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Etude de quelques systèmes d'ordre fractionnaire

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Contents

Résumé

Cette thèse est consacrée à l'étude des systèmes d'équations différentielles non-linéaires d'order fractionnaires. Elle est constituée de deux parties principales. Dans la première partie, nous nous intéressons à l'existence et l'unicité de la solution pour les systèmes d'équations différentielles dont les derivées fractionnaires sont de type Riemann-Liouville. Les démonstrations sont essentiellement basées sur les théorèmes du point fixe tels que le principe de contraction de Banach et l'alternative non linéaire de Leray Schauder. Dans la deuxième partie, nous étudions, en utilisant la théorie de Lyapunov, la stabilité de la solution pour des systèmes d'équations différentielles non linéaires à dérivées conformables dependant d'un paramètre, pour des systèmes d'équations différentielles perturbés à dérivées conformables et des systèmes pour d'équations différentielles non linéaires à dérivées conformables avec des incertitudes.

Mots clés: Systèmes fractionnaires, Dérivée fractionnaire de Riemann-Liouville, Existence et unicité de solution, Dérivée conformable, Stabilité, Théorème du point fixe, Méthode de Lyapunov.

Abstract

This thesis is devoted to the study of systems of fractional nonlinear differential equations. It contains two principle parts. In the first part, we focus on the existence and uniqueness of solution for systems of Riemann-Liouville fractional differential equations. The proofs are essentially based on some fixed point theorems such The Banach contraction principle and the Leray Schauder nonlinear alternative. In the second part, we study, by using Lyapunov theory, the stability of solution for systems of conformable nonlinear differential equations depending on a parameter, for a class of systems for perturbed conformable differential equations and of a class of systems for nonlinear conformable differential equations with uncertainties.

Key words: Fractional Systems, Riemann-Liouville fractional Derivative, Existence and uniqueness of solutions, Conformable derivative, Stability, Fixed point theorem, Lyapunov method.

Chapter 1

Introduction

The fractional calculus has been known for over three centuries, but is still not really popular in the scientific community. The importance of this subject is that the fractional integrals and derivatives do not have a local property, which is closer to real life since most processes are generally fractional.

Recently this subject has been exploited by several fields of engineering and sciences and economics and several applications based on this new subject have appeared and different definitions of fractional derivatives are involved.

The mathematicians Leibniz, Liouville, Riemann where the first who built the basic mathematical ideas of fractional calculus, but the first book on the topic was published by Oldham and Spanier, then the application of fractional calculus in sciences such in physics, continuum mechanics, signal processing, and electromagnetic have been the objective of many monographs and symposium. The application of fractional calculus is extended to many fields if sciences and engineering, for example it's used to describe the viscous interactions between fluid and solid structure, in modeling of speech signals, in rheology, in fluid mechanics,...see [6,8,17,18,20,28,33,37,38,43,53,56,57,59].

An other type of non classical derivative is the so called conformable derivative that was introduced by Khalil et al. in [31]. This interesting fractional derivative is based on a limit form as in the classical derivative and has similar properties than the classical one. The new conformable fractional derivative is now knowing a great interest and is the subject of several articles concerning boundary value problems, see [1,9,10,30].

Motived by works on non classical derivatives, the thesis has two objectives, the first one is the study of existence and uniqueness of solutions for systems of fractional differential equations with boundary conditions, to his end, some fixed point theorems are used. The second objective is to prove, by using Lyapunov techniques, the stability of conformable fractional nonlinear systems depending on a parameter, systems for perturbed conformable differential equations and systems for nonlinear conformable differential equations with uncertainties. The thesis consists of three chapters.

Chapter 1. is devoted to preliminaries. After giving some special functions, we introduce the concept of fractional integrals and derivatives and their properties. We cite some fixed point theorems, introduce the conformable derivative and their properties, then we give the theory concerning the stability of solutions for systems of differential equations.

Chapter 2. we study the following system of Riemann-Liouville fractional differential equations with boundary conditions:

$$(FS)\begin{cases} -D_{0^+}^{\alpha}u(t)=g(t)f(u(t)),0<t<1, \\ u(0)=u'(0)=0,\ au(1)+bu'(1)=0, \end{cases}$$

where $D_{0^+}^{\alpha}$ denotes the Riemann-Liouville fractional derivative, $2<\alpha<3$, $u=(u_1,u_2,...,u_n)^T$ is an unknown function with $u_i:[0,1]\to\mathbb{R}$, and $g:[0,1]\to\mathbb{R}$ is a given function, $f:\mathbb{R}^n\to\mathbb{R}^n$, $f(u)=(f_1(u_1,u_2,...,u_n),...,f_n(u_1,u_2,...,u_n))^T$, $f_i:\mathbb{R}^n\to\mathbb{R}$.

Using Banach contraction principle and nonlinear alternative of Leray-Schauder, we prove the existence and uniqueness of solution for problem (FS). The results of this chapter are published in:

A. Guezane-Lakoud, G. Rebiai and R. Khaldi, Existence of solutions for nonlinear fractional system with nonlocal boundary conditions, Proyecciones Journal of Mathematics, Volume 36, 4, december 2017.

Chapter 3. We study the stability of solution for systems of conformable nonlinear differential equations of order α, $0 < \alpha < 1$. Firstly, we consider systems of conformable nonlinear differential equations depending on a parameter

$$\begin{aligned} T_{t_0}^{\alpha} x &= f(t, x, \varepsilon),\ t > t_0 \\ x(t_0) &= x_0 \end{aligned}$$

where $t_0 > 0$, $x \in \mathbb{R}^n$, $f(.,.,\varepsilon) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a given nonlinear function. By using Lyapunov techniques, we prove that this system is ϵ^*-uniformly practically fractional exponentially stable.

Secondly, we study systems for perturbed conformable differential equations:

$$\begin{aligned} T_{t_0}^{\alpha} x &= Ax + Bu + g(t, x, u, \varepsilon),\ t > t_0, \\ x(t_0) &= x_0 \end{aligned}$$

where $0 < \alpha < 1$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^q$, A and B are respectively $(n \times n)$, $(n \times q)$ constant matrices, $g(.,.,.,\varepsilon) : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ is a given nonlinear function. Under some conditions on the perturbation term g and on the matrices A and B, we prove that the feedback law $u(x) = Kx$, uniformly practically fractional exponentially stabilizes the system.

Finally, we study the stability of a class of systems for nonlinear conformable differential equations with uncertainties

$$\begin{aligned} T_{t_0}^{\alpha} x &= Ax + B\left(\Phi\left(x, u\right) + u\right) + g(x, u), \\ x(t_0) &= x_0, \end{aligned}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^q$, A and B are respectively $(n \times n)$, $(n \times q)$ constant matrices, $\Phi : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$. The results of this chapter are submitted for publication.

Chapter 2

Preliminaries

2.1 Basic functions of fractional calculus

We give some special functions that will be used later, such as Gamma function, Beta function and Mittag-Leffler functions. These functions generalize the factorial and exponential functions and play an important role in the theory of fractional differential equations, see [32,34,35,42,47,48]

2.1.1 Gamma Function

Definition 1 *Gamma function $\Gamma(z)$ is defined by the integral*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad Re(z) > 0$$

which is the Euler integral of the second kind and converges in the right half of the complex plan $Re(z) > 0$.

The reduction formula of the Gamma function is

$$\Gamma(z+1) = z\Gamma(z), \qquad Re(z) > 0,$$

which can be proved by integrating by parts. Since, $\Gamma(1) = 1$, the recurrence shows that for any positive integer n, we have

$$\Gamma(n+1) = n!$$

The Gamma function can be represented for every $z \in \mathbb{C}$ by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^2}{z(z+1)\ldots(z+n)}, \ Re(z) > 0,$$

2.1.2 Beta Function

Definition 2 *Beta function $B(z,w)$ is defined by*

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,dt, \quad z,w \in \mathbb{C}, \ Re(z) > 0,$$

which is the Euler's integral of first kind.

The relation between Gamma function and Beta function is the following

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \ Re(z) > 0, \ Re(w) > 0.$$

2.1.3 Mittag-Leffler function

The Mittag-Leffler function plays the role of exponential function in the fractional calculus and arises naturally in the expression of solution of fractional order differential equations.

Definition 3 *For $\alpha > 0$, the Mittag-Leffler function for one parameter is defined by:*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \qquad z \in \mathbb{C}, \ Re(z) > 0,$$

when $\alpha = 1$ and $\alpha = 2$, we have, $E_1(z) = e^z$ and $E_2(z) = cosh(\sqrt{z})$.

The Mittag-Leffler type function with two parameter α, β is defined by the series expansion as follows,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}. \qquad (\alpha > 0, \beta > 0).$$

2.1.4 Riemann-Liouville fractional integral

Definition 4 *The Riemann-Liouville fractional integral of order α of a function $f : (a,\infty) \to \mathbb{R}$ is defined by*

$$I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, t \geq a$$

provided the right side is pointwise defined on (a,∞).

For $\alpha = 0$, we set $I_{a^+}^{\alpha} f(t) = I$, the identity operator. An important property of Riemman-Liouville integral is the following.

Theorem 1 *Let $\alpha, \beta \geq 0$ and $f \in L^1(a,b)$. Then*

$$I_{a^+}^{\alpha} I_{a^+}^{\beta} f(t) = I_{a^+}^{\beta} I_{a^+}^{\alpha} f(t) = I_{a^+}^{\alpha+\beta} f(t),$$

holds almost everywhere on $[a,b]$. In addition, if $f \in C[a,b]$ or $\alpha + \beta \geq 1$, then the identity holds everywhere on $[a,b]$.

2.1.5 Riemann-Liouville fractional derivative

Definition 5 *Let $p > 0$, the Riemann-Liouville fractional derivative of order p of a function $f \in C((a,\infty),\mathbb{R})$ is defined by*

$$D_{a^+}^{p} f(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-p-1} f(s)\, ds,$$

where $n = [p]+1$, provided the right side is pointwise defined on (a,∞).

