For $p = \infty$, the space $L^\infty(\Omega)$ consists of all essentially bounded measurable functions on Ω, with the norm:

$$\|f\|_{L^\infty} = ess\ sup_{x\in\Omega}|f(x)| = \inf\{M \geq 0 : |f(x)| \leq M \ almost\ everywhere\ on\ \Omega\}.$$

$L^p(\Omega)$ equipped with its norm is a Banach space for $1 \leq p \leq \infty$.

2.1.2 Space of Continuous Functions

Let $\Omega = [a, b]$ be a finite closed interval.

Definition 2.2 (Space $C(\Omega)$)**.** *The space $C(\Omega)$, also denoted $C([a, b])$, is the space of all continuous real-valued functions $f : \Omega \to \mathbb{R}$. Equipped with the supremum norm (or uniform norm):*

$$\|f\|_\infty = \sup_{x\in\Omega} |f(x)| = \max_{x\in\Omega} |f(x)|,$$

$C(\Omega)$ is a Banach space.

2.1.3 Absolutely Continuous Functions

Let $\Omega = [a, b]$ be a finite closed interval.

Definition 2.3 (Absolutely Continuous Functions $AC(\Omega)$ [54])**.** *A function $f : \Omega \to \mathbb{R}$ is said to be absolutely continuous on Ω if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for any finite collection of pairwise disjoint sub-intervals (a_k, b_k) of Ω satisfying $\sum_k (b_k - a_k) < \delta$, we have $\sum_k |f(b_k) - f(a_k)| < \epsilon$. We denote the space of absolutely continuous functions on Ω by $AC(\Omega)$ or $AC([a, b])$.*

Equivalently, a function f is in $AC(\Omega)$ if and only if it possesses a derivative f' almost everywhere on Ω, the derivative f' is Lebesgue integrable (i.e., $f' \in L^1(\Omega)$), and for all $x \in \Omega$,

$$f(x) = f(a) + \int_a^x f'(t)\, dt.$$

Definition 2.4 (Space $AC^n(\Omega)$)**.** *For $n \in \mathbb{N}$, $n \geq 1$, the space $AC^n(\Omega)$ is defined as the set of functions $f : \Omega \to \mathbb{R}$ such that f and its derivatives up to order $n - 1$, denoted $f^{(k)}$ for $k = 0, 1, \ldots, n - 1$, are continuous on Ω, and the derivative $f^{(n-1)}$ is absolutely continuous on Ω. That is, $f \in C^{n-1}(\Omega)$ and $f^{(n-1)} \in AC(\Omega)$. This implies that the n-th derivative $f^{(n)}$ exists almost everywhere and belongs to $L^1(\Omega)$.*

Lemma 2.5 (Integral Representation for AC^n functions)**.** *A function $f \in AC^n(\Omega)$ if and only if it can be represented by the Taylor-like integral formula:*

$$f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t - a)^k + \frac{1}{(n - 1)!} \int_a^t (t - s)^{n-1} f^{(n)}(s)\, ds, \quad \forall t \in \Omega.$$

2.2 Special Functions

In this section, we introduce several special functions that play a crucial role in the analysis of fractional calculus: the Gamma function, the Beta function, the Mittag-Leffler function, and the Lerch Transcendent function [70, 66].

2.2.1 Gamma Function

The Gamma function, denoted by $\Gamma(z)$, is a generalization of the factorial function to complex numbers [2].

Definition 2.6 (Gamma Function Integral Representation). *For complex numbers z with positive real part λ ($\Re(z) > 0$), the Gamma function is defined by the integral:*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt.$$

Domain: The Gamma function can be analytically continued to the entire complex plane, except for simple poles at the non-positive integers $z = 0, -1, -2, \ldots$ [2].

Properties: The Gamma function satisfies several important properties [2]:

- *Recursive relation:* $\Gamma(z+1) = z\Gamma(z)$.
- *Relation to factorial:* For any positive integer n, $\Gamma(n) = (n-1)!$.
- *Reflection formula (Euler):* $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ for non-integer z.
- *Multiplication formula (Gauss):* For any integer $n \geq 1$,

$$\Gamma(z)\Gamma\left(z + \frac{1}{n}\right) \cdots \Gamma\left(z + \frac{n-1}{n}\right) = (2\pi)^{(n-1)/2} n^{1/2-nz} \Gamma(nz).$$

2.2.2 Beta Function

The Beta function, denoted by $B(x, y)$, is closely related to the Gamma function [33].

Definition 2.7 (Beta Function Integral Representation). *For complex numbers x, y with positive real parts ($\Re(x) > 0$, $\Re(y) > 0$), the Beta function is defined as:*

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt.$$

Domain: The standard integral definition requires $\Re(x) > 0$ and $\Re(y) > 0$. The function can be extended via its relation to the Gamma function.

Properties: Key properties include [33]:

- *Relation to Gamma function:* $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. This relation defines the Beta function for a wider range of arguments.
- *Symmetry:* $B(x, y) = B(y, x)$.
- *Trigonometric integral representation:*

$$B(x, y) = 2 \int_0^{\pi/2} (\sin t)^{2x-1} (\cos t)^{2y-1} \, dt.$$

2.2.3 Mittag-Leffler Function

The Mittag-Leffler function generalizes the exponential function and appears frequently in the solutions of fractional differential equations [32, 70].

Definition 2.8 (Mittag-Leffler Function). *The one-parameter Mittag-Leffler function $E_\alpha(z)$ is defined by the series:*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0.$$

The two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0,\ \beta > 0 \ \textit{(sometimes } \beta \in \mathbb{C}\textit{)}.$$

Domain: For $\alpha > 0$, the Mittag-Leffler function $E_{\alpha,\beta}(z)$ is an entire function of z, meaning it is defined and analytic for all complex numbers z [32].

Properties: Important properties include [32]:

- *Generalization of e^z:* $E_1(z) = E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$.
- *Other special cases:* $E_{1,2}(z) = \frac{e^z - 1}{z}$, $E_{2,1}(z^2) = \cosh(z)$.
- *Role in fractional calculus:* The Mittag-Leffler function naturally arises when solving linear fractional differential equations with constant coefficients, playing a role analogous to the exponential function in integer-order equations.

2.2.4 Lerch Transcendent Function

The Lerch transcendent function provides a generalization for several functions in analysis, including the Riemann Zeta function and polylogarithms [66].

Definition 2.9 (Lerch Transcendent Series Representation [66]). *Let $v \in \mathbb{C}$ such that $v \neq 0, -1, -2, \ldots$. For complex parameters z and q, the Lerch transcendent function $\Phi(z, q, v)$ is defined by the series:*

$$\Phi(z, q, v) = \sum_{n=0}^{\infty} \frac{z^n}{(n + v)^q}.$$

Domain of Convergence: The convergence of the series depends on z and q:

- The series converges absolutely for $|z| < 1$.
- If $\Re(q) > 1$, the series converges absolutely for $|z| = 1$.
- If $0 < \Re(q) \leq 1$, the series converges for $|z| = 1$ provided $z \neq 1$.

Properties and Relevance:

- *Generalization:* It generalizes the Riemann-Zeta function and also the Hurwitz Zeta function $\zeta(q, v) = \Phi(1, q, v)$ (for $\Re(q) > 1$) and the polylogarithm $\mathrm{Li}_q(z) = z\Phi(z, q, 1)$.
- *Appearance in Fractional Calculus:* This function, particularly for specific values of q, can arise in the explicit representation of Green's functions or solutions for certain fractional boundary value problems, as encountered in the analysis in Chapter 5.
- *Singularity:* As will be discussed in the Riesz-Caputo thermostat chapter, for $q = 1$, $\Phi(z, 1, v)$ exhibits a logarithmic singularity as $z \to 1^-$.

2.3 Fractional Calculus Basics

In this section, we provide essential definitions related to fractional integrals and derivatives that form the basis of this thesis [52, 80]. These definitions rely on the concept of absolutely continuous functions, as introduced in Section 2.1.

Definition 2.10 (Riemann-Liouville Fractional Integral [80])**.** *Let $\alpha > 0$. The left-sided Riemann-Liouville fractional integral of order α of a function $\omega \in L^1([0,T])$ is defined as:*

$$({}_0I_\tau^\alpha \omega)(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - \varsigma)^{\alpha-1} \omega(\varsigma) d\varsigma, \quad \tau \in [0,T].$$

The right-sided Riemann-Liouville fractional integral of order α is defined as:

$$({}_\tau I_T^\alpha \omega)(\tau) = \frac{1}{\Gamma(\alpha)} \int_\tau^T (\varsigma - \tau)^{\alpha-1} \omega(\varsigma) d\varsigma, \quad \tau \in [0,T].$$

Here, Γ is the Euler Gamma function.

Definition 2.11 (Caputo Fractional Derivative [70, 23])**.** *Let $n-1 < \alpha \leq n$ with $n \in \mathbb{N}$. Assume $\omega \in AC^n([0,T])$. The left-sided Caputo fractional derivative of order α is defined as:*

$$({}_0^C D_\tau^\alpha \omega)(\tau) = \frac{1}{\Gamma(n-\alpha)} \int_0^\tau \frac{\omega^{(n)}(\varsigma)}{(\tau - \varsigma)^{\alpha-n+1}} d\varsigma = ({}_0 I_\tau^{n-\alpha} \omega^{(n)})(\tau), \quad \tau \in [0,T].$$

The right-sided Caputo fractional derivative of order α is defined as:

$$({}_\tau^C D_T^\alpha \omega)(\tau) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_\tau^T \frac{\omega^{(n)}(\varsigma)}{(\varsigma - \tau)^{\alpha-n+1}} d\varsigma = (-1)^n ({}_\tau I_T^{n-\alpha} \omega^{(n)})(\tau), \quad \tau \in [0,T].$$

Remark 2.12. *The Caputo derivative is often preferred in applications because its initial conditions involve integer-order derivatives, which often have clearer physical interpretations compared to the fractional-order initial conditions required for Riemann-Liouville derivatives and is more compatible with the classical derivative e.g. the derivative of the constant function is 0.*

Definition 2.13 (Riesz-Caputo Derivative [91])**.** *Let $n-1 < \alpha \leq n$ with $n \in \mathbb{N}$. The Riesz-Caputo derivative of order α for a function ω on $[0,T]$ is defined as a symmetric average combination of the left and right Caputo derivatives:*

$${}_0^{RC} D_{0,T}^\alpha \omega(\tau) = \frac{1}{2} \left({}_0^C D_\tau^\alpha \omega(\tau) + (-1)^n {}_\tau^C D_T^\alpha \omega(\tau) \right), \quad \tau \in (0,T).$$

For the case $1 < \alpha \leq 2$ $(n = 2)$, this simplifies to:

$${}_0^{RC} D_{0,T}^\alpha \omega(\tau) = \frac{1}{2} \left({}_0^C D_\tau^\alpha \omega(\tau) + {}_\tau^C D_T^\alpha \omega(\tau) \right).$$

An alternative integral representation for $n-1 < \alpha < n$ is sometimes given as:

$${}_0^{RC} D_{0,T}^\alpha \omega(\tau) = \frac{1}{2\Gamma(n-\alpha)} \int_0^T \frac{\omega^{(n)}(\varsigma)}{|\tau - \varsigma|^{\alpha-n+1}} sgn(\tau - \varsigma) d\varsigma.$$

This derivative considers contributions from both sides of the point τ, making it suitable for modeling phenomena with spatial symmetry or interaction across the domain [91].

Lemma 2.14 (Riesz-Caputo Inversion Formula [6]). *Let $n-1 < \alpha \leq n$. If $u \in C^n([0,1])$, then*

$$\mathcal{I}^\alpha({}^{RC}D_{0,1}^\alpha u)(t) = u(t) - \frac{1}{2}\sum_{k=0}^{n-1}\left(\frac{u^{(k)}(0)}{k!}t^k + \frac{u^{(k)}(1)}{k!}(1-t)^k\right).$$

For $0 < \alpha < 1$ $(n = 1)$, this simplifies to:

$$\mathcal{I}^\alpha({}^{RC}D_{0,1}^\alpha u)(t) = u(t) - \frac{u(0)+u(1)}{2}.$$

For $1 < \alpha \leq 2$ $(n = 2)$, this simplifies to:

$$\mathcal{I}^\alpha({}^{RC}D_{0,1}^\alpha u)(t) = u(t) - \frac{1}{2}\left(u(0) + u(1) + u'(0)t - u'(1)(1-t)\right).$$

2.4 The Nature of Fractional Calculus

This section delves into the conceptual underpinnings and diverse manifestations of fractional calculus, beginning with its historical roots and exploring its deeper meaning and application in various scientific and engineering disciplines.

2.4.1 Historical Background

The desire of mathematicians to generalize concepts [53] led L'Hopital in 1695 to contact Leibniz by a letter, wondering if the order of derivatives can be extended in a coherent rigorous manner to fractional numbers instead of integers. L'Hopital's exact letter was of content, whether the expression $\frac{d^{\frac{1}{2}}x}{dx}$ can have a mathematical meaning. Leibniz responded that it will lead to a paradox, from which one day useful applications may emerge [67]. And rightly so, many applications in different fields like electronics, optics, modeling fusion models, and even in quantum physics and artificial intelligence found the use of the fractional derivative to be of good help to improve the precision of their model.

In 1819, Lacroix was the first mathematician to publish a paper mentioning a fractional derivative [67]. He showed that for $y = x^m$, we have:

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$$

By letting $n = \frac{1}{2}$ and $m = 1$, he obtained:

$$\frac{d^{\frac{1}{2}}y}{dx} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

However, the primary application of fractional operations did not begin with Lacroix; instead, Abel was the first to employ them in 1823, as it emerged naturally when solving an integral equation that arises in the formulation of the Tautochrone problem.

Figure 2.1 shows particles moving along a tautochrone curve. For an animation, one can check this Wolfram Demonstration.

The Tautochrone Problem:

The tautochrone problem seeks to determine a curve along which a particle, sliding without friction under the influence of gravity, reaches the lowest point in a fixed time, irrespective of its starting position. The historical roots of this problem trace back to

.

Figure 2.1: The cycloid in which all balls arrive at the same time.

Christiaan Huygens, who, in 1673, demonstrated geometrically that the cycloid was the solution. However, the problem was revisited analytically by Abel in 1823 [70], leading to formulations involving integral equations now recognized as part of fractional calculus.

To analyze this problem, we consider a mass m constrained to move along a smooth curve $x = \Psi(y)$ under gravity. The arc-length differential is given by:

$$ds = \sqrt{1 + (\Psi'(y))^2}\, dy. \tag{2.1}$$

Assuming frictionless motion, we invoke the conservation of mechanical energy:

$$\frac{1}{2}mv^2 + mgy = mgy^*. \tag{2.2}$$

Solving for velocity,

$$v = \sqrt{2g(y^* - y)}. \tag{2.3}$$

Rewriting the time differential,

$$dt = \frac{ds}{v} = \frac{\sqrt{1 + (\Psi'(y))^2}}{\sqrt{2g(y^* - y)}} dy. \tag{2.4}$$

Defining the function $\phi(y)$ as:

$$\phi(y) = \frac{\sqrt{1 + (\Psi'(y))^2}}{\sqrt{2g}}, \tag{2.5}$$

we integrate from $y = y^*$ to $y = 0$:

$$\int_0^{t(y^*)} dt = \int_0^{y^*} \frac{\phi(y)}{\sqrt{y^* - y}} dy. \tag{2.6}$$

This integral equation takes the form of Abel's integral equation of the first kind:

$$t(y^*) = \int_0^{y^*} \frac{\phi(y)}{\sqrt{y^* - y}} dy. \tag{2.7}$$

Recognizing this as a convolution,

$$t(y^*) = \phi * y^{-1/2}, \tag{2.8}$$

we apply the Laplace transform,

$$\mathcal{L}\{g * h\}(s) = \mathcal{L}\{g\}(s)\mathcal{L}\{h\}(s). \tag{2.9}$$

Applying this identity,

$$T(s) = \Phi(s) \cdot \frac{\Gamma(1/2)}{s^{3/2}}. \tag{2.10}$$

Solving for $\Phi(s)$,

$$\Phi(s) = \frac{T(s)}{s^{1/2}\Gamma(1/2)}. \tag{2.11}$$

Taking the inverse Laplace transform,

$$\phi(y) = \mathcal{L}^{-1}\left\{\frac{T(s)}{s^{1/2}\Gamma(1/2)}\right\}(y). \tag{2.12}$$

Utilizing known Laplace transform properties,

$$\phi(y) = \frac{1}{\pi}\frac{d}{dy^*}\int_0^{y^*} \frac{t(y^*)}{(y^* - y)^{1/2}} dy. \tag{2.13}$$

For a constant descent time $t(y^*) = k$, we obtain:

$$\phi(y) = \frac{k}{\pi}\frac{d}{dy^*}\int_0^{y^*} (y^* - y)^{-1/2} dy = \frac{k}{\pi}\frac{d}{dy^*}(y^*)^{1/2} = \frac{k}{\pi}\frac{1}{\sqrt{y^*}}. \tag{2.14}$$

Equating this with the definition of $\phi(y)$,

$$\sqrt{1 + (\Psi'(y))^2} = \frac{2gk^2}{\pi^2}\frac{1}{y^*} = 2R\frac{1}{y^*}. \tag{2.15}$$

Solving for $\Psi'(y)$,

$$\Psi'(y) = \sqrt{\frac{2R}{y^*} - 1}. \tag{2.16}$$

This separable differential equation is integrated using the substitution $y = R(1 - \cos t)$, leading to the parametric equations:

$$x = R(t - \sin t), \tag{2.17}$$

$$y = R(1 - \cos t), \tag{2.18}$$

which describe a cycloid.

Fractional Calculus Interpretation

Comparing Abel's integral equation to the Riemann-Liouville fractional integral:

$$J^\gamma f(x) = \frac{1}{\Gamma(\gamma)}\int_0^x f(t)(x - t)^{\gamma - 1} dt, \tag{2.19}$$

we identify the tautochrone equation as the case $\gamma = 1/2$:

$$t(y^*) = \sqrt{\pi} J^{1/2} \phi(y^*). \tag{2.20}$$

Furthermore, differentiating,

$$\phi(y^*) = \frac{1}{\pi} D^1 J^{1/2} t(y^*) = \frac{1}{\pi} D^{1/2} t(y^*). \tag{2.21}$$

This represents the general solution to Abel's Integral Equation of the first kind, demonstrating an early natural appearance of fractional calculus in physics.

2.4.2 The Ontological Question: Fractional Calculus and the Geometry of Nature

The generalization of differential and integral operators from integer to arbitrary (real or complex) order, known as fractional calculus, presents an immediate epistemological challenge. Beyond the algebraic elegance of extending established formulae, what new descriptive power or fundamental insight does this mathematical leap offer? If classical, integer-order calculus is the language adept at describing phenomena in smooth, Euclidean spaces, what then is the quintessential domain, the intrinsic geometric or physical context, for which fractional calculus is not merely a useful tool, but the most natural and revealing mathematical dialect?

The most commonly cited utility of fractional operators lies in their inherent non-locality, enabling them to capture memory effects and hereditary characteristics in dynamical systems [70, 52]. This is a powerful application, as demonstrated in fields like viscoelasticity and anomalous diffusion that we will discuss in future sections. However, one might posit that this "memory" is itself a macroscopic manifestation of underlying microscopic complexities, perhaps geometric in nature.

Concurrently, the pioneering work of Mandelbrot [61] unveiled the ubiquity of fractal geometries in the natural world—objects and processes characterized by self-similarity across scales and, crucially, by non-integer (fractal) dimensions. As noted by Chen, Sun, and Li [25] (see Preface, p. vii, and Chapter 3), traditional integer-order calculus encounters significant difficulties when attempting to describe or analyze processes occurring on or within these fractal media. The very notion of a derivative, reliant on local smoothness, becomes problematic.

The Mandelbrot set, denoted by $\mathcal{M}$, is a renowned example of a fractal arising from a simple iterative process in the complex plane. It is defined based on the behavior of the quadratic recurrence relation:

$$Z_{n+1} = Z_n^2 + c, \tag{2.22}$$

where Z_n and c are complex numbers. For a given complex number c, the sequence $\{Z_n\}_{n=0}^{\infty}$ is generated starting with $Z_0 = 0$. The Mandelbrot set $\mathcal{M}$ consists of all complex values c for which the orbit of $Z_0 = 0$ under the iteration (2.22) remains bounded; i.e., there exists a real number M such that $|Z_n| \leq M$ for all $n \geq 0$.

$$\mathcal{M} = \{c \in \mathbb{C} : \{Z_n\}_{n=0}^{\infty} \text{ with } Z_0 = 0, Z_{n+1} = Z_n^2 + c, \text{ is bounded}\}.$$

The boundary of the Mandelbrot set is an infinitely complex fractal curve with a Hausdorff dimension of 2. The intricate patterns emerge from determining, for each point c in the complex plane, whether the corresponding sequence diverges or remains bounded (Figure 2.2). The coloring often seen in visualizations of the Mandelbrot set typically indicates the rate at which points outside the set diverge to infinity.

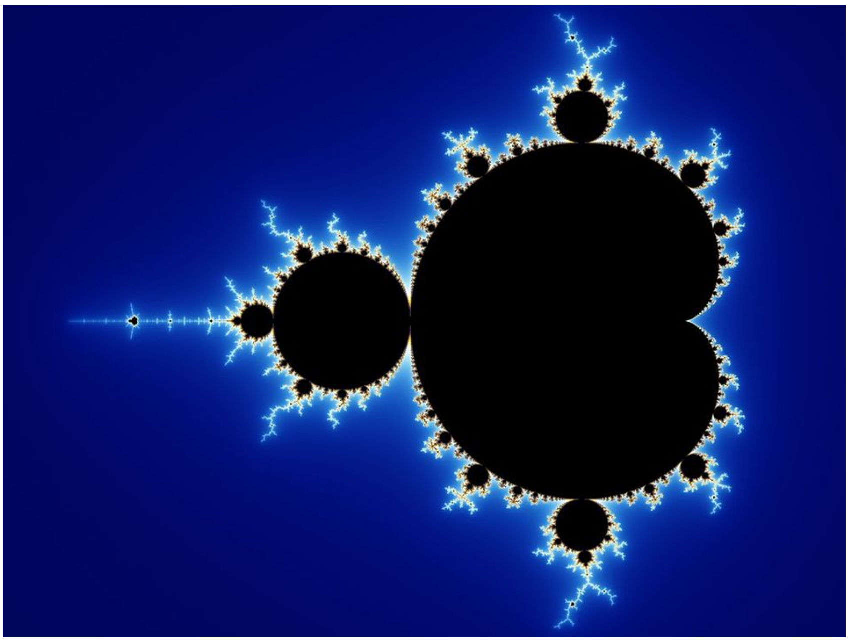


Figure 2.2: The Mandelbrot set. The black region consists of points c for which the iterative sequence $Z_{n+1} = Z_n^2 + c$ (starting with $Z_0 = 0$) remains bounded. The colors outside indicate the number of iterations required for $|Z_n|$ to exceed a certain threshold, signifying divergence.

This confluence begs a profound question: **Is fractional calculus the inherent mathematical language for describing physical laws and processes on fractal geometries?** Could the non-integer order α of a fractional derivative be intrinsically linked to the fractal dimension D_f of the space or manifold upon which a phenomenon unfolds?

Pioneering works have ventured into this conceptual territory. Tatom [88], for instance, investigated quantitative relationships for random fractal processes. A key focus is on fractional Brownian motion (fBm). For a function representing fBm, its fractal dimension D_f is related to the Hurst exponent H by $D_f = 2 - H$. The trail of a standard Brownian motion (where $H = 1/2$) has a fractal dimension $D_t = 2.0$, while for fBm, Tatom presents $D_f - D_t = H - 1/2$ [88, p. 220]. The power spectral density of an fBm signal $X(t)$ behaves as $S_X(\omega) \sim \omega^{-\alpha_{spec}}$, where the spectral exponent $\alpha_{spec} = 2H + 1$. Tatom then considers the Riemann-Liouville fractional integral of order $-q$ (for $q < 0$) or derivative of order q (for $q > 0$), denoted $d^q f/d(t-a)^q$. He argues that if white noise (with $\alpha_{spec} = 0$) is fractionally integrated with order $q = H - 1/2$, the resulting process is fBm with Hurst exponent H. This implies a transformation where the order of fractional integrodifferentiation q is directly linked to the resulting spectral characteristics and thus to H and D_f. For example, to generate a process with a power spectrum $S(\omega) \sim \omega^{-\alpha_{spec}}$ from white noise, one applies fractional integration of order q such that $2q = \alpha_{spec}$. Tatom's analysis (e.g., Fig. 3 in [88]) suggests a linear relationship: an operation of fractional integrodifferentiation of order q applied to a fractal process $X(t)$ with fractal dimension D_X could result in a new process $X'(t)$ with fractal dimension

$D_{X'} = D_X - q$ (for the function itself, interpretation based on his figure relating order of integrodifferentiation directly to the fractal dimension line). This work posits fractional calculus as a tool to *transform and generate* fractal processes with specific dimensional properties.

Building on a different foundation, Butera and Di Paola [20] argue for a "physically based connection" by analyzing viscous fluid flow through a self-similar porous medium modeled as a fractal structure (e.g., a Sierpinski carpet, their Fig. 2). They define the total flux $\phi(t)$ emerging from such a structure as a sum over n-order streams, which in the continuum limit becomes an integral (their Eq. 3):

$$\phi(t) = \chi \ell_0^2 \int_0^\infty d\alpha_s \epsilon^{(D-2)\alpha_s} v(\alpha_s, t),$$

where χ is a form factor, ℓ_0 is the initial stream characteristic length, ϵ is the inverse scaling factor, $D = \ln N / \ln \epsilon$ is the Hausdorff dimension of the fractal cross-section (with N being the branching factor), $v(\alpha_s, t)$ is the fluid velocity in a stream of "order" α_s. By performing a change of variable $x = \epsilon^{-\alpha_s}$ (their Eq. 5), the effective velocity $v_{eff}(t)$ driving the total flux is expressed as (their Eq. 6):

$$v_{eff}(t) = \frac{1}{\log \epsilon} \int_0^1 dx\, x^{1-D} v'(x, t),$$

where $v'(x, t)$ is the velocity in terms of the new variable x. Crucially, when they model the physics of flow using Bernoulli's equation with Darcy-Weisbach losses, they find that for small streams, the velocity $V(x, h)$ (at the exit, for a brick of height h) and the discharge time $T(x, h)$ scale as power laws of x: $V(x, h) = \bar{V}(h) x^\beta$ and $T(x, h) = \bar{T}(h) x^{-\beta}$ (where $\beta = 2$ for their specific physical model, see their Eq. 15). Substituting this into the integral for $v_{eff}(t)$ yields a power-law dependence for the flux itself for $t > 0$ (their Eq. 10):

$$v_{eff}(t) = \frac{\bar{V}'(h)}{\beta\gamma \log \epsilon} t^{-\gamma}, \quad \text{where } \gamma = 1 + \frac{2-D}{\beta}.$$

This directly links the exponent γ of the system's power-law response to the fractal dimension D of the flow geometry and the physical scaling exponent β. Furthermore, by linearizing this response around a fixed pressure p_0 and applying the Boltzmann superposition principle to a pressure history $\Delta p(t)$, they derive a constitutive law involving a Caputo fractional derivative of order γ relating the effective velocity (flux) to the pressure change (their Eq. 22):

$$v_{eff}(t) = C_1 \Delta p \left(t - \bar{T}(p_0)\right) + \frac{C_2}{\Gamma(1-\gamma)} \int_0^{t-\bar{T}(p_0)} \frac{\Delta \dot{p}(\tau)}{(t-\tau)^\gamma} d\tau = C_1 \Delta p + C_2 ({}^{C}_{T(p_0)} D_t^\gamma \Delta p)(t).$$

This demonstrates the *emergence* of fractional operators directly from the interplay of classical physics and the underlying fractal geometry, with the order γ being determined by the fractal dimension D and physical parameters β.

