

وزارة التعليم العالي والبحث العلمي  
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BADJI MOKHTAR-ANNABA  
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جامعة باجي مختار  
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## THESIS

Presented with a view to obtaining the doctorate degree

On the asymptotic behavior of some porous elastic systems.

Stream  
Applied Mathematics

Speciality  
Optimal control

By  
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## THÈSE

Présentée en vue de l'obtention du diplôme de Doctorat

**Sur le comportement asymptotique de certains systèmes élastiques  
poreux.**

Filière  
Mathématiques Appliquées

Spécialité  
Contrôle optimal

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## DEDICATION

*To **my mother**, who planted the seed of knowledge in my mind and nurtured it.*

*To the teachers who educate with passion and patience.*

*To the past that shaped me, the present that nurtures me, and the future that awaits me.*

## شكر وتقدير

الحمد لله حمدا كثيرا حتى يبلغ الحمد منتهاه والصلاة والسلام على أشرف مخلوق أناره الله بنوره واصطفاه.  
اتباعا لقوله صلى الله عليه وسلم: "لا يشكر الله من لا يشكر الناس" فلا بد لنا ونحن نخط آخر أسطر هاته الرسالة إلا وأن نتقدم بأسمى آيات الشكر، الامتنان، التقدير والمحبة إلى الذين حملوا أقدس رسالة في الحياة... إلى الذين مهدوا لنا طريق العلم والمعرفة... إلى جميع أساتذتنا الأفاضل.

وأخص بالتقدير والشكر أستاذي ومؤطري الأستاذ جبالة عبد الحق

الذي كفى ووفى في تقديم ما يجب لأجل إتمام هاته الرسالة

ولم يكتف فقط بدور الأستاذ فقد كان لي الناصح الأمين

المقوم لصقل الخبرة والمعرفة، المرشد حين ضللت الطريق

والنور في غياباتها، معلمي التفاؤل والمضي قدما مهما كانت النتائج بعد التعب المضي

فهل الشكر المقنن في هاته الأسطر يوفيك حقا؟

أكيد لا، فمهما أطلت مدحك وشكرك لن ابلغه منتهاه...

كذلك اشكر من ساعدني على إتمام هذا البحث وقدم لي العون ويد المساعدة وكان لي السند والداعم معنويا، الحريص على النجاح والتفوق دائما

الأستاذ كيلاني إبراهيم

كما أتقدم بالشكر، الاحترام والتقدير للسادة الأفاضل أعضاء لجنة المناقشة الموقرين على ما بذلوه من جهد في

قراءة رسالتي ومشاركتي آراءهم لتتقيحها.

يأتي الدور على من قال فيها أبو العلاء المعري:

العيشُ ماضٍ فأكرمُ والذِكُّ بهِ والأُمُّ أُولَى بِإِكْرَامِ وإِحْسَانِ

وحَسْبُهَا الحَمْلُ والإِرْضَاعُ تُدْمِئُهُ أَمْرَانِ بِالْفَضْلِ نَالَا كُلَّ إِنْسَانٍ...

وعليه شكرك وتقدير مجهودك في حثي على بلوغ هدفي والفوز بالمراتب العلا أولى أن يكون في الأسطر الأولى من انجازاتي، لكن لا عليك يا أماه ليس المكان والزمان والكلام من يحدد معيار تعبك ومقدار شقائك، ليست الحروف وأبيات الشعر من توفيك حقا من مدح وشكر تعبيراً عن تضحياتك، فمهما طال الخطاب لن يبلغ النصاب.

فهذا الشكر أيضا مخصوص به ذكر من زرعو التفاؤل في دربي وقدموا لي المساعدات، التسهيلات والأفكار. والمعلومات، ربما دون ان يشعروا... إليك جدي، جدتي، أخوالي وخالاتي.

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## ABSTRACT

This thesis is devoted to the study of the existence, uniqueness and the asymptotic behavior of some porous elastic systems.

The first system is a one-dimensional swelling porous elastic system with neutral delay and porous damping acting on the second equation. We prove that the porous damping dissipation is powerful enough to stabilize the system exponentially even in the presence of neutral delay.

The second system is a Lord-Shulman porous-elastic system with dissipation due to microtemperature effects, where the thermal conduction has a single-phase-lag that acts as a relaxation time. We show that the system is exponentially stable provided that the new stability number  $\chi = 0$ . Otherwise, we prove the lack of exponential stability under the assumption  $\chi \neq 0$ . Furthermore, in the last case, we show that the solution decays polynomially.

The final system is the same as the preceding system. However, the energy associated with the solution is not required to be positive definite ( $\xi\mu^* = \mu_0^2$ ). We introduce a stability number  $\chi$  and prove the exponential decay of the system if  $\chi = 0$  is valid. Otherwise, we show that the system decays polynomially.

The method we have used for studying the asymptotic behavior of solutions is the multiplier method based on the energy estimate and Lyapunov direct method. Semi-group and Faedo-Galerkin techniques are those used for the study of the existence and uniqueness.

**Key Words and Phrases:** Swelling porous; neutral delay; porous damping; Faedo-Galerkin method; Lyapunov functional; exponential stability; microtemperature effects; lack of exponential stability; polynomial stability; Lord-Shulman thermoelasticity; stability number.

## ملخص

هذه الأطروحة مكرسة لدراسة وجود ووحداية الحل والسلوك التقاربي لبعض جمل معادلات تعبر عن مسائل تنمذج لمواد مرنة مسامية في الفضاء المكاني ذو البعد الواحد.

الجملة الأولى هي عبارة عن جملة مرنة أحادية البعد ذو خاصية انتفاخيه، لها تبديد تأخير محايد وآخر مسامي. أثبتنا أن التبديد المسامي كان قوي بما يكفي لتحقيق الاستقرار في الجملة بشكل كبير، حتى في حالة وجود التأخير المحايد.

الجملة الثانية هي جملة لورد-شولمان المطاطية المسامية مع وجود تبديد وحيد بسبب درجة الحرارة الدقيقة، حيث يكون للتوصيل الحراري تأخر ذو طور واحد يعمل كوقت للاسترخاء. أظهرنا أن الجملة مستقرة بشكل كبير بشرط أن مؤشر الاستقرار الجديد  $\chi$  يحقق  $\chi = 0$ . وفي حالة عدم تحقق الشرط السابق، فقد أثبتنا عدم وجود استقرار أسي.

الجملة النهائية هي نفسها الجملة السابقة، غير أن الاختلاف يكمن في أن الطاقة المرتبطة بالحل معرفة وفق الشرط  $\xi\mu^* = \mu_0^2$ . قدمنا مؤشر الاستقرار  $\chi$  وأثبتنا أن الاستقرار الأسي لا يكون محققا إلا بتحقيق الشرط التالي:  $\chi = 0$ .

الطريقة التي استخدمناها لدراسة السلوك التقاربي للحلول هي طريقة الجداءات استنادا إلى تقدير الطاقة وطريقة ليابونوف المباشرة. تقنيات أنصاف الزمر وفايدو-غاليركين هي تلك المستخدمة لدراسة وجود ووحداية الحلول.

**الكلمات والجمل المفتاحية:** الانتفاخ المسامي؛ التأخير المحايد؛ التبديد المسامي؛ طريقة فايدو-غاليركين؛ دالة ليابونوف؛ الاستقرار الأسي؛ المرونة الحرارية للورد-شولمان؛ مؤشر الثبات.

## RÉSUMÉ

Cette thèse est consacrée à l'étude de l'existence, de l'unicité et du comportement asymptotique de certains systèmes élastiques poreux.

Le premier système est un système élastique poreux de gonflement unidimensionnel avec un retard neutre et un amortissement poreux agissant sur la deuxième équation. Nous prouvons que la dissipation d'amortissement poreuse est suffisamment puissante pour stabiliser le système de manière exponentielle, même en présence d'un retard neutre.

Le deuxième système est un système poreux-élastique de Lord-Shulman avec dissipation due aux effets de microtempérature, où la conduction thermique a un décalage monophasé qui agit comme un temps de relaxation. Nous montrons que le système est exponentiellement stable à condition que le nouveau nombre de stabilité  $\chi = 0$ . Sinon, nous prouvons le manque de stabilité exponentielle sous l'hypothèse  $\chi \neq 0$ . De plus, dans le dernier cas, nous montrons que la solution se désintègre polynomialement.

Le système final est le même que le système précédent. Cependant, l'énergie associée à la solution n'est pas nécessaire conçue pour être définie positive ( $\xi \mu^* = \mu_0^2$ ). Nous introduisons un nombre de stabilité  $\chi$  et prouvons la décroissance exponentielle du système si  $\chi = 0$  est valide. Sinon, nous montrons que le système se désintègre polynomialement. La méthode que nous avons utilisée pour étudier le comportement asymptotique des solutions est la méthode multiplicateur basée sur l'estimation de l'énergie et la méthode directe de Lyapunov. Les techniques de semi-groupes et de Faedo-Galerkin sont celles utilisées pour l'étude de l'existence et de l'unicité.

**Mots clés et phrases:** Gonflement poreux; retard neutre; amortissement poreux; méthode de Faedo-Galerkin; fonctionnelle de Lyapunov; stabilité exponentielle; effets de la microtempérature; absence de stabilité exponentielle; stabilité polynomiale; thermoélasticité de Lord-Shulman; nombre de stabilité.

## PUBLICATIONS

1. M. Boudeliou, A. Djebabla, and T. A. Apalara, Well posedness and stability result of Lord–Shulman system with microtemperature effects: The case  $\xi\mu^* = \mu_0^2$ , *Math. Meth. Appl. Sci.* 47 (2024), 8171–8186, DOI: 10.1002/mma.10009.
2. S. Meradji, M. Boudeliou, and A. Djebabla, New stability number of the Timoshenko system with only microtemperature effects and without thermal conductivity, *EURASIAN JOURNAL OF MATHEMATICAL AND COMPUTER APPLICATIONS*, 12 (2024), 94– 109, DOI: 10.32523/2306-6172-2024-12-1-94-109.
3. M. Boudeliou, B. Kilani, and A. Djebabla, Well-posedness and stability result for a swelling porous elastic system with neutral delay and porous damping, *Filomat*, 38(9) (2024), 3055-3067, DOI: 10.2298/FIL2409055B.

## THESIS STRUCTURE

This manuscript is divided into five chapters:

1. **The first chapter** reviews the fundamental concepts and mathematical techniques required for this thesis.
2. **The second chapter** is meant to provide a broad introduction.
3. **The third chapter** is devoted to the study of the existence, uniqueness, and exponential stability of a swelling porous elastic system with neutral delay and porous damping using the Faedo-Galerkin approach and the multiplier technique. This result has been published in the Filomat journal.
4. **The fourth chapter** focuses on the study of the well posedness and stability result of Lord-Shulman system with only microtemperature effects. A new number of stability was introduced and proved that the unique dissipation due to micro-temperatures is strong enough to drive the system to equilibrium exponentially. This study has been submitted for prospective publication.
5. **The fifth chapter** extends the results obtained in chapter 4 by studying the same Lord-Shulman system with only micro-temperature effect, but this time under different conditions. Exponential and polynomial stability were identified. This result has been published in the Mathematical Methods in the Applied Sciences journal.

## GENERAL INTRODUCTION

Understanding and predicting the long-term behavior of solutions to evolution equations and systems originating from mechanics and physics is crucial for practical applications, which makes the study of their asymptotic behavior an important topic. Asymptotic behavior of differential equations has been studied since Bellman [10, 11], Fubini [28], and Sansone [64] published their papers. The concept of asymptotic behavior and its relationship to stability and boundlessness were provided by Bellman [10]. Recall that the comparison of a differential equation's solution or its norm with elementary functions as  $t$  goes to infinity is known as the asymptotic behavior of the solution. As we will explain later, the stability notion introduced by A. Lyapunov in 1899 has a strong connection to the idea of asymptotic behavior of different systems. Here we will provide a brief overview of the ideas related to porous elastic systems with various damping circumstances.

### 1.1 Porous elastic materials

Elastic solids with voids are a straightforward extension of classical elasticity theory. This method treats porous materials with elastic matrix and void interstices. Goodman and Cowin [29] proposed a continuum theory of granular materials with interstitial spaces. Aside from the normal elastic effects, these materials contain a microstructure with an interesting property: the mass in each location can be calculated as the product of the mass density of the material matrix and the volume percentage. Nunziato and Cowin [52] utilized this concept to present a nonlinear theory of elastic materials with voids. This refers to materials with elastic skeleton or matrix material and interstices. Ieşan and others have spoken about these kinds of materials extensively [39]. The large number of papers that have been published in the previous three decades addressing applications to many sectors of physics and engineering (such as the petroleum industry, material science, biology, etc.) clearly indicate the relevance of elastic materials with microstructure. Solids

with tiny, dispersed pores, including rocks, soils, wood, ceramics, pressed powders, or biological materials like bones, should be treated according to the poroelasticity principle.

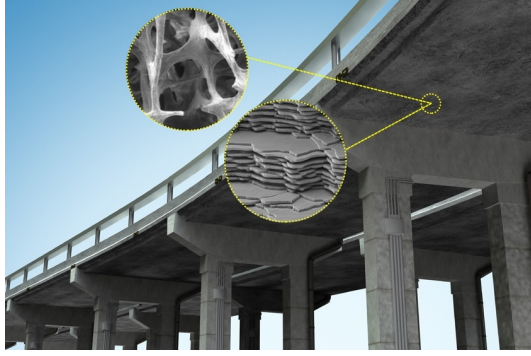


Figure 1: Building materials' pores.



Figure 2: Bone pores.

In [21], Cowin outlined the linear theory of elastic materials with voids. For an isotropic material, he emphasized that the equations are of the form

$$\begin{aligned}\rho \ddot{u}_i &= T_{ij,j} + \rho b_i, \\ \rho \kappa \ddot{\phi}_i &= h_{i,i} + g + \rho l,\end{aligned}$$

where  $T_{ij,j}$  represents the symmetric stress tensor,  $b_i$  is the body force vector,  $\kappa$  is the equilibrated inertia,  $h_i$  denotes the equilibrated stress vector,  $g$  and  $l$  are the intrinsic and the extrinsic equilibrated body forces, respectively. The theory's constitutive equations are provided by

$$\begin{aligned}T_{ij,j} &= \lambda \delta_{ij} E_{kk} + 2\mu E_{ij} + \beta \phi \delta_{ij}, \\ h_i &= \alpha \phi_i, \quad g = -\omega \phi_i - \xi \phi - \beta E_{kk},\end{aligned}$$

the details of this argument  $\lambda$ ,  $\mu$ ,  $\beta$ ,  $\alpha$ ,  $\omega$ ,  $\xi$ ,  $\phi$ ,  $\delta_{ij}$  and  $E_{kk}$  are described in [29], satisfying the relations

$$\mu \geq 0, \quad \alpha \geq 0, \quad \xi \geq 0, \quad \kappa = \lambda + 2\mu/3 \geq 0 \quad \text{and} \quad \kappa \xi - \beta^2 \geq 0.$$

The substitution of the constitutive equations into the evolution ones gives the following one-dimensional porous elastic system:

$$(1.1) \quad \begin{cases} \rho_0 u_{tt} = \mu u_{xx} + \beta \varphi_x, & \text{in } (0, L) \times (0, \infty), \\ \rho_0 \kappa \varphi_{tt} = \alpha \varphi_{xx} - \beta u_x - \xi \varphi, & \text{in } (0, L) \times (0, \infty). \end{cases}$$

In [56], Quintanilla studied the previous system in the presence of a viscoporous damp occurring in the second equation of (1.1) and found a slow decay, deducing that  $-\tau \varphi_t$  was insufficient to stabilise system (1.1) exponentially. In [3], Apalara demonstrated that the system studied in [56] is exponentially stable with identical wave propagation speeds. In [50], Magaña and Quintanilla considered the case where viscoelasticity and porous dissipation ( $\gamma u_{txx}$  and  $-\tau \varphi_t$ ) were both

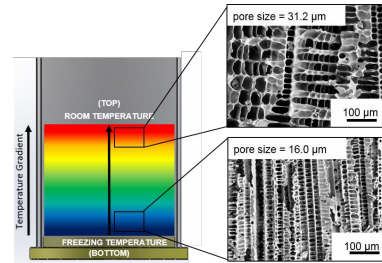
present. They proved by using the semigroup arguments, due to Liu and Zheng (1999), that the two dissipations were enough to have exponential stability. Determining the stability result for elastic systems with voids has generally been the subject of several articles [65, 66].

Eringen (1994) developed a complete formulation of mixture theory for a porous elastic solid filled with fluid and gas. This formulation has many applications in various practical problems, such as swelling fields, oil explanation, slurred, and consolidation. Thus, Eringen has classified expansive (swelling) soils under the classification of porous media theory. The discussion of the mathematical study of the swelling porous elastic systems is the primary focus of Chapter 3.

## 1.2 Thermoelasticity

Both mechanical and thermal loadings are applied to structures in many significant stress analysis issues. Heat transmission by conduction is caused by thermal processes in an elastic solid, and this movement of thermal energy creates a temperature field within the material. Since most solids experience a volumetric change in response to temperature changes, the existence of a temperature distribution often results in strains resulting from internal or boundary limitations. These stresses have the potential to cause structural damage if the temperature change is large enough, particularly for fragile materials. Thus, understanding thermal stress analysis may be crucial for many issues requiring large temperature variations.

Using the temperature effect on porous elastic materials (Thermo-Poro-Elasticity) as an example, the figure on the right illustrates the detrimental effects of temperature on pores: the higher the temperature, the more the pores enlarge and the more fragile the material.



Temperature effect on porous elastic materials.

Fourier law is the renowned law for modeling heat conduction, which is also known as the thermoelasticity law.

$$q = -\kappa\theta_x,$$

where  $\theta$  is the difference temperature,  $q$  is the heat conduction vector and  $\kappa$  is the coefficient of thermal conductivity. Several authors investigated thermo-porous system using Fourier's law. Where in [15], Casas and Quintanilla established system (1.1) with thermal effects, as follow

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x, \\ J\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi - \tau\varphi_t + m\theta, \\ c\theta_t = k^*\theta_{xx} - \beta u_x - m\varphi_t. \end{cases}$$

they proved that the combination of porous-viscosity and thermal factors results in exponential stability of solutions. In [50], Magaña and Quintanilla showed that viscoelasticity damping and temperature produced slow decay in time for the same system.

We observe that the previous studies applied the classical linear theory of thermoelasticity, allowing the traditional dilemma of unlimited propagation speeds of thermoelastic perturbation. The infinite speed of propagation phenomenon leads to an unrealistic property that a sudden disturbance at one point will be noticed instantly everywhere else in the materials. Experiments have shown that this paradox also produces inadequate or inaccurate representations of how a solid might behave to a short transient loading, such as that induced by brief laser pulses. Several experts have advanced numerous generalized thermoelasticity theories that use hyperbolic governing equations as a result of these drawbacks. In the footsteps of Maxwell and Cattaneo, Lord and Shulman's extended thermoelasticity theory, involving one relaxation time known as the single-phase-lag model and Green and Lindsay's temperature-rate-dependent thermoelasticity theory involving two relaxation times called dual-phase-lag model, are two major concepts of the generalized theory of thermoelasticity. For more details, we refer the reader to the introduction of chapter 5.

### 1.3 Microtemperature

The term microtemperature is a blend word that consists of two words: microscopic and temperature which means the effect of temperature on a micro-volume.

The latter idea was first introduced by Nunziato and Cowin [52] in their seminal work on elastic materials with voids. Temperature and microtemperatures were added to the concept by Ieşan [39–41] and Ieşan and Quintanilla [42]. In [16], Casas and Quintanilla utilized the semigroup technique for the following system

$$\begin{cases} \rho u_{tt} = \mu^* u_{xx} + b\varphi_x - \beta\theta_x, \\ J\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta, \\ c\theta_t = k\theta_{xx} - \gamma u_{tx} - l\varphi_t + \kappa_1 w_x, \\ \delta w_t = \kappa_4^* w_{xx} - d\varphi_{tx} - \kappa_3\theta_x - \kappa_2 w, \end{cases}$$

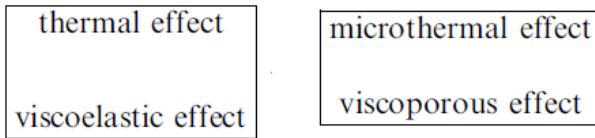
to demonstrate the exponential stability of solutions in a one-dimensional porous-elastic system with thermal effect and microtemperature damping. The solutions were stable regardless of wave propagation speed. In [50], viscoelasticity combined with microtemperatures created exponential stability, as shown by Magaña and Quintanilla in the following system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + \gamma u_{txx}, \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x, \\ \alpha w_t = \kappa_4 w_{xx} - d\varphi_{tx} - \kappa_2 w. \end{cases}$$

Similarly, for the case of porous-viscosity combined with microtemperatures and in the absence of elastic dissipation, as introduced below

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x, \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x - \tau\varphi_t, \\ \alpha w_t = \kappa_4 w_{xx} - d\varphi_{tx} - \kappa_2 w. \end{cases}$$

The same authors proved a slow decay when  $J\mu \neq \delta\rho$ . And they summarized their main results as below:



Combining effects from the right and left squares results in exponential stability. However, considering two simultaneous impacts from a single square results in slow decay. Of course, combining more than two of these processes results in exponential stability.

## PRELIMINARIES

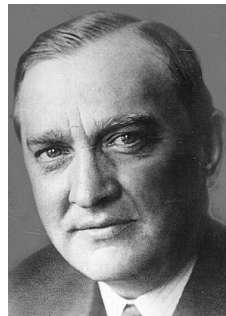
In this chapter, we provide different lemmas, inequalities and definitions of various basic tools that will be employed later. For more details see [14, 68–70]

## 2.1 Basic spaces

This section covers the basic spaces (Banach, Hilbert, Lebesgue and Sobolev).

### 2.1.1 Banach space

Banach space (pronounced [ˈbanax]), named after the Polish mathematician **Stefan Banach**, is a complete normed vector space. Thus, a Banach space is a vector space with a metric that allows the computation of vector length and distance between vectors and is complete in the sense that a Cauchy sequence of vectors always converges to a well-defined limit that is within the space.



**Stefan Banach.**

**Definition 2.1.1.** A Banach space is a complete normed linear space  $X$ . Its dual space  $X'$  is the linear space of all continuous linear functional  $f : X \rightarrow \mathbb{R}$ .

**Proposition 2.1.2.**  $X'$  equipped with the norm  $\|\cdot\|_{X'}$  defined by

$$(2.1) \quad \|f\|_{X'} = \sup\{|f(u)| : \|u\| \leq 1\}$$

is also a Banach space.

### 2.1.1.1 The weak and weak star topologies

Let  $X$  be a Banach space and  $f \in X'$ . Denote by

$$(2.2) \quad \begin{aligned} \phi_f : X &\rightarrow \mathbb{R} \\ x &\mapsto \phi_f(x), \end{aligned}$$

when  $f$  cover  $X'$ , we obtain a family  $(\phi_f)_{f \in X'}$  of applications to  $X$  in  $\mathbb{R}$ .

**Definition 2.1.3.** *The weak topology on  $X$ , denoted by  $\sigma(X, X')$ , is the weakest topology on  $X$  for which every  $(\phi_f)_{f \in X'}$  is continuous.*

We will define the third topology on  $X'$ , the weak star topology, denoted by  $\sigma(X', X)$ . For all  $x \in X$

$$(2.3) \quad \begin{aligned} \phi_x : X' &\rightarrow \mathbb{R} \\ f &\mapsto \phi_x(f) = \langle f, x \rangle_{X', X}, \end{aligned}$$

when  $x$  cover  $X$ , we obtain a family  $(\phi_x)_{x \in X}$  of applications to  $X'$  in  $\mathbb{R}$ .

**Definition 2.1.4.** *The weak star topology on  $X'$  is the weakest topology on  $X'$  for which every  $(\phi_x)_{x \in X}$  is continuous.*

**Remark 2.1.5.** *Since  $X \subset X''$ , it is clear that, the weak star topology  $\sigma(X', X)$  is weaker than  $\sigma(X', X'')$  topology, and this later is weaker than the strong topology.*

**Definition 2.1.6.** *A sequence  $(u_n)_{n \in \mathbb{N}}$  in  $X$  is weakly convergent to  $X$  if and only if*

$$\lim_{n \rightarrow +\infty} f(u_n) = f(u),$$

for every  $f \in X'$ , and this is denoted by  $u_n \rightharpoonup u$ .

**Remark 2.1.7.** *We have:*

1. *If the weak limit exist, it is unique.*
2. *If  $u_n \rightarrow u \in X$  (strongly), then  $u_n \rightharpoonup u$  (weakly).*
3. *If  $\dim X < +\infty$ , then the weak convergent implies the strong convergent.*

**Proposition 2.1.8.** *On the compactness in the three topologies in the Banach space  $X$ :*

1. *First, the unit ball  $B \equiv \{x \in X : \|x\| \leq 1\}$  in  $X$  is compact if and only if  $\dim(X) < \infty$ .*
2. *Second, the unit ball  $B'$  in  $X'$  (The closed subspace of a product of compact spaces) is weakly compact in  $X'$  if and only if  $X$  is reflexive.*
3. *Third,  $B'$  is always weakly star compact in the weak star topology of  $X'$ .*

**Proposition 2.1.9.** Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence in  $X'$ . We have:

1.  $[f_n \xrightarrow{*} f \text{ in } \sigma(X', X)] \Leftrightarrow [f_n(x) \rightarrow f(x), \forall x \in X]$ .
2. If  $f_n \rightarrow f$  (strongly), then  $f_n \rightarrow f$  in  $\sigma(X', X'')$ ,  
If  $f_n \rightarrow f$  in  $\sigma(X', X'')$ , then  $f_n \xrightarrow{*} f$  in  $\sigma(X', X)$ .
3. If  $f_n \xrightarrow{*} f$  in  $\sigma(X', X)$ , then  $\|f_n\|$  is bounded and  $\|f\| \leq \liminf \|f_n\|$ .
4. If  $f_n \xrightarrow{*} f$  in  $\sigma(X', X)$  and  $x_n \rightarrow x$  (strongly) in  $X$ , then  $f_n(x_n) \rightarrow f(x)$ .

### 2.1.2 Hilbert space

Hilbert space, named after **David Hilbert**, the most significant function space in contemporary physics and analysis, are crucial for rigorously studying partial differential equations.



**David Hilbert.**

**Definition 2.1.10.** A Hilbert space  $H$  is a vectorial space supplied with inner product  $\langle u, v \rangle$  such that  $\|u\| = \sqrt{\langle u, u \rangle}$  is the norm which let  $H$  complete.

**Theorem 2.1.11.** Let  $(u_n)_{n \in \mathbb{N}}$  be a bounded sequence in the Hilbert space  $H$ , then it possess a subsequence which converges in the weak topology of  $H$ .

**Theorem 2.1.12.** In Hilbert space, all sequence which converges in the weak topology is bounded.

**Theorem 2.1.13. (Lax Milgram)** Assume that  $a(\cdot, \cdot): H \times H \rightarrow \mathbb{R}$  is a continuous and coercive bilinear form, then for every linear continuous form  $b \in H'$ , there exists a unique element  $u \in H$  such that

$$a(u, v) = b(v), \forall v \in H.$$

### 2.1.3 Lebesgue space

$L^p$  spaces are an important type of Banach spaces in functional analysis, as well as topological vector spaces. Because of their importance in the mathematical study of measure and probability spaces, Lebesgue spaces, named after **Henri Lebesgue**, are utilized in the theoretical discussion of issues in physics, statistics, economics, finance, engineering, and other fields.



**Henri Lebesgue.**

**Definition 2.1.14.** Let  $\Omega$  be a domain in  $\mathbb{R}$ , the space  $L^p(\Omega)$  is defined as follow:

- For  $1 \leq p < \infty$ , the space  $L^p(\Omega)$  is for the real functions  $f$  on  $\Omega$ , such that  $f$  is measurable and

$$\int_{\Omega} |f(x)|^p dx < \infty.$$

If  $f \in L^p(\Omega)$ , the associated norm is defined by

$$\|f\|_p = \left( \int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

- For  $p = \infty$ , the space  $L^p(\Omega)$  is for the measurable functions  $f$  which are essentially bounded on  $\Omega$ .

If  $f \in L^\infty(\Omega)$ , the associated norm is defined by

$$\|f\|_\infty = \inf \{A \geq 0 : \mu\{x \in \Omega : f(x) > A\} = 0\}.$$

## 2.1.4 Sobolev space

Sobolev spaces are named after the Russian mathematician **Sergei Sobolev**. Their importance comes from the fact that weak solutions of some important partial differential equations exist in appropriate Sobolev spaces, even when there are no strong solutions in spaces of continuous functions with the derivatives understood in the classical sense.