2.1.6 Caputo fractional derivative

Definition 6 *Let $p > 0$ and $n = [p] + 1$, the Caputo's fractional derivative of a function $f \in C^n([a,b], \mathbb{R})$ is defined by*

$$^{C}D_{a^+}^{p} f(t) = \frac{1}{\Gamma(p-n)} \int_a^t (t-s)^{n-p-1} f^{(n)}(s)\, ds.$$

Suppose $p > 0$ and $n = [p] + 1$, then the relation between Riemman-Liouville, Caputo fractional derivatives and Riemann Liouville integral can be expressed by the theorem below.

Theorem 2 *Set $D = \frac{d}{dt}$, then we have for $p, q > 0$:*

$$D_{a^+}^{p} f(t) = ^{C} D_{a^+}^{p} f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-p}}{\Gamma(k-p+1)}.$$

$$D_{a^+}^{p} f(t) = D^n \left(I_{a^+}^{n-p} f\right)(t), ^{C} D_{a^+}^{p} f(t) = \left(I_{a^+}^{n-p} D^n f\right)(t)$$

$$\left(D_{a^+}^{p} I_{a^+}^{p} f\right)(t) = f(t), \left(^{C}D_{a^+}^{p} I_{a^+}^{p} f\right)(t) = f(t).$$

$$\left(I_{a^+}^{p} D_{a^+}^{p} f\right)(t) = f(t) - \sum_{j=1}^{n} \frac{\left(I_{a^+}^{n-p} f(a)\right)^{(n-j)}}{\Gamma(p-j+1)} (t-a)^{p-j}.$$

$$\left(I_{a^+}^{pC} D_{a^+}^{p} f\right)(t) = f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{k!} (t-a)^{j}.$$

$$\left(D_{a^+}^{p} I_{a^+}^{q} f\right)(t) = I_{a^+}^{q-p} f(t).$$

$$D^m D_{a^+}^{p} f(t) = D_{a^+}^{p+m} f(t), m \in N.$$

2.1.7 Grunwald-Letnikov fractional derivative

To solve fractional differential equations, Grunwald and Letnikov developed an other fractional derivative which is used to construct numerical methods for fractional differential equations.

Definition 7 *The Grunwald-Letnikov fractional derivative of order* $\alpha > 0$ *of a function* f *is defined by*

$$^{GL}D_a^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} (-1)^k C_k^{\alpha} f(t - kh), a \leq t \leq b.$$

2.2 Conformable derivative

Recently, Khalil et al. gave a new definition of integral and derivative of non-integer order [31]. This new definition is used as a limit form as in the case of the classical derivative. They proved the product rule, the fractional Rolle theorem and the mean value theorem. Later, this theory is developed by Abdeljawad who gave definitions of the left and right conformable derivatives of higher order, integration by part formulas, chain rule, Taylor power series representation, see [1].

In this section, some definitions, lemmas and theorems related to the conformable fractional calculus are given.

Definition 8 *Let* $n < \alpha < n+1$, *and set* $\beta = \alpha - n$. *For a function* g *defined on* $[a, \infty)$, *we define the conformable integral by*

$$I_a^{\alpha} g(t) = \int_a^t (s-a)^{\alpha-1} g(s) ds, 0 < \alpha < 1,$$

and

$$\begin{aligned} I_a^{\alpha} g(t) &= \frac{1}{n!} \int_a^t (t-s)^n g(s) d\beta(s,a) \\ &= \frac{1}{n!} \int_a^t (t-s)^n (s-a)^{\beta-1} g(s) ds, n \geq 1. \end{aligned}$$

Definition 9 *The conformable fractional derivative of order* $\alpha > 0$, $0 < \alpha < 1$, *of a* *function h defined on $[a,\infty)$ is given by*

$$T_a^{\alpha}h\left(t\right)=\lim_{\varepsilon\to 0}\frac{h\left(t+\varepsilon\left(t-a\right)^{1-\alpha}\right)-h\left(t\right)}{\varepsilon},$$

for all $t>a$. If $T_a^{\alpha}h\left(t\right)$ exists $\forall t\in(a,b)$, $b>a$ and $\lim_{t\to a^+}T_a^{\alpha}h\left(t\right)$ exists, then by definition

$$T_a^{\alpha}h\left(a\right)=\lim_{t\to a^+}T_a^{\alpha}h\left(t\right).$$

The conformable derivative of order α, $n<\alpha<n+1$ of a function h, when $h^{(n)}$ exists, is defined by

$$T_a^{\alpha}h(t)=T_{\beta}^{a}h^{(n)}(t),$$

where $\beta=\alpha-n\in(0,1)$.

In addition, if the conformable fractional derivative of h of order α exists, then we simply say that h is α-differentiable.

Lemma 1 *Assume that h is a continuous function on an (a,∞) and $0<\alpha<1$. Then for all $t>a$ we have*

$$T_a^{\alpha}I_a^{\alpha}h\left(t\right)=h\left(t\right).$$

For the properties of the conformable derivative, we state the following.

Proposition 1 *Let $n<\alpha<n+1$ and h be an $(n+1)$-differentiable at $t>a$, then we have*

$$T_a^{\alpha}h\left(t\right)=\left(t-a\right)^{n+1-\alpha}h^{(n+1)}\left(t\right)$$

and

$$I_a^{\alpha}T_a^{\alpha}h\left(t\right)=h\left(t\right)-\sum_{k=0}^{n}\frac{h^{(k)}\left(a\right)\left(t-a\right)^{k}}{k!}.$$

For $0<\alpha<1$, it yields

$$\lim_{\alpha\to 1}T_a^{\alpha}h(t)=h'(t)$$

and

$$\lim_{\alpha\to 0} T_a^{\alpha} h(t) = (t-a)h'(t),$$

i.e. the zero order derivative of a differentiable function does not return to the function itself.

If h is $(n+1)$-differentiable on (a,b), $b>a$ and $\lim\limits_{t\to a^+} h^{(n+1)}$ exists, then $T_a^{\alpha} h\,(a) = \lim\limits_{t\to a^+} T_a^{\alpha} h(t) = 0$.

If h is $(n+1)$-differentiable at $t>a$, then $T_a^{\alpha} h\,(t) = T_a^{\alpha-k} h^{(k)}\,(t)$ for all positive integer $k<\alpha$.

Similarly to the classical case, we give a property on the extremum of a function that has a conformable derivative:

Proposition 2 *Let $h:[a,\infty)\to\mathbb{R}$ be a continuous function such that $T_a^{\alpha} h\,(t)$ exists on (a,∞), if $T_a^{\alpha} h\,(t)\geq 0$ (respectively $T_a^{\alpha} h\,(t)\leq 0$), for all $t\in(a,\infty)$, then the graph of h is increasing (respectively decreasing).*

Proposition 3 *[30] Let $1<\alpha<2$, if a function $g\in C^1[a,b]$ attains a global maximum (respectively minimum) at some point $\xi\in(a,b)$, then $T_a^{\alpha} g\,(\xi)\leq 0$ (respectively $T_a^{\alpha} g\,(\xi)\geq 0$).*

Lemma 2 *Let $\alpha\in(0,1)$, $c_1, c_2, r, \in\mathbb{R}$, and the functions $f, g:[a,+\infty)\to\mathbb{R}$ be $\alpha-$differentiable on (a,∞). Then*

$$\begin{aligned} T_a^{\alpha}(c_1 f + c_2 g) &= c_1 T_a^{\alpha} f + c_2 T_a^{\alpha} g, \\ T_a^{\alpha}(t-a)^r &= r(t-a)^{r-1}, \\ T_a^{\alpha}\lambda &= 0, \\ T_a^{\alpha}(fg) &= f T_a^{\alpha} g + g T_a^{\alpha} f, \\ T_a^{\alpha}\left(\frac{f}{g}\right) &= \frac{f T_a^{\alpha} g - g T_a^{\alpha} f}{g^2}. \end{aligned}$$

for every function g such that $g(t)\neq 0$, $t>a$.

Remark 1 *Let $h : [a, \infty) \to \mathbb{R}$ such that $T_a^\alpha h(t)$ exists on (a, ∞). Then $T_a^\alpha h^2(t)$ exists on (a, ∞) and*

$$T_a^\alpha h^2(t) = 2h(t)T_a^\alpha h(t), \ \forall t > a.$$

Remark 2 *Let $h : [a, \infty) \to \mathbb{R}^n$ such that $T_a^\alpha h(t)$ exists on (a, ∞). Then $T_a^\alpha h^T h(t)$ exists on (a, ∞) and*

$$T_a^\alpha h^T h(t) = 2h(t)T_a^\alpha h^T(t), \ \forall t > a.$$

Remark 3 *Mittag-Leffler functions play important role in fractional calculus as generalization of exponential functions while the fractional conformable exponential function*

$$E_\alpha(\lambda, s) = \exp\left(\lambda \frac{s^\alpha}{\alpha}\right), 0 < \alpha < 1, s > 0, \lambda \in \mathbb{R},$$

appears in case of conformable fractional calculus.

2.3 Fixed point theorems

2.3.1 Banach's fixed point theorem

For a real-valued continuous function $f : \mathbb{R} \to \mathbb{R}$, we look at the problem to find a fixed point using Banach's fixed point theorem. For this we need f to be a contraction, that is there is a positive real number $c < 1$, such that

$$|f(x) - f(y)| \leq c|x - y|$$

for arbitrary x, $y \in \mathbb{R}$. The conclusion from Banach fixed point theorem is that there is a unique fixed point for f.

Theorem 3 *Let T be a contraction on a Banach space X. Then T has a unique fixed point.*

2.3.2 Brouwer and Schauder fixed point theorems

We recall that a set $K \subset X$ is compact if every sequence in K has a convergent subsequence in K. Moreover we say that K is relatively compact if every sequence in K has a subsequence that converges in X. We start by formulating Brouwer fixed point theorem.

