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Mathématiques Appliquées

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dedicatory

I dedicate this work to my parents, my brothers, my sisters, my wife, my children, my colleagues, and all my family.

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Abstract

This thesis investigates the qualitative aspects of solutions for a specific class of heat equations with logarithmic nonlinearity governed by the $p(x)$-Laplacian. The focus is on exploring stability, existence of global solutions, and blow-up phenomena in their behavior. The variable exponent $p(x)$ introduces spatial dependency, while the temporal evolution of the system is captured by the heat equation. These equations are both mathematically difficult and fascinating to study due to the presence of such variable exponents and the logarithmic nonlinearity in them. The main goals of this study are to investigate the stability characteristics of solutions, identify the prerequisites for their global existence, and examine the occurrence of blow-up in finite time.

Keywords: Global existence, Blow-up, logarithmic nonlinearity, variable exponents.

Résumé

Cette thèse examine la stabilité, l'existence globale et le phénomène d'explosion dans le cadre du comportement qualitatif des solutions pour une classe d'équations de la chaleur impliquant $p(x)-$Laplacian avec non-linéarité logarithmique. L'exposent variable $p(x)$ de l'opérateur $p(x)-$Laplacian introduit la dépendance spatiale, tandis que l'évolution temporelle du système est capturée par l'équation de la chaleur. Ces équations sont à la fois mathématiquement difficiles et fascinantes à étudier en raison de la présence de tels exposents variables et de la non-linéarité logarithmique. Les principaux objectifs de cette étude sont d'étudier les caractéristiques de stabilité des solutions, d'identifier les conditions préalables à leur existence globale et d'examiner l'apparition d'une explosion en temps fini.

Mots clés: Existence globale, Explosion en temps fini, non-linéarité logarithmique, exposants variables.

ملخص

في هذه الاطروحة، نقوم بدراسة مسألة القيمة الحدية ذات شرط ابتدائي لمعادلة شبه مكافئة باستخدام المتباينة اللوغاريتمية لسوبوليف وطريقة الآبار المحتملة. نحصل على نتائج الوجود الكلي للحلول الضعيفة ثم تقديرات الاضمحلال و انفجار الحلول الضعيفة في وجود شروط معينة نحددها لاحقا نتائجنا هي تمديد لنتائج عمل بحثي حديث ظهر في الأدبيات.

الكلمات المفتاحية: وجود كلي، انفجار في زمن محدود، لاخطية لوغاريتمية، أُس متغير.

List of Abbreviations

$\Delta_{p(x)}u = \operatorname{div}\left(\lvert\nabla u\rvert^{p(x)-2}\,\nabla u\right)$	$p(x)-$Laplacian operator
$\langle .,. \rangle$	scalar product in duality E^*, E
$\mathcal{N}$	$= \{u \in X_0 : I(u) = 0\}$ (the Nehari manifold)
$\mathcal{P}(\Omega)$	The collection of Lebesgue measurable functions defined from Ω to $[1, \infty]$.
$\mathcal{P}^{\log}(\Omega)$	$= \{q(.) \in \mathcal{P}(\Omega)$: the inverse of q is globally log-Hölder continuous$\}$
$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right)$	gradient of u
Ω	open subset and bounded in $\mathbb{R}^n$
Ω_∞	$= \{x \in \Omega : p(x) = \infty\}$
$\overline{\Omega}$	closure of Ω
$\partial\Omega$	boundary of Ω
$\rho_{p(x)}(u)$	modular of a measurable function u
$C\left(\overline{\Omega}\right)$	space of continuous functions in $\overline{\Omega}$
E^*	dual space of E
$p^*(x)$	The Sobolev exponent at the critical point
$p'(.)$	conjugate exponent function of $p(.)$, i.e. $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$
u_t	partial derivative of u in t

Contents

Chapter 1

Introduction

Partial differential equations (PDEs) are fundamental tools for describing a wide range of phenomena in the field of mathematical analysis. The heat equation, which represents the development of temperature distribution in a particular domain over time, is one significant class of PDEs. Numerous studies have looked into the characteristics and behavior of different kinds of heat equation solutions over the years. The study of nonlinear heat equations in particular has drawn a lot of interest because of their complex dynamics and difficult mathematical characteristics.

This thesis focuses on the $p(x)-$ Laplacian heat equations, which have a variable coefficient Laplacian operator and logarithmic nonlinearity. Due to their numerous applications in fields including mathematical physics, fluid dynamics, and image processing, these equations have recently attracted a great deal of interest. The solutions may exhibit fascinating behavior due to the nontrivial nonlinearity introduced by the logarithmic term. The problem is substantially more difficult than the traditional heat equation with a constant coefficient Laplacian operator because of the spatial variability. Additionally, the system is given a nonlinear feedback mechanism via the nonlinear term $f(u)$, which adds complex dynamics and the possibility of blow-up phenomena in the solutions.

1.1 Background and Motivation

The PDE class where the $p(x)-$ Laplacian specifies the type of nonlinear partial derivative equations is interested in heat equations with non-linear logarithmic. These equations have a nonlinear diffusion term which is represented by the operator $p(x)-$ Laplacian and a non-linear logarithmic term. In the analysis of this type of equations, the stabilization of solutions holds significant importance. The term "stabilization" describes the behavior of a solution over time as it approaches a stable state or equilibrium. For situations where it is important to have a stable and controllable behavior of the system, understanding the stabilization characteristics is essential. The study of stabilization in the context of the specified PDE involves looking at whether the solutions tend toward a constant balance or to a solution in the stationary state when time goes to infinity.

The global existence is a fundamental question in the analysis of these equations. Global existence describes the fact that PDE solutions always exist, regardless of the beginning condition inside a particular function space. It is crucial to establish global existence. It enables the study of long-term behavior and offers a deeper knowledge of the system dynamics.

However, blow-up phenomena are of interest in cases where solutions of the PDE might cease existing or can grow infinitely within a finite amount of time. Blow-up occurs when the solution experiences a singularity or reaches extreme values within a finite time interval. Understanding the conditions under which blow-up occurs is crucial to identify potential instabilities in the system and to establish limitations on the existence of solutions.

The study of these PDEs is made more difficult by the interaction of the logarithmic nonlinearity and the $p(x)-$ Laplacian operator. The solution's behavior can be considerably affected by the nonlinear growth introduced by the logarithmic nonlinearity and the variable exponent $p(x)$, which allows for spatially changeable diffusion qualities. For this class of equations, careful mathematical techniques including energy estimates, variational methods, and potential theory are needed to analyze stabilization, global existence, and blow-up.

1.2 Problem Statement

The problem is defined as follows:

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) = |u|^{p(x)-2} u \log|u|, & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a subset of $\mathbb{R}^n$ that is surrounded by a smooth boundary $\partial\Omega$, and $p(.)$ is continuous function on $\overline{\Omega}$ into $\mathbb{R}_+$ such that

$$2 < p_- \leq p(x) \leq p_+ < p^*(x), \tag{1.2}$$

where

$$p_- = ess\inf_{x\in\Omega} p(x), \quad p_+ = ess\sup_{x\in\Omega} p(x) \quad \text{and} \quad p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, & \text{if} \quad p_+ < n, \\ +\infty, & \text{if} \quad p_+ \geq n, \end{cases}$$

and $u_0 \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$.

1.3 Literature overview

According to recent studies, investigating problems in variable exponent spaces is a powerful tool for simulating a wide range of events in many scientific and technical fields. Its application to the analysis of problems with variable exponents has been motivated by their applications in the study of elastic mechanics, fluid dynamics, and the calculus of variations (see, for example, [1, 2, 36, 37]). Variable exponent spaces are regarded as a useful tool for studing problems of this type. Many theoretical studies have led to the development of the variable exponent Lebesgue and Sobolev spaces (see [13, 18, 19, 20, 22, 25, 32]). As a result, this subject is becoming more and more famous and significant; for further details on variable exponent spaces, see [10, 12, 33, 39].

We may find quite many global existence or nonglobality, stability and instability, blow-up, and long-time behavior of weak solution results on differential equations in the literature at the level of classical Lebesgue and Sobolev spaces; see, for instance [8, 11, 15, 17, 24, 29, 38].

When $p(x) \equiv 2$, the problem (1.1) has been considered in [9]; The researchers examined the behavior of the solutions, studying both their global existence and potential blow-up phenomena over an unbounded time interval. If $p(x) \equiv p$ a constant exponent, Cong Nhan Le and Xuan Truong Le established in [28] the global existence and potential blow-up phenomena. They showed that for values of p greater than 2, the solutions exhibit finite-time blow-up, and they derived specific conditions under which global weak solutions exist.

Conversely, there is a relatively limited amount of research focused on differential equations involving variable exponent Lebesgue and Sobolev spaces.

Hua Wang and Yijun He in [40] were interested in the case where

$$u_t = \Delta u + |u|^{p(x)}, \quad x \in \Omega, \quad t > 0.$$

They prouved that under condition $1 < p_- \leq p_+ \leq \frac{n+2}{n-2}$ and certain initial data, the solution experiences finite-time blow-up given a positive initial energy.

M. Kbiri Alaoui et al. [3] considered the equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(|\nabla u|^{m(x)-2} \nabla u\right) = |u|^{p(x)-2} u + f, \;\; \text{in } Q = \Omega \times (0, T).$$

They proved that any solution starting from a nontrivial initial data experiences finite-time blow-up whenever

$$\int\limits_{\Omega} u_0^2 dx > 0, \quad f \equiv 0 \quad \text{and} \quad \int\limits_{\Omega} \left(\frac{1}{p(x)} |u_0|^{p(x)} - \frac{1}{m(x)} |\nabla u_0|^{m(x)} \right) dx \geq 0.$$

Boudjriou in [7] studied the problem

$$u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = |u|^{q(x)-2}u\log|u|, \quad x \in \Omega, \quad t > 0.$$

Under suitable conditions, the author discusses the existence of global solutions and their blow-up by using the potential well method via the Pohozaev manifold and the concavity method.

1.4 Objectives and display results

The primary objective of this thesis is to investigate the long-term behavior of solutions to the p(x) Laplacian heat equation with logarithmic nonlinearity. Our analysis focuses on three fundamental aspects: stabilization, existence of global solutions and their blow-up.

The ability of a system to reach a stable equilibrium or steady state over time is referred to as stabilization. Understanding stabilizing behavior is essential for describing the solutions' long-term behavior and determining whether asymptotic states or attractors are present. We want to define the conditions in which the steady-state solutions to the p(x) Laplacian heat equation converge.

The question of global existence concerns the availability of solutions to the p(x) Laplacian heat equation throughout the course of all positive time. To ensure that the problem is well-posed and that there are solutions that last indefinitely, it is crucial to establish global existence results.

When the p(x) Laplacian heat equation solutions become unbounded in finite time, blow-up phenomena happen. To locate key regions of the parameter space where the solutions behave singularly, the study of blow-up phenomena is essential. To understand the dynamics and constraints of the system, it is imperative to understand the conditions in which blow-up occurs or is avoided.

So, we examine problem (1.1) by introducing a nonlinear diffusion term of variable exponent and a nonlinear logarithmic term. This extension expands upon the problem addressed in [28], which was limited to traditional Lebesgue and Sobolev spaces, now considering Lebesgue and Sobolev spaces with a variable exponent. Our objective is to establish the existence of global solutions, long-term decay, and blow-up solutions for problem (1.1) within the framework of Lebesgue and Sobolev spaces with variable exponents. To achieve this, we employ the potential well technique (see [34]) utilizing the Nehari manifold, as well as the concavity method (see [29]), to obtain the existence of global solutions and their blow-up for (1.1). In our work, we discuss the following details:

-The problem (1.1) solutions exists locally and globally in time if it holds the condition $p_+ < p_-^*$, but here we have discussed two cases

Case 1: If $2 < p_- \leq p_+ < \left(1 + \frac{2}{n}\right) p_-$,

Case 2: If $p_+ < \left(1 + \frac{2}{n}\right) p_-$ does note holds, i.e., $\left(\left(1 + \frac{2}{n}\right) p_- \leq p_+ < p_-^*\right)$.

We point out that the case 1 with $p_- = p$ and $p_+ = q$ has been discussed in [23] where the authors obtained results of decay and finite time blow-up of solutions for a parabolic equation of Pseudo type, including $p(x)-$Laplacian nonlinear logarithmic term.

$$u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) = |u|^{q(x)-2} u \log |u|, \quad x \in \Omega, \quad t > 0. \tag{1.3}$$

But case 2 is not studied in the previous cited papers in the literature.
We montion also that an equation like (1.3) has been considered in [5, 14], (see also [41])

An other two cases that we have discussed for prouving the coercivity of the energy functional defined in (3.3) (see the proof of Lemma 3.4 below) that are:

Case 3: If $2 < p_- \leq p_+ < \left(1 + \frac{p_-}{n}\right) p_-$,

Case 4: If $p_+ < \left(1 + \frac{p_-}{n}\right) p_-$ does note holds, i.e., $\left(\left(1 + \frac{p_-}{n}\right) p_- \leq p_+ < p_-^*\right)$.

Also, these cases are not considered in the literature.

We also note that the inequality (3.5), which we have used throughout the paper and without which the potential well method does not work, plays an important role in this inequality, as does the generic constant γ, which we defined for the first time in (a).

To the best of our knowledge, there is no existing outcome regarding the logarithmic inequality of Sobolev in spaces of variable exponents. However, this study represents the inaugural result in the literature on this topic that allows the treatment of non-standard growth parabolic equations by using the classical logarithmic Sobolev inequality, which is a fundamental inequality, to get the results in [28], for dealing with the logarithmic nonlinear term (see Lemma 3.3)

1.5 Thesis Organization

The rest of this thesis is divided into four chapters.

The second Chapter deals with some preliminaries about the variable exponent functions and spaces and some their properties.

Chapter three focuses on examining the potential well method, a widely utilized technique across multiple domains of mathematics and physics. This method finds application in studying ordinary and partial differential equations, primarily focusing on understanding solution behavior, particularly in scenarios where potential energy significantly influences the system's dynamics.

In Chapter four, our initial focus is on establishing the existence of local weak solutions to our problem. Subsequently, we provide proof for the existence of global solutions subject to specific conditions, which will be specified at a later stage. Next, we will demonstrate the stability conclusion.

The blowing up phenomena and the instability that results are the subjects of the fifth Chapter.

Chapter 2

Preliminaries

2.1 A brief introduction to variable exponent spaces

Consider the function on the real line.

$$f(x) = |x|^{-1/3}.$$

Despite being exceedingly well-behaved, the function f is not in $L^p(\mathbb{R})$ for any p, $1 \leq p \leq \infty$. When p is given a single value, it either increases at the origin too quickly or decomposes at infinity too slowly.

We need to consider two alternative L^p spaces, such as L^2 and L^4, in order to more clearly understand the behavior of f We can divide up the domain of f and say that $f \in L^2([-2,2])$ and $f \in L^4(\mathbb{R}\backslash[-2,2])$

The disadvantage of this strategy is that we must add more L^p spaces for more intricate functions or risk losing information. Indeed

$$\begin{aligned}
\|f(x)\|^2_{L^2([-2,2])} &= -\int_{-2}^{0} -(-x)^{-2/3}\,dx + \int_{0}^{2} x^{-2/3}dx \\
&= \left[-\frac{1}{-2/3+1}(-x)^{-2/3+1}\right]_{-2}^{0} + \left[\frac{1}{-2/3+1}x^{-2/3+1}\right]_{0}^{2} \\
&= \left[-3(-x)^{1/3}\right]_{-2}^{0} + \left[3x^{1/3}\right]_{0}^{2} \\
&= 3 \times 2^{1/3} + 3 \times 2^{1/3} = 6 \times 2^{1/3}.
\end{aligned}$$

and $f \in L^4(\mathbb{R}\backslash[-2,2])$, so we have

$$\|f(x)\|^4_{L^4(\mathbb{R}\backslash[-2,2])} = -\int_{-\infty}^{-2} -(-x)^{-4/3}\,dx + \int_{2}^{+\infty} x^{-4/3}dx$$

$$
\begin{aligned}
&= \left[-\frac{1}{-4/3+1}(-x)^{-4/3+1}\right]_{-\infty}^{-2} + \left[\frac{1}{-4/3+1}x^{-4/3+1}\right]_{2}^{+\infty} \\
&= \left[3(-x)^{-1/3}\right]_{-\infty}^{-2} + \left[-3x^{-1/3}\right]_{2}^{+\infty} \\
&= 3\times 2^{-1/3} - \lim_{X\to-\infty} 3(-X)^{-1/3} - \lim_{X\to+\infty} 3X^{-1/3} + 3\times 2^{-1/3} \\
&= 6\times 2^{-1/3}
\end{aligned}
$$

In general we can see that $f \in L^2([-\varepsilon,\varepsilon])$ and $f \in L^4(\mathbb{R}\setminus[-\varepsilon,\varepsilon])$ for any $\varepsilon > 0$

If we let

$$
g(x) = |x|^{-1/3} + |x-1|^{-1/4},
$$

then $g \in L^2([-2,2])$, or more generally in $L^p([-2,2])$ for any $p < 3$, however, information about the local behavior of the singularity at $x = 1$ has been lost. Nevertheless, g is no longer in $L^4(\mathbb{R}\setminus[-2,2])$: we have $g \in L^p(\mathbb{R}\setminus[-2,2])$ for $p > 4$. We must further partition the domain in order to capture this behavior, for instance, writing $g \in L^2([-1,1/2])$, $g \in L^3([1/2,2])$ and $g \in L^{9/2}(\mathbb{R}\setminus[-1,2])$. Indeed $g \in L^p([-2,2])$ for any $p < 3$.

$$
\left\|g(x)\right\|_{L^p([-2,2])}^p \leq \int_{-2}^{2} |x|^{-p/3}\,dx + \int_{-2}^{2} |x-1|^{-p/4}\,dx.
$$

The right-sided integrals make sense if the exponents $-p/3+1$ and $-p/4+1$ are positive, that means the little one must be positive

$$
\begin{aligned}
-p/3+1 = (-p+3)/3 > 0 \quad &\text{then} \quad p < 3, \\
-p/4+1 = (-p+4)/4 > 0 \quad &\text{then} \quad p < 4,
\end{aligned}
$$

so $p < 3$ but $g \notin L^4(\mathbb{R}\setminus[-2,2])$ we have $g \in L^p(\mathbb{R}\setminus[-2,2])$ for $p > 4$. Indeed

$$
\begin{aligned}
\left\|g(x)\right\|_{L^p(\mathbb{R}\setminus[-2,2])}^p &= \int_{\mathbb{R}\setminus[-2,2]} \left||x|^{-1/3} + |x-1|^{-1/4}\right|^p dx \\
&\leq \int_{\mathbb{R}\setminus[-2,2]} |x|^{-p/3}\,dx + \int_{\mathbb{R}\setminus[-2,2]} |x-1|^{-p/4}\,dx.
\end{aligned}
$$

The right-sided integrals make sense if the exponents $-p/3+1$ and $-p/4+1$ are negative, that means the great one must be negative

$$-p/3+1=(-p+3)/3<0 \quad \text{then} \quad p>3,$$
$$-p/4+1=(-p+4)/4<0 \quad \text{then} \quad p>4,$$

so $p>4$.