**Sergei Sobolev.**

### 2.1.4.1 Sobolev space $W^{1,p}(\Omega)$

Let  $C_0^\infty(\Omega)$  be the space of functions indefinitely derivable with compact support.

**Definition 2.1.15.** Let  $\Omega$  be an open domain in  $\mathbb{R}^n$  and  $1 \leq p \leq \infty$ . Then, the Sobolev space  $W^{1,p}(\Omega)$  is defined by

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega), \exists g \in L^p(\Omega), \text{ such that: } \int_{\Omega} u(x)\phi'(x)dx = - \int_{\Omega} g(x)\phi(x)dx, \forall \phi \in C_0^\infty(\Omega) \right\}.$$

- When  $p = 2$ , we prefer to denote  $W^{1,2}(\Omega)$  by  $H^1(\Omega)$ .

### 2.1.4.2 Sobolev space $W^{m,p}(\Omega)$

**Definition 2.1.16.** Let  $\Omega$  be an open domain in  $\mathbb{R}^n$ ,  $m \geq 2$  and  $1 \leq p \leq \infty$ . Then, the Sobolev space  $W^{m,p}(\Omega)$  is defined by

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega), \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq m, \exists g_\alpha \in L^p(\Omega), \text{ such that: } \int_{\Omega} u(x)D^\alpha \phi(x)dx = (-1)^{|\alpha|} \int_{\Omega} g_\alpha(x)\phi(x)dx, \forall \phi \in C_0^\infty(\Omega) \right\},$$

where  $|\alpha| = \sum_{i=1}^n \alpha_i$  and  $D^\alpha \phi = \frac{\partial^{|\alpha|} \phi}{\partial^{|\alpha|} x_1 \partial^{\alpha_2} x_2 \dots \partial^{\alpha_n} x_n}$ .

The Sobolev space  $W^{m,p}(\Omega)$  is equipped with the following norm

$$\|u\|_{W^{m,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{0 < \alpha \leq m} \|D^\alpha u\|_{L^p(\Omega)}.$$

- When  $p = 2$ , we prefer to denote  $W^{m,2}(\Omega)$  by  $H^m(\Omega)$ . The  $H^m(\Omega)$  space is equipped with the following scalar product

$$\langle u, v \rangle_{H^m(\Omega)} = \langle u, v \rangle_{L^2(\Omega)} + \sum_{\alpha=1}^m \langle D^\alpha u, D^\alpha v \rangle_{L^2(\Omega)}, \quad \forall u \text{ and } v \in H^m(\Omega).$$

### 2.1.4.3 Sobolev space $W_0^{1,p}(\Omega)$

Let  $C_c^k(\Omega)$  be the function space  $k$  times derivable and the order derivative  $k$  is continuous with compact support.

**Definition 2.1.17.** Let  $1 \leq p < \infty$ , we denote by  $W_0^{1,p}(\Omega)$  the closure of  $C_c^\infty(\Omega)$  in  $W^{1,p}(\Omega)$ .

- For  $p = 2$ , we note  $W_0^{1,2}(\Omega) = H_0^1(\Omega)$ .

**Notation 2.1.18.**  $W^{-1,p}(\Omega)$  is the dual space of  $W_0^{1,p}(\Omega)$ , where  $1 \leq p < \infty$ , and  $H^{-1}(\Omega)$  is the dual space of  $H_0^1(\Omega)$ .

**Theorem 2.1.19.** We have:

1.  $H^m(\Omega)$  equipped with inner product  $\langle \cdot, \cdot \rangle_{H^m(\Omega)}$  is a Hilbert space.
2. If  $m \geq m'$ ,  $H^m(\Omega) \hookrightarrow H^{m'}(\Omega)$ , with continuous embedding.

**Lemma 2.1.20.** Since  $C_0^\infty(\Omega)$  is dense in  $H_0^m(\Omega)$ , we identify a dual space  $H^{-m}(\Omega)$  of  $H_0^m(\Omega)$  in a weak subspace on  $\Omega$ , and we have

$$C_0^\infty(\Omega) \hookrightarrow H_0^m(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-m}(\Omega) \hookrightarrow (C_0^\infty(\Omega))'.$$

## 2.2 Useful inequalities

### 2.2.1 Hölder inequality

The Hölder inequality, so named in honor of **Otto Hölder**, is a fundamental inequality with regard to  $L_p$  function spaces, such as  $l_p$  sequence spaces.

This is a generalization of the Cauchy-Schwarz inequality.



**Otto Hölder.**

**Lemma 2.2.1.** For  $p$  and  $q$  two conjugated real numbers, i.e.  $\frac{1}{p} + \frac{1}{q} = 1$ , for  $f$  and  $g$  two positive measurable functions. Then, for all  $f \in L^p(\Omega)$ ,  $g \in L^q(\Omega)$ :  $fg \in L^1(\Omega)$ , we have

- If  $p, q \in [1, +\infty]$ , we have

$$\int_{\Omega} |f(x)g(x)| dx \leq \left( \int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |g(x)|^q dx \right)^{\frac{1}{q}}.$$

- If  $p = 1$ ,  $q = +\infty$ , we have

$$\int_{\Omega} |f(x)g(x)| dx \leq \left( \int_{\Omega} |f(x)| dx \right) \|g\|_{L^\infty(\Omega)}.$$

**Remark 2.2.2.** The Cauchy-Schwarz inequality is a special case of Hölder's inequality when  $p = 2$  and  $q = 2$ .

**Theorem 2.2.3.** Let  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  be real. Then

$$(a_1b_1 + \dots + a_nb_n)^2 \leq (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2),$$

with equality if and only if the vectors  $a = (a_1, \dots, a_n)$  and  $b = (b_1, \dots, b_n)$  are collinear, i.e.,

$$\exists k \in \mathbb{R}, \text{ such that: } a = kb.$$

- Particular case: If  $\forall i = 1, \dots, n : b_i = 1$ . Then  $(a_1b_1 + \dots + a_nb_n)^2 \leq n(a_1^2 + \dots + a_n^2)$ .

## 2.2.2 Young inequality

A mathematical inequality concerning the product of two integers is known as Young's inequality for products. **William Henry Young** is honored by the name of the inequality.

Hölder's inequality may be shown using Young's inequality for products. Since it enables one to estimate a product of two terms by a sum of the same terms raised to a power and scaled, it is also often used to estimate the norm of nonlinear terms in PDE theory.



**William Henry Young.**

**Lemma 2.2.4.** For  $p$  and  $q$  two conjugated real numbers. Then, for all  $f$  and  $g \in \mathbb{R}_+$ , we have

$$fg \leq \frac{1}{p}f^p + \frac{1}{q}g^q.$$

**Remark 2.2.5.** The above inequality is expressed as follows for  $p = q = 2$

$$\forall \varepsilon > 0, fg \leq \varepsilon f^2 + \frac{1}{4\varepsilon} g^2.$$

### 2.2.3 Poincaré inequality

Sobolev's theory of spaces leads to the mathematical inequality known as Poincaré. The French mathematician **Henri Poincaré** is honored by the name of the inequality.

A function may be limited using this inequality by estimating its derivatives and the geometry of its domain of definition. For the modern, direct approach of estimating variations, these estimations are crucial.



**Henri Poincaré.**

**Lemma 2.2.6.** *There exists a positive constant  $K$ , for  $\Omega$  a bounded domain in  $\mathbb{R}_+^n$ , such that*

$$\forall f \in H_0^1(\Omega), \quad \|f\|_{H^1(\Omega)} \leq K \|\nabla f\|_{L^2(\Omega)},$$

where  $\nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$ .

The following is the usual Poincaré inequality in  $L^2(\Omega)$ :

**Lemma 2.2.7.** *Let  $f \in H_0^1(\Omega)$ . Then, there exists a positive constant  $K$ , so that*

$$\|f\|_{L^2(\Omega)} \leq K \|\nabla f\|_{L^2(\Omega)}.$$

## 2.3 Convolution product

**Definition 2.3.1.** *The convolution product of two real or complex functions,  $f$  and  $g$ , results in another function, denoted as  $(f * g)$  and defined by:*

$$(f * g)(t) = \int_0^t f(r)g(t-r)dr = \int_0^t f(t-r)g(r)dr, \quad t \geq 0.$$

**Definition 2.3.2.** *We define the binary operators  $\square$  and  $\circ$  respectively by*

$$(h \square u_t)(t) = \int_0^t |h(t-r)| |u_t(t) - u_t(r)| dr,$$

$$(h \circ u_t)(t) = \int_0^t |h(t-r)| |u_t(t) - u_t(r)|^2 dr.$$

**Lemma 2.3.3.** *We have for  $t \geq 0$*

$$(2.4) \quad \int_0^1 u_t(t) \int_0^t h(t-r)u_{tt}(x,r)dr dx$$

$$= -\frac{1}{2} \int_0^1 h' \square u_t dx + \frac{1}{2} \frac{d}{dt} \int_0^1 \int_0^t h(t-r)u_t^2(x,r)dr dx + \frac{1}{2} h(t) \int_0^1 u_t^2 dx - h(t) \int_0^1 u_t(t)u_t(0)dx,$$

for all  $u_t \in C^1([0, \infty); L^2(0, 1))$  and  $h \in C^1([0, \infty))$ .

**Lemma 2.3.4.** *Let  $u_t \in C^1([0, \infty); L^2(0, 1))$  and  $h \in C^1([0, \infty))$ . Then, we have*

$$(2.5) \quad [(h \square u_t)(t)]^2 \leq \left( \int_0^t |h(t-r)| dr \right) (h \circ u_t)(t).$$

**Proof.** By applying Cauchy-Schwarz's inequality, we get

$$\begin{aligned} [(h \square u_t)(t)]^2 &= \left[ \int_0^t |h(t-r)| |u_t(t) - u_t(r)| dr \right]^2 \\ &= \left[ \int_0^t |h(t-r)|^{\frac{1}{2}} |h(t-r)|^{\frac{1}{2}} |u_t(t) - u_t(r)| dr \right]^2 \\ &\leq \left[ \left( \int_0^t |h(t-r)| dr \right)^{\frac{1}{2}} \left( \int_0^t |h(t-r)| |u_t(t) - u_t(r)|^2 dr \right)^{\frac{1}{2}} \right]^2 \\ &\leq \left( \int_0^t |h(t-r)| dr \right) (h \circ u_t)(t). \end{aligned}$$

■

**Lemma 2.3.5.** *Let  $u_t \in C^1([0, \infty); L^2(0, 1))$  and  $h \in C^1([0, \infty))$ . Then, we have*

$$(2.6) \quad [(u_t * h)(t)]^2 \leq 2 \left( \int_0^t h(t-r) dr \right)^2 u_t^2 + 2 \left( \int_0^t h(t-r) dr \right) (h \circ u_t)(t).$$

**Proof.** By using the fact that  $(f + g)^2 \leq 2f^2 + 2g^2$ , and (2.5), we obtain

$$\begin{aligned} [(u_t * h)(t)]^2 &= \left[ \int_0^t h(t-r) u_t(r) dr \right]^2 \\ &= \left\{ \int_0^t -h(t-r) [-u_t(r) + u_t(t) - u_t(t)] dr \right\}^2 \\ &= \left\{ \int_0^t -h(t-r) [u_t(t) - u_t(r)] dr + \int_0^t h(t-r) u_t(t) dr \right\}^2 \\ &\leq 2 \left[ \int_0^t h(t-r) u_t(r) dr \right]^2 + 2 \left\{ \int_0^t h(t-r) [u_t(t) - u_t(r)] dr \right\}^2 \\ &\leq 2 \left[ \int_0^t h(t-r) u_t(r) dr \right]^2 + 2 \left( \int_0^t h(t-r) dr \right) (h \circ u_t)(t). \end{aligned}$$

■

## 2.4 Integral inequalities

In this part, we review several well-known and extensively utilized integral inequalities for improving the efficiency of dissipative and non-dissipative evolution systems. A. Haraux [34, 35] and V. Komornik [46] proposed the following lemmas for estimating the energy of dissipative issues.

**Lemma 2.4.1.** *Let  $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a decreasing continuous function and  $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  a strictly increasing function of class  $C^1(\mathbb{R}_+)$  such as*

$$G(0) = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} G(t) = +\infty.$$

*We suppose that  $\exists p \geq 0$  and  $\exists q > 0$  such that*

$$\int_s^{+\infty} G'(t)E^{p+1}dt \leq \frac{1}{d}E^p(0)E(s), \quad \forall s > 0.$$

*Then,*

- $E(t) \leq E(0)e^{1-dG(t)}, \quad \forall t > 0, \text{ if } p = 0.$
- $E(t) \leq E(0)\left(\frac{1+p}{1+pdG(t)}\right)^{\frac{1}{p}}, \quad \forall t > 0, \text{ if } p > 0.$

*In the particular case where  $G(t) = t$ , we deduce the following inequalities:*

- $E(t) \leq E(0)e^{1-dt}, \quad \forall t > 0, \text{ if } p = 0.$
- $E(t) \leq E(0)\left(\frac{1+p}{1+pd t}\right)^{\frac{1}{p}}, \quad \forall t > 0, \text{ if } p > 0,$

*called respectively, exponential estimation and polynomial estimation.*

## 2.5 Operators notions

Let  $E$  and  $F$  be two spaces of Banach, and let  $B(E)$  be a space of bounded linear operators from  $E$  to  $E$ .

**Definition 2.5.1. (Unbounded linear operator)**

*A linear operator is a linear application defined on a vector subspace  $\mathbb{D}(\mathbb{A}) \subset E$  which has values in  $F$ , ( $\mathbb{D}(\mathbb{A})$  is called the operator domain of  $\mathbb{A}$ ).*

$$\begin{aligned} \mathbb{A} : \mathbb{D}(\mathbb{A}) \subset E &\rightarrow F, \\ \mathbb{D}(\mathbb{A}) &= \left\{ u \in E, \mathbb{A}u \in F \right\}, \end{aligned}$$

*and  $\forall u, v \in \mathbb{D}(\mathbb{A}), \forall \lambda \text{ in } \mathbb{C}$*

$$\begin{aligned} \mathbb{A}(u+v) &= \mathbb{A}u + \mathbb{A}v, \\ \mathbb{A}(\lambda u) &= \lambda \mathbb{A}u. \end{aligned}$$

**Definition 2.5.2.**  *$\mathbb{A}$  is a bounded linear operator if it exists  $c \geq 0$  such that*

$$\forall u \in \mathbb{D}(\mathbb{A}), \|\mathbb{A}u\|_F \leq c\|u\|_E.$$

*We note that  $\mathbb{A} \in B(E)$ .*

**Definition 2.5.3. (Graph-Kernel-Image )**

The graph of  $\mathbb{A}$  is the vector subspace of  $E \times F$  noted  $G_r(\mathbb{A})$ , defined by

$$G_r(\mathbb{A}) = \{(u, \mathbb{A}u) : u \in \mathbb{D}(\mathbb{A})\}.$$

The kernel of  $\mathbb{A}$  is the subspace of  $E$  noted  $\ker(\mathbb{A})$ , defined by

$$\ker(\mathbb{A}) = \{u \in \mathbb{D}(\mathbb{A}) : \mathbb{A}u = 0\}.$$

The image of  $\mathbb{A}$  is the subspace of  $F$  noted  $Im(\mathbb{A})$ , defined by

$$Im(\mathbb{A}) = \{\mathbb{A}u, u \in \mathbb{D}(\mathbb{A})\}.$$

- If  $\ker(\mathbb{A}) = 0$ , then  $\mathbb{A}$  is injective.
- If  $Im(\mathbb{A}) = F$ , then  $\mathbb{A}$  is surjective.
- If  $\mathbb{A}$  is simultaneously injective and surjective, it is bijective.

**Definition 2.5.4. (Invertible Operator)**

An operator  $\mathbb{A} : \mathbb{D}(\mathbb{A}) \subset E \rightarrow F$  is invertible if it is bijective and has a bounded inverse

$$\mathbb{A}^{-1} : F \rightarrow \mathbb{D}(\mathbb{A}) \subset E.$$

**Definition 2.5.5. (Maximal Monotone Operators)**

An unbounded linear operator  $\mathbb{A} : \mathbb{D}(\mathbb{A}) \subset H \rightarrow H$  is said to be monotone if it satisfies

$$\langle \mathbb{A}v, v \rangle \geq 0, \forall v \in \mathbb{D}(\mathbb{A}).$$

In the case of a complex Hilbert space, the previous condition will be

$$Re \langle \mathbb{A}v, v \rangle \geq 0, \forall v \in \mathbb{D}(\mathbb{A}).$$

It is called maximal monotone if in addition  $Im(I + \mathbb{A}) = H$ , i.e.,  $\forall f \in H, \exists u \in \mathbb{D}(\mathbb{A})$ , such that  $u + \mathbb{A}u = f$ .

**Proposition 2.5.6.** Let  $\mathbb{A}$  be a maximal monotone operator. Then

- $\mathbb{D}(\mathbb{A})$  is dense in  $H$ ,
- $\mathbb{A}$  is a closed operator;
- $\forall \lambda > 0, (I + \lambda\mathbb{A})$  is bijective from  $\mathbb{D}(\mathbb{A})$  onto  $H$ ,  $(I + \lambda\mathbb{A})^{-1}$  is a bounded operator, and  $\|(I + \lambda\mathbb{A})^{-1}\| \leq 1$ .

**Remark 2.5.7.** If  $-\mathbb{A}$  is monotone, we say that  $\mathbb{A}$  is dissipative.

**Definition 2.5.8. (Dissipative operator)**

The operator  $\mathbb{A}$  is called dissipative if

$$\forall x \in \mathbb{D}(\mathbb{A}), \langle \mathbb{A}x, x \rangle \leq 0.$$

In the case of a complex Hilbert space, the previous condition will be

$$\forall x \in \mathbb{D}(\mathbb{A}), \operatorname{Re} \langle \mathbb{A}x, x \rangle \leq 0.$$

**Definition 2.5.9.** An unbounded linear operator  $\mathbb{A}$  in  $E$  is said  $m$ -dissipative if

- $\mathbb{A}$  is dissipative.
- $\forall f \in E, \forall \lambda > 0, \exists x \in \mathbb{D}(\mathbb{A}),$  such that  $\lambda x - \mathbb{A}x = f$ .

**Theorem 2.5.10.** Let  $E$  be a Banach space. If  $\mathbb{A}$  is  $m$ -dissipative, then for all  $\lambda > 0$ , the operator  $\lambda I - \mathbb{A}$  has an inverse, where  $(\lambda I - \mathbb{A})^{-1}f$  belongs to  $\mathbb{D}(\mathbb{A})$ , for all  $f \in E$  and  $(\lambda I - \mathbb{A})^{-1}$  is a bounded linear operator on  $E$  verifying

$$\|(\lambda I - \mathbb{A})^{-1}\| \leq \frac{1}{\lambda}.$$

**2.5.1 The spectrum and the resolvent set of an operator**

Let  $H$  be a space of Hilbert, and  $B(H)$  be a space of bounded linear operators from  $H$  to  $H$ .

**Definition 2.5.11.** Let  $\mathbb{A}$  be an operator defined on  $H$ . The resolvent set of  $\mathbb{A}$ , denoted by  $\rho(\mathbb{A})$ , is the set of  $\lambda \in \mathbb{C}$  such that  $\lambda I - \mathbb{A}$  is invertible.

$$\rho(\mathbb{A}) = \left\{ \lambda \in \mathbb{C} : (\lambda I - \mathbb{A})^{-1} \text{ is invertible in } B(H) \right\}.$$

**Definition 2.5.12.** Let  $\mathbb{A}$  be an operator defined on  $H$ . The spectrum  $\sigma(\mathbb{A})$  of operator  $\mathbb{A}$  is the complement of the resolvent set, i.e.,

$$\sigma(\mathbb{A}) = \mathbb{C} \setminus \rho(\mathbb{A}).$$

**Definition 2.5.13. (Resolvent application)**

The resolvent application of  $\mathbb{A}$  is the one defined as follow:

$$\begin{aligned} R(\cdot, \mathbb{A}) : \rho(\mathbb{A}) &\rightarrow B(H) \\ \lambda &\rightarrow R(\lambda, \mathbb{A}) = (\lambda I - \mathbb{A})^{-1}, \forall \lambda \in \rho(\mathbb{A}). \end{aligned}$$

**Proposition 2.5.14.** For a bounded operator  $\mathbb{A}$  on a Banach space, the spectrum  $\sigma(\mathbb{A})$  is always compact and not empty, hence its spectral radius

$$r(\mathbb{A}) = \sup \left\{ |\lambda| : \forall \lambda \in \sigma(\mathbb{A}) \right\},$$

satisfies  $r(\mathbb{A}) \leq \|\mathbb{A}\|$ .

**Theorem 2.5.15.** If  $\mathbb{A} \in B(H)$  and  $\|\mathbb{A}\| < 1$ , then  $I - \mathbb{A}$  is invertible and

$$(I - \mathbb{A})^{-1} = \sum_{n=1}^{\infty} \mathbb{A}^n.$$

## 2.6 Exponential stability of $C_0$ -semigroup

Let  $L(H)$  be the space of all bounded linear operator over  $H$ .

### 2.6.1 $C_0$ -semigroup

**Definition 2.6.1.** A family  $\{S(t)\}_{t \geq 0}$  of linear continuous operator is called semigroup strongly continuous ( $C_0$ -semigroup) on  $H$  if it satisfies the following properties

- $S(0) = I$ , ( $I$  is the identity operator of  $x$ ).
- $S(t + s) = S(t)S(s)$ ,  $\forall t, s \geq 0$ .
- $\lim_{t \rightarrow 0^+} S(t)x = x$ ,  $\forall x \in H$ .

**Definition 2.6.2.** We call an infinitesimal generator of a  $C_0$ -semigroup  $\{S(t)\}_{t \geq 0}$  an operator  $\mathbb{A}$  defined on the set

$$\mathbb{D}(\mathbb{A}) = \left\{ x \in H, \lim_{t \rightarrow 0^+} \frac{S(t)x - x}{t} \exists \right\},$$

for

$$\mathbb{A}x = \lim_{t \rightarrow 0^+} \frac{S(t)x - x}{t}, \quad \forall x \in \mathbb{D}(\mathbb{A}).$$

Sometimes we note  $\{e^{\mathbb{A}t}\}_{t \geq 0}$  instead of  $\{S(t)\}_{t \geq 0}$ .

**Definition 2.6.3.**  $\{e^{\mathbb{A}t}\}_{t \geq 0}$  is said to be exponentially stable if there exist two positive constants,  $\alpha$  and  $M$ , such that

$$\|e^{\mathbb{A}t}\|_{L(H)} \leq M e^{-\alpha t}, \quad \forall t \geq 0.$$

### 2.6.2 $C_0$ -semigroup generated by a dissipative operator

We suppose that the linear operator  $\mathbb{A}$  generates a  $C_0$ -semigroup  $e^{\mathbb{A}t}$  on a space of Hilbert  $H$ .

**Theorem 2.6.4. (Hille-Yosida)** A linear operator (unbound)  $\mathbb{A} : \mathbb{D}(\mathbb{A}) \subset H \rightarrow H$  is the infinitesimal generator of a  $C_0$ -semigroup of contraction  $\{S(t)\}_{t \geq 0}$ , if and only if:

- $\mathbb{A}$  is closed and  $\overline{\mathbb{D}(\mathbb{A})} = H$ .
- The resolvent set  $\rho(\mathbb{A})$  of  $\mathbb{A}$  contains  $\mathbb{R}^+$ , and

$$\forall \lambda > 0, \quad \|(\lambda I - \mathbb{A})^{-1}\|_{L(H)} \leq \frac{1}{\lambda}.$$

**Theorem 2.6.5. (Lumer-Phillips)** An unbounded linear operator  $\mathbb{A}$  in  $H$  is the infinitesimal generator of a  $C_0$ -semigroup of contractions on  $H$ , if and only if  $\mathbb{A}$  is  $m$ -dissipative and dense domain in  $H$ .

**Theorem 2.6.6.** *Let  $\mathbb{A}$  be a linear operator with dense domain  $\mathbb{D}(\mathbb{A})$  in a Hilbert space  $H$ . If  $\mathbb{A}$  is dissipative and  $0 \in \rho(\mathbb{A})$ , the resolvent set of  $\mathbb{A}$ , then  $\mathbb{A}$  is the infinitesimal generator of a  $C_0$ -semigroup of contractions on  $H$ .*

**Theorem 2.6.7.** *Let  $\mathbb{A} : \mathbb{D}(\mathbb{A}) \subset H \rightarrow H$  is the infinitesimal generator of a  $C_0$ -semigroup of contraction  $\{S(t), t \geq 0\}$ . Then, for each  $\xi \in \mathbb{D}(\mathbb{A})$  and each  $t \geq 0$ , we have  $S(t)\xi \in \mathbb{D}(\mathbb{A})$  and the mapping*

$$(2.7) \quad t \rightarrow S(t)\xi$$

*is of class  $C^1$  on  $[0, +\infty)$  and satisfies*

$$(2.8) \quad \frac{d}{dt}(S(t)\xi) = \mathbb{A}S(t)\xi = S(t)\mathbb{A}\xi.$$

**Theorem 2.6.8. (Existence and uniqueness: Hille-Yosida)** *Let  $\mathbb{A}$  be a maximum operator monotone ( $m$ -dissipative) in a space of Hilbert  $H$ . Then for all  $u_0 \in \mathbb{D}(\mathbb{A})$  it exists a unique function*

$$u \in C^1([0, +\infty[; H) \cap C([0, +\infty[; \mathbb{D}(\mathbb{A})),$$

*such that*

$$\begin{cases} \frac{du}{dt} - \mathbb{A}u = 0, \text{ on } \Omega \times [0, +\infty[, \\ u(x, 0) = u_0(x). \end{cases}$$

*In addition, we have*

$$\|u(t)\| \leq \|u_0\|, \text{ and } \|\mathbb{A}u(t)\| \leq \|\mathbb{A}u_0\|, \forall t \geq 0.$$

### 2.6.3 Exponential stability

**Theorem 2.6.9.** *Let  $S(t) = e^{\mathbb{A}t}$  a  $C_0$ -semigroup on a Hilbert space. Then  $S(t)$  is exponentially stable if and only if*

$$\sup \{ \operatorname{Re}(\lambda), \lambda \in \rho(\mathbb{A}) \} < 0,$$

*and*

$$\sup \|(\lambda I - \mathbb{A})^{-1}\|_{L(H)} < \infty.$$

**Theorem 2.6.10. (Gearhart-Pruss)** *Let  $\{e^{\mathbb{A}t}\}_{t \geq 0}$  be a  $C_0$ -semigroup of contraction generated by operator  $\mathbb{A}$  in a Hilbert space  $H$ . Then the semigroup is exponentially stable if and only if*

$$i\mathbb{R} = \{i\lambda, \lambda \in \mathbb{R}\} \subset \rho(\mathbb{A}),$$

*and*

$$\overline{\lim}_{|\lambda| \rightarrow \infty} \|(i\lambda I - \mathbb{A})^{-1}\|_{L(H)} < \infty, \lambda \in \mathbb{R}.$$

### 2.6.4 Lack of exponential decay

According to Gerhart-Prüss-Huang theorem, to prove that a solution  $U$  of a linear system

$$\begin{cases} U_t(t) = \mathbb{A}U(t), & t \geq 0, \\ U(0) = U_0, \end{cases}$$

is not exponentially stable, it suffices to prove that there exists a sequence  $(F_n)_{n \in \mathbb{N}} \subset H$  with bounded norm  $\|F_n\| < 1$ , such that

$$\overline{\lim}_{|\lambda| \rightarrow \infty} \|(i\lambda I - \mathbb{A})^{-1} F_n\|_H = \overline{\lim}_{|\lambda| \rightarrow \infty} \|U_n\|_H = \infty.$$

## WELL-POSEDNESS AND STABILITY RESULT FOR A SWELLING POROUS ELASTIC SYSTEM WITH NEUTRAL DELAY AND POROUS DAMPING

### 3.1 Introduction

In recent decades, the theory of mixtures of solids has received a lot of attention by researchers and an increasing interest has been oriented to the study of the qualitative properties of solutions related to mixtures composed by two interacting continua, see [6]-[9]-[13]. Note that one of the first works in continuity theory applied to mixtures were the contributions of [72]-[31]. Eringen in [24] developed the first mathematical model consisting of three partial differential equations that give form to the problem of saturation of porous solids by the action of a gas or fluids. This mathematical model represents, in fact, the theory of mixtures for the saturation of porous solids by the action of a gas or fluid. Then, several mathematical results on the existence, uniqueness and asymptotic behaviour for this theory have been developed by many researchers see [58]-[5]. Alves et al in [2], considered the one-dimensional system composed of a mixture of two thermoelastic solids. By using the semigroup method, they established a necessary and sufficient condition over the coefficients of the system to get the exponential stability of the corresponding semigroup. As established by Ieşan [38], towards the end of the 19th century and simplified by Quintanilla in [55], the basic field equations for the theory of swelling of one-dimensional porous elastic soils are given by

$$(3.1) \quad \begin{aligned} \rho_z z_{tt} &= P_{1x} - G_1 + H_1 \\ \rho_u u_{tt} &= P_{2x} + G_2 + H_2, \end{aligned}$$

here  $P_i$  denote the partial tensions,  $H_i$  are the external forces,  $G_i$  are internal body forces associated with the dependent variables  $z$  and  $u$ , respectively.  $z$  is the displacement of the fluid,

$u$  is the elastic solid material and the coefficients  $\rho_z$  and  $\rho_u$  are the densities of each constituent.