Theorem 4 *Assume that K is a compact convex subset of $\mathbb{R}^n$ and that $T : K \to K$ is a continuous mapping. Then T has a fixed point in K.*

Remark 4 *Note that it does not follow from Brouwer fixed point theorem that the fixed point is unique.*

In the case of finite dimensional normed space, compactness is equivalent to closedness and boundedness. This is not the case in an infinite-dimensional normed space. We cite Schauder fixed point theorem:

Theorem 5 *Assume that K is a nonempty convex compact set in a Banach space X and that $T : K \to K$ is a continuous mapping. Then T has a fixed point.*

2.3.3 Other theorems

An other important fixed point theorem is the nonlinear alternative of Leray-Schauder:

Lemma 3 *Let F be a Banach space and Ω a bounded open subset of F, $0 \in \Omega$. Let $T : \Omega \to F$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x \in \Omega$ of T.*

We give some additional fixed point theorems, that is Schaefer's fixed point theorem which is a version of Schauder's theorem.

Theorem 6 *Assume that X is a Banach space and that $T : X \to X$ is a continuous compact mapping. Moreover assume that the set*

$$\Omega = \{x \in X : x = \lambda T(x),\ 0 \leq \lambda \leq 1\}$$

is bounded. Then T has a fixed point.

In particular, note that to apply Schaefer's theorem we do not need to prove that a certain set is convex or compact. The problem is reformulated as to show certain a priori estimates for the operator T. Let us give an other important fixed point theorem:

Theorem 7 ***(Krasnoselskii's fixed point theorem)*** *Assume that F is a closed bounded convex subset of a Banach space X . Furthermore assume that T_1 and T_2 are mappings from F into X such that*
1. *$T_1(x) + T_2(y) \in F$ for all $x, y \in F$,*
2. *T_1 is a contraction,*
3. *T_2 is continuous and compact.*
Then $T_1 + T_2$ has a fixed point in F.

Now, we give a criteria for compactness for sets in the space of continuous functions. We recall that the family of continuous functions $S \subset C([a, b])$ is uniformly bounded if that there exists $M > 0$ such that

$$\|f\| = \max_{x \in [a,b]} |f(x)| \leq M, \text{ for all } f \in S.$$

The family S is equicontinuous on $[a, b]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in [a, b]$ and for every $f \in S$, we have

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Theorem 8 ***(Arzela-Ascoli theorem).*** *A set $S \subset C([a,b])$ is relatively compact in $C([a,b])$ iff the functions in S are uniformly bounded and equicontinuous on $[a,b]$.*

We give Guo-Krasnoselskii theorem.

Theorem 9 *Let E be a Banach space and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$A : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to K$$

be a completely continuous operator. In addition suppose either

i) $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$, or

ii) $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$

holds. Then A has a fixed point in $K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$.

Next we formulate a criteria for compactness for sets of L^p -functions.

Theorem 10 ***(Riesz-Kolmogorov).*** *Assume that $1 \leq p < \infty$ and that $S \subset L^p(\mathbb{R}^n)$. Then S is relatively compact in $L^p(\mathbb{R}^n)$ iff the following conditions are satisfied:*

1. *S is a bounded set in $L^p(\mathbb{R}^n)$, i.e. there exists $M > 0$ such that*

$$\|f\|_{L^p} \leq M, \text{ for all } f \in S.$$

2. *$\lim_{x \to 0} \int_{\mathbb{R}^n} |f(y+x) - f(y)|^p dy = 0$ uniformly in S, i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|x| < \delta, \ f \in S \Rightarrow \|f(.+x) - f(.)\| \equiv (\int_{\mathbb{R}^n} |f(y+x) - f(y)|^p dy)^{1/p} < \varepsilon.$$

3. *$\lim_{R \to \infty} \|f\|_{L^p(\mathbb{R}^n \backslash B(0,R))} = \left(\int_{|x|>R} |f(x)|^p \, dx\right)^{\frac{1}{p}} = 0$ uniformly in S, i.e. for every $\varepsilon > 0$ there exists a $w > 0$ such that*

$$R > w, \ f \in S \Rightarrow (\int_{x|>R} |f(x)|^p dy)^{1/p} < \varepsilon.$$

2.4 Systems of differential equations and stability

Many real life situations are governed by systems of differential equations, so, they can arise quite easily from naturally occurring situations. For example if we consider the population problems then, to find the population of either the prey or the predator, we need to study a system of at least two differential equations that should be solved simultaneously in order to determine the population of the prey and the predator.

There are many interesting and important questions concerning systems of differential equations, such existence, uniqueness and stability of solutions. For systems of differential equations it's important to study the stability of solutions under small perturbations. Stability means that the trajectories do not change too much under small perturbations.

2.4.1 Stability theory of solutions for systems of differential equations

We begin by giving the theory concerning the stability of solutions of systems of differential equations [16]. Since a n th-order ordinary differential equation can be reduced through appropriate substitutions to a system of n first-order ordinary differential equations, the general system formulation can be written as

$$\dot{x}(t) = f(t, x(t)) \tag{2.1}$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$ may depends on the time t and the n-dimensional state variable x and is piecewise continuous in t and locally Lipschitz in x. D is a domain containing the origin.

Definition 10 *A solution $x(t)$ of (2.1) is said to be:*

- *Stable if, given any $\epsilon > 0$ and any $t_0 \geq 0$, there exists $\delta = \delta(\epsilon, t_0)$ such that*

$$|x(t_0) - y(t_0)| < \delta \Rightarrow |x(t) - y(t)| < \epsilon, \forall t \geq t_0 \geq 0, \tag{2.2}$$

for any solution $y(t)$ *of (2.1).* $|\cdot|$ *can be regarded as representing any norm on* $\mathbb{R}^n$.

• *Uniformly stable if, for every* $\epsilon > 0$, *there exits* $\delta = \delta(\epsilon)$, *independent of* t_0, *such that (2,2) is satisfied*

• *Unstable if it is not stable.*

• *Asymptotically stable if it is stable and for any* $t_0 \geq 0$ *there exists a positive constant* $c = c(t_0)$ *such that*

$$|x(t_0) - y(t_0)| < c \Rightarrow |x(t) - y(t)| \to 0 \text{ as }, t \to \infty,$$

for any solution $x(t)$ *of (2.1).*

• *Uniformly asymptotically stable if it is uniformly stable and there exists a positive constant* c, *independent of* t_0, *such that, for every* $\eta > 0$, *there exists* $T = T(\eta) > 0$ *such that, for all* $t_0 \geq 0$

$$|x(t_0) - y(t_0)| < c \Rightarrow |x(t) - y(t)| < \eta, \forall t \geq t_0 + T(\eta),$$

for any solution $x(t)$ *of (2.1).*

• *Globally uniformly asymptotically stable if it is uniformly stable with* $\delta = \delta(\epsilon)$ *satisfying* $\lim_{\epsilon \to \infty} \delta(\epsilon) = \infty$, *and for all positive* η *and* c, *there exists* $T = T(\eta, c) > 0$ *such that, for all* $t_0 \geq 0$

$$|x(t_0) - y(t_0)| < c \Rightarrow |x(t) - y(t)| < \eta, \forall t \geq t_0 + T(\eta, c),$$

for any solution $x(t)$ *of (2.1).*

Remark 5 *When* f *does not depend explicitly on the time* t, *(the system (2.1) is autonomous), the condition (2.2) in the definition of stability need only to be satisfied for* $t_0 = 0$, *then it will follow for all* $t_0 > 0$.

Definition 11 *The point* $\widetilde{x} \in \mathbb{R}^n$ *is an equilibrium point for (2.1), if* $f(t, \widetilde{x}) = 0$, *for all* $t \geq 0$.

If $\widetilde{x} \in \mathbb{R}^n$ is an equilibrium point for (2.1), then it is a solution for all t. It is important to know if this solution is stable, i.e., if it still unchanged on the interval $[0, \infty)$ under small changes in the initial data, this fact is important in applications, where in general the initial data aren't known perfectly.

2.4.2 Lyapunov's indirect method

Various criteria have been developed to prove stability. A more general method involves Lyapunov functions. Lyapunov in his Doctoral Thesis (1892) considered the system

$$\dot{x} = f(x), \tag{2.3}$$

where f is a nonlinear function, then he expanded the function f as a Taylor series about the equilibrium $x = 0$

$$f(x) \sim x \frac{\partial f}{\partial x}(0) + \bigcirc(x).$$

If the initial state $x(0) = x_0$ is chosen close enough to 0, then we can approximate the nonlinear system (2.3) by the linear system

$$\dot{x} = Ax, \text{ where } A = \frac{\partial f}{\partial x}(0).$$

The following theorem precise when this approximation can be used to determine the stability properties of the system (2.3).

Theorem 11 *Let $x = 0$ be an equilibrium point of the nonlinear system (2.3), where $f : D \to \mathbb{R}^n$ is continuously differentiable on D a domain containing the origin. Let $A = \frac{\partial f}{\partial x}(0)$, then*

- *$x = 0$ is asymptotically stable if $\mathrm{Re}(\lambda) < 0$ for all eigenvalues λ of A.*
- *$x = 0$ is unstable if $\mathrm{Re}(\lambda) > 0$ for some eigenvalue λ of A.*

Remark 6 *By calculating the eigenvalues of the Jacobian matrix at the equilibrium point, then applying Lyapunov's indirect method, we can test the stability of a nonlinear system. Unfortunately, this fails when all eigenvalues have* $\mathrm{Re}(\lambda) \leq 0$ *but some* $\mathrm{Re}(\lambda) = 0$ *and then higher-order terms in the series expansion of* f *become significant.*

2.4.3 Lyapunov's direct method

Lyapunov considered the system, see [16]

$$\dot{x}(t) = f(t, x(t)) \tag{2.4}$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$, piecewise continuous in t and locally Lipschitz in x, D is a domain containing the origin. The main idea of Lyapunov's direct method is to generate a function V, commonly known as a Lyapunov function, which is a generalization of a physical energy function.