The variable Lebesgue spaces offer an alternative strategy: we maintain the domain while allowing the exponent to vary. Let the exponent function,

$$p(x)=(9|x|+2)/(2|x|+1)=\frac{9}{2}-\frac{5/2}{2|x|+1}.$$

Then $p(0)=2$, $p(1)=11/3$ and $p(x)\to 9/2$ as $|x|\to+\infty$

for $x\geq 0$ we have

$$p(x)=\frac{9x+2}{2x+1} \quad \text{and} \quad \lim_{x\to+\infty}\frac{9x+2}{2x+1}=\frac{9}{2}$$

$$p'(x)=\left(\frac{9x+2}{2x+1}\right)'=\frac{9(2x+1)-2(9x+2)}{(2x+1)^2}=\frac{5}{(2x+1)^2}>0$$

then for $x\geq 0$, $p(x)$ is an increasing function on $[0,+\infty[$ and $p(0)=2\leq p(x)\leq \lim_{x\to+\infty}p(x)=9/2$

for $x\leq 0$ we have

$$p(x)=\frac{-9x+2}{-2x+1} \quad \text{and} \quad \lim_{x\to-\infty}\frac{-9x+2}{-2x+1}=\frac{9}{2}$$

$$p'(x)=\left(\frac{-9x+2}{-2x+1}\right)'=\frac{-9(-2x+1)-(-2)(-9x+2)}{(-2x+1)^2}=\frac{-5}{(2x+1)^2}<0$$

then for $x\leq 0$, $p(x)$ is a decreasing function on $]-\infty,0]$ and $\lim_{x\to-\infty}p(x)=9/2\geq p(x)\geq p(0)=2$. So we can see that

$$\int_{\mathbb{R}}|f(x)|^{p(x)}\,dx<+\infty \quad \text{and} \quad \int_{\mathbb{R}}|g(x)|^{p(x)}\,dx<+\infty.$$

In another meaning, we may describe the behavior of each function more clearly using the single variable exponent $p(x)$. Furthermore, by altering the exponent function, we can tell them apart at infinity. For example, let

$$q(x)=(8|x|+2)/(2|x|+1)=4-\frac{2}{2|x|+1},$$

then

$$\int_{\mathbb{R}} \left|f(x)\right|^{q(x)} dx < +\infty$$

and $\left|g(.)\right|^{q(.)}$ can be locally integrated, but

$$\int_{\mathbb{R}} \left|g(x)\right|^{q(x)} dx = +\infty.$$

We define $L^{p(.)}(\Omega)$ as the set of functions f such that

$$\int_{\Omega} \left|f(x)\right|^{p(x)} dx < \infty.$$

for some given a set Ω and a measurable function $p(.) : \Omega \to [1, +\infty)$.

Even L^{∞} can be included in the definition. We can redefine $L^{p(.)}(\Omega)$ as the collection of functions satisfying $p(.)$ is permitted to be infinite on sets of positive measures and take $\Omega_{\infty} = \{x \in \Omega : p(x) = \infty\}$.

$$\rho_{p(.)}(f) = \int_{\Omega} \left|f(x)\right|^{p(x)} dx + \left\|f\right\|_{L^{\infty}(\Omega_{\infty})} < \infty.$$

Even though this description strongly resembles the traditional definition, we quickly run into issues. It is simple to demonstrate that $L^{p(.)}(\Omega)$ is a vector space using this definition if $p(.)$ is a bounded function. Consider, for instance, that $\Omega = \mathbb{R}$ and $p(x) = 1+|x|$ are no longer true if $p(.)$ is unbounded. Next $\rho_{p(.)}(1/2) < \infty$, so $1/2 \in L^{p(.)}(\mathbb{R})$, but $\rho_{p(.)}(1) = \infty$. Additionally, the modular "modular" $\rho_{p(.)}$ does not instantly become a norm since, unlike in the case of constant exponents, we are unable to apply a power $1/p(x)$ to the integral's periphery.

The approach used to solve this issue is similar to that of Orlicz spaces. Remember that the Orlicz space $L^{\Phi}(\Omega)$ contains all functions f such that for some $\lambda > 0$ given Young function $\Phi : [0, +\infty) \to [0, +\infty)$, a continuous, strictly increasing, convex function, such that

$$\rho_{\Phi}(f/\lambda) = \int_{\Omega} \Phi\left(\frac{\left|f(x)\right|}{\lambda}\right) dx < \infty.$$

If outfitted with the Luxembourg norm, this space becomes a Banach space.

$$\left\|f\right\|_{L^{\Phi}(\Omega)} = \inf\left\{\lambda > 0 : \rho_{\Phi}\left(f/\lambda\right) \leq 1\right\}.$$

In the same way, we define $L^{p(.)}(\Omega)$ as the collection of functions f in which for some $\lambda > 0$, $\rho_{p(.)}(f/\lambda) < \infty$, and the norm is defined by

$$\left\|f\right\|_{L^{p(.)}(\Omega)} = \inf\left\{\lambda > 0 : \rho_{\Phi}\left(f/\lambda\right) \leq 1\right\}$$

This quickly reduces to the typical norm on the traditional Lebesgue spaces when $p(.)$ is a constant. When $p(.)$ is a bounded function, $L^{p(.)}(\Omega)$ turns into a Banach space and has many characteristics with the traditional Lebesgue spaces. The dual space of $L^{p(.)}(\Omega)$ is isomorphic to $L^{p'(.)}(\Omega)$, and the variable Lebesgue spaces are separable in this case. The exponent $p'(.)$ is the conjugate exponent function of $p(.)$ such that

$$\frac{1}{p(.)} + \frac{1}{p'(.)} = 1,$$

with the convention $1 + 1/\infty = 1$.

Now we give some results about the Lebesgue and Sobolev spaces with variable exponents, which are well-known in [12, 25, 32].

2.2 Variable exponents functions ($p(x)$)

Consider $\mathcal{P}(\Omega)$ as the collection of all Lebesgue measurable functions $p(.) : \Omega \to [1, \infty]$, where Ω represents a bounded domain within $\mathbb{R}^n$.

A function $p(.)$ is considered to fulfill the log-Hölder continuity condition in Ω if

$$\forall x, y \in \Omega \text{ with } \left|x - y\right| \leq \frac{1}{2}, \quad \left|p(x) - p(y)\right| \leq \frac{C_0}{-\log\left(\left|x - y\right|\right)}, \tag{2.1}$$

where $C_0 > 0$ is a constant.

We define the log-Hölder decay condition to hold for $p(.)$ in Ω if

$$\forall x \in \Omega, \quad \left|p(x) - p_{\infty}\right| \leq \frac{C_{\infty}}{\log(e + |x|)}, \tag{2.2}$$

where $p_{\infty} = \lim_{|x|\to\infty} p(x)$ and $C_{\infty} > 0$ are constants.

By $\mathcal{P}^{\log}(\Omega)$ we denote the class of variable exponents:

$$\mathcal{P}^{\log}(\Omega) = \{r(.) \in \mathcal{P}(\Omega) : \text{the inverse of } r \text{ is globally log-Hölder continuous}\}.$$

Note that $r(.) \in \mathcal{P}(\Omega)$ is globally log-Hölder continuous in Ω, if $r(.)$ satisfies both (2.1)-(2.2) conditions.

Proposition 2.1 (see [10]). *Given a domain* Ω
1) If $p(.)$ *fulfills* (2.1), *then it is uniformly continuous and fulfills* (2.2) *on every bounded subset.* $E \subset \Omega$.
2) If $p(.) \in \mathcal{P}(\Omega)$ *and* $p_+ < +\infty$, *then* $1/p(.)$ *satisfies either conditions* (2.1), (2.2) *or both if and only if* $p(.)$ *is also.*

Remark 2.1. *From Proposition 2.1 we deduce that if* Ω *is bounded,* $p(.) \in C\left(\overline{\Omega}\right)$ *and satisfies the conditions* (1.2), (2.1) *then* $p(.), 1/p(.) \in \mathcal{P}^{\log}(\Omega)$

2.3 Variable exponent spaces

Variable Lebesgue and Sobolev spaces are function spaces that accommodate variable exponents. In traditional Lebesgue and Sobolev spaces, the exponent in the defining norm is a constant. However, in the variable setting, the exponent is allowed to vary depending on the position in the domain.

Now we start by defining the $p(.)$ modular of a measurable function $u : \Omega \to \mathbb{R}$ as follows:

$$\rho_{p(x)}(u) = \int\limits_{\Omega \backslash \Omega_\infty} |u(x)|^{p(x)}\, dx + ess \sup_{x \in \Omega_\infty} |u(x)|,$$

where

$$\Omega_\infty = \{x \in \Omega : p(x) = \infty\}.$$

2.3.1 $L^{p(x)}$ spaces

The generalized Lebesgue space $L^{p(.)}(\Omega)$ is the class of those measurable functions u defined on Ω as follows

$$L^{p(x)}(\Omega) = \left\{ u : u \in \mathcal{P}(\Omega), \int\limits_{\Omega} |u(x)|^{p(x)}\, dx < \infty \right\}$$

We define the Luxembourg norm of the space $L^{p(.)}(\Omega)$ by

$$\|u\|_{p(x)} = \inf\left\{\kappa > 0 : \rho_{p(x)}(u/\kappa) \leq 1\right\}.$$

The space $L^{p(.)}(\Omega)$ equipped with this norm, is a Banach space.

Now we present some results that concern variable-exponent Lebesgue spaces

Proposition 2.2 (see [19, 20]). *Let* $u \in L^{p(x)}(\Omega)$, $(u_n)_{n\in\mathbb{N}} \subset L^{p(x)}(\Omega)$, *then*
1) $\|u\|_{p(x)} < 1\ (= 1; > 1) \Leftrightarrow \rho_{p(x)}(u) < 1\ (= 1; > 1)$,
2) If $\|u\|_{p(x)} > 1$, *then* $\|u\|_{p(x)}^{p_-} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p_+}$,
3) If $\|u\|_{p(x)} < 1$, *then* $\|u\|_{p(x)}^{p_+} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p_-}$,
4) $\|u_n\|_{p(x)} \underset{n\to+\infty}{\longrightarrow} 0 \Leftrightarrow \rho_{p(x)}(u_n) \underset{n\to+\infty}{\longrightarrow} 0$,
5) $\|u_n\|_{p(x)} \underset{n\to+\infty}{\longrightarrow} +\infty \Leftrightarrow \rho_{p(x)}(u_n) \underset{n\to+\infty}{\longrightarrow} +\infty$.

Proposition 2.3 (see [19]). *Let* u *and* u_n *be functions belonging to* $L^{p(x)}(\Omega)$ *for* $n = 1, 2, \ldots$. *The following claims are equivalent:*
1) The limit of $\|u_n - u\|_{p(x)}$ *as* n *approaches infinity is zero.*
2) The limit of $\rho_{p(x)}(u_n - u)$ *as* n *approaches infinity is zero.*
3) u_n *converges to* u *in measure in* Ω *and the limit of* $\rho_{p(x)}(u_n)$ *as* n *approaches infinity is equal to* $\rho_{p(x)}(u)$.

Proposition 2.4 (see [19]). *Suppose that* Ω *has finite measure,* $p_1(x), p_2(x) \in \mathcal{P}(\Omega)$. *If* $p_1(x) \leq p_2(x)$ *for almost all* $x \in \Omega$ *and* $1 \leq p_{i-} \leq p_{i+} < +\infty$, $(i = 1, 2)$, *then* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ *and the embedding is continuous.*

Proposition 2.5 (generalized Hölder inequality, see [25, 33]). *Let* $p(.), p'(.) \in \mathcal{P}(\Omega)$ *such that* $p_- > 1$ *and*

$$\frac{1}{p(.)} + \frac{1}{p'(.)} = 1, \quad a.e.\ x \in \Omega.$$

Then the inequality

$$\int_\Omega |u(x) v(x)|\, dx \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|u\|_{p(x)} \|v\|_{p'(x)},$$

holds for every $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{p'(x)}(\Omega)$.

The interpolation inequality given in [39, Lemma 8.2 page 37] is also needed

Lemma 2.1. *Let* $1 \leq p_0 < p_\theta < p_1 \leq \infty$. *For any* $u \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$ *and* $\theta \in (0, 1)$ *such that* $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, *the following inequality holds:*

$$\|u\|_{p_\theta} \leq \|u\|_{p_0}^{1-\theta} \|u\|_{p_1}^{\theta}$$

Generally, Lebesgue spaces of variable-exponent share numerous properties with traditional Lebesgue spaces. Moreover, several results that concern variable-exponent Lebesgue spaces have been obtained, see for instance [12, 25] for the following statements.

- The modular $\rho_{p(.)}$ and the norm $\|.\|_{p(x)}$ exhibit lower semi-continuity concerning (sequential) weak convergence and almost everywhere convergence.
- The space $L^{p(.)}(\Omega)$ possesses reflexivity if and only if $1 < p_- < p_+ < \infty$.
- Continuous functions form a dense set when $p_+ < \infty$.

2.3.2 Sobolev spaces

The space $W^{1,p(.)}(\Omega)$ can be defined as the variable-exponent Sobolev space, consisting of functions u in $L^{p(x)}(\Omega)$ satisfying $|\nabla u| \in L^{p(x)}(\Omega)$. Outfitted with the norm $\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}$, it forms a Banach space.

Moreover, $W_0^{1,p(x)}(\Omega)$ can be described as the completion of $C_0^\infty(\Omega)$ within $W^{1,p(x)}(\Omega)$, using the norm $\|u\|_{1,p(x)} = \|u\|_{W_0^{1,p(.)}(\Omega)} = \|\nabla u\|_{p(x)}$. This definition holds under the condition that $p(.)$ satisfies (1.2) and (2.1).

Additionally, $W^{-1,p'(.)}(\Omega)$ serves as the dual space of $W_0^{1,p(x)}(\Omega)$, where $p'(x)$ represents the conjugate exponent function of $p(x)$, satisfying $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

To denote the set $X_0 = W_0^{1,p(x)}(\Omega) \setminus \{0\}$, we employ the norm $\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)}$.

Proposition 2.6 (see [12, Theorem 8.1.13])**.** *Let* $p \in \mathcal{P}(\mathbb{R}^n)$. *The Banach space* $W_0^{1,p(.)}(\Omega)$ *possesses certain properties depending on the range of the function* $p(.)$. *If* $p(.)$ *is bounded, the space is separable. On the other hand, if* $1 < p_- \leq p_+ < +\infty$, *it becomes reflexive and uniformly convex.*

Proposition 2.7. *Assume that* $1 \leq \operatorname{ess\,inf}_{x\in\Omega} p_i(x) \leq p_i(x) \leq \operatorname{ess\,sup}_{x\in\Omega} p_i(x) < +\infty$, $(i = 1, 2)$. *If* $p_1(x) \leq p_2(x)$, *then* $W^{1,p_2(x)}(\Omega) \hookrightarrow W^{1,p_1(x)}(\Omega)$.

Proposition 2.8 (see [32])**.** *If* Ω *is bounded,* $p(x) \in C\left(\overline{\Omega}\right)$ *such that* $p_+ < n$ *and* $q(x)$ *defined in* Ω *with* $q_- \geq 1$ *and*

$$\operatorname*{ess\,inf}_{x\in\Omega} (p^*(x) - q(x)) > 0,$$

then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ *is compact.*

Proposition 2.9 (see [10, Theorem 6.29] and [12, Theorem 8.3.1])**.**
1) Let Ω *be a domain and* $p(.) \in \mathcal{P}(\Omega)$ *be a function such that* $p_+ < n$. *Suppose the maximal operator is bounded on* $L^{(p^*(.)/n')'}(\Omega)$. *Then, it follows that* $W_0^{1,p(.)}(\Omega) \subset L^{p^*(.)}(\Omega)$, *and we have the inequality*

$$\|u\|_{p^*(.)} \leq C \|\nabla u\|_{p(.)} .$$

2) Consider $p \in \mathcal{P}^{\log}(\Omega)$ *satisfying* $1 \leq p_- \leq p_+ < n$. *For any* $u \in W_0^{1,p(.)}(\Omega)$, *it holds the inequality*

$$\|u\|_{p^*(.)} \leq c \|\nabla u\|_{p(.)}$$

where c *is a constant dependent solely on* n, $c_{\log}(p)$, *and* p_+.