We assume that the constitutive equations of partial tensions are given by

$$(3.2) \quad \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}}_{=A} \begin{pmatrix} z_x \\ u_x \end{pmatrix},$$

where  $a_1, a_3 > 0$  and  $a_2 \neq 0$  is a real number. The matrix  $A$  is positive definite such that  $a_1 a_3 > a_2^2$ . Ramos et al. [60] studied the case  $H_1 = 0$  and  $H_2 = -\gamma(t)g(u_t)$  where  $\gamma(t)g(u_t)$  is a nonlinear damping term, which acts only in the second equation. Using the multiplier method and some properties of convex functions, they established an exponential decay rate provided that the wave speeds of the system are equal. Similarly, in [74] Wang and Guo considered (3.1) with initial and some mixed boundary conditions and took

$$(3.3) \quad G_1 = G_2 = 0, \quad H_1 = -\rho_u \gamma(x) u_t, \quad H_2 = 0,$$

where  $\gamma(x)$  is an internal viscous damping function with positive mean. Using the Riesz basis approach, they proved that the whole system can be exponentially stabilized by a single internal viscous damping. For more interesting results on swelling porous elastic soils, we refer the reader to [57]-[19].

Now, on the other hand, the scientific community is observing a considerable growth interest in problems involving time delays, because most phenomena naturally depend not only on the current state but also on some past events see [59]-[1]. Tatar in [71], considered the following damped wave equation with neutral delay

$$u_{tt} = u_{xx} - u_t - \int_0^t h(t-s) u_{tt}(s) ds, \quad x \in (0, 1), \quad t > 0,$$

with initial and boundary conditions

$$\begin{cases} u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (0, 1), \\ u(0, t) = u(1, t) = 0, \quad t \geq 0, \end{cases}$$

he demonstrated that the solution decays exponentially under certain conditions on the kernel  $h$ . In many cases, it has been shown that delay is a source of instability unless additional conditions or control terms are added, as in the work of Kerbal and Tatar [44], where they investigated the following neutrally delayed viscoelastic Timoshenko beam system

$$\begin{cases} \varphi_{tt} = (\varphi_x + \psi)_x, \\ \left( \psi_t + \int_0^t k(t-s) \psi_t(s) ds \right)_t = \psi_{xx} - \int_0^t g(t-s) \psi_{xx}(s) ds - (\varphi_x + \psi), \end{cases}$$

for  $x \in (0, 1), t > 0$  with initial and boundary conditions

$$\begin{cases} \varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \quad x \in (0, 1), \\ \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), \quad x \in (0, 1). \\ \varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t) = 0, \quad t \geq 0, \end{cases}$$

they obtained an exponential stability result.

Our present work focuses on the study of system (3.1) with internal and external body forces, which act only on the elastic solid, present the porous dissipation and the neutral delay term respectively.

$$(3.4) \quad G_1 = 0, \quad G_2 = -\beta u_t, \quad H_1 = 0, \quad H_2 = - \left[ \int_0^t h(t-r)u_t(r)dr \right]_t,$$

where  $\beta$  is a positive constant and  $h$  is a given kernel.

Thus, when we substitute (3.4) into (3.1), our system becomes

$$(3.5) \quad \begin{aligned} \rho_z z_{tt} &= a_1 z_{xx} + a_2 u_{xx}, & \text{in } \Omega \\ \rho_u \left[ u_t + \int_0^t h(t-r)u_t(x,r)dr \right]_t &= a_3 u_{xx} + a_2 z_{xx} - \beta u_t, & \text{in } \Omega \end{aligned}$$

with initial conditions

$$(3.6) \quad \begin{aligned} z(x, 0) &= z_0(x), \quad z_t(x, 0) = z_1(x), \\ u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in [0, 1] \end{aligned}$$

and boundary conditions given by

$$(3.7) \quad z(0, t) = z(1, t) = u(0, t) = u(1, t) = 0, \quad t \geq 0,$$

where  $\Omega = (0, 1) \times (0, \infty)$ .

**(H)** The kernel  $h$  is a nonnegative continuously differentiable and summable function satisfying

$$(3.8) \quad h'(t) \leq -\eta h(t), \quad \int_0^{+\infty} e^{\tau r} |h(t)| dr < \infty, \quad t \geq 0,$$

for some positive constants  $\eta$  and  $\tau$ .

The associated energy  $E(t)$  is a nonnegative functional defined as

$$E(t) = \frac{1}{2} \int_0^1 (\rho_z z_t^2 + a_1 z_x^2 + 2a_2 z_x u_x + \rho_u u_t^2 + a_3 u_x^2) dx + \frac{\rho_u}{2} \int_0^1 \left( \int_0^t h(t-r)u_t^2(x,r)dr \right) dx.$$

Observing that

$$a_3 u_x^2 + 2a_2 u_x z_x + a_1 z_x^2 = \frac{1}{2} \left[ a_3 \left( u_x + \frac{a_2}{a_3} z_x \right)^2 + a_1 \left( z_x + \frac{a_2}{a_1} u_x \right)^2 + \left( a_3 - \frac{a_2^2}{a_1} \right) u_x^2 + \left( a_1 - \frac{a_2^2}{a_3} \right) z_x^2 \right],$$

with the use of assumption  $a_3 a_1 > a_2^2$ , we get

$$a_3 u_x^2 + 2a_2 u_x z_x + a_1 z_x^2 > \frac{1}{2} \left[ \left( a_3 - \frac{a_2^2}{a_1} \right) u_x^2 + \left( a_1 - \frac{a_2^2}{a_3} \right) z_x^2 \right].$$

Now, we conclude that

$$E(t) > \frac{1}{2} \int_0^1 [\rho_z z_t^2 + a'_1 z_x^2 + \rho_u u_t^2 + a'_3 u_x^2 + \rho_u \int_0^t h(t-r)u_t^2(x,r)dr] dx,$$

where  $2a'_1 = a_1 - \frac{a_2^2}{a_3}$  and  $2a'_3 = a_3 - \frac{a_2^2}{a_1}$ .

## 3.2 Well posedness result

In this part, we will use Faedo-Galerkin technique to demonstrate the global existence and uniqueness of system's solutions (3.5)-(3.7), where we shall utilize the regular Lebesgue space  $L^2(0, 1)$  and the Sobolev space  $H_0^1(0, 1)$  with their typical scalar products and norms . The space  $\mathbb{H}$  is defined as

$$(3.9) \quad \mathbb{H} = [H_0^1(0, 1) \times L^2(0, 1)]^2.$$

The following theorem is used as the first outcome.

**Theorem 3.2.1.** *For all  $(z_0, z_1, u_0, u_1) \in \mathbb{H}$ ,  $(\mathbf{H})$  is verified, and  $T > 0$ , there existes a unique weak solution of problem (3.5)-(3.7) on  $(0, T)$ , such that*

$$\begin{aligned} (z, u) &\in C([0, T], H_0^1(0, 1)) \cap C^1([0, T], L^2(0, 1)), \\ (z_t, u_t) &\in L^2([0, T], (0, 1)) \cap L^2([0, T], H_0^1(0, 1)). \end{aligned}$$

**Proof. Existence :** The main tool of our proof is the use of Faedo-Galerkin's method, which base on the construction of approximations of the solutions, then we obtain an energy estimates proving that  $t_n = T$  for  $n \in \mathbb{N}$ . Finally, we pass to the limit of the approximations, for more details see [23]-[26].

**Step 1 : Faedo-Galerkin approximations.** For every  $n \geq 1$ , let  $V_n = Span\{w_1, \dots, w_n\}$ ,  $1 \leq i \leq n$ , be a Hilbert basis of the space  $H_0^1(0, 1)$  and  $L^2(0, 1)$ . As Hilbert space is a separable space, we can choose  $z_0^n, z_1^n, u_0^n$  and  $u_1^n \in [w_1, \dots, w_n]$  such that

$$(3.10) \quad \begin{aligned} z_0^n &= \sum_{k=1}^n \alpha_k^n w_k \rightarrow z_0 \text{ in } H_0^1(0, 1), \\ z_1^n &= \sum_{k=1}^n \beta_k^n w_k \rightarrow z_1 \text{ in } H_0^1(0, 1), \\ u_0^n &= \sum_{k=1}^n \bar{\alpha}_k^n w_k \rightarrow u_0 \text{ in } H_0^1(0, 1), \\ u_1^n &= \sum_{k=1}^n \bar{\beta}_k^n w_k \rightarrow u_1 \text{ in } H_0^1(0, 1). \end{aligned}$$

Now, we search for solution having the following form

$$\begin{aligned} z^n &= \sum_{k=1}^n g_k^n(t) w_k(x), \\ u^n &= \sum_{k=1}^n \bar{g}_k^n(t) w_k(x), \end{aligned}$$

of the following approximate system, for  $k = 1, \dots, n$

$$(3.11) \quad \begin{cases} \rho_z \int_0^1 z_{tt}^n w_k dx + a_1 \int_0^1 z_x^n w_{kx} dx + a_2 \int_0^1 u_x^n w_{kx} dx = 0, \\ \left[ \rho_u \int_0^1 \left[ u_t^n + \int_0^t h(t-r) u_t^n(x, r) dr \right] w_k dx + a_3 \int_0^1 u_x^n w_{kx} dx + a_2 \int_0^1 z_x^n w_{kx} dx + \beta \int_0^1 u_t^n w_k dx = 0, \right. \end{cases}$$

with initial data

$$(3.12) \quad z^n(x, 0) = z_0^n(x), z_t^n(x, 0) = z_1^n(x), u^n(x, 0) = u_0^n(x), u_t^n(x, 0) = u_1^n(x).$$

By using the Caratheodory theorem for an ordinary differential equation, we derive that the aforementioned Cauchy problem (3.11) – (3.12) has a unique global solution  $(g_k^n(t), \bar{g}_k^n(t))_{k=1, \dots, n}$  defined on  $[0, t_n]$ .

**Step 2 : Energy estimates.**

The main purpose of this step is to prove that  $t_n = T$ , we obtain this result by multiplying in  $L^2$  the first and the second equation of system (3.11) by  $((g_k^n(t))', (\bar{g}_k^n(t))')$  respectively, and by using integration by parts, boundary-initial conditions and (2.4), we find for all  $t > 0$

$$(3.13) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \left[ \rho_z (z_t^n)^2 + a_1 (z_x^n)^2 + 2a_2 u_x^n z_x^n + \rho_u (u_t^n)^2 + a_3 (u_x^n)^2 + \rho_u \int_0^t h(t-r) (u_t^n(r))^2 dr \right] dx \\ & = - \left( \frac{\rho_u}{2} h(t) + \beta \right) \int_0^1 (u_t^n)^2 dx + \frac{\rho_u}{2} \int_0^1 h' \square u_t^n dx, \end{aligned}$$

for every  $n \geq 1$ . From the hypotheses on the function  $h$ , and by integrating (3.13) over  $(0, t)$ , we obtain

$$(3.14) \quad \begin{aligned} & \frac{1}{2} \int_0^1 \left[ \rho_z (z_t^n)^2 + a_1 (z_x^n)^2 + 2a_2 u_x^n z_x^n + \rho_u (u_t^n)^2 + a_3 (u_x^n)^2 + \rho_u \int_0^t h(t-r) (u_t^n(r))^2 dr \right] dx \\ & \leq \frac{1}{2} \int_0^1 \left[ \rho_z (z_1^n)^2 + a_1 (z_0^n)^2 + 2a_2 u_0^n z_0^n + \rho_u (u_1^n)^2 + a_3 (u_0^n)^2 + \rho_u \int_0^t h(t-r) (u_1^n(r))^2 dr \right] dx. \end{aligned}$$

(3.14) may be noted as

$$(3.15) \quad E^n(t) \leq E^n(0).$$

Now, since the energy is nonnegative, from (3.10) and the hypotheses on the function  $h$ , we can write (3.15) as

$$(3.16) \quad E^n(t) \leq E(0) \leq C,$$

where  $C$  is a positive constant independent of  $n$  and  $t$ . From (3.16), we deduce that

$$(3.17) \quad \begin{aligned} & (z^n, u^n) \text{ are bounded in } L^\infty([0, T]; H_0^1(0, 1)), \\ & (z_t^n, u_t^n) \text{ are bounded in } L^\infty([0, T]; H_0^1(0, 1)). \end{aligned}$$

**Step 3 : The limit process.**

Now, by using Aubin-Lions theorem [47] and up to the subsequence, we can observe from (3.17) that there exists a subsequences  $(z^m, u^m)$  of  $(z^n, u^n)$ , such that

$$(3.18) \quad \begin{aligned} & (z^m, u^m) \rightarrow (z, u) \text{ weak } * \text{ in } L^\infty([0, T]; H_0^1(0, 1)), \\ & (z_t^m, u_t^m) \rightarrow (z_t, u_t) \text{ weak } * \text{ in } L^\infty([0, T]; H_0^1(0, 1)), \end{aligned}$$

and by using the fact that

$$L^\infty([0, T]; H_0^1(0, 1)) \hookrightarrow L^2([0, T]; H_0^1(0, 1)).$$

We get

$$(3.19) \quad \begin{aligned} (z^n, u^n) & \text{ are bounded in } L^2([0, T]; H_0^1(0, 1)), \\ (z_t^n, u_t^n) & \text{ are bounded in } L^2([0, T]; H_0^1(0, 1)). \end{aligned}$$

Therefore,

$$(3.20) \quad (z^n, u^n) \text{ are bounded in } H^1([0, T]; H^1(0, 1)).$$

Since the embedding  $H^1([0, T]; H^1(0, 1)) \hookrightarrow L^2([0, T]; L^2(0, 1))$  is compact, then the subsequences

$$(3.21) \quad \begin{aligned} (z^m, u^m) & \rightarrow (z, u) \text{ strongly in } L^2([0, T]; L^2(0, 1)), \\ (z_t^m, u_t^m) & \rightarrow (z_t, u_t) \text{ strongly in } L^2([0, T]; L^2(0, 1)). \end{aligned}$$

**Uniqueness :** Let  $(z^1, u^1)$  and  $(z^2, u^2)$  two solutions of (3.5)-(3.7), and let  $\tilde{z} = z^1 - z^2$  and  $\tilde{u} = u^1 - u^2$  satisfy

$$(3.22) \quad \begin{cases} \rho_z \tilde{z}_{tt} - a_1 \tilde{z}_{xx} - a_2 \tilde{u}_{xx} = 0, \\ \rho_u \left[ \tilde{u}_t + \int_0^t h(t-r) \tilde{u}_t(x, r) dr \right]_t - a_3 \tilde{u}_{xx} - a_2 \tilde{z}_{xx} + \beta \tilde{u}_t = 0. \end{cases}$$

Multiplying (3.22) by  $\tilde{z}_t, \tilde{u}_t$  respectively, then integrating over  $(0, 1)$  and by using (2.4), we get

$$(3.23) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \left[ \rho_z \tilde{z}_t^2 + a_1 \tilde{z}_x^2 + 2a_2 \tilde{u}_x \tilde{z}_x + \rho_u \tilde{u}_t^2 + a_3 \tilde{u}_x^2 + \rho_u \int_0^t h(t-r) \tilde{u}_t^2(r) dr \right] dx \\ & = - \left( \frac{\rho_u}{2} h(t) + \beta \right) \int_0^1 \tilde{u}_t^2 dx + \frac{\rho_u}{2} \int_0^1 h' \square \tilde{u}_t dx. \end{aligned}$$

Now, we integrate (3.23) over  $(0, t)$ , we obtain :

$$(3.24) \quad \begin{aligned} & \frac{1}{2} \int_0^1 \left[ \rho_z \tilde{z}_t^2 + a_1 \tilde{z}_x^2 + 2a_2 \tilde{u}_x \tilde{z}_x + \rho_u \tilde{u}_t^2 + a_3 \tilde{u}_x^2 + \rho_u \int_0^t h(t-r) \tilde{u}_t^2(r) dr \right] dx \\ & = - \left( \frac{\rho_u}{2} \int_0^t h(r) dr + \beta t \right) \int_0^1 \tilde{u}_t^2 dx ds - \frac{\rho_u \eta}{2} \int_0^t \int_0^1 h \square \tilde{u}_t dx ds \leq 0. \end{aligned}$$

From estimate (3.24), we deduce that  $(\tilde{z}, \tilde{u}) = (0, 0)$ , which implies that problem (3.5)-(3.7) has a unique solution.

**Continuous dependence :** Multiplying the first and the second equation of system (3.5) by  $z_t, u_t$  respectively, we obtain

$$(3.25) \quad \frac{1}{2} \frac{d}{dt} \int_0^1 \left[ \rho_z z_t^2 + a_1 z_x^2 + 2a_2 u_x z_x + \rho_u u_t^2 + a_3 u_x^2 + \rho_u \int_0^t h(t-r) u_t^2(r) dr \right] \leq 0,$$

integrating (3.25) over  $(0, t)$ , and from the positivity of the energy, we can get the following estimate:

$$(3.26) \quad E(t) \leq E(0) + \frac{1}{2} \int_0^t \left[ \int_0^1 \left( \rho_z z_t^2 + a_1 z_x^2 + \underbrace{2a_2 u_x z_x}_{f_0} + \rho_u u_t^2 + \rho_u \int_0^t h(t-r) u_t^2(r) dr + a_3 u_x^2 \right) dx \right] d\tau.$$

Now, applying Young's inequality to  $f_0$  with the fact that  $a_1 a_3 > a_2^2$ , gives

$$(3.27) \quad \begin{aligned} E(t) &\leq E(0) + \frac{1}{2} \int_0^t \left[ \int_0^1 \left( \rho_z z_t^2 + 2a_1 z_x^2 + \rho_u u_t^2 + 2a_3 u_x^2 + \rho_u \int_0^t h(t-r) u_t^2(r) dr \right) dx \right] d\tau \\ &\leq E(0) + \rho_1 \int_0^t \left[ \int_0^1 \left( z_t^2 + z_x^2 + u_t^2 + u_x^2 + \int_0^t h(t-r) u_t^2(r) dr \right) dx \right] d\tau, \end{aligned}$$

where  $2\rho_1 = \max(\rho_z, 2a_1, \rho_u, 2a_3)$ . On the other side, we know that

$$(3.28) \quad \begin{aligned} E(t) &> \frac{1}{2} \int_0^1 \left[ \rho_z z_t^2 + a'_1 z_x^2 + \rho_u u_t^2 + a'_3 u_x^2 + \rho_u \int_0^t h(t-r) u_t^2(x, r) dr \right] dx, \\ &> \rho_2 \int_0^1 \left[ z_t^2 + z_x^2 + u_t^2 + \int_0^t h(t-r) u_t^2(x, r) dr + u_x^2 \right] dx, \end{aligned}$$

where  $2\rho_2 = \max(\rho_z, a'_1, \rho_u, a'_3)$ ,  $2a'_1 = a_1 - \frac{a_2^2}{a_3}$  and  $2a'_3 = a_3 - \frac{a_2^2}{a_1}$ . From (3.27) – (3.28), we get

$$(3.29) \quad \begin{aligned} &\rho_2 \int_0^1 \left[ z_t^2 + z_x^2 + u_t^2 + \int_0^t h(t-r) u_t^2(x, r) dr + u_x^2 \right] dx \\ &< E(0) + \rho_1 \int_0^t \left[ \int_0^1 \left( z_t^2 + z_x^2 + u_t^2 + u_x^2 + \int_0^t h(t-r) u_t^2(r) dr \right) dx \right] d\tau. \end{aligned}$$

Applying Gronwall's inequality on (3.29), we obtain

$$(3.30) \quad \int_0^1 \left[ z_t^2 + z_x^2 + u_t^2 + \int_0^t h(t-r) u_t^2(x, r) dr + u_x^2 \right] dx \leq E(0) e^{\rho_3 t},$$

where  $\rho_3$  is a positive constant. From (3.30), we deduce that the solution of problem (3.5)-(3.7) continuously depends on initial data.  $\blacksquare$

### 3.3 Stability Result

In this section, we set out and prove several fundamental lemmas. Then, we end with the confirmation of our stability result.

**Lemma 3.3.1.** *The energy functional  $E$ , defined by*

$$(3.31) \quad E(t) = \frac{1}{2} \int_0^1 \left( \rho_z z_t^2 + a_1 z_x^2 + 2a_2 z_x u_x + \rho_u u_t^2 + a_3 u_x^2 \right) dx + \frac{\rho_u}{2} \int_0^1 \left( \int_0^t h(t-r) u_t^2(x, r) dr \right) dx, \quad t > 0,$$

satisfies

$$(3.32) \quad E'(t) \leq -\beta \int_0^1 u_t^2 dx + \frac{\rho_u}{2} \int_0^1 h' \square u_t dx, \quad t > 0.$$

**Proof.** Multiplying the first equation of (3.5) by  $z_t$ , and integrating by parts over  $(0, 1)$ , we obtain

$$(3.33) \quad \frac{\rho_z}{2} \frac{d}{dt} \int_0^1 z_t^2 dx + \frac{a_1}{2} \frac{d}{dt} \int_0^1 z_x^2 dx + a_2 \int_0^1 u_x z_{tx} dx = 0.$$

Multiplying the second equation of (3.5) by  $u_t$ , and integrating by parts over  $(0, 1)$ , we obtain

$$(3.34) \quad \frac{\rho_u}{2} \frac{d}{dt} \int_0^1 u_t^2 dx + \rho_u \int_0^1 \left[ \int_0^t h(t-r) u_t(r) dr \right]_t u_t dx + \frac{a_3}{2} \frac{d}{dt} \int_0^1 u_x^2 dx + a_2 \int_0^1 z_x u_{tx} dx + \beta \int_0^1 u_t^2 dx = 0.$$

We have

$$(3.35) \quad \left[ \int_0^t h(t-r) u_t(r) dr \right]_t = \int_0^t h(t-r) u_{tt}(r) dr + h(t) u_t(0).$$

By using (2.4) and (3.35), we get

$$(3.36) \quad \begin{aligned} & \frac{\rho_u}{2} \frac{d}{dt} \int_0^1 u_t^2 dx - \frac{\rho_u}{2} \int_0^1 h' \square u_t dx + \frac{\rho_u}{2} h(t) \int_0^1 u_t^2 dx + \frac{\rho_u}{2} \frac{d}{dt} \int_0^1 \int_0^t h(t-r) u_t^2(r) dr dx + \frac{a_3}{2} \frac{d}{dt} \int_0^1 u_x^2 dx \\ & + a_2 \int_0^1 z_x u_{tx} dx + \beta \int_0^1 u_t^2 dx = 0. \end{aligned}$$

Now adding (3.33) to (3.36) and by using the fact that the kernel  $h$  is a nonnegative function, we get, for any  $t > 0$ ,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 [\rho_z z_t^2 + a_1 z_x^2 + 2a_2 z_x u_x + \rho_u u_t^2 + a_3 u_x^2 + \rho_u \int_0^t h(t-r) u_t^2(x, r) dr] dx \\ & \leq -\beta \int_0^1 u_t^2 dx + \frac{\rho_u}{2} \int_0^1 h' \square u_t dx. \end{aligned}$$

This completes the proof of the lemma. ■

**Lemma 3.3.2.** *The functional*

$$(3.37) \quad I_1(t) = \rho_u \int_0^1 \left( u_t + \int_0^t h(t-r) u_t(r) dr \right) u dx + \frac{\beta}{2} \int_0^1 u^2 dx - \frac{a_2}{a_1} \rho_z \int_0^1 z_t u dx, \quad t > 0,$$

satisfies, for any  $\varepsilon_1 > 0$ ,

$$(3.38) \quad I_1'(t) \leq - \left( a_3 - \frac{a_2^2}{a_1} \right) \int_0^1 u_x^2 dx + \varepsilon_1 \int_0^1 z_t^2 dx + C_1 \int_0^1 u_t^2 dx + \frac{\rho_u}{2} \int_0^t h(r) dr \int_0^1 h \circ u_t dx, \quad t > 0,$$

where  $C_1 = \rho_u \left[ \frac{a_2^2 \rho_z^2}{4\rho_u \varepsilon_1 a_1^2} + \int_0^t h(r) dr + \frac{3}{2} \right]$ .

**Proof.** A simple differentiation of (3.37) and by using the first and the second equation of (3.5), we find

$$(3.39) \quad I_1'(t) = - \left( a_3 - \frac{a_2^2}{a_1} \right) \int_0^1 u_x^2 dx + \rho_u \int_0^1 u_t^2 dx - \underbrace{\frac{a_2}{a_1} \rho_z \int_0^1 z_t u_t dx}_{f_1} + \underbrace{\rho_u \int_0^1 u_t \int_0^t h(t-r) u_t(r) dr dx}_{f_2}.$$

Applying Young's inequality to  $f_1$  and  $f_2$  with the help of (2.6), gives

$$(3.40) \quad f_1 \leq \varepsilon_1 \int_0^1 z_t^2 dx + \frac{a_2^2 \rho_z^2}{4\varepsilon_1 a_1^2} \int_0^1 u_t^2 dx.$$

$$(3.41) \quad \begin{aligned} f_2 &= \rho_u \int_0^t h(r) dr \int_0^1 u_t^2 dx - \rho_u \int_0^1 u_t \int_0^t h(t-r)(u_t(t) - u_t(r)) dr dx \\ &\leq \rho_u \left( \int_0^t h(r) dr + \frac{\rho_u}{2} \right) \int_0^1 u_t^2 dx + \frac{1}{2} \int_0^t h(r) dr \int_0^1 h \circ u_t dx. \end{aligned}$$

Substituting (3.40) and (3.41) into (3.39), we find (3.38).  $\blacksquare$

**Lemma 3.3.3.** *The functional*

$$(3.42) \quad I_2(t) = a_2 \left[ \int_0^1 \left( u_t + \int_0^t h(t-r)u_t(r) dr \right) z dx - \int_0^1 z_t u dx + \frac{\beta}{\rho_u} \int_0^1 z u dx \right], \quad t > 0,$$

satisfies, for any  $\varepsilon_3 > 0$ ,

$$(3.43) \quad \begin{aligned} I_2'(t) &\leq -\frac{a_2^2}{2\rho_u} \int_0^1 z_x^2 dx + a_2^2 \varepsilon_3 \int_0^1 z_t^2 dx + \frac{1}{\varepsilon_3} \left( \int_0^t h(r) dr \right)^2 \int_0^1 u_t^2 dx + \frac{1}{\varepsilon_3} \left( \int_0^t h(r) dr \right) \int_0^1 h \circ u_t dx \\ &+ C_2(\varepsilon_3) \int_0^1 u_x^2 dx, \quad t > 0, \end{aligned}$$

where  $C_2(\varepsilon_3) = \frac{a_2^2}{\rho_z} + \frac{\rho_u}{2} \left( \frac{a_1}{\rho_z} - \frac{a_3}{\rho_u} \right)^2 + \frac{\beta^2}{2\varepsilon_3 \rho_u^2}$ .