Theorem 12 *Let* $x = 0$ *be an equilibrium point of the system (2.4) and* $U \subset D$ *be a domain containing* $x = 0$. *Suppose that there exists a continuous function* $V : [0, \infty) \times \mathbb{R}^n \to R$, *such that, with the time derivative along the system trajectories defined as*

$$\dot{V}(t, x(t)) = \lim \sup_{h \to 0^+} \frac{1}{h} \{V(t + h, x(t + h)) - V(t, x(t))\}$$

V satisfies:

$$V(t, 0) = 0, \forall t \geq 0.$$

$$V(t, x) \geq W_1(x), \forall t \geq 0, \forall x \in U,$$

for some continuous positive definite function W_1 *on* U.

$$\dot{V}(t, x) \leq 0, \forall t \geq t_0, \forall x \in U.$$

Then the equilibrium point $x = 0$ *is stable.*

Finally, notice that Lyapunov's direct method provides a way to analyze the stability of a system without explicitly solving the differential equations.

Chapter 3

Existence of Solutions for a Nonlinear Fractional System with Nonlocal Boundary Conditions

3.1 Introduction

In recent years the theory of differential fractional equations has become an interesting field to explore as long as this theory has many applications in several real world events as well as in many sciences, such as in engineering, physics, chemistry, biology, etc. [28,32,37,38,57]. Moreover, the study of the systems of fractional differential equations has become more and more popular tool for controlling and modeling different systems. Different methods are used in the study of the existence, uniqueness and positivity of solutions for systems of fractional differential equations, such fixed point theorems, upper and lower solutions method, Mawhin method, iterative approximation method... [2-5,19,21-24,46,51,52,54,55].

Systems for fractional differential equations have been studied by many authors. Ahmad et al, in [4] considered an initial value problem for a coupled differential system of fractional order given by

$$\begin{aligned} {}^{C}D_{0+}^{\rho}u(t) &= f\left(t, {}^{C}D_{0+}^{\beta}v(t)\right), u^{(k)}(0) = \eta_k, 0 < t \leq 1 \\ {}^{C}D_{0+}^{\sigma}v(t) &= g\left(t, {}^{C}D_{0+}^{\alpha}u(t)\right), v^{(k)}(0) = \xi_k, 0 < t \leq 1 \end{aligned}$$

where $f, g : [0,1] \times R \to R$ are given functions, ${}^{C}D_{0+}$ denotes the Caputo fractional derivative, $\rho, \sigma \in (m-1, m)$, $\alpha, \beta \in (n-1, n)$, $m, n \in N$, $\rho > \beta$, $\sigma > \alpha$, $k = 0, 1, 2, ..., m-1$, $\rho, \sigma, \alpha, \beta \notin N$, and η_k, ξ_k are suitable real constants. By the nonlinear alternative of Leray-Schauder, the authors proved the existence of solutions, then they established the uniqueness of solutions of the fractional differential system by applying Banach contraction principle.

By means of upper and lower solutions method, Guezane-Lakoud et al. in [25], proved the existence of positive solutions for a system of multi-order fractional differential equations with nonlocal boundary conditions, where each equation has an order that may be different from the order of the other equations, that is:

$$(P)\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0,\ 0 < t < 1, \\ u(0) = u'(0) = 0,\ Au(1) = Bu'(1), \end{cases}$$

where the function

$$u = (u_1, u_2, ..., u_n), u_i : [0,1] \to \mathbb{R},$$

$$D_{0+}^{\alpha}u(t) = \left(D_{0+}^{\alpha_1}u_1(t), D_{0+}^{\alpha_2}u_2(t), ..., D_{0+}^{\alpha_n}u_n(t)\right),$$

$D_{0+}^{\alpha_i}$ denotes the Riemann-Liouville fractional derivative of order α_i, $2 < \alpha_i < 3$, $i \in \{1, .., n\}$, $n \geq 2$, the function f is such that

$$\begin{aligned} f(t, u) &= (f_1(t, u), ..., f_n(t, u)), \\ u &= (u_1, u_2, ..., u_n), \end{aligned}$$

$$f_i \in C([0,1] \times \mathbb{R}^n, \mathbb{R}_+), A = (a_1, ..., a_n), B = (b_1, ..., b_n) \in \mathbb{R}^n.$$

In [21], Henderson et al. considered the system of nonlinear fractional differential equations

$$\begin{aligned}
D_{0+}^{\alpha}u(t)+\lambda f(t,u(t),v(t)) &= 0, 0<t<1, n-1<\alpha\leq n \\
D_{0+}^{\beta}u(t)+\mu g(t,u(t),v(t)) &= 0,\ 0<t<1, m-1<\beta\leq m \\
u^{(k)}(0) &= 0, k=0,...n-2 \\
u(1) &= \int_0^1 v(s)\,dH(s) \\
v^{(k)}(0) &= 0, k=0,...m-2 \\
v(1) &= \int_0^1 u(s)\,dH(s)
\end{aligned}$$

where $n,m\in N$, $n,m\geq 3$, D_{0+}^{α} and D_{0+}^{β} denote the Riemann–Liouville derivatives of orders α and β respectively, and the integrals in the boundary conditions are Riemann–Stieltjes integrals. By applying Guo–Krasnosel'skii fixed point theorem, the authors proved the existence of positive solutions.

In this work, we consider the following system of fractional differential equations with boundary conditions:

$$(FS)\begin{cases} -D_{0+}^{\alpha}u(t)=g(t)f(u(t)), 0<t<1, \\ u(0)=u'(0)=0,\ au(1)+bu'(1)=0, \end{cases}$$

where D_{0+}^{α} denotes the Riemann-Liouville fractional derivative, $2<\alpha<3$,

$u=(u_1,u_2,...,u_n)^T$ is an unknown function with

$$u_i:[0,1]\to\mathbb{R},\ \text{and } g:[0,1]\to\mathbb{R} \text{ is a given function,}$$

$$\begin{aligned}
f &: \mathbb{R}^n\to\mathbb{R}^n, \\
f(u) &= (f_1(u_1,u_2,...,u_n),...,f_n(u_1,u_2,...,u_n))^T, \\
f_i &: \mathbb{R}^n\to\mathbb{R}.
\end{aligned}$$

Using Banach contraction principle and the nonlinear alternative of Leray-Schauder, we prove the existence and uniqueness of solution for problem (FS).

3.2 Existence and uniqueness of solution

We need the following Lemma

Lemma 4 *For $\alpha > 0$, the general solution of the homogeneous equation*

$$D_{0^+}^{\alpha} u(t) = 0,$$

is given by

$$u(t) = c_0 t^{\alpha-n} + c_1 t^{\alpha-n-1} + \cdots + c_{n-2} t^{\alpha-2} + c_{n-1} t^{\alpha-1},$$

where $c_i, i = 1, 2, ..., n-1$, are arbitrary real constants, $n = [\alpha] + 1$ ($[\alpha]$ denotes the integer part of the real number α).

Lemma 5 *Let $y \in C([0,1], \mathbb{R})$. Assume that $a, b \in \mathbb{R}$, such that $a - b(\alpha - 1) \neq 0$, then for $i \in \{1, .., n\}$, the linear nonhomogeneous problem*

$$(S_i) = \begin{cases} -D_{0^+}^{\alpha} u_i(t) = y(t), 0 < t < 1, \\ u_i(0) = u_i'(0) = 0, \ a u_i(1) - b u_i'(1) = 0, \ i \in \{1, .., n\}, \end{cases} \tag{3.1}$$

has the following solution

$$u_i(t) = \int_0^1 G_i(t,s) y(s) ds, i \in \{1, .., n\} \tag{3.2}$$

where

$$G_i(t,s) = \begin{cases} -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{t^{\alpha-1}}{a - b(\alpha-1)} \left(\dfrac{a}{\Gamma(\alpha)} (1-s)^{\alpha-1} - \dfrac{b}{\Gamma(\alpha-1)} (1-s)^{\alpha-2} \right), \ s \leq t, \\ \dfrac{t^{\alpha-1}}{a - b(\alpha-1)} \left(\dfrac{a}{\Gamma(\alpha)} (1-s)^{\alpha-1} - \dfrac{b}{\Gamma(\alpha-1)} (1-s)^{\alpha-2} \right), \ s \geq t. \end{cases} \tag{3.3}$$

Proof: Let u_i be a solution of the fractional boundary value problem (S_i), $i \in \{1, .., n\}$. Using Lemma 4, we obtain

$$u_i(t) = -I_{0^+}^{\alpha} y(t) + At^{\alpha-1} + Bt^{\alpha-2} + Ct^{\alpha-3}, \tag{3.4}$$

by multiplying (3.4) by $t^{3-\alpha}$, we get

$$t^{3-\alpha} u_i(t) = -I_{0^+}^{\alpha} y(t)\, t^{\alpha-3} + At^2 + Bt + C.$$

According to the condition $u(0) = 0$, we obtain $C = 0$. Therefore, differentiating (3.4), it yields

$$u_i'(t) = -I_{0^+}^{\alpha-1} y(t) + (\alpha-1) At^{\alpha-2} + (\alpha-2) B. \tag{3.5}$$

Multiplying (3.5) by $t^{3-\alpha}$, we obtain

$$t^{3-\alpha} u_i'(t) = -I_{0^+}^{\alpha-1} y(t)\, t^{3-\alpha} + (\alpha-1) At + (\alpha-2) B. \tag{3.6}$$

It follows from condition $u_i'(0) = 0$ that $B = 0$, so,

$$u_i(t) = -I_{0^+}^{\alpha} y(t) + At^{\alpha-1}. \tag{3.7}$$

Since $au_i(1) - bu_i'(1) = 0$, then

$$A = \frac{a}{a - b(\alpha-1)} I_{0^+}^{\alpha} y(1) - \frac{b}{a - b(\alpha-1)} I_{0^+}^{\alpha-1} y(1). \tag{3.8}$$

Substituting A in (3.7), we get what follows

$$u_i(t) = \int_0^1 G_i(t,s)\, y(s)\, ds.$$

Lemma 6 *If $a > 0$ and $b < 0$, then the functions G_i are nonnegative, continuous and*

$$G_i(t,s) \leq \frac{1}{\Gamma(\alpha-1)}, \forall s,t \in [0,1], i \in \{1,..,n\}. \tag{3.9}$$

Let X be the Banach space of all functions

$$u \in C^n[0,1] = C[0,1] \times ... \times C[0,1]$$

with the norm $\|.\|$ defined by

$$\|u\| = \sum_{i=1}^{i=n} \max_{t\in[0,1]} |u_i(t)|.$$

Define the integral operator $T : X \to X$ by

$$T(u) = (T_1 u, T_2 u, ..., T_n u)$$

where

$$(T_i u)(t) = \int_0^1 G_i(t,s)\, g(s)\, f_i(u(s))\, ds. \tag{3.10}$$

Lemma 7 *The function $u \in X$ is a solution of the system (FS) if and only if $T_i u(t) = u(t)$, for all $t \in [0,1]$, $\forall i \in \{1,...,n\}$.*

Consequently, the existence of solutions for the system (FS) can be turned into a fixed point problem in X for the operator T.