The monograph by Chen, Sun, and Li [25] provides a comprehensive treatment of "Fractional Derivative Modeling in Mechanics and Engineering," with their Chapter 3 dedicated to "Fractal and Fractional Calculus." They underscore the limitations of integer-order calculus for history-dependent processes and for media with fractal structure (Preface, p. vii; Chapter 1, Section 1.2.2). In Section 3.2.2, they examine functions like the Besicovitch function, whose Hausdorff dimension is $D_H(B)$. Applying a fractional integral of order v to obtain $g(t) = D^{-v}B(t)$ or a fractional derivative of order u to get $m(t) = D^u B(t)$, they present relationships suggesting a linear transformation of the dimension (their Fig. 3.20 and surrounding text based on Liang and Su's work):

$$\dim_H(g(t)) \approx \dim_H(B(t)) - v \tag{2.23}$$
$$\dim_H(m(t)) \approx \dim_H(B(t)) + u \quad \text{(their Eq. 3.2.11 for this case).} \tag{2.24}$$

This indicates that fractional operators systematically alter the fractal dimension of such functions. Furthermore, in Chapter 3, Section 3.2.4, "The Kinetic Equations of Fractal Media," they generalize the concept of mass for fractal media. Starting with the integer-dimensional mass $M_3(W) = \int_W \rho(\mathbf{r})d^3\mathbf{r}$, they propose a fractional extension using the Riesz integral of order D (their Eq. 3.2.22, $(I^\alpha f)(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{f(t)}{|x-t|^{n-\alpha}}dt$) to define the mass of a fractal medium of dimension D within a region W as (their Eq. 3.2.23):

$$(I^D\rho)(\mathbf{r}_0) = \int_W \rho(\mathbf{r})dV_D,$$

where dV_D is a fractional volume element. For a homogeneous fractal medium with constant density ρ_0, this leads to $M_D(W) = \rho_0 \pi \frac{2^{5-D}\Gamma(3/2)}{\Gamma(D/2)} R^D$ (their Eq. 3.2.26, for a spherical region of radius R), directly linking the order of the fractional integral D to the mass dimension D. Additionally, in Chapter 2, Section 2.5.1, Chen et al. introduce a "fractal derivative" based on spacetime transformations $\hat{x} = x^\beta, \hat{t} = t^\alpha$, defining it as $\frac{du(t)}{dt^\alpha} = \lim_{t' \to t} \frac{u(t)-u(t')}{t^\alpha - (t')^\alpha}$ (their Eq. 2.5.6).

This formulation leads to fractional diffusion equations like $\frac{\partial^\alpha s}{\partial t^\alpha} + \gamma(-\nabla^2)^\beta s = 0$ (their Eq. 1.2.3 and 2.5.2), where the orders α and β are directly interpreted as scaling indices of the "fractal metric spacetime."

These varied approaches from Tatom, Butera & Di Paola, and Chen, Sun & Li, while distinct in their methodologies—stochastic process analysis, physical modeling on fractal domains, and direct generalization of calculus for fractal functions/media—converge on the theme that non-integer orders in fractional calculus are deeply intertwined with the non-integer dimensions and scaling properties characteristic of fractals.

The pursuit of a fundamental link between fractional calculus and fractal geometry is vital. If established, it would frame fractional calculus as essential for describing a fractal reality, suggesting classical smoothness and integer-dimensionality are idealized limits. While fractional calculus is a powerful modeling tool, its intrinsic geometric connection to fractals remains a key research objective to unify these mathematical generalizations.[1]

2.4.3 Fractional Analysis Inside and Outside of Mathematics

Inside of Mathematics:

Fourier's Definition of Fractional Derivatives

In 1822, Joseph Fourier presented a novel approach to defining fractional derivatives, leveraging his influential work on heat conduction and trigonometric series. Instead of relying on integer-order differentiation, Fourier's method extended the concept to non-integer orders by considering the Fourier transform of a function [85]. The core idea is that

[1] Stephen Wolfram discussed the potential, yet unestablished, link between fractional calculus and fractal geometry in an interview where I asked him the question: (https://youtu.be/Wh-g-TQ2VlY?t=1213).

differentiation, in the Fourier domain, corresponds to multiplication by $(i\omega)^n$ for integer-order n. Fourier generalized this to define the α-th order derivative as multiplication by $(i\omega)^\alpha$ for any real α [85]. Specifically, given a function $u(t)$, with Fourier transform $\hat{u}(\omega)$:

$$\hat{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t) e^{-i\omega t} dt \tag{2.25}$$

Fourier's fractional derivative of order α, denoted by $D^\alpha u(t)$, is defined through its Fourier transform as:

$$\widehat{D^\alpha u}(\omega) = (i\omega)^\alpha \hat{u}(\omega). \tag{2.26}$$

Similarly, the fractional integral of order $\alpha > 0$ is defined by:

$$\widehat{D^{-\alpha} u}(\omega) = (i\omega)^{-\alpha} \hat{u}(\omega). \tag{2.27}$$

This elegant approach makes differentiation an algebraic operation. The definitions imply that the composition formula $D^{\alpha_1}(D^{\alpha_2} u) = D^{\alpha_1+\alpha_2} u$, the fundamental theorem of fractional calculus and the consistency limits such as $\lim_{\alpha\to 0} D^\alpha u = u$, $\lim_{\alpha\to 1} D^\alpha u = u'$ and $\lim_{\alpha\to n} D^\alpha u = u^{(n)}$ hold [85].

The Method of Semigroups for Fractional Derivatives

The Fourier-based definition of fractional derivatives, though powerful, raises several questions, such as for which class of functions $D^\alpha u$ is well-defined. These questions can be addressed by considering the fractional derivative as an operator, specifically, a fractional power of the derivative operator. This is where the concept of semigroups comes into play [85].

The method of semigroups is a powerful tool to generalize operators and their fractional powers. The integer-order derivative can be seen as the infinitesimal generator of the left-translation semigroup $T_\tau u(t) = u(t-\tau)$, $\tau > 0$ [85]. In this view, fractional derivatives and integrals are fractional powers of the derivative operator [85]. By using the Fourier transform, it can be shown that,

$$(i\omega)^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} \frac{e^{-i\tau\omega} - 1}{\tau^{1+\alpha}} d\tau \tag{2.28}$$

This key identity allows to derive an expression for the fractional derivative of order α, $0 < \alpha < 1$ which is:

$$D^\alpha u(t) = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} \frac{u(t-\tau) - u(t)}{\tau^{1+\alpha}} d\tau \tag{2.29}$$

This is an important formula that allows us to compute the fractional derivative by comparing $u(t)$ with past values $u(t-\tau)$. This means fractional derivatives are non-local operators [85]. Moreover, this formula reveals that the correct choice of the phase of $(i\omega)^\alpha$ is the one given by the principal branch of z^α [85]. The semigroup method also highlights the one-sided nature of fractional derivatives, distinguishing between left and right derivatives, which have implications in numerical analysis due to their connection to implicit and explicit methods respectively [85].

Theory of Left-Sided Fractional Derivatives

From the semigroup point of view, the left-sided fractional derivative (also known as the Marchaud-Weyl derivative), denoted as $(D^{\alpha}_{\text{left}}u)(t)$, is defined through the formula [85]:

$$(D^{\alpha}_{\text{left}}u)(t) = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} \frac{u(t-\tau) - u(t)}{\tau^{1+\alpha}} d\tau. \tag{2.30}$$

The fractional integral (also known as the Weyl integral) is given by:

$$(D^{-\alpha}_{\text{left}}u)(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} \frac{u(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{2.31}$$

This operator acts on past values of the function, emphasizing its memory . The theory of left-sided fractional derivatives has been developed rigorously in functional analysis, particularly in the context of weighted Lebesgue spaces. It has been shown that the operators D^{α}_{left} and $D^{-\alpha}_{\text{left}}$ behave like differentiation and integration in terms of their effect on smoothness in the scale of Hölder spaces . The left fractional derivative of a constant vanishes, while the left fractional derivative of a polynomial function $u(t) = (t+\lambda)^{\beta}$, $\lambda > 0$, is given by:

$$(D^{\alpha}_{\text{left}}(t+\lambda)^{\beta})(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t+\lambda)^{\beta-\alpha}. \tag{2.32}$$

Note that if we set $\lambda = 0$, and focus on positive times $t > 0$ we have:

$$(D^{\alpha}_{\text{left}}t^{\beta})(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}, \tag{2.33}$$

which is analogous to the result for the integer-order derivative. The analysis of fractional derivatives has given rise to one-sided Sobolev spaces that appropriately characterize the one-sided structure of fractional derivatives. In particular, the fundamental theorem of fractional calculus holds in the almost everywhere sense and in one-sided weighted Lebesgue spaces, where the weight must satisfy a certain condition known as the A_p condition of Sawyer. Finally, the nonlocal structure of the fractional derivative D^{α} has also been studied in relation to PDEs using the extension method. It has been shown that $D^{\alpha}u$ is the trace of a degenerate two dimensional PDE whose solution is defined in a half space, therefore allowing for the use of PDE methods for the study of fractional order operators.

Outside of Mathematics:

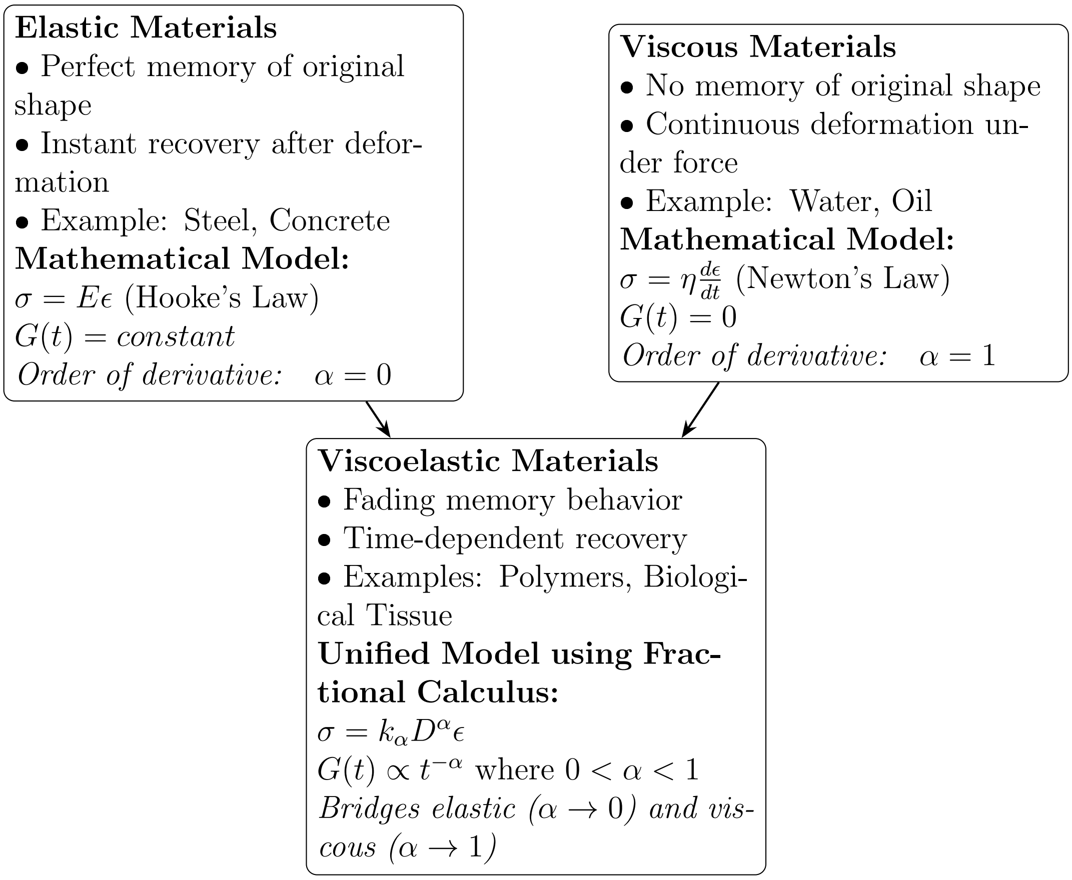
Viscoelasticity: Materials with Memory and Boltzmann

Viscoelasticity: a property of materials that exhibit both elastic (solid-like) and viscous (fluid-like) behavior. Elastic materials, such as steel or concrete, can fully recover their original shape once external forces are removed. In contrast, viscous materials, like honey or oil, deform continuously under force and do not regain their original shape, effectively "remembering" the applied forces [59]. Viscoelastic materials, such as polymers, wood, and human tissue, blend elastic recovery with time-dependent flow, demonstrating mechanical memory under stress [59].

Elastic and Viscous Behavior Contrast: Elastic materials (e.g., steel) follow Hooke's law ($\sigma = E\epsilon$, where σ is stress, ϵ is strain, and E is a material contsant called elastic or Young's modulus), while **viscous fluids** (e.g., water) adhere to Newton's law of viscosity $\sigma = \eta\frac{d\epsilon}{dt}$, where η is the viscosity constant and is fluid-dependent and experimentally-observed [59].

Viscoelasticity and Fading Memory: Viscoelastic materials exhibit *fading memory* [59]. This means their current state of stress and strain depends on the history of applied loads and deformations, but the influence of the distant past gradually fades [59]. Intuitively, ideal liquids "instantly forget" their previous states, and the Navier–Stokes equations that describe them include only local time derivatives of the velocity field. In contrast, ideal elastic materials possess a "perfect memory," as they consistently return to their original configuration and therefore have "no memory" of any forces previously applied and released. Viscoelastic materials fall in between: they retain a memory of their recent states, but as their internal molecular structure undergoes irreversible changes, they lose the memory of their initial configuration [59].

Because fractional derivatives bridge the identity operator and the classical derivative, an intermediate model can be written as: $\sigma = k_\alpha D^\alpha \epsilon$ where k_α is a material constant and α is the fractional order of the derivative $0 < \alpha < 1$. (when $\alpha = 0$: elastic and we have $k_0 = E$ and when $\alpha = 1$: viscous we have $k_1 = \eta$) [59] This model actually comes from experimental data; Viscoelastic material behavior can be studied through stress relaxation tests. These tests apply a constant strain $\epsilon(t) = \epsilon_0 H(t)$, where H(t) is the Heaviside function, and measure the resulting time-dependent stress $\sigma(t)$. For linear materials, $\sigma(t) = \epsilon_0 G(t)$, where G(t) is the stress relaxation modulus. G(t) represents the material's fading memory and is often experimentally determined [59]. For example for polymer gels $G(t) = kt^{-0.5}$ and flour dough,$G(t) = kt^{-0.36}$ [59]. Many viscoelastic materials exhibit G(t) as a constant multiple of t^α, where $0 < \alpha < 1$ [59].



Applying Boltzmann's principle of superposition i.e for a linear viscoelastic material, the total stress at a given time is the sum of the stresses caused by each individual strain increment applied throughout its history. Mathematically: we know from the relaxation test : $\sigma(t) = \epsilon(0)G(t)$, G(t) represents the stress response at time t resulting from a unit step strain applied at t = 0. and for $\Delta t > 0$: $\sigma(t) = \epsilon(0)G(t) + (\epsilon(\Delta\tau) - \varepsilon(0))G(t - \Delta\tau)$

After N increments and taking $\Delta\tau \to 0$ we get:

$$\sigma(t) = \epsilon(0)G(t) + \int_0^t \epsilon'(\tau)G(t-\tau)d\tau \tag{2.34}$$

From experiment: $G(t) = kt^{-\alpha}$, for $0 < \alpha < 1$. A more flexible model that doesn't include the derivative of the stress and accounting for the material's entire history can be obtained by extending $\epsilon(\tau) \equiv \epsilon(0)$ for $\tau < 0$, extending the integral to $(-\infty, t)$, and integrating by parts:

$$\begin{aligned}\sigma(t) &= \epsilon(0)G(t) + \int_{-\infty}^t (\epsilon(\tau) - \epsilon(t))G'(t-\tau)d\tau \\ &= \epsilon(0)G(t) + c_{k,\alpha}(D^\alpha_{\text{left}}\epsilon)(t),\end{aligned}$$

where $c_{k,\alpha}$ depends on k and α. The fractional derivative model's kernel decay accounts for the material's fading memory, providing an adjustable "material memory" parameter α for describing viscoelastic stress/strain behavior.

Experimental Evidence and Fractional Calculus: For many materials, the relaxation modulus follows a power-law: $G(t) \propto t^{-\alpha}$, where $0 < \alpha < 1$ [26, 59]. This behavior naturally leads to the incorporation of fractional derivatives into the constitutive equations [19, 26, 59].

The use of fractional derivatives offers a compact and elegant way to model viscoelasticity [59]. Fractional-order models can capture a wide range of material responses by adjusting the fractional order, effectively representing the material's memory [59, 95, 87].

Anomalous Diffusion: Departures from Normality

The heat equation, a cornerstone of classical physics, provides a powerful model for describing diffusion processes in various systems [30]. For an ideal one-dimensional metal rod, the heat equation is given by:

$$\frac{\partial u}{\partial t} = \frac{k}{2}\frac{\partial^2 u}{\partial x^2}, \tag{2.35}$$

where $u(x,t)$ represents the absolute temperature at position x and time t, and $k/2 > 0$ is the diffusivity constant. Fourier's insightful observation underlying this equation is that the heat flow between two adjacent molecules is proportional to the minuscule temperature difference between them [30, 43]. This implies a flow of caloric energy from regions of higher concentration to those of lower concentration—an experimentally verified phenomenon. Intriguingly, this process parallels the random movement of a pollen particle suspended on water's surface, a phenomenon meticulously observed under the microscope by the Scottish botanist Robert Brown in 1827.

Einstein, in his seminal work on Brownian motion, derived the heat equation from first principles. A crucial assumption in his derivation is that the direction of a particle's motion is "forgotten" after an infinitesimally short period.

Let us derive the heat equation by considering a simple one-dimensional random walk. Assume a small step size $\Delta x > 0$ and a small time interval $\Delta\tau > 0$. Consider a random walker moving along the x-axis according to these rules: During the time interval $\Delta\tau$, the walker takes one step of size Δx, starting from, for instance, $x = 0$. The walker moves either to the left or the right with a probability of 1/2, independently of previous steps. We aim to determine the probability $u(x,t)$ of finding the walker at position x at time t. Since each step is independent of previous ones, the law of total probability dictates:

$$u(x,t) = \frac{1}{2}u(x-\Delta x, t-\Delta\tau) + \frac{1}{2}u(x+\Delta x, t-\Delta\tau). \tag{2.36}$$

Indeed, at time t, the walker could have arrived at position x from either the previous position $x - \Delta x$ or $x + \Delta x$, each with a probability of 1/2. Denoting the second-order incremental quotient of u with respect to the spatial variable as:

$$\delta_x^2 u(x,t) = \frac{u(x-\Delta x,t) + u(x+\Delta x,t) - 2u(x,t)}{(\Delta x)^2}, \tag{2.37}$$

and subtracting $u(x, t-\Delta\tau)$ from both sides of the probability equation, we obtain:

$$\frac{u(x,t) - u(x,t-\Delta\tau)}{\Delta\tau} = \frac{(\Delta x)^2}{2\Delta\tau}\delta_x^2 u(x,t-\Delta\tau). \tag{2.38}$$

Taking the limit as $\Delta x, \Delta\tau \to 0$, assuming that $\frac{(\Delta x)^2}{\Delta\tau} \to k$, leads to the familiar heat equation:

$$\frac{\partial u}{\partial t} = \frac{k}{2}\frac{\partial^2 u}{\partial x^2}. \tag{2.39}$$

To find the fundamental solution, consider the initial condition $u(x, 0) = \delta_0(x)$, a Dirac delta function or unit impulse concentrated at the origin. Applying the Fourier transform with respect to the spatial variable x, denoted by $\hat{u}(\omega, t)$, yields a family of ordinary differential equations parameterized by the Fourier variable $\omega \in \mathbb{R}$:

$$\frac{\partial \hat{u}}{\partial t}(\omega, t) = -\frac{k}{2}|\omega|^2 \hat{u}(\omega, t), \quad t > 0$$
$$\hat{u}(\omega, 0) = 1.$$

The solution to this initial value problem is $\hat{u}(\omega, t) = e^{-\frac{k}{2}|\omega|^2 t}$. Inverting the Fourier transform gives the fundamental solution:

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{k}{2}\omega^2 t} e^{ix\omega} d\omega = \frac{1}{\sqrt{2\pi k t}} e^{-\frac{x^2}{2kt}}. \tag{2.40}$$

This is the classical Gaussian or normal distribution, representing the probability of finding the (memoryless) random walker at position x at time t. The mean or average position is $\mu = 0$, and the standard deviation is $\sigma = \sqrt{kt}$. Notably, as time progresses, the walker deviates from the origin by an average distance of $\sqrt{kt}$. The mean square displacement, or second moment $\langle x^2 \rangle$, is precisely kt. In fact, the scaling relationship $\langle x^2 \rangle \propto \Delta\tau$, implied by $\frac{(\Delta x)^2}{\Delta \tau} = k$, indicates that the mean square displacement is proportional to the waiting time $\Delta\tau$ between steps.

However, there exist numerous situations where the distribution of a quantity deviates from the normal distribution. Wealth distribution, for example, does not follow a Gaussian distribution; the vast gap between extreme poverty and wealth renders the concept of average wealth almost meaningless. Such phenomena exhibit Pareto distributions, where the probability of encountering extremely wealthy individuals remains positive, and the mean becomes infinite.

In the mid-1970s, researchers began to focus more intently on these non-Gaussian processes, identifying instances where Einstein's assumptions break down. Scher and Montroll, in their influential work [82], observed that electron transport in photocopiers and laser printer machines does not adhere to the diffusion equation. They hypothesized that electrons become trapped in "holes" within the surface of amorphous semiconductors for a certain duration before being released due to a temperature potential. Physicists refer to this phenomenon as diffusion on disordered media, or simply anomalous diffusion. Crucially, they also observed that the probability distribution of waiting times between steps, denoted by $\psi(\tau)$, follows a Pareto power law: $\psi(\tau) \propto \tau^{-(1+\alpha)}$ for large waiting times τ, where $0 < \alpha < 1$. Intuitively, if the waiting times between steps are considerably longer compared to the step size, the random walker will not deviate as much from its starting position as a Gaussian random walker would—a process known as subdiffusion [63, 95]. In these scenarios, the relationship $\langle x^2 \rangle = kt$ no longer holds. Instead, a new subdiffusion power law emerges: $\langle x^2 \rangle = k_\alpha t^\alpha$, where $0 < \alpha < 1$ and $k_\alpha > 0$ [63, 95].

Many experiments and natural observations have revealed the prevalence of anomalous diffusion, encompassing phenomena like the diffusion of lipids and receptors in cell membranes, the transport of molecules within the cytosol and the nucleus, the movement patterns of wild animals [87], sleep-wake transitions [87], the propagation of electric currents in cardiac tissue [87], the avalanche-like behavior of plasma particles [87], and the fluctuations observed in stock markets [87]. Indeed, Klafter and Sokolov assert in [56] that "the clear picture that has emerged over the last few decades is that although these phenomena are called anomalous, they are abundant in everyday life: anomalous is the new normal!"

To model these departures from classical diffusion, new mathematical frameworks have been developed. These include continuous-time random walks with power-law waiting times [63], elephant random walks that incorporate memory [83], and nonlocal master equations [84].

Let's consider a random walker exhibiting memory effects. The walker follows the same spatial dynamics as before, taking steps of size $\Delta x > 0$ to the left or right with probability $1/2$, but now also pauses at each location for a random period. Thus, there is a random waiting time between steps. Let $\psi(n)$ represent the probability of waiting for $n\Delta\tau$ before the next step. We are interested in the probability $u(x,t)$ that the walker has just arrived at position x at time t. The law of total probability gives:

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{1}{2} u(x - \Delta x, t - n\Delta\tau) + \frac{1}{2} u(x + \Delta x, t - n\Delta\tau) \right] \psi(n). \tag{2.41}$$

The term in brackets relates to the probability of arriving at x from either $x - \Delta x$ or $x + \Delta x$. These events occur with probability $1/2$. The infinite sum accounts for the fact that the walker could have been at those positions not only at the previous time $t - \Delta\tau$, but could have been waiting there for a period of time $t - n\Delta\tau$, with the probability of a waiting time of length $n\Delta\tau$ being $\psi(n)$. In accordance with many experimental observations, we assume $\psi(n) = d_\alpha n^{-(1+\alpha)}$, where $0 < \alpha < 1$ and $d_\alpha > 0$ is chosen such that $\sum_{n=1}^{\infty} \psi(n) = 1$. Using this, we can rewrite the equation above in terms of second-order incremental quotients in space as:

$$\frac{u(x,t) - \sum_{n=1}^{\infty} u(x, t - n\Delta\tau)\psi(n)}{\Delta\tau} = \frac{(\Delta x)^2}{2\Delta\tau} \sum_{n=1}^{\infty} \delta_x^2 u(x, t - n\Delta\tau)\psi(n). \tag{2.42}$$

In the limit $\Delta x, \Delta\tau \to 0$, assuming that $\frac{(\Delta x)^2}{2d_\alpha(\Delta\tau)^\alpha} \to k_\alpha|\Gamma(-\alpha)|$, we arrive at the time-fractional equation:

$$(D^\alpha_{\text{left}} u)(x,t) = k_\alpha \frac{\partial^2 u}{\partial x^2}. \tag{2.43}$$

To find the fundamental solution, suppose we are given the past condition $u(x,t) = \delta_0(x)$, a unit impulse concentrated at the origin, for all times $t \leq 0$. Applying the Fourier transform in space yields:

$$\begin{aligned} (D^\alpha_{\text{left}} \hat{u})(\omega,t) &= -k_\alpha|\omega|^2 \hat{u}(\omega,t), \quad t > 0 \\ \hat{u}(\omega,0) &= 1. \end{aligned}$$

We have already encountered this problem in the population growth model example. The solution is given in terms of the Mittag-Leffler function as:

$$\hat{u}(\omega,t) = E_{\alpha,1}(-k_\alpha|\omega|^2(t_+)^\alpha). \tag{2.44}$$

Using the scaling properties of the Fourier transform, for $t > 0$, we find:

$$u(x,t) = \frac{1}{(k_\alpha t^\alpha)^{1/2}} H_\alpha \left(\frac{|x|}{(k_\alpha t^\alpha)^{1/2}} \right). \tag{2.45}$$

The profile function $H_\alpha(r)$, $r > 0$, is a Fox–Wright function. This fractional approach successfully models the memory-dependent random walks observed in complex systems

where classical diffusion theory fails.
As we can see, fractional calculus extends beyond a mere mathematical curiosity to provide essential tools for modeling complex phenomena in physics, engineering, and biology. The non-local nature of fractional derivatives naturally captures memory effects and power-law behaviors observed in various systems. From its historical origins in questions posed by L'Hopital to Leibniz, through rigorous mathematical development by Fourier and others, to modern applications in materials science and anomalous diffusion, fractional calculus demonstrates the profound connection between abstract mathematical generalizations and physical reality. The unified framework offered by fractional operators bridges previously disparate models, allowing for more accurate descriptions of viscoelastic materials, complex diffusion processes, and other phenomena where memory and history dependence play crucial roles. As Klafter and Sokolov noted, "the clear picture that has emerged over the last few decades is that although these phenomena are called anomalous, they are abundant in everyday life: anomalous is the new normal!" [56]. Future research directions include developing more efficient computational methods for fractional differential equations, exploring applications in emerging fields like complex networks and artificial intelligence, and further understanding the fundamental connections between fractional operators and the physical world.