Lemma 2.2. *Let $\varrho(.)$ be a continuous function from $\overline{\Omega}$ into $(0,+\infty)$ such that $0 < \text{ess}\inf_{x\in\Omega}\varrho(x) = \varrho_- \leq \varrho(x) \leq \varrho_+ = \text{ess}\sup_{x\in\Omega}\varrho(x) < +\infty$. Then the following inequalities hold*

$$\log s \leq \frac{e^{-1}}{\varrho(x)} s^{\varrho(x)},\ \text{for all } s \in [1,+\infty) \quad \text{and} \quad -\frac{e^{-1}}{\varrho(x)} \leq s^{\varrho(x)} \log s \leq 0,\ \text{for all } s \in (0,1].$$

Proof. The proof follows directly by studying the variations of the function $\varphi(s) = \log s - \frac{e^{-1}}{\varrho(x)} s^{\varrho(x)}$, for $s \in [1,+\infty)$ and the function $\phi(s) = s^{\varrho(x)} \log s$, for $s \in (0,1]$. □

Remark 2.2. *It can be derived from Lemma 2.2, that*

$$s^{p(x)} \log s \leq \frac{e^{-1}}{\varrho(x)} s^{p(x)+\varrho(x)}, \quad \text{for all } s \in [1,+\infty).$$

Lemma 2.3 (see [27]).
(a) Let $u \in W_0^{1,p}(\Omega)$, then it holds:

$$\|u\|_q \leq B_{q,p} \|\nabla u\|_p,$$

the constant $B_{q,p}$, denoted as B_p when $q = p$,
(b) Let $2 \leq s \leq p < q < p^*$, $u \in W_0^{1,p}(\Omega)$, it holds:

$$\|u\|_q \leq C \|\nabla u\|_p^{\alpha} \|u\|_s^{1-\alpha},$$

where $C > 0$ and

$$\alpha = \left(\frac{1}{s} - \frac{1}{q}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{s}\right)^{-1}.$$

2.4 Weak solutions

If a function satisfies an integral equation derived from the original PDE, it is regarded as a weak solution to the PDE when measured against a sufficient set of test functions. Weak derivatives and distributions are frequently applied to this integral equation. Now we define what a weak solution of problem (1.1) means.

Definition 2.1 (Weak solution). *A function u in the space $L^{\infty}(0,T;X_0)$, where u_t belongs to $L^{p'(x)}\left(0,T;W^{-1,p'(x)}(\Omega)\right) \cap L^2(0,T;L^2(\Omega))$, is considered a weak solution of problem (1.1) on $\Omega \times [0,T)$ if it fulfills the initial condition $u(.,0) = u_0(.) \in X_0$. and*

$$\int_{\Omega} u_t w dx + \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla w dx = \int_{\Omega} |u|^{p(x)-2} u \log|u| w dx, \tag{2.3}$$

for all $w \in W_0^{1,p(x)}(\Omega)$, and for almost every $t \in (0,T)$.

Chapter 3

Potential well

The potential well method is a mathematical approach widely employed to investigate the behavior of solutions and analyze associated problems in the realm of differential equations. It is particularly utilized when potential energy plays a crucial role, allowing for a comprehensive understanding of solution dynamics.

3.1 $p(x)-$ Laplacian Operator

The operator $p(x)-$ Laplacian is an extension of the conventional operator Laplacian, which introduces a variable exponent $p(x)$ depending on the spatial position x within the domain. It emerges in the analysis of nonlinear PDEs and serves to characterize phenomena exhibiting spatially varying properties or non-uniform behavior.

The $p(x)-$ Laplacian operator is mathematically defined as:

$$\Delta_{p(x)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\,\nabla u\right).$$

Here, u represents a function defined on the domain, ∇u represents the gradient of u, and $|\nabla u|$ is the magnitude of the gradient. The exponent $p(x)$ is a function that governs the rate of operator's growth.

The $p(x)-$ Laplacian operator exhibits different behavior depending on the properties of the exponent function $p(x)$. When $p(x)$ is constant, the $p(x)-$ Laplacian reduces to the $p-$ Laplacian operator.

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right).$$

When $p(x) = 2$, the $p(x)-$ Laplacian reduces to the standard Laplacian operator.

$$\Delta u = \operatorname{div}\left(\nabla u\right),$$

which is commonly used in elliptic PDEs. However, by allowing $p(x)$ to vary, the $p(x)-$ Laplacian captures spatial variations in the equation, enabling the study of non-uniform phenomena and accommodating different types of behavior in different regions of the domain. The $p(x)-$ Laplacian operator has found applications in various fields, including mathematical physics, image processing, fluid dynamics, and materials science. It appears in equations such as the $p(x)-$ Laplacian equation, $p(x)-$ Laplacian heat equation, or $p(x)-$ Laplacian eigenvalue problems. The study of PDEs including the operator $p(x)-$ Laplacian involves investigating existence, regularity, and qualitative properties of solutions, as well as developing numerical methods and techniques for their analysis. The $p(x)-$ Laplacian operator provides a flexible mathematical tool to describe and analyze phenomena with spatially varying properties or non-uniform behavior. It extends the capabilities of the standard Laplacian operator to capture more complex and diverse phenomena, making it a valuable tool in the study of nonlinear PDEs.

3.2 Logarithmic nonlinearity

Mathematicians and physicists have expressed a strong interest in logarithmic nonlinearity in light of the extensive literature on polynomial nonlinear terms. By introducing the logarithmic nonlinearity, it was possible to study both the relativistic wave equation for spinless particles and the non-relativistic wave equation for spinning particles moving via an external electromagnetic field (see [4]). There is also much interest in the well-posedness of the global-in-time solution to the evolution equation with such logarithmic-type nonlinearity. Numerous other areas of physics also encounter this type of nonlinearity, including including Solitary waves with incoherent white light characteristics can be observed in logarithmically saturable nonlinear media that do not exhibit instantaneous response. [8], inflationary cosmology [16], nuclear physics [25], optics [26], and geophysics [30].

It is worth mentioning that the inclusion of logarithmic nonlinearity presents certain challenges when implementing the potential well method.

Lemma 3.1. *Let* $q > 1, \mu > 0$, *and* $u \in W^{1,q}(\mathbb{R}^n) \setminus \{0\}$. *Then we have*

$$q \int\limits_{\mathbb{R}^n} |u(x)|^q \log\left(\frac{|u(x)|}{\|u(x)\|_{L^q(\mathbb{R}^n)}}\right) dx + \frac{n}{q} \log\left(\frac{q\mu e}{n\mathcal{L}_q}\right) \int\limits_{\mathbb{R}^n} |u(x)|^q \, dx \leq \mu \int\limits_{\mathbb{R}^n} |\nabla u(x)|^q \, dx,$$

where

$$\mathcal{L}_q = \frac{q}{n}\left(\frac{q-1}{e}\right)^{q-1} \pi^{-\frac{q}{2}} \left[\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n\frac{q-1}{q}+1\right)}\right]^{\frac{q}{n}}.$$

Remark 3.1. *If* $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$, *when specifying* $u(x) = 0$ *for* $x \in \mathbb{R}^n \backslash \Omega$ *it holds*

$$p_- \int\limits_\Omega |u|^{p_-} \log\left(\frac{|u|}{\|u\|_{p_-}}\right) dx + \frac{n}{p_-} \log\left(\frac{p_- \mu e}{n \mathcal{L}_{p_-}}\right) \int\limits_\Omega |u|^{p_-}\, dx \le \mu \int\limits_\Omega |\nabla u|^{p_-}\, dx, \tag{3.1}$$

$$p_+ \int\limits_\Omega |u|^{p_+} \log\left(\frac{|u|}{\|u\|_{p_+}}\right) dx + \frac{n}{p_+} \log\left(\frac{p_+ \mu e}{n \mathcal{L}_{p_+}}\right) \int\limits_\Omega |u|^{p_+}\, dx \le \mu \int\limits_\Omega |\nabla u|^{p_+}\, dx. \tag{3.2}$$

for all real number $\mu > 0$, *where*

$$\mathcal{L}_{p_-} = \frac{p_-}{n}\left(\frac{p_- - 1}{e}\right)^{p_- - 1} \pi^{-\frac{p_-}{2}} \left[\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n\frac{p_- - 1}{p_-}+1\right)}\right]^{\frac{p_-}{n}}, \quad \mathcal{L}_{p_+} = \frac{p_+}{n}\left(\frac{p_+ - 1}{e}\right)^{p_+ - 1} \pi^{-\frac{p_+}{2}} \left[\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n\frac{p_+ - 1}{p_+}+1\right)}\right]^{\frac{p_+}{n}}.$$

3.3 Energy and Nehari Functionals

On X_0, we define the two functionals J and I as follows:

$$\begin{aligned} J(u) &= \int\limits_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx - \int\limits_\Omega \frac{1}{p(x)} |u|^{p(x)} \log |u|\, dx + \int\limits_\Omega \frac{1}{p^2(x)} |u|^{p(x)}\, dx, \\ &= J_1(u) - \int\limits_\Omega \frac{1}{p(x)} |u|^{p(x)} \log |u|\, dx + \int\limits_\Omega \frac{1}{p^2(x)} |u|^{p(x)}\, dx, \end{aligned} \tag{3.3}$$

$$I(u) = \int\limits_\Omega |\nabla u|^{p(x)}\, dx - \int\limits_\Omega |u|^{p(x)} \log |u|\, dx = I_1(u) - \int\limits_\Omega |u|^{p(x)} \log |u|\, dx. \tag{3.4}$$

The functionals I and J are defined as in [28] with some modifications, they are well-defined in X_0. Furthermore, the following proposition given in [22] characterizes the functionals J_1 and I_1 in $W_0^{1,p(x)}(\Omega)$.

Proposition 3.1. *Let Ω be a domain and $p(x)$ be a function defined on Ω. Define the functional $J_1(u)$ as follows: $J_1(u) = \int\limits_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx$ where u belongs to the function space $W_0^{1,p(x)}(\Omega)$. Then, J_1 is a continuously differentiable function from $W_0^{1,p(x)}(\Omega)$ to $\mathbb{R}$, and the operator $p(x)-$ Laplacian is its derivative. We define the operator $J_1' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ as follows:*

$$\langle J_1' (u), v\rangle = \int\limits_{\Omega} |\nabla u|^{p(x)-2}\, \nabla u \nabla v dx,$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$. Moreover, J_1' fulfils
(1) J_1' is a bounded, continuous, homeomorphism, and strictly monotone operator.

(2) J_1' is a (S_+) type mapping, meaning that if $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and $\overline{\lim}_{n\to\infty}\langle J_1'(u_n) - J_1'(u), u_n - u\rangle \leq 0$, then $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$.

Remark 3.2. *Note that in Proposition 3.1, $\langle J_1' (u), u\rangle = I_1 (u)$, for all $u \in X_0$, and then $\langle J' (u), u\rangle = I (u)$, for all $u \in X_0$. Indeed*

$$\begin{aligned}\langle J' (u), u\rangle &= \langle J_1' (u), u\rangle - \int\limits_{\Omega} |u|^{p(x)-2}\, u \log(|u|)\, u dx - \int\limits_{\Omega} \frac{1}{p(x)} |u|^{p(x)}\, dx + \int\limits_{\Omega} \frac{1}{p(x)} \left(|u|^{p(x)-2}\, u\right) u dx \\ &= I_1(u) - \int\limits_{\Omega} |u|^{p(x)} \log|u|\, dx = I(u).\end{aligned}$$

Remark 3.3. *It is easy to show by lemma 2.2 the continuity of $u \mapsto \int\limits_{\Omega} |u|^{p(x)} \log|u|\, dx$ on X_0, and then by Proposition 3.1 and Remark 3.2 we deduce that the functionals J and I are continuous from X_0 into $\mathbb{R}$. Furthermore we have $J \in C^1(X_0, \mathbb{R})$.*

On the other hand, since $I(u)$ changes sign (see Lemma 3.3 below), so we denote by $\gamma_{I(u)} \equiv \gamma$ a generic constant, i.e. a constant changing value according to the sign of $I(u)$, such that

$$\gamma = \frac{1}{2}\left(\frac{1}{p_+} - \operatorname{sgn}(I(u))\frac{1}{p_-}\right) + \frac{1}{2}\left(\frac{1}{p_-} + \operatorname{sgn}(I(u))\frac{1}{p_+}\right) = \begin{cases} 1/p_-, & \text{if } I(u) \leq 0 \\ 1/p_+, & \text{if } I(u) > 0 \end{cases} \tag{a}$$

then from (3.3)-(3.4) yields

$$J(u) \geq \gamma I(u) + \frac{1}{p_+^2}\int_\Omega |u|^{p(x)}\,dx. \tag{3.5}$$

Defining for $u \in X_0$ and $\lambda > 0$, the function $j : \lambda \to J(\lambda u)$, by

$$\begin{aligned} j(\lambda) = J(\lambda u) &= \int_\Omega \frac{\lambda^{p(x)}}{p(x)} |\nabla u|^{p(x)}\,dx - \int_\Omega \frac{\lambda^{p(x)}}{p(x)} |u|^{p(x)} \log|u|\,dx \\ &- \log\lambda \int_\Omega \frac{\lambda^{p(x)}}{p(x)} |u|^{p(x)}\,dx + \int_\Omega \frac{\lambda^{p(x)}}{p^2(x)} |u|^{p(x)}\,dx. \end{aligned} \tag{3.6}$$

In the subsequent lemma, we demonstrate the existence of a solitary positive critical point $\lambda^* = \lambda^*(u)$ for $j(\lambda)$ see [15, 28, 34, 35].

Lemma 3.2. *Let $u \in X_0$. The following statements hold:*
(1) $\lim_{\lambda\to 0^+} j(\lambda) = 0$ and $\lim_{\lambda\to+\infty} j(\lambda) = -\infty$.
(2) There exists a unique positive value $\lambda^* = \lambda^*(u)$ for which $j'(\lambda^*) = 0$.
(3) $j(\lambda)$ is increasing on the interval $(0, \lambda^*)$, decreasing on the interval $(\lambda^*, +\infty)$, and attains its maximum value at λ^*.
(4) For $0 < \lambda < \lambda^*$, $I(\lambda u) > 0$; for $\lambda^* < \lambda < +\infty$, $I(\lambda u) < 0$; and $I(\lambda^* u) = 0$.

Proof. Let $u \in X_0$, by (3.6) it is easy to show that (1) holds since $\int_\Omega \frac{\lambda^{p(x)}}{p(x)} |u|^{p(x)}\,dx \neq 0$, $(\lambda \neq 0)$. By simple calculation, we get

$$\frac{d}{d\lambda} j(\lambda) = \int_\Omega \lambda^{p(x)-1} |\nabla u|^{p(x)}\,dx - \int_\Omega \lambda^{p(x)-1} |u|^{p(x)} \log|u|\,dx$$

$$-\log\lambda\int\limits_{\Omega}\lambda^{p(x)-1}\left|u\right|^{p(x)}dx, \tag{3.7}$$

to show (2) and (3), it suffices to take

$$\lambda^{*}=\lambda^{*}(u)=\exp\left(\frac{\int_{\Omega}(\lambda^{*})^{p(x)-1}\left(|\nabla u|^{p(x)}-|u|^{p(x)}\log|u|\right)dx}{\int_{\Omega}(\lambda^{*})^{p(x)-1}|u|^{p(x)}dx}\right)$$

implicitly. The last property (4) follows from the relationship.

$$I(\lambda u)=\lambda(\int\limits_{\Omega}\lambda^{p(x)-1}|\nabla u|^{p(x)}dx-\int\limits_{\Omega}\lambda^{p(x)-1}|u|^{p(x)}\log|u|\,dx-\log\lambda\int\limits_{\Omega}\lambda^{p(x)-1}|u|^{p(x)}dx)$$

$$=\lambda j'(\lambda).$$

Thus the proof. □

Lemma 3.3. *Let* $u_0\in X_0$. *Suppose that*

$$\min_{\overline{\Omega}}\int\limits_{\Omega}|u_0|^{p(x)}\log(|u_0|)\,dx\geq 0, \tag{3.8}$$

then a positive real number R exists and fulfils

(1) if $0<\max\left\{\|u\|_{p(x)},\|u\|_{p(x)}^{p_+/p_-}\right\}<R$ *then* $I(u)>0$.

(2) if $I(u)<0$ *then* $\min\left\{\|u\|_{p(x)},\|u\|_{p(x)}^{p_+/p_-}\right\}>R$.

(3) if $I(u)=0$ *then* $\min\left\{\|u\|_{p(x)},\|u\|_{p(x)}^{p_+/p_-}\right\}\geq R$.