The first main statement in this work is the uniqueness of solution of the boundary problem (FS).

Theorem 13 *Assume that*

i) $f_i \in C(\mathbb{R}^n, \mathbb{R})$, $g \in L^1([0,1], \mathbb{R})$

ii) There exists a constant $L > 0$ such that

$$|f_i(x_1, .., x_n) - f_i(y_1, ..., y_n)| \leq L \sum_{i=1}^{n} |x_i - y_i| \tag{3.11}$$

and

$$K = \frac{nL \|g\|_{L^1[0,1]}}{\Gamma(\alpha - 1)} < 1, \tag{3.12}$$

for all $t \in [0, 1]$ and for all $x_i, y_i \in \mathbb{R}$, $i = 1, ...n$. Then, the boundary value problem (FS) has a unique solution in X.

Proof: We will use the Banach contraction principle to prove that the operator T has a fixed point. Using the properties of the functions G_i, it yields

$$\begin{aligned} |T_i x(t) - T_i y(t)| &\leq \int_0^1 |G_i(t,s)| |g(s)| |f_i(x(s)) - f_i(y(s))| ds \\ &\leq \frac{L}{\Gamma(\alpha - 1)} \int_0^1 |g(s)| \sum_{i=1}^{n} |x_i(s) - y_i(s)| ds \\ &\leq \frac{L}{\Gamma(\alpha - 1)} \|g\|_{L^1[0,1]} \|x - y\|, \end{aligned}$$

then by taking the maximum over $t \in [0, 1]$, it follows

$$\max_{t \in [0,1]} |T_i x(t) - T_i y(t)| \leq \frac{L}{\Gamma(\alpha - 1)} \|g\|_{L^1[0,1]} \|x - y\|. \tag{3.13}$$

Summing the n inequalities in (3.13), it yields

$$\begin{aligned} \|Tx - Ty\| &\leq \frac{nL}{\Gamma(\alpha - 1)} \|g\|_{L^1[0,1]} \|x - y\| \\ &= K \|x - y\|. \end{aligned}$$

So, T is a contraction. As a consequence of Banach fixed-point theorem, we deduce that T has a fixed point that is the unique solution of the (FS), this achieves the proof.

Now we give an existence result for the boundary problem (FS).

Theorem 14 *Assume that* $f_i(0) \neq 0, i \in \{1,..,n\}$, $\exists \eta > 0$ *and there exists a nonnegative function* $\Psi \in C(\mathbb{R}^n, (0,\infty))$ *satisfying*

$$\Psi(x_1,...,x_n) \leq \Psi(y_1,...,y_n), \ 0 \leq x_i \leq y_i, \ i = 1,...,n.$$

If

$$|f_i(u)| \leq \Psi(|u|), t \in [0,1], \ u \in \mathbb{R}^n \tag{3.14}$$

and

$$\frac{n}{\Gamma(\alpha-1)} \Psi(\eta,...,\eta) \|g\|_{L^1[0,1]} \leq \eta, \tag{3.15}$$

then, the problem (FS) has at least one nontrivial solution $u^* \in X$.

For the proof of Theorem we need the nonlinear alternative of Leray-Schauder that we recall:

Lemma 8 *Let* F *be a Banach space and* Ω *a bounded open subset of* F, $0 \in \Omega$. *Let* $T : \Omega \to F$ *be a completely continuous operator. Then, either there exists* $x \in \partial\Omega$, $\lambda > 1$ *such that* $T(x) = \lambda x$, *or there exists a fixed point* $x \in \Omega$ *of* T.

Proof: of Theorem 14. The continuity of the operator T follows from the continuity of f. Set $B_\eta = \{u \in X : \|u\| \leq \eta\}$. Let us prove that $T : B_\eta \to X$ is completely continuous operator. By assumptions, we have for each $t \in [0,1]$

$$\begin{aligned} |T_i u(t)| &\leq \int_0^1 |G_i(t,s)| |g(s)| |f_i(u(s))| ds \\ &\leq \frac{1}{\Gamma(\alpha-1)} \int_0^1 |g(s)| \Psi(|u(s)|) ds \\ &= \frac{1}{\Gamma(\alpha-1)} \int_0^1 |g(s)| \Psi(|u_1(s)|,...,|u_2(s)|) ds \end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha-1)} \Psi(\eta,...,\eta) \|g\|_{L^1[0,1]}.$$

Taking the supremum over $[0,1]$, then summing the obtained inequalities according to i from 1 to n, we get

$$\|Tu\| \leq \frac{n\Psi(\eta,...,\eta)\|g\|_{L^1[0,1]}}{\Gamma(\alpha-1)}$$

which implies that $T(B_\eta)$ is uniformly bounded. Let us show that (Tu) is equicontinuous, $u \in B_\eta$. Let $t_1,\, t_2 \in [0,1]$, $t_1 < t_2$, then

$$\begin{aligned}
|T_i u(t_1) - T_i u(t_2)| &\leq \int_0^1 |G_i(t_1,s) - G_i(t_2,s)|\,|g(s)|\,|f_i(u(s))|\,ds \\
&\leq \int_0^{t_1} |G_i(t_1,s) - G_i(t_2,s)|\,|g(s)|\,|f_i(u(s))|\,ds \\
&\quad + \int_{t_1}^{t_2} |G_i(t_1,s) - G_i(t_2,s)|\,|g(s)|\,|f_i(u(s))|\,ds \\
&\quad + \int_{t_2}^{1} |G_i(t_1,s) - G_i(t_2,s)|\,|g(s)|\,|f_i(u(s))|\,ds
\end{aligned}$$

then

$$|T_i u(t_1) - T_i u(t_2)| \leq$$

$$\begin{aligned}
&\frac{\Psi(\eta,...,\eta)}{\Gamma(\alpha)}\Bigg[\int_0^{t_1}\left[\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) + \left((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right)\right]|g(s)|\,ds \\
&+\int_{t_1}^{t_2}\left[\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) + (t_2-s)^{\alpha-1}\right]|g(s)|\,ds \\
&+\int_{t_2}^{1}\left[\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)\right]|g(s)|\,ds\Bigg]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\Psi(\eta,...,\eta)}{\Gamma(\alpha)}\Bigg[\left[\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) + (t_2-t_1)^{\alpha-1}\right]\int_0^{t_1}|g(s)|\,ds \\
&\quad + \left[\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) + (t_2-t_1)^{\alpha-1}\right]\int_{t_1}^{t_2}|g(s)|\,ds \\
&\quad + \left[\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)\right]\int_{t_2}^{1}|g(s)|\,ds \\
&\leq \frac{\Psi(\eta,...,\eta)}{\Gamma(\alpha)}\left[3\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) + 2(t_2-t_1)^{\alpha-1}\right]\int_0^1|g(s)|\,ds.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. By Ascoli-Arzela theorem, we conclude that the operator $T : X \to X$ is completely continuous. Now we apply the nonlinear alternative of Leray-Schauder. Let $u \in \partial B_\eta$, such that $u = \lambda Tu$ for some $0 < \lambda < 1$. We have

$$\begin{aligned} u_i(t) &= \lambda T_i u(t) \leq \max_{t\in[0,1]} |T_i u(t)| \\ &\leq \frac{1}{\Gamma(\alpha-1)} \Psi(\eta, ..., \eta) \|g\|_{L^1[0,1]}. \end{aligned}$$

Taking the supremum over $[0, 1]$, then summing the obtained inequalities according to i from 1 to n, we get

$$\|u\| \leq \frac{n}{\Gamma(\alpha-1)} \Psi(\eta, ..., \eta) \|g\|_{L^1[0,1]}.$$

taking (3.15) into account, we conclude

$$\|u\| < \eta,$$

that contradicts the fact $u \in \partial B_\eta$. Hence T has at least one fixed point $u^* \in B_\eta$ and then the (FS) has a nontrivial solution $u^* \in B_\eta$.

3.3 Examples

We give examples to illustrate the usefulness of the main results.