2.5 Fixed Point Theorems

Fixed point theorems are fundamental tools for proving the existence (and sometimes uniqueness) of solutions to differential and integral equations. These theorems provide powerful frameworks for transforming a differential equation problem into an equivalent fixed point problem for a suitable operator in a Banach space.

Theorem 2.15 (Banach Fixed Point Theorem (Contraction Mapping Principle) [16])**.** *Let (X, d) be a non-empty complete metric space. Let $T : X \to X$ be a contraction mapping, i.e., there exists a constant $k \in [0, 1)$ such that for all $x, y \in X$,*

$$d(T(x), T(y)) \leq k\, d(x, y).$$

Then T has a unique fixed point $x^* \in X$ (i.e., $T(x^*) = x^*$). Furthermore, for any initial point $x_0 \in X$, the sequence $x_{n+1} = T(x_n)$ converges to x^*.

Theorem 2.16 (Schaefer's Fixed Point Theorem [81])**.** *Let E be a Banach space and let $\mathcal{T} : E \to E$ be a completely continuous operator (i.e., continuous and maps bounded sets into relatively compact sets). Define the set*

$$\mathcal{E} = \{u \in E \,:\, u = \lambda \mathcal{T} u \text{ for some } \lambda \in [0, 1]\}.$$

If the set $\mathcal{E}$ is bounded (i.e., there exists a constant $M > 0$ such that $\|u\| \leq M$ for all $u \in \mathcal{E}$), then the operator $\mathcal{T}$ has at least one fixed point in E.

Theorem 2.17 (Krasnoselskii Fixed Point Theorem [55])**.** *Let M be a closed, convex, and nonempty subset of a Banach space X. Suppose that operators A and B map M into X such that:*

1. *$Ax + By \in M$ for all $x, y \in M$.*

2. *A is a contraction mapping on M.*

3. *B is compact and continuous on M.*

Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 2.18 (Guo-Krasnoselskii Fixed Point Theorem on Cones [41]). *Let X be a Banach space, and let $K \subset X$ be a cone. Assume Ω_1, Ω_2 are open bounded subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either:*

1. *$\|Tx\| \leq \|x\|$ for $x \in K \cap \partial\Omega_1$ and $\|Tx\| \geq \|x\|$ for $x \in K \cap \partial\Omega_2$; or*

2. *$\|Tx\| \geq \|x\|$ for $x \in K \cap \partial\Omega_1$ and $\|Tx\| \leq \|x\|$ for $x \in K \cap \partial\Omega_2$.*

Then T has at least one fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Remark 2.19. *The Guo-Krasnoselskii theorem is particularly useful for proving the existence of positive solutions when the operator T maps a cone (often the cone of non-negative functions) into itself.*

Theorem 2.20 (Arzelà-Ascoli Theorem [9, 76]). *Let Ω be a compact metric space (e.g., $\Omega = [a, b]$). A subset F of the space $C(\Omega)$ of continuous functions on Ω is relatively compact (i.e., its closure $\overline{F}$ is compact in $C(\Omega)$) if and only if F is uniformly bounded and equicontinuous.*

- ***Uniformly bounded:*** *There exists a constant $M > 0$ such that $\|f\|_\infty \leq M$ for all $f \in F$.*

- ***Equicontinuous:*** *For every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $f \in F$ and all $x, y \in \Omega$ with $d(x, y) < \delta$, we have $|f(x) - f(y)| < \epsilon$.*

Remark 2.21. *The Arzelà-Ascoli theorem provides the criterion for compactness in $C(\Omega)$, which is essential for verifying the compactness condition in Schauder's theorem, Schaefer's theorem, and Krasnoselskii's theorem (condition 3).*

Theorem 2.22 (Leray-Schauder Nonlinear Alternative [34]). *Let E be a Banach space, C be a closed convex subset of E, and U be an open subset of C with $0 \in U$. Suppose $T : \overline{U} \to C$ is a compact and continuous map (i.e., completely continuous). Then either:*

1. *T has a fixed point in $\overline{U}$; or*

2. *There exists $x \in \partial U$ (the boundary of U relative to C) and $\lambda \in (0, 1)$ such that $x = \lambda T(x)$.*

Remark 2.23. *This theorem is often used to prove existence by showing that the second alternative cannot occur, typically by deriving an a priori bound for all possible solutions x to $x = \lambda T(x)$.*

2.6 Stability Concepts

We introduce basic definitions of stability for dynamical systems, which are relevant when analyzing the behavior of solutions to differential equations.

Definition 2.24 (Lyapunov Stability [51]). *An equilibrium point x_e of a dynamical system $\dot{x} = f(x)$ is said to be **Lyapunov stable** if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that if the initial state $x(0)$ satisfies $\|x(0) - x_e\| < \delta$, then the resulting trajectory $x(t)$ satisfies $\|x(t) - x_e\| < \epsilon$ for all $t \geq 0$.*

Definition 2.25 (Asymptotic Stability [51]). *An equilibrium point x_e is **asymptotically stable** if it is Lyapunov stable and, additionally, it is locally attractive. That is, there exists a $\delta > 0$ such that if $\|x(0) - x_e\| < \delta$, then $\lim_{t\to\infty} x(t) = x_e$ (which is equivalent to $\lim_{t\to\infty} \|x(t) - x_e\| = 0$).*

Definition 2.26 (Hyers-Ulam Stability [44, 92, 72]). *Consider a functional equation (e.g., a differential equation) $\mathcal{D}(u) = 0$. The equation is said to possess **Hyers-Ulam stability** if for every $\epsilon > 0$ and every function v that satisfies the equation approximately, i.e., $\|\mathcal{D}(v)\| \leq \epsilon$, there exists an exact solution u of $\mathcal{D}(u) = 0$ such that the distance between v and u is bounded by $K\epsilon$, where K is a constant independent of v and ϵ. That is, $\|v - u\| \leq K\epsilon$.*

Remark 2.27. *Hyers-Ulam stability essentially means that if a function approximately satisfies a given functional equation, then it is close to an exact solution of that equation [44, 92, 72]. This is a desirable property in mathematical modeling, indicating robustness of the solutions against small errors or perturbations in the equation itself [46]. The concept originated from a question posed by Ulam concerning the stability of group homomorphisms, which was affirmatively answered by Hyers for additive Cauchy functional equations in Banach spaces [92, 44].*

Chapter 3

Non Linear Multi-Points BVPs Involving Riesz-Caputo derivative

3.1 Introduction and Motivation

This chapter is dedicated to the exploration of a specific class of nonlinear multi-point boundary value problems (BVPs). The focus of our investigation lies in problems involving Riesz-Caputo fractional differential equations, coupled with integral boundary conditions. The central aim is to establish the existence of solutions under precisely defined conditions. We employ two powerful analytical tools to achieve this: Krasnoselskii's fixed-point theorem and the Leray-Schauder nonlinear alternative. Moreover, to demonstrate the practical relevance of our findings, we present numerical examples plotting the solution using Python. Specifically, we are concerned with establishing the existence of solutions for the following problem:

$$
\begin{aligned}
&{}_{0}^{RC}D_{0,1}^{\alpha}w(s) = A(s, w(s)), \quad 0 < s < 1, \\
&w(0) = 0, \\
&w(1) = \sum_{j=1}^{k} \lambda_j \int_0^{c_j} w(s)ds.
\end{aligned} \tag{3.1}
$$

Here, we have $1 < \alpha \leq 2$, $\lambda_j > 0$, and $0 < c_j < 1$. The term ${}_{0}^{RC}D_{1}^{\alpha}$ represents the Riesz-Caputo derivative. The function $A : [0, 1] \times \mathbb{R} \to \mathbb{R}$ will be fully defined later. It is assumed that the constants λ_j and c_j adhere to the following restriction:

$$
\sigma = \sum_{j=1}^{k} \lambda_j \frac{c_j^2}{2} \neq 2.
$$

The field of fractional calculus has recently emerged as an indispensable tool for modelling systems characterized by non-locality and memory. Fractional derivatives are capable of representing long-range correlations and non-local interactions that are prevalent in such systems. There has been considerable effort dedicated to the study of this class of differential equations, reflected in numerous studies as indicated in references [3, 14, 48, 50, 39, 36].

To provide a broader context, some relevant works on fractional differential equations are outlined below:

3.1.1 Prior Research and Motivation

In the paper [36], the authors examined a problem involving a p-Laplacian operator with Caputo fractional derivatives and integral boundary conditions, establishing the existence of at least one positive solution using the Guo-Krasnoselskii theorem. Another study, [39], investigated a nonlinear fractional Euler-Lagrange equation incorporating left Riemann-Liouville and right Caputo derivatives. The authors employed the lower and upper solutions method and the Laplace transform to analyze the equation. Moreover, the work [39] also delves into nonlinear Euler-Bernoulli Beam type equations involving both left and right Caputo fractional derivatives.

Among the various types of fractional derivatives, the Riesz-Caputo fractional derivative has gained attraction. In comparison with the Caputo derivative, the Riesz-Caputo offers certain advantages such as being symmetric in nature, demonstrating enhanced stability and being useful for modelling systems with memory by including both left and right side fractional integrals.

In contrast with these prior works, we here aim to provide a solution to a problem encompassing integral boundary conditions. We will utilize Krasnoselskii's fixed point theorem and the Leray-Schauder alternative to demonstrate theexistence. Moreover, we provide detailed numerical examples and illustrations to visualize the solutions, enriching the overall understanding of the system dynamics.

3.1.2 Intuition Behind the Boundary Condition

In our investigation of this class of fractional differential equations, a key element lies in the choice of the multi-point integral boundary condition:

$$w(1) = \sum_{j=1}^{k} \lambda_j \int_0^{c_j} w(s), ds. \tag{3.2}$$

This condition provides a compelling framework for modeling systems with intricate memory effects. The integral term, $\int_0^{c_j} w(s)ds$, naturally serves as a representation of the accumulated history or the average state of the system up to a specific intermediate time c_j. By incorporating a sum of such integrals evaluated at multiple points, we are effectively "sampling" the system's memory at distinct moments in its past. The coefficients λ_j then act as weights, dictating the relative importance of the system's history up to each point c_j in determining the final state $w(1)$. This allows for a nuanced representation where certain periods in the system's history exert a greater influence on its ultimate behavior.

This choice of boundary condition resonates deeply with the nature of the Riesz-Caputo derivative, which inherently considers influences from both the past and the future, reflecting a non-local dependence on the function's values over an interval. Our boundary condition, by connecting the final state to the integrated history across multiple points, complements this non-local characteristic. It establishes a constraint that links the endpoint of the system's evolution to its behavior throughout its history, mirroring how the Riesz-Caputo derivative itself links the rate of change at a point to the function's values over an interval. The Riesz-Caputo derivative at a given moment considers both the progress made so far and the anticipated path ahead.

In terms of system memory, our boundary condition provides a mechanism for modeling **fading memory** where the final state is influenced by a weighted recall of past states at specific time instances. The integrals represent the accumulated memory up to those points, and the weights allow for certain historical periods to be more strongly "remembered" than others. This contrasts with simpler boundary conditions that might only consider the initial state or a single point in the past.

This framework, combining the Riesz-Caputo derivative with our chosen boundary condition, opens avenues for modeling diverse phenomena. Consider, for example, viscoelastic materials where the final deformation depends on the material's configuration at specific earlier times due to its complex relaxation processes. Similarly, in heat conduction within heterogeneous media, the final temperature distribution might be influenced by the average temperatures in different regions at intermediate times. Other potential applications include population dynamics where the final population size is related to population levels at critical earlier stages, control systems where the final control action is based on past measurements, and financial modeling where asset values are influenced by historical market conditions at specific reference points. Further investigation into the specific parameter choices within such models will be crucial for accurately representing the nuances of real-world phenomena.

3.1.3 Important lemmas

Lemma 3.1. *For $n-1<\alpha\leq n$, $n\in\mathbb{N}$, and $w\in C^n([0,1])$, the following hold:*

$$ {}_0I_{s0}^{\alpha C}D_s^{\alpha}w(s) = w(s) - \sum_{i=0}^{n-1}\frac{w^{(i)}(0)s^i}{i!} $$

and

$$ {}_sI_{10}^{\alpha C}D_1^{\alpha}w(s) = w(s) - \sum_{i=0}^{n-1}\frac{(-1)^i w^{(i)}(1)(1-s)^i}{i!}. $$

Thus, we derive:

$$ \begin{aligned} {}_0I_{10}^{\alpha RC}D_1^{\alpha}w(s) &= \frac{1}{2}\left[\left({}_0I_{s0}^{\alpha\ C}D_s^{\alpha} + {}_sI_{10}^{\alpha C}D_s^{\alpha}\right)w(s)\right.\\ &\quad \left. + \left({}_0I_{ss}^{\alpha C}D_s^{\alpha} + {}_sI_{11}^{\alpha C}D_1^{\alpha}\right)w(s)\right]\\ &= \frac{1}{2}\left[{}_0I_{s0}^{\alpha C}D_s^{\alpha} + {}_sI_{11}^{\alpha C}D_1^{\alpha}\right]w(s). \end{aligned} $$

Given that $1<\alpha\leq 2$ and $w\in C^1([0,1])$, we have:

$$ \begin{aligned} {}_0I_{10}^{\alpha RC}D_1^{\alpha}w(s) &= \frac{1}{2}\left(2w(s) - w(0) - w(1) - w'(0)s + w'(1)(1-s)\right) \qquad (3.3)\\ &= w(s) - \frac{1}{2}\left(w(0)+w(1)\right) - \frac{1}{2}(w'(0)s - w'(1)(1-s)). \end{aligned} $$

Lemma 3.2. *Given a function $f\in L^1[0,1]$, the linear boundary value problem:*

$$ \begin{aligned} &{}_0^{RC}D_1^{\alpha}w(s) = f(s), \quad s\in[0,1]\\ &w(0) = 0,\\ &w(1) = \sum_{j=1}^{k}\lambda_j\int_0^{c_j}w(s)ds, \end{aligned} $$

has a unique solution, given by:

$$w(s) = \int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}dx \tag{3.4}$$
$$+ \frac{2s}{\sigma-2}\left[\int_0^1 B(x)\, f(x)(1-x)^{\alpha-2}dx\right.$$
$$\left. + \sum_{j=1}^{k}\lambda_j\left(-\int_0^{c_j}\frac{(c_j-x)^\alpha}{\Gamma(\alpha+1)}f(x)dx + \int_{c_j}^1\frac{(x-c_j)^\alpha}{\Gamma(\alpha+1)}f(x)dx\right)\right],$$

where

$$k(s,x) = \begin{cases} \frac{(s-x)^\alpha}{\Gamma(\alpha+1)} + \frac{(x-s)^\alpha}{\Gamma(\alpha+1)}, & 0 \le x \le s \le 1, \\ \frac{(x-s)^\alpha}{\Gamma(\alpha+1)}, & 0 \le s \le x \le 1, \end{cases}$$

and

$$B(x) = \int_0^1 k(s,x)ds.$$

Proof. Taking the Riesz-Caputo fractional integral of order α from both sides of the fractional differential equation:

$$_0^{RC}D_{0,1}^{\alpha}w(s) = f(s),$$

where $s \in [0,1]$, and by the aid of Lemma 2.1, and recalling that $w(0)=0$, we have:

$$_0I_1{}_0^{\alpha RC}D_1^{\alpha}w(s) = \frac{1}{2}\left(2w(s) - w(1) - w'(0)s + w'(1)(1-s)\right) \tag{3.5}$$
$$= {}_0I_1 f(s).$$

Thus we get:

$$w(s) = \frac{w(1)}{2} + \frac{w'(0)}{2}s - \frac{w'(1)}{2}(1-s) + {}_0I_1 f(s)$$
$$= \frac{w(1)}{2} + \frac{w'(0)}{2}s - \frac{w'(1)}{2}(1-s) + \int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}dx,$$

where ${}_0I_1 f(s)$ can be expressed as the convolution: ${}_0I_1 f(s) = \int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}dx$ with:

$$k(s,x) = \begin{cases} \frac{(s-x)^\alpha}{\Gamma(\alpha+1)} + \frac{(x-s)^\alpha}{\Gamma(\alpha+1)}, & 0 \le x \le s \le 1, \\ \frac{(x-s)^\alpha}{\Gamma(\alpha+1)}, & 0 \le s \le x \le 1, \end{cases}$$

Next, we apply the boundary condition in equation (3.1), obtaining the expression:

$$w(1) = \sum_{j=1}^{k}\lambda_j\int_0^{c_j} w(s)ds,$$

and the expression for $w(s)$:

$$w(s) = \frac{w(1)}{2} + \frac{w'(0)}{2}s - \frac{w'(1)}{2}(1-s) + {}_0I_1 f(s),$$

so we get:

$$w(1) = \sum_{j=1}^{k} \lambda_j \int_0^{c_j} \left(\frac{w(1)}{2} + \frac{w'(0)}{2}s - \frac{w'(1)}{2}(1-s) + \int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}\,dx \right) ds,$$

$$w(1)\left(1 - \sum_{j=1}^{k} \lambda_j \frac{c_j}{2}\right) = \sum_{j=1}^{k} \lambda_j \int_0^{c_j} \left(\frac{w'(0)}{2}s - \frac{w'(1)}{2}(1-s) + \int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}\,dx \right) ds.$$

From this, we get:

$$w(1) = \frac{2}{\sigma - 2} \sum_{j=1}^{k} \lambda_j \int_0^{c_j} \left(\frac{w'(0)}{2}s - \frac{w'(1)}{2}(1-s) + \int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}dx \right) ds, \tag{3.6}$$

where $\sigma = \sum\limits_{j=1}^{k} \lambda_j \frac{c_j^2}{2}$. Next, evaluating:

$$w'(0) = ({}_0I_1 f(s))'|_{s=0}. \tag{3.7}$$

Using the chain rule for the fractional derivative, we have that:

$$(1-s)w'(1) = (\sum_{j=1}^{k} \lambda_j \int_0^{c_j} w(s)ds)|_{s=1}. \tag{3.8}$$

Plugging (3.6),(3.7), and (3.8) into the expression:

$$w(s) = \frac{w(1)}{2} + \frac{w'(0)}{2}s - \frac{w'(1)}{2}(1-s) + {}_0I_1 f(s),$$

we get:

$$w(s) = \frac{s}{\sigma - 2} \sum_{j=1}^{k} \lambda_j \int_0^{c_j} \left(\frac{w'(0)}{2}s - \frac{w'(1)}{2}(1-s) + \int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}\,dx \right) ds + \int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}\,dx.$$

Then solving for w(s) we finally get:

$$w(s) = \int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}dx + \frac{2s}{\sigma-2}\left[\int_0^1 B(x)\, f(x)(1-x)^{\alpha-2}dx + \sum_{j=1}^{k} \lambda_j \left(-\int_0^{c_j} \frac{(c_j - x)^\alpha}{\Gamma(\alpha+1)} f(x)dx + \int_{c_j}^1 \frac{(x-c_j)^\alpha}{\Gamma(\alpha+1)} f(x)dx \right) \right].$$

□

3.2 Existence Results

Let $\mathcal{C} = C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the supremum norm:

$$\|w\| = \sup_{s\in[0,1]} |w(s)|.$$

Define an operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ as follows:

$$\begin{aligned}(\mathcal{F}w)(s) = & \int_0^1 k(s,x)A(x,w(x))(1-x)^{\alpha-2}dx \\ & + \frac{2s}{\sigma-2}\Bigg[\int_0^1 B\,(x)\,A(x,w(x))(1-x)^{\alpha-2}dx \\ & + \sum_{j=1}^{k}\lambda_j\left(-\int_0^{c_j}\frac{(c_j-x)^\alpha}{\Gamma(\alpha+1)}A(x,w(x))dx + \int_{c_j}^1\frac{(x-c_j)^\alpha}{\Gamma(\alpha+1)}A(x,w(x))dx\right)\Bigg].\end{aligned} \tag{3.9}$$

Observe that the existence of solutions to the fractional boundary value problem (3.1) is equivalent to finding fixed points of the operator $\mathcal{F}$.

To establish the existence of at least one solution to the BVP (3.1), we decompose the operator $\mathcal{F}$ into two operators $\mathcal{F}_1$ and $\mathcal{F}_2$, where

$$(\mathcal{F}_1 w)(s) = \int_0^1 k(s,x)A(x,w(x))(1-x)^{\alpha-2}dx, \tag{3.10}$$

and

$$\begin{aligned}(\mathcal{F}_2 w)(s) = & \frac{2s}{\sigma-2}\Bigg[\int_0^1 B(x)A(x,w(x))(1-x)^{\alpha-2}\,dx \\ & + \sum_{j=1}^{k}\lambda_j\left(-\int_0^{c_j}\frac{(c_j-x)^\alpha}{\Gamma(\alpha+1)}A(x,w(x))\,dx + \int_{c_j}^1\frac{(x-c_j)^\alpha}{\Gamma(\alpha+1)}A(x,w(x))\,dx\right)\Bigg].\end{aligned} \tag{3.11}$$

Thus, $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$. We will employ Krasnoselskii's fixed point theorem to prove the existence of at least one solution to the fractional boundary value problem (3.1).

Theorem 3.3. *Assume that:*

1. $|A(s,w)| \le M$ *for all* $s \in [0,1]$*, and all* $w \in \mathbb{R}$.
2. *There exists a constant* $L > 0$ *such that* $|A(s,w_1) - A(s,w_2)| \le L|w_1 - w_2|$*, for each* $s \in [0,1]$*, and all* $w_1, w_2 \in \mathbb{R}$.

Then the fractional boundary value problem (3.1) *has at least one solution on* $[0,1]$ *if*

$$\frac{L}{\Gamma(\alpha+1)} < 1,$$

Proof. Define a closed, bounded, convex, and nonempty subset of a Banach space $\mathcal{C}$ as:

$$M = \{w \in \mathcal{C} : \|w\| \le r\},$$

where the constant r will be defined later. We will show that the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ defined in (3.10) and (3.11) satisfy the conditions of Theorem 2.17. First, we prove that $\mathcal{F}_1 w + \mathcal{F}_2 w \in M$ for all $w \in M$. For any $w_1, w_2 \in M$ we have:

$$(\mathcal{F}_1 w_1)(s) + (\mathcal{F}_2 w_2)(s) = \int_0^1 k(s,x)A(x,w_1(x))(1-x)^{\alpha-2}dx$$

$$+\frac{2s}{\sigma-2}\left[\int_0^1 B\,(x)\,A(x,w_2(x))(1-x)^{\alpha-2}|dx\right.$$

$$\left.+\sum_{j=1}^{k}\lambda_j\left(-\int_0^{c_j}\frac{(c_j-x)^\alpha}{\Gamma(\alpha+1)}A(x,w_2(x))dx+\int_{c_j}^1\frac{(x-c_j)^\alpha}{\Gamma(\alpha+1)}A(x,w_2(x))dx\right)\right].$$

Then for $s \in [0,1]$,

$$|(\mathcal{F}_1 w_1)(s) + (\mathcal{F}_2 w_2)(s)| \le \int_0^1 |k(s,x)A(x,w_1(x))(1-x)^{\alpha-2}|dx$$

$$+|\frac{2s}{\sigma-2}|\left[\int_0^1 |B\,(x)\,A(x,w_2(x))(1-x)^{\alpha-2}|dx\right.$$

$$\left.+\sum_{j=1}^{k}\lambda_j\left(\int_0^{c_j}\frac{|(c_j-x)^\alpha A(x,w_2(x))|}{\Gamma(\alpha+1)}dx+\int_{c_j}^1\frac{|(x-c_j)^\alpha A(x,w_2(x))|}{\Gamma(\alpha+1)}dx\right)\right].$$

Since $|A(s,w)| \le M$ for all $s \in [0,1]$, and all $w \in \mathbb{R}$, we get:

$$|(\mathcal{F}_1 w_1)(s) + (\mathcal{F}_2 w_2)(s)| \le M\int_0^1 |k(s,x)(1-x)^{\alpha-2}|dx$$

$$+|\frac{2M}{\sigma-2}|\left[\int_0^1 |B\,(x)\,(1-x)^{\alpha-2}|dx\right.$$

$$\left.+\sum_{j=1}^{k}\lambda_j\left(\int_0^{c_j}\frac{|(c_j-x)^\alpha|}{\Gamma(\alpha+1)}dx+\int_{c_j}^1\frac{|(x-c_j)^\alpha|}{\Gamma(\alpha+1)}dx\right)\right].$$

Then, evaluating all integrals we finally get:

$$|(\mathcal{F}_1 w_1)(s) + (\mathcal{F}_2 w_2)(s)| \le \frac{M}{\Gamma(\alpha+1)}$$

$$+|\frac{2M}{\sigma-2}|\left[\int_0^1 |B\,(x)\,(1-x)^{\alpha-2}|dx\right.$$

$$\left.+\sum_{j=1}^{k}\lambda_j\left(\frac{c_j^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{(1-c_j)^{\alpha+1}}{\Gamma(\alpha+2)}\right)\right] \le r,$$

by choosing the radius r of the ball $\mathcal{B}_r$ to satisfy $r \ge L_0$, where L_0 is the total bound defined by:

$$L_0 = \frac{M}{\Gamma(\alpha+1)} + |\frac{2M}{\sigma-2}|\left[\int_0^1 |B\,(x)\,(1-x)^{\alpha-2}|dx + \sum_{j=1}^{k}\lambda_j\left(\frac{c_j^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{(1-c_j)^{\alpha+1}}{\Gamma(\alpha+2)}\right)\right].$$

We select $r = L_0$. Since the first step of the proof (which precedes this section) established that $|(\mathcal{F}_1 w_1)(s) + (\mathcal{F}_2 w_2)(s)| \leq L_0$, choosing $r = L_0$ ensures that $\mathcal{F}_1 w_1 + \mathcal{F}_2 w_2 \in \mathcal{B}_r$ for all $w_1, w_2 \in \mathcal{B}_r$.