**Proof.** A simple differentiation of (3.42) and using the first and the second equation of (3.5), we obtain

$$(3.44) \quad \begin{aligned} I_2'(t) &= -\frac{a_2^2}{\rho_u} \int_0^1 z_x^2 dx + a_2 \underbrace{\left( \frac{a_1}{\rho_z} - \frac{a_3}{\rho_u} \right) \int_0^1 z_x u_x dx}_{f_3} + \frac{a_2^2}{\rho_z} \int_0^1 u_x^2 dx + a_2 \underbrace{\int_0^1 z_t \int_0^t h(t-r)u_t(r) dr dx}_{f_4} \\ &+ \underbrace{\frac{a_2 \beta}{\rho_u} \int_0^1 z_t u dx}_{f_5}. \end{aligned}$$

Now, by using Poincaré's and Young's inequalities, we get

$$(3.45) \quad f_3 \leq \frac{a_2^2 \varepsilon_2}{2} \int_0^1 z_x^2 dx + \frac{1}{2\varepsilon_2} \left( \frac{a_1}{\rho_z} - \frac{a_3}{\rho_u} \right)^2 \int_0^1 u_x^2 dx,$$

$$(3.46) \quad \begin{aligned} f_4 &\leq \frac{a_2^2 \varepsilon_3}{2} \int_0^1 z_t^2 dx + \frac{1}{2\varepsilon_3} \int_0^1 \left( \int_0^t h(t-r)u_t(r) dr \right)^2 dx \\ &\leq \frac{a_2^2 \varepsilon_3}{2} \int_0^1 z_t^2 dx + \frac{1}{2\varepsilon_3} \int_0^1 \left( - \int_0^t h(t-r)[u_t(t) - u_t(r) - u_t(t)] dr \right)^2 dx \\ &\leq \frac{a_2^2 \varepsilon_3}{2} \int_0^1 z_t^2 dx + \frac{1}{\varepsilon_3} \int_0^1 \left( \int_0^t h(t-r)(u_t(t) - u_t(r)) dr \right)^2 dx + \frac{1}{\varepsilon_3} \left( \int_0^t h(r) dr \right)^2 \int_0^1 u_t^2 dx \\ &\leq \frac{a_2^2 \varepsilon_3}{2} \int_0^1 z_t^2 dx + \frac{1}{\varepsilon_3} \left( \int_0^t h(r) dr \right) \int_0^1 h \circ u_t dx + \frac{1}{\varepsilon_3} \left( \int_0^t h(r) dr \right)^2 \int_0^1 u_t^2 dx, \end{aligned}$$

$$(3.47) \quad f_5 \leq \frac{\varepsilon_3 a_2^2}{2} \int_0^1 z_t^2 dx + \frac{\beta^2}{2\varepsilon_3 \rho_u^2} \int_0^1 u_x^2 dx.$$

We end up with (3.43), by substituting (3.45), (3.46) and (3.47) into (3.44) and taking  $\varepsilon_2 = \frac{1}{\rho_u}$ . ■

**Lemma 3.3.4.** *The functional*

$$(3.48) \quad I_3(t) = -\rho_z \int_0^1 z_t z dx, \quad t > 0,$$

satisfies

$$(3.49) \quad I_3'(t) \leq -\rho_z \int_0^1 z_t^2 dx + 2a_1 \int_0^1 z_x^2 dx + \frac{a_3}{4} \int_0^1 u_x^2 dx, \quad t > 0.$$

**Proof.** A simple differentiation of (3.48) and using the second equation of (3.5), we find

$$(3.50) \quad I_3'(t) = -\rho_z \int_0^1 z_t^2 dx + a_1 \int_0^1 z_x^2 dx + \underbrace{a_2 \int_0^1 u_x z_x dx}_{f_5}.$$

Since  $a_1 a_3 > a_2^2$  and thanks to Young's inequality, we find

$$(3.51) \quad f_5 \leq a_1 \int_0^1 z_x^2 dx + \frac{a_3}{4} \int_0^1 u_x^2 dx.$$

Substituting (3.51) into (3.50), we end up with (3.49). ■

**Lemma 3.3.5.** *The functional*

$$(3.52) \quad I_4(t) = e^{-\delta t} \int_0^1 \int_0^t e^{\delta r} K(t-r) u_t^2(r) dr dx, \quad t > 0,$$

where

$$K(t) = \int_t^{+\infty} e^{\tau r} |h(t)| dr,$$

satisfies

$$(3.53) \quad I_4'(t) = -\delta I_4(t) + \int_0^1 K(0) u_t^2 dx - \int_0^1 \int_0^t h(t-r) u_t^2(r) dr dx, \quad t > 0.$$

Now, by using the previous lemmas and the Lyapunov functional  $F(t)$  defined by

$$(3.54) \quad F(t) = NE(t) + N_1 I_1(t) + N_2 I_2(t) + I_3(t) + N_3 I_4(t), \quad t > 0,$$

where  $N, N_1, N_2$  and  $N_3$  are positive constants. Then, our stability result reads.

**Lemma 3.3.6.** *For some positive constants  $\alpha_1$  and  $\alpha_2$ , the Lyapunov functional  $F(t)$  introduced by (3.54) is equivalent to  $NE(t) + N_3 I_4(t)$ , such that*

$$(3.55) \quad \alpha_1 E(t) \leq F(t) \leq \alpha_2 [E(t) + I_4(t)], \quad t > 0,$$

**Proof.** By exploiting relation (3.54) with the use of (3.37), (3.42) and (3.48), we get

$$\begin{aligned}
 |F(t) - NE(t) - N_3 I_4(t)| &= |N_1 I_1(t) + N_2 I_2(t) + I_3(t)| \\
 &\leq \rho_u N_1 \int_0^1 \left| u_t + \int_0^t h(t-r) u_t(r) dr \right| |u| dx + \frac{\beta N_1}{2} \int_0^1 u^2 dx \\
 &\quad + \frac{N_2 \beta a_2}{\rho_u} \int_0^1 |z| |u| dx + N_2 a_2 \int_0^1 \left| u_t + \int_0^t h(t-r) u_t(r) dr \right| |z| dx \\
 &\quad - a_2 \left( \frac{N_1 \rho_z}{a_1} + N_2 \right) \int_0^1 |z_t| |u| dx - \rho_z \int_0^1 |z_t| |z| dx,
 \end{aligned}$$

by using Poincaré's, Cauchy-Schwarz and Young's inequalities, we obtain

$$(3.56) \quad |F(t) - NE(t) - N_3 I_4(t)| \leq \lambda E(t), \quad \lambda > 0.$$

Thus, (4.26) leads to

$$(N - \lambda)E(t) + N_3 I_4(t) \leq F(t) \leq (N + \lambda)E(t) + N_3 I_4(t),$$

we conclude (5.45), when we set  $\alpha_1 = \min\{N - \lambda, N_3\}$  and  $\alpha_2 = \max\{N + \lambda, N_3\}$  such that  $N$  is sufficiently large.  $\blacksquare$

**Theorem 3.3.7.** *Under the above assumptions (H), we have*

$$(3.57) \quad E(t) \leq M e^{-\delta t}, \quad t > 0,$$

for some positive constants  $M$  and  $\delta$ .

**Proof.** Recalling (3.32), (3.38), (3.43), (3.49) and (3.53), we obtain

$$\begin{aligned}
 F'(t) &\leq - \left[ N\beta - N_1 C_1 - \frac{N_2}{\varepsilon_3} \left( \int_0^t h(r) dr \right)^2 - N_4 K(0) \right] \int_0^1 u_t^2 dx + \frac{N\rho_u}{2} \int_0^1 h' \square u_t dx \\
 &\quad - \left[ N_1 \left( a_3 - \frac{a_2^2}{a_1} \right) - N_2 C_2(\varepsilon_3) - \frac{N_3 a_3}{4} \right] \int_0^1 u_x^2 dx \\
 &\quad + \left[ \left( \frac{N_1}{2} + \frac{N_2}{\varepsilon_3} \right) \int_0^t h(r) dr \right] \int_0^1 h \circ u_t dx \\
 &\quad - \left[ \frac{N_2 a_2^2}{2\rho_u} - 2N_3 a_1 \right] \int_0^1 z_x^2 dx - [N_3 \rho_z - N_1 \varepsilon_1 - N_2 a_2^2 \varepsilon_3] \int_0^1 z_t^2 dx \\
 &\quad - \delta N_4 I_4(t) - N_4 \int_0^1 \int_0^t h(t-r) u_t^2(r) dr dx,
 \end{aligned}$$

by taking  $\varepsilon_1 = \frac{\rho_z}{4N_1}$  and  $\varepsilon_3 = \frac{\rho_z}{4a_2^2 N_2}$ , we end up with

$$\begin{aligned}
 F'(t) &\leq - \left[ N\beta - N_1 C_1 - \frac{4(a_2 N_2)^2}{\rho_z} h_0^2 - N_4 K(0) \right] \int_0^1 u_t^2 dx + \frac{N\rho_u}{2} \int_0^1 h' \square u_t dx \\
 &\quad + \left[ \frac{N_1}{2} h_0 + \frac{4(a_2 N_2)^2}{\rho_z} h_0 \right] \int_0^1 h \circ u_t dx - \left[ N_1 \left( a_3 - \frac{a_2^2}{a_1} \right) - N_2 C_2 \left( \frac{\rho_z}{4a_2^2 N_2} \right) - \frac{N_3 a_3}{4} \right] \int_0^1 u_x^2 dx \\
 (3.58) \quad &- \left[ \frac{N_2 a_2^2}{2\rho_u} - 2N_3 a_1 \right] \int_0^1 z_x^2 dx - \frac{\rho_z}{2} \int_0^1 z_t^2 dx - \delta N_4 I_4(t) - N_4 \int_0^1 \int_0^t h(t-r) u_t^2(r) dr dx,
 \end{aligned}$$

where, for all  $t \geq t_0 > 0$ ,

$$\int_0^t h(r)dr \geq \int_0^{t_0} h(r)dr = h_0.$$

Now, we choose  $N_2$  large enough such that

$$N_2 > \frac{4N_3\rho_u a_1}{a_2^2},$$

then, we select  $N_1$  large enough such that

$$N_1 \left( a_3 - \frac{a_2^2}{a_1} \right) - N_2 C_2 \left( \frac{\rho_z}{4a_2^2 N_2} \right) - \frac{N_3 a_3}{4} > 0.$$

Finally

$$N\beta - N_1 C_1 - \frac{N_2}{\varepsilon_3} h_0^2 - N_4 K(0) > 0.$$

As a result, the relation (3.58) becomes

$$(3.59) \quad F'(t) \leq -C_3 \int_0^1 (u_t^2 + u_x^2 + z_t^2 + z_x^2 + h * u_t^2) dx - C_4 I_4(t), \quad \forall t > 0,$$

where  $C_3$  and  $C_4$  are positive constants. We also have from (3.31), after using Young's inequality, that

$$(3.60) \quad \begin{aligned} E(t) &\leq \frac{1}{2} \int_0^1 [\rho_z z_t^2 + (a_1 + a_2) z_x^2 + \rho_u u_t^2 + (a_3 + a_2) u_x^2] dx + \frac{\rho_u}{2} \int_0^1 \left( \int_0^t h(t-r) u_t^2(x,r) dr \right) dx \\ &\leq \lambda_1 \int_0^1 \left[ z_t^2 + z_x^2 + u_t^2 + u_x^2 + \left( \int_0^t h(t-r) u_t^2(x,r) dr \right) \right] dx, \quad \lambda_1 > 0. \end{aligned}$$

The combination of (3.59) and (3.60) results in

$$(3.61) \quad F'(t) \leq -C_5 [E(t) + I_4(t)], \quad C_5 > 0.$$

From (3.61) and the right side of (3.55), we obtain

$$(3.62) \quad F(t) \leq C_6 e^{-\frac{C_5}{\alpha_2} t}, \quad t > 0,$$

where  $C_6$  is a positive constant. Which yields the desired result (3.57) by using the other side of the equivalence relation again. ■

## STABILITY RESULT OF LORD-SHULMAN SYSTEM WITH ONLY MICROTEMPERATURE EFFECTS

### 4.1 Introduction

The most well-known law for modeling heat conduction is the Fourier law. However, this law of thermoelasticity based on a parabolic-type heat-conduction presents a physical incoherence known as infinite heat propagation velocity, which is in disagreement with physical experiments. Many theories emerged over this time period to overcome this anomaly, we want to focus on the theory of Lord Shulman [49] which take into account the acceleration of the heat flow. From there, many researchers have been interested in the formulation of alternative constitutive relations to prevail this physical paradox (see [7, 17, 18, 20, 30, 73]).

Lord-Shulman thermoelasticity has received a lot of attention among scientists in the last few years and the amount of contributions to explain this theory is enormous. This theory consists in studying a system of four hyperbolic equations with heat dissipation. In this case, the heat equation is also hyperbolic unlike the one obtained for the Fourier law which is parabolic. The evolution equations for the thermo-poro-elasticity with microtemperatures in the case of Cattaneo-Maxwell law are

$$(4.1) \quad \rho u_{tt} = S_x, \quad J \varphi_{tt} = H_x + G, \quad \rho E_t = P_x + q - Q,$$

where  $\rho$  is the mass density,  $J$  is the equilibrated inertia,  $S$  is the stress tensor,  $q$  is the heat flux vector,  $P$  is the first heat flux moment,  $H$  is the equilibrated stress,  $G$  is the equilibrated body force,  $Q$  is the mean heat flux and  $E$  is the first moment of energy. The variables  $u$  and  $\varphi$  are, respectively, the displacement of the solid elastic material and the volume fraction. The

constitutive equations are

$$(4.2) \quad \begin{aligned} S &= (\lambda + 2\mu)u_x + \mu_0\varphi, \quad G = -\mu_0u_x - \xi\varphi, \quad q = k_1W, \\ H &= a_0\varphi_x - \mu_2(\tau W_t + W), \quad Q = k_3W, \quad P = -k_2W_x, \\ \rho E &= -b(\tau W_t + W) - \mu_2\varphi_x, \end{aligned}$$

where  $W$  is the microtemperature,  $\lambda$  and  $\mu$  are the Lamé's parameters,  $\mu_0$  is the coupling between the displacement and the porosity,  $\xi$  and  $\mu_2$  denote a porosity coefficients,  $a_0$  is the porosity diffusion coefficient, and the other parameters  $k_1$ ,  $k_2$ ,  $k_3$  and  $b$  are usual in the study of the present theory and satisfy the following inequalities

$$(4.3) \quad k_1 - k_3 > 0.$$

Now, by substituting (4.2) into (4.1), we obtain the following system

$$(4.4) \quad \begin{cases} \rho u_{tt} = \mu^* u_{xx} + \mu_0\varphi_x, & \text{in } (0, 1) \times (0, \infty) \\ J\varphi_{tt} = a_0\varphi_{xx} - \mu_2(\tau W_{tx} + W_x) - \mu_0u_x - \xi\varphi, & \text{in } (0, 1) \times (0, \infty) \\ b(\tau W_t + W)_t = k_2W_{xx} - \mu_2\varphi_{tx} - kW, & \text{in } (0, 1) \times (0, \infty) \end{cases}$$

with  $\mu^* = \lambda + 2\mu$ ,  $k = k_1 - k_3 > 0$  and  $\mu^* > \frac{\mu_0^2}{\xi}$ .

We consider the boundary conditions

$$(4.5) \quad \begin{aligned} u(0, t) = u(1, t) = W(0, t) = W(1, t) &= 0, \quad t > 0, \\ \varphi_x(0, t) = \varphi_x(1, t) &= 0, \quad t > 0, \end{aligned}$$

and initial data

$$(4.6) \quad \begin{aligned} u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad \varphi_t(x, 0) = \varphi^1(x), \quad x \in (0, 1), \\ \varphi(x, 0) = \varphi^0(x), \quad W(x, 0) = W^0(x), \quad W_t(x, 0) = W^1(x), \quad x \in (0, 1), \end{aligned}$$

here  $u^0$ ,  $u^1$ ,  $\varphi^0$ ,  $\varphi^1$ ,  $W^0$  and  $W^1$  are given functions. Note that for the case  $\tau = 0$ , system (4.4) takes the form

$$(4.7) \quad \begin{cases} \rho u_{tt} = \mu^* u_{xx} + \mu_0\varphi_x, & \text{in } (0, 1) \times (0, \infty), \\ J\varphi_{tt} = a_0\varphi_{xx} - \mu_2W_x - \mu_0u_x - \xi\varphi, & \text{in } (0, 1) \times (0, \infty), \\ bW_t = k_2W_{xx} - \mu_2\varphi_{tx} - kW, & \text{in } (0, 1) \times (0, \infty). \end{cases}$$

This system was studied by Apalara in [4] where he showed that the system is stable if and only if

$$(4.8) \quad \chi_0 = \frac{\mu^*}{\rho} - \frac{a_0}{J} = 0$$

holds. Furthermore, if (4.8) does not hold, the system is polynomially stable. For more details on elastic porous systems with microtemperature effects, we refer the reader to [22, 32, 45, 51, 61–63] and the references therein.

Here in this work, our objective, is to complete this result by giving a necessary and sufficient condition on the coefficients to drive the system to the equilibrium state in an exponential manner. More precisely, if the coefficients satisfy the following condition

$$(4.9) \quad \chi = 0,$$

where

$$(4.10) \quad \chi = \mu_2^2 \tau - \left( \frac{a_0 \mu^* b \tau}{\rho} - a_0 k_2 \right) \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right).$$

Meanwhile, from the second equation of (4.4) and the boundary conditions in (4.5), we get

$$(4.11) \quad \frac{d^2}{dt^2} \int_0^1 \varphi(x, t) dx + \frac{\xi}{J} \int_0^1 \varphi(x, t) dx = 0, \quad \forall t \geq 0.$$

So, by solving (4.11) and using  $\varphi$ 's initial data, we arrive at

$$\int_0^1 \varphi(x, t) dx = \left( \int_0^1 \varphi^0(x) dx \right) \cos \left( \sqrt{\frac{\xi}{J}} t \right) + \sqrt{\frac{J}{\xi}} \left( \int_0^1 \varphi^1(x) dx \right) \sin \left( \sqrt{\frac{\xi}{J}} t \right).$$

Consequently, if we set

$$\bar{\varphi}(x, t) = \varphi(x, t) - \int_0^1 \varphi(x, t) dx, \quad t \geq 0, \quad x \in (0, 1),$$

by integrating the previous equation over  $(0, 1)$  with respect to  $x$ , we find

$$\int_0^1 \bar{\varphi}(x, t) dx = 0, \quad t \geq 0.$$

$(u, \bar{\varphi}, W)$  satisfies the same equations in (4.4)-(4.6). In what follows we will work with  $\bar{\varphi}$  but, for convenience, we write  $\varphi$  instead of  $\bar{\varphi}$ .

## 4.2 Existence and uniqueness

Our goal in this part is to apply the semigroup theory to transform our problem into a Cauchy problem in an appropriate Hilbert space. First, we define the vector function  $U = (u, v, \varphi, \phi, W, \vartheta)^T$ ,  $v = u_t$ ,  $\phi = \varphi_t$  and  $\vartheta = W_t$ . System (4.4)-(4.6) may therefore be rewritten as follows

$$(4.12) \quad \begin{cases} U_t = \mathbb{A}U, & t > 0, \\ U(0) = U_0 = (u_0, u_1, \varphi_0, \varphi_1, W_0, W_1)^T, \end{cases}$$

where the operator  $\mathbb{A} : \mathbb{D}(\mathbb{A}) \subset \mathbb{H} \rightarrow \mathbb{H}$  is defined by

$$(4.13) \quad \mathbb{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{\mu^*}{\rho} \partial_x^2(\cdot) & 0 & \frac{\mu_0}{\rho} \partial_x(\cdot) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{\mu_0}{J} \partial_x(\cdot) & 0 & \frac{a_0 \partial_x^2(\cdot)}{J} - \frac{\xi}{J} & 0 & -\frac{\mu_2}{J} \partial_x(\cdot) & -\frac{\mu_2 \tau}{J} \partial_x(\cdot) \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{\mu_2}{\tau b} \partial_x(\cdot) & \frac{k_2 \partial_x^2(\cdot)}{\tau b} - \frac{k}{\tau b} & -\frac{1}{\tau} \end{pmatrix}.$$

We take into account the following spaces

$$(4.14) \quad \begin{aligned} L_*^2(0, 1) &= \left\{ \Psi \in L^2(0, 1), \int_0^1 \Psi(x) dx = 0 \right\}, \\ H_*^2(0, 1) &= \left\{ \Psi \in H^2(0, 1), \Psi_x(0) = \Psi_x(1) = 0 \right\}, \\ H_*^1(0, 1) &= H^1(0, 1) \cap L_*^2(0, 1), \end{aligned}$$

and let

$$\mathbb{H} = H_0^1(0, 1) \times L^2(0, 1) \times H_*^1(0, 1) \times L_*^2(0, 1) \times H_0^1(0, 1) \times L^2(0, 1)$$

be the Hilbert space with the inner product shown below

$$(4.15) \quad \begin{aligned} \langle U, \tilde{U} \rangle &= \frac{\rho}{2} \int_0^1 v \tilde{v} dx + \frac{J}{2} \int_0^1 \phi \tilde{\phi} dx + \frac{1}{2} \left( \mu^* - \frac{\mu_0^2}{\xi} \right) \int_0^1 u_x \tilde{u}_x dx + \frac{k_2 \tau}{2} \int_0^1 W_x \tilde{W}_x dx \\ &+ \frac{1}{2} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \phi \right) \left( \frac{\mu_0}{\sqrt{\xi}} \tilde{u}_x + \sqrt{\xi} \tilde{\phi} \right) dx + \frac{a_0}{2} \int_0^1 \phi_x \tilde{\phi}_x dx \\ &+ \frac{k \tau}{2} \int_0^1 W \tilde{W} dx + \frac{b}{2} \int_0^1 (W + \tau \vartheta) (\tilde{W} + \tau \tilde{\vartheta}) dx, \end{aligned}$$

for  $U = (u, v, \phi, \phi, W, \vartheta)^T \in \mathbb{H}$ ,  $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\phi}, \tilde{\phi}, \tilde{W}, \tilde{\vartheta})^T \in \mathbb{H}$ .

**Remark 4.2.1.** Relation (4.15) clearly establishes an inner product, and according to assumption  $\mu^* > \frac{\mu_0^2}{\xi}$ , we have

$$(U, U)_{\mathbb{H}} = \frac{1}{2} \int_0^1 \left[ \rho v^2 + J \phi^2 + a_0 \phi_x^2 + \left( \mu^* - \frac{\mu_0^2}{\xi} \right) u_x^2 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \phi \right)^2 + b(\tau \vartheta + W)^2 + k_2 \tau W_x^2 + k \tau W^2 \right] dx,$$

which yields  $(U, U)_{\mathbb{H}} > 0$ .

Consequently, we conclude that  $(U, \tilde{U})_{\mathbb{H}}$  defines an inner product on  $\mathbb{H}$  and the associated norm  $\|\cdot\|_{\mathbb{H}}$  is equivalent to the usual one. The domain of  $\mathbb{A}$  is given by

$$(4.16) \quad \mathbb{D}(\mathbb{A}) = \left( \begin{array}{l} U \in \mathbb{H} / u \in H^2(0, 1) \cap H_0^1(0, 1), v \in H_0^1(0, 1), \\ \phi \in H_*^2(0, 1) \cap H_*^1(0, 1), \phi \in H_*^1(0, 1), \\ W \in H^2(0, 1) \cap H_0^1(0, 1), \vartheta \in H_0^1(0, 1) \end{array} \right).$$

The domain  $\mathbb{D}(\mathbb{A})$  is clearly dense in  $\mathbb{H}$ . So, based on inner product (4.15), we have

$$\langle \mathbb{A}U, U \rangle_{\mathbb{H}} = -k_2 \int_0^1 W_x^2 dx - k \int_0^1 W^2 dx \leq 0,$$

because of this, the operator  $\mathbb{A}$  is dissipative. We can now argue that the operator  $I - \mathbb{A}$  is surjective using the Lax–Milgram lemma and classical regularity arguments. So, using the Lumer–Phillips theorem, we can assert that  $\mathbb{A}$  is an infinitesimal generator of a linear  $C_0$ –semigroup on  $\mathbb{H}$ , and the following well-posedness result follows [53].

**Theorem 4.2.2.** Let  $U_0 \in \mathbb{D}(\mathbb{A})$ , problem (5.15) has a unique solution  $U \in C(\mathbb{R}_+, \mathbb{D}(\mathbb{A})) \cap C^1(\mathbb{R}_+, \mathbb{H})$ . Moreover if  $U_0 \in \mathbb{H}$ , then the solution  $U \in C(\mathbb{R}_+, \mathbb{H})$ .

### 4.3 Exponential stability result

In this section, we use the energy method to prove the exponential stability result for (4.4)-(4.6).

First, we introduce the energy of our system as:

$$(4.17) \quad E(t) = \frac{1}{2} \int_0^1 \left[ \rho u_t^2 + \left( \mu^* - \frac{\mu_0^2}{\xi} \right) u_x^2 + J \varphi_t^2 + \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 + a_0 \varphi_x^2 + b(\tau W_t + W)^2 + k_2 \tau W_x^2 + k \tau W^2 \right] dx, \quad t > 0.$$

Next, we state the main result of this section

**Theorem 4.3.1.** *Let  $(u, \varphi, W)$  be a solution of problem (4.4)-(4.6) and assume that the coefficients of the system satisfy (4.9). Then  $E(t)$  decays exponentially, i.e., there exist two positive constants  $c_1$  and  $c_2$ , where  $c_2$  depends on the initial data, such that*

$$E(t) \leq c_2 e^{-c_1 t}, \quad \forall t \geq 0.$$

The proof of theorem 4.3.1 will be achieved with the help of a sequence of lemmas. The idea is to construct a Lyapunov functional  $L(t)$  equivalent to  $E(t)$ . For this end, we use the multiplier technique.

**Lemma 4.3.2.** *The related energy of system (4.4)-(4.6) defined by (4.17) satisfies*

$$(4.18) \quad E'(t) = -k_2 \int_0^1 W_x^2 dx - k \int_0^1 W^2 dx \leq 0, \quad t > 0.$$

**Lemma 4.3.3.** *Let  $(u, \varphi, W)$  be a solution of (4.4)-(4.6). Then the functional*

$$(4.19) \quad \Gamma_1(t) = -\rho \int_0^1 u_t u dx, \quad t > 0,$$

*satisfies*

$$(4.20) \quad \Gamma_1'(t) \leq -\rho \int_0^1 u_t^2 dx + \frac{\mu_0}{2\sqrt{\xi}} \int_0^1 u_x^2 dx + \frac{\mu_0}{2\sqrt{\xi}} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx, \quad t > 0.$$

**Proof.** By differentiating the functional  $\Gamma_1(t)$ , using (4.4)<sub>1</sub>, then integrating by parts, we get

$$(4.21) \quad \Gamma_1'(t) = -\rho \int_0^1 u_t^2 dx + \mu^* \int_0^1 u_x^2 dx + \mu_0 \int_0^1 \varphi u_x dx.$$

Now, by adding and subtracting the term  $\frac{\mu_0^2}{\xi} \int_0^1 u_x^2 dx$  into (4.21) with respect to  $\mu^* > \frac{\mu_0^2}{\xi}$ , we obtain

$$(4.22) \quad \Gamma_1'(t) = -\rho \int_0^1 u_t^2 dx + \left( \mu^* - \frac{\mu_0^2}{\xi} \right) \int_0^1 u_x^2 dx + \frac{\mu_0}{\sqrt{\xi}} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) u_x dx.$$

Applying Young's inequality, we find

$$(4.23) \quad \frac{\mu_0}{\sqrt{\xi}} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) u_x dx \leq \frac{\mu_0}{2\sqrt{\xi}} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx + \frac{\mu_0}{2\sqrt{\xi}} \int_0^1 u_x^2 dx.$$

Then, the relationship (4.20) follows easily, when we substitute (4.23) into (4.22).  $\blacksquare$

**Lemma 4.3.4.** *Let  $(u, \varphi, W)$  be a solution of (4.4)-(4.6). Then, we have*

$$(4.24) \quad \Gamma_2(t) = -bJ \int_0^1 (\tau W_t + W) \left( \int_0^x \varphi_t dy \right) dx, \quad t > 0,$$

satisfies, for any  $\varepsilon_1, \varepsilon_2 > 0$

$$(4.25) \quad \begin{aligned} \Gamma_2'(t) \leq & -\frac{\mu_2 J}{2} \int_0^1 \varphi_t^2 dx + \varepsilon_1 \int_0^1 u_x^2 dx + \varepsilon_2 \int_0^1 \varphi_x^2 dx + \frac{k_2^2 J}{\mu_2} \int_0^1 W_x^2 dx + C(\varepsilon_1, \varepsilon_2) \int_0^1 (\tau W_t + W)^2 dx \\ & + \frac{k^2 J}{\mu_2} \int_0^1 W^2 dx, \quad t > 0, \end{aligned}$$

where  $C(\varepsilon_1, \varepsilon_2) = \frac{b^2 a_0^2}{2\varepsilon_2} + \frac{\mu_0^2 b^2}{4\varepsilon_1} + \frac{\xi^2 b^2}{2\varepsilon_2} + \mu_2 b$ .