Example 1. Consider the following two-dimensional fractional order system

$$(S_i) = \begin{cases} D_{0^+}^{\frac{5}{2}} u_1(t) = 2t\frac{e^{-\left(u_1^2+u_2^2\right)}}{1+u_1^2+u_2^2}, \quad D_{0^+}^{\frac{5}{2}} u_2(t) = 2t\frac{e^{-u_1^2}}{1+u_1^2+u_2^2}, \\ u_1(0) = 0, \ u_1'(0) = 0, \quad u_2(0) = 0, \ u_2'(0) = 0, \\ au_1(1) - bu_1'(0) = 0, \quad au_2(1) - bu_2'(0) = 0. \end{cases} \tag{3.16}$$

We have

$$\begin{aligned}\alpha &= \frac{5}{2}, g(t) = 2t, f_1(u_1,u_2) = \frac{e^{-\left(u_1^2+u_2^2\right)}}{1+u_1^2+u_2^2},\\ f_2(u_1,u_2) &= \frac{e^{-u_1^2}}{1+u_1^2+u_2^2}, f_i \in C\left(\mathbb{R}^2,\mathbb{R}\right), f_i(0) \neq 0.\end{aligned}$$

If we choose

$$\Psi(u_1,u_2) = \frac{1}{1+u_1^2+u_2^2},$$

then

$$|f_i(u_1,u_2)| \leq \frac{1}{1+u_1^2+u_2^2} = \Psi(|u_1|,|u_2|).$$

For $\eta = 2$, we get

$$\frac{n}{\Gamma(\alpha-1)} \Psi(\eta,\eta) \|g\|_{L^1[0,1]} \leq \frac{2}{\Gamma\left(\frac{3}{2}\right)(1+2\eta^2)} = 0.250\,75 \leq \eta.$$

Then, according to Theorem 14, the boundary value problem (3.16) has at least one fixed point $u^* \in B_2$.

Example 2. Consider the following two-dimensional fractional order system

$$(S_i) = \begin{cases} D_{0+}^{\frac{5}{2}} u_1(t) = \frac{e^{-t}}{10}(u_1-u_2), \quad D_{0+}^{\frac{5}{2}} u_2(t) = \frac{e^{-t}}{10}(u_1+1),\\ u_1(0)=0,\ u_1'(0)=0, \quad u_2(0)=0,\ u_2'(0)=0,\\ au_1(1) - bu_1'(0) = 0, \quad au_2(1) - bu_2'(0) = 0.\end{cases} \tag{3.17}$$

We have

$$\begin{aligned}\alpha &= \frac{5}{2}, g(t) = \frac{e^{-t}}{10}, f_1(u_1,u_2) = \frac{e^{-t}}{10}(u_1-u_2),\\ f_2(u_1,u_2) &= \frac{e^{-t}}{10}(u_1+1), f_i \in C\left(\mathbb{R}^2,\mathbb{R}\right),\end{aligned}$$

then

$$\left|f_i\left(x_1, x_2\right) - f_i\left(y_1, y_2\right)\right| \leq L \sum_{i=1}^{2} \left|x_i - y_i\right|$$

with $L = 1$ and

$$K = \frac{2\left(1 - e^{-1}\right)}{10\Gamma\left(\frac{3}{2}\right)} = 0.142\,65 < 1,$$

then hypotheses of Theorem 13 are satisfied. So, the boundary value problem (3.17) has a unique solution $u \in X$.

Chapter 4

Stability Analysis of Conformable Nonlinear Systems

4.1 Introduction

Conformable derivative is attracting more attention and many papers on this subject appeared in the literature. This new derivative is introduced recently by Khalil et al. [31], and is based on a limit form as in the case of the classical derivative. Later, Abdeljawad developed this theory [1]. Boundary value problems for conformable derivatives have been studied by several authors.

In [30], Khaldi et al. proved by using the method of upper and lower solutions and Schauder's fixed point theorem, the existence of solutions for a conformable boundary value problem, then established a Lyapunov type inequality for the corresponding problem. Precisely the studied problem was

$$T_a^{\alpha} u\,(t) + f(t, u\,(t)) = 0, a < t < b,$$

$$u\,(a) = u\,(b) = 0$$

where $1 < \alpha \leq 2$, T_a^{α} denotes the conformable fractional derivative of order α, u is the unknown function and $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function.

In [9], Batarfi et al. studied the following conformable fractional boundary value problem:

$$D^{\alpha}(D + \lambda) x(t) = f(t, x(t)), 0 \leq t \leq 1$$

$$\begin{aligned} x(0) &= x'(0) = 0, \\ x(1) &= \beta x(\eta) \end{aligned}$$

where D^{α} is the conformable fractional derivative of order $\alpha \in (1, 2]$, D is the ordinary derivative, $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is a known continuous function, $\lambda, \beta \in \mathbb{R}$, $\lambda > 0$, $\eta \in (0, 1)$. The authors proved the existence and uniqueness of solution by using some fixed point theorems.

In [10], Bayour et al., solved an initial conformable fractional boundary value problem by applying a generalization of the lower and upper solutions method:

$$T_a^{\alpha} x(t) + f(t, x(t)) = 0, a \leq t \leq b, a > 0$$

$$x(a) = x_0$$

where T_a^{α} denotes the conformable fractional derivative of order α, $\alpha \in (0, 1)$, $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

An important question in the study of differential equations is the stability of solutions. The theory of stability of the motion of systems with a finite number of degrees of freedom which was created by Lyapunov. Since then the subject knew an increasing development and is widely used in the engineering problems.

Since the dynamics of many systems involve non integer derivatives, then systems are better described by fractional order differential equations. For this reason several research works were done to solve stability problems for fractional order nonlinear systems [7,11-16,27,29,36,39-41,44,45,49,50,58].

As the physical parameters in a real system can change their values in the process of the latter, then this phenomenon is adequately reflected in the mathematical model of the process. Changing the parameters causes new states of equilibrium to occur of the system. Therefore, the theory of the stability of systems with uncertain parameters has been developed intensively and lot of investigations on stability and stabilization for nonlinear integer-order dynamic systems exist in literature [9,11-16,27,29,36,39-41,44,45,49,50,58].

In [11], Ben Hamed et al. considered the following perturbed system:

$$\dot{x} = f(t, x, \varepsilon), \quad x(t_0, \varepsilon) = \eta(\varepsilon),$$

where $t \in [t_0, t_1]$ is the time, $x \in D \subset \mathbb{R}^n$ is the state, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and $f : [t_0, t_1] \times D \times [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}^n$ is continuous in (t, x, ε) and locally Lipschitz in (x, ε), uniformly in t , and η is locally Lipschitz in ε.

The authors presented a converse Lyapunov theorem for the notion of uniform practical exponential stability of the nonlinear differential equations in presence of small perturbation.

In a recent paper [12], Ben Makhlouf et al. considered a parameterized family of fractional differential equations with a Caputo derivative (P)

$$\begin{aligned} {}^{C}D^{\alpha}_{t_0} x(t) &= f(t, x, \varepsilon), t \geq t_0 \geq 0 \\ x(t_0) &= x_0 \end{aligned}$$

where $0 < \alpha < 1$, $\varepsilon \in \mathbb{R}^{*}_{+}$, $x(t) \in \mathbb{R}^n$, $f(.,.,\varepsilon) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and locally Lipschitz in x. The authors studied the ε^*-practical Mittag Leffler stability of (P) by using a Lyapunov function.

Then they considered a perturbed system (PE):

$$\begin{aligned} {}^{C}D^{\alpha}_{t_0}x(t) &= Ax(t) + g(t,x,\varepsilon), t \geq t_0 \geq 0 \\ x(t_0) &= x_0 \end{aligned}$$

where $0 < \alpha < 1$, $\varepsilon \in \mathbb{R}^{*}_{+}$, $x(t) \in \mathbb{R}^n$, $g(.,.,\varepsilon) : \mathbb{R}_{+} \times \mathbb{R}^n \to \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ a constant matrix. The authors gave sufficient condition and proved that the system (PE) is ϵ-uniformly practically Mittag Leffler stable.

Finally, they discussed the problem of stabilization for a class of nonlinear fractional-order systems with uncertainties:

$$\begin{aligned} {}^{C}D^{\alpha}_{t_0}x(t) &= Ax(t) + B(\phi(x,u) + u), t \geq t_0 \geq 0 \\ x(t_0) &= x_0 \end{aligned}$$

where $0 < \alpha < 1$, $x(t) \in \mathbb{R}^n$, $u \in \mathbb{R}^n$ $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$ are two constant matrices, $\phi : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$. Sufficiently conditions are given by using Lyapunov theory.

Similarly to the above study we will consider a conformable fractional system of order α, $0 < \alpha < 1$ having the following form:

$$T^{\alpha}_{t_0}x = f(t,x,\varepsilon), t > t_0$$

$$x(t_0) = x_0$$

where $t_0 > 0$, $x \in \mathbb{R}^n$, $f(.,.,\varepsilon) : \mathbb{R}^{+} \times \mathbb{R}^n \to \mathbb{R}^n$ is a given nonlinear function. We will focus on the ε^{*}-uniformly practically fractional exponentially stable. Then we will consider the perturbed system as well as the conformable fractional systems with uncertainties.

4.2 Stability of conformable nonlinear systems depending on a parameter

In this section, we shall give sufficient conditions on the stability of conformable nonlinear systems depending on a parameter.

Consider the following system for differential equations involving conformable derivative of order α:

$$T_{t_0}^{\alpha} x = f(t, x, \varepsilon), \ t > t_0 \tag{4.1}$$

$$x(t_0) = x_0 \tag{4.2}$$

where $0 < \alpha < 1$, $t_0 > 0$, $x \in \mathbb{R}^n$, $f(.,.,\varepsilon) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a given nonlinear function.

Our objective is to establish the stability of system $(4.1)-(4.2)$ by using Lyapunov techniques. We will assume that for any $\varepsilon > 0$ and initial data $(t_0, x_0) \in [0, \infty) \times \mathbb{R}^n$, the system $(4.1)-(4.2)$ has a unique solution $x_\varepsilon(t) \in C([t_0, +\infty), \mathbb{R}^n)$.

Definition 12 *The system* $(4.1)-(4.2)$ *is said to be* ε^*- *uniformly practically fractional exponentially stable if for all* $0 < \varepsilon < \varepsilon^*$ *there exists positive scalars* $K(\varepsilon)$, $\lambda\,(\varepsilon)$ *and* $\rho\,(\varepsilon)$ *such that*

$$\|x_{\varepsilon}\,(t)\| \leq k\,(\varepsilon)\,\|x_{\varepsilon}\,(t_0)\|\,E_\alpha\,(\lambda\,(t-t_0)) + \rho\,(\varepsilon)\,,\ \forall t \geq t_0 \geq 0, \tag{4.3}$$

with $\rho\,(\varepsilon) \to 0$ *as* $\varepsilon \to 0^+$ *and there exists* K, λ_1, $\lambda_2 > 0$ *such that* $\lambda_1 \leq \lambda\,(\varepsilon) \leq \lambda_2$, $0 < K(\varepsilon) \leq K$ *for all* $\varepsilon \in]0, \varepsilon^*]$.