Next, we show that $\mathcal{F}_1$ is a contraction mapping. Consider $w_1, w_2 \in \mathcal{B}_r$ then for $s \in [0,1]$, we have:

$$|(\mathcal{F}_1 w_1)(s) - (\mathcal{F}_1 w_2)(s)| \leq \int_0^1 |k(s,x)(A(x, w_1(x)) - A(x, w_2(x)))(1-x)^{\alpha-2}| dx$$

$$\leq L \|w_1 - w_2\| \int_0^1 |k(s,x)(1-x)^{\alpha-2}| dx \leq \frac{L \|w_1 - w_2\|}{\Gamma(\alpha+1)},$$

which means that:

$$|(\mathcal{F}_1 w_1)(s) - (\mathcal{F}_1 w_2)(s)| \leq \frac{L \|w_1 - w_2\|}{\Gamma(\alpha+1)}.$$

Thus, $\mathcal{F}_1$ is a contraction mapping, given that

$$\frac{L}{\Gamma(\alpha+1)} < 1,$$

Finally, we show that the operator $\mathcal{F}_2$ is completely continuous. We have to show that $\mathcal{F}_2$ is uniformly bounded and equicontinuous. Then, for each $w \in M$ we get that:

$$|(\mathcal{F}_2 w)(s)| = \left|\frac{2s}{\sigma - 2}\right| \left[\int_0^1 \left|B(x)\, A(x, w(x))(1-x)^{\alpha-2}\right| dx\right.$$

$$\left. + \sum_{j=1}^{k} \lambda_j \left(\int_0^{c_j} \left|\frac{(c_j - x)^\alpha A(x, w(x))}{\Gamma(\alpha+1)}\right| dx + \int_{c_j}^1 \left|\frac{(x - c_j)^\alpha A(x, w(x))}{\Gamma(\alpha+1)}\right| dx\right)\right],$$

Since $|A(s, w)| \leq M$ for all $s \in [0,1]$, and all $w \in \mathbb{R}$, we get:

$$|(\mathcal{F}_2 w)(s)| \leq \left|\frac{2M}{\sigma - 2}\right| \left[\int_0^1 \left|B(x)\,(1-x)^{\alpha-2}\right| dx\right.$$

$$\left. + \sum_{j=1}^{k} \lambda_j \left(\int_0^{c_j} \frac{|(c_j - x)^\alpha|}{\Gamma(\alpha+1)} dx + \int_{c_j}^1 \frac{|(x - c_j)^\alpha|}{\Gamma(\alpha+1)} dx\right)\right] < \infty.$$

Thus, $\mathcal{F}_2$ is uniformly bounded. Next, we show that $\mathcal{F}_2$ is equicontinuous. For $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ we get:

$$|(\mathcal{F}_2 w)(t_2) - (\mathcal{F}_2 w)(t_1)| = |\frac{2t_2}{\sigma - 2}| \left[\int_0^1 |B(x)\, A(x, w(x))(1-x)^{\alpha-2}| dx\right.$$

$$\left. + \sum_{j=1}^{k} \lambda_j \left(\int_0^{c_j} \frac{|(c_j - x)^\alpha A(x, w(x))|}{\Gamma(\alpha+1)} dx + \int_{c_j}^1 \frac{|(x - c_j)^\alpha A(x, w(x))|}{\Gamma(\alpha+1)} dx\right)\right],$$

$$-|\frac{2t_1}{\sigma - 2}| \left[\int_0^1 |B(x)\, A(x, w(x))(1-x)^{\alpha-2}| dx\right.$$

$$\left. + \sum_{j=1}^{k} \lambda_j \left(\int_0^{c_j} \frac{|(c_j - x)^\alpha A(x, w(x))|}{\Gamma(\alpha+1)} dx + \int_{c_j}^1 \frac{|(x - c_j)^\alpha A(x, w(x))|}{\Gamma(\alpha+1)} dx\right)\right],$$

$$= |\frac{2(t_2 - t_1)}{\sigma - 2}| \left[\int_0^1 |B(x)\, A(x, w(x))(1-x)^{\alpha-2}| dx \right.$$

$$\left. + \sum_{j=1}^{k} \lambda_j \left(\int_0^{c_j} \frac{|(c_j - x)^\alpha A(x, w(x))|}{\Gamma(\alpha+1)} dx + \int_{c_j}^1 \frac{|(x - c_j)^\alpha A(x, w(x))|}{\Gamma(\alpha+1)} dx \right) \right] \to 0,$$

independently of w as $t_2 \to t_1$. Thus, by the Arzelà-Ascoli theorem, $\mathcal{F}_2$ is completely continuous.

Therefore, by Theorem 2.17, the fractional boundary value problem (3.1) has at least one solution on $[0, 1]$. □

3.2.1 Existence via Leray-Schauder Alternative

We now present an alternative existence result for (3.1) using the Leray-Schauder nonlinear alternative.

Let's express the integral equation (3.4) in the form:

$$\begin{aligned} w(s) = &\int_0^1 \mathcal{H}(s,x) A(x, w(x))(1-x)^{\alpha-2}\, dx \\ &+ \frac{2s}{\sigma-2} \sum_{j=1}^{k} \lambda_j \left(- \int_0^{c_j} \frac{(c_j - x)^\alpha}{\Gamma(\alpha+1)} A(x, w(x))\, dx \right. \\ &\left. + \int_{c_j}^1 \frac{(x - c_j)^\alpha}{\Gamma(\alpha+1)} A(x, w(x))\, dx \right) \end{aligned}$$

where

$$\begin{aligned} \mathcal{H}(s,x) = &\frac{1}{\Gamma(\alpha)} \left[\left(\frac{4s}{\sigma-2} + 1 \right) (1-x) - (\alpha-1)(1-s) + |s-x|^{\alpha-1}(1-x)^{2-\alpha} \right. \\ &\left. + \frac{2s}{\sigma-2} \sum_{j=1}^{k} \lambda_j \left(c_j(1-x) - (\alpha-1)\left(c_j - \frac{c_j^2}{2} \right) + \frac{(1-x)^{2-\alpha} x^\alpha}{\alpha} \right) \right] \end{aligned}$$

Lemma 3.4. *The inequality $|\mathcal{H}(s,x)| \le \kappa$ holds, where κ is given by:*

$$\kappa = \frac{1+\alpha}{\Gamma(\alpha)} + \frac{2}{|\sigma-2|} \left(2\alpha + \sum_{j=1}^{k} \lambda_j \left(\alpha^2 c_j + 1 \right) \right).$$

Let us define the integral operator $\mathcal{T}$ as:

$$\begin{aligned} \mathcal{T} w(s) = &\int_0^1 \mathcal{H}(s,x) A(x, w(x))(1-x)^{\alpha-2}\, dx \\ &+ \frac{2s}{\sigma-2} \sum_{j=1}^{k} \lambda_j \left(- \int_0^{c_j} \frac{(c_j - x)^\alpha}{\Gamma(\alpha+1)} A(x, w(x))\, dx \right. \\ &\left. + \int_{c_j}^1 \frac{(x - c_j)^\alpha}{\Gamma(\alpha+1)} A(x, w(x))\, dx \right). \end{aligned}$$

We aim to demonstrate that $\mathcal{T}$ possesses a fixed point, which corresponds to a solution of (3.1).

Theorem 3.5. *Assume $A(\cdot,0)$ is non-identically zero and that there exist non-negative functions $\psi \in L^1_{\alpha-1}[0,1]$ and $\varphi \in C[0,+\infty)$ with φ non-decreasing, and $\eta > 0$, such that*

$$|A(x,t)| \leq \psi(x)\,\varphi(|t|),$$

for all $0 \leq x \leq 1$ and $t \in \mathbb{R}$, with

$$\left(\kappa + \sum_{j=1}^{k} \frac{2\lambda_j}{|\sigma-2|}\right) \|\psi\|_{L^1_{\alpha-1}} \varphi(\eta) < \eta. \tag{3.12}$$

Then, the nonlinear problem (3.1) admits at least one non-trivial solution in X.

Proof. Let $\mathcal{B}_\eta = \{w \in X : ||w||_X < \eta\}$ where $\eta > 0$. The set $\mathcal{T}(\mathcal{B}_\eta)$ is uniformly bounded. For $w \in \mathcal{B}_\eta$, we have:

$$\begin{aligned}
|\mathcal{T}w(s)| \leq & \int_0^1 |\mathcal{H}(s,x)|\,|A(x,w(x))|\,(1-x)^{\alpha-2}dx \\
& + \frac{2}{|\sigma-2|}\sum_{j=1}^{k}\lambda_j \left(\int_0^{c_j} \frac{(c_j-x)^\alpha}{\Gamma(\alpha+1)}|A(x,w(x))|\,dx\right. \\
& \left. + \int_{c_j}^1 \frac{(x-c_j)^\alpha}{\Gamma(\alpha+1)}|A(x,w(x))|\,dx\right) \\
\leq & \int_0^1 \left(|\mathcal{H}(s,x)| + \frac{2}{|\sigma-2|}\sum_{j=1}^{k}\lambda_j \frac{|c_j-x|^\alpha}{\Gamma(\alpha+1)}\right) \\
& \times |A(x,w(x))|\,(1-x)^{\alpha-2}dx \\
\leq & \left(\kappa + \sum_{j=1}^{k}\frac{2\lambda_j}{|\sigma-2|}\right)\int_0^1 \psi(x)\,\varphi(|w(x)|)\,(1-x)^{\alpha-2}dx \\
\leq & \left(\kappa + \sum_{j=1}^{k}\frac{2\lambda_j}{|\sigma-2|}\right)\|\psi\|_{L^1_{\alpha-1}}\varphi(\eta) \\
< & +\infty.
\end{aligned}$$

The set $\mathcal{T}(\mathcal{B}_\eta)$ is also equicontinuous. For $w \in \mathcal{B}_\eta$ and $0 \leq s_1 \leq s_2 \leq 1$, we have:

$$
\begin{aligned}
&|\mathcal{T}w(s_1) - \mathcal{T}w(s_2)| \\
&\leq \int_0^1 |\mathcal{H}(s_1,x) - \mathcal{H}(s_2,x)|\, |A(x,w(x))|\,(1-x)^{\alpha-2}\,dx \\
&\quad + \frac{2\,(s_2-s_1)}{|\sigma-2|}\sum_{j=1}^{k}\lambda_j \int_0^1 \frac{|c_j-x|^{\alpha}}{\Gamma(\alpha+1)}\,|A(x,w(x))|\,(1-x)^{\alpha-2}\,dx \\
&\leq \left[\frac{4\,(s_2-s_1)}{|\sigma-2|\,\Gamma(\alpha)} + \frac{(\alpha-1)\,(s_2-s_1)}{\Gamma(\alpha)}\right]\int_0^1 |A(x,w(x))|\,(1-x)^{\alpha-2}\,dx \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^1 \left||s_2-x|^{\alpha-1} - |s_1-x|^{\alpha-1}\right|\,|A(x,w(x))|\,(1-x)^{\alpha-2}\,dx \\
&\quad + \frac{2\,(s_2-s_1)}{|\sigma-2|}\sum_{j=1}^{k}\lambda_j\int_0^1 |A(x,w(x))|\,(1-x)^{\alpha-2}\,dx \\
&\leq \left(\frac{4}{|\sigma-2|\,\Gamma(\alpha)} + \frac{(\alpha-1)}{\Gamma(\alpha)} + \frac{2}{|\sigma-2|}\sum_{j=1}^{k}\lambda_j\right)(s_2-s_1)\,\|\psi\|_{L^1_{\alpha-1}}\,\varphi(\eta) \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{s_1}\left[(s_2-x)^{\alpha-1} - (s_1-x)^{\alpha-1}\right]|A(x,w(x))|\,(1-x)^{\alpha-2}\,dx \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{s_1}^{s_2}\left||s_1-x|^{\alpha-1} - |s_2-x|^{\alpha-1}\right|\,|A(x,w(x))|\,(1-x)^{\alpha-2}\,dx \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{s_2}^{1}\left[(x-s_1)^{\alpha-1} - (x-s_2)^{\alpha-1}\right]|A(x,w(x))|\,(1-x)^{\alpha-2}\,dx.
\end{aligned}
$$

This expression tends to 0 as $s_1 \to s_2$.

Hence, $\mathcal{T}(\mathcal{B}_\eta)$ is relatively compact. By Arzela-Ascoli theorem, $\mathcal{T}$ is compact.

Now, setting $\mathcal{O} = \mathcal{B}_\eta$, consider $w \in \partial\mathcal{O}$ such that $w = \lambda\mathcal{T}w$ for $0 < \lambda < 1$. Then,

$$
\begin{aligned}
|w\,(x)| &\leq \lambda\,|\mathcal{T}w(s)| \\
&\leq \left(\kappa + \sum_{j=1}^{k}\frac{2\lambda_j}{|\sigma-2|}\right)\|\psi\|_{L^1_{\alpha-1}}\,\varphi\,(\eta)\,.
\end{aligned}
$$

Using (3.12), we have

$$\|w\|_X = \eta < \eta,$$

which is a contradiction. By the Leray-Schauder nonlinear alternative, the operator $\mathcal{T}$ has a fixed point w^* in $\overline{\mathcal{O}}$. Moreover, since $A(\cdot,0)$ is not identically zero, w^* is a non-trivial solution of (3.1). □

3.2.2 Illustrative Examples

3.2.3 Example 1

Consider the problem (3.1) with $\alpha = \frac{3}{2}$, $\lambda_j = 2^{j+1}$, $c_j = \frac{1}{j+1}$ for $j = 1,2,3$ and

$$A(x,t) = \frac{1}{60}\sqrt{1-x}e^{-10x}\left(t - \frac{x}{2(t^2+1)}\right).$$

The function $A(x,0) = -\frac{x}{120}\sqrt{1-x}e^{-10x}$ is continuous and not identically zero on $[0,1]$. Moreover,

$$\begin{aligned}|A(x,t) - A(x,r)| &= \frac{1}{60}\sqrt{1-x}e^{-10x}\left|t - r + \frac{x}{2(r^2+1)} - \frac{x}{2(t^2+1)}\right| \\ &\leq \frac{1}{60}\sqrt{1-x}e^{-10x}|t-r|\left|1 + \frac{x}{2}\frac{t+r}{(r^2+1)(t^2+1)}\right| \\ &\leq \psi(x)|t-r|,\end{aligned}$$

where $\psi(x) = \frac{1}{40}\sqrt{1-x}e^{-10x}$. We compute:

$$\begin{aligned}\|\psi\|_{L^1_{\alpha-1}} &= \int_0^1 \frac{1}{40}e^{-10x}dx \\ &= 2.49991 \times 10^{-3}\end{aligned}$$

and

$$\begin{aligned}\sigma &= \sum_{j=1}^{3} 2^{j+1}\left(\frac{1}{j+1}\right)^2 \\ &= \frac{26}{9}\end{aligned}$$

We also have

$$\begin{aligned}L &= \max\left\{\frac{\alpha+1}{\Gamma(\alpha)}, \frac{2}{|2-\sigma|}\left(\frac{2}{\Gamma(\alpha)} + \sum_{j=1}^{3} 2^{j+1}\frac{\alpha^2 c_j + 2}{\Gamma(\alpha+1)}\right)\right\} \\ &= 132.8666.\end{aligned}$$

Thus,

$$\frac{1}{2L} = 3.7631 \times 10^{-3}.$$

We observe that $\|\psi\|_{L^1_{\alpha-1}} < \frac{1}{2L}$. As all conditions of the theorem are satisfied, the problem (3.1) has at least one solution in X.

3.2.4 Example 2

Consider the same problem setup as above, but with:

$$A(x,t) = (1-x)e^{-x}\left(\frac{1}{10} + \arctan t\right).$$

Then, $A(x,0) = (1-x)e^{-x} \neq 0$ and

$$\begin{aligned}|A(x,t)| &= (1-x)e^{-x}\left|\frac{1}{10} + \arctan t\right| \\ &\leq \psi(x)\varphi(|t|),\end{aligned}$$

where $\psi(x) = \sqrt{1-x}e^{-x}$ and $\varphi(t) = \frac{1}{10} + \arctan t$. We calculate:

$$\begin{aligned}\sigma &= \sum_{j=1}^{3} 2^{j+1}\left(\frac{1}{j+1}\right)^2 = \frac{26}{9} \\ \kappa &= \frac{5}{\sqrt{\pi}} + \frac{909}{8} \simeq 116.45 \\ \|\psi\|_{L^1_{\alpha-1}} &\simeq 5.2677 \times 10^{-3}.\end{aligned}$$

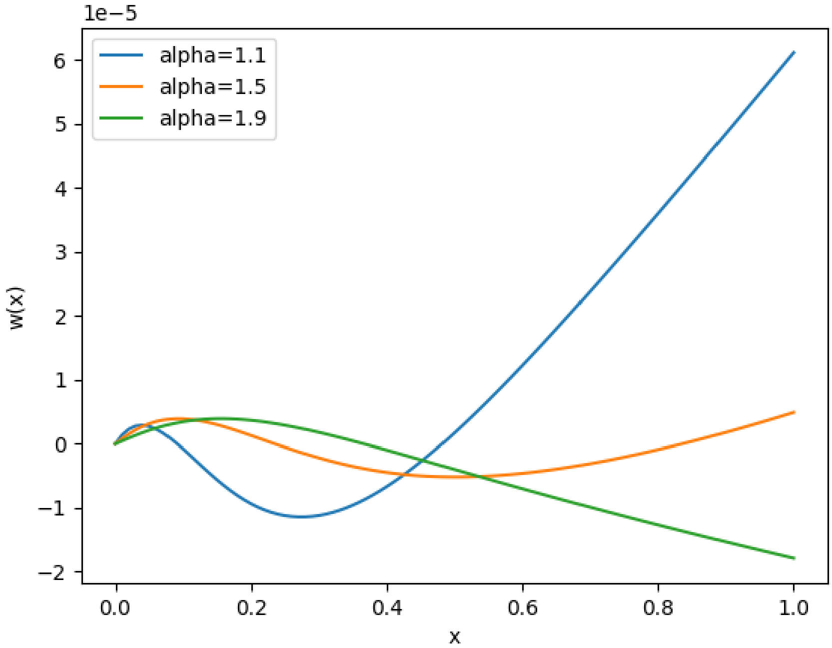


Figure 3.1: Plot of $w(s)$ for different values of α

With $\eta = 1$, we get:

$$\left(\kappa + \sum_{j=1}^{k} \frac{2\lambda_j}{|\sigma - 2|}\right) \|\psi\|_{L^1_{\alpha-1}} \varphi(\eta) = 0.83693 < \eta = 1.$$

As all of Theorem's conditions are satisfied, we conclude that (3.1) has at least one solution in X.

3.2.5 Numerical Discretization Scheme

To numerically solve the given problem, we consider the spatial domain as $[0, 1]$. We create a mesh of N equal intervals with $h = 1/N$ and $x_l = lh$ for $0 \leq l \leq N$. Following [90], we can approximate the left Caputo fractional derivative as:

$$\begin{aligned}
&\frac{1}{\Gamma(2-\alpha)}\int_0^x \frac{u^{(2)}(\xi)d\xi}{(x-\xi)^{\alpha-1}} \\
&= \frac{1}{\Gamma(2-\alpha)}\int_0^x \frac{u^{(2)}(x-\xi)d\xi}{\xi^{\alpha-1}} \\
&= \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{l-1}\int_{jh}^{(j+1)h} \frac{u^{(2)}(x-\xi)d\xi}{\xi^{\alpha-1}} \\
&\approx \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{l-1} \frac{u(x_l-(j-1)h)-2u(x_l-(j)h)+u(x_l-(j+1)h)}{h^2} \\
&\quad \times \int_{jh}^{(j+1)h} \frac{d\xi}{\xi^{\alpha-1}} \\
&= \frac{h^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{l-1} (u_{l-j+1}-2u_{l-j}+u_{l-j-1})\left[(j+1)^{2-\alpha}-j^{2-\alpha}\right],
\end{aligned}$$

Where $u(x_l-(j)h)=u_{l-j}$. We used Python to plot solutions based on these approximations, with the plots presented in the published article.

3.3 Conclusion and Future Directions

In this chapter, we have successfully established the existence of solutions for a specific class of Riesz-Caputo fractional boundary value problems (BVPs) with integral boundary conditions, using both the Krasnoselskii fixed-point Theorem and the Leray-Schauder nonlinear alternative. The numerical examples, alongside the accompanying visualisations, demonstrate the practical relevance of our findings.

Chapter 4

Riesz-Caputo Pantograph Equation

This chapter aims to study a Riesz-Caputo pantograph equation, proving the existence, uniqueness, positivity and stability of the solution.

The structure of this chapter is as follows: Section 4.1 formally introduces the boundary value problem and derives its equivalent integral formulation. Section 4.2 defines the relevant functional space and the associated fixed-point operator. Section 4.3 presents the conditions for existence and uniqueness of solutions. Section 4.4 investigates the conditions for obtaining positive solutions. Section 4.5 analyzes the Ulam-Hyers stability. Section 5.6 outlines a numerical approach. Section 4.7 provides a concrete example, and Section 4.8 summarizes the chapter's contributions.

4.1 Problem Formulation and Equivalent Integral Equation

We consider the following (BVP) for a nonlinear fractional pantograph-type delay differential equation involving the Riesz-Caputo derivative:

$$\begin{aligned} {}^{RC}D^{\alpha}_{0,1}w(s) &= A(s, w(s), w(qs)), \quad s \in (0,1) \\ w(0) &= a, \quad w(1) = b, \end{aligned} \tag{4.1}$$

where $0 < \alpha < 1$, $0 < q < 1$, $a, b \in \mathbb{R}$ are given constants, the order α lies in $[0,1]$, and $A : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

$$ {}^{RC}D^{\alpha}_{0,s}w(s) = \frac{1}{2}\left({}^{C}D^{\alpha}_{0,s}w(s) - {}^{C}D^{\alpha}_{s,1}w(s) \right), \quad s \in (0,1), $$

where ${}^{C}D^{\alpha}_{0,s}$ and ${}^{C}D^{\alpha}_{s,1}$ are the left and right Caputo derivatives, respectively (Definitions 2.3 and 2.4, Chapter 2).

To analyze the BVP (4.1), we convert it into an equivalent integral equation using the properties of the Riesz-Caputo operator. We use the integral operator $\mathcal{I}^{\alpha}$ associated with the Riesz-Caputo derivative:

Definition 4.1 (Integral Operator $\mathcal{I}^{\alpha}$)**.** *For $\alpha > 0$ and $h \in L^1([0,1])$, the operator $\mathcal{I}^{\alpha}$ is defined as:*

$$ (\mathcal{I}^{\alpha}h)(s) := \frac{1}{\Gamma(\alpha)} \int_0^1 |s - x|^{\alpha-1} h(x)dx, \quad s \in [0,1]. \tag{4.2}$$

The relationship between the Riesz-Caputo derivative and this integral operator is crucial:

Lemma 4.2 ([6]). *Let $0 < \alpha < 1$. If $w \in C^1([0,1])$ (or a suitable space where ${}^{RC}D^{\alpha}_{0,s}w$ is defined), then*

$$\mathcal{I}^{\alpha}({}^{RC}D^{\alpha}_{0,s}w)(s) = w(s) - \frac{w(0) + w(1)}{2}. \tag{4.3}$$

Using this lemma, we establish the equivalence between the BVP and an integral equation:

Lemma 4.3 (Integral Equation Equivalence). *Let $h \in C([0,1])$. A function $w \in C([0,1])$ (possessing sufficient regularity for ${}^{RC}D^{\alpha}_{0,s}w$ to be defined and equal to h) is a solution to the linear fractional boundary value problem*

$$\begin{gathered} {}^{RC}D^{\alpha}_{0,1}w(s) = h(s), \quad s \in (0,1), \\ w(0) = a, \quad w(1) = b, \end{gathered}$$

if and only if w satisfies the integral equation

$$w(s) = \frac{a+b}{2} + \frac{1}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1} h(x)dx. \tag{4.4}$$

Proof. Assume w solves the linear BVP. Applying the operator $\mathcal{I}^{\alpha}$ to ${}^{RC}D^{\alpha}_{0,1}w(s) = h(s)$ and using Lemma 4.2, we get:

$$w(s) - \frac{w(0) + w(1)}{2} = \mathcal{I}^{\alpha}h(s).$$

Substituting the boundary conditions $w(0) = a$ and $w(1) = b$ directly yields (4.4). Conversely, assume w satisfies (4.4). Under suitable regularity conditions on w (which are often implied by the properties of the operator $\mathcal{I}^{\alpha}$ and the space of continuous functions), applying ${}^{RC}D^{\alpha}_{0,s}$ to both sides yields ${}^{RC}D^{\alpha}_{0,s}w(s) = {}^{RC} D^{\alpha}_{0,s}(\mathcal{I}^{\alpha}h)(s)$.

For $\alpha \in (0,1)$, ${}^{RC}D^{\alpha}_{0,1}$ is the inverse operator of $\mathcal{I}^{\alpha}$ on suitable spaces, and the derivative of a constant is zero, so ${}^{RC}D^{\alpha}_{0,s}w(s) = h(s)$. The boundary conditions $w(0) = a$ and $w(1) = b$ can be verified by carefully evaluating (4.4) at $s = 0$ and $s = 1$, assuming the integral definition extends appropriately to the boundaries. Thus, the equivalence holds. □

Applying Lemma 4.3 to the nonlinear BVP (4.1) with $h(s) = A(s, w(s), w(qs))$, we conclude that solving the BVP (4.1) is equivalent to finding a solution w to the integral equation:

$$w(s) = \frac{a+b}{2} + \frac{1}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1} A(x, w(x), w(qx))dx. \tag{4.5}$$

4.2 Fixed point formulation

We analyze the existence and uniqueness of solutions to the integral equation derived in the previous section. The appropriate functional space for searching for continuous solutions is the Banach space $X = C([0,1])$, consisting of all continuous real-valued functions defined on $[0,1]$, equipped with the supremum norm $\|w\| = \max_{s\in[0,1]} |w(s)|$. This space is defined in Section 2.1 of Chapter 2.

Solving the integral equation is equivalent to finding a fixed point $w \in X$ for the operator $T : X \to X$ defined by:

$$(Tw)(s) = \frac{a+b}{2} + \frac{1}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1} A(x, w(x), w(qx))dx. \tag{4.6}$$

Let $\gamma = a + b$. For convenience in bounding integral terms involving the kernel $|s - x|^{\alpha-1}$, we use the constant Λ_α. Recall that for $s \in [0,1]$, $\int_0^1 |s-x|^{\alpha-1}dx = \frac{s^\alpha + (1-s)^\alpha}{\alpha}$, and the maximum value of $\frac{s^\alpha+(1-s)^\alpha}{\alpha}$ on $[0,1]$ for $0 < \alpha < 1$ is $\frac{2^{1-\alpha}}{\alpha}$. Using $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, we define:

$$\Lambda_\alpha = \frac{2^{1-\alpha}}{\alpha\Gamma(\alpha)} = \frac{2^{1-\alpha}}{\Gamma(\alpha+1)}. \tag{4.7}$$

Thus, $\max_{s\in[0,1]} \frac{1}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1} dx = \frac{1}{\Gamma(\alpha)} \frac{2^{1-\alpha}}{\alpha} = \Lambda_\alpha$.

We introduce several hypotheses on the function A which will be used in the subsequent analysis. Recall that $A : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H1) There exists a constant $L > 0$ such that for all $x \in [0,1]$ and $y_1, y_2, z_1, z_2 \in \mathbb{R}$:

$$|A(x, y_1, z_1) - A(x, y_2, z_2)| \leq L(|y_1 - y_2| + |z_1 - z_2|).$$

Furthermore, the constant L satisfies:

$$K := 2L\Lambda_\alpha = \frac{L \cdot 2^{2-\alpha}}{\Gamma(\alpha+1)} < 1.$$

(H2) There exist non-decreasing, non-negative continuous functions $f, g : [0,\infty) \to [0,\infty)$ such that for all $x \in [0,1], y, z \in \mathbb{R}$:

$$|A(x, y, z)| \leq f(|y|) + g(|z|) + M_0,$$

where $M_0 = \sup_{s\in[0,1]} |A(s, 0, 0)|$. We assume M_0 is finite.

(H3) There exists a constant $c > 0$ such that $c > |\gamma/2|$ and

$$\frac{c - |\gamma/2|}{f(c) + g(c) + M_0} > \Lambda_\alpha.$$

4.3 Existence and Uniqueness of Solutions

In this section, we employ fixed-point theorems from nonlinear analysis (Section 2.5, Chapter 2) to establish conditions for the existence and uniqueness of solutions to the BVP (4.1).

First, we show that the operator T defined by (4.6) is completely continuous.

Lemma 4.4. *Assume hypothesis (H2) holds. Then the operator $T : X \to X$ defined by (4.6) is completely continuous.*

Proof. The proof relies on standard arguments for Hammerstein-type integral operators and the Arzelà-Ascoli theorem (Theorem 2.20, Chapter 2).

1. *Continuity*: Let $\{w_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $w_n \to w$ in X. Since w_n converges uniformly to w, the sequence $\{w_n\}$ is uniformly bounded. By the continuity of A, $A(x, w_n(x), w_n(qx))$ converges pointwise to $A(x, w(x), w(qx))$ for each $x \in [0,1]$. Furthermore, by the uniform boundedness of w_n and continuity of A, the sequence $\{A(x, w_n(x), w_n(qx))\}$ is uniformly bounded on $[0,1]$. The kernel $|s-x|^{\alpha-1}$ is integrable. By the dominated convergence theorem, the integral converges:

$$\lim_{n\to\infty} \int_0^1 |s-x|^{\alpha-1} A(x, w_n(x), w_n(qx)) dx = \int_0^1 |s-x|^{\alpha-1} A(x, w(x), w(qx)) dx.$$

Moreover, the convergence is uniform with respect to $s \in [0,1]$. Thus, $Tw_n(s) \to Tw(s)$ uniformly, which implies $\|Tw_n - Tw\| \to 0$ as $n \to \infty$. T is continuous.