Proof. Divided Ω into l subsets in the following way $\overline{\Omega}=\cup_{i=1}^{l}\overline{\Omega_i}$ such that in every subset Ω_i we have $p_{i-}\leq p(x)\leq p_{i+}$, for all $1\leq i\leq l$, where $p_{i-}=p_-(\Omega_i)$ and $p_{i+}=p_+(\Omega_i)$. Here we have used the assumption (1.2) and the continuity of the function $p(x)$. Now for l large enough we suppose that

$$\max_{\overline{\Omega}_i}\int\limits_{\Omega_i}|u|^{p(x)}dx\leq 1,\quad i=1,2,...,l, \tag{3.9}$$

and by assumption (3.8) that

$$\int_{\Omega_i} |u|^{p_{i-}} \log(|u|)\, dx \geq 0, \quad i = 1, 2, \ldots, l. \tag{3.10}$$

On one hand it follows by Proposition 2.2 (with Ω is replaced by Ω_i) that

$$\|u\|_{p(x),\Omega_i}^{p_{i+}} \leq \int_{\Omega_i} |u|^{p(x)}\, dx \leq \|u\|_{p(x),\Omega_i}^{p_{i-}}. \tag{3.11}$$

On the other hand by (3.9) we may write

$$\min\left\{\|u\|_{p_{i-},\Omega_i}^{p_{i-}}, \|u\|_{p_{i+},\Omega_i}^{p_{i+}}\right\} \leq \int_{\Omega_i} |u|^{p(x)}\, dx \leq \max\left\{\|u\|_{p_{i-},\Omega_i}^{p_{i-}}, \|u\|_{p_{i+},\Omega_i}^{p_{i+}}\right\}. \tag{3.12}$$

According to (3.4), (3.11) and (3.1)-(3.2) (with p_-, p_+ and Ω are replaced by p_{i-}, p_{i+} and Ω_i respectively), we get

$$\begin{aligned}
\int_{\Omega_i} |\nabla u|^{p(x)}\, dx - \int_{\Omega_i} |u|^{p(x)} \log|u|\, dx \geq{}& \int_{\Omega_i} |\nabla u|^{p(x)}\, dx - \frac{\mu_i}{p_{i-}} \int_{\Omega_i} |\nabla u|^{p_{i-}}\, dx - \frac{\mu_i}{p_{i+}} \int_{\Omega_i} |\nabla u|^{p_{i+}}\, dx \\
& - \int_{\Omega_i} |u|^{p(x)} \left(\log\left(\frac{|u|}{\|u\|_{p(x),\Omega_i}} \right) + \frac{1}{p_{i+}} \log\left(\int_{\Omega_i} |u|^{p(x)}\, dx \right) \right) dx \\
& + \int_{\Omega_i} |u|^{p_{i-}} \log\left(\frac{|u|}{\|u\|_{p_{i-},\Omega_i}} \right) dx + \frac{n}{p_{i-}^2} \log\left(\frac{p_{i-}\mu_i e}{n \mathcal{L}_{p_{i-}}} \right) \int_{\Omega} |u|^{p_{i-}}\, dx \\
& + \int_{\Omega_i} |u|^{p_{i+}} \log\left(\frac{|u|}{\|u\|_{p_{i+},\Omega_i}} \right) dx + \frac{n}{p_{i+}^2} \log\left(\frac{p_{i+}\mu_i e}{n \mathcal{L}_{p_{i+}}} \right) \int_{\Omega} |u|^{p_{i+}}\, dx,
\end{aligned}$$

Choose $\mu_i = \int\limits_{\Omega_i} |\nabla u|^{p(x)}\, dx \Big/ \left(p_{i-}^{-1} \int\limits_{\Omega_i} |\nabla u|^{p_{i-}}\, dx + p_{i+}^{-1} \int\limits_{\Omega_i} |\nabla u|^{p_{i+}}\, dx \right)$.

Since by (3.10), $\int\limits_{\Omega_i} |u|^{p_-} \log(|u|)\, dx \geq 0$ for all $i = 1, 2, \ldots, l$, we may write

$$\begin{aligned}\int\limits_{\Omega_{i-}} \left(|u|^{p(x)} - |u|^{p_{i-}} - |u|^{p_{i+}} \right) \log(|u|)\, dx &\leq - \int\limits_{\Omega_{i-}} |u|^{p_{i-}} \log(|u|)\, dx \\ &\leq \int\limits_{\Omega_{i+}} \left(|u|^{p_{i-}} + |u|^{p_{i+}} - |u|^{p(x)} \right) \log(|u|)\, dx\end{aligned}$$

which means that

$$\int\limits_{\Omega_i} |u|^{p_{i-}} \log(|u|)\, dx + \int\limits_{\Omega_i} |u|^{p_{i+}} \log(|u|)\, dx - \int\limits_{\Omega_i} |u|^{p(x)} \log(|u|)\, dx \geq 0. \tag{3.13}$$

It follows from (3.9), (3.11) and (3.12) that

$$\|u\|_{p_{i-},\Omega_i} \|u\|_{p_{i+},\Omega_i} \leq \min\left\{ \|u\|_{p_{i-},\Omega_i}, \|u\|_{p_{i+},\Omega_i} \right\} \leq \|u\|_{p(x),\Omega_i}.$$

There is $\delta \in (0, 1)$ such that by (3.12) we find

$$\begin{aligned}&\log\left(\|u\|_{p_{i-},\Omega_i}\right) \|u\|_{p_{i-},\Omega_i} + \log\left(\|u\|_{p_{i+},\Omega_i}\right) \|u\|_{p_{i+},\Omega_i} \\ &\leq \left(\delta \max\left\{ \|u\|_{p_{i-},\Omega_i}, \|u\|_{p_{i+},\Omega_i} \right\} + (1-\delta) \min\left\{ \|u\|_{p_{i-},\Omega_i}, \|u\|_{p_{i+},\Omega_i} \right\} \right) \log\left(\|u\|_{p_{i-},\Omega_i} \|u\|_{p_{i+},\Omega_i} \right) \\ &\leq \log\left(\|u\|_{p(x),\Omega_i}\right) \int\limits_{\Omega_i} |u|^{p(x)}\, dx.\end{aligned} \tag{3.14}$$

So (3.13) and (3.14) give

$$0 \leq \int\limits_{\Omega_i} |u|^{p_{i-}} \log\left(\frac{|u|}{\|u\|_{p_{i-},\Omega_i}} \right) dx + \int\limits_{\Omega_i} |u|^{p_{i+}} \log\left(\frac{|u|}{\|u\|_{p_{i+},\Omega_i}} \right) dx - \int\limits_{\Omega_i} |u|^{p(x)} \log\left(\frac{|u|}{\|u\|_{p(x),\Omega_i}} \right) dx$$

Therefore we obtain

$$\begin{aligned}
&\int\limits_{\Omega_i} |\nabla u|^{p(x)}\, dx - \int\limits_{\Omega_i} |u|^{p(x)} \log |u|\, dx \\
&\geq a\left[\log\left(\left(\frac{p_{i-}\mu_i e}{n\mathcal{L}_{p_{i-}}}\right)^{\frac{n}{p_{i-}^2}} \left(\frac{p_{i+}\mu_i e}{n\mathcal{L}_{p_{i+}}}\right)^{\frac{n}{p_{i+}^2}}\right) - \frac{1}{p_{i+}} \log\left(\int\limits_{\Omega_i} |u|^{p(x)}\, dx\right)\right],
\end{aligned}$$

where $a = \min\left\{\|u\|_{p_{i-},\Omega_i}, \|u\|_{p_{i+},\Omega_i}\right\}$.

By applying the well known property of the logarithmic function $\sum\limits_{i\geq 1} \log(\delta_i) \leq \log\left(\sum\limits_{i\geq 1} \delta_i\right)$ for $\delta_i \leq 1$, we get

$$\begin{aligned}
I(u) &= \sum_{i=1}^{l}\left(\int\limits_{\Omega_i} |\nabla u|^{p(x)}\, dx - \int\limits_{\Omega_i} |u|^{p(x)} \log |u|\, dx\right) \\
&\geq a\left[\log \prod_{i=1}^{l}\left(\left(\frac{p_{i-}\mu_i e}{n\mathcal{L}_{p_{i-}}}\right)^{\frac{n}{p_{i-}^2}} \left(\frac{p_{i+}\mu_i e}{n\mathcal{L}_{p_{i+}}}\right)^{\frac{n}{p_{i+}^2}}\right) - \sum_{i=1}^{l} \frac{1}{p_{i+}} \log\left(\int\limits_{\Omega_i} |u|^{p(x)}\, dx\right)\right] \\
&\geq a\left[\log \prod_{i=1}^{l}\left(\left(\frac{p_{i-}\mu_i e}{n\mathcal{L}_{p_{i-}}}\right)^{\frac{n}{p_{i-}^2}} \left(\frac{p_{i+}\mu_i e}{n\mathcal{L}_{p_{i+}}}\right)^{\frac{n}{p_{i+}^2}}\right) - \frac{1}{p_-} \log\left(\int\limits_{\Omega} |u|^{p(x)}\, dx\right)\right]. \qquad (3.15)
\end{aligned}$$

If $\|u\|_{p(x)} \leq 1$ then from (3.15) we get

$$I(u) \geq a\left[\log \prod_{i=1}^{l}\left(\left(\frac{p_{i-}\mu_i e}{n\mathcal{L}_{p_{i-}}}\right)^{\frac{n}{p_{i-}^2}} \left(\frac{p_{i+}\mu_i e}{n\mathcal{L}_{p_{i+}}}\right)^{\frac{n}{p_{i+}^2}}\right) - \log\left(\|u\|_{p(x)}\right)\right], \qquad (3.16)$$

if $\|u\|_{p(x)} > 1$ then (3.15) gives

$$I(u) \geq a\left[\log \prod_{i=1}^{l}\left(\left(\frac{p_{i-}\mu_i e}{n\mathcal{L}_{p_{i-}}}\right)^{\frac{n}{p_{i-}^2}} \left(\frac{p_{i+}\mu_i e}{n\mathcal{L}_{p_{i+}}}\right)^{\frac{n}{p_{i+}^2}}\right) - \log\left(\|u\|_{p(x)}^{p_+/p_-}\right)\right]. \qquad (3.17)$$

Seting

$$R = \prod_{i=1}^{l} \left(\left(\frac{p_{i-}\mu_i e}{n\mathcal{L}_{p_{i-}}} \right)^{\frac{n}{p_{i-}^2}} \left(\frac{p_{i+}\mu_i e}{n\mathcal{L}_{p_{i+}}} \right)^{\frac{n}{p_{i+}^2}} \right).$$

(1) From (3.16)-(3.17) we may deduce that for $0 < \max\left\{\|u\|_{p(x)}, \|u\|_{p(x)}^{p_+/p_-}\right\} < R$, then

$$I(u) > 0.$$

(2) suppose that $I(u) < 0$. then from (3.16)-(3.17) we find

$$\log\left(R/\|u\|_{p(x)}\right) < 0, \quad \text{and} \quad \log\left(R/\|u\|_{p(x)}^{p_+/p_-}\right) < 0,$$

this means that

$$R/\|u\|_{p(x)} < 1, \quad \text{and} \quad R/\|u\|_{p(x)}^{p_+/p_-} < 1$$

which implies

$$\min\left\{\|u\|_{p(x)}, \|u\|_{p(x)}^{p_+/p_-}\right\} > R.$$

To prove (3), we proceed in the same way as in (2), which completes the proof of the lemma. □

3.4 Nehari manifold

Let us denote by $\mathcal{N}$ the manifold of Nehari

$$\mathcal{N} = \{u \in X_0 : I(u) = 0\}.$$

Clearly, $\mathcal{N}$ is not an empty set in accordance with Lemma 3.2 (Notice that by the definition of $\mathcal{N}$, $I(\lambda^* u) = 0$ then $\lambda^* u$ is in $\mathcal{N}$). Furthermore, we have the following result.

Lemma 3.4.
1) Assume that $p_+ + \varrho_+ < p_-^*$, then the functional J is coercive on $\mathcal{N}$,
2) The functionals J and I are weakly lower semicontinuous.

Proof. 1) Remember that the coercivity of J on $\mathcal{N}$ means $\lim\limits_{u\in\mathcal{N},\|u\|_{W_0^{1,p(x)}}\to\infty} J(u)=\infty.$

For this we assume that

$$\|u\|_{p(x)}, \|\nabla u\|_{p(x)} > 1. \tag{3.18}$$

Now suppose that $u\in\mathcal{N}$, from formula (3.5) we get

$$J(u) \geq \frac{1}{p_+^2}\int\limits_{\Omega} |u|^{p(x)}\,dx. \tag{3.19}$$

According to Remark 2.2, we may write

$$\begin{aligned}\int\limits_{\Omega} |u(x)|^{p(x)}\log|u(x)|\,dx &\leq \int\limits_{\Omega_-} |u(x)|^{p_+}\log|u(x)|\,dx + \int\limits_{\Omega_+} |u(x)|^{p_+}\log|u(x)|\,dx\\ &\leq \int\limits_{\Omega_+} |u(x)|^{p_+}\log|u(x)|\,dx\\ &\leq \frac{1}{\varrho_-}\int\limits_{\Omega} |u(x)|^{p_++\varrho_+}\,dx.\end{aligned}$$

Therefore we apply the Lemma 2.3 by using the assumption $p_- < p_+ + \varrho_+ < p_-^*$, we obtain

$$\int\limits_{\Omega} |u(x)|^{p(x)}\log|u(x)|\,dx \leq C\,\|u\|_{p_-}^{(1-\alpha)(p_++\varrho_+)}\,\|\nabla u\|_{p_-}^{\alpha(p_++\varrho_+)},$$

where

$$p_-^* = \frac{np_-}{n-p_-} \quad \text{and} \quad \alpha = n\left(\frac{1}{p_-}-\frac{1}{p_++\varrho_+}\right) = \frac{n(p_++\varrho_+-p_-)}{p_-(p_++\varrho_+)}.$$

By Proposition 2.4 and Proposition 2.7, the continuous embeddings $L^{p(x)} \hookrightarrow L^{p_-}(\Omega)$, $W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,p_-}(\Omega)$ ensure that

$$\int\limits_{\Omega} |u(x)|^{p(x)}\log|u(x)|\,dx \leq C\,\|u\|_{p(x)}^{(1-\alpha)(p_++\varrho_+)}\,\|\nabla u\|_{p(x)}^{\alpha(p_++\varrho_+)}. \tag{3.20}$$

To complete the proof of 1) we have two cases for p_+.

Case 1 : if $p_+ < p_- + \frac{p_-^2}{n}$, in this case choose $0 < \varrho_+ < \left(p_- + \frac{p_-^2}{n}\right) - p_+$, then $p_- > \alpha\left(p_+ + \varrho_+\right)$, by Young's inequality, we get

$$\int\limits_{\Omega} |u(x)|^{p(x)} \log |u(x)|\, dx \leq C_\varepsilon \left(\|u\|_{p(x)}^{p_-}\right)^{\beta} + \varepsilon \|\nabla u\|_{p(x)}^{p_-},$$

where $\varepsilon > 0$ and $\beta = \frac{(1-\alpha)\left(p_+ + \varrho\right)}{p_- - \alpha\left(p_+ + \varrho\right)} > 1$. Since $u \in \mathcal{N}$, so if $\|\nabla u\|_{p(x)} > 1$, we find from Proposition 2.2 (with u is replaced by ∇u) that

$$\|\nabla u\|_{p(x)}^{p_-} \leq \int\limits_{\Omega} |\nabla u|^{p(x)}\, dx = \int\limits_{\Omega} |u|^{p(x)} \log |u|\, dx \leq C_\varepsilon \left(\|u\|_{p(x)}^{p_-}\right)^{\beta} + \varepsilon \|\nabla u\|_{p(x)}^{p_-}.$$

Taking $\varepsilon < 1$ so from (3.19) and Proposition 2.2 by using the assumption $\|u\|_{p(x)} > 1$, yield

$$J(u) \geq c_\varepsilon \left(\|\nabla u\|_{p(x)}^{p_-}\right)^{\frac{1}{\beta}}, \tag{3.21}$$

Hence, J is coercive on $\mathcal{N}$.

Case 2 : if $p_+ < p_- + \frac{p_-^2}{n}$ does not hold. So divided Ω into l subsets in the following way $\overline{\Omega} = \cup_{j=1}^{l} \overline{\Omega}_j$ and in every subset Ω_j we have $p_{j+} < p_{j-} + \frac{p_{j-}^2}{n}$, $(j = 1, 2, ..., l)$ and $\varrho_{j+} < p_{j-}^{*} - p_{j+}$, $(j = 1, 2, ..., l)$. Here we have used the assumption (1.2) and the continuity of the functions $p(x)$ and $\varrho(x)$. For any $u \in \mathcal{N}$, we shall show that (3.21) holds, for this we assume that $\|u\|_{p(x),\Omega_j} \leq 1$ and $\|\nabla u\|_{p(x),\Omega_j} \leq 1$, for all $j = 1, 2, ..., l$. Therefore, choose $\varrho_{j+} < \left(p_{j-} + \frac{p_{j-}^2}{n}\right) - p_{j+}$. Then $p_{j-} > \alpha_j\left(p_{j+} + \varrho_{j+}\right)$, $(j = 1, 2, ..., l)$. As above we have

$$\int\limits_{\Omega_j} |u(x)|^{p(x)} \log |u(x)|\, dx \leq C_{j\varepsilon} \|u\|_{p(x),\Omega_j}^{\beta_j p_{j-}} + \varepsilon \|\nabla u\|_{p(x),\Omega_j}^{p_{j-}}$$

$$\le C_{j\varepsilon}\left(\int\limits_{\Omega_j}|u|^{p(x)}\,dx\right)^{\beta_j p_{j-}/p_{j+}}+\varepsilon\left(\int\limits_{\Omega_j}|\nabla u|^{p(x)}\,dx\right)^{p_{j-}/p_{j+}}$$

for every $j=1,2,...,l$, where $\beta_j=\frac{(1-\alpha_j)(p_{j+}+\varrho_{j+})}{p_{j-}-\alpha_j(p_{j+}+\varrho_{j+})}>1$, here we have used Proposition 2.2. Because of

$$\int\limits_{\Omega}|u(x)|^{p(x)}\log|u(x)|\,dx\le\sum_{j=1}^{l}\int\limits_{\Omega_j}|u(x)|^{p(x)}\log|u(x)|\,dx$$

$$\le\sum_{j=1}^{l}\left[C_{j\varepsilon}\left(\int\limits_{\Omega_j}|u|^{p(x)}\,dx\right)^{\beta_j p_{j-}/p_{j+}}+\varepsilon\left(\int\limits_{\Omega_j}|\nabla u|^{p(x)}\,dx\right)^{p_{j-}/p_{j+}}\right].$$

Set $\beta=\min_{1\le j\le l}\beta_j>1$, since $\|u\|_{p(x),\Omega_j},\|\nabla u\|_{p(x),\Omega_j}\le 1$ and $1<p_-\le p_{j-}$, $\forall j=1,2,...,l$, then

$$\int\limits_{\Omega}|u(x)|^{p(x)}\log|u(x)|\,dx\le C_{\varepsilon}\sum_{j=1}^{l}\left(\int\limits_{\Omega_j}|u|^{p(x)}\,dx\right)^{\beta p_-/p_+}+\varepsilon\sum_{j=1}^{l}\left(\int\limits_{\Omega_j}|\nabla u|^{p(x)}\,dx\right)^{p_-/p_+}$$

$$\le C_{\varepsilon}\left(\sum_{j=1}^{l}\int\limits_{\Omega_j}|u|^{p(x)}\,dx\right)^{\beta p_-/p_+}+\varepsilon\left(\sum_{j=1}^{l}\int\limits_{\Omega_j}|\nabla u|^{p(x)}\,dx\right)^{p_-/p_+}$$

$$=C_{\varepsilon}\left(\left(\int\limits_{\Omega}|u|^{p(x)}\,dx\right)^{p_-/p_+}\right)^{\beta}+\varepsilon\left(\int\limits_{\Omega}|\nabla u|^{p(x)}\,dx\right)^{p_-/p_+}.$$

If (3.18) holds, Proposition 2 gives us

$$\int\limits_{\Omega}|u(x)|^{p(x)}\log|u(x)|\,dx\le C_{\varepsilon}\left(\|u\|_{p(x),\Omega}^{p_-}\right)^{\beta}+\varepsilon\,\|\nabla u\|_{p(x),\Omega}^{p_-}.$$

Similarly, we obtain (3.21) as described earlier. Therefore, J exhibits coercivity on $\mathcal{N}$.