**Proof.** By exploiting the functional  $\Gamma_2(t)$  using (4.4)<sub>2</sub> and (4.4)<sub>3</sub>, then integrating by parts over  $(0, 1)$  and recalling the fact that  $\int_0^1 \varphi_t dx = 0$ , we obtain

$$\begin{aligned} \Gamma_2'(t) = & \mu_2 J \int_0^1 \varphi_{tx} \left( \int_0^x \varphi_t dy \right) dx - b a_0 \int_0^1 (\tau W_t + W) \left( \int_0^x \varphi_{xx} dy \right) dx - J k_2 \int_0^1 W_{xx} \left( \int_0^x \varphi_t dy \right) dx \\ & + b \mu_0 \int_0^1 (\tau W_t + W) \left( \int_0^x u_x dy \right) dx + b \xi \int_0^1 (\tau W_t + W) \left( \int_0^x \varphi dy \right) dx \\ & + k J \int_0^1 W \left( \int_0^x \varphi_t dy \right) dx + b \mu_2 \int_0^1 (\tau W_t + W) \left[ \int_0^x (\tau W_{tx} + W_x) dy \right] dx, \end{aligned}$$

consequently, we get

$$(4.26) \quad \begin{aligned} \Gamma_2'(t) = & -\mu_2 J \int_0^1 \varphi_t^2 dx - b a_0 \int_0^1 \varphi_x (\tau W_t + W) dx + k_2 J \int_0^1 \varphi_t W_x dx + \mu_0 b \int_0^1 u (\tau W_t + W) dx \\ & + \xi b \int_0^1 \left( \int_0^x \varphi dy \right) (\tau W_t + W) dx + k J \int_0^1 \left( \int_0^x \varphi_t dy \right) W dx + \mu_2 b \int_0^1 (\tau W_t + W)^2 dx, \end{aligned}$$

by using Young, Poincaré and Cauchy-Schwarz inequalities

$$\begin{aligned} b a_0 \int_0^1 \varphi_x (\tau W_t + W) dx & \leq \frac{\varepsilon_2}{2} \int_0^1 \varphi_x^2 dx + \frac{b^2 a_0^2}{2\varepsilon_2} \int_0^1 (\tau W_t + W)^2 dx, \\ \mu_0 b \int_0^1 u (\tau W_t + W) dx & \leq \varepsilon_1 \int_0^1 u_x^2 dx + \frac{\mu_0^2 b^2}{4\varepsilon_1} \int_0^1 (\tau W_t + W)^2 dx, \\ \xi b \int_0^1 \left( \int_0^x \varphi dy \right) (\tau W_t + W) dx & \leq \frac{\varepsilon_2}{2} \int_0^1 \varphi_x^2 dx + \frac{\xi^2 b^2}{2\varepsilon_2} \int_0^1 (\tau W_t + W)^2 dx, \end{aligned}$$

$$(4.27) \quad \begin{aligned} k_2 J \int_0^1 \varphi_t W_x dx &\leq \frac{\mu_2 J}{4} \int_0^1 \varphi_t^2 dx + \frac{k_2^2 J}{\mu_2} \int_0^1 W_x^2 dx, \\ k J \int_0^1 \left( \int_0^x \varphi_t dy \right) W dx &\leq \frac{\mu_2 J}{4} \int_0^1 \varphi_t^2 dx + \frac{k^2 J}{\mu_2} \int_0^1 W^2 dx. \end{aligned}$$

We conclude (4.25), when we estimate (4.26) by (4.27)'s quantities.  $\blacksquare$

**Lemma 4.3.5.** *Let  $(u, \varphi, W)$  be a solution of (4.4)-(4.6). Then, we have*

$$(4.28) \quad \Gamma_3(t) = -b \int_0^1 \tau^2 W_t W dx - \frac{b\tau}{2} \int_0^1 W^2 dx, \quad t > 0,$$

satisfies, for any  $\varepsilon_3 > 0$

$$(4.29) \quad \Gamma_3'(t) \leq -\frac{b}{2} \int_0^1 (\tau W_t + W)^2 dx + \varepsilon_3 \int_0^1 \varphi_t^2 dx + (b + k\tau) \int_0^1 W^2 dx + \left( \tau k_2 + \frac{\tau^2 \mu_2^2}{4\varepsilon_3} \right) \int_0^1 W_x^2 dx.$$

**Proof.** Differentiating  $\Gamma_3(t)$  using (4.4)<sub>3</sub>, we obtain

$$(4.30) \quad \begin{aligned} \Gamma_3'(t) &= b\tau \int_0^1 W_t W dx - \tau k_2 \int_0^1 W_{xx} W dx + \mu_2 \tau \int_0^1 \varphi_{tx} W dx + k\tau \int_0^1 W^2 dx - b \int_0^1 \tau^2 W_t^2 dx \\ &\quad - b\tau \int_0^1 W_t W dx, \end{aligned}$$

by integrating by parts (4.30), we find

$$\Gamma_3'(t) = -b \int_0^1 (\tau W_t)^2 dx + \tau k_2 \int_0^1 W_x^2 dx - \tau \mu_2 \int_0^1 \varphi_t W_x dx + k\tau \int_0^1 W^2 dx.$$

By using the fact that

$$-b \int_0^1 (\tau W_t)^2 dx \leq -\frac{b}{2} \int_0^1 (\tau W_t + W)^2 dx + b \int_0^1 W^2 dx,$$

and Young's inequality

$$\tau \mu_2 \int_0^1 \varphi_t W_x dx \leq \varepsilon_3 \int_0^1 \varphi_t^2 dx + \frac{\tau^2 \mu_2^2}{4\varepsilon_3} \int_0^1 W_x^2 dx.$$

Thus, estimate (4.29) easily follows.  $\blacksquare$

**Lemma 4.3.6.** *Let  $(u, \varphi, W)$  be a solution of (4.4)-(4.6). Then the functional*

$$(4.31) \quad \begin{aligned} \Gamma_4(t) &= J \int_0^1 \varphi_t \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) dx - \frac{\alpha_0 b \mu_0}{\mu_2 \sqrt{\xi}} \left( \frac{J}{\alpha_0} - \frac{\rho}{\mu^*} \right) \int_0^1 (\tau W_t + W) u_t dx \\ &\quad + \frac{\mu_0 \alpha_0 \rho}{\mu^* \sqrt{\xi}} \int_0^1 u_t \varphi_x dx - \frac{\alpha_0}{\mu_2^2} \left( \frac{J}{\alpha_0} - \frac{\rho}{\mu^*} \right) \frac{k_2 \mu_2 \mu_0}{\sqrt{\xi}} \int_0^1 W_x u_x dx - \mu_2 \tau \sqrt{\xi} \int_0^1 W \varphi_x dx, \quad t > 0, \end{aligned}$$

satisfies, for any  $\varepsilon_4, \varepsilon_5 > 0$

$$\Gamma_4'(t) \leq -\frac{\sigma}{2} \int_0^1 \varphi_x^2 dx - \frac{\sqrt{\xi}}{2} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx + \varepsilon_4 \int_0^1 u_t^2 dx + \varepsilon_5 \int_0^1 u_x^2 dx$$

$$(4.32) \quad \begin{aligned} & + \left( \frac{\mu_2 \tau \sqrt{\xi}}{2} + \sqrt{\xi} J \right) \int_0^1 \varphi_t^2 dx + C(\varepsilon_5) \int_0^1 W_x^2 dx + \frac{\sigma_1^2}{4\varepsilon_4} \int_0^1 W^2 dx \\ & + \frac{\sigma_3^2}{\sigma} \int_0^1 (\tau W_t + W)^2 dx - \frac{\mu_0}{\mu_2 \sqrt{\xi}} \chi \int_0^1 W_{tx} u_x dx, \quad t > 0, \end{aligned}$$

where  $\sigma = \frac{a_0}{\sqrt{\xi}} \left( \xi - \frac{\mu_0^2}{\mu^*} \right)$  and  $C(\varepsilon_5) = \frac{\mu_2 \tau \sqrt{\xi}}{2} + \frac{\mu_2^2}{2\sqrt{\xi}} + \frac{\sigma_2^2}{4\varepsilon_5}$ .

**Proof.** Direct differentiation, using (4.4)<sub>1</sub>, (4.4)<sub>2</sub> and (4.4)<sub>3</sub>, gives

$$\begin{aligned} \Gamma_4'(t) = & a_0 \int_0^1 \varphi_{xx} \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) dx - \mu_2 \int_0^1 (\tau W_{tx} + W_x) \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) dx \\ & - \mu_0 \int_0^1 u_x \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) dx - \xi \int_0^1 \varphi \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) dx + \frac{\mu_0^2 a_0}{\mu^* \sqrt{\xi}} \int_0^1 \varphi_x^2 dx \\ & + J \int_0^1 \varphi_t \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) dx + \frac{k a_0 \mu_0}{\mu_2 \sqrt{\xi}} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 W u_t dx + \frac{\mu_0 a_0}{\sqrt{\xi}} \int_0^1 u_{xx} \varphi dx \\ & - \frac{k_2 a_0 \mu_0}{\mu_2 \sqrt{\xi}} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 W_{xx} u_t dx + \frac{a_0 \mu_2 \mu_0}{\mu_2 \sqrt{\xi}} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 \varphi_{tx} u_t dx \\ & - \frac{a_0 b \mu^* \mu_0}{\rho \mu_2 \sqrt{\xi}} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 (\tau W_t + W) u_{xx} dx + \frac{a_0 b \mu_0^2}{\rho \mu_2 \sqrt{\xi}} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 (\tau W_t + W) \varphi_x dx \\ & - \mu_2 \tau \sqrt{\xi} \int_0^1 W \varphi_{tx} dx + \frac{\mu_0 a_0 \rho}{\mu^* \sqrt{\xi}} \int_0^1 u_t \varphi_{tx} dx - \frac{a_0}{\mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \frac{k_2 \mu_2 \mu_0}{\sqrt{\xi}} \int_0^1 W_{tx} u_x dx \\ & - \frac{a_0}{\mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \frac{k_2 \mu_2 \mu_0}{\sqrt{\xi}} \int_0^1 W_x u_{tx} dx - \mu_2 \tau \sqrt{\xi} \int_0^1 W_t \varphi_x dx. \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} \Gamma_4'(t) = & - \frac{a_0}{\sqrt{\xi}} \left( \xi - \frac{\mu_0^2}{\mu^*} \right) \int_0^1 \varphi_x^2 dx - \frac{\mu_0^2}{\sqrt{\xi}} \int_0^1 u_x^2 dx - \xi \int_0^1 \varphi \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) dx + \sqrt{\xi} J \int_0^1 \varphi_t^2 dx \\ & - \mu_0 \sqrt{\xi} \int_0^1 u_x \varphi dx - \mu_2 \int_0^1 W_x \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) dx + \frac{\mu_0 a_0 k}{\sqrt{\xi} \mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 W u_t dx \\ & + \frac{a_0 \mu^* b \mu_0}{\mu_2 \rho \sqrt{\xi}} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 W_x u_x dx - \frac{a_0 b \mu_0^2}{\rho \mu_2 \sqrt{\xi}} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 (\tau W_t + W) \varphi_x dx \\ & + \mu_2 \tau \sqrt{\xi} \int_0^1 W_x \varphi_t dx - \frac{\mu_0}{\mu_2 \sqrt{\xi}} \left[ \mu_2^2 \tau - \left( \frac{a_0 \mu^* b \tau}{\rho} - a_0 k_2 \right) \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \right] \int_0^1 W_{tx} u_x dx. \end{aligned}$$

The use of the fact that,

$$\begin{aligned} - \sqrt{\xi} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx = & - \frac{\mu_0^2}{\sqrt{\xi}} \int_0^1 u_x^2 dx - \xi \int_0^1 \varphi \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) dx \\ & - \mu_0 \sqrt{\xi} \int_0^1 u_x \varphi dx, \end{aligned}$$

results in

$$\begin{aligned}
 \Gamma'_4(t) = & - \underbrace{\frac{a_0}{\sqrt{\xi}} \left( \xi - \frac{\mu_0^2}{\mu^*} \right)}_{=\sigma} \int_0^1 \varphi_x^2 dx - \sqrt{\xi} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx + \mu_2 \tau \sqrt{\xi} \int_0^1 W_x \varphi_t dx \\
 & + \sqrt{\xi} J \int_0^1 \varphi_t^2 dx - \mu_2 \int_0^1 W_x \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) dx + \underbrace{\frac{k a_0 \mu_0}{\mu_2 \sqrt{\xi}} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right)}_{=\sigma_1} \int_0^1 W u_t dx \\
 & + \underbrace{\frac{\mu^* a_0 b \mu_0}{\rho \mu_2 \sqrt{\xi}} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right)}_{=\sigma_2} \int_0^1 W_x u_x dx - \underbrace{\frac{a_0 b \mu_0^2}{\rho \mu_2 \sqrt{\xi}} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right)}_{=\sigma_3} \int_0^1 (\tau W_t + W) \varphi_x dx \\
 (4.33) \quad & - \underbrace{\frac{\mu_0}{\mu_2 \sqrt{\xi}} \left[ \mu_2^2 \tau - \left( \frac{a_0 \mu^* b \tau}{\rho} - a_0 k_2 \right) \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \right]}_{=\chi} \int_0^1 W_{tx} u_x dx.
 \end{aligned}$$

Applying Young's inequality, we get

$$\begin{aligned}
 \mu_2 \tau \sqrt{\xi} \int_0^1 W_x \varphi_t dx & \leq \frac{\mu_2 \tau \sqrt{\xi}}{2} \int_0^1 \varphi_t^2 dx + \frac{\mu_2 \tau \sqrt{\xi}}{2} \int_0^1 W_x^2 dx, \\
 \mu_2 \int_0^1 W_x \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) dx & \leq \frac{\sqrt{\xi}}{2} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx + \frac{\mu_2^2}{2 \sqrt{\xi}} \int_0^1 W_x^2 dx, \\
 \sigma_1 \int_0^1 W u_t dx & \leq \frac{\sigma_1^2}{4 \varepsilon_4} \int_0^1 W^2 dx + \varepsilon_4 \int_0^1 u_t^2 dx, \\
 \sigma_2 \int_0^1 W_x u_x dx & \leq \frac{\sigma_2^2}{4 \varepsilon_5} \int_0^1 W_x^2 dx + \varepsilon_5 \int_0^1 u_x^2 dx, \\
 \sigma_3 \int_0^1 (\tau W_t + W) \varphi_x dx & \leq \frac{\sigma}{2} \int_0^1 \varphi_x^2 dx + \frac{\sigma_3^2}{\sigma} \int_0^1 (\tau W_t + W)^2 dx.
 \end{aligned}$$

The substitution of the previous estimates into (4.33), yields (4.32). ■

**Lemma 4.3.7.** *Let  $(u, \varphi, W)$  be a solution of (4.4)-(4.6). Then the functional*

$$\begin{aligned}
 \Gamma_5(t) = & \frac{a_0 \rho}{\mu^*} \int_0^1 \varphi_x u_t dx + J \int_0^1 \varphi_t u_x dx - \frac{a_0 b}{\mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 (\tau W_t + W) u_t dx \\
 (4.34) \quad & - \frac{a_0 k_2}{\mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 W_x u_x dx, \quad t > 0,
 \end{aligned}$$

satisfies, for any  $\varepsilon_6 > 0$

$$\begin{aligned}
 \Gamma'_5(t) \leq & - \frac{\mu_0}{2} \int_0^1 u_x^2 dx + \tilde{C} \int_0^1 \varphi_x^2 dx + \frac{\sigma_4^2}{4 \varepsilon_6} \int_0^1 W^2 dx + \varepsilon_6 \int_0^1 u_t^2 dx + \frac{\sigma_5^2}{\mu_0} \int_0^1 W_x^2 dx \\
 (4.35) \quad & + \frac{\sigma_6}{2} \int_0^1 (\tau W_t + W)^2 dx + \frac{\chi}{\mu_2} \int_0^1 W_{tx} u_x dx, \quad t > 0,
 \end{aligned}$$

where  $\tilde{C} = \frac{a_0 \mu_0}{\mu^*} + \frac{\xi^2}{\mu_0} + \frac{\sigma_6}{2}$ .

**Proof.** By differentiating expression (4.34), using (4.4)<sub>1</sub>, (4.4)<sub>2</sub> and (4.4)<sub>3</sub>, we obtain

$$\begin{aligned}
 \Gamma'_5(t) = & -\mu_0 \int_0^1 u_x^2 dx + \frac{\mu_0 a_0}{\mu^*} \int_0^1 \varphi_x^2 dx - \xi \int_0^1 \varphi u_x dx + \frac{a_0 k}{\mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 W u_t dx \\
 & - \mu_2 \int_0^1 (\tau W_t + W_x) u_x dx - \frac{\mu_0 a_0 b}{\rho \mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 \varphi_x (\tau W_t + W) dx + a_0 \int_0^1 \varphi_{xx} u_x dx \\
 & + J \int_0^1 \varphi_t u_{tx} dx - \frac{a_0 k_2}{\mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 W_{tx} u_x dx + \frac{a_0 \rho}{\mu^*} \int_0^1 \varphi_{tx} u_t dx + a_0 \int_0^1 u_{xx} \varphi_x dx \\
 & - \frac{k_2 a_0}{\mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 W_{xx} u_t dx - \frac{a_0 b \mu^*}{\rho \mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 u_{xx} (\tau W_t + W) dx \\
 & + \frac{\mu_2 a_0}{\mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 \varphi_{tx} u_t dx - \frac{a_0 k_2}{\mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 W_x u_{tx} dx.
 \end{aligned}$$

Integrating by parts provides

$$\begin{aligned}
 \Gamma'_5(t) = & -\mu_0 \int_0^1 u_x^2 dx + \frac{a_0 \mu_0}{\mu^*} \int_0^1 \varphi_x^2 dx - \xi \int_0^1 \varphi u_x dx + \underbrace{\frac{a_0 k}{\mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right)}_{=\sigma_4} \int_0^1 W u_t dx \\
 & + \underbrace{\left[ \frac{b a_0 \mu^*}{\rho \mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) - \mu_2 \right]}_{=\sigma_5} \int_0^1 W_x u_x dx - \underbrace{\frac{b a_0 \mu_0}{\rho \mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right)}_{=\sigma_6} \int_0^1 \varphi_x (\tau W_t + W) dx \\
 (4.36) \quad & - \frac{1}{\mu_2} \underbrace{\left[ \mu_2^2 \tau - \left( \frac{a_0 \mu^* b \tau}{\rho} - a_0 k_2 \right) \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \right]}_{=\chi} \int_0^1 W_{tx} u_x dx.
 \end{aligned}$$

Using Young's and Poincaré's inequalities, we get

$$\begin{aligned}
 \xi \int_0^1 \varphi u_x dx & \leq \frac{\mu_0}{4} \int_0^1 u_x^2 dx + \frac{\xi^2}{\mu_0} \int_0^1 \varphi_x^2 dx, \\
 \sigma_4 \int_0^1 W u_t dx & \leq \frac{\sigma_4^2}{4\epsilon_6} \int_0^1 W^2 dx + \epsilon_6 \int_0^1 u_t^2 dx, \\
 \sigma_5 \int_0^1 W_x u_x dx & \leq \frac{\mu_0}{4} \int_0^1 u_x^2 dx + \frac{\sigma_5^2}{\mu_0} \int_0^1 W_x^2 dx, \\
 \sigma_6 \int_0^1 (\tau W_t + W) \varphi_x dx & \leq \frac{\sigma_6}{2} \int_0^1 \varphi_x^2 dx + \frac{\sigma_6}{2} \int_0^1 (\tau W_t + W)^2 dx.
 \end{aligned}$$

The substitution of the aforementioned inequalities into (4.36), gives (4.35). ■

#### 4.4 Proof of theorem 4.3.1

Using the preceding lemmas and the Lyapunov functional  $L(t)$ , defined by

$$L(t) = NE(t) + \Gamma_1(t) + N_2 \Gamma_2(t) + N_3 \Gamma_3(t) + N_4 \Gamma_4(t) + N_5 \Gamma_5(t), \quad t > 0,$$

for  $N$  and  $N_i > 0$ ,  $i = \overline{2, 5}$ , we have

$$\begin{aligned}
 L'(t) \leq & - \left[ k_2 N - \frac{k_2^2 J}{\mu_2} N_2 - \left( \tau k_2 + \frac{\tau^2 \mu_2^2}{4 \varepsilon_3} \right) N_3 - C(\varepsilon_5) N_4 - \frac{\sigma_5^2}{\mu_0} N_5 \right] \int_0^1 W_x^2 dx \\
 & - \left[ kN - \frac{k^2 J}{\mu_2} N_2 - (b + k\tau) N_3 - \frac{\sigma_1^2}{4 \varepsilon_4} N_4 - \frac{\sigma_4^2}{4 \varepsilon_6} N_5 \right] \int_0^1 W^2 dx \\
 & - [\rho - \varepsilon_4 N_4 - \varepsilon_6 N_5] \int_0^1 u_t^2 dx - \left[ \frac{\mu_2 J}{2} N_2 - \varepsilon_3 N_3 - \left( \frac{\mu_2 \tau \sqrt{\xi}}{2} + \sqrt{\xi} J \right) N_4 \right] \int_0^1 \varphi_t^2 dx \\
 & - \left[ \frac{b}{2} N_3 - C(\varepsilon_1, \varepsilon_2) N_2 - \frac{\sigma_3^2}{\sigma} N_4 - \frac{\sigma_6}{2} N_5 \right] \int_0^1 (\tau W_t + W)^2 dx \\
 & - \left[ \frac{\sqrt{\xi}}{2} N_4 - \frac{\mu_0}{2 \sqrt{\xi}} \right] \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx - \left[ \frac{\sigma}{2} N_4 - \varepsilon_2 N_2 - \tilde{C} N_5 \right] \int_0^1 \varphi_x^2 dx \\
 & - \left[ \frac{\mu_0}{2} N_5 - \varepsilon_1 N_2 - \varepsilon_5 N_4 - \frac{\mu_0}{2 \sqrt{\xi}} \right] \int_0^1 u_x^2 dx.
 \end{aligned}$$

By setting  $N_5 = \frac{2}{\sqrt{\xi}}$ ,  $\varepsilon_1 = \frac{\mu_0}{8 \sqrt{\xi} N_2}$ ,  $\varepsilon_2 = \frac{\tilde{C}}{\sqrt{\xi} N_2}$ ,  $\varepsilon_3 = \frac{1}{N_3}$ ,  $\varepsilon_4 = \frac{\rho}{4 N_4}$ ,  $\varepsilon_5 = \frac{\mu_0}{8 \sqrt{\xi} N_4}$  and  $\varepsilon_6 = \frac{\rho}{4 N_5}$ , we get

$$\begin{aligned}
 L'(t) \leq & - \left[ k_2 N - \frac{k_2^2 J}{\mu_2} N_2 - \left( \tau k_2 + \frac{\tau^2 \mu_2^2 N_3}{4} \right) N_3 - \left( \frac{\mu_2 \tau \sqrt{\xi}}{2} + \frac{\mu_2^2}{2 \sqrt{\xi}} + \frac{2 \sqrt{\xi} \sigma_2^2 N_4}{\mu_0} \right) N_4 \right. \\
 & \left. - \frac{2 \sigma_5^2}{\sqrt{\xi} \mu_0} \right] \int_0^1 W_x^2 dx - \left[ kN - \frac{k^2 J}{\mu_2} N_2 - (b + k\tau) N_3 - \frac{\sigma_1^2}{\rho} N_4^2 - \frac{4 \sigma_4^2}{\rho \xi} \right] \int_0^1 W^2 dx \\
 & - \left[ \frac{\mu_2 J}{2} N_2 - \left( \frac{\mu_2 \tau \sqrt{\xi}}{2} + \sqrt{\xi} J \right) N_4 - 1 \right] \int_0^1 \varphi_t^2 dx - \frac{\mu_0}{4 \sqrt{\xi}} \int_0^1 u_x^2 dx - \frac{\rho}{2} \int_0^1 u_t^2 dx \\
 & - \left[ \frac{b}{2} N_3 - b^2 \sqrt{\xi} \left( \frac{\alpha_0^2 + \xi^2}{2 \tilde{C}} + \frac{\mu_2}{b \sqrt{\xi} N_2} + 2 \mu_0 \right) N_2^2 - \frac{\sigma_3^2}{\sigma} N_4 - \frac{\sigma_6}{\sqrt{\xi}} \right] \int_0^1 (\tau W_t + W)^2 dx \\
 & - \left[ \frac{\sigma}{2} N_4 - \frac{3 \tilde{C}}{\sqrt{\xi}} \right] \int_0^1 \varphi_x^2 dx - \left[ \frac{\sqrt{\xi}}{2} N_4 - \frac{\mu_0}{2 \sqrt{\xi}} \right] \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx.
 \end{aligned}$$

Now, all the terms on the right-hand side of the last inequality become positive if we select our parameters in an appropriate manner. First, we choose  $N_4$  large enough, so that

$$N_4 > \max \left( \frac{\mu_0}{\xi}, \frac{6 \tilde{C}}{\sigma \sqrt{\xi}} \right).$$

Then,  $N_2$  must be selected such that

$$\frac{\mu_2 J}{2} N_2 - \left( \frac{\mu_2 \tau \sqrt{\xi}}{2} + \sqrt{\xi} J \right) N_4 - 1 > 0.$$

Also, we select  $N_3$  large enough so that

$$\frac{b}{2} N_3 - b^2 \sqrt{\xi} \left( \frac{\alpha_0^2 + \xi^2}{2 \tilde{C}} + \frac{\mu_2}{b \sqrt{\xi} N_2} + 2 \mu_0 \right) N_2^2 - \frac{\sigma_3^2}{\sigma} N_4 - \frac{\sigma_6}{\sqrt{\xi}} > 0.$$

Finally, we choose  $N$  sufficiently large such that

$$k_2 N - \frac{k_2^2 J}{\mu_2} N_2 - \left( \tau k_2 + \frac{\tau^2 \mu_2^2 N_3}{4} \right) N_3 - \left( \frac{\mu_2 \tau \sqrt{\xi}}{2} + \frac{\mu_2^2}{2 \sqrt{\xi}} + \frac{2 \sqrt{\xi} \sigma_2^2 N_4}{\mu_0} \right) N_4 - \frac{2 \sigma_5^2}{\sqrt{\xi} \mu_0} > 0,$$

$$k N - \frac{k^2 J}{\mu_2} N_2 - (b + k \tau) N_3 - \frac{\sigma_1^2}{\rho} N_4^2 - \frac{4 \sigma_4^2}{\rho \xi} > 0.$$

All these choices lead to

$$L'(t) \leq -c_3 E(t), \quad t > 0.$$

Moreover, it is easy to see that there exist two positive constants  $\eta_1$  and  $\eta_2$  such that

$$\eta_1 E(t) \leq L(t) \leq \eta_2 E(t), \quad t > 0,$$

and then we conclude for  $\gamma = \eta_2 / \eta_1$  and  $c_1 = \frac{c_3}{\eta_2}$  that

$$E(t) \leq \gamma E(0) e^{-c_1 t}, \quad t > 0.$$

## 4.5 The lack of exponential decay

Now, we prove the lack of exponential decay under the assumption  $\chi \neq 0$  by using Gearhart-Prüss theorem [48]. We can write  $\chi$  as

$$\chi = \zeta \left( \frac{\rho}{\mu^*} - \frac{J}{\alpha_0} \right) - \frac{\mu_2^2 \tau}{\alpha_0},$$

where  $\zeta = k_2 - \frac{\mu^* b \tau}{\rho}$ .