Proposition 4 *Let* $p \geq 1$ *and* $\varepsilon^* > 0$. *Assume that for all* $\varepsilon \in]0, \varepsilon^*]$ *there exist a continuous function* $V_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, *a continuous function:* $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ *and positive constants scalar* $a_1(\varepsilon)$, $a_2(\varepsilon)$, $a_3(\varepsilon)$, $\eta_1\,(\varepsilon)$ *and* $\eta_2\,(\varepsilon)$ *such that*

(A_1)

$$a_1(\varepsilon)\,\|x\|^p \leq V_\varepsilon\,(t, x) \leq a_2(\varepsilon)\,\|x\|^p + \eta_1\,(\varepsilon)\,, \forall t \geq t_0, x \in \mathbb{R}^n$$

(A_2) *$V_\varepsilon(t,x)$ has a conformable derivative of order α for all $t>t_0$, $x\in\mathbb{R}^n$*

(A_3)

$$T^{\alpha}_{t_0}V_\varepsilon(t,x_\varepsilon(t)) \le -a_3(\varepsilon)\|x\|^p + \eta_2(\varepsilon)\mu(t), \forall t\ge t_0, x\in\mathbb{R}^n$$

with

$$\frac{a_3(\varepsilon)}{a_2(\varepsilon)} \ge \lambda,\ 0<\frac{a_2(\varepsilon)}{a_1(\varepsilon)}\le k,\ \lambda,k>0$$

There exists $M_1\ge 0$ such that

$$\int_0^t (s-t_0)^{\alpha-1}E_\alpha(-\lambda(t-t_0))E_\alpha(\lambda(s-t_0))\mu(s)\,ds \le M_1, \forall t\ge 0$$

- *$C(\varepsilon)\to 0$ as $\varepsilon\to 0^+$, where*

$$C(\varepsilon)=\eta_1(\varepsilon)\frac{a_2(\varepsilon)+Ma_3(\varepsilon)}{a_1(\varepsilon)a_2(\varepsilon)}+\eta_2(\varepsilon)\frac{M}{a_1(\varepsilon)} \tag{4.4}$$

with $M=M_1+\dfrac{1}{\lambda}$.
Then, the system $(4.1)-(4.2)$ is ε^*-uniformly practically fractional exponentially stable.

Proof: Taking (A_1) and (A_3) into account, it yields

P roof.

$$T^{\alpha}_{t_0}V(t,x_\varepsilon(t)) \le -\lambda V(t,x_\varepsilon(t))+\rho(t)l(\varepsilon), \forall t\ge t_0, \tag{4.5}$$

where

$$\rho(t)=(1+\mu(t)),\ l(\varepsilon)=\eta_2(\varepsilon)+\eta_1(\varepsilon)\frac{a_3(\varepsilon)}{a_2(\varepsilon)}.$$

Let

$$h(t)=E_\alpha(\lambda(t-t_0))V(t,x_\varepsilon(t))-l(\varepsilon)I^{\alpha}_{t_0}\left[E_\alpha(\lambda(t-t_0))\rho(t)\right].$$

Applying Lemma 1 on conformable calculus, (see Cahpter 1), we get

$$\begin{aligned} T^{\alpha}_{t_0}h(t) &= \lambda E_\alpha(\lambda(t-t_0))V(t,x_\varepsilon(t)) \\ &\quad + E_\alpha(\lambda(t-t_0))T^{\alpha}_{t_0}V(t,x_\varepsilon(t)) - l(\varepsilon)E_\alpha(\lambda(t-t_0))\rho(t). \end{aligned}$$

It follows from (4.5) that

$$T_{t_0}^{\alpha} h(t) \leq 0, \ \forall t \geq t_0.$$

Thus

$$h(t) \leq V(t_0, x_\varepsilon(t_0)), \forall t \geq t_0,$$

consequently

$$\begin{aligned} V(t, x_\varepsilon(t)) &\leq E_\alpha(-\lambda(t-t_0)) V(t_0, x_\varepsilon(t_0)) \\ &+ l(\varepsilon) \int_{t_0}^{t} (s-t_0)^{\alpha-1} E_\alpha(-\lambda(t-t_0)) E_\alpha(\lambda(s-t_0)) \rho(s) ds. \end{aligned}$$

Using the change of variable $u = \dfrac{(s-t_0)^\alpha}{\alpha}$, we obtain

$$\begin{aligned} & \int_{t_0}^{t} (s-t_0)^{\alpha-1} E_\alpha(-\lambda(t-t_0)) E_\alpha(\lambda(s-t_0)) ds \\ = \ & E_\alpha(-\lambda(t-t_0)) \int_0^{\frac{(t-t_0)^\alpha}{\alpha}} \exp(\lambda\mu) d\mu \\ = \ & E_\alpha(-\lambda(t-t_0)) \frac{E_\alpha(\lambda(t-t_0)-1)}{\lambda} \\ \leq \ & \frac{1}{\lambda}. \end{aligned}$$

Then

$$V(t, x_\varepsilon(t)) \leq E_\alpha(-\lambda(t-t_0)) V(t_0, x_\varepsilon(t_0)) M l(\varepsilon).$$

We need the following inequalities

$$\begin{aligned} (a+b)^p &\leq 2^{p-1}(a^p + b^p), \\ (a+b)^{\frac{1}{p}} &\leq a^{\frac{1}{p}} + b^{\frac{1}{p}}, \forall p \geq 1, a, b > 0 \end{aligned}$$

It follows from the above inequalities and (A_1) that

$$\|x_{\varepsilon}(t)\| \leq \left(\frac{a_2(\varepsilon)}{a_1(\varepsilon)}\right)^{\frac{1}{p}} E_\alpha\left(-\frac{\lambda}{p}(t-t_0)\right) V(t_0, x_\varepsilon(t_0)) + \eta(\varepsilon), \ \forall t \geq t_0$$

where

$$\eta(\varepsilon) = C(\varepsilon)^{\frac{1}{\varepsilon}}.$$

Hence, the system $(4.1)-(4.2)$ is ε^*-uniformly practically fractional exponentially stable.

■

4.3 Stability for perturbed conformable systems

Let us consider a physical input system modeled by finite dimensional ordinary differential equations [7]

$$\dot{x} = f(x, u) \tag{4.6}$$

where $x = (x_1, ..., x_n) \in \mathbb{R}^n$ represents the state variables, $u \in \mathbb{R}^q$ represents the input variables and $f : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$. When $u = 0$, then the system $\dot{x} = f(x, 0)$ describes the natural dynamics of (4.6) in the case where no energy is supplied by the input channels. To study the stability behavior by means of well designed feedback laws, it is convenient to choose the input as a sum $u = u_e + u_c$. Here u_e represents external forces and u_e is actually available for control action. The system (4.6) is said to be stabilizable if there exists a map $u_c = k(x)$ such that the closed loop system

$$\dot{x} = f(x, k(x) + u_e)$$

presents improved stability performances.

The term feedback is used to designate a situation in which two or more dynamic systems are interconnected so that each system influences the other and their dynamics are thus strongly coupled. Feedback linearization is a common approach used to control nonlinear systems. The approach consists in proposing a transformation of the nonlinear system into an equivalent linear system by a change of variables and an appropriate control input. To ensure that the transformed system is an equivalent representation of the original system, the transformation must be a diffeomorphism.

Now, let us consider the following perturbed system

$$T_{t_0}^{\alpha}x = Ax + Bu + g(t, x, u, \varepsilon), t > t_0 \tag{4.7}$$

$$x(t_0) = x_0 \tag{4.8}$$

where $0 < \alpha < 1$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^q$, A and B are respectively $(n \times n)$, $(n \times q)$ constant matrices, $g(., ., ., \varepsilon) : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ is a given nonlinear function.

Let us give the following assumptions:

(H_1) The perturbation term $g(t, x, u, \varepsilon)$ satisfies, for all $t \geq 0, \varepsilon > 0, x \in \mathbb{R}^n$ and $u \in \mathbb{R}^q$.

$$\|g(t, x, u, \varepsilon)\| \leq \delta_1(\varepsilon)\nu(t) + \delta_2(\varepsilon)\|x\| + \delta_3(\varepsilon)\|u\|,$$

here $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$, $\delta_3(\varepsilon) > 0$, $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$, $\delta_3(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ and ν is a nonnegative continuous function.

(H_2) There exists a constant, square, symmetric and positive definite matrix P, a constant matrix $K(q \times n)$ and a positive constant η such that

$$(A + BK)^T P + P(A + BK) + \eta I < 0, \tag{4.9}$$

and $t \to \int_{t_0}^{t} (s - t_0)^{\alpha - 1} E_\alpha(-\lambda(t - t_0)) E_\alpha(\lambda(s - t_0)) \nu^2(s)\, ds$ is a bounded function.

Theorem 15 *Assume that (H_1) and (H_2) hold, then the feedback law*

$$u(x) = Kx \tag{4.10}$$

ε^**-uniformly practically fractional exponentially stabilizes the system* $(4.7) - (4.8)$.