2) Let us consider the sequence $(u_n)_{n\in\mathbb{N}}$ in X_0, satisfying $u_n \rightharpoonup u$ weakly in X_0. By Proposition 2.3, Proposition 2.6, and Proposition 2.7, we can conclude that $(u_n)_{n\in\mathbb{N}}$ is bounded in X_0. Consequently, there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$, also denoted as $(u_n)_{n\in\mathbb{N}}$, such that $u_n \to u$ almost everywhere in Ω. By utilizing the Lebesgue dominated convergence theorem, taking into account (1.2), Lemma 2.1, and Remark 2.2, we get

$$\lim_{n\to\infty} \int_\Omega \frac{1}{p(x)} |u_n|^{p(x)} \log |u_n| \, dx = \int_\Omega \frac{1}{p(x)} |u|^{p(x)} \log |u| \, dx, \tag{3.22}$$

and

$$\lim_{n\to\infty} \int_\Omega |u_n|^{p(x)} \log |u_n| \, dx = \int_\Omega |u|^{p(x)} \log |u| \, dx. \tag{3.23}$$

Moreover from Fatou Lemma we have

$$\int_\Omega \liminf_{n\to\infty} \frac{1}{p(x)} |\nabla u_n|^{p(x)} \, dx \le \liminf_{n\to\infty} \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} \, dx, \tag{3.24}$$

and

$$\int_\Omega \liminf_{n\to\infty} \frac{1}{p^2(x)} |u_n|^{p(x)} \, dx \le \liminf_{n\to\infty} \int_\Omega \frac{1}{p^2(x)} |u_n|^{p(x)} \, dx, \tag{3.25}$$

which means by (3.22), (3.24) and (3.25) that

$$\begin{aligned} J(u) &\le \liminf_{n\to\infty} \left(\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} \, dx - \int_\Omega \frac{1}{p(x)} |u_n|^{p(x)} \log |u_n| \, dx + \int_\Omega \frac{1}{p^2(x)} |u_n|^{p(x)} \, dx \right) \\ &= \liminf_{n\to\infty} J(u_n). \end{aligned}$$

For the funtional I we use (3.23) and the modular's weak lower semicontinuity $\rho_{p(.)}$ (see [12, Theorem 3.2.9 p. 77]) that is

$$\begin{aligned} I(u) &= \int_\Omega |\nabla u|^{p(x)} \, dx - \int_\Omega |u|^{p(x)} \log |u| \, dx \\ &\le \liminf_{n\to\infty} \left(\int_\Omega |\nabla u_n|^{p(x)} \, dx - \int_\Omega |u_n|^{p(x)} \log |u_n| \, dx \right) \\ &= \liminf_{n\to\infty} I(u_n). \end{aligned}$$

Thus 2). This completes the proof. $\square$

Now, define

$$d = \inf_{u \in \mathcal{N}} J(u), \tag{3.26}$$

and in the subsequent lemma, we establish the existence of a nontrivial critical minimizing point $u \in \mathcal{N}$ for J defined on X_0. This point serves as a solution to the equilibrium problem (3.26) related to (1.1).

Lemma 3.5. *Let $u \in X_0$, and $p(.) \in \mathcal{P}^{\log}(\Omega)$, then the following assertions hold*
(1) $d = \inf_{u \in X_0} \sup_{\lambda > 0} J(\lambda u)$.

(2) There exists a positive lower bound for d, that is

$$d \geq \frac{R^{p_-}}{p_+^2} = M, \tag{3.27}$$

(3) The problem (3.26) has a positive extremal solution $u \in \mathcal{N}$, In other words, it means $J(u) = d$.

Proof. Let $u \in X_0$, then by lemma 3.2, Proposition 2.2 and (3.5) (with u is replaced by $\lambda^* u$) we may write

$$\begin{aligned} \sup_{\lambda>0} J(\lambda u) = J(\lambda^* u) \geq \gamma I(\lambda^* u) + \frac{1}{p_+^2} \int_\Omega |\lambda^* u|^{p(x)} \\ \geq \frac{1}{p_+^2} \min\left\{ \|\lambda^* u\|_{p(x)}^{p_-}, \|\lambda^* u\|_{p(x)}^{p_+} \right\}. \end{aligned} \tag{3.28}$$

Firstly, we prove (1). By the definition of $\mathcal{N}$, and lemma 3.2 we can deduce that $\lambda^* u \in \mathcal{N}$. Consequently,

$$J(\lambda^* u) \geq \inf_{u \in \mathcal{N}} J(u) = d. \tag{3.29}$$

So (3.28) together with (3.29) yield that

$$\inf_{u \in X_0} \sup_{\lambda > 0} J(\lambda u) \geq d. \tag{3.30}$$

In addition, if $u \in \mathcal{N}$ then it follows from (3.7) that the critical point $\lambda^* = 1$ is the sole one within the interval $(0, \infty)$ for the mapping $j(\lambda)$. Consequently, $\sup J(\lambda u) = J(u)$, where the supremum is taken over all λ greater than zero. Hence

$$\inf_{u \in X_0} \sup_{\lambda > 0} J(\lambda u) \leq \inf_{u \in \mathcal{N}} \sup_{\lambda > 0} J(\lambda u) = \inf_{u \in \mathcal{N}} J(u) = d. \tag{3.31}$$

Thus, (3.30) and (3.31) lead to the desired result.
(2) In accordance with lemma 3.2, for each $u \in X_0$, then $I(\lambda^* u) = 0$. Which means by lemma 3.3 that

$$\min\left\{\|\lambda^* u\|_{p(x)}, \|\lambda^* u\|_{p(x)}^{p_+/p_-}\right\} \geq R. \tag{3.32}$$

So (3.28) and (3.32) give

$$\sup_{\lambda>0} J(\lambda u) \geq \frac{R^{p_-}}{p_+^2} = M.$$

Hence, (3.27) arises from assertion (1) and therefore the assertion (2) is proved.
(3) Consider a sequence $\{u_m\}_m^\infty$ contained within $\mathcal{N}$, which serves as a sequence minimizing the functional J, and suppose that $\{u_m\}_m^\infty$ is of positive terms ($u_m > 0$) a.e. Ω for all $m \in \mathbb{N}$, such that

$$\lim_{m\to\infty} J(u_m) = d.$$

Moreover the sequence $\{|u_m|\}_m^\infty$ contained within $\mathcal{N}$, serves also as a sequence minimizing the functional J and $J(|u_m|) = J(u_m)$ because of $u_m > 0$. In addition, we have previously seen the coercivity of J on $\mathcal{N}$ so $\{u_m\}_m^\infty$ is bounded in $W_0^{1.p(x)}(\Omega)$.

Since $p(.) \in \mathcal{P}^{\log}(\Omega)$, then the compact embedding $W_0^{1.p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{p(.)}(\Omega)$ (see [12, Theorem 8.4.2]), guarantees that there exists u and $\{u_m\}_m^\infty$, anyway denoted by $\{u_m\}_m^\infty$, so that

$$u_m \to u \quad \text{weakly in} \quad W_0^{1.p(x)}(\Omega),$$

$$u_m \to u \quad \text{strongly in} \quad L^{p(x)}(\Omega),$$
$$u_m(x) \to u(x) \quad \text{almost everywhere in} \quad \Omega.$$

so that $u \geq 0$ almost everywhere in Ω. Since $p(.) \in \mathcal{P}(\Omega)$, since J is a weak lower semicontinuous (see Lemma 3.4), then

$$J(u) \leq \liminf_{m\to\infty} J(u_m) = d.$$

since $u_m \in \mathcal{N}$ then $u_m \in X_0$ and $I(u_m) = 0$ this means by Lemma 3.3 that

$$\|u_m\|_{p(x)} \geq R.$$

This mean that $\|u_m\|_{p(x)} \neq 0$ by strong convergence in $L^{p(x)}(\Omega)$, that is, $u \in X_0$. In addition, from weak lower semicontinuity of $I(u)$ (see Lemma 3.4), we find

$$I(u) \leq \liminf_{k\to\infty} I(u_k) = 0.$$

So, to complete the proof of (3), we must show that $I(u) = 0$. Indeed, suppose that $I(u) < 0$, then, by Lemma 3.2, there is $\lambda^* > 0$,

$$\lambda^* = \lambda^*(u) = \exp\left(\frac{\int_\Omega (\lambda^*)^{p(x)-1}\left(|\nabla u|^{p(x)} - |u|^{p(x)}\log|u|\right)dx}{\int_\Omega (\lambda^*)^{p(x)-1}|u|^{p(x)}\,dx}\right) < 1$$

and λ^* fulfils $I(\lambda^* u) = 0$. so,

$$0 < d \leq J(\lambda^* u) \leq \int\limits_\Omega \frac{1}{p^2(x)}|\lambda^* u|^{p(x)}\,dx \leq (\lambda^*)^{p_-}\liminf_{k\to\infty} J(u_k) = (\lambda^*)^{p_-} d < d.$$

This is not possible, and the lemma is thus established. □

3.5 Potential well sets

Now we presnt the potential well sets that were introduced in [28] (see also [34])

$$\begin{aligned}
&\mathcal{W}_1 = \{u \in X_0 : J(u) < d\}, && \mathcal{W}_2 = \{u \in X_0 : J(u) = d\}, && \mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2,\\
&\mathcal{W}_1^+ = \{u \in \mathcal{W}_1 : I(u) > 0\}, && \mathcal{W}_2^+ = \{u \in \mathcal{W}_2 : I(u) > 0\}, && \mathcal{W}^+ = \mathcal{W}_1^+ \cup \mathcal{W}_2^+,\\
&\mathcal{W}_1^- = \{u \in \mathcal{W}_1 : I(u) < 0\}, && \mathcal{W}_2^- = \{u \in \mathcal{W}_2 : I(u) < 0\}, && \mathcal{W}^- = \mathcal{W}_1^- \cup \mathcal{W}_2^-.
\end{aligned}$$

It is clear that the sets $\mathcal{W}^+$ and $\mathcal{W}^-$ are mutually exclusive, i.e., they have no common elements, and their union covers the entire set $\mathcal{W}$.

It is clear that, $\mathcal{W}^+ \cap \mathcal{W}^- = \emptyset$ and $\mathcal{W}^+ \cup \mathcal{W}^- = \mathcal{W}$. Note that $\mathcal{W}^+$ is the best part of the well, so we will prove that if the initial datum belongs to $\mathcal{W}^+$ then every weak solution of the problem (1.1) exists globally in time. On the other hand, a result of blow up for weak solutions may be obtained if the initial datum belongs to $\mathcal{W}^-$.

Chapter 4

Stabilization and Global existence

In this chapter, our initial focus is on establishing the existence of the local solutions for problem (1.1). Subsequently, we provide a proof for the global existence of weak solutions for the same problem, under the condition that the initial data belong to $\mathcal{W}^{+}$. Furthermore, we present a stability result similar to the one found in [28], demonstrating that the decay of the norm $\|u(t)\|_2$ is polynomial rather than exponential, as indicated in [9], considering the case where $p(x) \equiv 2$. To accomplish this, we rely on a lemma introduced by Martinez [31].

4.1 Preliminaries

Lemma 4.1. *Let $f : \mathbb{R}^{+} \to \mathbb{R}^{+}$ be a non-increasing function, and σ be a non-negative constant satisfying*

$$\int_{t}^{+\infty} f^{1+\sigma}(s)\,ds \leq \frac{1}{\omega} f^{\sigma}(0)\, f(t), \quad \forall t \geq 0.$$

where ω is a positive constant. Then the following statements hold:
(1) For $\sigma = 0$, we have $f(t) \leq f(0)e^{1-\omega t}$ for all $t \geq 0$,

(2) For $\sigma > 0$, we have $f(t) \leq f(0)\left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{\frac{1}{\sigma}}$ for all $t \geq 0$.

Lemma 4.2 (Minty trick, see [21]). *Let Ω be a bounded open set of $\mathbb{R}^{n}$, $n \geq 1$ and φ be a continuous nondecreasing function from $\mathbb{R}$ to $\mathbb{R}$. We assume that there exists $C \in \mathbb{R}^{+}$ such that*

$$\left|\varphi(s)\right| \leq C(|s| + 1) \ \textit{for all } s \in \mathbb{R}.$$

(Note that the existence of C is true if φ is Lipschitz continuous.) Let $(\rho_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ be sequences such that $\rho_n \to \rho$ weakly in $L^2(\Omega)$, $u_n \to u$ weakly in $L^2(\Omega)$. We assume that $u_n = \varphi(\rho_n)$ a.e., for all n, and that

$$\lim_{n\to\infty} \int_\Omega \rho_n(s)u_n(s)ds = \int_\Omega \rho(s)u(s)ds.$$

Then, $u = \varphi(\rho)$ a.e..

4.2 Local existence

Now we start with the existence of local of solutions to our problem.

Theorem 4.1 (Existence of a local solutions)**.** *Under the assumptions that $u_0 \in X_0$, $p \in C\left(\overline{\Omega}\right)$ satisfying $2 < p_- \le p(x) \le p_+ < p_-^*$, and the log-Hölder continuous condition (2.1), the problem (1.1) possesses a weak local solution $u(x,t)$ on $\Omega \times (0, T_0)$, where T_0 is a positive constant.*

Proof. By using The Faedo-Galerkin's methods: We consider in the space $W_0^{1,p(x)}(\Omega)$, the basis $\left\{w_j\right\}_{j=1}^{\infty}$ and let V_m be the finite dimensional space defined by

$$V_m = \text{span}\left\{w_1, w_2, ..., w_m\right\}.$$

Suppose that u_{0m} an element of V_m such that

$$u_{0m} = \sum_{j=1}^{m} a_{mj}w_j \to u_0 \quad \text{strongly in } W_0^{1,p(x)}(\Omega). \tag{4.1}$$

when $m \to +\infty$. Defining the approximate solution $u_m(x,t)$ of the problem (1.1) as follows

$$u_m(x,t) = \sum_{j=1}^{m} \alpha_{mj}(t)\, w_j(x),$$

with coefficients α_{mj} $(1 \leq j \leq m)$ satisfying the following equations

$$\int\limits_{\Omega} u_{mt}(t)\, w_i dx + \int\limits_{\Omega} |\nabla u_m(t)|^{p(x)-2} \nabla u_m(t)\, \nabla w_i dx = \int\limits_{\Omega} |u_m(t)|^{p(x)-2} u_m(t) \log|u_m(t)|\, w_i dx, \tag{4.2}$$

$1 \leq i \leq m$, with the initial conditions

$$\alpha_{mj}(0) = a_{mj}, \quad 1 \leq j \leq m. \tag{4.3}$$

The Theorem of Peano guarantees the local existence of the solution of a system (4.2)-(4.3). Let us multiply the i^{th} equation in (4.2) by $\alpha_{mi}(t)$ and take the sum over $1 \leq i \leq m$, we get

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_2^2 + \int\limits_{\Omega} |\nabla u_m(t)|^{p(x)} dx = \int\limits_{\Omega} |u_m(t)|^{p(x)} \log|u_m(t)|\, dx. \tag{4.4}$$