**Theorem 4.5.1.** *The  $C_0$ -semigroup of contractions associated with the operator  $\mathbb{A}$  on the Hilbert space  $\mathbb{H}$  is exponentially stable if and only if*

$$i\mathbb{R} \equiv \{i\alpha, \alpha \in \mathbb{R}\} \subset \rho(\mathbb{A})$$

and

$$\overline{\lim}_{|\alpha| \rightarrow \infty} \|(i\alpha I - \mathbb{A})^{-1}\|_{\mathbb{H}} < +\infty.$$

The main result of this section is as follow :

**Theorem 4.5.2.** *Let  $\mathbb{A}$  be the operator defined by (4.13) and we assume that  $\chi \neq 0$ . Then the semigroup associated with  $\mathbb{A}$  is non-exponentially decaying.*

**Proof.** [67] It is sufficient to demonstrate the existence of  $F \in \mathbb{H}$ , with  $\|F\|_{\mathbb{H}} < \infty$  such that

$$\|U_\alpha\|_{\mathbb{H}} = \|(i\alpha I - \mathbb{A})^{-1} F\|_{\mathbb{H}} \xrightarrow{\alpha \rightarrow \infty} \infty.$$

Let as assume that there exists  $U \in \mathbb{H}$ ,  $\|U\|_{\mathbb{H}} \neq 0$  so that

$$(i\alpha I - \mathbb{A})U = F,$$

therefore, for fixed  $F = \left(0, \frac{\sin(n\pi x)}{\rho}, 0, 0, 0, 0\right) \in \mathbb{H}$  we get

$$(4.37) \quad v = i\alpha u, \quad \phi = i\alpha\varphi, \quad \vartheta = i\alpha W,$$

and

$$(4.38) \quad \begin{cases} i\alpha\rho v - \mu^* u_{xx} - \mu_0\varphi_x = \sin(n\pi x), \\ i\alpha J\phi - \alpha_0\varphi_{xx} + \tau\mu_2\vartheta_x + \mu_2 W_x + \mu_0 u_x + \xi\varphi = 0, \\ i\alpha b\tau\vartheta + b\vartheta - k_2 W_{xx} + kW + \mu_2\phi_x = 0. \end{cases}$$

Using (4.37), the system (4.38) can be written as

$$(4.39) \quad \begin{cases} -\alpha^2\rho u - \mu^* u_{xx} - \mu_0\varphi_x = \sin(n\pi x), \\ (\xi - \alpha^2 J)\varphi - \alpha_0\varphi_{xx} + (1 + i\alpha\tau)\mu_2 W_x + \mu_0 u_x = 0, \\ (bi\alpha + k - \alpha^2 b\tau)W - k_2 W_{xx} + i\alpha\mu_2\varphi_x = 0, \end{cases}$$

we take into account the boundary conditions and we set

$$u = X \sin(n\pi x), \quad \varphi = Y \cos(n\pi x), \quad W = Z \sin(n\pi x),$$

then we can write (4.39) as

$$(4.40) \quad \begin{pmatrix} D_1 & \mu_0 n\pi & 0 \\ \mu_0 \pi n & D_2 & D_4 \\ 0 & -i\pi\alpha\mu_2 n & D_3 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

where

$$(4.41) \quad \begin{cases} D_1 = (\mu^* \pi^2 n^2 - \alpha^2 \rho), \\ D_2 = (\xi - \alpha^2 J) + \pi^2 \alpha_0 n^2, \\ D_3 = (bi\alpha + k - \alpha^2 b\tau) + k_2 \pi^2 n^2, \\ D_4 = (1 + i\alpha\tau)\mu_2 \pi n. \end{cases}$$

By solving (4.40), the solutions  $X$ ,  $Y$  and  $Z$  are as follow

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \frac{D_2 D_3 + i\pi n \alpha \mu_2 D_4}{-\pi^2 D_3 n^2 \mu_0^2 + i\pi \alpha \mu_2 D_1 D_4 n + D_1 D_2 D_3} \\ -\pi n \mu_0 \frac{D_3}{-\pi^2 D_3 n^2 \mu_0^2 + i\pi \alpha \mu_2 D_1 D_4 n + D_1 D_2 D_3} \\ -i\pi^2 n^2 \alpha \mu_0 \frac{\mu_2}{-\pi^2 D_3 n^2 \mu_0^2 + i\pi \alpha \mu_2 D_1 D_4 n + D_1 D_2 D_3} \end{pmatrix}.$$

We can write the solution  $X$  as

$$X = \frac{Q}{D_1 Q - \pi^2 n^2 \mu_0^2},$$

where

$$(4.42) \quad Q = D_2 + i\pi \alpha \mu_2 n \frac{D_4}{D_3}.$$

After that, for some selected  $r \in \mathbb{R}$ , we take  $\alpha$  such that  $D_1(\alpha) = r$ . Then, we have

$$(4.43) \quad -\rho\alpha^2 = r - \mu^* \pi^2 n^2,$$

by substituting  $D_3$  and  $D_4$  into (4.42) and using (4.43), we get

$$(4.44) \quad Q = D_2 + (\mu_2 \pi n)^2 \frac{-\alpha^2 \tau + i\alpha}{\pi^2 n^2 \zeta + k + \frac{br\tau}{\rho} + i\alpha b}.$$

We discuss two cases :

**The first case :** Let  $\zeta = 0$  and by choosing  $r = 0$ ,  $Q$  can be written as

$$Q = \xi + (\alpha_0 \rho - J\mu^*) \frac{\pi^2 n^2}{\rho} + (\mu_2 \pi n)^2 \frac{-\alpha^2 \tau + i\alpha}{k + i\alpha b},$$

and as  $\alpha = \alpha_n = \sqrt{\mu^*} \pi n$ . Since  $r = 0$ , then  $D_1 = 0$ . Therefore for  $\alpha_n$  large

$$\begin{aligned} X_n &= \frac{Q}{-\pi^2 n^2 \mu_0^2} \\ &= \alpha_n \frac{\mu_2^2}{\mu_0^2} \frac{\alpha_n \tau - i}{k + i\alpha_n b} - \frac{\xi}{\pi^2 n^2 \mu_0^2} + \frac{J\mu^* - \alpha_0 \rho}{\rho \mu_0^2} \\ &\approx c\alpha_n + c', \end{aligned}$$

where  $c = \frac{\tau \mu_2^2}{ib \mu_0^2}$  and  $c' = \frac{J\mu^* - \alpha_0 \rho}{\rho \mu_0^2}$ . Then, we have

$$\begin{aligned} \|U_n\|_{\mathbb{H}^1}^2 &\geq \rho \|v_n\|^2 = \rho \alpha_n^2 \|X_n\|^2 \int_0^1 (\sin(\pi n x))^2 dx \\ &= \frac{\rho \alpha_n^2 \|X_n\|^2}{2} \geq \frac{\rho}{2} \alpha_n^2 \|c\alpha_n + c'\|^2. \end{aligned}$$

This implies that

$$\lim_{n \rightarrow \infty} \|U_n\|_{\mathbb{H}^1} \rightarrow \infty.$$

**The second case:** Let  $\chi \neq 0$  and  $\zeta \neq 0$ . Then from (4.44) we get

$$\begin{aligned} Q &= D_2 - (\mu_2 \pi n)^2 \frac{\frac{\tau}{\rho} (\mu^* \pi^2 n^2 + r) - i\alpha_n}{\pi^2 n^2 \zeta + k + \frac{br\tau}{\rho} + i\alpha_n c_0} \\ &= D_2 - (\mu_2 \pi n)^2 \frac{\mu^* \tau}{\rho \zeta} - (\mu_2 \pi n)^2 \frac{\frac{\mu^* \tau}{\rho \zeta} (k + i\alpha_n c_0 + r \frac{k_2}{\mu^*}) - i\alpha_n}{\pi^2 n^2 \zeta + \frac{br\tau}{\rho} + k + i\alpha_n c_0}, \end{aligned}$$

where  $c_0$  is constant independent of  $n$ . Using (4.41), we have

$$Q = \xi + \frac{r}{\rho} J + \left( \alpha_0 - \frac{\mu^*}{\rho} J - \mu_2^2 \frac{\mu^* \tau}{\rho \zeta} \right) \pi^2 n^2 - (\mu_2 \pi n)^2 \frac{\frac{\mu^* \tau}{\rho \zeta} (k + i\alpha_n c_0 + r \frac{k_2}{\mu^*}) - i\alpha_n}{\pi^2 n^2 \zeta + \frac{br\tau}{\rho} + k + i\alpha_n c_0}.$$

We take  $r$  such that

$$(4.45) \quad \left( \alpha_0 - \frac{\mu^*}{\rho} J - \mu_2^2 \frac{\mu^* \tau}{\rho \zeta} \right) r = \mu_0^2 \Rightarrow r = \frac{\rho \zeta \mu_0^2}{\mu^* \alpha_0 \chi},$$

by utilizing (4.45), we obtain

$$(4.46) \quad \begin{aligned} Q &= \frac{\pi^2 n^2 \mu_0^2}{r} + \xi + \frac{r}{\rho} J - (\mu_2 \pi n)^2 \frac{\frac{\mu^* \tau}{\rho \zeta} \left( k + i \alpha_n c_0 + r \frac{k_2}{\mu^*} \right) - i \alpha_n}{\pi^2 n^2 \zeta + \frac{b r \tau}{\rho} + k + i \alpha_n c_0} \\ &\approx \frac{\pi^2 n^2 \mu_0^2}{r} + \xi + \frac{r}{\rho} J + \frac{\mu^* \tau}{\rho \zeta^2} k + r \frac{k_2 \tau}{\rho \zeta^2} + i \alpha_n c'_0. \end{aligned}$$

Now for the solution  $X_n$  we have

$$X_n = \frac{Q}{D_1 Q - \pi^2 n^2 \mu_0^2} = \frac{1}{D_1} \left( 1 + \frac{\pi^2 n^2 \mu_0^2}{D_1 Q - \pi^2 n^2 \mu_0^2} \right),$$

then using (4.46) and  $D_1 = r = \frac{\rho \zeta \mu_0^2}{\chi}$ , thus

$$\begin{aligned} X_n &= \frac{1}{r} \left( 1 + \frac{\mu_0^2}{\mu^*} \frac{r + \rho \alpha_n^2}{\xi + \frac{r}{\rho} J + \frac{\mu^* \tau}{\rho \zeta^2} k + r \frac{k_2 \tau}{\rho \zeta^2} + i \alpha_n c'_0} \right) \\ &= \frac{\mu^* \alpha_0 \chi}{\rho \zeta \mu_0^2} \left( 1 + \frac{\mu_0^2}{\mu^*} \frac{r + \rho \alpha_n^2}{\xi + \frac{r}{\rho} J + \frac{\mu^* \tau}{\rho \zeta^2} k + r \frac{k_2 \tau}{\rho \zeta^2} + i \alpha_n c'_0} \right). \end{aligned}$$

Therefore

$$\begin{aligned} \|U_n\|_{\mathbb{H}}^2 &\geq \rho \|v_n\|^2 = \rho \alpha_n^2 \|X_n\|^2 \int_0^1 (\sin(\pi n x))^2 dx \\ &= \frac{\rho \alpha_n^2 \|X_n\|^2}{2} \approx C \chi \alpha_n^2. \end{aligned}$$

This means that if  $\chi \neq 0$ , then

$$\lim_{n \rightarrow \infty} \|U_n\|_{\mathbb{H}} \rightarrow \infty.$$

As a result, our conclusion follows. We direct the reader to other related research for further information on [4, 27, 43]. ■

## 4.6 Polynomial decay

In this part, we investigate problem (4.4)-(4.6) that arises when

$$(4.47) \quad \chi = \mu_2^2 \tau - \left( \frac{\alpha_0 \mu^* b \tau}{\rho} - \alpha_0 k_2 \right) \left( \frac{J}{\alpha_0} - \frac{\rho}{\mu^*} \right) \neq 0$$

and provide a polynomial decay result.

We need to apply second order energy  $E_*(t)$  to get the intended outcome of this section, which is given by

$$(4.48) \quad E_*(t) = \frac{1}{2} \int_0^1 \left[ \rho u_{tt}^2 + \left( \mu^* - \frac{\mu_0^2}{\xi} \right) u_{xt}^2 + J \varphi_{tt}^2 + \left( \frac{\mu_0}{\sqrt{\xi}} u_{xt} + \sqrt{\xi} \varphi_t \right)^2 + a_0 \varphi_{xt}^2 + b (\tau W_{tt} + W_t)^2 + k_2 \tau W_{xt}^2 + k \tau W_t^2 \right] dx, \quad t > 0.$$

**Theorem 4.6.1.** *Let  $(u, \varphi, W)$  be a solution of problem (4.4)-(4.6) and assume that the coefficients of the system satisfy (4.47). Then  $E(t)$  decays polynomially, i.e., there exists a positive constant  $\lambda$  depending on the initial data, such that*

$$(4.49) \quad E(t) \leq \frac{\lambda}{t}, \quad t > 0.$$

This theorem requires an extra lemma to be proven.

**Lemma 4.6.2.** *Since  $(u, \varphi, W)$  is the solution of (4.4)-(4.6), the functional  $E_*(t)$  satisfies the following estimate*

$$(4.50) \quad E'_*(t) = -k_2 \int_0^1 W_{xt}^2 dx - k \int_0^1 W_t^2 dx \leq 0, \quad t > 0.$$

**Lemma 4.6.3.** *Given that  $(u, \varphi, W)$  is the solution of (4.4)-(4.6), the functional  $\Gamma_4(t)$  defined by (4.31) meets the following estimate*

$$(4.51) \quad \begin{aligned} \Gamma'_4(t) \leq & -\frac{\sigma}{2} \int_0^1 \varphi_x^2 dx - \frac{\sqrt{\xi}}{2} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx + \varepsilon_4 \int_0^1 u_t^2 dx + \frac{3\varepsilon_5}{2} \int_0^1 u_x^2 dx \\ & + \left( \frac{\mu_2 \tau \sqrt{\xi}}{2} + \sqrt{\xi} J \right) \int_0^1 \varphi_t^2 dx + C(\varepsilon_5) \int_0^1 W_x^2 dx + \frac{\sigma_1^2}{4\varepsilon_4} \int_0^1 W^2 dx + \frac{\mu_0^2 \chi^2}{2\varepsilon_5 \mu_2^2 \xi} \int_0^1 W_{tx}^2 dx \\ & + \frac{\sigma_3^2}{\sigma} \int_0^1 (\tau W_t + W)^2 dx, \quad t > 0. \end{aligned}$$

**Proof.** By doing the lemma 4.3.6 proof calculations again and accounting for (4.47), we obtain

$$(4.52) \quad \begin{aligned} \Gamma'_4(t) \leq & -\frac{\sigma}{2} \int_0^1 \varphi_x^2 dx - \frac{\sqrt{\xi}}{2} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx + \varepsilon_4 \int_0^1 u_t^2 dx + \varepsilon_5 \int_0^1 u_x^2 dx \\ & + \left( \frac{\mu_2 \tau \sqrt{\xi}}{2} + \sqrt{\xi} J \right) \int_0^1 \varphi_t^2 dx + C(\varepsilon_5) \int_0^1 W_x^2 dx + \frac{\sigma_1^2}{4\varepsilon_4} \int_0^1 W^2 dx + \frac{\sigma_3^2}{\sigma} \int_0^1 (\tau W_t + W)^2 dx \\ & - \frac{\mu_0}{\mu_2 \sqrt{\xi}} \chi \int_0^1 W_{tx} u_x dx. \end{aligned}$$

By applying Young inequality on  $-\frac{\mu_0}{\mu_2\sqrt{\xi}}\chi\int_0^1 W_{tx}u_x dx$ , we find

$$(4.53) \quad -\frac{\mu_0}{\mu_2\sqrt{\xi}}\chi\int_0^1 W_{tx}u_x dx \leq \frac{\varepsilon_5}{2}\int_0^1 u_x^2 dx + \frac{\mu_0^2\chi^2}{2\varepsilon_5\mu_2^2\xi}\int_0^1 W_{tx}^2 dx.$$

We conclude (4.51), when we replace (4.53) into (4.52).  $\blacksquare$

**Lemma 4.6.4.** *Given that  $(u, \varphi, W)$  is the solution of (4.4)-(4.6), the functional  $\Gamma_5(t)$  defined by (4.34) meets the following estimate*

$$(4.54) \quad \begin{aligned} \Gamma'_5(t) \leq & -\frac{\mu_0}{4}\int_0^1 u_x^2 dx + \tilde{C}\int_0^1 \varphi_x^2 dx + \frac{\sigma_4^2}{4\varepsilon_6}\int_0^1 W^2 dx + \varepsilon_6\int_0^1 u_t^2 dx + \frac{\sigma_5^2}{\mu_0}\int_0^1 W_x^2 dx \\ & + \frac{\sigma_6}{2}\int_0^1 (\tau W_t + W)^2 dx + \frac{\chi^2}{\mu_0\mu_2^2}\int_0^1 W_{tx}^2 dx, \quad t > 0, \end{aligned}$$

where  $\tilde{C} = \frac{a_0\mu_0}{\mu^*} + \frac{\xi^2}{\mu_0} + \frac{\sigma_6}{2}$ .

**Proof.** By following lemma 4.3.7 proof steps and accounting for (4.47), we get

$$(4.55) \quad \begin{aligned} \Gamma'_5(t) \leq & -\frac{\mu_0}{2}\int_0^1 u_x^2 dx + \left(\frac{a_0\mu_0}{\mu^*} + \frac{\xi^2}{\mu_0} + \frac{\sigma_6}{2}\right)\int_0^1 \varphi_x^2 dx + \frac{\sigma_4^2}{4\varepsilon_6}\int_0^1 W^2 dx + \varepsilon_6\int_0^1 u_t^2 dx \\ & + \frac{\sigma_5^2}{\mu_0}\int_0^1 W_x^2 dx + \frac{\sigma_6}{2}\int_0^1 (\tau W_t + W)^2 dx + \frac{\chi}{\mu_2}\int_0^1 W_{tx}u_x dx. \end{aligned}$$

By applying Young's inequality, we have

$$(4.56) \quad \frac{\chi}{\mu_2}\int_0^1 W_{tx}u_x dx \leq \frac{\mu_0}{4}\int_0^1 u_x^2 dx + \frac{\chi^2}{\mu_0\mu_2^2}\int_0^1 W_{tx}^2 dx.$$

The substitution of (4.56) into (4.55) yields (4.54).  $\blacksquare$

We define the Lyapunov functional  $\check{c}(t)$  by

$$(4.57) \quad \check{c}(t) = N[E(t) + E_*(t)] + \Gamma_1(t) + N_2\Gamma_2(t) + N_3\Gamma_3(t) + N_4\Gamma_4(t) + N_5\Gamma_5(t), \quad t > 0,$$

where  $N, N_2, N_3, N_4$  and  $N_5 > 0$ . Additionally, it is simple to establish that two positive constants  $\bar{\eta}_1$  and  $\bar{\eta}_2$  exist such that

$$(4.58) \quad \bar{\eta}_1[E(t) + E_*(t)] \leq \check{c}'(t) \leq \bar{\eta}_2[E(t) + E_*(t)], \quad t > 0.$$

Now, by recalling (4.18), (4.50), (4.20), (4.25), (4.29), (4.51) and (4.54), we obtain

$$\begin{aligned} \check{c}'(t) \leq & -\left[k_2N - \frac{k_2^2J}{\mu_2}N_2 - \left(\tau k_2 + \frac{\tau^2\mu_2^2}{4\varepsilon_3}\right)N_3 - C(\varepsilon_5)N_4 - \frac{\sigma_5^2}{\mu_0}N_5\right]\int_0^1 W_x^2 dx \\ & -\left[kN - \frac{k^2J}{\mu_2}N_2 - (b + k\tau)N_3 - \frac{\sigma_1^2}{4\varepsilon_4}N_4 - \frac{\sigma_4^2}{4\varepsilon_6}N_5\right]\int_0^1 W^2 dx - Nk\int_0^1 W_t^2 dx \\ & -\left[k_2N - \frac{\mu_0^2\chi^2}{2\varepsilon_5\mu_2^2\xi}N_4 - \frac{\chi^2}{\mu_0\mu_2^2}N_5\right]\int_0^1 W_{xt}^2 dx - [\rho - \varepsilon_4N_4 - \varepsilon_6N_5]\int_0^1 u_t^2 dx \\ & -\left[\frac{\mu_2J}{2}N_2 - \varepsilon_3N_3 - \left(\frac{\mu_2\tau\sqrt{\xi}}{2} + \sqrt{\xi}J\right)N_4\right]\int_0^1 \varphi_t^2 dx \end{aligned}$$

$$\begin{aligned}
 & - \left[ \frac{b}{2} N_3 - C(\varepsilon_1, \varepsilon_2) N_2 - \frac{\sigma_3^2}{\sigma} N_4 - \frac{\sigma_6}{2} N_5 \right] \int_0^1 (\tau W_t + W)^2 dx \\
 & - \left[ \frac{\sigma}{2} N_4 - \varepsilon_2 N_2 - \tilde{C} N_5 \right] \int_0^1 \varphi_x^2 dx - \left[ \frac{\sqrt{\xi}}{2} N_4 - \frac{\mu_0}{2\sqrt{\xi}} \right] \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx \\
 (4.59) \quad & - \left[ \frac{\mu_0}{4} N_5 - \frac{\mu_0}{2\sqrt{\xi}} - \varepsilon_1 N_2 - \frac{3\varepsilon_5}{2} N_4 \right] \int_0^1 u_x^2 dx,
 \end{aligned}$$

a similar selection of  $N_2$ ,  $\varepsilon_1$ ,  $\varepsilon_2$ ,  $\varepsilon_3$ ,  $\varepsilon_4$  and  $\varepsilon_6$  as in the proof of theorem 4.3.1, where  $\varepsilon_1 = \frac{\mu_0}{8\sqrt{\xi}N_2}$ ,  $\varepsilon_2 = \frac{\tilde{C}}{\sqrt{\xi}N_2}$ ,  $\varepsilon_3 = \frac{1}{N_3}$ ,  $\varepsilon_4 = \frac{\rho}{4N_4}$  and  $\varepsilon_6 = \frac{\rho}{4N_5}$ , besides by setting  $N_5 = \frac{4}{\sqrt{\xi}}$  and  $\varepsilon_5 = \frac{\mu_0}{12\sqrt{\xi}N_4}$ , (4.59) will be

$$\begin{aligned}
 \check{c}'(t) \leq & - \left[ k_2 N - \frac{k_2^2 J}{\mu_2} N_2 - \left( \tau k_2 + \frac{\tau^2 \mu_2^2 N_3}{4} \right) N_3 - \left( \frac{\mu_2 \tau \sqrt{\xi}}{2} + \frac{\mu_2^2}{2\sqrt{\xi}} + \frac{3\sqrt{\xi} \sigma_2^2 N_4}{\mu_0} \right) N_4 \right. \\
 & \left. - \frac{4\sigma_5^2}{\sqrt{\xi} \mu_0} \right] \int_0^1 W_x^2 dx - \left[ kN - \frac{k^2 J}{\mu_2} N_2 - (b + k\tau) N_3 - \frac{\sigma_1^2}{\rho} N_4^2 - \frac{16\sigma_4^2}{\rho\xi} \right] \int_0^1 W^2 dx \\
 & - \left[ \frac{\mu_2 J}{2} N_2 - \left( \frac{\mu_2 \tau \sqrt{\xi}}{2} + \sqrt{\xi} J \right) N_4 - 1 \right] \int_0^1 \varphi_t^2 dx - \frac{\mu_0}{4\sqrt{\xi}} \int_0^1 u_x^2 dx - \frac{\rho}{2} \int_0^1 u_t^2 dx \\
 & - \left[ \frac{b}{2} N_3 - \sqrt{\xi} b^2 \left( \frac{\alpha_0^2 + \xi^2}{2\tilde{C}} + \frac{\mu_2}{\sqrt{\xi} b N_2} + 2\mu_0 \right) N_2^2 - \frac{\sigma_3^2}{\sigma} N_4 - \frac{2\sigma_6}{\sqrt{\xi}} \right] \int_0^1 (\tau W_t + W)^2 dx \\
 (4.60) \quad & - \left[ \frac{\sigma}{2} N_4 - \frac{5\tilde{C}}{\sqrt{\xi}} \right] \int_0^1 \varphi_x^2 dx - \left[ \frac{\sqrt{\xi}}{2} N_4 - \frac{\mu_0}{2\sqrt{\xi}} \right] \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx,
 \end{aligned}$$

much farther, we select  $N_4$  such that

$$N_4 > \max \left( \frac{10\tilde{C}}{\sigma\sqrt{\xi}}, \frac{\mu_0}{\xi} \right),$$

and we choose  $N_3$  so that

$$\frac{b}{2} N_3 - \sqrt{\xi} b^2 \left( \frac{\alpha_0^2 + \xi^2}{2\tilde{C}} + \frac{\mu_2}{\sqrt{\xi} b N_2} + 2\mu_0 \right) N_2^2 - \frac{\sigma_3^2}{\sigma} N_4 - \frac{2\sigma_6}{\sqrt{\xi}} > 0.$$

Finally,  $N$  must be selected such that

$$\begin{aligned}
 & kN - \frac{k^2 J}{\mu_2} N_2 - (b + k\tau) N_3 - \frac{\sigma_1^2}{\rho} N_4^2 - \frac{16\sigma_4^2}{\rho\xi} > 0, \\
 & k_2 N - \frac{k_2^2 J}{\mu_2} N_2 - \left( \tau k_2 + \frac{\tau^2 \mu_2^2 N_3}{4} \right) N_3 - \left( \frac{\mu_2 \tau \sqrt{\xi}}{2} + \frac{\mu_2^2}{2\sqrt{\xi}} + \frac{3\sqrt{\xi} \sigma_2^2 N_4}{\mu_0} \right) N_4 - \frac{4\sigma_5^2}{\sqrt{\xi} \mu_0} > 0.
 \end{aligned}$$

Although, (4.60) leads, for  $\bar{\lambda}$  a positive constant, to

$$(4.61) \quad \check{c}'(t) \leq -\bar{\lambda} E(t), \quad \forall t \geq 0.$$

Once we integrate (4.61) over  $(0, t)$ , we get

$${}^t E(t) \leq \int_0^t E(s) ds \leq \frac{1}{\bar{\lambda}} (\check{c}(0) - \check{c}(t)), \quad \forall t \geq 0.$$

The preceding inequality has the following form while keeping relation (4.58) in mind

$$E(t) \leq \frac{\check{c}(0)}{\bar{\lambda}t} \leq \frac{\bar{\eta}_2(E(0) + E_*(0))}{\bar{\lambda}t},$$

by posing  $\lambda = \frac{\bar{\eta}_2(E(0) + E_*(0))}{\bar{\lambda}}$ , we conclude (4.49).

## WELL POSEDNESS AND STABILITY RESULT OF LORD-SHULMAN SYSTEM WITH MICROTHERMATURE EFFECTS: THE CASE $\xi\mu^* = \mu_0^2$

### 5.1 Introduction

The most well-known law for modeling heat conduction is the Fourier law. This law of thermoelasticity, based on a parabolic-type heat-conduction equation, is characteristic of the paradoxical behavior of thermoelastic waves with discontinuities and infinite velocities as a function of frequency. Precisely,

$$(5.1) \quad q = -\kappa\theta_x,$$

where  $\kappa > 0$  is the thermal conductivity which depends on the properties of the material. However, this law may not be applicable in thermal non-equilibrium and/or non-local situations. Thus, several attempts have been proposed to overcome this physical paradox such as Cattaneo's law [17] in which the heat is assumed to be transported by a wave propagation process rather than diffusion,

$$(5.2) \quad \tau_0 q_t + q = -\kappa\theta_x,$$

where the parameter  $\tau_0$  is the relaxation time called single-phase-lag. Other modifications of Fourier's law has been proposed by Tzou [73] which take into account the acceleration of the heat flow:

$$(5.3) \quad q + \tau_q q_t + \frac{\tau_q^2}{2} q_{tt} = -\kappa_1 (\tau_\theta \theta_{tx} + \theta_x),$$

here  $\tau_q$  and  $\tau_\theta$  are the thermal phase lags of heat flux vector and temperature gradient, respectively, and by Choudhuri [20] which consider the three-phase-lag thermoelastic model. Directly

from (5.3), we can easily derive the Fourier's law (5.1) for the special case  $\tau_q = \tau_\theta = 0$  and by neglecting the terms  $O(\tau_q^2)$ , we find the Cattaneo law (5.2) in the case  $\tau_\theta = 0$ .

Note that, the well-known Lord-Shulman thermoelasticity has attracted a lot of attention from scientists in recent years, and a considerable number of results have been obtained see [25, 33] and the references therein. This study builds on linear thermoelasticity with finite wave speeds, which deals with the paradox of infinite propagation speeds of thermal signals caused by the Fourier model of heat conduction. For more results in the case of the thermoelasticity with microtemperature see [22, 32, 36, 62].

The basic evolution equations for one-dimensional theories of porous materials with microtemperatures in the case of the relaxation time are:

$$(5.4) \quad \rho u_{tt} = S_x, \quad J\varphi_{tt} = H_x + G, \quad \rho E_t = P_x + q - Q,$$

where  $\rho$  is the mass density,  $J$  is the equilibrated inertia,  $S$  is the stress tensor,  $q$  is the heat flux vector,  $P$  is the first heat flux moment,  $H$  is the equilibrated stress,  $G$  is the equilibrated body force,  $Q$  is the mean heat flux and  $E$  is the first moment of energy. The variables  $u$  and  $\varphi$  are respectively, the displacement of the solid elastic material and the volume fraction. The constitutive equations are:

$$(5.5) \quad \begin{aligned} S &= \mu^* u_x + \mu_0 \varphi, & H &= a_0 \varphi_x - \mu_2 (\tau T_t + T), \\ G &= -\mu_0 u_x - \xi \varphi, & \rho E &= -b (\tau T_t + T) - \mu_2 \varphi_x, \\ Q &= k_3 T, \quad q = k_1 T, & P &= -k_2 T_x, \end{aligned}$$

where  $T$  denotes the microtemperature vector,  $\tau > 0$  is the relaxation parameter and the rest of the parameters, namely  $\xi$  and  $\mu_2$  denote a porosity coefficients,  $a_0$  is the porosity diffusion coefficient and the other parameters  $b$ ,  $k_1$ ,  $k_2$  and  $k_3$  are usual in the study of the present theory. Among the precedent coefficients, that which satisfy the following inequalities

$$k = k_1 - k_3 > 0,$$

with  $\mu^* = \lambda + 2\mu$  and  $\lambda$ ,  $\mu$  are the Lamé's parameters,  $\mu_0$  is the coupling between the displacement and the porosity which must be different from zero and satisfies the relation

$$(5.6) \quad \mu^* > \frac{\mu_0^2}{\xi}.$$

Now, by substituting Eq. (5.5) into Eq. (5.4), we obtain the following system

$$(5.7) \quad \begin{cases} \rho u_{tt} = \mu^* u_{xx} + \mu_0 \varphi_x, & \text{in } (0, 1) \times (0, \infty), \\ J\varphi_{tt} = a_0 \varphi_{xx} - \mu_2 (\tau T_{tx} + T_x) - \mu_0 u_x - \xi \varphi, & \text{in } (0, 1) \times (0, \infty), \\ b (\tau T_t + T)_t = k_2 T_{xx} - \mu_2 \varphi_{tx} - k T, & \text{in } (0, 1) \times (0, \infty). \end{cases}$$

In order to make system (5.7) well determined, we impose the following initial and boundary conditions:

$$(5.8) \quad \begin{aligned} u(0, t) = u(1, t) = T(0, t) = T(1, t) = 0, & \quad t > 0, \\ \varphi_x(0, t) = \varphi_x(1, t) = 0, & \quad t > 0, \end{aligned}$$

and

$$(5.9) \quad \begin{aligned} u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad \varphi_t(x, 0) = \varphi^1(x), & \quad x \in (0, 1), \\ \varphi(x, 0) = \varphi^0(x), \quad T(x, 0) = T^0(x), \quad T_t(x, 0) = T^1(x), & \quad x \in (0, 1), \end{aligned}$$

here  $u^0, u^1, \varphi^0, \varphi^1, T^0$  and  $T^1$  are given functions.