2. *Mapping Bounded Sets to Bounded Sets*: Let $\mathcal{B}_R = \{w \in X : \|w\| \leq R\}$ be a bounded set in X. For any $w \in \mathcal{B}_R$, using (H2):

$$\begin{aligned}|(Tw)(s)| &\leq \left|\frac{\gamma}{2}\right| + \frac{1}{\Gamma(\alpha)}\int_0^1 |s-x|^{\alpha-1}|A(x,w(x),w(qx))|dx \\ &\leq \left|\frac{\gamma}{2}\right| + \frac{1}{\Gamma(\alpha)}\int_0^1 |s-x|^{\alpha-1}(f(|w(x)|)+g(|w(qx)|)+M_0)dx \\ &\leq \left|\frac{\gamma}{2}\right| + \frac{f(\|w\|)+g(\|w\|)+M_0}{\Gamma(\alpha)}\int_0^1 |s-x|^{\alpha-1}dx \\ &\leq \left|\frac{\gamma}{2}\right| + (f(R)+g(R)+M_0)\Lambda_\alpha.\end{aligned}$$

Thus, $\|Tw\| \leq |\gamma/2| + (f(R)+g(R)+M_0)\Lambda_\alpha$. This upper bound is independent of $w \in \mathcal{B}_R$, so $T(\mathcal{B}_R)$ is bounded.

3. *Equicontinuity*: Let $\mathcal{B}_R$ be a bounded set as above. For $w \in \mathcal{B}_R$, let $A_w(x) = A(x,w(x),w(qx))$. By (H2), $|A_w(x)| \leq f(R)+g(R)+M_0 := N_R$ for all $x \in [0,1]$. For any $s_1, s_2 \in [0,1]$, $s_1 < s_2$:

$$\begin{aligned}|(Tw)(s_2)-(Tw)(s_1)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^1 ||s_2-x|^{\alpha-1}-|s_1-x|^{\alpha-1}||A_w(x)|dx \\ &\leq \frac{N_R}{\Gamma(\alpha)}\int_0^1 ||s_2-x|^{\alpha-1}-|s_1-x|^{\alpha-1}|dx.\end{aligned}$$

The function $k(s,x) = |s-x|^{\alpha-1}$ is continuous for $s \neq x$. The integral $\int_0^1 |k(s_2,x) - k(s_1,x)|dx$ converges to 0 as $s_2 \to s_1$, which can be shown by considering the change of variables or by invoking standard results on continuity of integrals with parameters, accounting for the singularity. This convergence is independent of $w \in \mathcal{B}_R$. Thus, $T(\mathcal{B}_R)$ is equicontinuous.

Since T is continuous, maps bounded sets to bounded sets, and maps bounded sets to equicontinuous sets, the Arzelà-Ascoli Theorem (Theorem 2.20, Chapter 2) implies that $T(\mathcal{B}_R)$ is relatively compact for any bounded set $\mathcal{B}_R$. Therefore, T is completely continuous. □

Using the complete continuity of T, we can prove the existence of at least one solution.

Theorem 4.5 (Existence)**.** *Assume hypotheses (H2) and (H3) hold. Then the BVP* (4.1) *has at least one solution* $w \in X$.

Proof. We apply the Leray-Schauder Alternative Theorem (Theorem 2.22, Chapter 2). From Lemma 4.4, the operator $T: X \to X$ is completely continuous under hypothesis (H2). Let $c > 0$ be the constant specified in hypothesis (H3). Define the open ball $\mathcal{O} = \{w \in X : \|w\| < c\}$. $\mathcal{O}$ is a bounded open subset of the Banach space X, and $0 \in \mathcal{O}$.

The Leray-Schauder Alternative states that either T has a fixed point in $\overline{\mathcal{O}}$, or there exists $w \in \partial\mathcal{O}$ and $\lambda \in (0,1)$ such that $w = \lambda Tw$. Assume the second alternative holds. Then there exists $w \in X$ with $\|w\| = c$ and $\lambda \in (0,1)$ such that $w(s) = \lambda(Tw)(s)$ for all $s \in [0,1]$.

Taking the absolute value, we have $|w(s)| = \lambda|(Tw)(s)|$. Using the definition of T and hypothesis (H2):

$$\begin{aligned}|w(s)| &\leq \lambda\left(\left|\frac{\gamma}{2}\right| + \frac{1}{\Gamma(\alpha)}\int_0^1 |s-x|^{\alpha-1}|A(x,w(x),w(qs))|dx\right) \\ &\leq \lambda\left(\left|\frac{\gamma}{2}\right| + \frac{f(\|w\|)+g(\|w\|)+M_0}{\Gamma(\alpha)}\int_0^1 |s-x|^{\alpha-1}dx\right) \\ &\leq \lambda\left(\left|\frac{\gamma}{2}\right| + (f(\|w\|)+g(\|w\|)+M_0)\Lambda_\alpha\right).\end{aligned}$$

Since $\|w\| = c$, we have $f(\|w\|) = f(c)$ and $g(\|w\|) = g(c)$. Taking the maximum over $s \in [0,1]$:

$$\|w\| \leq \lambda \left(\left| \frac{\gamma}{2} \right| + (f(c) + g(c) + M_0)\Lambda_\alpha \right).$$

Since $\|w\| = c$ and $\lambda < 1$, this implies:

$$c < \left| \frac{\gamma}{2} \right| + (f(c) + g(c) + M_0)\Lambda_\alpha.$$

Rearranging the inequality:

$$c - \left| \frac{\gamma}{2} \right| < (f(c) + g(c) + M_0)\Lambda_\alpha.$$

From hypothesis (H3), $c > |\gamma/2|$ and $f(c) + g(c) + M_0 > 0$ (as $f, g \geq 0$ and $M_0 \geq 0$, and $c > 0$ implies $f(c) + g(c) + M_0$ might be zero only if $A(s, y, z) = 0$ for $|y|, |z| \leq c$, but H3 implies a constraint that makes the denominator non-zero in the relevant context). We can divide by $(f(c) + g(c) + M_0)$:

$$\frac{c - |\gamma/2|}{f(c) + g(c) + M_0} < \Lambda_\alpha.$$

This inequality directly contradicts hypothesis (H3). Therefore, the second alternative of the Leray-Schauder Theorem is false. The first alternative must be true: T has a fixed point w in $\overline{\mathcal{O}}$. This fixed point w is a solution to the integral equation, and thus to the BVP (4.1). □

Next, we establish conditions for the uniqueness of the solution using the Banach Contraction Principle (Theorem 2.15, Chapter 2).

Theorem 4.6 (Uniqueness)**.** *Assume hypothesis (H1) holds. Then the BVP (4.1) has a unique solution in X.*

Proof. We show that the operator $T : X \to X$ is a contraction mapping under hypothesis (H1). Let $w_1, w_2 \in X$. For any $s \in [0, 1]$, using the definition of T and hypothesis (H1):

$$\begin{aligned}
|(Tw_1)(s) - (Tw_2)(s)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^1 |s - x|^{\alpha-1} |A(x, w_1(x), w_1(qs)) - A(x, w_2(x), w_2(qs))| dx \\
&\leq \frac{L}{\Gamma(\alpha)} \int_0^1 |s - x|^{\alpha-1} (|w_1(x) - w_2(x)| + |w_1(qx) - w_2(qx)|) dx.
\end{aligned}$$

Since $|w_1(x) - w_2(x)| \leq \|w_1 - w_2\|$ and $|w_1(qx) - w_2(qx)| \leq \|w_1 - w_2\|$ for all $x \in [0, 1]$ (because $q \in (0, 1)$ maps $[0, 1]$ to $[0, q] \subset [0, 1]$), the inequality continues:

$$\begin{aligned}
|(Tw_1)(s) - (Tw_2)(s)| &\leq \frac{L}{\Gamma(\alpha)} \int_0^1 |s - x|^{\alpha-1} (\|w_1 - w_2\| + \|w_1 - w_2\|) dx \\
&= \frac{2L \|w_1 - w_2\|}{\Gamma(\alpha)} \int_0^1 |s - x|^{\alpha-1} dx \\
&\leq \frac{2L \|w_1 - w_2\|}{\Gamma(\alpha)} \frac{s^\alpha + (1 - s)^\alpha}{\alpha} \\
&\leq \frac{2L \|w_1 - w_2\|}{\Gamma(\alpha)} \max_{s \in [0,1]} \frac{s^\alpha + (1 - s)^\alpha}{\alpha} \\
&= \frac{2L \|w_1 - w_2\|}{\Gamma(\alpha)} \frac{2^{1-\alpha}}{\alpha} \\
&= \frac{L \cdot 2^{2-\alpha}}{\Gamma(\alpha + 1)} \|w_1 - w_2\| = K \|w_1 - w_2\|.
\end{aligned}$$

Taking the maximum over $s \in [0, 1]$, we obtain:

$$\|Tw_1 - Tw_2\| \leq K \|w_1 - w_2\|.$$

By hypothesis (H1), $K < 1$. Therefore, T is a contraction mapping on the complete metric space X. By the Banach Contraction Principle (Theorem 2.15, T has a unique fixed point in X. This unique fixed point is the unique solution to the BVP (4.1). □

4.4 Positivity of Solutions

In many applied problems, the physical meaning of the solution requires it to be non-negative (e.g., population size, concentration, temperature). This section investigates conditions under which the BVP (4.1) possesses a non-negative solution. We utilize the Guo-Krasnoselskii fixed-point Theorem on cones (Theorem 2.18).

Let $\Upsilon = \{w \in X : w(s) \geq 0 \text{ for all } s \in [0, 1]\}$ be the standard cone of non-negative functions in the Banach space $X = C([0, 1])$.

We introduce additional hypotheses on the non-negativity of the boundary data and the function A:

(H4) $\gamma = a + b > 0$.

(H5) $A(x, y, z) \geq 0$ for all $x \in [0, 1]$ and for all $y, z \geq 0$.

Under hypotheses (H4) and (H5), the operator T maps the cone Υ into itself. Indeed, if $w \in \Upsilon$, then $w(s) \geq 0$ and $w(qs) \geq 0$ for all $s \in [0, 1]$ (since $q \in (0, 1)$ maps $[0, 1]$ to $[0, q] \subset [0, 1]$). By (H5), $A(x, w(x), w(qx)) \geq 0$ for all $x \in [0, 1]$. The integral term in the definition of Tw is then non-negative: $\frac{1}{\Gamma(\alpha)} \int_0^1 |s - x|^{\alpha-1} A(x, w(x), w(qs))dx \geq 0$. Since $\gamma/2 = (a + b)/2 > 0$ by (H4), we have $(Tw)(s) = \gamma/2 + (\mathcal{I}^\alpha A_w)(s) \geq 0$ for all $s \in [0, 1]$. Thus, $Tw \in \Upsilon$, meaning $T : \Upsilon \to \Upsilon$.

We require further conditions for the Guo-Krasnoselskii Theorem:

(H6) There exist constants ϱ, ς such that $\varsigma > \gamma/2$ and $\varrho > \max(\varsigma, \gamma/2)$.

(H7) The function A satisfies the following conditions:

  (i) For all $x \in [0, 1]$ and $0 \leq y, z \leq \varrho$: $A(x, y, z) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}}(\varrho - \gamma/2)$.

  (ii) For all $x \in [0, 1]$ and $0 \leq y, z \leq \varsigma$: $A(x, y, z) \geq \Gamma(\alpha + 1)(\varsigma - \gamma/2)$.

Theorem 4.7 (Existence of Positive Solution)**.** *Assume hypotheses (H2), (H4), (H5), (H6), and (H7) hold. Then the BVP (4.1) has at least one positive solution $w \in \Upsilon$ such that $\varsigma \leq \|w\| \leq \varrho$.*

Proof. We apply the Guo-Krasnoselskii fixed-point Theorem (Theorem 2.18, Chapter 2). From Lemma 4.4, T is completely continuous on X under (H2). Since T maps the cone Υ into itself (under (H4) and (H5)), its restriction to Υ is also completely continuous on Υ.

Let $\Omega_1 = \{w \in X : \|w\| < \varsigma\}$ and $\Omega_2 = \{w \in X : \|w\| < \varrho\}$. These are bounded open sets in X. From (H6), $\varrho > \varsigma > \gamma/2 > 0$, which implies $0 < \varsigma < \varrho$, so $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. We check the conditions of the Guo-Krasnoselskii theorem on the boundaries $\partial\Omega_1$ and $\partial\Omega_2$.

1. Consider $w \in \Upsilon \cap \partial\Omega_2$. This means $w \in \Upsilon$ and $\|w\| = \varrho$. Since $w \in \Upsilon$, $w(s) \geq 0$ for all $s \in [0, 1]$. The condition $\|w\| = \varrho$ implies $0 \leq w(s) \leq \varrho$ for all s. Also, $0 \leq w(qs) \leq \varrho$

for all s. Using hypothesis (H7)(i) (which holds for $0 \leq y, z \leq \varrho$) and $\varrho - \gamma/2 > 0$ (from (H6)):

$$\begin{aligned}
(Tw)(s) &= \frac{\gamma}{2} + \frac{1}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1} A(x, w(x), w(qs))dx \\
&\leq \frac{\gamma}{2} + \frac{1}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1} \left(\frac{\Gamma(\alpha+1)}{2^{1-\alpha}} (\varrho - \gamma/2) \right) dx \\
&= \frac{\gamma}{2} + \frac{\Gamma(\alpha+1)(\varrho-\gamma/2)}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1} dx \\
&= \frac{\gamma}{2} + \alpha(\varrho - \gamma/2) \frac{s^\alpha + (1-s)^\alpha}{\alpha} \\
&= \frac{\gamma}{2} + (\varrho - \gamma/2)(s^\alpha + (1-s)^\alpha).
\end{aligned}$$

Taking the maximum over $s \in [0,1]$, $\max_{s\in[0,1]}(s^\alpha + (1-s)^\alpha) = 2^{1-\alpha}$. However, this bound seems to lead to $\varrho - \gamma/2$ rather than the constant Λ_α. Let's re-evaluate the bound using Λ_α:

$$\begin{aligned}
(Tw)(s) &= \frac{\gamma}{2} + \frac{1}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1} A(x, w(x), w(qs))dx \\
&\leq \frac{\gamma}{2} + \frac{1}{\Gamma(\alpha)} \left(\frac{\Gamma(\alpha+1)}{2^{1-\alpha}} (\varrho - \gamma/2) \right) \int_0^1 |s-x|^{\alpha-1} dx \\
&= \frac{\gamma}{2} + \frac{\alpha\Gamma(\alpha)(\varrho-\gamma/2)}{2^{1-\alpha}\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1} dx \\
&\leq \frac{\gamma}{2} + \frac{\alpha(\varrho-\gamma/2)}{2^{1-\alpha}} \frac{2^{1-\alpha}}{\alpha} \\
&= \frac{\gamma}{2} + \varrho - \gamma/2 = \varrho.
\end{aligned}$$

Thus, $\|Tw\| = \max_{s\in[0,1]}(Tw)(s) \leq \varrho = \|w\|$ for $w \in \Upsilon \cap \partial\Omega_2$. This satisfies the second part of the Guo-Krasnoselskii condition (part 2).

2. Consider $w \in \Upsilon \cap \partial\Omega_1$. This means $w \in \Upsilon$ and $\|w\| = \varsigma$. Since $w \in \Upsilon$, $w(s) \geq 0$ for all $s \in [0,1]$. The condition $\|w\| = \varsigma$ implies $0 \leq w(s) \leq \varsigma$ for all s. Also, $0 \leq w(qs) \leq \varsigma$ for all s. Using hypothesis (H7)(ii) (which holds for $0 \leq y, z \leq \varsigma$) and $\varsigma - \gamma/2 > 0$ (from (H6)):

$$\begin{aligned}
(Tw)(s) &= \frac{\gamma}{2} + \frac{1}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1} A(x, w(x), w(qs))dx \\
&\geq \frac{\gamma}{2} + \frac{1}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1} (\Gamma(\alpha+1)(\varsigma - \gamma/2)) dx \\
&= \frac{\gamma}{2} + \frac{\Gamma(\alpha+1)(\varsigma-\gamma/2)}{\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-1} dx \\
&= \frac{\gamma}{2} + \alpha(\varsigma - \gamma/2) \cdot \frac{1}{\alpha} \\
&= \frac{\gamma}{2} + \varsigma - \gamma/2 = \varsigma.
\end{aligned}$$

Since $(Tw)(s) \geq 0$ for all s (as $T : \Upsilon \to \Upsilon$) and $(Tw)(1) \geq \varsigma$, we have $\|Tw\| = \max_{s\in[0,1]}(Tw)(s) \geq (Tw)(1) \geq \varsigma = \|w\|$ for $w \in \Upsilon \cap \partial\Omega_1$. This satisfies the first part of the Guo-Krasnoselskii condition (part 2).

Both conditions of part 2 of the Guo-Krasnoselskii fixed-point theorem are satisfied. Therefore, T has at least one fixed point w in $\Upsilon \cap (\overline{\Omega}_2 \setminus \Omega_1)$. This fixed point is a non-negative solution to the BVP (4.1) such that $\varsigma \leq \|w\| \leq \varrho$. Since $\varsigma > \gamma/2 > 0$ by (H6) and (H4), the solution is non-trivial and positive. □

4.5 Ulam-Hyers Stability

Stability analysis is crucial in assessing the robustness of mathematical models to small perturbations. We now investigate the Ulam-Hyers stability for the BVP (4.1). This type of stability concerns the difference between an approximate solution and the exact solution when the approximate solution satisfies a perturbed version of the original equation.

Definition 4.8 (Ulam-Hyers Stability). *The BVP (4.1) is said to be Ulam-Hyers (UH) stable if there exists a constant $c_A > 0$ such that for every $\epsilon > 0$ and every function $v \in C([0,1])$ satisfying $v(0) = a, v(1) = b$ and the inequality*

$$|{}^{RC}D^{\alpha}_{0,s}v(s) - A(s, v(s), v(qs))| \leq \epsilon, \quad \text{for all } s \in (0,1), \tag{4.8}$$

there exists the unique solution $w \in C([0,1])$ of (4.1) such that

$$|v(s) - w(s)| \leq c_A\epsilon, \quad \text{for all } s \in [0,1].$$

Theorem 4.9 (Ulam-Hyers Stability). *Assume hypothesis (H1) holds. Then the BVP (4.1) is Ulam-Hyers stable.*

Proof. Let $\epsilon > 0$ be given, and let $v \in C([0,1])$ be a function satisfying $v(0) = a, v(1) = b$ and the Ulam-Hyers inequality (4.8). Define the perturbation function $h(s) ={}^{RC} D^{\alpha}_{0,s}v(s) - A(s, v(s), v(qs))$ for $s \in (0,1)$. By assumption, $|h(s)| \leq \epsilon$ for all $s \in (0,1)$.

The function v can be viewed as a solution to the perturbed BVP:

$$\begin{aligned} {}^{RC}D^{\alpha}_{0,s}v(s) &= A(s, v(s), v(qs)) + h(s), \quad s \in (0,1), \\ v(0) &= a, \quad v(1) = b. \end{aligned}$$

Applying Lemma 4.3 to this perturbed problem, the function v must satisfy the integral equation:

$$v(s) = \frac{a+b}{2} + \frac{1}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1}(A(x, v(x), v(qs)) + h(x))dx.$$

By Theorem 4.6, hypothesis (H1) guarantees the existence of a unique solution $w \in C([0,1])$ to the original BVP (4.1). This solution w satisfies the integral equation:

$$w(s) = \frac{a+b}{2} + \frac{1}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1} A(x, w(x), w(qs))dx.$$

Consider the difference between $v(s)$ and $w(s)$:

$$\begin{aligned} |v(s) - w(s)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1} \left[(A(x, v(x), v(qs)) + h(x)) - A(x, w(x), w(qs)) \right] dx \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1} |A(x, v(x), v(qs)) - A(x, w(x), w(qs))| dx \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1} |h(x)| dx. \end{aligned}$$

Using hypothesis (H1) for the first integral and the bound $|h(x)| \le \epsilon$ for the second integral:

$$\begin{aligned}
|v(s) - w(s)| &\le \frac{L}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1}(|v(x)-w(x)| + |v(qx)-w(qx)|)dx \\
&\quad + \frac{\epsilon}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1} dx \\
&\le \frac{L}{\Gamma(\alpha)}(\|v-w\| + \|v-w\|) \int_0^1 |s-x|^{\alpha-1}dx + \epsilon\Lambda_\alpha \\
&= \frac{2L\,\|v-w\|}{\Gamma(\alpha)} \int_0^1 |s-x|^{\alpha-1}dx + \epsilon\Lambda_\alpha \\
&\le \frac{2L\,\|v-w\|}{\Gamma(\alpha)} \frac{s^\alpha + (1-s)^\alpha}{\alpha} + \epsilon\Lambda_\alpha \\
&\le \frac{2L\,\|v-w\|}{\Gamma(\alpha)} \max_{s\in[0,1]} \frac{s^\alpha + (1-s)^\alpha}{\alpha} + \epsilon\Lambda_\alpha \\
&= 2L\,\|v-w\|\,\Lambda_\alpha + \epsilon\Lambda_\alpha.
\end{aligned}$$

Taking the maximum over $s \in [0,1]$ on the left side:

$$\|v-w\| \le 2L\Lambda_\alpha \|v-w\| + \epsilon\Lambda_\alpha.$$

Let $K = 2L\Lambda_\alpha$. By hypothesis (H1), $K < 1$. Rearranging the inequality:

$$\|v-w\| - K\,\|v-w\| \le \Lambda_\alpha \epsilon$$

$$(1-K)\,\|v-w\| \le \Lambda_\alpha \epsilon.$$

Since $1-K > 0$, we can divide:

$$\|v-w\| \le \frac{\Lambda_\alpha}{1-K}\epsilon.$$

This establishes the inequality (5.20) with the Hyers-Ulam stability constant $K = \frac{G^*}{1-LG^*}$. The existence of a unique solution $u(t)$ is guaranteed under the condition $LG^* < 1$, as noted in Remark 5.7 below. Thus, the definition of Hyers-Ulam stability is satisfied. □

Remark 4.10. *The condition $LG^* < 1$ used in Theorem 5.6 implies, via the Banach Fixed Point Theorem (Theorem 2.15, Chapter 2) applied to the operator $\mathcal{T}u(t) = \int_0^1 G(t,s)f(s,u(s))ds$, that the boundary value problem ${}_0^{RC}D_1^\alpha u(t) = f(t,u(t))$ with conditions (5.2) has a unique solution $u(t)$ in $C([0,1])$. This uniqueness is a necessary component of the definition of Hyers-Ulam stability used here.*

4.6 Numerical Approach

Solving fractional differential equations numerically often involves discretizing the equivalent integral equation. In this section, we outline a numerical approach for solving the BVP (4.1) based on the integral formulation (4.6).

We discretize the interval $[0,1]$ into N equally spaced points $s_i = i/N$ for $i = 0, 1, \ldots, N$, with a step size $h = 1/N$. We seek approximate values $w_i \approx w(s_i)$ for the solution w. The integral equation (4.6) is evaluated at the grid points s_i:

$$w(s_i) = \frac{a+b}{2} + \frac{1}{\Gamma(\alpha)} \int_0^1 |s_i - x|^{\alpha-1} A(x, w(x), w(qx))dx.$$

The integral is approximated using a numerical quadrature rule, such as the composite trapezoidal rule. The discrete approximation becomes:

$$w_i \approx \frac{a+b}{2} + \frac{h}{\Gamma(\alpha)} \sum_{j=0}^{N} \omega_j |s_i - s_j|^{\alpha-1} A(s_j, w_j, w(qs_j)), \tag{4.9}$$

where ω_j are the weights of the trapezoidal rule ($\omega_0 = \omega_N = 1$, $\omega_j = 2$ for $j = 1, \ldots, N-1$). Note that the kernel $|s_i - s_j|^{\alpha-1}$ has a singularity at $i = j$. For the trapezoidal rule, this point corresponds to $|0|^{\alpha-1}$, which requires special handling or adapting the quadrature rule near the diagonal. Assuming a suitable treatment of the singularity or choice of quadrature, we proceed.

The term $w(qs_j)$ involves evaluating the solution at points qs_j which may not coincide with the grid points s_k. We approximate $w(qs_j)$ using interpolation based on the grid values w_k. We chose linear interpolation as it is the common choice:

If $s_k \le qs_j < s_{k+1}$, then $w(qs_j) \approx w_k + (qs_j - s_k)\frac{w_{k+1}-w_k}{s_{k+1}-s_k}$. Since $0 < q < 1$, $qs_j \in [0, q]$, which is within the domain $[0, 1]$.

This leads to a system of nonlinear algebraic equations for the values w_i. For the fixed-point operator T, the iterative scheme $w^{(k+1)} = T(w^{(k)})$ can be implemented. Starting with an initial guess, say $w^{(0)}(s) = a + (b-a)s$ (linear interpolation of boundary values), we compute $w_i^{(k+1)}$ using the discrete approximation of the integral:

$$w_i^{(k+1)} = \frac{a+b}{2} + \frac{h}{\Gamma(\alpha)} \sum_{j=0}^{N} \omega_j |s_i - s_j|^{\alpha-1} A(s_j, w_j^{(k)}, \text{interpolate}(w^{(k)}, qs_j)).$$

This iterative process continues until a desired level of convergence $\left\| w^{(k+1)} - w^{(k)} \right\| < \varepsilon$ is achieved. The convergence of this iterative scheme depends on the properties of the operator T and the chosen discretization and interpolation methods. If T is a contraction (under H1), the iteration $w_{k+1} = Tw_k$ in the continuous setting converges to the unique solution. The discrete scheme aims to approximate this process.

The global error of this approach depends on the order of the quadrature rule and the interpolation method. For the trapezoidal rule and linear interpolation, an error of $O(h^2)$ might be expected in the absence of singularities. However, the kernel $|s - x|^{\alpha-1}$ has a singularity at $s = x$, which can reduce the convergence rate, particularly for smaller values of α. Special quadrature rules for singular integrals may improve accuracy.

4.7 Illustrative Example

To illustrate the application of our theoretical results and the numerical approach, consider the following linear boundary value problem:

$$\begin{aligned} {}^{RC}D_{0,s}^{\alpha} w(s) &= 0.2e^{-s} + 0.1\ln(e^{-w(s)} + 1) + 0.05w(qs), \quad s \in [0, 1], \\ w(0) &= 0.5, \quad w(1) = 0.1. \end{aligned}$$

Here, the parameters are fixed at $a = 0.5$, $b = 0.1$, so $\gamma = a + b = 0.6$. The pantograph delay factor is $q = 0.5$. The function A is given by $A(s, y, z) = 0.2e^{-s} + 0.1\ln(e^{-y} + 1) + 0.05z$. This function is continuous for $(s, y, z) \in [0, 1] \times \mathbb{R}^2$.

We verify the hypotheses (H1)-(H7) for this example for different values of $\alpha \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

4.8 Conclusion

This chapter presented a comprehensive analysis of a boundary value problem for a nonlinear fractional pantograph-type delay differential equation involving the Riesz-Caputo derivative.