In view of Remark 2.2, we have

$$\int\limits_{\Omega} |u_m(t)|^{p(x)} \log|u_m(t)|\, dx \leq \frac{1}{\varrho_-}\int\limits_{\Omega} |u_m(t)|^{p_+ + \varrho_+} dx, \tag{4.5}$$

for some ϱ_+ chosen small enough such that $2 < p_- < p_+ + \varrho_+ \leq p_-^* = \frac{np_-}{n-p_-}$ with $\Omega_- = \{x \in \Omega \;\; |u_m(t)| \leq 1\}$ and $\Omega_+ = \{x \in \Omega \;\; |u_m(t)| > 1\}$. By Lemma 2.1 we get from (4.5),

$$\int\limits_{\Omega} |u_m(t)|^{p(x)} \log|u_m(t)|\, dx \leq C\|u_m(t)\|_{p_-^*}^{\theta(p_+ + \varrho_+)} \|u_m(t)\|_2^{(1-\theta)(p_+ + \varrho_+)}$$

with $0 < \theta < 1$ and $\frac{1}{p_+ + \varrho_+} = \frac{\theta(n-p_-)}{np_-} + \frac{1-\theta}{2}$. From the continuous embeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega) \hookrightarrow L^{p_-^*}(\Omega)$, we have

$$\int\limits_{\Omega} |u_m(t)|^{p(x)} \log|u_m(t)|\, dx \leq C\|\nabla u_m(t)\|_{p(x)}^{\theta(p_+ + \varrho_+)} \|u_m(t)\|_2^{(1-\theta)(p_+ + \varrho_+)}. \tag{4.6}$$

We distinguishe two cases
Case 1: if $\|\nabla u_m\|_{p(x)} > 1$. Assume that $p_+ < (1+\frac{2}{n})p_-$, it follows from (4.6) that

$$\int\limits_{\Omega} |u_m(t)|^{p(x)} \log |u_m(t)| \, dx \le C \left(\int\limits_{\Omega} |\nabla u_m(t)|^{p(x)} \, dx \right)^{\frac{\theta(p_+ + \varrho_+)}{p_-}} \|u_m(t)\|_2^{(1-\theta)\left(p_+ + \varrho_+\right)},$$

Young's inequality gives

$$\int\limits_{\Omega} |u_m(t)|^{p(x)} \log |u_m(t)| \, dx \le \varepsilon \int\limits_{\Omega} |\nabla u_m(t)|^{p(x)} \, dx + C_\varepsilon \left(\|u_m(t)\|_2^2 \right)^{\nu}, \tag{4.7}$$

where we have chosen $0 < \varrho_+ < (1+\frac{2}{n})p_- - p_+$ so that $\theta(p_+ + \varrho_+) < p_-$, with

$$\theta = \left(\frac{1}{2} - \frac{1}{p_+ + \varrho_+} \right) \left(\frac{1}{n} - \frac{1}{p_-} + \frac{1}{2} \right)^{-1} \quad \text{and} \quad \nu = \frac{p_-(1-\theta)(p_+ + \varrho_+)}{2\left[p_- - \theta(p_+ + \varrho_+)\right]} > 1.$$

If $p_+ < (1+\frac{2}{n})p_-$ does not hold. here we use the same as in the proof of Lemma 3.4. By dividing Ω into l subsets such that $\overline{\Omega} = \cup_{j=1}^{l} \overline{\Omega}_j$ with $p_{j+} < (1+\frac{2}{n})p_{j-}$, $(j = 1, 2, ..., l)$ and $\varrho_{j+} < p_{j-}^* - p_{j+}$, $(j = 1, 2, ..., l)$. For l large enough, assume that $\|\nabla u_m(t)\|_{p(x),\Omega_j} \le 1$, for all $j = 1, 2, ..., l$. So, choose $\varrho_{j+} < (1+\frac{2}{n})p_{j-} - p_{j+}$. Then $p_{j-} > \theta_j\left(p_{j+} + \varrho_{j+}\right)$, $(j = 1, 2, ..., l)$. By Proposition 2.2 we find

$$\int\limits_{\Omega_j} |u_m(t)|^{p(x)} \log |u_m(t)| \, dx \le C_j \left(\int\limits_{\Omega_j} |\nabla u_m(t)|^{p(x)} \, dx \right)^{\frac{\theta_j\left(p_{j+} + \varrho_{j+}\right)}{p_{j+}}} \|u_m(t)\|_{2,\Omega_j}^{(1-\theta_j)\left(p_{j+} + \varrho_{j+}\right)}$$

Young's inequality gives

$$\int\limits_{\Omega_j} |u_m(t)|^{p(x)} \log |u_m(t)| \, dx \le \varepsilon \left(\int\limits_{\Omega_j} |\nabla u_m(t)|^{p(x)} \, dx \right)^{p_{j-}/p_{j+}} + C_{j\varepsilon} \left(\|u_m(t)\|_{2,\Omega_j}^2 \right)^{\nu_j}$$

where

$$\theta_j = \left(\frac{1}{2} - \frac{1}{p_{j+} + \varrho_{j+}}\right)\left(\frac{1}{n} - \frac{1}{p_{j-}} + \frac{1}{2}\right)^{-1} \text{ and } \nu_j = \frac{p_{j-}\left(1-\theta_j\right)\left(p_{j+} + \varrho_{j+}\right)}{2\left[p_{j-} - \theta_j\left(p_{j+} + \varrho_{j+}\right)\right]} > 1.$$

We can see that

$$\begin{aligned} \int_{\Omega} |u(x)|^{p(x)} \log|u(x)|\, dx &\leq \sum_{j=1}^{l} \int_{\Omega_j} |u(x)|^{p(x)} \log|u(x)|\, dx \\ &\leq \sum_{j=1}^{l} C_{j\varepsilon}\left(\int_{\Omega_j} |u_m(t)|^2\, dx\right)^{\nu_j} + \varepsilon \sum_{j=1}^{l}\left(\int_{\Omega_j} |\nabla u_m(t)|^{p(x)}\, dx\right)^{p_{j-}/p_{j+}}. \end{aligned}$$

Set $\nu = \min_{1\leq j\leq l} \nu_j > 1$, since $\|\nabla u_m(t)\|_{p(x),\Omega_j} \leq 1$ and $1 < p_- \leq p_{j-} \leq p_{j+} \leq p_+$, $\forall j = 1, 2, ..., l$, then

$$\begin{aligned} \int_{\Omega} |u(x)|^{p(x)} \log|u(x)|\, dx &\leq C_\varepsilon \sum_{j=1}^{l}\left(\int_{\Omega_j} |u_m(t)|^2\, dx\right)^{\nu} + \varepsilon \sum_{j=1}^{l}\left(\int_{\Omega_j} |\nabla u_m(t)|^{p(x)}\, dx\right)^{p_-/p_+} \\ &\leq C_\varepsilon\left(\sum_{j=1}^{l}\int_{\Omega_j} |u_m(t)|^2\, dx\right)^{\nu} + \varepsilon\left(\sum_{j=1}^{l}\int_{\Omega_j} |\nabla u_m(t)|^{p(x)}\, dx\right)^{p_-/p_+} \\ &= C_\varepsilon\left(\|u_m(t)\|_{2,\Omega}^2\right)^{\nu} + \varepsilon \int_{\Omega} |\nabla u_m(t)|^{p(x)}\, dx, \end{aligned}$$

and (4.7) follows again
Case 2: if $\|\nabla u_m\|_{p(x)} \leq 1$, so (4.6) becomes

$$\int_{\Omega} |u_m(t)|^{p(x)} \log|u_m(t)|\, dx \leq C\left(\|u_m(t)\|_2^2\right)^{\frac{(1-\theta)(p_+ + \varrho_+)}{2}}$$

Since $\frac{p_-}{p_- - \theta\left(p_+ + \varrho_+\right)} > 1$ then $\frac{2\nu}{(1-\theta)\left(p_+ + \varrho_+\right)} > 1$, Young's inequality gives

$$\int\limits_{\Omega} |u_m(t)|^{p(x)} \log |u_m(t)|\, dx \leq C + \left(\|u_m(t)\|_2^2\right)^{\nu}. \tag{4.8}$$

We combine (4.7) and (4.8) we find

$$\int\limits_{\Omega} |u_m(t)|^{p(x)} \log |u_m(t)|\, dx \leq \varepsilon \int\limits_{\Omega} |\nabla u_m(t)|^{p(x)}\, dx + c_{\varepsilon}\left(\|u_m(t)\|_2^2\right)^{\nu}. \tag{4.9}$$

This combined with (4.4) yields

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_2^2 + (1-\varepsilon)\int\limits_{\Omega} |\nabla u_m(t)|^{p(x)}\, dx \leq c_{\varepsilon}\left(\|u_m(t)\|_2^2\right)^{\nu}.$$

Choose $0 < \varepsilon < 1$, then for all $t \in [0, T_0]$ with $T_0 > 0$, we find

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_2^2 \leq c_{\varepsilon}\left(\|u_m(t)\|_2^2\right)^{\nu}.$$

setting $\eta = \max\limits_{m\in\mathbb{N}} \|u_{0m}\|_2^2$, $r(s) = s^{\nu}$, $g(s) = 2c_{\varepsilon}$ and choose T_0 such that there is $C_0 > 0$ large enough satisfying $T_0 \leq \frac{\eta^{1-\nu} - C_0^{1-\nu}}{2c_{\varepsilon}(\nu-1)}$, so that Bihari's integral inequality yields

$$\|u_m(t)\|_2^2 \leq C_0, \quad \forall t \in [0, T_0]. \tag{4.10}$$

Let us multiply again the two sides of (4.2) by $\alpha'_{mi}(t)$, and compute the summation over $i = 1, 2 \ldots m$, and integrate in time over the interval $[0, t]$. We obtain

$$\int\limits_0^t \|u_{ms}(s)\|_2^2\, ds + J(u_m(t)) = J(u_m(0)). \tag{4.11}$$

Then (4.1) means that there is a positive constant C_1 such that

$$J(u_m(0)) \leq C_1, \quad \text{for all } m. \tag{4.12}$$

On the other hand, (4.9) and (4.10) with the help of (3.5) (where u is replaced by $u_m(t)$), derive that

$$J(u_m(t)) \geq \gamma(1-\varepsilon)\int\limits_{\Omega} |\nabla u_m(t)|^{p(x)}\,dx + \frac{1}{p_+^2}\int\limits_{\Omega} |u_m(t)|^{p(x)}\,dx - \gamma c_\varepsilon (C_0)^{\nu}, \tag{4.13}$$

where $C_0 > 0$ and depends on T_0

So (4.11) and (4.13) give

$$\int\limits_0^t \|u_{ms}(s)\|_2^2\,ds + \frac{1-\varepsilon}{p_+}\int\limits_{\Omega} |\nabla u_m(t)|^{p(x)}\,dx \leq C_1 + \frac{c_\varepsilon (C_0)^{\nu}}{p_-}$$

This means, for $\varepsilon < 1$ that

$$\int\limits_{\Omega} |\nabla u_m(t)|^{p(x)}\,dx \leq C, \tag{4.14}$$

and

$$\|u_{mt}\|_{L^2\left(0,T_0;L^2(\Omega)\right)} \leq C. \tag{4.15}$$

If we combine a priori estimates (4.14), (4.15) we conclude that there exists u and a subsequence of $\{u_m\}_{m=1}^{\infty}$ again denoted by $\{u_m\}_{m=1}^{\infty}$ such that

$$u_m \to u \quad \text{weakly}^* \text{ in } L^{\infty}\left(0,T_0;W_0^{1,p(x)}(\Omega)\right), \tag{4.16}$$

$$u_{mt} \to u_t \quad \text{weakly in } L^2\left(0,T_0;L^2(\Omega)\right), \tag{4.17}$$

$$|\nabla u_m|^{p(x)-2}\nabla u_m \to \chi \quad \text{weakly}^* \text{ in } L^{\infty}\left(0,T_0;W_0^{-1,p'(x)}(\Omega)\right). \tag{4.18}$$

Due to the compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$ given in Proposition 2.8 and by the compactness theorem of Aubin-Lions-Simon, it follows from (4.16) and (4.17) that

$$u_m \to u \text{ strongly in } C\left([0,T_0];L^{r(x)}(\Omega)\right),$$

for all function $r(.)$ such that $2 \leq r(x) \ll p^*(x) = \frac{np(x)}{n-p(x)}$. Obviously, this means by the continuity of the function $u_m \mapsto |u_m|^{p(x)-2} u_m \log|u_m|$ that

$$|u_m|^{p(x)-2} u_m \log|u_m| \to |u|^{p(x)-2} u \log|u| \quad \text{a.e. } (x,t) \in \Omega \times (0, T_0). \tag{4.19}$$

On the other hand, a simple calculation, gives

$$\begin{aligned}\int_\Omega \left|\psi_m(x,t)\right|^{p'(x)} dx &\leq \int_{\Omega_-} \left|\psi_m(x,t)\right|^{p'_+} dx + \int_{\Omega_+} \left|\psi_m(x,t)\right|^{p'_+} dx \\ &\leq \left(\frac{e^{-1}}{p_- - 1}\right)^{p'_+} |\Omega| + \left(\frac{p_+}{q_-}\right)^{p'_+} \int_{\Omega_+} |u_m(t)|^{q(x)} dx\end{aligned}$$

where $q(.) : \Omega \to \mathbb{R}$ is a measurable function satisfies $p_+ \leq q_- \leq q(x) \leq q_+ < p^*(x)$, $\psi_m(x,t) = |u_m(x,t)|^{p(x)-1} \log|u_m(x,t)|$ and

$$\Omega_- = \{x \in \Omega : |u_m(x,t)| \leq 1\}, \quad \Omega_+ = \{x \in \Omega : |u_m(x,t)| > 1\},$$

by Proposition 2.2 we get

$$\int_\Omega \left|\psi_m(x,t)\right|^{p'(x)} dx \leq \left(\frac{e^{-1}}{p_- - 1}\right)^{p'_+} |\Omega| + \left(\frac{p_+}{q_-}\right)^{p'_+} \max\left\{\|u_m\|_{q(x)}^{q_-}, \|u_m\|_{q(x)}^{q_+}\right\}$$

Using the embeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ (see Proposition 2.4 and Proposition 2.9) we find

$$\int_\Omega \left|\psi_m(x,t)\right|^{p'(x)} dx \leq \left(\frac{1}{p_- - 1}\right)^{p'_+} |\Omega| + \left(\frac{p_+}{q_-}\right)^{p'_+} \max\left\{S^{q_-} \|\nabla u_m(t)\|_{p(x)}^{q_-}, S^{q_+} \|\nabla u_m(t)\|_{p(x)}^{q_+}\right\} \leq C_0,$$

since by (4.14) and S represent the optimal constant in the Sobolev embedding. Then, using Lions lemma (see [6, Lemma 1.3 p. 12]), we deduce from (4.17) and (4.18) that

$$|u_m|^{p(x)-2} u_m \log|u_m| \to |u|^{p(x)-2} u \log|u| \quad \text{weakly}^* \text{ in } L^\infty\left(0, T_0; L^{p'(x)}(\Omega)\right). \tag{4.20}$$

Taking, in (4.2) and (4.3) the limit as $m \to +\infty$, and then by using (4.16)-(4.18) and (4.20), it is readily shown that u satisfies the initial condition $u(0) = u_0$ and

$$\int_\Omega u_t(t)\, w dx + \int_\Omega \chi(t)\, \nabla w dx = \int_\Omega |u(t)|^{p(x)-2} u(t) \log|u(t)|\, w dx, \tag{4.21}$$

for all $w \in W_0^{1,p(x)}(\Omega)$ and for almost every $t \in [0, T_0]$. Finally, by means of well-known arguments from the theory of monotone operators in variable exponent spaces (see [13]) and Minty's trick we obtain

$$\chi = |\nabla u|^{p(x)-2} \nabla u$$

Indeed, we demonstrate that

$$\limsup_{m\to\infty} \int_0^{T_0} \int_\Omega |\nabla u_m(t)|^{p(x)} dxdt \leq \int_0^{T_0} \int_\Omega \chi(t) \nabla u(t)\, dxdt.$$

So on one hand because of u_m is a test function we have from (4.2)

$$\int_0^{T_0} \int_\Omega |\nabla u_m(t)|^{p(x)} dxdt = -\int_0^{T_0} \int_\Omega u_{mt}(t)\, u_m(t)\, dxdt + \int_0^{T_0} \int_\Omega |u_m(t)|^{p(x)} \log|u_m(t)|\, dxdt$$

an integration by parts gives

$$\int_0^{T_0} \int_\Omega |\nabla u_m(t)|^{p(x)} dxdt = -\frac{1}{2} \|u_m(T_0)\|_2^2 + \frac{1}{2} \|u_m(0)\|_2^2 + \int_0^{T_0} \int_\Omega |u_m(t)|^{p(x)} \log|u_m(t)|\, dxdt$$

Taking $\limsup_{m\to\infty}$ on both sides and using the fact that V_m is dense in $W_0^{1,p(x)}(\Omega)$ and the lower semicontinuity of the norm as well as (4.16)-(4.18) and (4.20) yields

$$\begin{aligned}\limsup_{m\to\infty} \int_0^{T_0} \int_\Omega |\nabla u_m(t)|^{p(x)} dxdt &\leq -\frac{1}{2} \|u(T_0)\|_2^2 + \frac{1}{2} \|u(0)\|_2^2 + \int_0^{T_0} \int_\Omega |u(t)|^{p(x)} \log|u(t)|\, dxdt \\ &= -\int_0^{T_0} \int_\Omega u_t(t)\, u(t)\, dxdt + \int_0^{T_0} \int_\Omega |u(t)|^{p(x)} \log|u(t)|\, dxdt \\ &= \int_0^{T_0} \int_\Omega \chi(t)\, \nabla u(t)\, dxdt\end{aligned}$$

since by (4.21) (with w is replaced by u). Thus, the function u is a desirable solution of the problem (1.1). This concludes the proof. □

Remark 4.1. *Note that the continuous embedding* $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega)$ *in Proposition 2.9 (see [10, 12]) can be also obtained from the embeddings* $W_0^{1,p(x)}(\Omega) \hookrightarrow W^{1,p(x)}(\Omega)$ *which is straightforward and the continuous one* $W^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega)$ *given in [18, Theorem 1.1, Theorem 1.2].*

4.3 Global existence

Now we present our main theorem of this section.