In [4], Apalara investigated system (5.7) for the case  $\tau = 0$ , he showed that the unique dissipation due to the microtemperatures is strong enough to exponentially stabilize the system if and only if the wave speeds of the system are equal

$$(5.10) \quad \chi_0 = \frac{\mu^*}{\rho} - \frac{\alpha_0}{J} = 0.$$

Additionally, if (5.10) does not hold, he proved that the system is polynomially stable. In [8], Bazarra et al. generalized thermoelasticity theory acting on a poroelastic problem with temperature and microtemperature and established the exponential decay using the method developed by Liu and Zheng [48]. Recently, Ramos et al. [61] considered system (5.7) under the condition (5.6) and by using the method based on the Gearhart–Herbst–Prüss–Huang theorem to dissipative systems [37, 54], they seeked a necessary and sufficient condition to obtain the exponential or polynomial stability of the system.

In this work, we aim to complement the work of [61] by giving a necessary condition on the coefficients to drive the system to the equilibrium state in an exponential manner. More precisely, if the coefficients satisfy the following condition

$$(5.11) \quad \chi = 0,$$

where

$$(5.12) \quad \chi = \mu_2^2 \tau - \left( \frac{\alpha_0 \mu^* b \tau}{\rho} - \alpha_0 k_2 \right) \left( \frac{J}{\alpha_0} - \frac{\rho}{\mu^*} \right),$$

and under

$$(5.13) \quad \xi \mu^* = \mu_0^2.$$

To the best of our knowledge, Lord–Shulman with porosity and microtemperature but without temperature with coefficients satisfying (5.13) have not been discussed before.

Meanwhile, from the second equation of (5.7) and the boundary conditions (5.8), we get

$$(5.14) \quad \frac{d^2}{dt^2} \int_0^1 \varphi(x, t) dx + \frac{\xi}{J} \int_0^1 \varphi(x, t) dx = 0, \quad \forall t \geq 0.$$

So, by solving (5.14) and using the initial data of  $\varphi$ , we arrive at

$$\int_0^1 \varphi(x, t) dx = \left( \int_0^1 \varphi^0(x) dx \right) \cos \left( \sqrt{\frac{\xi}{J}} t \right) + \sqrt{\frac{J}{\xi}} \left( \int_0^1 \varphi^1(x) dx \right) \sin \left( \sqrt{\frac{\xi}{J}} t \right).$$

Consequently, if we set

$$\bar{\varphi}(x, t) = \varphi(x, t) - \int_0^1 \varphi(x, t) dx, \quad t \geq 0, \quad x \in [0, 1],$$

we find

$$\int_0^1 \bar{\varphi}(x, t) dx = 0, \quad t \geq 0.$$

$(u, \bar{\varphi}, \theta)$  satisfies the same equations in (5.7)-(5.9) with

$$\begin{aligned} \bar{\varphi}(x, 0) &= \bar{\varphi}^0(x) = \varphi^0(x) - \int_0^1 \varphi^0(x) dx, \\ \bar{\varphi}_t(x, 0) &= \bar{\varphi}^1(x) = \varphi^1(x) - \int_0^1 \varphi^1(x) dx. \end{aligned}$$

In what follows we will work with  $\bar{\varphi}$  but, for convenience, we write  $\varphi$  instead of  $\bar{\varphi}$ .

## 5.2 Well posedness

In this section, we will use the semigroup theory to transform our problem into a Cauchy problem in the proper Hilbert space. First, we define the vector function  $U = (u, \Psi, \varphi, \Xi, T, \Theta)^T$  where  $\Psi = u_t$ ,  $\Xi = \varphi_t$  and  $\Theta = T_t$ , leading to the following rewriting of system (5.7)-(5.9)

$$(5.15) \quad \begin{cases} U_t = \mathbb{A}U, & t > 0, \\ U(0) = U_0 = (u_0, u_1, \varphi_0, \varphi_1, T_0, T_1)^T, \end{cases}$$

where the definition of the operator  $\mathbb{A} : \mathbb{D}(\mathbb{A}) \subset \mathbb{H} \rightarrow \mathbb{H}$  is

$$\mathbb{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ \frac{\mu^*}{\rho} D^2 & 0 & \frac{\mu_0}{\rho} D & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ -\frac{\mu_0}{J} D & 0 & \frac{\alpha_0}{J} D^2 - \frac{\xi}{J} I & 0 & -\frac{\mu_2}{J} D & -\frac{\mu_2 \tau}{J} D \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & -\frac{\mu_2}{b\tau} D & \frac{k_2}{b\tau} D^2 - \frac{k}{b\tau} I & -\frac{1}{\tau} I \end{pmatrix},$$

where  $D^i = \frac{\partial^i}{\partial x^i}$  and we denote by  $I$  the identity operator. The domain of  $\mathbb{A}$  is given by:

$$(5.16) \quad \mathbb{D}(\mathbb{A}) = \left( \begin{array}{l} U = (u, \Psi, \varphi, \Xi, T, \Theta)^T \in \mathbb{H} / u, T \in H^2(0, 1) \cap H_0^1(0, 1); \\ \varphi \in H_*^2(0, 1) \cap H_*^1(0, 1); \Psi, \Theta \in H_0^1(0, 1); \Xi \in H_*^1(0, 1) \end{array} \right).$$

Let

$$\mathbb{H} = H_0^1(0, 1) \times L^2(0, 1) \times H_*^1(0, 1) \times L_*^2(0, 1) \times H_0^1(0, 1) \times L^2(0, 1)$$

be the Hilbert space, where

$$(5.17) \quad \begin{aligned} L_*^2(0, 1) &= \left\{ \Psi \in L^2(0, 1), \int_0^1 \Psi(x) dx = 0 \right\}, \\ H_*^1(0, 1) &= H^1(0, 1) \cap L_*^2(0, 1), \\ H_*^2(0, 1) &= \left\{ \Psi \in H^2(0, 1), \Psi_x(0) = \Psi_x(1) = 0 \right\}, \end{aligned}$$

with the following inner product

$$(5.18) \quad \begin{aligned} \langle U, \tilde{U} \rangle &= \rho \int_0^1 \Psi \tilde{\Psi} dx + \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) \left( \frac{\mu_0}{\sqrt{\xi}} \tilde{u}_x + \sqrt{\xi} \tilde{\varphi} \right) dx \\ &+ a_0 \int_0^1 \varphi_x \tilde{\varphi}_x dx + J \int_0^1 \Xi \tilde{\Xi} dx + k_2 \tau \int_0^1 T_x \tilde{T}_x dx + k \tau \int_0^1 T \tilde{T} dx \\ &+ b \int_0^1 (\tau \Theta + T)(\tau \tilde{\Theta} + \tilde{T}) dx, \end{aligned}$$

with  $\xi \mu^* = \mu_0^2$  and for all  $U = (u, \Psi, \varphi, \Xi, T, \Theta)^T \in \mathbb{H}$  and  $\tilde{U} = (\tilde{u}, \tilde{\Psi}, \tilde{\varphi}, \tilde{\Xi}, \tilde{T}, \tilde{\Theta})^T \in \mathbb{H}$ . The corresponding norm in  $\mathbb{H}$  is

$$(5.19) \quad \|U\|_{\mathbb{H}} = \int_0^1 \left[ \rho \Psi^2 + \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 + a_0 \varphi_x^2 + J \Xi^2 + k_2 \tau T_x^2 + k \tau T^2 + b(\tau \Theta + T)^2 \right] dx.$$

The well-posedness of problem (5.7)-(5.9) is given in the following theorem.

**Theorem 5.2.1.** *Let  $U_0 \in \mathbb{H}$ , then there exists a unique solution  $U \in C(\mathbb{R}_+, \mathbb{H})$  to Problem (5.7)-(5.9). Moreover, if  $U_0 \in \mathbb{D}(\mathbb{A})$ , then the solution  $U \in C(\mathbb{R}_+, \mathbb{D}(\mathbb{A})) \cap C^1(\mathbb{R}_+, \mathbb{H})$ .*

**Proof.** First, we would prove that  $\mathbb{A}$  is a maximal dissipatif operator. So, by using (5.15) and the boundary conditions, we have for  $U \in \mathbb{D}(\mathbb{A})$ :

$$\langle \mathbb{A}U, U \rangle_{\mathbb{H}} = -k_2 \int_0^1 T_x^2 dx - k \int_0^1 T^2 dx \leq 0.$$

Next, to prove that the operator  $(I - \mathbb{A})$  is surjective, we follow the same steps as in [61]. Consequently, using the Lumer–Phillips theorem [53], it follows that  $\mathbb{A}$  generates a  $C_0$ -semigroup of contractions on  $\mathbb{H}$ . So, the conclusion of theorem 5.1 follows.  $\blacksquare$

### 5.3 Exponential stability

In this part, we demonstrate the exponential stability result for system (5.7)-(5.9), by using the energy technique where we introduce a stability number  $\chi$ . We begin by establishing our system's energy as follows

$$(5.20) \quad E(t) = \frac{1}{2} \int_0^1 \left[ \rho u_t^2 + J \varphi_t^2 + a_0 \varphi_x^2 + b(\tau T_t + T)^2 + k_2 \tau T_x^2 + k \tau T^2 + \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 \right] dx.$$

The main result of this section is given by the following theorem.

**Theorem 5.3.1.** *Let  $(u, \varphi, T)$  be a solution of problem (5.7)-(5.9) and assume that the coefficients of the system satisfy (5.11) and (5.13). Then the energy  $E(t)$  decays exponentially, i.e., there exist two positive constants  $\alpha$  and  $\gamma$ , where  $\alpha$  is dependent of the initial data such that*

$$(5.21) \quad E(t) \leq \alpha e^{-\gamma t}, \quad \forall t \geq 0.$$

Many lemmas will be used to establish the proof of this theorem.

**Lemma 5.3.2.** *Let  $(u, \varphi, T)$  be a solution of (5.7)-(5.9). Then, the energy functional  $E(t)$ , defined by (5.20) satisfies*

$$(5.22) \quad E'(t) = -k_2 \int_0^1 T_x^2 dx - k \int_0^1 T^2 dx \leq 0, \quad t > 0.$$

**Proof.** Let's multiply the three equations of system (5.7), respectively by  $u_t$ ,  $\varphi_t$  and  $(\tau T_t + T)$ . Then, integrating by parts over  $(0, 1)$ , we get

$$\begin{aligned} \frac{\rho}{2} \frac{d}{dt} \int_0^1 u_t^2 dx + \frac{\mu^*}{2} \frac{d}{dt} \int_0^1 u_x^2 dx &= -\mu_0 \int_0^1 \varphi u_{tx} dx, \\ \frac{1}{2} \frac{d}{dt} \int_0^1 (J\varphi_t^2 + a_0\varphi_x^2 + \xi\varphi^2) dx &= \mu_2 \int_0^1 (\tau T_t + T) \varphi_{tx} dx - \mu_0 \int_0^1 u_x \varphi_t dx, \\ \frac{1}{2} \frac{d}{dt} \int_0^1 [b(\tau T_t + T)^2 + k_2\tau T_x^2 + k\tau T^2] dx &= -k_2 \int_0^1 T_x^2 dx - k \int_0^1 T^2 dx - \mu_2 \int_0^1 \varphi_{tx} (\tau T_t + T) dx, \end{aligned}$$

by summing the above equations, we obtain

$$(5.23) \quad \begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_0^1 [\rho u_t^2 + \mu^* u_x^2 + J\varphi_t^2 + a_0\varphi_x^2 + \xi\varphi^2 + 2\mu_0\varphi u_x + b(\tau T_t + T)^2 + k_2\tau T_x^2 + k\tau T^2] dx \\ &= -k \int_0^1 T^2 dx - k_2 \int_0^1 T_x^2 dx. \end{aligned}$$

Condition (5.13) clearly shows that

$$(5.24) \quad \mu^* \int_0^1 u_x^2 dx + \xi \int_0^1 \varphi^2 dx + 2\mu_0 \int_0^1 u_x \varphi dx = \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx.$$

Now, the relation (5.22) is proved by substituting (5.24) into (5.23). ■

**Lemma 5.3.3.** *The functional  $\Gamma_1$  defined by*

$$(5.25) \quad \Gamma_1(t) = -\rho \int_0^1 u_t u dx, \quad t > 0,$$

satisfies

$$(5.26) \quad \Gamma_1'(t) \leq -\rho \int_0^1 u_t^2 dx + \frac{5}{2} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx + \frac{7\xi}{2} \int_0^1 \varphi_x^2 dx, \quad t > 0.$$

**Proof.** By differentiating  $\Gamma_1$  and integrating by parts over  $(0, 1)$ , we obtain

$$(5.27) \quad \Gamma_1'(t) = \mu^* \int_0^1 u_x^2 dx - \mu_0 \int_0^1 \varphi_x u dx - \rho \int_0^1 u_t^2 dx.$$

Recalling (5.13) and using Poincaré's inequality, we arrive at

$$(5.28) \quad \mu^* \int_0^1 u_x^2 dx = \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x \right)^2 dx \leq 2 \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx + 2\xi \int_0^1 \varphi_x^2 dx,$$

in the same way, after using the inequality of Young

$$(5.29) \quad \begin{aligned} |\mu_0 \int_0^1 \varphi_x u dx| &\leq \frac{1}{4} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x \right)^2 dx + \xi \int_0^1 \varphi_x^2 dx \\ &\leq \frac{1}{2} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx + \frac{3\xi}{2} \int_0^1 \varphi_x^2 dx. \end{aligned}$$

The substitution of (5.28) and (5.29) into (5.27), yields (5.26). ■

**Lemma 5.3.4.** *The functional  $\Gamma_2$  defined by*

$$(5.30) \quad \Gamma_2(t) = -b \int_0^1 \tau^2 T_t T dx - \frac{b\tau}{2} \int_0^1 T^2 dx, \quad t > 0,$$

satisfies, for any  $\varepsilon_1 > 0$ ,

$$(5.31) \quad \Gamma_2'(t) \leq \frac{-b}{2} \int_0^1 (\tau T_t + T)^2 dx + (b + k\tau) \int_0^1 T^2 dx + \varepsilon_1 \int_0^1 \varphi_t^2 dx + \left( \tau k_2 + \frac{\tau^2 \mu_2^2}{4\varepsilon_1} \right) \int_0^1 T_x^2 dx, \quad t > 0.$$

**Proof.** A simple differentiation of  $\Gamma_2$ , using (5.7)<sub>3</sub> and then integrating by parts, we get

$$(5.32) \quad \Gamma_2'(t) = -b \int_0^1 (\tau T_t)^2 dx + \tau k_2 \int_0^1 T_x^2 dx - \tau \mu_2 \int_0^1 \varphi_t T_x dx + k\tau \int_0^1 T^2 dx.$$

Applying Young's inequality

$$(5.33) \quad -\tau \mu_2 \int_0^1 \varphi_t T_x dx \leq \varepsilon_1 \int_0^1 \varphi_t^2 dx + \frac{\tau^2 \mu_2^2}{4\varepsilon_1} \int_0^1 T_x^2 dx,$$

and using the fact that

$$(5.34) \quad -b \int_0^1 (\tau T_t)^2 dx \leq \frac{-b}{2} \int_0^1 (\tau T_t + T)^2 dx + b \int_0^1 T^2 dx,$$

we obtain (5.31). ■

**Lemma 5.3.5.** *The functional  $\Gamma_3$  defined by*

$$(5.35) \quad \Gamma_3(t) = \frac{1}{\sqrt{\xi}} \left[ J \int_0^1 \varphi_t \varphi dx - \frac{\rho\mu_0}{\mu^*} \int_0^1 u_t \left( \int_0^x \varphi dy \right) dx \right], \quad t > 0,$$

satisfies the following estimate, for any  $\varepsilon_2 > 0$ ,

$$(5.36) \quad \begin{aligned} \Gamma_3'(t) \leq & -\frac{\alpha_0}{2\sqrt{\xi}} \int_0^1 \varphi_x^2 dx + \frac{\mu_2^2}{2\sqrt{\xi}\alpha_0} \int_0^1 (\tau T_t + T)^2 dx + \left( \frac{J}{\sqrt{\xi}} + \frac{\rho^2}{4\varepsilon_2\mu^*} \right) \int_0^1 \varphi_t^2 dx \\ & + \varepsilon_2 \int_0^1 u_t^2 dx, \quad t > 0. \end{aligned}$$

**Proof.** Direct computation, using equation (5.7)<sub>1</sub> and (5.7)<sub>2</sub>, we find

$$\begin{aligned} \Gamma_3'(t) = & \frac{1}{\sqrt{\xi}} \int_0^1 [a_0 \varphi_{xx} - \mu_2 (\tau T_{tx} + T_x) - \mu_0 u_x - \xi \varphi] \varphi dx + \frac{J}{\sqrt{\xi}} \int_0^1 \varphi_t^2 dx \\ & - \frac{\mu_0}{\sqrt{\xi}\mu^*} \int_0^1 (\mu^* u_{xx} + \mu_0 \varphi_x) \left( \int_0^x \varphi dy \right) dx - \frac{\rho\mu_0}{\sqrt{\xi}\mu^*} \int_0^1 u_t \left( \int_0^x \varphi_t dy \right) dx. \end{aligned}$$

Thus,

$$\begin{aligned} \Gamma_3'(t) = & \frac{\alpha_0}{\sqrt{\xi}} \int_0^1 \varphi_{xx} \varphi dx - \frac{\mu_2}{\sqrt{\xi}} \int_0^1 (\tau T_{tx} + T_x) \varphi dx + \frac{J}{\sqrt{\xi}} \int_0^1 \varphi_t^2 dx - \frac{\rho\mu_0}{\sqrt{\xi}\mu^*} \int_0^1 u_t \left( \int_0^x \varphi_t dy \right) dx \\ & - \frac{1}{\sqrt{\xi}} \int_0^1 (\mu_0 u_x + \xi \varphi) \varphi dx - \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \frac{\mu_0^2}{\sqrt{\xi}\mu^*} \varphi \right) \left( \int_0^x \varphi dy \right) dx, \end{aligned}$$

by integrating by parts over  $(0, 1)$ , using the fact that  $\xi = \frac{\mu_0^2}{\mu^*}$  and  $\int_0^1 \varphi dx = 0$ , we arrive at

$$(5.37) \quad \begin{aligned} \Gamma_3'(t) = & -\frac{\alpha_0}{\sqrt{\xi}} \int_0^1 \varphi_x^2 dx + \frac{\mu_2}{\sqrt{\xi}} \int_0^1 (\tau T_t + T) \varphi_x dx - \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) \varphi dx \\ & + \frac{J}{\sqrt{\xi}} \int_0^1 \varphi_t^2 dx + \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) \varphi dx - \frac{\rho\mu_0}{\sqrt{\xi}\mu^*} \int_0^1 u_t \left( \int_0^x \varphi_t dy \right) dx. \end{aligned}$$

Young's and Cauchy Schwarz's inequalities, lead

$$\begin{aligned} \frac{\mu_2}{\sqrt{\xi}} \int_0^1 (\tau T_t + T) \varphi_x dx & \leq \frac{\alpha_0}{2\sqrt{\xi}} \int_0^1 \varphi_x^2 dx + \frac{\mu_2^2}{2\sqrt{\xi}\alpha_0} \int_0^1 (\tau T_t + T)^2 dx, \\ -\frac{\rho\mu_0}{\sqrt{\xi}\mu^*} \int_0^1 u_t \left( \int_0^x \varphi_t dy \right) dx & \leq \varepsilon_2 \int_0^1 u_t^2 dx + \frac{\rho^2}{4\varepsilon_2\mu^*} \int_0^1 \varphi_t^2 dx. \end{aligned}$$

By inserting the previous inequalities into (5.37), we conclude (5.36). ■

**Lemma 5.3.6.** *The functional  $\Gamma_4$  defined by*

$$(5.38) \quad \begin{aligned} \Gamma_4(t) = & J \int_0^1 \varphi_t \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) dx - \frac{a_0 b \mu_0}{\mu_2 \sqrt{\xi}} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 (\tau T_t + T) u_t dx \\ & + \frac{\mu_0 a_0 \rho}{\mu^* \sqrt{\xi}} \int_0^1 u_t \varphi_x dx - \frac{k_2 \mu_0 a_0}{\sqrt{\xi} \mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 T_x u_x dx - \mu_2 \tau \sqrt{\xi} \int_0^1 T \varphi_x dx, \quad t > 0, \end{aligned}$$

satisfies, for any  $\varepsilon_3, \varepsilon_4 > 0$ ,

$$(5.39) \quad \begin{aligned} \Gamma'_4(t) \leq & -\frac{\sqrt{\xi}}{2} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx + \left( \frac{\sqrt{\xi}^3}{4} + \frac{\sigma^2 a_0^2 b^2 \mu^{*2} \xi}{4 \varepsilon_4 \rho^2 \mu_2^2} \right) \int_0^1 \varphi_x^2 dx \\ & + \left( \frac{\mu_2^2 \tau^2 \sqrt{\xi}}{4J} + \frac{\mu_2^2}{\sqrt{\xi}} + \frac{2\sigma^2 \mu^{*2} a_0^2 b^2}{\rho^2 \mu_2^2 \sqrt{\xi}} \right) \int_0^1 T_x^2 dx + \frac{\sigma^2 k^2 a_0^2 \mu^*}{4 \varepsilon_3 \mu_2^2} \int_0^1 T^2 dx + \varepsilon_3 \int_0^1 u_t^2 dx \\ & + 2 \sqrt{\xi} J \int_0^1 \varphi_t^2 dx + \varepsilon_4 \int_0^1 (\tau T_t + T)^2 dx - \frac{\mu_0}{\mu_2 \sqrt{\xi}} \chi \int_0^1 T_{tx} u_x dx, \quad t > 0. \end{aligned}$$

**Proof.** Differentiating functional (5.38), using (5.7)<sub>1</sub>, (5.7)<sub>2</sub> and (5.7)<sub>3</sub>. Then, integrating by parts, we obtain

$$\begin{aligned} \Gamma'_4(t) = & -\frac{a_0}{\sqrt{\xi}} \left( \xi - \frac{\mu_0^2}{\mu^*} \right) \int_0^1 \varphi_x^2 dx - \frac{\mu_0^2}{\sqrt{\xi}} \int_0^1 u_x^2 dx + \mu_2 \tau \sqrt{\xi} \int_0^1 T_x \varphi_t dx \\ & - \mu_2 \int_0^1 T_x \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) dx + \frac{a_0 \mu^* b \mu_0}{\mu_2 \rho \sqrt{\xi}} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 T_x u_x dx \\ & + \frac{\mu_0 a_0 k}{\sqrt{\xi} \mu_2} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 T u_t dx - \frac{a_0 b \mu_0^2}{\rho \mu_2 \sqrt{\xi}} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 (\tau T_t + T) \varphi_x dx \\ & - \mu_0 \sqrt{\xi} \int_0^1 u_x \varphi dx + \sqrt{\xi} J \int_0^1 \varphi_t^2 dx - \xi \int_0^1 \varphi \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) dx \\ & - \frac{\mu_0}{\mu_2 \sqrt{\xi}} \left[ \mu_2^2 \tau - \left( \frac{a_0 \mu^* b \tau}{\rho} - a_0 k_2 \right) \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \right] \int_0^1 T_{tx} u_x dx, \end{aligned}$$

and noting that

$$-\sqrt{\xi} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx = -\frac{\mu_0^2}{\sqrt{\xi}} \int_0^1 u_x^2 dx - \mu_0 \sqrt{\xi} \int_0^1 u_x \varphi dx - \xi \int_0^1 \varphi \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) dx,$$

we arrive at

$$(5.40) \quad \begin{aligned} \Gamma'_4(t) = & -\sqrt{\xi} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx + \mu_2 \tau \sqrt{\xi} \int_0^1 T_x \varphi_t dx + \sqrt{\xi} J \int_0^1 \varphi_t^2 dx \\ & - \frac{a_0 b \mu_0^2}{\rho \mu_2 \sqrt{\xi}} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 (\tau T_t + T) \varphi_x dx + \frac{\mu^* a_0 b \mu_0}{\rho \mu_2 \sqrt{\xi}} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 T_x u_x dx \\ & + \frac{k a_0 \mu_0}{\mu_2 \sqrt{\xi}} \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \int_0^1 T u_t dx - \mu_2 \int_0^1 T_x \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right) dx \\ & - \frac{\mu_0}{\mu_2 \sqrt{\xi}} \underbrace{\left[ \mu_2^2 \tau - \left( \frac{a_0 \mu^* b \tau}{\rho} - a_0 k_2 \right) \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \right]}_{=\chi} \int_0^1 T_{tx} u_x dx. \end{aligned}$$

Applying Young's and Poincaré's inequalities with the use of  $\mu^*\xi = \mu_0^2$ , we get

$$\begin{aligned}
 & \mu_2\tau\sqrt{\xi}\int_0^1 T_x\varphi_t dx \leq \sqrt{\xi}J\int_0^1 \varphi_t^2 dx + \frac{\mu_2^2\tau^2\sqrt{\xi}}{4J}\int_0^1 T_x^2 dx, \\
 & -\mu_2\int_0^1 T_x\left(\frac{\mu_0}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right) dx \leq \frac{\mu_2^2}{\sqrt{\xi}}\int_0^1 T_x^2 dx + \frac{\sqrt{\xi}}{4}\int_0^1 \left(\frac{\mu_0}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx, \\
 & \frac{ka_0\mu_0}{\mu_2\sqrt{\xi}}\underbrace{\left(\frac{J}{a_0} - \frac{\rho}{\mu^*}\right)}_{=\sigma}\int_0^1 Tu_t dx \leq \frac{\sigma^2k^2a_0^2\mu^*}{4\varepsilon_3\mu_2^2}\int_0^1 T^2 dx + \varepsilon_3\int_0^1 u_t^2 dx, \\
 & \frac{\mu^*a_0b\mu_0}{\rho\mu_2\sqrt{\xi}}\left(\frac{J}{a_0} - \frac{\rho}{\mu^*}\right)\int_0^1 T_xu_x dx \leq \frac{2\sigma^2\mu^*a_0^2b^2}{\rho^2\mu_2^2\sqrt{\xi}}\int_0^1 T_x^2 dx + \frac{\sqrt{\xi}}{8}\int_0^1 \left(\frac{\mu_0}{\sqrt{\xi}}u_x\right)^2 dx \\
 & \leq \frac{2\sigma^2\mu^*a_0^2b^2}{\rho^2\mu_2^2\sqrt{\xi}}\int_0^1 T_x^2 dx + \frac{\sqrt{\xi}^3}{4}\int_0^1 \varphi_x^2 dx \\
 & \quad + \frac{\sqrt{\xi}}{4}\int_0^1 \left(\frac{\mu_0}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx, \\
 & -\frac{a_0b\mu_0^2}{\rho\mu_2\sqrt{\xi}}\left(\frac{J}{a_0} - \frac{\rho}{\mu^*}\right)\int_0^1 (\tau T_t + T)\varphi_x dx \leq \frac{\sigma^2a_0^2b^2\mu^*2\xi}{4\varepsilon_4\rho^2\mu_2^2}\int_0^1 \varphi_x^2 dx + \varepsilon_4\int_0^1 (\tau T_t + T)^2 dx.
 \end{aligned}$$