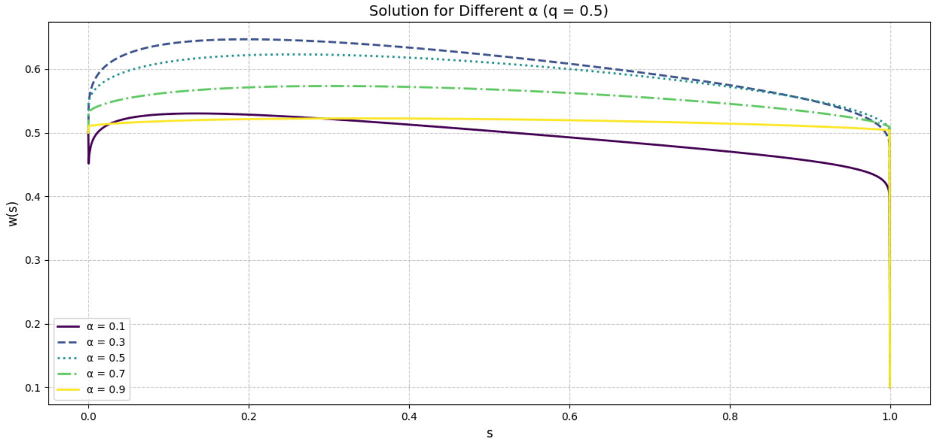


Figure 4.1: Numerical solution profiles for the example BVP with different values of α, $q = 0.5$, $w(0) = 0.5$, $w(1) = 0.1$.

We successfully transformed the problem into an equivalent Volterra-type integral equation, which served as the basis for our theoretical investigation.

By employing standard fixed-point theorems, including the Banach Contraction Principle and the Leray-Schauder Alternative (from Chapter 2), we established verifiable conditions on the nonlinearity A and the fractional order α that guarantee the existence and uniqueness of continuous solutions. Furthermore, we used the Guo-Krasnoselskii fixed-point theorem on cones (from Chapter 2) to identify sufficient conditions for the existence of positive solutions, which are particularly relevant in many physical and biological applications.

The Ulam-Hyers stability analysis demonstrated that, under a Lipschitz condition (which also ensures uniqueness), the solution is stable under small perturbations of the equation, highlighting the robustness of the model.

A numerical discretization strategy based on the integral equation was outlined, and an illustrative example was provided. This example numerically validated the theoretical conditions for existence, uniqueness, and positivity for various fractional orders and visually demonstrated the form of the solutions.

The choice of the symmetric Riesz-Caputo derivative allows for modeling memory and delay effects across the entire domain, making this framework suitable for problems in anomalous diffusion or viscoelasticity within bounded spatial regions.

Future work could extend this analysis to different types of boundary conditions (e.g., nonlocal, fractional), investigate the problem with other types of fractional derivatives, or delve deeper into the numerical analysis of the singular integral equation, potentially exploring higher-order methods or spectral approaches. The application of these results to specific modeling problems in science and engineering also represents a promising direction.

Chapter 5

Riesz-Caputo Thermostat Equation

5.1 Introduction

The thermostat, a quintessential example of a control system [10], has long served as a paradigmatic model for understanding feedback mechanisms in cybernetics [94]. At its core, a thermostat measures an environmental variable (e.g., temperature), compares it to a desired set point, and actuates a response (e.g., heating/cooling) to regulate the system. Classical thermostat models operate on instantaneous measurements, translating them into binary actions—a simplification that neglects the temporal dynamics and memory effects inherent in real-world systems. Modern control theory [10], however, demands more sophisticated frameworks that account for *memory* (dependence on past states) and *anticipation* (projection of future trajectories), particularly when optimizing performance over extended time horizons. Fractional calculus [52], with its nonlocal operators, provides a natural mathematical language to encode these features, bridging the gap between idealized instantaneous control and realistic, history-dependent regulation.

The thermostat model, providing a concrete physical context for nonlocal BVPs, involves regulating temperature based on internal sensor measurements. Classical models utilize second-order ODEs with nonlocal boundary conditions [40, 45] of the form $-u''(t) = f(t, u(t))$. An early contribution in the analysis of such problems was made by Guidotti and Merino in [40], who examined positivity in a model subject to the boundary conditions $u'(0) = 0$ and $\beta u'(1)+u(0) = 0$, demonstrating that a decrease in β affects the positivity of solutions. Later, in [45] the authors introduced a more general thermostat model characterized by an insulated boundary at one end and a feedback control mechanism at the other, which injects or extracts heat depending on the temperature measured at an interior sensor point η. This leads to the boundary condition $\beta u'(1)+u(\eta) = 0$. They proved the existence of positive solutions to the corresponding boundary value problem by applying fixed point index theory in a cone to a Hammerstein integral equation, particularly under the assumption $\beta \geq 1-\eta$ along with additional conditions on the nonlinearity f.

Extending these models to fractional order, as initiated by works like [65], incorporates memory effects into the thermal process or control mechanism. Previous fractional thermostat studies have primarily focused on the left-sided Caputo derivative with various nonlocal conditions [21, 42, 78, 12, 31, 74, 29, 15, 8].

In this chapter, we investigate a nonlinear BVP for a fractional thermostat model that employs the symmetric *Riesz-Caputo (RC) derivative* for the fractional order $1 < \alpha \leq 2$. Unlike unidirectional derivatives, the RC derivative accounts for spatial interactions across the entire interval $[0, 1]$, suitable for systems with global dependencies [52]. Analyzing BVPs with the RC derivative often introduces complexities, especially in the structure of the Green's function

[96, 24, 64]. We consider the following BVP:

$$ {}^{RC}_{0}D^{\alpha}_{1}u(t) = -f(t, u(t)), \quad t \in [0, 1], \quad 1 < \alpha \leq 2, \tag{5.1} $$

$$ u'(0) = 0, \quad \beta\, {}^{C}D^{\alpha-1}u(1) + u(\eta) = 0, \quad \beta > 0, \quad 0 \leq \eta \leq 1. \tag{5.2} $$

Here, f is a continuous function, and the boundary conditions include an insulation condition at $t = 0$ and a nonlocal condition linking an interior point temperature $u(\eta)$ to a fractional derivative of the temperature ${}^{C}D^{\alpha-1}u(1)$ at $t = 1$. For $\alpha = 2$, this recovers the classical problem $-u'' = f$ with $u'(0) = 0, \beta u'(1) + u(\eta) = 0$. The combination of the symmetric RC operator and the fractional nonlocal boundary condition (5.2) poses significant analytical challenges.

The primary objective of this chapter is to provide a rigorous mathematical analysis of the existence, positivity, and stability of solutions to the BVP (5.1)-(5.2). Our main contributions are:

- Deriving the explicit Green's function for the linear problem and analyzing its structure, revealing a logarithmic singularity near $s = 1$.
- Establishing the existence of solutions using Schaefer's fixed-point theorem under general growth conditions on f.
- Developing conditions and employing cone-theoretic methods (Guo-Krasnosel'skii) to prove the existence of positive solutions, addressing the challenges posed by the Green's function's negativity.
- Demonstrating the Hyers-Ulam stability of the problem under a Lipschitz condition on f.
- Including illustrative examples.

The analytical intricacies arising from the Riesz-Caputo derivative and the specific nonlocal fractional boundary condition (5.2) necessitate careful treatment throughout our analysis.

5.2 Solution Representation via Green Function

In this section, we derive the Green's function for the linear version of the BVP (5.1)-(5.2), i.e., ${}^{RC}_{0}D^{\alpha}_{1}u(t) = y(t)$ with boundary conditions (5.2), where $y \in C([0, 1])$. This allows us to represent the solution of the nonlinear BVP (5.1)-(5.2) as an integral equation $u(t) = \int_0^1 G(t, s)(-f(s, u(s)))ds$, where $G(t, s)$ is the Green's function.

Theorem 5.1. *Let $y \in C([0, 1])$. A function $u \in C^2([0, 1])$ is a solution to the linear boundary value problem*

$$ {}^{RC}_{0}D^{\alpha}_{1}u(t) = y(t), \quad t \in [0, 1], \quad 1 < \alpha \leq 2, $$
$$ u'(0) = 0, \quad \beta\, {}^{C}D^{\alpha-1}u(1) + u(\eta) = 0, \quad \beta > 0, \quad 0 \leq \eta \leq 1, $$

if and only if it can be represented as an integral equation:

$$ u(t) = \int_0^1 G(t, s)y(s)ds, \tag{5.3} $$

where $G(t,s)$ is the Green's function given by:

$$
\begin{aligned}
G(t,s) &= \frac{1}{2\Gamma(\alpha)}\left(\frac{\beta}{\Gamma(3-\alpha)}+\eta-t\right)\left[s^{\alpha-1}+(1-s)^{\alpha-1}\right] \\
&+\frac{\beta}{2}\left[1-\frac{1}{\Gamma(2-\alpha)\Gamma(\alpha-1)}\sum_{n=0}^{\infty}\frac{s^{n+\alpha-1}}{n+\alpha-1}\right] \\
&+\frac{1}{2\Gamma(\alpha)}\begin{cases}(\eta-s)^{\alpha-1}-(s-t)^{\alpha-1}, & \text{if } t<s \text{ and } s\le\eta,\\ (s-\eta)^{\alpha-1}-(s-t)^{\alpha-1}, & \text{if } t<s \text{ and } s>\eta,\\ (\eta-s)^{\alpha-1}-(t-s)^{\alpha-1}, & \text{if } t\ge s \text{ and } s\le\eta,\\ (s-\eta)^{\alpha-1}-(t-s)^{\alpha-1}, & \text{if } t\ge s \text{ and } s>\eta.\end{cases}
\end{aligned}
\tag{5.4}
$$

Proof. We construct the Green's function for the linear problem:

$$ {}^{RC}_{0}D^{\alpha}_{1}u(t)=y(t), \quad t\in(0,1), \quad \text{for } 1<\alpha<2. \tag{5.5}$$

Applying the Riesz-Caputo inversion formula (see Lemma 2.14), yields:

$$\mathcal{I}^{\alpha}(y(t))=u(t)-\frac{1}{2}\Big[u(0)+u(1)+u'(0)t-u'(1)(1-t)\Big].$$

Let $\mathcal{Y}(t)=\mathcal{I}^{\alpha}y(t)$ denote the Riesz integral of $y(t)$:

$$\mathcal{Y}(t)=\frac{1}{2\Gamma(\alpha)}\int_0^1|t-s|^{\alpha-1}y(s)\,ds.$$

The general solution is then:

$$u(t)=\frac{1}{2}\Big[u(0)+u(1)+u'(0)t-u'(1)(1-t)\Big]+\mathcal{Y}(t). \tag{5.6}$$

Applying the first boundary condition, $u'(0)=0$, simplifies (5.6) to:

$$u(t)=\frac{1}{2}\Big[u(0)+u(1)-u'(1)(1-t)\Big]+\mathcal{Y}(t). \tag{5.7}$$

Evaluating (5.7) at $t=0$ and $t=1$:

$$u(0)=\frac{1}{2}\Big[u(0)+u(1)-u'(1)\Big]+\mathcal{Y}(0), \tag{5.8}$$

$$u(1)=\frac{1}{2}\Big[u(0)+u(1)\Big]+\mathcal{Y}(1). \tag{5.9}$$

From (5.9), we find $u(1)=u(0)+2\mathcal{Y}(1)$. Substituting this into (5.8) yields:

$$
\begin{aligned}
u(0)&=\frac{1}{2}\Big[u(0)+(u(0)+2\mathcal{Y}(1))-u'(1)\Big]+\mathcal{Y}(0)\\
&=u(0)+\mathcal{Y}(1)-\frac{u'(1)}{2}+\mathcal{Y}(0).
\end{aligned}
$$

This simplifies to $\frac{u'(1)}{2}=\mathcal{Y}(0)+\mathcal{Y}(1)$, thus:

$$
\begin{aligned}
u'(1)&=2\Big(\mathcal{Y}(0)+\mathcal{Y}(1)\Big)\\
&=\frac{1}{\Gamma(\alpha)}\int_0^1\Big[s^{\alpha-1}+(1-s)^{\alpha-1}\Big]y(s)\,ds.
\end{aligned}
\tag{5.10}
$$

Next, we apply the second boundary condition:

$$\beta\, {}^C D^{\alpha-1} u(1) + u(\eta) = 0. \tag{5.11}$$

We compute ${}^C D^{\alpha-1} u(1)$. From (5.7), for $1 < \alpha < 2$ (so $0 < \alpha - 1 < 1$):

$${}^C D^{\alpha-1} u(t) = \frac{1}{2} {}^C D^{\alpha-1} \Big[u(0) + u(1) - u'(1)(1-t) \Big] + {}^C D^{\alpha-1} \mathcal{Y}(t).$$

Let $P(t) = \frac{1}{2}\Big[u(0) + u(1) - u'(1)(1-t)\Big]$. The Caputo derivative of the constant part $u(0)+u(1)$ is zero. For the term involving $(1-t)$:

$${}^C D^{\alpha-1}(1-t) = \frac{-(1-t)^{1-(\alpha-1)}}{\Gamma(2-(\alpha-1))} = \frac{-(1-t)^{2-\alpha}}{\Gamma(3-\alpha)}.$$

So,

$${}^C D^{\alpha-1} P(t) = \frac{1}{2}(-u'(1)) \left(\frac{-(1-t)^{2-\alpha}}{\Gamma(3-\alpha)} \right) = \frac{u'(1)}{2} \frac{(1-t)^{2-\alpha}}{\Gamma(3-\alpha)}.$$

Since $1 < \alpha < 2$, we have $2 - \alpha > 0$. Thus, at $t = 1$, ${}^C D^{\alpha-1} P(1) = 0$. Therefore, for $1 < \alpha < 2$:

$${}^C D^{\alpha-1} u(1) = {}^C D^{\alpha-1} \mathcal{Y}(1).$$

Let $\Theta = {}^C D^{\alpha-1} \mathcal{Y}(1)$. We have:

$$\Theta = \frac{1}{\Gamma(2-\alpha)} \int_0^1 (1-s)^{1-\alpha} \mathcal{Y}'(s)\, ds.$$

Calculating Θ: Step 1: Compute $\mathcal{Y}'(s)$. Differentiating $\mathcal{Y}(t) = \frac{1}{2\Gamma(\alpha)} \int_0^1 |t-\xi|^{\alpha-1} y(\xi)\, d\xi$ (for $1 < \alpha < 2$, so $\alpha - 1 > 0$) yields:

$$\mathcal{Y}'(t) = \frac{\alpha-1}{2\Gamma(\alpha)} \left[\int_0^t (t-\xi)^{\alpha-2} y(\xi)\, d\xi - \int_t^1 (\xi - t)^{\alpha-2} y(\xi)\, d\xi \right].$$

Using $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$, this simplifies to:

$$\mathcal{Y}'(t) = \frac{1}{2\Gamma(\alpha-1)} \left[\int_0^t (t-\xi)^{\alpha-2} y(\xi)\, d\xi - \int_t^1 (\xi - t)^{\alpha-2} y(\xi)\, d\xi \right]. \tag{5.12}$$

Substituting $\mathcal{Y}'(s)$ into Θ, and letting $B = \Gamma(2-\alpha)\Gamma(\alpha-1)$, we have $\Theta = I_1 + I_2$, where

$$I_1 = \frac{1}{2B} \int_0^1 (1-s)^{1-\alpha} \left(\int_0^s (s-t)^{\alpha-2} y(t)\, dt \right) ds,$$
$$I_2 = -\frac{1}{2B} \int_0^1 (1-s)^{1-\alpha} \left(\int_s^1 (t-s)^{\alpha-2} y(t)\, dt \right) ds.$$

Step 2: Simplification of I_1 via Fubini's Theorem and Beta Function Identity.

Recall that:

$$I_1 = \frac{1}{2B} \int_0^1 (1-s)^{1-\alpha} \left(\int_0^s (s-t)^{\alpha-2} y(t)\, dt \right) ds.$$

The domain of integration for I_1 is $0 \le t \le s \le 1$. By Fubini's theorem, we can interchange the order of integration. The new limits will be $0 \le t \le 1$ for the outer integral and $t \le s \le 1$ for the inner integral:

$$I_1 = \frac{1}{2B} \int_0^1 y(t) \left[\int_t^1 (1-s)^{1-\alpha} (s-t)^{\alpha-2}\, ds \right] dt.$$

Let $K_1(t)$ be the inner integral:

$$K_1(t) = \int_t^1 (1-s)^{1-\alpha}(s-t)^{\alpha-2}\,ds.$$

To evaluate $K_1(t)$, we perform a change of variable. Let $s = t + z(1-t)$. Then, $ds = (1-t)dz$. The limits of integration for z are:

- When $s = t$: $t = t + z(1-t) \implies z(1-t) = 0$. If $t \neq 1$, then $z = 0$.
- When $s = 1$: $1 = t + z(1-t) \implies 1 - t = z(1-t)$. If $t \neq 1$, then $z = 1$.

(If $t = 1$, the integral $K_1(1)$ is over an empty range $[1, 1]$ and is 0, which is consistent as $(1-t)^0 = 1$ later if handled by limit). The terms in the integral become:

- $1 - s = 1 - (t + z(1-t)) = 1 - t - z(1-t) = (1-t)(1-z)$.
- $s - t = (t + z(1-t)) - t = z(1-t)$.

Substituting these into $K_1(t)$:

$$\begin{aligned}
K_1(t) &= \int_0^1 [(1-t)(1-z)]^{1-\alpha}\,[z(1-t)]^{\alpha-2}\,(1-t)dz \\
&= \int_0^1 (1-t)^{1-\alpha}(1-z)^{1-\alpha}z^{\alpha-2}(1-t)^{\alpha-2}(1-t)^1 dz \\
&= (1-t)^{(1-\alpha)+(\alpha-2)+1}\int_0^1 (1-z)^{1-\alpha}z^{\alpha-2}dz \\
&= (1-t)^0\int_0^1 (1-z)^{(2-\alpha)-1}z^{(\alpha-1)-1}dz.
\end{aligned}$$

For $t \neq 1$, $(1-t)^0 = 1$. The integral is the definition of the Beta function $B(X,Y) = \int_0^1 v^{X-1}(1-v)^{Y-1}dv$. Here, let $v = z$, then $X - 1 = \alpha - 2 \implies X = \alpha - 1$, and $Y - 1 = 1 - \alpha \implies Y = 2 - \alpha$. So,

$$K_1(t) = \int_0^1 z^{(\alpha-1)-1}(1-z)^{(2-\alpha)-1}dz = B(\alpha-1, 2-\alpha).$$

Using the property $B(X,Y) = \Gamma(X)\Gamma(Y)/\Gamma(X+Y)$:

$$B(\alpha-1, 2-\alpha) = \frac{\Gamma(\alpha-1)\Gamma(2-\alpha)}{\Gamma((\alpha-1)+(2-\alpha))} = \frac{\Gamma(\alpha-1)\Gamma(2-\alpha)}{\Gamma(1)}.$$

Since $\Gamma(1) = 1$, we have $K_1(t) = \Gamma(\alpha-1)\Gamma(2-\alpha)$. Recall that we defined $B = \Gamma(2-\alpha)\Gamma(\alpha-1)$. Thus, $K_1(t) = B$. Substituting this back into the expression for I_1:

$$I_1 = \frac{1}{2B}\int_0^1 y(t)[B]dt = \frac{1}{2}\int_0^1 y(t)\,dt.$$

This simplification holds for $1 < \alpha < 2$, which ensures $\alpha - 1 > 0$ and $2 - \alpha > 0$, so the arguments of the Gamma functions in the Beta function are positive.

Step 3: Simplification of I_2 using Binomial Expansion and Beta Function. Recall that:

$$I_2 = -\frac{1}{2B}\int_0^1 y(t)\left(\int_0^t (1-s)^{1-\alpha}(t-s)^{\alpha-2}\,ds\right)dt.$$

Let $J(t)$ be the inner integral:

$$J(t) = \int_0^t (1-s)^{1-\alpha}(t-s)^{\alpha-2}\,ds.$$

For this integral, $0 \le s < t$. If $t < 1$, then s is also less than 1. We use the generalized binomial theorem (Newton's series) for $(1-s)^{1-\alpha}$, which states that for any real number A and $|x| < 1$:

$$(1-x)^A = \sum_{n=0}^{\infty} \binom{A}{n} (-x)^n = \sum_{n=0}^{\infty} \frac{A(A-1)\dots(A-n+1)}{n!} (-1)^n x^n.$$

Here, $A = 1-\alpha$. Since $0 \le s < t < 1$, the condition $|s| < 1$ for the expansion of $(1-s)^{1-\alpha}$ is met.

$$(1-s)^{1-\alpha} = \sum_{n=0}^{\infty} \binom{1-\alpha}{n} (-s)^n = \sum_{n=0}^{\infty} \binom{1-\alpha}{n} (-1)^n s^n.$$

Substituting this into $J(t)$:

$$J(t) = \int_0^t \left(\sum_{n=0}^{\infty} \binom{1-\alpha}{n} (-1)^n s^n \right) (t-s)^{\alpha-2}\, ds.$$

Assuming uniform convergence of the series multiplied by $(t-s)^{\alpha-2}$ on the interval of integration (which holds for $s \in [0,t]$ as $t < 1$), we can interchange summation and integration:

$$J(t) = \sum_{n=0}^{\infty} \binom{1-\alpha}{n} (-1)^n \int_0^t s^n (t-s)^{\alpha-2}\, ds.$$

The integral $\int_0^t s^n (t-s)^{\alpha-2}\, ds$ is a form of the Beta function integral. Let $s = t\xi$, so $ds = td\xi$. When $s = 0, \xi = 0$; when $s = t, \xi = 1$.

$$\begin{aligned}
\int_0^t s^n (t-s)^{\alpha-2}\, ds &= \int_0^1 (t\xi)^n (t-t\xi)^{\alpha-2} (td\xi) \\
&= \int_0^1 t^n \xi^n t^{\alpha-2} (1-\xi)^{\alpha-2} t d\xi \\
&= t^{n+\alpha-2+1} \int_0^1 \xi^n (1-\xi)^{\alpha-2} d\xi \\
&= t^{n+\alpha-1} \int_0^1 \xi^{(n+1)-1} (1-\xi)^{(\alpha-1)-1} d\xi \\
&= t^{n+\alpha-1} B(n+1, \alpha-1).
\end{aligned}$$

This requires $n+1 > 0$ (true since $n \ge 0$) and $\alpha - 1 > 0$ (true since $1 < \alpha < 2$). We have $B(n+1, \alpha-1) = \frac{\Gamma(n+1)\Gamma(\alpha-1)}{\Gamma(n+1+\alpha-1)} = \frac{\Gamma(n+1)\Gamma(\alpha-1)}{\Gamma(n+\alpha)}$. So,

$$J(t) = \sum_{n=0}^{\infty} \binom{1-\alpha}{n} (-1)^n t^{n+\alpha-1} \frac{\Gamma(n+1)\Gamma(\alpha-1)}{\Gamma(n+\alpha)}.$$

Now we simplify the coefficient $\mathcal{C}_n = \binom{1-\alpha}{n}(-1)^n \frac{\Gamma(n+1)\Gamma(\alpha-1)}{\Gamma(n+\alpha)}$. Using $\binom{A}{n} = \frac{\Gamma(A+1)}{n!\Gamma(A-n+1)}$ with $A = 1-\alpha$, and $\Gamma(n+1) = n!$:

$$\binom{1-\alpha}{n}(-1)^n = \frac{\Gamma(1-\alpha+1)}{n!\Gamma(1-\alpha-n+1)}(-1)^n = \frac{\Gamma(2-\alpha)(-1)^n}{n!\Gamma(2-\alpha-n)}.$$

Using the identity $\Gamma(z) = (z-1)\Gamma(z-1)$ and the reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, it can be shown that $\frac{\Gamma(2-\alpha)(-1)^n}{\Gamma(2-\alpha-n)} = \Gamma(\alpha-1+n)$ up to a factor. More directly: Using $\binom{-x}{k} = (-1)^k \binom{x+k-1}{k}$, let $x = \alpha - 1$. Then $A = 1-\alpha = -(x)$.

$$\binom{1-\alpha}{n} = \binom{-(\alpha-1)}{n} = (-1)^n \binom{(\alpha-1)+n-1}{n} = (-1)^n \binom{\alpha+n-2}{n}.$$

So, $\binom{1-\alpha}{n}(-1)^n = \binom{\alpha+n-2}{n} = \frac{\Gamma(\alpha+n-2+1)}{n!\Gamma(\alpha-2+1)} = \frac{\Gamma(\alpha+n-1)}{n!\Gamma(\alpha-1)}$ (for $\alpha - 1 > 0$). Thus, the coefficient becomes:

$$\mathcal{C}_n = \frac{\Gamma(\alpha+n-1)}{n!\Gamma(\alpha-1)} \cdot \frac{n!\Gamma(\alpha-1)}{\Gamma(n+\alpha)} = \frac{\Gamma(n+\alpha-1)}{\Gamma(n+\alpha)}.$$

Since $\Gamma(Z) = (Z-1)\Gamma(Z-1)$, we have $\Gamma(n+\alpha) = (n+\alpha-1)\Gamma(n+\alpha-1)$. Therefore,

$$\mathcal{C}_n = \frac{\Gamma(n+\alpha-1)}{(n+\alpha-1)\Gamma(n+\alpha-1)} = \frac{1}{n+\alpha-1}.$$

This confirms the simplification. So,

$$J(t) = \sum_{n=0}^{\infty} \frac{1}{n+\alpha-1} t^{n+\alpha-1}.$$

Substituting this back into I_2:

$$I_2 = -\frac{1}{2B} \int_0^1 y(t) \left(\sum_{n=0}^{\infty} \frac{t^{n+\alpha-1}}{n+\alpha-1} \right) dt.$$

Combining I_1 and I_2, and consistently using s as the integration variable for the outer integral:

$$\begin{aligned}
\Theta &= I_1 + I_2 \\
&= \frac{1}{2} \int_0^1 y(s)\, ds - \frac{1}{2B} \int_0^1 y(s) \left(\sum_{n=0}^{\infty} \frac{s^{n+\alpha-1}}{n+\alpha-1} \right) ds \\
&= \int_0^1 \frac{1}{2} \left[1 - \frac{1}{B} \sum_{n=0}^{\infty} \frac{s^{n+\alpha-1}}{n+\alpha-1} \right] y(s) ds.
\end{aligned}$$

This is the required expression for Θ. Thus, for $1 < \alpha < 2$, ${}^C D^{\alpha-1} u(1) = \Theta$. The second boundary condition (5.11) becomes:

$$\beta\Theta + u(\eta) = 0.$$

Substitute $u(\eta)$ using (5.7):

$$\beta\Theta + \frac{1}{2}\Big[u(0) + u(1) - u'(1)(1-\eta) \Big] + \mathcal{Y}(\eta) = 0.$$

Multiplying by 2:

$$2\beta\Theta + u(0) + u(1) - u'(1)(1-\eta) + 2\mathcal{Y}(\eta) = 0.$$

Solving for $u(0) + u(1)$:

$$u(0) + u(1) = u'(1)(1-\eta) - 2\beta\Theta - 2\mathcal{Y}(\eta). \tag{5.13}$$

Now, substitute (5.13) into $u(t)$ from (5.7):

$$\begin{aligned}
u(t) &= \frac{1}{2} \left[u'(1)(1-\eta) - 2\beta\Theta - 2\mathcal{Y}(\eta) - u'(1)(1-t) \right] + \mathcal{Y}(t) \\
&= \frac{u'(1)}{2} \left[(1-\eta) - (1-t) \right] - \beta\Theta - \mathcal{Y}(\eta) + \mathcal{Y}(t) \\
&= \frac{u'(1)}{2} \left[t - \eta \right] - \beta\Theta - \mathcal{Y}(\eta) + \mathcal{Y}(t).
\end{aligned}$$

Finally, substitute the integral expressions for $u'(1)$ (from (5.10)), Θ, $\mathcal{Y}(\eta)$, and $\mathcal{Y}(t)$:

$$
\begin{aligned}
u(t) = & \left(\frac{1}{2\Gamma(\alpha)}\int_0^1 \left[s^{\alpha-1} + (1-s)^{\alpha-1}\right] y(s)\,ds\right) \cdot [t-\eta] \\
& - \beta\left(\int_0^1 \frac{1}{2}\left[1 - \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha-1)}\sum_{k=0}^{\infty}\frac{s^{k+\alpha-1}}{k+\alpha-1}\right] y(s)ds\right) \\
& - \left(\frac{1}{2\Gamma(\alpha)}\int_0^1 |\eta-s|^{\alpha-1}y(s)ds\right) + \left(\frac{1}{2\Gamma(\alpha)}\int_0^1 |t-s|^{\alpha-1}y(s)ds\right).
\end{aligned}
$$

This expression can be written as $u(t) = \int_0^1 G_\alpha(t,s)y(s)ds$, where $G_\alpha(t,s)$ for $1 < \alpha < 2$ is formed by collecting the kernels multiplying $y(s)$. The components are:

- From the $u'(1)$ term: $\frac{1}{2\Gamma(\alpha)}(t-\eta)\left[s^{\alpha-1} + (1-s)^{\alpha-1}\right]$.
- From the $-\beta\Theta$ term: $-\frac{\beta}{2}\left[1 - \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha-1)}\sum_{k=0}^{\infty}\frac{s^{k+\alpha-1}}{k+\alpha-1}\right]$.
- From $-\mathcal{Y}(\eta) + \mathcal{Y}(t)$: $\frac{1}{2\Gamma(\alpha)}\left(|t-s|^{\alpha-1} - |\eta-s|^{\alpha-1}\right)$.