Theorem 4.2. *Assume that* $u_0 \in \mathcal{W}^+$. *Then the weak solution of the problem (1.1) is globally in time and fulfils:*

$$u(t) \in \overline{\mathcal{W}^+} \quad \textit{for} \quad 0 \leq t < +\infty.$$

Proof. Two cases can be distinguished

4.3.1 Case I: the initial datum $u_0 \in \mathcal{W}_1^+$

As in the theorem's 4.1 proof, let the same sequences $\left\{w_j\right\}_{j=1}^{+\infty}$, $\{u_{0m}\}_{m=1}^{+\infty}$, and $\{u_m\}_{m=1}^{+\infty}$.

Let us multiply the two sides of (4.2) by $\alpha'_{mi}(t)$, and computing the summation over $i = 1, 2...m$, and then integrating with respect to time over the interval $[0, t]$, we have

$$\int_0^t \|u_{ms}(s)\|_2^2 \, ds + J(u_m(t)) = J(u_m(0)), \quad 0 \leq t < T_m, \tag{4.22}$$

here, T_m represents the maximum duration for which the solution $u_m(x, t)$ exists. Since J is continuous, then from (4.1), (4.3) and (4.22) we get

$$J(u_m(0)) \longrightarrow J(u_0) \quad \text{as } m \longrightarrow +\infty,$$

with $J(u_0) < d$ and

$$\int_0^t \|u_{ms}(s)\|_2^2 \, ds + J(u_m(t)) < d, \quad 0 \leq t < T_m, \tag{4.23}$$

for some m large enough. We shall show that

$$u_m(t) \in \mathcal{W}_1^+, \quad \forall t \geq 0, \tag{4.24}$$

for some m large enough. Indeed, suppose that (4.24) is not true and that t_* be the smallest time such that $u_m(t_*) \notin \mathcal{W}_1^+$. Then, by the continuity of $u_m(t)$, we have $u_m(t_*) \in \partial\mathcal{W}_1^+$. Therefore

$$J(u_m(t_*)) = d, \tag{4.25}$$

or

$$I(u_m(t_*)) = 0. \tag{4.26}$$

However, it is obvious that (4.25) could not arise from (4.23) whereas if (4.26) is verified then, through the formula (3.26), we have

$$J(u_m(t_*)) \geq \inf_{u\in\mathcal{N}} J(u) = d,$$

this contradicts (4.23). then (4.24) holds.

On the other hand, because of $u_m(t) \in \mathcal{W}_1^+$ so that $I(u_m(t)) > 0$ and

$$J(u_m(t)) \geq \frac{1}{p_+} I(u_m(t)) + \frac{1}{p_+^2} \int_\Omega |u_m(t)|^{p(x)}\, dx, \quad \forall t \in [0, T_m), \tag{4.27}$$

then through (4.23) we derive,

$$\int_\Omega |u_m(t)|^{p(x)}\, dx < p_+^2 d, \quad \text{and} \quad \int_0^t \|u_{ms}(s)\|_2^2\, ds < d, \tag{4.28}$$

for some m large enough and $t \in [0, T_m)$. Moreover, from the fact that $I(u_m(t)) > 0$ there is $0 < \delta < 1$ such that

$$\int_\Omega |u_m(t)|^{p(x)} \log|u_m(t)|\, dx \leq \delta \int_\Omega |\nabla u_m(t)|^{p(x)}\, dx$$

this combined with (4.27), we may write

$$\int_\Omega |\nabla u_m(t)|^{p(x)}\, dx \leq p_+ J(u_m(t)) + \int_\Omega |u_m(t)|^{p(x)} \log|u_{,m}(t)|\, dx - \frac{1}{p_+} \int_\Omega |u_m(t)|^{p(x)}\, dx$$

$$\leq p_+ J(u_m(t)) + \delta \int\limits_\Omega |\nabla u_m(t)|^{p(x)}\, dx - \frac{1}{p_+} \int\limits_\Omega |u_m(t)|^{p(x)}\, dx,$$

since $0 < \delta < 1$ we conclude from (4.23) and (4.28) that

$$\int\limits_\Omega |\nabla u_m(t)|^{p(x)}\, dx \leq C_d, \tag{4.29}$$

for every t belonging to the interval $[0, T_m)$, the aforementioned inequalities indicate that we can select $T_m = T$ for all m, with any positive value of T.

According to (4.28) and (4.29), it can be seen by proceeding in a similar approach to the proof of Theorem 1 yields a weak solution u for the problem (1.1) in the interval $[0, \infty]$.

4.3.2 Case II: the Initial Datum $u_0 \in \mathcal{W}_2^+$

In order to establish global time existence for the solution of problem (1.1), it is necessary to introduce the sequence

$$\{\gamma_m\}_{m=1}^{+\infty} \subset (0, 1)$$

with $\lim_{m\to+\infty} \gamma_m = 1$. And let us examine the following problem

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) = |u|^{p(x)-2} u \log|u|, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_{0m}(x), & x \in \Omega, \end{cases} \tag{4.30}$$

where $u_{0m} = \gamma_m u_0$. Owing to $I(u_0) \geq 0$ then from Lemma 3.2 we have

$$\lambda^* = \lambda^*(u_0) = \exp\left(\frac{\int_\Omega \lambda^{*p(x)-1} \left(|\nabla u_0|^{p(x)} - |u_0|^{p(x)} \log|u_0| \right) dx}{\int_\Omega \lambda^{*p(x)-1} |u_0|^{p(x)}\, dx} \right) \geq 1.$$

Therefore, we obtain

$$I(u_{0m}) = I(\gamma_m u_0) > 0 \quad \text{and} \quad J(u_{0m}) = J(\gamma_m u_0) < J(u_0) = d$$

it means by the definition of $\mathcal{W}_1^+$ that $u_{0m} \in \mathcal{W}_1^+$. Proceeding in the same way as in the previous subsection, leads to the problem (4.30) having a global weak solution u_m as follows

$$u_m \in L^\infty\left(0,T;W_0^{1,p(x)}(\Omega)\right), \quad u_{mt} \in L^{p'(x)}\left(0,T;W_0^{-1,p'(x)}(\Omega)\right) \cap L^2\left(0,T;L^2(\Omega)\right)$$

and

$$\int_\Omega u_{mt}(t)\,w\,dx + \int_\Omega |\nabla u_m(t)|^{p(x)-2}\nabla u_m(t)\nabla w\,dx = \int_\Omega |u_m(t)|^{p(x)-2}u_m(t)\log|u_m(t)|\,w\,dx,$$

for all $w \in W_0^{1,p(x)}(\Omega) \cap L^2(\Omega)$ and for a.e. $t > 0$ as well, we have

$$u_m(t) \in \overline{\mathcal{W}^+},$$

for all $t \in [0,+\infty)$, and

$$\int_0^t \|u_{ms}(s)\|_2^2\,ds + J(u_m(t)) \leq J(u_{0m}) < d, \quad t \in [0,+\infty).$$

The remaining part of the proof can be formulated in a similar way as earlier. □

4.4 Stability (Polynomial decay)

Research on the stability, global existence, and blow-up of solutions for this type of equation involves studying the qualitative behavior of solutions, investigating their long-term properties, and understanding the conditions under which solutions exist globally or blow up in finite time.

Theorem 4.3. *Assume that $u_0 \in X_0$, $p \in C\left(\overline{\Omega}\right)$ satisfying the condition $2 < p_- \leq p(x) \leq p_+ < p_-^*$ and the log-Hölder continuous condition (2.1). Then the local weak solution of problem satisfies the following energy inequality*

$$\int_0^t \|u_s(s)\|_2^2\,ds + J(u(t)) \leq J(u_0), \quad t \in [0,T_0). \tag{4.31}$$

where T_0 is a positive constant.

Proof. We demonstrate that the solution u fulfils the inequality of energy (4.31). To this end, we consider the positive continuous function $\theta \in C([0, T_0])$. Then, from (4.11) we get

$$\int_0^{T_0} \theta(t)\, dt \int_0^t \|u_{ms}(s)\|_2^2\, ds + \int_0^{T_0} J(u_m(t))\, \theta(t)\, dt = \int_0^{T_0} J(u_m(0))\, \theta(t)\, dt. \tag{4.32}$$

The term on the right side of (4.32) converges to

$$\int_0^{T_0} J(u_0)\, \theta(t)\, dt$$

as $m \to +\infty$. The lower semi-continuity of The second component on the left side of (4.32) pertaining to the weak topology of $W_0^{1,p(x)}(\Omega)$, means that

$$\int_0^{T_0} J(u(t))\, \theta(t)\, dt \leq \liminf_{m \to +\infty} \int_0^{T_0} J(u_m(t))\, \theta(t)\, dt.$$

so, we get

$$\int_0^{T_0} \theta(t)\, dt \int_0^t \|u_s(s)\|_2^2\, ds + \int_0^{T_0} J(u(t))\, \theta(t)\, dt \leq \int_0^{T_0} J(u_0)\, \theta(t)\, dt.$$

θ is arbitrary, then, it holds the inequality of energy

$$\int_0^t \|u_s(s)\|_2^2\, ds + J(u(t)) \leq J(u_0), \quad t \in [0, T_0].$$

□

Theorem 4.4. *Assume that* $u_0 \in \mathcal{W}^+$*. Then the global weak solution of the problem* (1.1) *satisfies the following estimate of energy*

$$\int_0^t \|u_s(s)\|_2^2 \, ds + J(u(t)) \le J(u_0), \quad a.e.\ t \in [0, +\infty). \tag{4.33}$$

Proof. According to (4.28) and (4.29), we can see by proceeding in a similar approach to the proof of Theorem 4.3 that (4.33) is fulfilled. □

Theorem 4.5. *Assume that* $u_0 \in \mathcal{W}^+$*. Then the decay of the solution is polynomial, as follows,*
(i) if $J(u_0) < M$*, then it holds the estimate*

$$\|u(t)\|_2 \le \|u_0\|_2 \left(\frac{p_-}{2\left(1 + \zeta (p_- - 2) \|u_0\|_2^{p_- - 2} t\right)} \right)^{1/(p_- - 2)}, \quad t \ge 0,$$

where $\zeta = \frac{a}{p_- R^{p_-}} \log \frac{M}{J(u_0)} > 0$*;*

(ii) if $J(u_0) = M$*, then there is a time* $t_\varepsilon > 0$ *such that the norm* $\|u(t)\|_2$ *satisfies the estimate*

$$\|u(t)\|_2 \le \|u(t_\varepsilon)\|_2 \left(\frac{p_-}{2\left(1 + \zeta_\varepsilon (p_- - 2) \|u(t_\varepsilon)\|_2^{p_- - 2} t\right)} \right)^{1/(p_- - 2)}, \quad t \ge t_\varepsilon,$$

where $\zeta_\varepsilon = \frac{a}{p_- R^{p_-}} \log \frac{M}{J(u(t_\varepsilon))} > 0$.

Proof.

Case $J(u_0) < M$: due to $u(t) \in \overline{\mathcal{W}_1^+}$, through (3.5) and the energy inequality we can conclude that

$$\int_\Omega |u_m(t)|^{p(x)} \, dx \le p_+^2 J(u(t)) \le p_+^2 J(u_0).$$

utilising (3.15), Proposition 2.2 and Lemma 3.3, because of $I(u(t)) \geq 0$ we have

$$\begin{aligned} I(u(t)) &\geq \frac{a}{R^{p_-}}\left(\log(R) - \frac{1}{p_-}\log\left(\int_\Omega |u|^{p(x)}\,dx\right)\right)\max\left\{\|u\|_{p(x)}^{p_-}, \|u\|_{p(x)}^{p_+}\right\} \\ &\geq \frac{a}{R^{p_-}}\left(\log(R) - \frac{1}{p_-}\log\left(p_+^2 J(u_0)\right)\right)\int_\Omega |u|^{p(x)}\,dx \\ &\geq \zeta\,\|u(t)\|_2^{p_-}, \end{aligned} \tag{4.34}$$

because of $p_- > 2$, where $\zeta = \frac{a}{p_- R^{p_-}} \log \frac{M}{J(u_0)} > 0$.

Alternatively, utilizing the initial equation of problem (1.1), we get

$$\int_t^T I(u(s))\,ds = -\int_t^T \int_\Omega u_s(s)\,u(s)\,dxds \leq \frac{1}{2}\|u(t)\|_2^2, \tag{4.35}$$

for all $t \in [0, T]$.

We Combine (4.28) and (4.29), resulting in

$$\int_t^T \|u(t)\|_2^{p_-}\,ds \leq \frac{1}{2\zeta}\|u(t)\|_2^2, \quad \forall t \in [0, T].$$

Tending T to $+\infty$ then by Lemma 4.1 (with $f(t)$ replaced by $\|u(t)\|_2^2$, $\sigma = (p_- - 2)/2$ and $\omega = 2\zeta\|u_0\|_2^{p_- - 2}$), we obtain the following decay estimate

$$\|u(t)\|_2 \leq \|u_0\|_2 \left(\frac{p_-}{2\left(1 + \zeta(p_- - 2)\|u_0\|_2^{p_- - 2}\, t\right)}\right)^{1/(p_- - 2)}, \quad t \geq 0.$$

Case $J(u_0) = M$: in view of $I(u(t)) > 0$ for all $t \geq 0$ we may write

$$\int_\Omega u_t(t)\,u(t)\,dx = -I(u(t)) < 0, \quad \forall t > 0.$$

And as $\|u_t(t)\|_2^2$ should be positive $\left(\|u_t(t)\|_2^2 > 0\right)$, for all $t > 0$. Since $t \to \int\limits_0^t \|u_s(s)\|_2^2\, ds$ is a continuous function and by the help of the energy inequality (4.33), thus, for any number $\varepsilon > 0$ small enough, there exists $t_\varepsilon > 0$ so that

$$J(u(t_\varepsilon)) \leq J(u_0) - \int\limits_0^{t_\varepsilon} \|u_s(s)\|_2^2\, ds = M - \varepsilon.$$

Finally, we consider the initial time t_ε and we proceed in the same way as in the case $0 < J(u_0) < M$ then, we find the estimate

$$\|u(t)\|_2 \leq \|u(t_\varepsilon)\|_2 \left(\frac{p_-}{2\left(1 + \zeta_\varepsilon (p_- - 2) \|u(t_\varepsilon)\|_2^{p_- - 2}\, t\right)} \right)^{1/(p_- - 2)}, \quad t \geq t_\varepsilon,$$

where $\zeta_\varepsilon = \frac{a}{p_- R^{p_-}} \log \frac{M}{J(u(t_\varepsilon))} > 0$. $\square$

Chapter 5

Instability and Weak solutions Blow-up

5.1 Introduction

In the study of differential equations or mathematical models, blow-up criteria frequently come up. For instance, a blow-up criteria in the context of partial differential equations refers to a condition that establishes whether an equation's solution will become unbounded or discontinuous in a finite amount of time. The behavior of solutions can be understood, and their stability can be predicted, using these criteria.

In this section, our objective is to establish the finite-time blow-up and instability results for weak solutions to problem (1.1), under the condition that the initial datum u_0 belongs to $\mathcal{W}^-$ and satisfies $J(u_0) \le M$. To accomplish this, we will rely on several lemmas, which are provided in references [9, 28].

5.2 Instability (Polynomial growth)

Theorem 5.1. *Let $u_0 \in \mathcal{W}^-$ with $J(u_0) \le 0$. Suppose that the local weak solution $u(x,t)$ of problem (1.1) which corresponds to u_0 and for which it holds the energy inequality*

$$\int_0^t \|u_s(s)\|_2^2 \, ds + J(u(t)) \le J(u_0), \quad \forall t \in [0,T) . \tag{5.1}$$

Then, the growth of the solution is polynomial as follows

$$\|u(t)\|_2^2 \ge \left(\frac{1}{\|u_0\|_2^{2-p_-} - \left(p_-/p_+^2 l_{2,p(x)}^{p_-}\right)(p_- - 2)\, t} \right)^{2/(p_- - 2)} .$$

Proof. We start by proving that $u(t) \in \mathcal{W}_1^-$, for all $t \geq 0$, when $u_0 \in \mathcal{W}_1^-$. Indeed, reasoning by absurd, we suppose that there exists a time $t_0 \in (0, T)$ for which

$$u(., t) \in \mathcal{W}_1^- \quad \text{for all } t \in [0, t_0), \quad \text{and} \quad u(., t_0) \in \partial\mathcal{W}_1^-,$$

so

$$I(u(t_0)) = 0 \quad \text{or} \quad J(u(t_0)) = d.$$

Nevertheless, owing to (3.27) and the energy inequality (5.1) it is obvious that the identity $J(u(t_0)) = d$ can not arise. However, if $I(u(t_0)) = 0$, it follows that $\min\left\{\|u\|_{p(x)}, \|u\|_{p(x)}^{p_+/p_-}\right\} \geq R > 0$ in view of Lemma 3.3. We conclude that $u(t_0) \in \mathcal{N}$ so, through the formula (3.26), we find $J(u(t_0)) \geq d$ which contracts the inequality of energy (5.1).

After that, weak solutions to problem (1.1) blow up at finite time. Thus we may assume that

$$\|u(t)\|_{p(x)} > 1. \tag{5.2}$$

since by the embedding $L^{p(x)}(\Omega) \hookrightarrow L^2(\Omega), p(x) > 2$.