The substitution of the previous estimates into (5.40), yields (5.39). ■

**Lemma 5.3.7.** *The functional  $\Gamma_5$  defined by*

$$(5.41) \quad \Gamma_5(t) = -\frac{1}{\sqrt{\xi}}\left[bJ\int_0^1 (\tau T_t + T)\left(\int_0^x \varphi_t dy\right) dx\right], \quad t > 0,$$

satisfies, for any  $\varepsilon_5 > 0$ ,

$$\begin{aligned}
 (5.42) \quad \Gamma_5'(t) & \leq -\frac{\mu_2J}{2\sqrt{\xi}}\int_0^1 \varphi_t^2 dx + \frac{k_2^2J}{\sqrt{\xi}\mu_2}\int_0^1 T_x^2 dx + \frac{\varepsilon_5}{3\xi}\int_0^1 \left(\frac{\mu_0}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx + \varepsilon_5\int_0^1 \varphi_x^2 dx \\
 & \quad + \left(\frac{3a_0^2b^2}{4\varepsilon_5\xi} + \frac{\mu_2b}{\sqrt{\xi}} + \frac{9b^2\xi}{4\varepsilon_5}\right)\int_0^1 (\tau T_t + T)^2 dx + \frac{k^2J}{\sqrt{\xi}\mu_2}\int_0^1 T^2 dx, \quad t > 0.
 \end{aligned}$$

**Proof.** By exploiting the functional  $\Gamma_5(t)$  using (5.7)<sub>2</sub> and (5.7)<sub>3</sub>. Then integrating by parts with the relation  $\int_0^1 \varphi_t dx = 0$ , we get

$$\begin{aligned}
 (5.43) \quad \Gamma_5'(t) & = -\frac{\mu_2J}{\sqrt{\xi}}\int_0^1 \varphi_t^2 dx + \frac{k_2J}{\sqrt{\xi}}\int_0^1 T_x\varphi_t dx + \frac{kJ}{\sqrt{\xi}}\int_0^1 T\left(\int_0^x \varphi_t dy\right) dx + \frac{\mu_2b}{\sqrt{\xi}}\int_0^1 (\tau T_t + T)^2 dx \\
 & \quad - \frac{a_0b}{\sqrt{\xi}}\int_0^1 \varphi_x(\tau T_t + T) dx + \frac{b\mu_0}{\sqrt{\xi}}\int_0^1 u(\tau T_t + T) dx + \sqrt{\xi}b\int_0^1 \left(\int_0^x \varphi dy\right)(\tau T_t + T) dx,
 \end{aligned}$$

by using Young's, Cauchy Schwarz's and Poincaré's inequalities

$$\begin{aligned}
-\frac{a_0 b}{\sqrt{\xi}} \int_0^1 \varphi_x (\tau T_t + T) dx &\leq \frac{\varepsilon_5}{3} \int_0^1 \varphi_x^2 dx + \frac{3a_0^2 b^2}{4\varepsilon_5 \xi} \int_0^1 (\tau T_t + T)^2 dx, \\
\frac{b \mu_0}{\sqrt{\xi}} \int_0^1 u (\tau T_t + T) dx &\leq \frac{\varepsilon_5}{6\xi} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x \right)^2 dx + \frac{3\xi b^2}{2\varepsilon_5} \int_0^1 (\tau T_t + T)^2 dx \\
&\leq \frac{\varepsilon_5}{3\xi} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx + \frac{\varepsilon_5}{3} \int_0^1 \varphi_x^2 dx \\
&\quad + \frac{3\xi b^2}{2\varepsilon_5} \int_0^1 (\tau T_t + T)^2 dx, \\
b \sqrt{\xi} \int_0^1 \left( \int_0^x \varphi dy \right) (\tau T_t + T) dx &\leq \frac{\varepsilon_5}{3} \int_0^1 \varphi_x^2 dx + \frac{3b^2 \xi}{4\varepsilon_5} \int_0^1 (\tau T_t + T)^2 dx, \\
\frac{k_2 J}{\sqrt{\xi}} \int_0^1 T_x \varphi_t dx &\leq \frac{\mu_2 J}{4\sqrt{\xi}} \int_0^1 \varphi_t^2 dx + \frac{k_2^2 J}{\sqrt{\xi} \mu_2} \int_0^1 T_x^2 dx, \\
\frac{k J}{\sqrt{\xi}} \int_0^1 T \left( \int_0^x \varphi_t dy \right) dx &\leq \frac{\mu_2 J}{4\sqrt{\xi}} \int_0^1 \varphi_t^2 dx + \frac{k^2 J}{\sqrt{\xi} \mu_2} \int_0^1 T^2 dx,
\end{aligned}$$

we conclude (5.42), when we estimate (5.43) by the previous quantities.  $\blacksquare$

## 5.4 Proof of theorem 5.3.1

Utilizing the previous lemmas with the following Lyapunov functional  $L(t)$

$$L(t) = NE(t) + \Gamma_1(t) + N_1 \Gamma_2 + N_2 \Gamma_3 + N_3 \Gamma_4 + N_4 \Gamma_5, \quad t > 0,$$

for  $N, N_1, N_2, N_3$  and  $N_4 > 0$ , we have

$$\begin{aligned}
L'(t) &\leq -C_{T_x} \int_0^1 T_x^2 dx - C_T \int_0^1 T^2 dx - C_{u_t} \int_0^1 u_t^2 dx - C_{\varphi_t} \int_0^1 \varphi_t^2 dx - C_{\varphi_x} \int_0^1 \varphi_x^2 dx \\
&\quad - C_{(\tau T_t + T)} \int_0^1 (\tau T_t + T)^2 dx - C_{\left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx,
\end{aligned}$$

where

$$\begin{aligned}
C_{T_x} &= N k_2 - N_1 \left( \tau k_2 + \frac{\tau^2 \mu_2^2}{4\varepsilon_1} \right) - N_3 \left( \frac{\mu_2^2 \tau^2 \sqrt{\xi}}{4J} + \frac{\mu_2^2}{\sqrt{\xi}} + \frac{2\sigma^2 \mu^{*2} a_0^2 b^2}{\rho^2 \mu_2^2 \sqrt{\xi}} \right) - N_4 \frac{k_2^2 J}{\sqrt{\xi} \mu_2}, \\
C_T &= N k - N_1 (b + k\tau) - N_3 \frac{\sigma^2 k^2 a_0^2 \mu^*}{4\varepsilon_3 \mu_2^2} - N_4 \frac{k^2 J}{\sqrt{\xi} \mu_2}, \\
C_{u_t} &= \rho - N_2 \varepsilon_2 - N_3 \varepsilon_3, \quad C_{\left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)} = N_3 \frac{\sqrt{\xi}}{2} - N_4 \frac{\varepsilon_5}{3\xi} - \frac{5}{2}, \\
C_{\varphi_t} &= N_4 \frac{\mu_2 J}{2\sqrt{\xi}} - N_1 \varepsilon_1 - N_2 \left( \frac{J}{\sqrt{\xi}} + \frac{\rho^2}{4\varepsilon_2 \mu^*} \right) - 2N_3 \sqrt{\xi} J,
\end{aligned}$$

$$C_{\varphi_x} = N_2 \frac{a_0}{2\sqrt{\xi}} - N_3 \left( \frac{\sqrt{\xi}^3}{4} + \frac{\sigma^2 a_0^2 b^2 \mu^{*2} \xi}{4\varepsilon_4 \rho^2 \mu_2^2} \right) - N_4 \varepsilon_5 - \frac{7\xi}{2},$$

$$C_{(\tau T_t + T)} = N_1 \frac{b}{2} - N_2 \frac{\mu_2^2}{2\sqrt{\xi} a_0} - N_3 \varepsilon_4 - N_4 \left( \frac{3a_0^2 b^2}{4\varepsilon_5 \xi} + \frac{\mu_2 b}{\sqrt{\xi}} + \frac{9b^2 \xi}{4\varepsilon_5} \right).$$

By setting  $N_3 = \frac{7}{\sqrt{\xi}}$ ,  $\varepsilon_1 = \frac{1}{N_1}$ ,  $\varepsilon_2 = \frac{\rho}{4N_2}$ ,  $\varepsilon_3 = \frac{\rho}{4N_3}$ ,  $\varepsilon_4 = 1$  and  $\varepsilon_5 = \frac{3\xi}{2N_4}$ , we obtain

$$C_{T_x} = Nk_2 - N_1 \left( \tau k_2 + \frac{N_1 \tau^2 \mu_2^2}{4} \right) - \frac{7}{\sqrt{\xi}} \left( \frac{\mu_2^2 \tau^2 \sqrt{\xi}}{4J} + \frac{\mu_2^2}{\sqrt{\xi}} + \frac{2\sigma^2 \mu^{*2} a_0^2 b^2}{\rho^2 \mu_2^2 \sqrt{\xi}} \right) - N_4 \frac{k_2^2 J}{\sqrt{\xi} \mu_2},$$

$$C_T = Nk - N_1(b + k\tau) - \frac{49\sigma^2 k^2 a_0^2 \mu^*}{\xi \rho \mu_2^2} - N_4 \frac{k^2 J}{\sqrt{\xi} \mu_2},$$

$$C_{u_t} = \frac{\rho}{2}, \quad C_{\left(\frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi\right)} = \frac{1}{2}, \quad C_{\varphi_t} = N_4 \frac{\mu_2 J}{2\sqrt{\xi}} - 1 - N_2 \left( \frac{J}{\sqrt{\xi}} + \frac{N_2 \rho}{\mu^*} \right) - 14J,$$

$$C_{\varphi_x} = N_2 \frac{a_0}{2\sqrt{\xi}} - \frac{7}{\sqrt{\xi}} \left( \frac{\sqrt{\xi}^3}{4} + \frac{\sigma^2 a_0^2 b^2 \mu^{*2} \xi}{4\rho^2 \mu_2^2} \right) - 5\xi,$$

$$C_{(\tau T_t + T)} = N_1 \frac{b}{2} - N_2 \frac{\mu_2^2}{2\sqrt{\xi} a_0} - \frac{7}{\sqrt{\xi}} - N_4 \left( \frac{N_4 a_0^2 b^2}{2\xi^2} + \frac{\mu_2 b}{\sqrt{\xi}} + \frac{3N_4 b^2}{2} \right).$$

If we choose our parameters correctly, all of the terms on the right-hand side of the final inequality turn positive. First, we select  $N_2$  large enough, so that

$$C_{\varphi_x} = N_2 \frac{a_0}{2\sqrt{\xi}} - \frac{7}{\sqrt{\xi}} \left( \frac{\sqrt{\xi}^3}{4} + \frac{\sigma^2 a_0^2 b^2 \mu^{*2} \xi}{4\rho^2 \mu_2^2} \right) - 5\xi > 0,$$

and we pick  $N_4$  large enough, such that

$$C_{\varphi_t} = N_4 \frac{\mu_2 J}{2\sqrt{\xi}} - 1 - N_2 \left( \frac{J}{\sqrt{\xi}} + \frac{N_2 \rho}{\mu^*} \right) - 14J > 0.$$

Then,  $N_1$  must be large enough so that

$$C_{(\tau T_t + T)} = N_1 \frac{b}{2} - N_2 \frac{\mu_2^2}{2\sqrt{\xi} a_0} - \frac{7}{\sqrt{\xi}} - N_4 \left( \frac{N_4 a_0^2 b^2}{2\xi^2} + \frac{\mu_2 b}{\sqrt{\xi}} + \frac{3N_4 b^2}{2} \right) > 0.$$

Finally, we choose  $N$  sufficiently large such that

$$C_{T_x} = Nk_2 - N_1 \left( \tau k_2 + \frac{N_1 \tau^2 \mu_2^2}{4} \right) - \frac{7}{\sqrt{\xi}} \left( \frac{\mu_2^2 \tau^2 \sqrt{\xi}}{4J} + \frac{\mu_2^2}{\sqrt{\xi}} + \frac{2\sigma^2 \mu^{*2} a_0^2 b^2}{\rho^2 \mu_2^2 \sqrt{\xi}} \right) - N_4 \frac{k_2^2 J}{\sqrt{\xi} \mu_2} > 0,$$

$$C_T = Nk - N_1(b + k\tau) - \frac{49\sigma^2 k^2 a_0^2 \mu^*}{\xi \rho \mu_2^2} - N_4 \frac{k^2 J}{\sqrt{\xi} \mu_2} > 0.$$

All of these options yield to

$$(5.44) \quad L'(t) \leq -\beta E(t), \quad t > 0, \quad \text{for some } \beta > 0.$$

Moreover, it is simple to establish that two positive constants,  $\eta_1$  and  $\eta_2$ , exist such that

$$(5.45) \quad \eta_1 E(t) \leq L(t) \leq \eta_2 E(t), \quad t > 0.$$

From (5.44) and the right side of (5.45), we obtain

$$(5.46) \quad L(t) \leq C_1 e^{-\frac{\beta}{\eta_2} t}, \quad t > 0,$$

where  $C_1$  is a positive constant. which yields the desired result (5.21) by using the other side of the equivalence relation again.

## 5.5 Polynomial decay

In this part, we study the polynomial decay result of system (5.7)-(5.9) in the case when

$$(5.47) \quad \chi = \mu_2^2 \tau - \left( \frac{a_0 \mu^* b \tau}{\rho} - a_0 k_2 \right) \left( \frac{J}{a_0} - \frac{\rho}{\mu^*} \right) \neq 0,$$

and prove a polynomial decay result. In addition to the first-order energy, given by (5.20), we define the second-order energy  $E_*(t)$  by

$$(5.48) \quad E_*(t) = \frac{1}{2} \int_0^1 \left[ \rho u_{tt}^2 + J \varphi_{tt}^2 + \left( \frac{\mu_0}{\sqrt{\xi}} u_{xt} + \sqrt{\xi} \varphi_t \right)^2 + a_0 \varphi_{xt}^2 + b (\tau T_{tt} + T_t)^2 + \kappa_2 \tau T_{xt}^2 + \kappa \tau T_t^2 \right] dx.$$

We can see that

$$(5.49) \quad E'_*(t) = -k_2 \int_0^1 T_{xt}^2 dx - k \int_0^1 T_t^2 dx \leq 0, \quad t > 0.$$

**Lemma 5.5.1.** *The functional  $\Gamma_4$  defined by (5.38) satisfies the following estimate*

$$(5.50) \quad \begin{aligned} \Gamma'_4(t) \leq & -\frac{\sqrt{\xi}}{4} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx + \left( \frac{\sqrt{\xi}^3}{2} + \frac{\sigma^2 a_0^2 b^2 \mu^{*2} \xi}{4 \varepsilon_4 \rho^2 \mu_2^2} \right) \int_0^1 \varphi_x^2 dx \\ & + \left( \frac{\mu_2^2 \tau^2 \sqrt{\xi}}{4J} + \frac{\mu_2^2}{\sqrt{\xi}} + \frac{2\sigma^2 \mu^{*2} a_0^2 b^2}{\rho^2 \mu_2^2 \sqrt{\xi}} \right) \int_0^1 T_x^2 dx + \frac{2\chi^2}{\mu_2^2 \sqrt{\xi}} \int_0^1 T_{tx}^2 dx + \varepsilon_3 \int_0^1 u_t^2 dx \\ & + \frac{\sigma^2 k^2 a_0^2 \mu^*}{4 \varepsilon_3 \mu_2^2} \int_0^1 T^2 dx + 2\sqrt{\xi} J \int_0^1 \varphi_t^2 dx + \varepsilon_4 \int_0^1 (\tau T_t + T)^2 dx, \quad t > 0. \end{aligned}$$

**Proof.** Following the steps of lemma 5.3.6 and recalling (5.47), we get

$$(5.51) \quad \begin{aligned} \Gamma'_4(t) \leq & -\frac{\sqrt{\xi}}{2} \int_0^1 \left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)^2 dx + \left( \frac{\sqrt{\xi}^3}{4} + \frac{\sigma^2 a_0^2 b^2 \mu^{*2} \xi}{4 \varepsilon_4 \rho^2 \mu_2^2} \right) \int_0^1 \varphi_x^2 dx \\ & + \left( \frac{\mu_2^2 \tau^2 \sqrt{\xi}}{4J} + \frac{\mu_2^2}{\sqrt{\xi}} + \frac{2\sigma^2 \mu^{*2} a_0^2 b^2}{\rho^2 \mu_2^2 \sqrt{\xi}} \right) \int_0^1 T_x^2 dx + \frac{\sigma^2 k^2 a_0^2 \mu^*}{4 \varepsilon_3 \mu_2^2} \int_0^1 T^2 dx \\ & + \varepsilon_3 \int_0^1 u_t^2 dx + 2\sqrt{\xi} J \int_0^1 \varphi_t^2 dx + \varepsilon_4 \int_0^1 (\tau T_t + T)^2 dx - \frac{\mu_0}{\mu_2 \sqrt{\xi}} \chi \int_0^1 T_{tx} u_x dx. \end{aligned}$$

Applying Young's inequality to the last term of (5.51), we find

$$(5.52) \quad \begin{aligned} -\frac{\mu_0}{\mu_2\sqrt{\xi}}\chi\int_0^1 T_{tx}u_x dx &\leq \frac{2\chi^2}{\mu_2^2\sqrt{\xi}}\int_0^1 T_{tx}^2 dx + \frac{\sqrt{\xi}}{8}\int_0^1 \left(\frac{\mu_0}{\sqrt{\xi}}u_x\right)^2 dx \\ &\leq \frac{2\chi^2}{\mu_2^2\sqrt{\xi}}\int_0^1 T_{tx}^2 dx + \frac{\sqrt{\xi}^3}{4}\int_0^1 \varphi_x^2 dx + \frac{\sqrt{\xi}}{4}\int_0^1 \left(\frac{\mu_0}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx. \end{aligned}$$

We end up with (5.50), by substituting (5.52) into (5.51).  $\blacksquare$

**Theorem 5.5.2.** *Suppose that (5.47) is valid. Problem (5.7)-(5.9) is then polynomially stable, when  $\lambda$  is a positive constant and*

$$(5.53) \quad E(t) \leq \frac{\lambda}{t}, \quad t > 0.$$

**Proof.** We define the Lyapunov functional  $\check{c}(t)$  by

$$\check{c}(t) = N[E(t) + E_*(t)] + \Gamma_1(t) + N_1\Gamma_2(t) + N_2\Gamma_3(t) + N_3\Gamma_4(t) + N_4\Gamma_5(t), \quad t > 0,$$

where  $N, N_1, N_2, N_3$  and  $N_4 > 0$ . Additionally, it is simple to establish that two positive constants,  $\bar{\eta}_1$  and  $\bar{\eta}_2$ , exist such that

$$(5.54) \quad \bar{\eta}_1[E(t) + E_*(t)] \leq \check{c}(t) \leq \bar{\eta}_2[E(t) + E_*(t)], \quad t > 0.$$

Now, by recalling the previous lemmas with (5.49) and (5.50), we obtain

$$\begin{aligned} \check{c}'(t) &\leq -C_{T_x}\int_0^1 T_x^2 dx - C_T\int_0^1 T^2 dx - C_{u_t}\int_0^1 u_t^2 dx - C_{\varphi_t}\int_0^1 \varphi_t^2 dx - C_{T_t}\int_0^1 T_t^2 dx - C_{\varphi_x}\int_0^1 \varphi_x^2 dx \\ &\quad - C_{(\tau T_t + T)}\int_0^1 (\tau T_t + T)^2 dx - C_{\left(\frac{\mu_0}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)}\int_0^1 \left(\frac{\mu_0}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)^2 dx - C_{T_{xt}}\int_0^1 T_{tx}^2 dx, \end{aligned}$$

where

$$(5.55) \quad \begin{aligned} C_{T_x} &= Nk_2 - N_1\left(\tau k_2 + \frac{\tau^2\mu_2^2}{4\varepsilon_1}\right) - N_3\left(\frac{\mu_2^2\tau^2\sqrt{\xi}}{4J} + \frac{\mu_2^2}{\sqrt{\xi}} + \frac{2\sigma^2\mu^{*2}a_0^2b^2}{\rho^2\mu_2^2\sqrt{\xi}}\right) - N_4\frac{k_2^2J}{\sqrt{\xi}\mu_2}, \\ C_{\left(\frac{\mu_0}{\sqrt{\xi}}u_x + \sqrt{\xi}\varphi\right)} &= N_3\frac{\sqrt{\xi}}{4} - N_4\frac{\varepsilon_5}{3\xi} - \frac{5}{2}, \\ C_T &= Nk - N_1(b + k\tau) - N_3\frac{\sigma^2k^2a_0^2\mu^*}{4\varepsilon_3\mu_2^2} - N_4\frac{k^2J}{\sqrt{\xi}\mu_2}, \\ C_{T_{xt}} &= Nk_2 - N_3\frac{2\chi^2}{\mu_2^2\sqrt{\xi}}, \quad C_{u_t} = \rho - N_2\varepsilon_2 - N_3\varepsilon_3, \quad C_{T_t} = Nk, \end{aligned}$$

$$\begin{aligned}
C_{\varphi_t} &= N_4 \frac{\mu_2 J}{2\sqrt{\xi}} - N_1 \varepsilon_1 - N_2 \left( \frac{J}{\sqrt{\xi}} + \frac{\rho^2}{4\varepsilon_2 \mu^*} \right) - 2N_3 \sqrt{\xi} J, \\
C_{\varphi_x} &= N_2 \frac{a_0}{2\sqrt{\xi}} - N_3 \left( \frac{\sqrt{\xi}^3}{2} + \frac{\sigma^2 a_0^2 b^2 \mu^{*2} \xi}{4\varepsilon_4 \rho^2 \mu_2^2} \right) - N_4 \varepsilon_5 - \frac{7\xi}{2}, \\
(5.56) \quad C_{(\tau T_t + T)} &= N_1 \frac{b}{2} - N_2 \frac{\mu_2^2}{2\sqrt{\xi} a_0} - N_3 \varepsilon_4 - N_4 \left( \frac{3a_0^2 b^2}{4\varepsilon_5 \xi} + \frac{\mu_2 b}{\sqrt{\xi}} + \frac{9b^2 \xi}{4\varepsilon_5} \right).
\end{aligned}$$

A similar selection of  $\varepsilon_{j_{j=1,4}}$  as in the proof of theorem 5.3.1 with  $N_3 = \frac{16}{\sqrt{\xi}}$  and  $\varepsilon_5 = \frac{3\xi}{2N_4}$ , then the relation (5.56) takes the form

$$\begin{aligned}
C_{T_x} &= Nk_2 - N_1 \left( \tau k_2 + \frac{N_1 \tau^2 \mu_2^2}{4} \right) - N_3 \left( \frac{\mu_2^2 \tau^2 \sqrt{\xi}}{4J} + \frac{\mu_2^2}{\sqrt{\xi}} + \frac{2\sigma^2 \mu^{*2} a_0^2 b^2}{\rho^2 \mu_2^2 \sqrt{\xi}} \right) - N_4 \frac{k_2^2 J}{\sqrt{\xi} \mu_2}, \\
C_{T_{tx}} &= Nk_2 - N_3 \frac{2\chi^2}{\mu_2^2 \sqrt{\xi}}, \quad C_{T_t} = Nk, \\
C_T &= Nk - N_1(b + k\tau) - \frac{256\sigma^2 k^2 a_0^2 \mu^*}{\sqrt{\xi} \rho \mu_2^2} - N_4 \frac{k^2 J}{\sqrt{\xi} \mu_2}, \\
C_{(\tau T_t + T)} &= N_1 \frac{b}{2} - N_2 \frac{\mu_2^2}{2\sqrt{\xi} a_0} - \frac{16}{\sqrt{\xi}} - N_4 \left( \frac{N_4 a_0^2 b^2}{2\xi^2} + \frac{\mu_2 b}{\sqrt{\xi}} + \frac{3N_4 b^2}{2} \right), \\
C_{\varphi_t} &= N_4 \frac{\mu_2 J}{2\sqrt{\xi}} - 1 - N_2 \left( \frac{J}{\sqrt{\xi}} + \frac{4N_2 \rho}{\mu^*} \right) - 32J, \\
C_{\varphi_x} &= N_2 \frac{a_0}{2\sqrt{\xi}} - \frac{16}{\sqrt{\xi}} \left( \frac{\sqrt{\xi}^3}{2} + \frac{\sigma^2 a_0^2 b^2 \mu^{*2} \xi}{4\rho^2 \mu_2^2} \right) - 5\xi, \\
(5.57) \quad C_{u_t} &= \frac{\rho}{2}, \quad C_{\left( \frac{\mu_0}{\sqrt{\xi}} u_x + \sqrt{\xi} \varphi \right)} = 1.
\end{aligned}$$

Also as in the proof of theorem 5.3.1, we select  $N_2$  large enough, such that

$$C_{\varphi_x} = N_2 \frac{a_0}{2\sqrt{\xi}} - \frac{16}{\sqrt{\xi}} \left( \frac{\sqrt{\xi}^3}{2} + \frac{\sigma^2 a_0^2 b^2 \mu^{*2} \xi}{4\rho^2 \mu_2^2} \right) - 5\xi > 0,$$

in addition,  $N_4$  must be large enough so that

$$C_{\varphi_t} = N_4 \frac{\mu_2 J}{2\sqrt{\xi}} - 1 - N_2 \left( \frac{J}{\sqrt{\xi}} + \frac{4N_2 \rho}{\mu^*} \right) - 32J > 0.$$

Now, we fixed  $N_1$  large enough, so that

$$C_{(\tau T_t + T)} = N_1 \frac{b}{2} - N_2 \frac{\mu_2^2}{2\sqrt{\xi} a_0} - \frac{16}{\sqrt{\xi}} - N_4 \left( \frac{N_4 a_0^2 b^2}{2\xi^2} + \frac{\mu_2 b}{\sqrt{\xi}} + \frac{3N_4 b^2}{2} \right) > 0.$$

Finally, we pick  $N$  large enough so that

$$\begin{aligned} C_{T_x} &= Nk_2 - N_1 \left( \tau k_2 + \frac{N_1 \tau^2 \mu_2^2}{4} \right) - N_3 \left( \frac{\mu_2^2 \tau^2 \sqrt{\xi}}{4J} + \frac{\mu_2^2}{\sqrt{\xi}} + \frac{2\sigma^2 \mu^{*2} a_0^2 b^2}{\rho^2 \mu_2^2 \sqrt{\xi}} \right) - N_4 \frac{k_2^2 J}{\sqrt{\xi} \mu_2} > 0, \\ C_T &= Nk - N_1(b + k\tau) - \frac{256\sigma^2 k^2 a_0^2 \mu^*}{\sqrt{\xi} \rho \mu_2^2} - N_4 \frac{k^2 J}{\sqrt{\xi} \mu_2} > 0, \\ C_{T_{xx}} &= Nk_2 - N_3 \frac{2\chi^2}{\mu_2^2 \sqrt{\xi}} > 0. \end{aligned}$$

Although, (5.57) leads, for  $\bar{\lambda}$  a positive constant, to

$$(5.58) \quad \check{c}'(t) \leq -\bar{\lambda}E(t), \quad \forall t \geq 0.$$

Once we integrate (5.58) over  $(0, t)$ , we get

$$tE(t) \leq \int_0^t E(s) ds \leq \frac{1}{\bar{\lambda}} (\check{c}(0) - \check{c}(t)), \quad \forall t \geq 0.$$

The preceding inequality has the following form while keeping the relation (5.54) in mind

$$E(t) \leq \frac{\check{c}(0)}{\bar{\lambda}t} \leq \frac{\bar{\eta}_2(E(0) + E_*(0))}{\bar{\lambda}t},$$

by posing  $\lambda = \frac{\bar{\eta}_2(E(0) + E_*(0))}{\bar{\lambda}}$ , we conclude (4.49). ■

## CONCLUSION

Based on the preceding studies and research, we derive numerous conclusions.

In chapter 3, we studied the asymptotic behavior of a swelling porous elastic system in the presence of porous damping and neutral delay. Our findings show that the porous damping dissipation is powerful enough to stabilise the system exponentially even with neutral delay.

In chapter 4, We were able to study the stability result of Lord-Shulman system with microtemperature effect, where we achieved a positive result in just one damping owing to microtemperature and when  $\mu^* > \frac{\mu_0^2}{\xi}$ ; we gain exponential stability when (4.9) prevails. While its absence results in a lack of exponential decay on one side, we get polynomial decay on the other, which is a slower sort of stability. For the optimality of theorem 4.6.1, we cite the result of Ramos et al. [61], where they used theorem 2.4 in Borichev and Tomilov [12] to prove the polynomial decay of order 1/2 for the same system (4.4).

In chapter 5, we were also able to prove the exponential stability of the same system as in the previous chapter but with a non-positive energy, i.e., when  $\mu^* = \frac{\mu_0^2}{\xi}$ , and under an appropriate new assumption about the coefficients of the system.

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