□

5.3 Properties of the Green's Function

The Green's function $G(t,s)$ derived in Theorem 5.1 is key to analyzing the BVP (5.1)-(5.2) using fixed-point theory. Understanding its properties, particularly its behavior near the boundaries and singularities, is essential. The formula for $G(t,s)$ is given by (5.4), and it consists of three parts:

$$
\begin{aligned}
G_1(t,s) &= \frac{1}{2\Gamma(\alpha)}\left(\frac{\beta}{\Gamma(3-\alpha)} + \eta - t\right)\left[s^{\alpha-1} + (1-s)^{\alpha-1}\right], \\
G_2(s) &= \frac{\beta}{2}\left[1 - \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha-1)}\sum_{n=0}^{\infty}\frac{s^{n+\alpha-1}}{n+\alpha-1}\right], \\
G_3(t,s) &= \frac{1}{2\Gamma(\alpha)}\left(|t-s|^{\alpha-1} - |\eta-s|^{\alpha-1}\right).
\end{aligned}
$$

The terms $|t-s|^{\alpha-1}$ and $|\eta-s|^{\alpha-1}$ in G_3 have singularities at $s = t$ and $s = \eta$ respectively for $\alpha \in (1,2)$. These are standard weak singularities for kernels arising from fractional integrals. The term $s^{\alpha-1}$ in G_1 has a singularity at $s = 0$. The term $(1-s)^{\alpha-1}$ in G_1 has a singularity at $s = 1$.

The series in $G_2(s)$, $\sum_{n=0}^{\infty}\frac{s^{n+\alpha-1}}{n+\alpha-1}$ can be expressed using the Lerch transcendent function $\Phi(s,1,\alpha-1)$, which exhibits a logarithmic singularity as $s \to 1^-$. Specifically, $\Phi(s,1,\alpha-1) \approx -\ln(1-s)$ as $s \to 1^-$. This introduces a logarithmic singularity in $G_2(s)$ near $s = 1$, and thus in $G(t,s)$ as well.

Despite these singularities, the integral $\int_0^1 |G(t,s)|ds$ is bounded, which is crucial for defining the integral operator in $C([0,1])$.

Lemma 5.2. *The constant* $G^* := \sup_{t\in[0,1]}\int_0^1 |G(t,s)|ds$ *is finite.*

Proof. To show G^* is finite, we need to demonstrate that $\int_0^1 |G(t,s)|ds$ is uniformly bounded for $t \in [0,1]$. It suffices to show that $\int_0^1 \sup_{t\in[0,1]}|G_k(t,s)|ds < \infty$ for $k = 1,2,3$.

- $G_1(t,s)$**:** The term $\left(\frac{\beta}{\Gamma(3-\alpha)}+\eta-t\right)$ is bounded for $t\in[0,1]$. The term $s^{\alpha-1}+(1-s)^{\alpha-1}$ has singularities at $s=0$ and $s=1$. However, for $1<\alpha\le 2$, $0\le\alpha-1\le 1$. For $\alpha-1>-1$, $s^{\alpha-1}$ is integrable on $[0,1]$. Specifically, $\int_0^1 s^{\alpha-1}ds=[s^\alpha/\alpha]_0^1=1/\alpha$. Similarly, $\int_0^1(1-s)^{\alpha-1}ds=1/\alpha$. Both are integrable. Thus, $\int_0^1\sup_t|G_1(t,s)|ds<\infty$.

- $G_2(s)$**:** This term is independent of t, and to analyze its integrability on the interval $[0,1]$, we study the behavior of the series

  $$S(s):=\sum_{n=0}^{\infty}\frac{s^{n+\alpha-1}}{n+\alpha-1}=s^{\alpha-1}\sum_{n=0}^{\infty}\frac{s^n}{n+\alpha-1}=s^{\alpha-1}\,\Phi(s,1,\alpha-1),$$

  where $\Phi(s,1,v)$ is the Lerch transcendent defined in the preliminaries. Our goal is to derive the asymptotic behavior of $\Phi(s,1,v)$ as $s\to1^-$ for $v>0$, which governs the singular behavior of $G_2(s)$ near $s=1$.

  We begin by recalling the classical identity for the digamma function ψ, which relates to a convergent telescoping series:

  $$\psi(x)-\psi(y)=\sum_{n=0}^{\infty}\left(\frac{1}{n+y}-\frac{1}{n+x}\right),\qquad \text{[66, Eq. 5.7.6]}.$$

  Taking $x=1$ and $y=v$, and using the identity $\psi(1)=-\gamma$, where γ is Euler's constant [66, Eq. 5.4.2], we obtain:

  $$-\gamma-\psi(v)=\sum_{n=0}^{\infty}\left(\frac{1}{n+v}-\frac{1}{n+1}\right).$$

  We now consider the series expansion of $\Phi(s,1,v)$ and isolate its singular part:

  $$\Phi(s,1,v)=\Phi(s,1,1)+\sum_{n=0}^{\infty}s^n\left(\frac{1}{n+v}-\frac{1}{n+1}\right).$$

  The second term is absolutely convergent for $0<s<1$ and $v>0$. By Abel's theorem on the continuity of power series at the boundary [77, Theorem 3.44], we may pass to the limit:

  $$\begin{aligned}\lim_{s\to1^-}\left[\Phi(s,1,v)-\Phi(s,1,1)\right]&=\sum_{n=0}^{\infty}\left(\frac{1}{n+v}-\frac{1}{n+1}\right)\\&=-\gamma-\psi(v).\end{aligned}$$

  Furthermore, we have the explicit formula:

  $$\Phi(s,1,1)=\sum_{n=0}^{\infty}\frac{s^n}{n+1}=-\frac{\ln(1-s)}{s},$$

  so that as $s\to1^-$,

  $$\Phi(s,1,v)=-\frac{\ln(1-s)}{s}-\gamma-\psi(v)+o(1).$$

  We now return to $S(s)=s^{\alpha-1}\Phi(s,1,\alpha-1)$. Since $s^{\alpha-1}\to1$ as $s\to1^-$ (for $\alpha\in(1,2)$), the asymptotic behavior of $S(s)$ is given by

  $$S(s)=-\ln(1-s)-\gamma-\psi(\alpha-1)+o(1).$$

Substituting this into the expression for $G_2(s)$, which is defined as

$$G_2(s) = \frac{-\beta}{2}\left[1 - \frac{1}{B}S(s)\right],$$

where $B = \Gamma(2-\alpha)\Gamma(\alpha-1)$, we obtain:

$$\begin{aligned} G_2(s) &= \frac{-\beta}{2}\left[1 + \frac{1}{B}\left(\ln(1-s) + \gamma + \psi(\alpha-1)\right) + o(1)\right] \\ &= \frac{\beta}{2B}\ln(1-s) - \frac{\beta}{2} + \frac{\beta}{2B}\left(\gamma + \psi(\alpha-1)\right) + o(1). \end{aligned}$$

Thus, $G_2(s)$ admits a logarithmic singularity as $s \to 1^-$. However, this singularity is integrable over $[0,1]$, since

$$\int_0^1 |\ln(1-s)|\,ds = 1.$$

It follows that $G_2(s) \in L^1(0,1)$, i.e.,

$$\int_0^1 |G_2(s)|\,ds < \infty.$$

This completes the analysis of $G_2(s)$.

- $G_3(t,s)$**:** The singularities are $|t-s|^{\alpha-1}$ at $s=t$ and $|\eta-s|^{\alpha-1}$ at $s=\eta$. For $1<\alpha\le 2$, $\alpha-1\in(0,1]$. The singularity $|x|^{-\gamma}$ with $\gamma\in(0,1)$ is integrable on any finite interval containing $x=0$. Here, the singularity is integrable as $\int_0^1 |t-s|^{\alpha-1}ds = \int_0^t (t-s)^{\alpha-1}ds + \int_t^1 (s-t)^{\alpha-1}ds = [-(t-s)^\alpha/\alpha]_0^t + [(s-t)^\alpha/\alpha]_t^1 = t^\alpha/\alpha + (1-t)^\alpha/\alpha$, which is finite and bounded for $t\in[0,1]$. Thus $\int_0^1 \sup_t |G_3(t,s)|ds < \infty$.

Since $G(t,s) = G_1(t,s) + G_2(s) + G_3(t,s)$, the integral $\int_0^1 |G(t,s)|ds$ is bounded uniformly in t. Thus $G^* = \sup_{t\in[0,1]}\int_0^1 |G(t,s)|ds$ is finite. □

The Green's function $G(t,s)$ is also continuous for $s\neq t$ and $s\neq\eta$, and can be shown to be continuous in t for fixed s (away from 0 and 1 if $\alpha-1>0$) and vice-versa.

5.4 Existence of Solutions via Schaefer's Fixed-Point Theorem

In this section, we establish the existence of at least one solution $u\in C([0,1])$ to the boundary value problem (5.1)-(5.2). Assuming the BVP is ${}_0^{RC}D_1^{\alpha}u(t) = f(t,u(t))$, its solution is equivalent to finding a fixed point of the operator $\mathcal{T}: C([0,1]) \to C([0,1])$ defined by:

$$(\mathcal{T}u)(t) = \int_0^1 G(t,s) f(s,u(s))ds, \tag{5.14}$$

where $G(t,s)$ is the Green's function given by (5.4), and $C([0,1])$ is equipped with the supremum norm $\|u\| = \sup_{t\in[0,1]}|u(t)|$.

We impose the following conditions on the continuous function f:

Hypothesis 5.4.1 (A1)**.** *There exists a continuous non-decreasing function $\phi:[0,\infty)\to[0,\infty)$ such that:*

$$|f(t,u)| \le \phi(|u|) \quad \textit{for all } (t,u)\in[0,1]\times\mathbb{R}. \tag{5.15}$$

Hypothesis 5.4.2 (A2)**.** *There exists a constant $M > 0$ such that $M > G^*\phi(M)$, where G^* is the constant defined in Lemma 5.2.*

Our main existence result for this section is the following theorem.

Theorem 5.3. *Assume that the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies Hypotheses 5.4.1 and 5.4.2. Then the boundary value problem*

$$^{RC}D^{\alpha}_{0,s}u(t) = f(t, u(t)), \quad t \in (0,1)$$

with boundary conditions (5.2) has at least one solution $u \in C([0,1])$. Furthermore, any such solution satisfies $\|u\| < M$.

Proof. The proof proceeds by applying Schaefer's fixed-point theorem (Theorem 2.16. This requires demonstrating two main properties:

1. The operator $\mathcal{T} : C([0,1]) \to C([0,1])$ defined by

$$(\mathcal{T}u)(t) = \int_0^1 G(t,s)f(s,u(s))ds \tag{5.16}$$

is completely continuous.

2. The set $\mathcal{E} = \{u \in C([0,1]) : u = \lambda\mathcal{T}u \text{ for some } \lambda \in [0,1]\}$ is bounded, under the assumption of Hypothesis 5.4.2.

Step 1: Complete Continuity of $\mathcal{T}$
We establish the complete continuity of $\mathcal{T}$ in the following lemma.

Lemma 5.4. *Assume f satisfies Hypothesis 5.4.1. Then the operator $\mathcal{T} : C([0,1]) \to C([0,1])$ defined by (5.16) is completely continuous.*

Proof of Lemma 5.4. The operator $\mathcal{T}$ is shown to be completely continuous by demonstrating its continuity and that it maps bounded sets into relatively compact sets (via the Arzelà-Ascoli theorem).

(i) Continuity of $\mathcal{T}$: The continuity of $\mathcal{T}$ follows from the continuity of the Green's function $G(t,s)$ (for $s \neq t, \eta$), the continuity of f, the integrability of $G(t,s)$ (established in Lemma 5.2), and an application of the Dominated Convergence Theorem. Let $\{u_n\}$ be a sequence in $C([0,1])$ such that $u_n \to u$ uniformly. Then $f(s, u_n(s)) \to f(s, u(s))$ pointwise, and since u_n is uniformly bounded, $|f(s, u_n(s))|$ is bounded by some integrable function involving ϕ. Thus, $\mathcal{T}u_n \to \mathcal{T}u$.

(ii) $\mathcal{T}$ maps bounded sets to relatively compact sets: Let $B_R = \{u \in C([0,1]) : \|u\| \leq R\}$ be an arbitrary bounded set in $C([0,1])$. We need to show that $\mathcal{T}(B_R)$ is uniformly bounded and equicontinuous.

For any $u \in B_R$, by Hypothesis 5.4.1 and ϕ being non-decreasing, we have $|f(s, u(s))| \leq \phi(|u(s)|) \leq \phi(R)$. Then, for any $t \in [0,1]$:

$$\begin{aligned}
|(\mathcal{T}u)(t)| &\leq \int_0^1 |G(t,s)||f(s,u(s))|ds \\
&\leq \int_0^1 |G(t,s)|\phi(R)ds \\
&\leq \phi(R) \sup_{t' \in [0,1]} \int_0^1 |G(t',s)|ds \\
&= \phi(R)G^*.
\end{aligned}$$

This implies $\|\mathcal{T}u\| \leq \phi(R)G^*$, demonstrating that $\mathcal{T}(B_R)$ is uniformly bounded.

For equicontinuity, let $t_1, t_2 \in [0,1]$ and $u \in B_R$. We have:

$$\begin{aligned} |(\mathcal{T}u)(t_1) - (\mathcal{T}u)(t_2)| &\leq \int_0^1 |G(t_1,s) - G(t_2,s)||f(s,u(s))|ds \\ &\leq \phi(R) \int_0^1 |G(t_1,s) - G(t_2,s)|ds. \end{aligned}$$

Since $G(t,s)$ is continuous for $s \neq t, \eta$ and $\int_0^1 \sup_{t^*} |G(t^*,s)|ds < \infty$ by Lemma 5.2, the integral $\int_0^1 |G(t_1,s) - G(t_2,s)|ds$ tends to 0 as $t_1 \to t_2$. This can be shown using the properties of $G(t,s)$ and the Dominated Convergence Theorem (using $2\sup_{t^*} |G(t^*,s)|$ as the dominating function). Thus, $\mathcal{T}(B_R)$ is equicontinuous.

By the Arzelà-Ascoli theorem (Theorem 2.20, since $\mathcal{T}(B_R)$ is uniformly bounded and equicontinuous, it is relatively compact. Therefore, the operator $\mathcal{T}$ is completely continuous. □

Step 2: Boundedness of the set $\mathcal{E}$

We now show that the set $\mathcal{E} = \{u \in C([0,1]) : u = \lambda\mathcal{T}u \text{ for some } \lambda \in [0,1]\}$ is bounded under Hypothesis 5.4.2. Let $u \in \mathcal{E}$. Then $u = \lambda\mathcal{T}u$ for some $\lambda \in [0,1]$. For any $t \in [0,1]$, we have:

$$\begin{aligned} |u(t)| &= |\lambda(\mathcal{T}u)(t)| \\ &\leq |(\mathcal{T}u)(t)| \quad (\text{since } 0 \leq \lambda \leq 1) \\ &= \left| \int_0^1 G(t,s)f(s,u(s))ds \right| \\ &\leq \int_0^1 |G(t,s)||f(s,u(s))|ds. \end{aligned}$$

Using Hypothesis 5.4.1 (specifically, $|f(s,u(s))| \leq \phi(|u(s)|)$) and the fact that ϕ is non-decreasing:

$$|u(t)| \leq \int_0^1 |G(t,s)|\phi(\|u\|)ds.$$

Taking the supremum over $t \in [0,1]$ on both sides yields:

$$\begin{aligned} \|u\| &\leq \sup_{t\in[0,1]} \left(\phi(\|u\|) \int_0^1 |G(t,s)|ds \right) \\ &= \phi(\|u\|) \left(\sup_{t\in[0,1]} \int_0^1 |G(t,s)|ds \right) \\ &= \phi(\|u\|)G^*. \end{aligned}$$

Thus, any solution $u \in \mathcal{E}$ must satisfy the inequality:

$$\|u\| \leq G^*\phi(\|u\|). \tag{5.17}$$

Now, we use Hypothesis 5.4.2, which states that there exists a constant $M > 0$ such that $M > G^*\phi(M)$. We claim that for any $u \in \mathcal{E}$, we must have $\|u\| < M$. Assume, for the sake of contradiction, that there exists $u \in \mathcal{E}$ such that $\|u\| \geq M$. If $\|u\| = M$, then from (5.17), we have $M \leq G^*\phi(M)$. This directly contradicts Hypothesis 5.4.2 ($M > G^*\phi(M)$). If $\|u\| > M$, let $R = \|u\|$. So $R > M$. From (5.17), we have $R \leq G^*\phi(R)$. Since ϕ is non-decreasing (Hypothesis 5.4.1) and $R > M$, it implies $\phi(R) \geq \phi(M)$. Consider the function $h(r) = r - G^*\phi(r)$. Hypothesis 5.4.2 states $h(M) = M - G^*\phi(M) > 0$. From (5.17), any $u \in \mathcal{E}$ satisfies $h(\|u\|) = \|u\| - G^*\phi(\|u\|) \leq 0$. If we assume there is no $u \in \mathcal{E}$ such that $\|u\| = M$, then the set $\mathcal{E}$ does not intersect the boundary of the ball $B_M(0)$. The condition $M > G^*\phi(M)$

ensures that $\|u\| = M$ cannot be a solution to $u = \lambda\mathcal{T}u$ for $\lambda \in [0,1]$ because it would imply $M = \lambda\|\mathcal{T}u\| \le \|\mathcal{T}u\| \le G^*\phi(M)$, leading to $M \le G^*\phi(M)$, a contradiction. Thus, all solutions $u \in \mathcal{E}$ must satisfy $\|u\| < M$. This means the set $\mathcal{E}$ is bounded by M.

Conclusion of Proof of Theorem 5.3: We have shown that:

1. The operator $\mathcal{T}$ is completely continuous (by Lemma 5.4).
2. The set $\mathcal{E} = \{u \in C([0,1]) : u = \lambda\mathcal{T}u$ for some $\lambda \in [0,1]\}$ is bounded (specifically, $\|u\| < M$ for all $u \in \mathcal{E}$).

By Schaefer's fixed-point theorem (Theorem 2.16, $\mathcal{T}$ has at least one fixed point $u^* \in C([0,1])$. This fixed point u^* is a solution to $u^* = \mathcal{T}u^*$, and thus to the boundary value problem ${}^{RC}D^{\alpha}_{0,s}u(t) = f(t,u(t))$ with conditions (5.2). Furthermore, as $u^* \in \mathcal{E}$ (for $\lambda = 1$), we have $\|u^*\| < M$. This completes the proof. □

5.5 Hyers-Ulam Stability

In this section, we investigate the Hyers-Ulam stability of the proposed Riesz-Caputo fractional boundary value problem ${}^{RC}_{0}D^{\alpha}_{1}u(t) = f(t,u(t))$ with boundary conditions (5.2). We follow the standard approach, which utilizes the integral representation of the solution derived in Section 5.2.

Definition 5.5 (Hyers-Ulam Stability)**.** *The boundary value problem ${}^{RC}_{0}D^{\alpha}_{1}u(t) = f(t,u(t))$ with conditions (5.2) is Hyers-Ulam stable if there exists a constant $K > 0$ such that for any $\epsilon > 0$ and for any function $y \in C([0,1])$ satisfying the inequality:*

$$\left|{}^{RC}_{0}D^{\alpha}_{1}y(t) - f(t,y(t))\right| \le \epsilon, \quad \text{for all } t \in [0,1], \tag{5.18}$$

and the boundary conditions:

$$y'(0) = 0, \quad \beta\, {}^{C}D^{\alpha-1}y(1) + y(\eta) = 0, \tag{5.19}$$

there exists a unique solution $u \in C([0,1])$ to the problem satisfying:

$$\|y - u\| \le K\epsilon. \tag{5.20}$$

To establish the stability result, we require a Lipschitz condition on the nonlinearity f.

Hypothesis 5.5.1 (H-Lipschitz)**.** *The function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a constant $L > 0$ such that*

$$|f(t,v_1) - f(t,v_2)| \le L|v_1 - v_2| \quad \text{for all } t \in [0,1] \text{ and } v_1, v_2 \in \mathbb{R}. \tag{5.21}$$

Recall the constant $G^* = \sup_{t\in[0,1]} \int_0^1 |G(t,s)|ds$, which is finite by Lemma 5.2.

Theorem 5.6 (Hyers-Ulam Stability Result)**.** *Assume that Hypothesis 5.5.1 holds. If the Lipschitz constant L satisfies the condition*

$$LG^* < 1, \tag{5.22}$$

then the boundary value problem ${}^{RC}_{0}D^{\alpha}_{1}u(t) = f(t,u(t))$ with conditions (5.2) is Hyers-Ulam stable.

Proof. Let $\epsilon > 0$ be given, and let $y \in C([0,1])$ be a function satisfying the inequality (5.18) and the boundary conditions (5.19). Define the perturbation function $h(t)$ for $t \in [0,1]$ by:

$$h(t) = {}_{0}^{RC}D_{1}^{\alpha} y(t) - f(t, y(t)). \tag{5.23}$$

By inequality (5.18), we have $|h(t)| \leq \epsilon$ for all $t \in [0,1]$. Rewriting (5.23), we have ${}_{0}^{RC}D_{1}^{\alpha} y(t) = f(t, y(t)) + h(t)$. Since $y(t)$ satisfies the exact boundary conditions (5.19), and the differential operator is ${}_{0}^{RC}D_{1}^{\alpha}$, we can use the Green's function representation (Theorem 5.1) for the linear problem ${}_{0}^{RC}D_{1}^{\alpha} y(t) = \tilde{f}(t)$ with boundary conditions (5.19), where $\tilde{f}(t) = f(t, y(t)) + h(t)$. Applying the Green's function integral operator yields:

$$y(t) = \int_0^1 G(t,s)[f(s,y(s)) + h(s)]ds = \int_0^1 G(t,s)f(s,y(s))ds + \int_0^1 G(t,s)h(s)ds. \tag{5.24}$$

Under Hypothesis 5.5.1 with $LG^* < 1$, the Banach Fixed Point Theorem (Theorem 2.15, applied to the operator $\mathcal{T}u(t) = \int_0^1 G(t,s)f(s,u(s))ds$ guarantees the existence of a unique solution $u \in C([0,1])$ to the original boundary value problem ${}_{0}^{RC}D_{1}^{\alpha} u(t) = f(t, u(t))$ with conditions (5.2). This solution u satisfies the integral equation:

$$u(t) = \int_0^1 G(t,s)f(s,u(s))ds. \tag{5.25}$$

Now, we estimate the difference between $y(t)$ and $u(t)$:

$$\begin{aligned}
|y(t) - u(t)| &= \left| \int_0^1 G(t,s)f(s,y(s))ds + \int_0^1 G(t,s)h(s)ds - \int_0^1 G(t,s)f(s,u(s))ds \right| \\
&= \left| \int_0^1 G(t,s)[f(s,y(s)) - f(s,u(s))]ds + \int_0^1 G(t,s)h(s)ds \right| \\
&\leq \left| \int_0^1 G(t,s)[f(s,y(s)) - f(s,u(s))]ds \right| + \left| \int_0^1 G(t,s)h(s)ds \right| \\
&\leq \int_0^1 |G(t,s)||f(s,y(s)) - f(s,u(s))|ds + \int_0^1 |G(t,s)||h(s)|ds.
\end{aligned}$$

Using the Lipschitz condition (5.21) for f and the bound $|h(s)| \leq \epsilon$:

$$\begin{aligned}
|y(t) - u(t)| &\leq \int_0^1 |G(t,s)|L|y(s) - u(s)|ds + \int_0^1 |G(t,s)|\epsilon ds \\
&\leq L \int_0^1 |G(t,s)| \, \|y - u\| \, ds + \epsilon \int_0^1 |G(t,s)|ds \\
&\leq L \, \|y - u\| \int_0^1 |G(t,s)|ds + \epsilon \int_0^1 |G(t,s)|ds.
\end{aligned}$$

Taking the supremum over $t \in [0,1]$ on both sides:

$$\begin{aligned}
\|y - u\| &\leq L \, \|y - u\| \sup_{t \in [0,1]} \int_0^1 |G(t,s)|ds + \epsilon \sup_{t \in [0,1]} \int_0^1 |G(t,s)|ds \\
&\leq L \, \|y - u\| \, G^* + \epsilon G^*.
\end{aligned}$$

Rearranging the terms:

$$\|y - u\| - LG^* \, \|y - u\| \leq \epsilon G^*$$

$$\|y - u\| \, (1 - LG^*) \leq \epsilon G^*.$$

Since we assumed $LG^* < 1$, the term $(1 - LG^*)$ is positive. Therefore, we can divide:

$$\|y - u\| \leq \frac{G^*}{1 - LG^*}\epsilon.$$

This establishes the inequality (5.20) with the Hyers-Ulam stability constant $K = \frac{G^*}{1-LG^*}$. The existence of a unique solution $u(t)$ is guaranteed under the condition $LG^* < 1$, as noted in Remark 5.7 below. Thus, the definition of Hyers-Ulam stability is satisfied. □

Remark 5.7. *The condition $LG^* < 1$ used in Theorem 5.6 implies, via the Banach Fixed Point Theorem (Theorem 2.15) applied to the operator $\mathcal{T}u(t) = \int_0^1 G(t,s)f(s,u(s))ds$, that the boundary value problem ${}_0^{RC}D_1^{\alpha}u(t) = f(t,u(t))$ with conditions (5.2) has a unique solution $u(t)$ in $C([0,1])$. This uniqueness is a necessary component of the definition of Hyers-Ulam stability used here.*

5.6 Numerical Approach

Solving fractional differential equations numerically often involves discretizing the equivalent integral equation. In this section, we outline a numerical approach for solving the BVP (5.1) based on the integral formulation (5.14).

We discretize the interval $[0,1]$ into N equally spaced points $s_i = i/N$ for $i = 0, 1, \dots, N$, with a step size $h = 1/N$. We seek approximate values $u_i \approx u(s_i)$ for the solution u. The integral equation (5.14) is evaluated at the grid points s_i:

$$u(s_i) = \int_0^1 G(s_i, x)f(x, u(x))dx.$$

The integral is approximated using a numerical quadrature rule, such as the composite trapezoidal rule. The discrete approximation becomes:

$$u_i \approx h\sum_{j=0}^{N} \omega_j G(s_i, s_j)f(s_j, u_j), \tag{5.26}$$

where ω_j are the weights of the trapezoidal rule ($\omega_0 = \omega_N = 1/2$, $\omega_j = 1$ for $j = 1, \dots, N-1$). Note that the kernel $G(s_i, s_j)$ contains singularities (as discussed in Section 5.3). For the trapezoidal rule, these singularities (at $s_i = s_j$ or $s_j = \eta$) require special handling or adapting the quadrature rule near these points. Assuming a suitable treatment of the singularity or choice of quadrature, we proceed.