Define the functional

$$\Gamma(t) = \int_0^t \|u(s)\|_2^2 \, ds + (T - t)\|u_0\|_2^2 \quad t \in [0, T]. \tag{5.3}$$

So we have

$$\Gamma'(t) = \|u(t)\|_2^2 - \|u_0\|_2^2 = \int_0^t \frac{d}{ds}\left(\|u(s)\|_2^2\right) ds = 2\int_0^t \int_\Omega u_s(s)\, u(s)\, dx ds. \tag{5.4}$$

From (5.4) and by Setting $u = w$ in (2.3) we get

$$\Gamma''(t) = 2\int_\Omega u_t(t)\, u(t)\, dx = -2\int_\Omega |\nabla u(t)|^{p(x)} dx + 2\int_\Omega |u(t)|^{p(x)} \log|u(t)|\, dx = -2I(u(t)). \tag{5.5}$$

By Proposition 2.2, (3.5), (5.2), and the inequality of energy (5.1), the formula (5.5) becomes

$$\begin{aligned}\Gamma''(t) &\geq -\frac{2}{\gamma}J(u(t)) + \frac{2}{\gamma p_+^2}\int_\Omega |u(t)|^{p(x)}\,dx \\ &\geq \frac{2}{\gamma}\int_0^t \|u_s(s)\|_2^2\,ds + \frac{2}{\gamma p_+^2}\|u(t)\|_{p(x)}^{p_-} - \frac{2}{\gamma}J(u_0).\end{aligned} \tag{5.6}$$

So, if $J(u_0) \leq 0$. we use the embedding $L^{p(x)}(\Omega) \hookrightarrow L^2(\Omega)$, $p(x) > 2$. Then (5.2), (5.4) and (5.6) give us

$$\begin{aligned}\Gamma''(t) &\geq \frac{2p_-}{p_+^2}\|u(t)\|_{p(x)}^{p_-} \\ &\geq \frac{2p_-}{p_+^2 l_{2,p(x)}^{p_-}}\left(\|u(t)\|_2^2\right)^{p_-/2} = \frac{2p_-}{p_+^2 l_{2,p(x)}^{p_-}}\left(\Gamma'(t) + \|u_0\|_2^2\right)^{p_-/2},\end{aligned}$$

since by the definition of γ, where $l_{2,p(x)}^{p_-}$ is the embedding constant. Therefore we can apply [28, Lemma 4.1] (with $\psi(t)$ replaced by $\Gamma'(t) + \|u_0\|_2^2$, $c = 2p_-/p_+^2 l_{2,p(x)}^{p_-}$ and $\sigma = p_-/2$) to obtain the estimate

$$\|u(t)\|_2^2 \geq \left(\frac{1}{\|u_0\|_2^{2-p_-} - \left(p_-/p_+^2 l_{2,p(x)}^{p_-}\right)(p_- - 2)\,t}\right)^{2/(p_- - 2)}.$$

□

5.3 Blow-up phenomena

The main result of this section is the following theorem

Theorem 5.2. *Let $u_0 \in \mathcal{W}^-$ with $J(u_0) \leq M$. Suppose that the local weak solution $u(x,t)$ of problem (1.1) which corresponds to u_0 and for which the energy inequality*

$$\int_0^t \|u_s(s)\|_2^2\, ds + J(u(t)) \leq J(u_0), \quad \forall t \in [0,T)\,.$$

holds. Then, it follow the assertions
(i) When $J(u_0) \leq 0$, The solution $u(x,t)$ blows up at finite time, so we have

$$\lim_{t\to T_*^-} \|u(t)\|_2^2 = +\infty, \quad where \quad T_* = \frac{p_+^2 l_{2,p(x)}^{p_-} \|u_0\|_2^{2-p_-}}{p_-(p_- - 2)}. \tag{5.7}$$

(ii) The solution $u(x,t)$ blows up at finite time when $0 < J(u_0) \leq M$, which means that there exists a time $T_* > 0$ for which

$$\lim_{t\to T_*^-} \|u(t)\|_2^2 = +\infty.$$

Proof. Three cases can be considered:
(i) Case $J(u_0) \leq 0$.
In the present case, we use the obtained estimate

$$\|u(t)\|_2^2 \geq \left(\frac{1}{\|u_0\|_2^{2-p_-} - \left(p_-/p_+^2 l_{2,p(x)}^{p_-}\right)(p_- - 2)\, t} \right)^{2/(p_- - 2)},$$

this demonstrates that

$$\lim_{t\to T_*^-} \|u(t)\|_2^2 = +\infty, \quad \text{where} \quad T_* = \frac{p_+^2 l_{2,p(x)}^{p_-} \|u_0\|_2^{2-p_-}}{p_-(p_- - 2)}.$$

(ii) Case $0 < J(u_0) < M$.
Since $u(t) \in \mathcal{W}_1^-$, for all $t \in [0,T]$, we have $I(u(t)) < 0$ this involves by Lemma 3.3 and (5.2) that

$$\min\left\{\|u\|_{p(x)}^{p_-}, \|u\|_{p(x)}^{p_+}\right\} = \|u\|_{p(x)}^{p_-} \geq R^{p_-} \quad \text{for all } t \in [0,T],$$

Then, through (5.6) by using the fact that $M - J(u_0) > 0$, and the definition of γ we have

$$\begin{aligned}\Gamma''(t) &\geq \frac{2}{\gamma}\int_0^t \|u_s(s)\|_2^2\, ds + \frac{2}{\gamma}\left(M - J(u_0)\right) \\ &\geq 2p_- \int_0^t \|u_s(s)\|_2^2\, ds + 2p_-\left(M - J(u_0)\right), \quad t \in [0,T]. \end{aligned} \tag{5.8}$$

So we get

$$\Gamma'(t) = \Gamma'(0) + \int_0^t \Gamma''(s)ds \geq 2p_-\left(M - J(u_0)\right)t \geq 0, \quad t \in [0,T]. \tag{5.9}$$

Therefore, (5.4) and (5.9) give us by using Hölder inequality that

$$\frac{1}{4}\left(\Gamma'(t)\right)^2 \leq \left(\int_0^t \int_\Omega u_s(s)\, u(s)\, dxds\right)^2 \leq \int_0^t \|u_s(s)\|_2^2\, ds \int_0^t \|u(s)\|_2^2\, ds, \tag{5.10}$$

for all $t \in [0,T]$.

We combine (5.3), (5.8) and (5.10), we find

$$\begin{aligned}\Gamma(t)\Gamma''(t) &\geq 2p_- \int_0^t \|u_s(s)\|_2^2\, ds \int_0^t \|u(s)\|_2^2\, ds + 2p_-\left(M - J(u_0)\right)\Gamma(t) \\ &\geq \frac{p_-}{2}\left(\Gamma'(t)\right)^2 + 2p_-\left(M - J(u_0)\right)\Gamma(t)\end{aligned}$$

for all $t \in [0,T]$. Consequently

$$\Gamma(t)\Gamma''(t) - \frac{p_-}{2}\left(\Gamma'(t)\right)^2 \geq 2p_-\left(M - J(u_0)\right)\Gamma(t) > 0. \quad \text{for all } t \in [0,T]$$

In accordance with [28, Lemma 4.2], there exists $T_* > 0$ such that

$$\lim_{t\to T_*^-} \Gamma(t) = +\infty,$$

which means that $\lim_{t\to T_*^-} \int\limits_0^t \|u(s)\|_2^2\, ds = +\infty$. And then, we obtain

$$\lim_{t\to T_*^-} \|u(t)\|_2^2 = +\infty.$$

(iii) Case $J(u_0) = M$.

Because $I(u(t)) < 0$ for all $t \geq 0$ this means that

$$\int\limits_\Omega u_t(t)\, u(t)\, dt = -\int\limits_\Omega |\nabla u(t)|^{p(x)}\, dx + \int\limits_\Omega |u(t)|^{p(x)} \log |u(t)|\, dx = -I(u(t)) > 0, \quad \forall t > 0.$$

And as $\|u_t(t)\|_2^2$ should be positive, for all $t > 0$. Since $t \to \int\limits_0^t \|u_s(s)\|_2^2\, ds$ is continuous function and by the help of the energy inequality (5.1), thus, for any number $\varepsilon > 0$ small enough, there exists $t_\varepsilon > 0$ such that

$$J(u(t_\varepsilon)) \leq J(u_0) - \int\limits_0^{t_\varepsilon} \|u_s(s)\|_2^2\, ds = M - \varepsilon$$

Finally, we consider the initial time, t_ε and we proceed in the same way as in the case $0 < J(u_0) < M$ then we obtain the finite blow up result

$$\lim_{t\to T_*^-} \|u(t)\|_2^2 = +\infty.$$

which complete the proof of the theorem. □

Conclusion

Researchers have made significant advances in understanding and controlling complex systems in their investigations of the stabilization of nonlinear evolutionary systems as well as the global existence and finite-time blow-up of solutions. These investigations have clarified the complex dynamics of nonlinear systems and prompted the creation of techniques to stabilize such systems under specific conditions.

These studies generally show that stabilization of nonlinear evolutionary systems can be accomplished successfully in many situations, but it can also present substantial problems because of the intrinsic complexity of these systems. According to research findings, stabilizing control rules can be created to guarantee the global existence of solutions, meaning that the system's solutions remain bounded over time.

It is crucial to remember that, in certain cases, solutions can nevertheless blow up in a finite amount of time. This suggests that there may exist trajectories for some system configurations where the solutions diverge to infinity within a finite time. Complex nonlinear processes like positive feedback, saturation, or unstable interactions between system components are frequently linked to finite-time blows-ups.

The research on global existence and the finite-time blow-up of solutions emphasizes the importance of doing a thorough analysis of system dynamics, looking into their mathematical characteristics, and coming up with the best possible control schemes. Theoretical frameworks for addressing these problems are provided by mathematical tools including stability analysis, dynamical systems theory, and control techniques.

Finally, studies on global existence, finite-time blow-up of solutions, and stability of nonlinear evolutionary systems are challenging and interesting subjects that continue to inspire research and the creation of new control strategies. The performance and reliability of nonlinear systems are improved by an understanding of these phenomena in a variety of application areas, including engineering, physics, biology, and many more.

Bibliography

[1] Aberqi, A., Bennouna, J., Benslimane, O., and Ragusa, M. A. On $p(z)$-laplacian system involving critical nonlinearities. *Journal of Function Spaces 2022* (2022).

[2] Acerbi, E., and Mingione, G. Regularity results for a class of functionals with non-standard growth. *Archive for Rational Mechanics and Analysis 156*, 2 (2001), 121–140.

[3] Alaoui, M. K., Messaoudi, S. A., and Khenous, H. A blow-up result for nonlinear generalized heat equation. *Computers & Mathematics with Applications 68*, 12 (2014), 1723–1732.

[4] Bialynicki-Birula, I., and Mycielski, J. Nonlinear wave mechanics. *Annals of Physics 100*, 1-2 (1976), 62–93.

[5] Biccari, U. Blow-up results for a logarithmic pseudo-parabolic $p(.)$-laplacian type equation. *arXiv preprint arXiv:2106.11620* (2021).

[6] Boschi, E. Recensioni: Jl lions-quelques méthodes de résolution des problèmes aux limites non linéaires. dunod, gauthier-vi;; ars, paris, 1969, 1971.

[7] Boudjeriou, T. On the diffusion $p(x)$-laplacian with logarithmic nonlinearity. *Journal of Elliptic and Parabolic Equations 6*, 2 (2020), 773–794.

[8] Buljan, H., Šiber, A., Soljačić, M., Schwartz, T., Segev, M., and Christodoulides, D. N. Incoherent white light solitons in logarithmically saturable noninstantaneous nonlinear media. *Physical Review E 68*, 3 (2003), 036607.

[9] Chen, H., Luo, P., and Liu, G. Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity. *Journal of Mathematical Analysis and Applications 422*, 1 (2015), 84–98.

[10] Cruz-Uribe, D. V., and Fiorenza, A. *Variable Lebesgue spaces: Foundations and harmonic analysis*. Springer Science & Business Media, 2013.

[11] Del Pino, M., and Dolbeault, J. Nonlinear diffusions and optimal constants in sobolev type inequalities: asymptotic behaviour of equations involving the $p-$laplacian. *Comptes Rendus Mathematique 334*, 5 (2002), 365–370.

[12] Diening, L., Harjulehto, P., Hästö, P., and Ruzicka, M. *Lebesgue and Sobolev spaces with variable exponents*. Springer, 2011.

[13] Diening, L., Nägele, P., and Růžička, M. Monotone operator theory for unsteady problems in variable exponent spaces. *Complex variables and elliptic equations 57*, 11 (2012), 1209–1231.

[14] Ding, H., and Zhou, J. Global existence and blow-up for a mixed pseudo-parabolic $p-$laplacian type equation with logarithmic nonlinearity. *Journal of Mathematical Analysis and Applications 478*, 2 (2019), 393–420.

[15] Drábek, P., and Pohozaev, S. I. Positive solutions for the p-laplacian: application of the fibrering method. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics 127*, 4 (1997), 703–726.

[16] Enqvist, K., and McDonald, J. Q-balls and baryogenesis in the mssm. *Physics Letters B 425*, 3-4 (1998), 309–321.

[17] Esquivel-Avila, J. A. The dynamics of a nonlinear wave equation. *Journal of mathematical analysis and applications 279*, 1 (2003), 135–150.

[18] Fan, X., Shen, J., and Zhao, D. Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$. *Journal of Mathematical Analysis and Applications 262*, 2 (2001), 749–760.

[19] Fan, X., and Zhao, D. On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. *J. Math. Anal. Appl 263*, 2 (2001), 424–446.

[20] Fan, X., Zhao, Y., and Zhao, D. Compact imbedding theorems with symmetry of strauss–lions type for the space $W^{1,p(x)}(\Omega)$. *Journal of Mathematical Analysis and Applications 255*, 1 (2001), 333–348.

[21] Gallouët, T. Discrete functional analysis tools for some evolution equations. *Computational Methods in Applied Mathematics 18*, 3 (2018), 477–493.

[22] Gao, J., Zhao, P., and Zhang, Y. Compact sobolev embedding theorems involving symmetry and its application. *Nonlinear Differential Equations and Applications NoDEA 17* (2010), 161–180.

[23] He, Y., Gao, H., and Wang, H. Blow-up and decay for a class of pseudo-parabolic $p-$laplacian equation with logarithmic nonlinearity. *Computers & Mathematics with Applications 75*, 2 (2018), 459–469.

[24] Ioku, N. The cauchy problem for heat equations with exponential nonlinearity. *Journal of Differential Equations 251*, 4-5 (2011), 1172–1194.

[25] Kováčik, O., and Rákosník, J. On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak mathematical journal 41*, 4 (1991), 592–618.

[26] Królikowski, W., Edmundson, D., and Bang, O. Unified model for partially coherent solitons in logarithmically nonlinear media. *Physical Review E 61*, 3 (2000), 3122.

[27] Ladyženskaja, O. A., Solonnikov, V. A., and Ural'ceva, N. N. *Linear and quasi-linear equations of parabolic type*, vol. 23. American Mathematical Soc., 1988.

[28] Le, C. N., and Le, X. T. Global solution and blow-up for a class of $p-$laplacian evolution equations with logarithmic nonlinearity. *Acta Applicandae Mathematicae 151*, 1 (2017), 149–169.

[29] Levine, H. A., and Payne, L. E. Nonexistence of global weak solutions for classes of nonlinear wave and parabolic equations. *Journal of Mathematical Analysis and Applications 55*, 2 (1976), 329–334.

[30] Linde, A. Strings, textures, inflation and spectrum bending. *Physics Letters B 284*, 3-4 (1992), 215–222.

[31] Martinez, P. A new method to obtain decay rate estimates for dissipative systems. *ESAIM: Control, Optimisation and Calculus of Variations 4* (1999), 419–444.

[32] Mizuta, Y., Ohno, T., Shimomura, T., and Shioji, N. Compact embeddings for sobolev spaces of variable exponents and existence of solutions for nonlinear elliptic problems involving the $p(x)-$laplacian and its critical exponent. *Ann. Acad. Sci. Fenn. Math 35*, 1 (2010), 115–130.

[33] Musielak, J. *Orlicz spaces and modular spaces*, vol. 1034. Springer, 2006.

[34] Payne, L. E., and Sattinger, D. H. Saddle points and instability of nonlinear hyperbolic equations. *Israel Journal of Mathematics 22*, 3-4 (1975), 273–303.

[35] Peral, I., and Vázquez, J. On the stability or instability of the singular solution of the semilinear heat equation with exponential reaction term. *Archive for rational mechanics and analysis 129*, 3 (1995), 201–224.

[36] Ragusa, M. A., and Tachikawa, A. Correction and addendum to "boundary regularity of minimizers of $p(x)-$energy functionals"[ann. inst. henri poincaré, anal. non linéaire 33 (2)(2016) 451–476]. In *Annales de l'Institut Henri Poincaré C, Analyse non linéaire* (2017), vol. 34, Elsevier, pp. 1633–1637.

[37] Ruzicka, M. *Electrorheological fluids: modeling and mathematical theory*. Springer, 2007.

[38] Tan, Z. Global solution and blow-up of semilinear heat equation with critical sobolev exponent. *Commun. in Partial Differential Equations* (2001).

[39] Tartar, L. *An introduction to Sobolev spaces and interpolation spaces*, vol. 3. Springer Science & Business Media, 2007.

[40] Wang, H., and He, Y. On blow-up of solutions for a semilinear parabolic equation involving variable source and positive initial energy. *Applied Mathematics Letters 26*, 10 (2013), 1008–1012.

[41] Zeng, F., Deng, Q., and Wang, D. Global existence and blow-up for the pseudo-parabolic $p(x)-$laplacian equation with logarithmic nonlinearity. *Journal of Nonlinear Mathematical Physics* (2022), 1–17.