Proof: It follows from the assumption (H_1) that

$$\begin{aligned} 2x^T Pg(t,x,u,\varepsilon) &\leq 2\|x\|\|P\|\|g(t,x,u,\varepsilon)\| \\ &\leq 2\|x\|\|P\|\delta_1(\varepsilon)\nu(t) + 2\|x\|^2\|P\|\delta_2(\varepsilon) + 2\|x\|^2\|P\|\|K\|\delta_3(\varepsilon). \end{aligned} \tag{4.11}$$

Let $0 < \lambda_1 < \lambda$, we have

$$2\|x\|\|P\|\delta_1(\varepsilon)\nu(t) \leq \lambda_1\|x\|^2 + \frac{\|P\|^2\delta_1^2(\varepsilon)\nu(t)^2}{\lambda_1}. \tag{4.12}$$

Substituting (4.11) into (4.12) yields

$$\begin{aligned} 2x^T Pg(t,x,u,\varepsilon) &\leq (\lambda_1 + 2\delta_2(\varepsilon)\|P\| + 2\delta_3(\varepsilon)\|P\|\|K\|)\|x\|^2 \\ &\quad + \frac{\|P\|^2\delta_1^2(\varepsilon)\nu(t)^2}{\lambda_1}. \end{aligned}$$

Since $\delta_2(\varepsilon)$, $\delta_3(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ then there exists $\varepsilon^* > 0$ such for all $0 < \varepsilon \leq \varepsilon^*$,

$$\lambda_1 + 2\delta_2(\varepsilon)\|P\| + 2\delta_3(\varepsilon)\|P\|\|K\| \leq \lambda.$$

Then, for $0 < \varepsilon \leq \varepsilon^*$, we have

$$2x^T Pg(t,x,u,\varepsilon) \leq \lambda_1\|x\|^2 + \frac{\|P\|^2\delta_1^2(\varepsilon)\nu(t)^2}{\lambda_1}. \tag{4.13}$$

Let $0 < \varepsilon \leq \varepsilon^*$. Chosing the Lyapunov function $V(t,x) = x^T Px$, it yields

$$\begin{aligned} T_{t_0}^{\alpha} V(t,x_\varepsilon) &= x_\varepsilon(t)^T((A+BK)TP + P(A+BK))x_\varepsilon(t) \\ &\quad + 2x_\varepsilon(t)^T Pg(t,x,u,\varepsilon) \\ &\leq -\eta_1\|x\|^2 + \frac{\|P\|^2\delta_1^2(\varepsilon)\nu(t)^2}{\lambda_1}, \end{aligned}$$

where $\eta_1 = \eta - \lambda$.
Since all hypotheses of Proposition 4 are satisfied, then the system $(4.7) - (4.8)$ is ε^*-uniformly practically fractional exponentially stable.

4.4 Stability of nonlinear conformable systems with uncertainties

We discuss the problem of stabilization for a class of nonlinear conformable systems with uncertainties. Consider the system

$$T_{t_0}^{\alpha} x = Ax + B\left(\Phi\left(x,u\right) + u\right) + g(x,u), \tag{4.14}$$

$$x(t_0) = x_{_0}, \tag{4.15}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^q$, A and B are respectively $(n \times n)$, $(n \times q)$ constant matrices, $\Phi : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$.

Assume that the following assumptions are satisfied.

(H_3) There exists a square, symmetric and a positive definite matrix P and $\eta > 0$ such that the following inequality holds:

$$A^T P + PA + \eta I < 0. \tag{4.16}$$

(H_4) There exists a nonnegative continuous function $\Psi : \mathbb{R}^n \to \mathbb{R}$ such that:

$$\left\|\Phi\left(x,u\right)\right\| \leq \Psi\left(x\right), \ \forall x \in \mathbb{R}^n, u \in \mathbb{R}^q. \tag{4.17}$$

(H_5) There exists a constant $k > 0$, with $2k\left\|P\right\| < \eta$ and such that

$$\left\|g\left(t,u\right)\right\| \leq k\left\|x\right\|, \forall x \in \mathbb{R}^n, u \in \mathbb{R}^q$$

Theorem 16 *Suppose that the assumptions* (H_1), (H_3), (H_4) *and* (H_5) *hold, then the feed-back law*

$$u(\varepsilon, x) = -\frac{B^T P x \Psi(x)^2}{\|B^T P x\| \Psi(x) + \rho(\varepsilon)}, \tag{4.18}$$

where $\rho(\varepsilon) \to 0$ *as* $\varepsilon \to 0^+$, $\rho(\varepsilon) > 0$, $\forall \varepsilon > 0$, ε^**-uniformly practically fractional exponentially stabilizes the system* $(4.14) - (4.15)$.

Proof: Choose the Lyapunov function $V(t,x) = x^T P x$, we have

$$\begin{aligned} T_{t_0}^{\alpha} V(t, x_\varepsilon(t)) &= 2x_\varepsilon(t)^T P[A x_\varepsilon + B((x_\varepsilon, u) + u) + g(x_\varepsilon, u) \\ &\leq x_\varepsilon(t)^T \left(A^T P + PA + I\right) x_\varepsilon(t) - \frac{2x_\varepsilon(t)^T P B B^T P x_\varepsilon \Psi(x)^2}{\|B^T P x_\varepsilon\| \Psi(x) + \rho(\varepsilon)} \\ &\quad + 2\left\|B^T P x_\varepsilon\right\| \Psi(x) + 2\|P\| \|x_\varepsilon\| \|g(x_\varepsilon, u)\| \\ &\leq -\eta \|x_\varepsilon\|^2 + \frac{2\left\|B^T P x_\varepsilon\right\| \Psi(x_\varepsilon) \rho(\varepsilon)}{\|B^T P x_\varepsilon\| \Psi(x) + \rho(\varepsilon)} + 2k\|P\| \|x\|^2 \\ &\leq -\eta_1 \|x_\varepsilon(t)\|^2 + 2\rho(\varepsilon), \end{aligned}$$

where $\eta_1 = \eta - 2k\|P\|$. Hence, all hypotheses of proposition 4 are satisfied . Then the system $(4.14) - (4.15)$ is ε^*-uniformly practically fractional exponentially stable.

4.4.1 Examples

Example 1 *Consider the following conformable fractional-order system:*

$$\begin{cases} T_{t_0}^{\alpha} x_1(t) = -x_1 + x_2 + \varepsilon e^{-t}(x_1 + u) + \varepsilon^2 \dfrac{1}{1+t^2}, \\ T_{t_0}^{\alpha} x_2(t) = x_1 - 2x_2 + \varepsilon e^{-t} x_2 + \varepsilon^2 \dfrac{2t}{1+t^2} + u, \\ T_{t_0}^{\alpha} x_3(t) = 3x_3 + \dfrac{\varepsilon^2}{1+t^2} + 1.5u, \end{cases} \tag{4.19}$$

where, $0 < \alpha < 1$ *and* $x(t) = (x_1(t), x_2(t), x_3(t)) \in \mathbb{R}^3$.

This system has the same form as (4.7) *with*

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Example 2

$$B = \begin{pmatrix} 0 \\ 1 \\ 1.5 \end{pmatrix}$$

and

$$g(t, x, u, \varepsilon) = \varepsilon e^{-t}(x_1, x_2, x_3) + \left(\frac{\varepsilon^2}{1+t^2}, \frac{2\varepsilon^2 t}{1+t^2}, \frac{\varepsilon^2}{1+t^2} \right).$$

The perturbation term $g(t, x, u, \varepsilon)$ *satisfies* (H_1) *and* (H_2)*with* $\delta_1(\varepsilon) = \varepsilon^2$, $\delta_2(\varepsilon) = \delta_3(\varepsilon) = \varepsilon$ *and* $\nu(t) = \sqrt{3}$.

Select $P = 2I$, *since*

$$A^T P + PA + I = \begin{pmatrix} -3 & 4 & 0 \\ 4 & -7 & 0 \\ 0 & 0 & 13 \end{pmatrix}.$$

After solving the inequality (4.9) *via the Matlab LMI toolbox, we can obtain* $\eta = 2.0381$, *and*

$$K = (-0.2844 \qquad 1.8377 \qquad -3.3003)$$

The chosen gain K *confirm the* ε^**-uniformly practical fractional exponential stability of the closed-loop system* (4.19).

Example 3 *Consider the following conformable fractional-order system:*

$$\begin{cases} T_{t_0}^{\alpha} x_1(t) = -3x_1 + x_2 + \frac{1}{4}\sin(u)\,x_1, \\ T_{t_0}^{\alpha} x_2(t) = x_1 - 3x_2 + 1.5u + \frac{x_1 x_2}{1+u^2}, \end{cases} \tag{4.20}$$

where $0 < \alpha < 1$ *and* $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$. *This system has the same form as* (4.14) *with*

$$A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix},$$

$$\Phi(x,u) = \frac{x_1 x_2}{1+u^2}.$$

and

$$g(x,u) = \frac{1}{4}\sin(u)\,x.$$

Choose $P = I$ *and* $\eta = 1$, *then*

$$A^T P + PA + I < 0.$$

The perturbation term g *satisfies* (H_1) *with* $\delta_1(\varepsilon) = \varepsilon^2$, $\delta_2(\varepsilon) = \varepsilon$ *and* $\nu(t) = \sqrt{2}$. *Select* $P = 2I$, *since*

$$A^T P + PA + I = \begin{pmatrix} -7 & 4 \\ 4 & -3 \end{pmatrix} < 0.$$

Then, the assumptions of Theorem 15 are satisfied. Hence, we obtain the ε^*-*practical fractional exponential stability of the closed loop fractional-order system* (4.20) *for some* $\varepsilon^* > 0$ *with*

$$u(\varepsilon, x_1, x_2) = \frac{1.5x_2^3 x_1^2}{1.5x_2^2\,|x_1| + \varepsilon}.$$

Conclusion

In this thesis, we study two fractional systems. The first concerns the existence and uniqueness of the solution for a fractional system of Riemann Liouville. To prove the main results, some fixed point theorems are used. The expression of the Green function for the systems posed was complicated and it does not make it possible to find its sign to study the positivity of the solution. The second problem studied is the stability of the solution for a system involving conformabe derivative. The Lyapunov theory demonstrates the $\varepsilon-$practical fractional exponential stability of the solution for the conformal nonlinear system, the perturbed conformable system and the conformable systems with uncertainties.

The results obtained are either published [26] or submitted for publication. As perspectives, similar problems can be considered with different types of fractional derivatives or using numerical methods. We believe that the results of this thesis contribute to the development of fractional differential equations.

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