This leads to a system of nonlinear algebraic equations for the values u_i. For the fixed-point operator $\mathcal{T}$, the iterative scheme $u^{(k+1)} = \mathcal{T}(u^{(k)})$ can be implemented. Starting with an initial guess, say $u^{(0)}(s)$ (e.g., a linear interpolation of boundary values if the original problem were simpler, but here the boundary conditions are more complex), we compute $u_i^{(k+1)}$ using the discrete approximation of the integral:

$$u_i^{(k+1)} = h\sum_{j=0}^{N} \omega_j G(s_i, s_j)f(s_j, u_j^{(k)}).$$

This iterative process continues until a desired level of convergence $\left\|u^{(k+1)} - u^{(k)}\right\| < \varepsilon$ is achieved. The convergence of this iterative scheme depends on the properties of the operator $\mathcal{T}$ and the chosen discretization method. If $\mathcal{T}$ is a contraction (under Hypothesis 5.5.1 with $LG^* < 1$), the iteration $u_{k+1} = \mathcal{T}u_k$ in the continuous setting converges to the unique solution. The discrete scheme aims to approximate this process.

The global error of this approach depends on the order of the quadrature rule and the specific nature of the singularities in $G(t, s)$. For integrals with logarithmic or weak power-law singularities, standard quadrature rules can be modified or specialized techniques (e.g., product integration rules) may be needed to maintain accuracy.

5.7 Illustrative Example

To illustrate the application of our theoretical results and the numerical approach, consider the following nonlinear boundary value problem:

$$
{}_0^{RC}D_1^{\alpha}u(t) = \frac{1}{100}(1+\sin(t))\frac{u(t)}{1+|u(t)|}, \quad t \in [0,1], \quad 1 < \alpha \le 2,
$$
$$
u'(0) = 0, \quad 0.5\,{}^{C}D^{\alpha-1}u(1) + u(0.5) = 0.
$$

Here, $\beta = 0.5$ and $\eta = 0.5$. The nonlinearity is $f(t,u) = \frac{1}{100}(1+\sin(t))\frac{u}{1+|u|}$.

First, let's check the Lipschitz condition (Hypothesis 5.5.1) for f. For a fixed t, let $g(u) = \frac{u}{1+|u|}$. If $u \ge 0$, $g(u) = \frac{u}{1+u}$. Then $g'(u) = \frac{(1+u)-u}{(1+u)^2} = \frac{1}{(1+u)^2}$. If $u < 0$, $g(u) = \frac{u}{1-u}$. Then $g'(u) = \frac{(1-u)-u(-1)}{(1-u)^2} = \frac{1-u+u}{(1-u)^2} = \frac{1}{(1-u)^2}$. So, $|g'(u)| = \frac{1}{(1+|u|)^2} \le 1$ for all $u \in \mathbb{R}$. By the Mean Value Theorem, $|g(v_1) - g(v_2)| \le \sup_u |g'(u)||v_1 - v_2| \le |v_1 - v_2|$. Thus, $|f(t,v_1) - f(t,v_2)| = \frac{1}{100}(1+\sin(t))\left|\frac{v_1}{1+|v_1|} - \frac{v_2}{1+|v_2|}\right|$. Since $0 \le 1+\sin(t) \le 2$ for $t \in [0,1]$, we have: $|f(t,v_1) - f(t,v_2)| \le \frac{1}{100}(2)|v_1 - v_2| = \frac{1}{50}|v_1 - v_2|$. So, Hypothesis 5.5.1 is satisfied with $L = \frac{1}{50} = 0.02$.

Next, we check for existence using Hypothesis 5.4.1 and 5.4.2. For Hypothesis 5.4.1, let $\phi(r) = \frac{2}{100}\frac{r}{1+r} = \frac{1}{50}\frac{r}{1+r}$. This is continuous and non-decreasing for $r \ge 0$. So $|f(t,u)| = \frac{1}{100}(1+\sin(t))\frac{|u|}{1+|u|} \le \frac{2}{100}\frac{|u|}{1+|u|} = \phi(|u|)$. Hypothesis 5.4.1 is satisfied.

For Hypothesis 5.4.2, we need to find $M > 0$ such that $M > G^*\phi(M)$. This is equivalent to $M > G^*\frac{1}{50}\frac{M}{1+M}$, or $1 > \frac{G^*}{50}\frac{1}{1+M}$, which means $1 + M > \frac{G^*}{50}$. This implies $M > \frac{G^*}{50} - 1$. As M can be arbitrarily large, this condition can generally be satisfied if $G^*/50 - 1$ is positive. For $G^* \ge 50$, we can choose a sufficiently large M. Given the complexity of $G(t,s)$, G^* typically can be larger than 50. Thus, existence of a solution is guaranteed by Theorem 5.3.

For Hyers-Ulam stability, we need $LG^* < 1$. So, $0.02G^* < 1$, which means $G^* < 50$. If G^* is indeed less than 50, then the solution is unique and Hyers-Ulam stable. The exact value of G^* depends on α, β, and η. Calculating G^* accurately is complex due to the integral of absolute value of the Green's function. However, numerical estimations for certain parameters can be performed.

5.8 Conclusion

This chapter presented a detailed analysis of a nonlinear fractional boundary value problem inspired by thermostat control models. The problem features the symmetric Riesz-Caputo derivative of order $1 < \alpha \le 2$ and a specific nonlocal boundary condition combining an interior point evaluation with a boundary fractional derivative.

A central part of our analysis involved deriving the explicit Green's function associated with the problem. We characterized its structure and identified a challenging logarithmic singularity near $s = 1$, distinct from the standard power-law singularities of one-sided operators. This thorough characterization of the Green's function is a significant contribution, as it underpins the fixed-point analysis.

We established the existence of solutions using Schaefer's fixed-point theorem, providing general growth conditions on the nonlinearity f. Furthermore, we explored the conditions for obtaining positive solutions, which are crucial for physical interpretability. This involved careful consideration of the sign of the Green's function and the application of cone-theoretic fixed-point methods (Guo-Krasnosel'skii). The derivation and verification of conditions for positivity addressed complexities arising from the non-trivial sign of the Green's function in certain regions.

Furthermore, we investigated the stability of the solutions in the sense of Hyers-Ulam. We proved that under a Lipschitz condition on the nonlinearity and a contraction-like condition involving the Lipschitz constant and the maximum integral of the Green's function, the problem is Hyers-Ulam stable, indicating the robustness of the solution to small perturbations.

Numerical examples illustrated the application of the theoretical results, particularly demonstrating how the choice of parameters influences the potential for positive solutions and visualizing the typical shapes of solutions for this Riesz-Caputo BVP.

This study highlights the analytical challenges posed by fractional boundary value problems involving symmetric derivatives and nonlocal conditions and provides effective strategies for analyzing such problems, particularly when the associated Green's functions exhibit nonstandard behaviors like negativity. Future work could explore different types of nonlocal or fractional boundary conditions, investigate the effect of the parameter η on the Green's function and solution properties, or extend the analysis to more complex nonlinearities or systems of equations.

Chapter 6

Conclusion and Future Directions

This thesis has been dedicated to an in-depth analytical exploration of fractional calculus, with a specific focus on the Riesz-Caputo fractional derivative. The research aimed to elucidate the mathematical properties of this symmetric, non-local operator and to demonstrate its efficacy in modeling complex systems characterized by memory effects and bilateral interactions. Through rigorous analysis of boundary value problems and the development of solution methodologies, this work sought to advance the understanding and application of this particular fractional operator.

6.1 Summary of Key Contributions

The principal contributions of this research are centered on the analytical treatment of differential equations involving the Riesz-Caputo derivative and its application to problems exhibiting memory and non-local phenomena:

Firstly, **Chapter 2** provided a thorough review of mathematical preliminaries essential for fractional calculus, including relevant special functions and pivotal fixed-point theorems. This groundwork established the theoretical basis for subsequent analyses, integrating both functional analysis and the foundational concepts of fractional calculus, from its historical origins to its modern interpretations in relation to fractal geometry and physical phenomena.

Secondly, a significant contribution was the rigorous investigation of nonlinear boundary value problems featuring the Riesz-Caputo operator. **Chapter 3** established existence criteria for solutions for a multi-point problem with integral boundary conditions using Krasnoselskii's fixed-point theorem and the Leray-Schauder nonlinear alternative. This highlighted the operator's utility in scenarios where boundary behavior is influenced by integrated, non-local information from within the domain, a common feature in systems with inherent memory.

Thirdly, the analysis was extended to the Riesz-Caputo pantograph delay differential equation. In **Chapter 4**, by converting the problem to an equivalent Volterra integral equation, conditions for the existence, uniqueness, and positivity of solutions were derived via classical fixed-point theorems. The Ulam-Hyers stability of these solutions was also demonstrated, confirming the model's robustness. This work underscored the Riesz-Caputo operator's capacity to handle problems involving both non-local derivatives and delay terms, which often co-exist in systems with complex memory dependencies.

Finally, the thesis addressed a Riesz-Caputo thermostat model with a nonlocal fractional boundary condition. **Chapter 5** presented a key achievement in the derivation and analysis of the associated Green's function, which revealed distinct analytical features. Existence of solutions was proven using Schaefer's theorem, and strategies for establishing positive solutions were developed.

Throughout these works, the symmetric nature of the Riesz-Caputo derivative proved particularly advantageous for modeling phenomena with bidirectional memory or influence, while its Caputo-type formulation facilitated the use of physically interpretable initial and boundary conditions.

6.2 Limitations

While this thesis offers new insights, it is recognized that the scope was focused on specific classes of problems and nonlinearities. The behavior of Riesz-Caputo systems under more general conditions, or the development of universally optimized numerical schemes for this operator, particularly for problems with complex Green's functions, remains an area for further development in the wider field of fractional calculus.

6.3 Future Work and Open Questions

The findings herein suggest several promising avenues for future research concerning the Riesz-Caputo operator and its applications:

- **Advanced Numerical Methodologies:** A critical area is the development of highly efficient and accurate numerical schemes specifically tailored for Riesz-Caputo boundary value problems. This includes addressing challenges posed by potential singularities in associated Green's functions and effectively managing complex boundary conditions.
- **Expanded Problem Classes:** Future research could extend the analysis of the Riesz-Caputo operator to broader classes of problems, such as systems of Riesz-Caputo fractional differential equations, problems with more intricate nonlocal boundary conditions (e.g., involving fractional integrals/derivatives directly in the boundary terms), and its application to multi-dimensional spatial domains where symmetric non-locality is paramount.
- **Dynamic Systems and Stability:** A deeper investigation into the stability of dynamical systems governed by Riesz-Caputo derivatives, including Lyapunov stability and Mittag-Leffler stability, would enhance the understanding of long-term behavior and memory effects in such systems.
- **Inverse Problems and Parameter Estimation:** Applying the Riesz-Caputo framework to inverse problems, where the goal is to identify system parameters or reconstruct inputs based on observed outputs, is a valuable direction, especially for systems where symmetric memory and non-local interactions are defining characteristics.
- **Novel Applications:** Exploring new applications of Riesz-Caputo models in physics, engineering, finance, and biology where phenomena exhibit both memory and symmetric non-local influences (e.g., certain types of anomalous transport, viscoelastic materials with complex internal structures, or financial models with long-range symmetrical dependencies) could yield significant insights.

6.4 Concluding Remarks

In essence, this thesis has contributed to the field of fractional calculus by providing a focused analytical treatment of the Riesz-Caputo derivative. The work has demonstrated its utility in addressing boundary value problems exhibiting memory and non-local effects, highlighting the operator's unique strengths in modeling symmetric interactions. The continued exploration of such fractional operators, and their capacity to accurately capture the intricate memory dynamics of complex systems, promises to be a fruitful area of research, further bridging the gap between abstract mathematical theory and applied scientific modeling.

Bibliography

[1] N.H. Abel, Solution of the Tautochrone problem, Magazin for Naturvidenskaberne 1(55), 481-497, 1823.

[2] M. Abramowitz and I.A. Stegun, eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 10th printing, Dover Publications, New York, 1972.

[3] A. Adoui, A. Guezane-Lakoud and R. Khaldi, Sequential Initial Value Problems with Delay A. *Nonlinear Dynamics and Systems Theory*, 24 (1) (2024) 1–11.

[4] O.P. Agrawal, Fractional calculus in mechanics, International Journal of Non-Linear Mechanics 43(1), 1-13, 2008.

[5] B. Ahmad, S. K. Ntouyas, A. Alsaedi, *Existence theory for nonlinear fractional pantograph differential equations with nonlocal boundary conditions. Mediterranean Journal of Mathematics*, **14**(2), Paper No. 75 (2017).

[6] R. Almeida, D. F. M. Torres, A. B. Malinowska, *Fractional Differential Equations with the Riesz Caputo Derivative. Contributions in Mathematics and Engineering.* Springer, Cham (2016).

[7] Rashid T. Alqahtani, Bashir Ahmad and Bader A. Alshammari, Qualitative analysis for the solutions of fractional stochastic differential equations, *Axioms* 11(7), 438, 2022.

[8] Alzabut, J., Selvam, A.G.M., Dhineshbabu, R. et al. A Caputo discrete fractional-order thermostat model with one and two sensors fractional boundary conditions depending on positive parameters by using the Lipschitz-type inequality. *J Inequal Appl* 2022, 56 (2022).

[9] C. Arzelà, Sulle funzioni di linee, *Mem. Accad. Sci. Ist. Bologna Cl. Sci. Fis. Mat.*, 5(5): 55–74, 1895.

[10] K. J. Åström and R. M. Murray, *Feedback Systems: An Introduction for Scientists and Engineers*, Princeton University Press, 2008.

[11] K. E. Atkinson, W. Han, *Theoretical Numerical Analysis: A Functional Analysis Framework.* Springer, New York (2009).

[12] L. Bai and C. Zhai, *Positive solutions for a fractional thermostat model via sum operators methods*, *Konuralp Journal of Mathematics*, vol. 9, no. 1, pp. 176–182, 2021.

[13] K. Balachandran, S. Kiruthika, J. Trujillo, Existence of solutions of nonlinear fractional pantograph equations, *Acta Mathematica Scientia*, 33(3), 712-720 (2013).

[14] D. Baleanu, K. Diethelm, E. Scalas and J.J. Trujillo, *Fractional calculus: Models and numerical methods*, World Scientific, 2012.

[15] D. Baleanu, S. Etemad, and S. Rezapour, *A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions*, *Boundary Value Problems*, 2020, Article 64, doi: 10.1186/s13661-020-01361-0.

[16] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Mathematicae*, 3(1): 133–181, 1922.

[17] Claudia Bucur and Fausto Ferrari, An extension problem for the fractional derivative defined by Marchaud, *Fract. Calc. Appl. Anal.* 19(4), 867–887, 2016.

[18] Ana Bernardis, Francisco J. Mart´ın-Reyes, Pablo Ra´ul Stinga and Jos´e L. Torrea, Maximum principles, extension problem and inversion for nonlocal one-sided equations, *J. Differential Equations* 260(7), 6333–6362, 2016.

[19] Ludwig Eduard Boltzmann, Zur theorie der elastischen nachwirkung, *Sitzungsberichte Kaiserliche Akademie Wissenhaft Wien Mathematische-Naturwissenhaft* 70, 275–306, 1874.

[20] S. Butera and M. Di Paola, A physically based connection between fractional calculus and fractal geometry, *Annals of Physics*, 350 (2014): 146–158.

[21] A. Cabada and G. Infante. Positive solutions of a nonlocal Caputo fractional BVP. *Dynamic Systems and Applications*, 23:715–722, 2014. https://arxiv.org/abs/1307.7651

[22] E. F. Camacho and C. Bordons, *Model Predictive Control*, Springer, 2007.

[23] M. Caputo, Linear models of dissipation whose Q is almost frequency independent—II, *Geophysical Journal International*, 13(5): 529–539, 1967.

[24] F. Chen, D. Baleanu, and G.-C. Wu, Existence results of fractional differential equations with Riesz–Caputo derivative, *Eur. Phys. J. Spec. Top.*, 226 (2017), pp. 3411–3425.

[25] W. Chen, H. Sun, and X. Li, *Fractional Derivative Modeling in Mechanics and Engineering*, Science Press Beijing; Springer, 2022.

[26] Bernard D. Coleman and Walter Noll, Foundations of linear viscoelasticity, *Rev. Modern Phys.* 33, 239–249, 1961.

[27] S. Das, *Functional Fractional Calculus for Dynamical Systems: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, 2011.

[28] K. Diethelm, *The Analysis of Fractional Differential Equations. Lecture Notes in Mathematics*, Vol. 2004, Springer-Verlag, Berlin Heidelberg (2010).

[29] S. Etemad, B. Tellab, J. Alzabut, and others, *Approximate solutions and Hyers–Ulam stability for a system of the coupled fractional thermostat control model via the generalized differential transform*, *Advances in Differential Equations*, vol. 2021, no. 428, 2021.

[30] Joseph Fourier, *Théorie analytique de la chaleur*, Firmin Didot, Paris, 1822. (Reprint: Editions Jacques Gabay, Paris, 1988).

[31] F. Chen, Y. Li, and H. Wei, *Stability Analysis of Fractional Thermostat Models with Memory Effects*, *Fractional Calculus and Applied Analysis*, vol. 26, no. 3, pp. 123–145, 2023.

[32] R. Gorenflo, Y. Luchko, and F. Mainardi, Analytical properties and applications of the Wright function, *Fractional Calculus and Applied Analysis*, 2(4): 383–414, 1999.

[33] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Edited by A. Jeffrey and D. Zwillinger, 8th ed., Academic Press, 2014.

[34] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[35] A.Guezane-Lakoud and Rabah Khaldi, Solvability of a Three-Point Fractional Nonlinear Boundary Value Problem, *Differential Equations and Dynamical Systems* 20(4), 395–403, 2012.

[36] A. Guezane-Lakoud and R. Rodríguez-López, Positive solutions to mixed fractional p-Laplacian boundary value problems. *J. Appl. Anal. Berlin, 29 (2023), pp. 49-58.*

[37] A.Guezane-Lakoud, Rabah Khaldi and Abdullah Kılıçman, Solvability of a Boundary Value Problem at Resonance, *SpringerPlus* 5(1), 1–9, 2016.

[38] Assia Guezane-Lakoud, Rabah Khaldi and Delfim F. M. Torres, Lyapunov-Type Inequality for a Fractional Boundary Value Problem with Natural Conditions, *SeMA Journal* 75, 157–162, 2018.

[39] A. Guezane-Lakoud, R. Khaldi, Solutions for a nonlinear fractional Euler–Lagrange type equation, *SeMA J. Madrid*, 76, 195-202 (2019).

[40] P. Guidotti and S. Merino, "Gradual loss of positivity and hidden invariant cones in a scalar heat equation." *Differential Integral Equations* 13 (10-12): 1551–1568, 2000.

[41] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones.* Academic Press, Boston, 1988.

[42] J. Henderson and R. Luca, *Positive solutions for systems of coupled fractional thermostat models*, *Results in Nonlinear Analysis* **6** (2023), no. 2, 110–127.

[43] J. Herivel, *Joseph Fourier: the man and the physicist*, Oxford, 1975.

[44] D. H. Hyers, On the stability of the linear functional equation, *Proceedings of the National Academy of Sciences of the United States of America*, 27(4) (1941): 222–224.

[45] G. Infante and J.R.L. Webb, Loss of positivity in a nonlinear scalar heat equation. *NoDEA Nonlinear Differ. Equ. Appl.* 13(2), 249–261 (2006).

[46] S.-M. Jung, Hyers-Ulam stability of functional equations: a fixed point approach, *Fixed Point Theory and Applications*, 2009, Article ID 570140 (2009).

[47] S.G. Kebede and A. Guezane Lakoud, Existence and Stability of Solution for Time-Delayed Nonlinear Fractional Differential Equations, *Applied Mathematics in Science and Engineering*, 2024, 32, NO. 1.

[48] S.G. Kebede, A. Guezane Lakoud, and H.E. Yesuf, Solution analysis for non-linear fractional differential equations. *Front. Appl. Math. Stat.* 10:1499179. (2024).

[49] Rabah Khaldi and Assia Guezane-Lakoud, Higher Order Fractional Boundary Value Problems for Mixed Type Derivatives, *Journal of Nonlinear Functional Analysis* 2017(30), 1–9, 2017.

[50] Rabah Khaldi and Assia Guezane-Lakoud, Minimal and Maximal Solutions for a Fractional Boundary Value Problem at Resonance on the Half Line, *Fractional Differential Calculus* 8(2), 299–307, 2018.

[51] H.K. Khalil, *Nonlinear Systems*, 3rd ed., Prentice Hall, Upper Saddle River, NJ, 2002.

[52] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies*, Vol. 204, Elsevier Science B.V., Amsterdam (2006).

[53] Morris Kline, *Mathematical thought from ancient to modern times*, Oxford University Press, 1972.

[54] A.N. Kolmogorov and S.V. Fomin, *Introductory Real Analysis*, Revised English ed., Dover Publications, New York, 1975.

[55] M.A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, Oxford, 1964.

[56] Joseph Klafter and Igor M. Sokolov, Anomalous diffusion spreads its wings, *Physics World* 18, 29–32, 2005.

[57] Lei Li, Jian-Guo Liu and Jianfeng Lu, Fractional stochastic differential equations satisfying fluctuation-dissipation theorem, *Journal of Statistical Physics* 168(3), 632-653, 2017.

[58] L. Ljung, *System Identification: Theory for the User*, 2nd ed., Prentice Hall, 1999.

[59] R.L. Magin, *Fractional calculus in bioengineering*, Begell House, 2006.

[60] F. Mainardi, *Fractional calculus and waves in linear viscoelasticity: An introduction to mathematical models*, World Scientific, 2010.

[61] B. B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman and Company, New York, 1982.

[62] W. Michiels and S. I. Niculescu, *Stability and Stabilization of Time-Delay Systems: An Eigenvalue-Based Approach*, SIAM, 2007.

[63] Ralf Metzler and Joseph Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.* 339(1), 77, 2000.

[64] A. Naas, M. Benbachir, M. S. Abdo, and A. Boutiara, Analysis of a fractional boundary value problem involving Riesz-Caputo fractional derivative, *Adv. Theory Nonlinear Anal. Appl.*, 1(1) (2022), pp. 14–27.

[65] J. J. Nieto and J. Pimentel. Positive solutions of a fractional thermostat model. *Boundary Value Problems*, 2013(1):5, 2013.

[66] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds. *NIST Digital Library of Mathematical Functions.* http://dlmf.nist.gov/, Release 1.1.8 of 2022-12-15.

[67] K.B. Oldham and J. Spanier, *The fractional calculus: Theory and applications of differentiation and integration to arbitrary order*, Academic Press, 1974.

[68] M. D. Ortigueira, *Riesz potential operators and derivatives. Fractional Calculus and Applied Analysis*, **14**(4), 646-657 (2011).

[69] K. M. Owolabi, A. Atangana, *Numerical Methods for Fractional Differentiation.* Springer Singapore (2019).

[70] I. Podlubny, *Fractional Differential Equations. Mathematics in Science and Engineering*, Vol. 198, Academic Press, San Diego (1999).

[71] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proceedings of the American Mathematical Society*, 72(2): 297–300, 1978.

[72] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, 2003.

[73] J. B. Rawlings and D. Q. Mayne, *Model Predictive Control: Theory and Design*, Nob Hill Publishing, 2009.

[74] S. Rezapour, M. S. Souid, Z. Bouazza, A. Hussain, and S. Etemad, "On the fractional variable order thermostat model: existence theory on cones via piece-wise constant functions," *Journal of Function Spaces*, 2022.

[75] R. Rosen, *Anticipatory Systems: Philosophical, Mathematical, and Methodological Foundations*, Pergamon Press, 1985.

[76] H.L. Royden and P.M. Fitzpatrick, *Real Analysis*, 4th ed., Pearson, Boston, 2010.

[77] W. Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, 3rd edition, 1976.

[78] K. K. Saha and N. Sukavanam, *Existence and uniqueness of blow-up solution to a fully fractional thermostat model*, *Chaos, Solitons and Fractals*, vol. 174, 2023.

[79] A. Salem, S. Djebali, D. Boucenna, R. P. Agarwal, *Existence and Ulam stability for Caputo–Hadamard fractional pantograph differential equations. Fractional Calculus and Applied Analysis*, **21**(5), 1354–1373 (2018).

[80] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Yverdon, 1993.

[81] H. H. Schaefer. Über die Methode der a priori-Schranken. *Mathematische Annalen*, 129:415–416, 1955.

[82] Harvey Scher and Elliot W. Montroll, Anomalous transit-time dispersion in amorphous solids, *Physical Review B* 12(6), 2455–2477, 1975.

[83] Gunter M. Sch¨utz and Steffen Trimper, Elephants can always remember: exact long-range memory effects in a non-markovian random walk., *Physical review. E, Statistical, nonlinear, and soft matter physics* 70(issue 4 Pt 2), 045101, 2004.

[84] Pablo Ra´ul Stinga and Jos´e L. Torrea, Regularity theory and extension problem for fractional nonlocal parabolic equations and the master equation, *SIAM J. Math. Anal.* 49(5), 3893–3924, 2017.

[85] Pablo Ra´ul Stinga, Fractional derivatives: Fourier, elephants, memory effects, viscoelastic materials and anomalous diffusions, *Preprint or unpublished notes, "To be added"*, 2010.

[86] Pablo Raúl Stinga and Mary Vaughan, One-sided fractional derivatives, fractional Laplacians, and weighted Sobolev spaces, *Nonlinear Anal.* 193, 111505, 2020.

[87] HongGuang Sun, Yong Zhang, Dumitru Baleanu, Wen Chen and YangQuan Chen, A new collection of real world applications of fractional calculus in science and engineering, *Communications in Nonlinear Science and Numerical Simulation* 64, 213-231, 2018.

[88] F. B. Tatom, The Relationship Between Fractional Calculus and Fractals, *Fractals*, 3(1) (1995): 217–229.

[89] M.S. Tavazoei and R. Safari, Fractional-order controller design for an electro-hydraulic servo system, *ISA transactions* 47(3), 241-251, 2008.

[90] Q. Yang, F. Liu, I. Turner, Numerical methods for fractional partial differential equations with Riesz space fractional derivatives, *Applied Mathematical Modelling*, 34(1), 200-218 (2010).

[91] V.V. Uchaikin, *Fractional Derivatives for Physicists and Engineers*, Volume I: Background and Theory, Nonlinear Physical Science, Springer, Berlin, Heidelberg, 2013.

[92] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, 1960.

[93] Bruce J. West, V. Seshadri and Katja Lindenberg, Fractional Langevin model of memory in financial markets, *Physical Review E* 65(3), 037106, 2002.

[94] N. Wiener. *Cybernetics: Or Control and Communication in the Animal and the Machine.* MIT Press, 1948.

[95] G. M. Zaslavsky, Chaos, fractional kinetics, and anomalous transport, *Phys. Rep.* 371(6), 461–580, 2002.

[96] T. Zeghida, R. Khaldi and A. Guezane-Lakoud, *Existence Analysis of Multi-Point Boundary Value Problems with Riesz-Caputo Fractional Derivatives*, *Sahand Commun. Math. Anal.*, Vol. 21 No. 3 (2024), pp. 301-322.